EE698G – Assignment 1

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1

Let P is the set of mutually orthogonal non-zero vectors.

$$P = \{p_i, i \in [1, n] \quad \text{and} \ < p_i,\ p_j > \ = 0 \,\forall\, i \neq j\} \tag{1}$$

Now, assume P is linearly dependent.

$$p_j = \sum_i a_i p_i, i \in [1, n] - \{j\}, j \in [1, n], \text{ where } a_i \neq 0 \tag{2}$$

Take inner product with any p_k

$$< p_j,\ p_k > = \sum_i a_i < p_i,\ p_k >,\ k \ \in [1, n] \tag{3}$$

from equation (1) and (3),

$$\Rightarrow a_i = 0 \tag{4}$$

This contradicts our assumption of S being linearly dependent. So S is linearly independent. Hence, orthogonality implies linear independence.

2

Measurements(X) can be represented in the terms of Heights(H) and errors(e)

$$X_1 \ = 1 * H_A + 0 * H_B + 0 * H_C + \ e_1 \ = 24.64\, m, \tag{5}$$
$$X_2 \ = 0 * H_A + 1 * H_B + 0 * H_C \ + e_2 \ = 38.80\, m, \tag{6}$$
$$X_3 \ = 0 * H_A + 0 * H_B + 1 * H_C \ + e_3 \ = 48.30\, m, \tag{7}$$
$$X_4 \ = (-1) * H_A + 1 * H_B + 0 * H_C \ + e_4 \ = 14.22\, m, \tag{8}$$
$$X_5 = (-1) * H_A + 0 * H_B + 1 * H_C + \ e_5 \ = 23.55\, m, \tag{9}$$
$$X_6 \ = 0 * H_A + (-1) * H_B + 1 * H_C + \ e_6 \ = 9.5\, m \tag{10}$$

Now, converting these values to a system of matrices.

$$X = \left\{ \begin{matrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{matrix} \right\} \quad A = \left\{ \begin{matrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \\ -1\ 1\ 0 \\ -1\ 0\ 1 \\ 0\ -1\ 1 \end{matrix} \right\} \quad H = \left\{ \begin{matrix} H_A \\ H_B \\ H_C \end{matrix} \right\} \quad e = \left\{ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} \right\}$$

$$X = A\,H \,+\, e$$

$$\Rightarrow\; e \;= X \,-\, A\,H$$

from least square method,

$$\Rightarrow\; H \;= (\,A^T\,A)^{\,-1}\,A^T\,X$$

Using MATLAB function pinv(A)

$$(\,A^T\,A)^{\,-1}\,A^T = \left\{ \begin{array}{cccccc} 0.50 & 0.25 & 0.25 & -0.25 & -0.25 & 0.00 \\ 0.25 & 0.50 & 0.25 & 0.25 & -0.00 & -0.25 \\ 0.25 & 0.25 & 0.50 & -0.00 & 0.25 & 0.25 \end{array} \right\}$$

$$\Rightarrow\; H \;= \left\{ \begin{array}{c} 24.6525 \\ 38.8150 \\ 48.2725 \end{array} \right\}$$

$\Rightarrow H_A = 24.65\,m,\; H_B = 38.82\,m\;and\;H_B = 48.27\,m$

3

$$Q \;= \left\{ \begin{array}{cc} 1.0 & 0.5 \\ 0.5 & 1.0 \end{array} \right\}$$

for maximizing $(\,x^T\,Q\,x)$, ensuring $\|x\| = 1.$ maximizing the following Lagrangian

$$L \;= x^T\,Q\,x - \lambda\,(\,x^T x - 1) \tag{11}$$

$$\frac{\partial L}{\partial x} \;= 0,$$

$$\Rightarrow Qx = \lambda x$$

Required vector will be the eigen vector corresponding to the maximum eigen value.

$$det(\,Q - \lambda I) \;\; = 0$$

$$\Rightarrow (\,\lambda - 1.5)\,(\,\lambda - 0.5) \;= 0$$

$\therefore$ The required eigenvalue is λ = 1.5 and corresponding eigenvector is

$$x \;= \left\{ \begin{array}{c} 0.7071 \\ 0.7071 \end{array} \right\}$$

4

4.1 Part a

For the given equation to be a valid pdf, it should integrate to 1 in the limit of $-\infty$ to ∞

$$I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})} dx$$

let $w = \frac{x-\mu}{\sigma}$

$$\Rightarrow I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-\frac{w^2}{2})} dw$$

take square,

$$I^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-\frac{x^2}{2})} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-\frac{y^2}{2})} dy$$
$$\Rightarrow I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(-\frac{x^2+y^2}{2})} dx dy$$

put $x = r\cos\theta$ and $y = r\sin\theta$

$$I^2 = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{(-\frac{r^2}{2})} r dr d\theta$$

Put $u = \frac{r^2}{2} \Rightarrow du = rdr$

$$I^2 = \frac{1}{2\pi} \int_{0}^{2\pi} (\int_{0}^{\infty} e^{(-u)} du) d\theta$$
$$\Rightarrow I^2 = \frac{1}{2\pi} \int_{0}^{2\pi} (\lim_{b\to\infty} [e^{-u}]_0^b) d\theta$$
$$\Rightarrow I^2 = \frac{1}{2\pi} \inf_0 2\pi - (0-1) d\theta = 1$$

Hence $I^2 = 1$. So $I = \pm 1$. Since I cannot be negative, so $I = 1$

4.2 Part b

$$f_X(\mu + x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{(x)^2}{\sigma^2})}$$
$$= f_X(\mu - x)$$

$\therefore$ f is symmetric about μ

4.3 Part c

4.3.1 Mean

$$mean = E(x) \tag{12}$$
$$mean = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})} dx$$

let $z = x + \mu$

$$\Rightarrow mean = \int_{-\infty}^{\infty} (z+\mu) \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{z^2}{2\sigma^2})} dzz$$

replace variable z by x

$$\Rightarrow mean = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{z^2}{2\sigma^2})} dx + \int_{-\infty}^{\infty} \mu \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{x^2}{2\sigma^2})} dx \tag{13}$$

Now,

$$I_1 = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{z^2}{2\sigma^2})} dx$$

$$\Rightarrow I_1 = \int_{-\infty}^{0} x \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{z^2}{2\sigma^2})} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{z^2}{2\sigma^2})} dx$$

swapping the limits of first part

$$I_1 = -\int_{0}^{-\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

put x = -x in the first part

$$I_1 = \int_{0}^{\infty} (-x) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(-x)^2}{2\sigma^2}} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$\Rightarrow I_1 = -\int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 0 \tag{14}$$

Thus,

$$mean = 0 + \int_{-\infty}^{\infty} \mu \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

replace $w = \sigma x$

$$mean = \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{2\pi}} e^{-} \frac{w^2}{2} dw$$

from Part a,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-} \frac{w^2}{2} dw = 1$$

$$\Rightarrow mean = E[x] = \mu$$

4.3.2 Variance

$$Variance = E[(x - E[x])^2] \tag{15}$$

$$Variance = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e - \frac{(x-\mu)^2}{2\sigma^2} dx$$

$$\Rightarrow \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \sigma\sqrt{2} \int_{-\infty}^{\infty} (\sigma\sqrt{2}x)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\sigma\sqrt{2}x)^2}{2\sigma^2}} dx$$

$$= \sigma^2 \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} x^2 e^{-x^2} dx$$

Put $x = \sqrt{t}$

$$\Rightarrow Variance = \sigma^2 \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} (\sqrt{t})^2 (2\sqrt{t})^{-1} e^{-t} dt = \sigma^2 \frac{4}{\sqrt{\pi}} \frac{1}{2} \int_{0}^{\infty} t^{\frac{3}{2}-1} e^{-t} dt$$

$$\Rightarrow Variance = \sigma^2 \frac{4}{\sqrt{\pi}} \frac{1}{2} \frac{\sqrt{\pi}}{2} = \sigma^2$$

5

5.1 Part a

Let the pdf of X (Uniform distribution) be denoted by $f_X(x)$ such that,

$$f_X(x) = \begin{cases} 1 & \forall\, x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

and let $z = g(x) = log(1/x)$
$\Rightarrow g^{-1}(z) = e^{-z}$
and $\frac{\partial(g^{-1}(z))}{\partial x} = -e^{-z}$
hence, $f_Z(z) = f_X(e^z))| - e^{-z}|$ where $e^{-z} \in [0, 1]$ and 0 otherwise $\Rightarrow z \in [0, \infty]$
hence

$$f_Z(z) = \begin{cases} e^{-z} & \forall\, z \in [0, \infty] \\ 0 & \text{otherwise.} \end{cases}$$

5.2 Part b

let $z = g(x) = e^x$
$\Rightarrow g^{-1}(z) = log(z)$
and $\frac{\partial(g^{-1}(z))}{\partial x} = \frac{1}{z}$
hence, $f_Z(z) = f_X(log(z))|\frac{1}{z}|$ where $log(z) \in [0, 1]$ and 0 otherwise $\Rightarrow \in [1, e]$
hence

$$f_Z(z) = \begin{cases} \frac{1}{z} & \forall\, z \in [1, e] \\ 0 & \text{otherwise.} \end{cases}$$

5.3 Part c

let Z = X+Y,
$f_Z(z) = f_X(x) * f_Y(y)$, where * denotes convolution and f denotes pdf

$$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

5.3.1 $z \leq 0$

$$f_Z(z) = 0$$

5.3.2 $0 < z \leq 1$

for z < x,
$f_Y(z - x) = 0 \Rightarrow f_Z(z) = 0$
for x < 0,
$f_X(x) = 0 \Rightarrow f_Z(z) = 0$

so the limits of integration will be 0 to z
and for $z \geq x$, $f_Y(z-x) = 1$

$$\Rightarrow f_Z(z) = \int_0^z 1\,dx = z$$

5.3.3 $1 < z \leq 2$

Similarly,

$$f_Z(z) = \int_{z-1}^1 1\,dx = 2 - z$$

5.3.4 $z > 2$

$$f_Z(z) = 0$$

$$\Rightarrow f_Z(z) = \begin{cases} z & \text{for } 0 < z < 1 \\ 2 - z & \text{for } 1 \leq z < 2 \\ 0 & \text{otherwise.} \end{cases}$$

6

6.1 Part a

L.H.S

$$P(A \cap B|C)$$
$$\Rightarrow \frac{P(A \cap B \cap C)}{P(C)}$$

R.H.S

$$P(A|B \cup C)P(B|C)$$
$$\Rightarrow \frac{P(A \cap B \cap C)}{\cancel{P(B \cap C)}} \frac{\cancel{P(B \cap C)}}{P(C)}$$

because $P(B \cap C) > 0$

$$\Rightarrow \frac{P(A \cap B \cap C)}{P(C)} = \textbf{L.H.S}$$

6.2 Part b

Let us assume that the part b is correct.
$P(A \cup B|C) = P(A|C)P(B|C)$
$\Rightarrow \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C)}{P(C)} \frac{P(B \cap C)}{P(C)}$
A and B are independent.
So $(A \cap C) and (B \cap C)$ will also be independent. i.e. $P(A \cap C)P(B \cap C) = P(A \cap B \cap C)$
$\Rightarrow P(C)^2 = P(C)$
So part b will only be true if $P(C) = 0 \; or \; 1.$ or A,B and C all three are independent.
Thus part B is not correct according to given information.

7

Let us assume that the direction along which there is maximum variance is given by a d dimensional unit vector a. Thus, the projections of the points along this direction are:

$$p_i = a \cdot x_i$$

$$\Rightarrow p_i = a^T x_i$$

$$\text{variance along the line } Var(\ \{p_i\}_{i=1}^n) \ = \frac{1}{n} \sum_{i=1}^{n} (\, p_i - p_\mu)^{\,2}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\, a^T (x_i - \mu))^{\,2}$$

$$= \frac{1}{n} \sum_{i=1}^{n} (\, a^T (x_i - \mu)(x_i - \mu)^T a)$$

$$= \frac{1}{n} a^T (\sum_{i=1}^{n} (\, (x_i - \mu)(x_i - \mu)^T) a$$

$$= a^T (\frac{1}{n} \sum_{i=1}^{n} (\, (x_i - \mu)(x_i - \mu)^T) a$$

$$\Rightarrow Var(\ \{p_i\}_{i=1}^n) \ = a^T R a$$

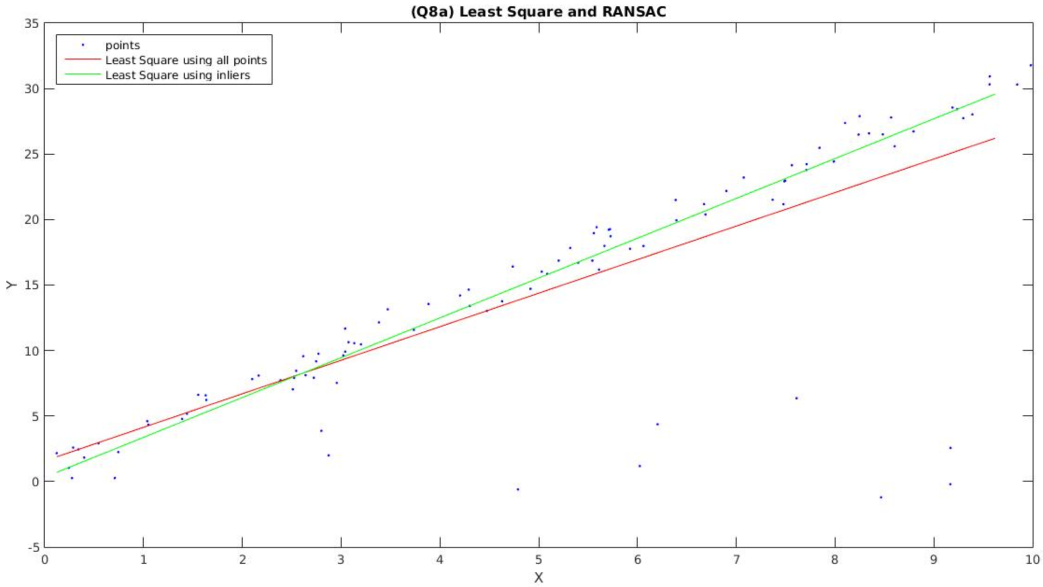
where $R = (\frac{1}{n} \sum_{i=1}^{n} (\, (x_i - \mu)(x_i - \mu)^T)$ (sample covariance matrix)
Size of the R = $d \times d$ where d = dimension of the points $x_1, ..., x_n$

Now, the problem is to maximise Variance ($a^T R a$) such that $||a|| = 1$
Similar to question 3, first Principal component is **eigenvector corresponding to the maximum eigenvalue**.
Total no of principal components are equal to the number of **non zero eigen values**.

8

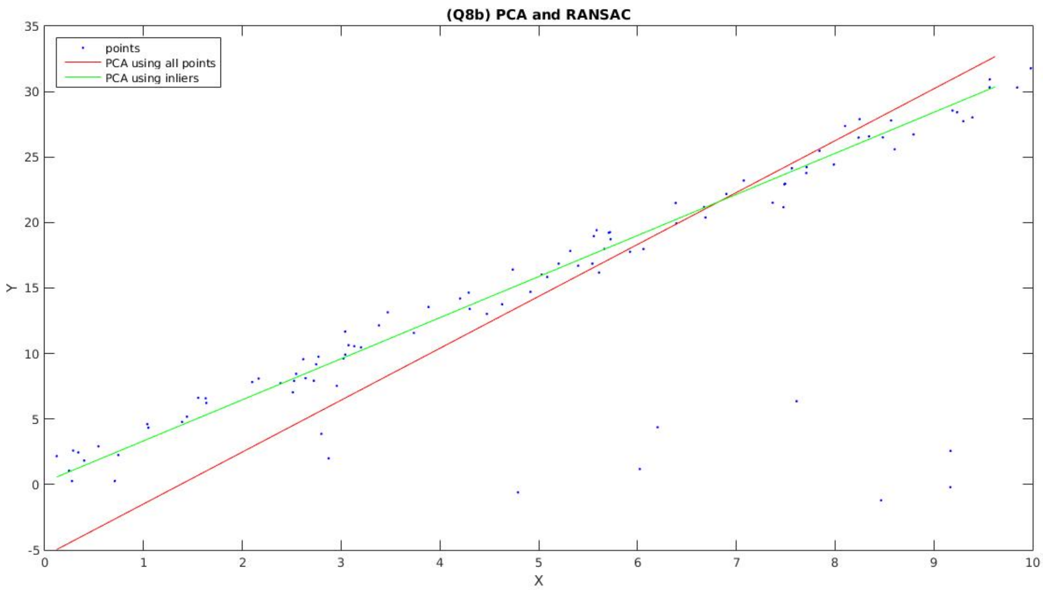
8.1 Part a

It can be easily seen that Least square fit of the whole points is not as good as Least square fit(OLS) of the inliers points obtained from RANSAC.



8.2 Part b

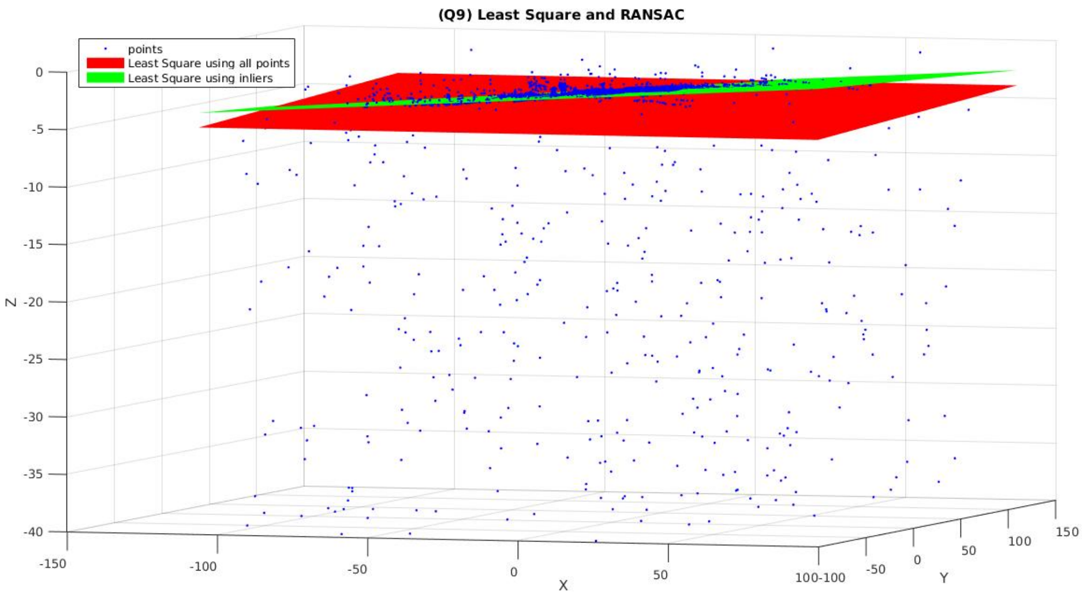
Principal Component analysis of all the points is not acceptable while PCA of inliers points is very good.



For given data set we can clearly say that,
$OLS of inliers \simeq PCA of inliers > OLS of all points > PCA of all points$

9

It can be easily seen that Least square fit of all the points is not as good as Least square fit(OLS) of the inliers points obtained from RANSAC.



Source: math stackexchange for some Tex commands in Question 4