23/4/2017

Practice Problems — Hints

Q1) Radar detection:

let $A = \{$ an aircraft is present $\}$.

$B = \{$ the radar registers presence of a aircraft $\}$

$$P(\text{false alarm}) = P(A^c \cap B) = \underset{0.95}{P(A^c)}\, \underset{0.01}{P(B/A^c)} = 0.095$$

$$P(\text{miss detection}) = P(A \cap B^c) = \underbrace{\underset{0.05}{P(A)}\, \underset{0.01}{P(B^c/A)}}_{\text{(given)}} = 0.0005.$$

Q2) Transformation of random variables:

Let 'X' be the speed & $Y = g(X)$ be the duration

$$\Rightarrow \quad g(x) = \left(\frac{180}{x}\right) \qquad \left(f_Y(y) = ?\right)$$

first Calculate $F_Y(y) = P(Y \leq y) = P\left(\frac{180}{X} \leq y\right) = P\left(X \geq \frac{180}{y}\right)$

To calculate $P\left(X \geq \frac{180}{y}\right)$ — i.e. find the CDF of 'X'.

Note: Given X is uniformly distributed in $[30 \;\; 60]$ miles/hour.

You should get $F_Y(y) = \begin{cases} 0 & \text{if } y \leq 3 \\ 2 - 6/y & 3 \leq y \leq 6 \\ 1 & 6 \leq y. \end{cases} \longrightarrow (1)$

differentiate (1) to get $f_Y(y) \Rightarrow f_Y(y) = \begin{cases} 0 & y \leq 3 \\ 6/y^2 & 3 \leq y \leq 6 \\ 0 & 6 \leq y. \end{cases}$

3. A matrix (square) is said to be diagonalizable if it is similar to a diagonal matrix.

(Note ie A, B are said to be similar if $\exists\, T \ni A = TBT^{-1}$)

Yes symmetric matrix is always diagonalizable since the eigen vector matrix of any symmetric matrix is always a orthogonal.

The following statements are equivalent (always holds for symmetric matrix)

(i) Geometric multiplicity = Algebraic multiplicity ~~for sym~~ a sym

(ie the dimension of eigen space corresponding to a repeated eigen value is equal to the no. of times that eigen value is repeated).

(ii) Eigen vectors for a symmetric matrix are always orthogonal.

(iii) ~~Therefore~~ ~~eve~~ Since (AM = GM) we can always choose orthogonal eigen vectors from the eigen space (corresponding to the repeated eigen value).

$\therefore$ Always any symmetric matrix is diagonalizable

ie $= U \Lambda U^T$

U: eigenvector matrix

Λ: diagonal matrix formed with eigen values.

Proof Proof follows from "Schur's Lemma".

i.e If 'A' is a square matrix then $\exists$ an upper triangular matrix $'T' = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ & \ddots & & \\ 0 & & & T_{nn} \end{bmatrix}$ and a unitary matrix 'U' $(U^H U = I = U U^H)$

$\ni \quad A = U T U^H$

Now when $A = A^H \Rightarrow U T U^H = U T^H U \Rightarrow T = T^H$.

for $T = T^H \Rightarrow T$ should be ~~sq~~ diagonal with real elements in the diagonal.

($\therefore$ ~~when~~ Any symmetric matrix is diagonalizable)

4) a) Let $\lambda_1, \lambda_2, \ldots \lambda_n$ are distinct eigen values of A

$\Rightarrow \quad A \lambda_i = \lambda_i \bar{v}_i$

L.I of eigen vectors (are independent) $\Rightarrow \quad c_1 \bar{v}_1 + \cdots + c_n \bar{v}_n = 0$ (1) (when $= 0$ holds only when $\forall c_i = 0$)

multiply (1) with $(A - \lambda_2 I)(A - \lambda_3 I) \cdots (A - \lambda_n I)$

$\Rightarrow$ we get $\quad c_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_n) \bar{v}_1 = 0$

$\Rightarrow \because$ eigen values are distinct $\Rightarrow \boxed{c_1 = 0}$ $\quad$ $||^{ly}$ holds ~~for~~ $\forall c_i$

5) Already discussed in Class.

6) Graph SLAM:

$$\cancel{\arg}\quad X^*, L^* = \underset{X,L}{\operatorname{argmin}} \left\{ \sum_{i=1}^{M} \| f_i(x_{i-1}, u_i) - x_i \|^2_{\Lambda_i} + \sum_{k=1}^{K} \| h_k(x_{i_k}, l_{j_k}) - z_k \|^2_{\Gamma_k} \right. \quad \rightarrow (1)$$

X - entire trajectory

L - Landmarks.

Z - Measurements (landmark).

x_i - odometry / scan matching.

$\Rightarrow$ linearize ~~both~~ all the non-linear functions (i.e 1st order approximation of Taylor series)

Hints

a) collect all the unknown parameters into a vector i.e 'θ'.

b) " " Jacobians into a large sparse matrix - '$\underline{A}$'.

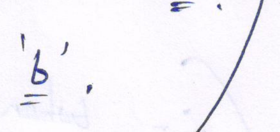
c) " " constants into a vector i.e '$\underline{b}$'.

$$\Rightarrow (1) \equiv \theta^* = \underset{\theta}{\operatorname{argmin}} \| A\theta - b \|^2.$$

8) $$P(\text{faulty} \mid \text{value} < 1m) = \frac{P(\text{value} < 1m \mid \text{faulty}) \times P(\text{faulty})}{P(\text{value} < 1m)}$$

$P(\text{faulty}) = 0.01$.

$P(\text{value} < 1M \mid \text{faulty}) = 1$. $P(\text{value} < 1M \mid \text{not faulty}) = 1/3$.



$P(\text{value} < 1) = P(v < 1 \mid \text{faulty}) + P(v < 1 \mid \text{not faulty})$.

Q-9) Given environment is linear.

$\Delta t = 1$ (for simplicity).

Acceleration $a \sim \mathcal{N}(0, 1)$.

a) If $[x_t] \rightarrow$ the position of car at time 't' is only one involved in state vector then

$$x_t = x_{t-1} + \left(\dot{x}_{t-1}\Delta t + \tfrac{1}{2}\ddot{x}_{t-1}\Delta t^2\right) \longrightarrow \left(\text{from linear dynamics}\right)$$

$$= x_{t-1} + \left(\left(\dot{x}_{t-2} + \ddot{x}_{t-2}\Delta t\right)(\Delta t) + \tfrac{1}{2}\ddot{x}_{t-1}\Delta t^2\right)$$

Clearly 'x_t' depends not only on the info available at time $(t-1)$ but also info at time $(t-2) \rightarrow$ this violates Markovian.

Let see what if state vector is $[x_t, \dot{x}_t]$.

$$\Rightarrow \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} = \begin{pmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{pmatrix} + \begin{pmatrix} \dot{x}_{t-1}\Delta t + \frac{1}{2}\ddot{x}_{t-1}\Delta t^2 \\ \ddot{x}_{t-1}\Delta t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} = \begin{pmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{pmatrix} + \begin{pmatrix} \dot{x}_{t-1} + \frac{1}{2}\ddot{x}_{t-1} \\ \ddot{x}_{t-1} \end{pmatrix}.$$

$$\Rightarrow \underbrace{\begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix}}_{\text{State at time 't'}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{pmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{pmatrix}}_{\text{State at time } (t-1)} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \underbrace{\ddot{x}_{t-1}}_{\text{Control at time } (t-1)}$$

(Clearly ~~the~~ Markovian assumption ~~is~~ holds here)

∴ Minimal state vector is $[x_t, \dot{x}_t]$.

b) let $X_t = \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix}$, $X_{t-1} = \begin{pmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{pmatrix}$.

$$\Rightarrow \quad X_t = A X_{t-1} + B a_{t-1}.$$

where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$$a_{t-1} \sim \mathcal{N}(0, 1).$$

let $X_{t-1} \sim \mathcal{N}(\mu_{t-1}, \Sigma_{t-1})$.

$$\Rightarrow X_t \sim \mathcal{N}(A\mu_{t-1}, A\Sigma_{t-1}A^T + BB^T).$$

c) Correlation b/w x_t & $\dot{x}_t$ is $E[x_t \dot{x}_t]$.

We generally start with 0 correlation between the state vector components $\Sigma_0 = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$

But the correlation between the components grows and the covariance matrix starts to fill

$$\Sigma_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} [1/2 \quad 1]$$

$$= \begin{bmatrix} \sigma_1^2 + \sigma_2^2 + 1/4 & \sigma_2^2 + 1/2 \\ \sigma_2^2 + 1/2 & \sigma_2^2 + 1 \end{bmatrix}$$

As $t \to \infty$ correlation $\to \infty$ and variance of components of state vectors also $\to \infty$

d) Given $z_t = x_t + n_t$ where $n_t \sim \mathcal{N}(0, 10)$

$$E[z_t] = x_t \rightarrow \text{true location at time } 't'.$$

$$\Rightarrow z_t \sim \mathcal{N}(\underbrace{x_t}_{\text{true location}}, 10).$$

Q-10)

$$w_t^{(i)} = \frac{p(x_{1:t}^{(i)} \mid z_{1:t}, u_{1:t-1}) \longrightarrow \text{Target distribution}}{\pi(x_{1:t}^{(i)} \mid z_{1:t}, u_{1:t-1}) \longrightarrow \text{proposal distribution}}$$

($i \rightarrow$ particle)

$\Downarrow$ (Bayes rule).

$$\Rightarrow w_t^{(i)} = \frac{\eta\, p(z_t \mid x_{1:t}^{(i)}, z_{1:t-1})\, p(x_t^{(i)} \mid x_{t-1}^{(i)}, u_{t-1})}{\pi(x_t^{(i)} \mid x_{1:t-1}^{(i)}, z_{1:t}, u_{1:t-1})}$$

$$\times \underbrace{\frac{p(x_{1:t-1}^{(i)} \mid z_{1:t-1}, u_{1:t-2})}{\pi(x_{1:t-1}^{(i)} \mid z_{1:t-1}, u_{1:t-2})}}_{w_{t-1}^{(i)}}$$

$$\therefore w_t^{(i)} = \frac{\eta\, p(z_t \mid x_{1:t}^{(i)}, z_{1:t-1})\, p(x_t^{(i)} \mid x_{t-1}^{(i)}, u_{t-1})}{\pi(x_t^{(i)} \mid x_{1:t-1}^{(i)}, z_{1:t}, u_{1:t-1})} \times w_{t-1}.$$

where $\eta = 1 / p(z_t \mid z_{1:t-1}, u_{1:t-1})$.

If the proposal distribution is chosen to be the motion model

i.e $p(x_t | x_{t-1}, u_{t-1})$ then

$$w_t^{(i)} = \frac{\eta\, p(z_t / x_{1:t}^{(i)}\ z_{1:t-1})\ \cancel{p(x_t^{(i)} / x_{t-1}^{(i)}, u_{t-1})}}{\cancel{p(x_t^{(i)} / x_{t-1}^{(i)}\ u_{t-1})}}.$$

$$= \eta\ p\left(z_t / x_{1:t}^{(i)}\ z_{1:t-1}\right)$$

$$\propto \cancel{\eta}\ p\left(z_t / x_t\right) \quad \left(\because \text{'}x_t\text{' captures the whole information}\right)$$

$M = 1$

In this case you have only one particle, and once we executed the motion model (i.e propogating the particle through motion model) in the resampling step this one particle is deterministically accepted irrespective of its importance weight. Therefore measurement plays no role in such cases.

i.e $p(\text{draw } x_t^{[m]}) = \dfrac{w_t^{[m]}}{w_t^{[m]}} = 1$

$\therefore$ As $M \uparrow$, this loss of degrees of freedom & dimensionality becomes less pronounced. (In general if M particles are used $\overline{w}_t^{i} \in R^M$ After normalization weights reside in $R^{(M-1)}$).