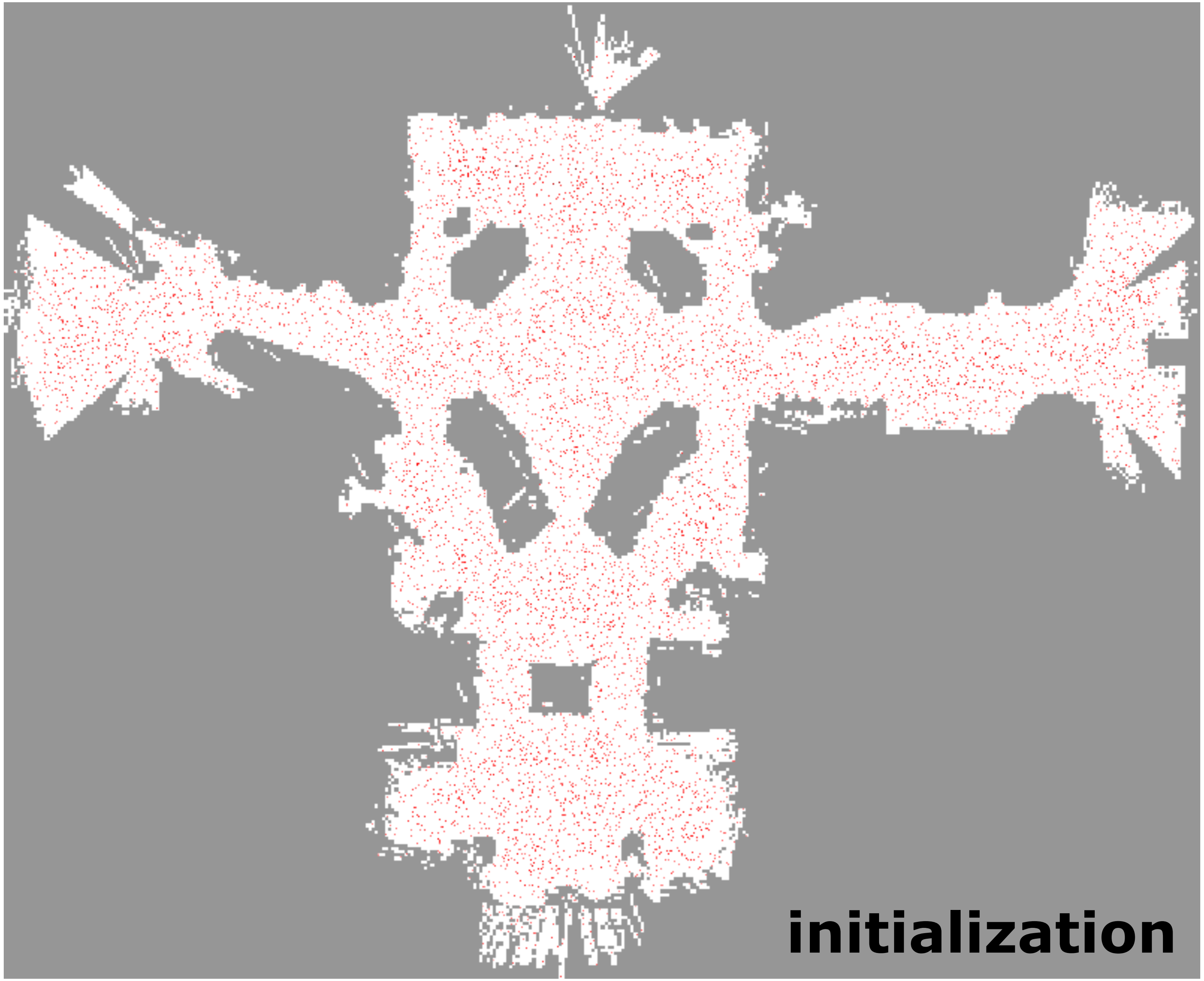
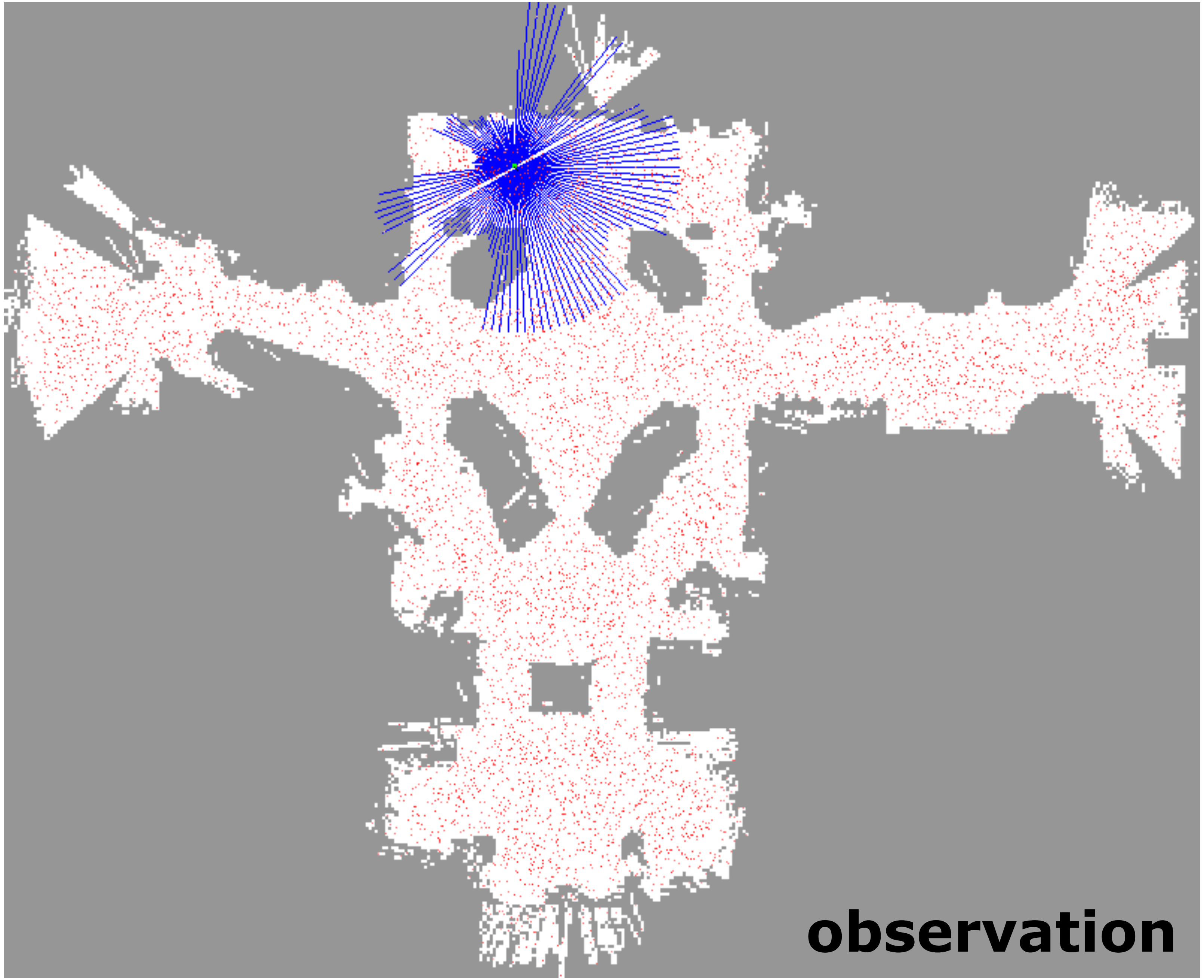
Particle Filter

Dr. Gaurav Pandey

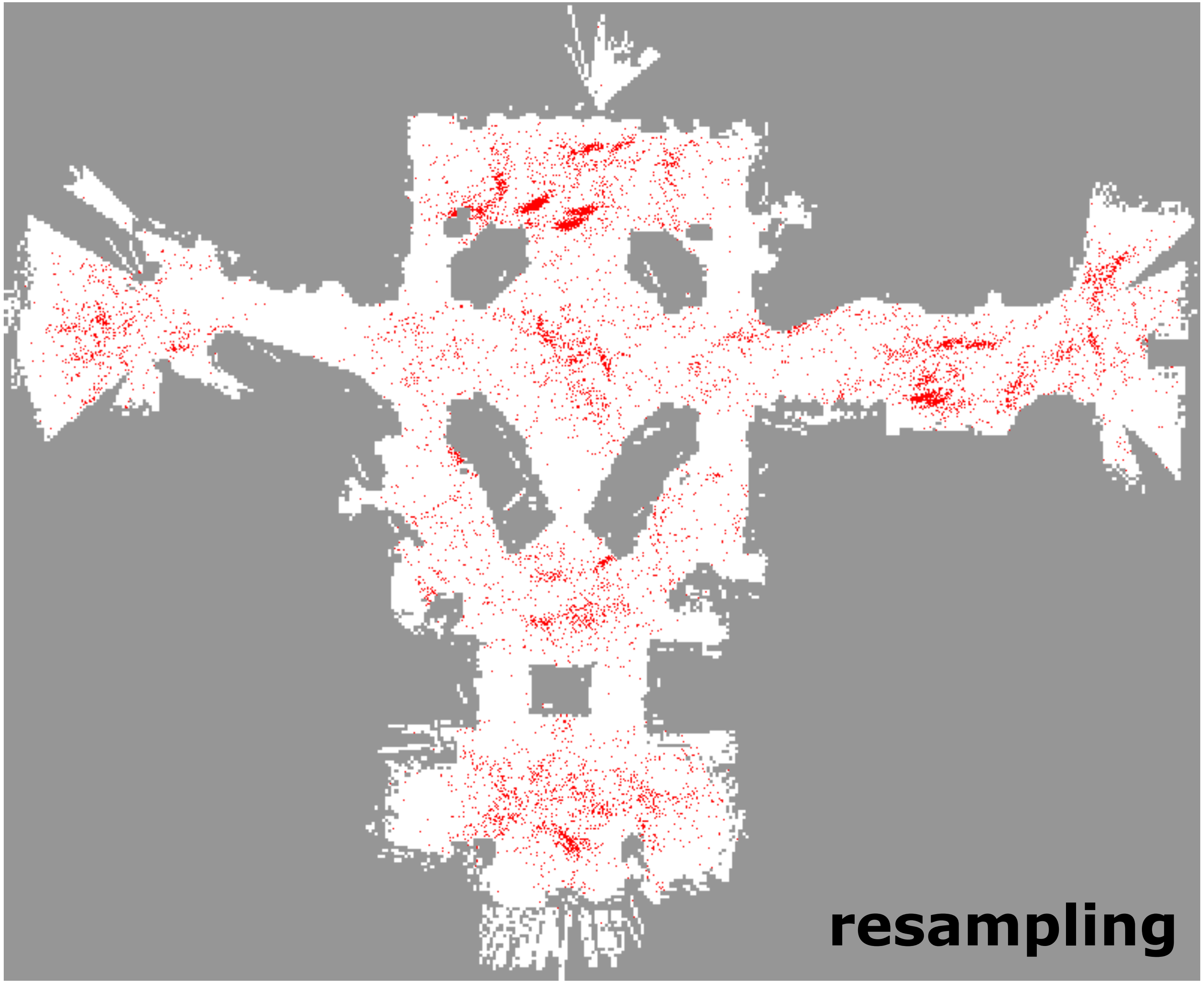
Reference: Probabilistic Robotics by Sebastian Thrun and Wolfram Burgard, Deiter Fox. MIT Press



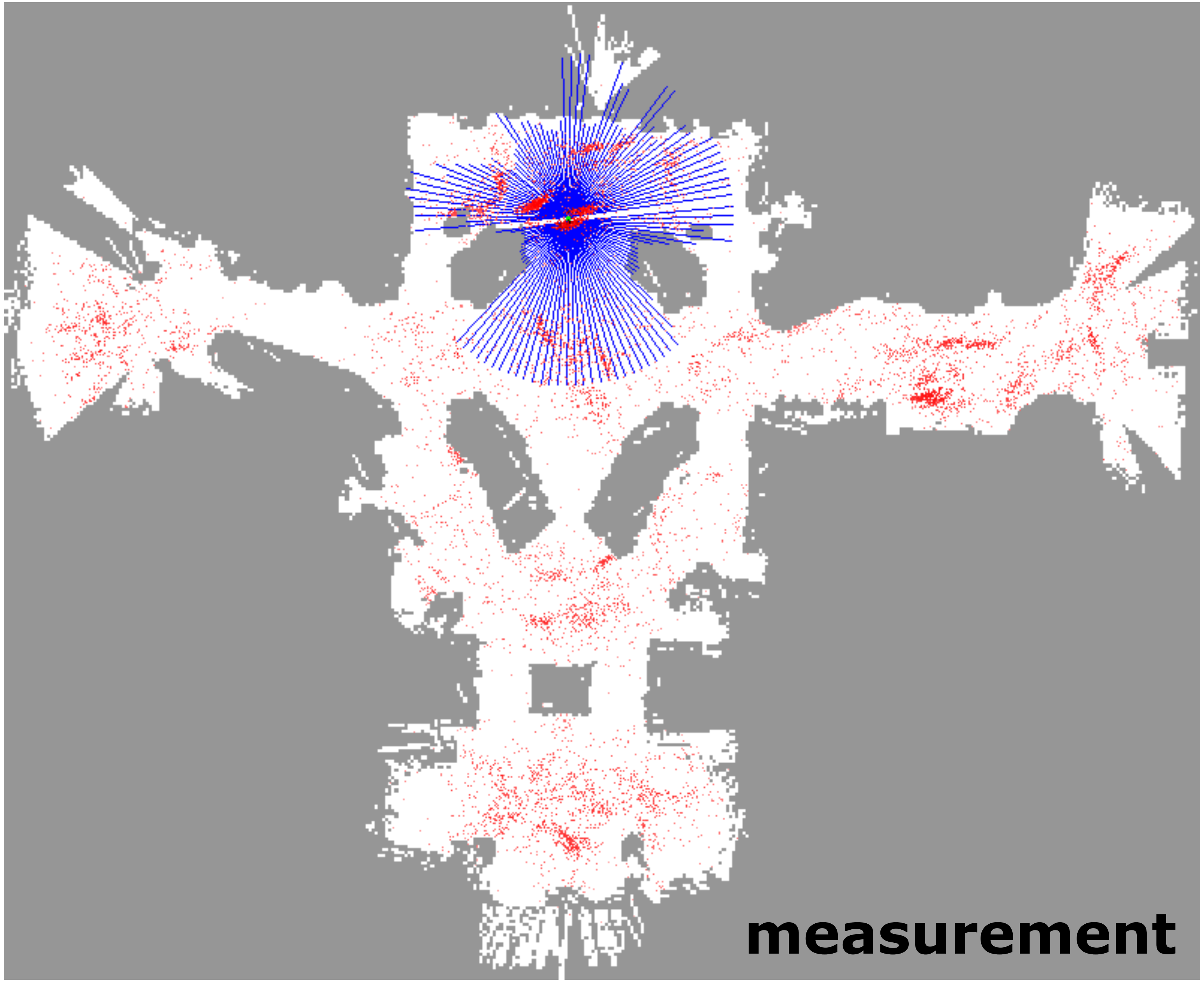
Slide Credit: Cyrill Stachnis



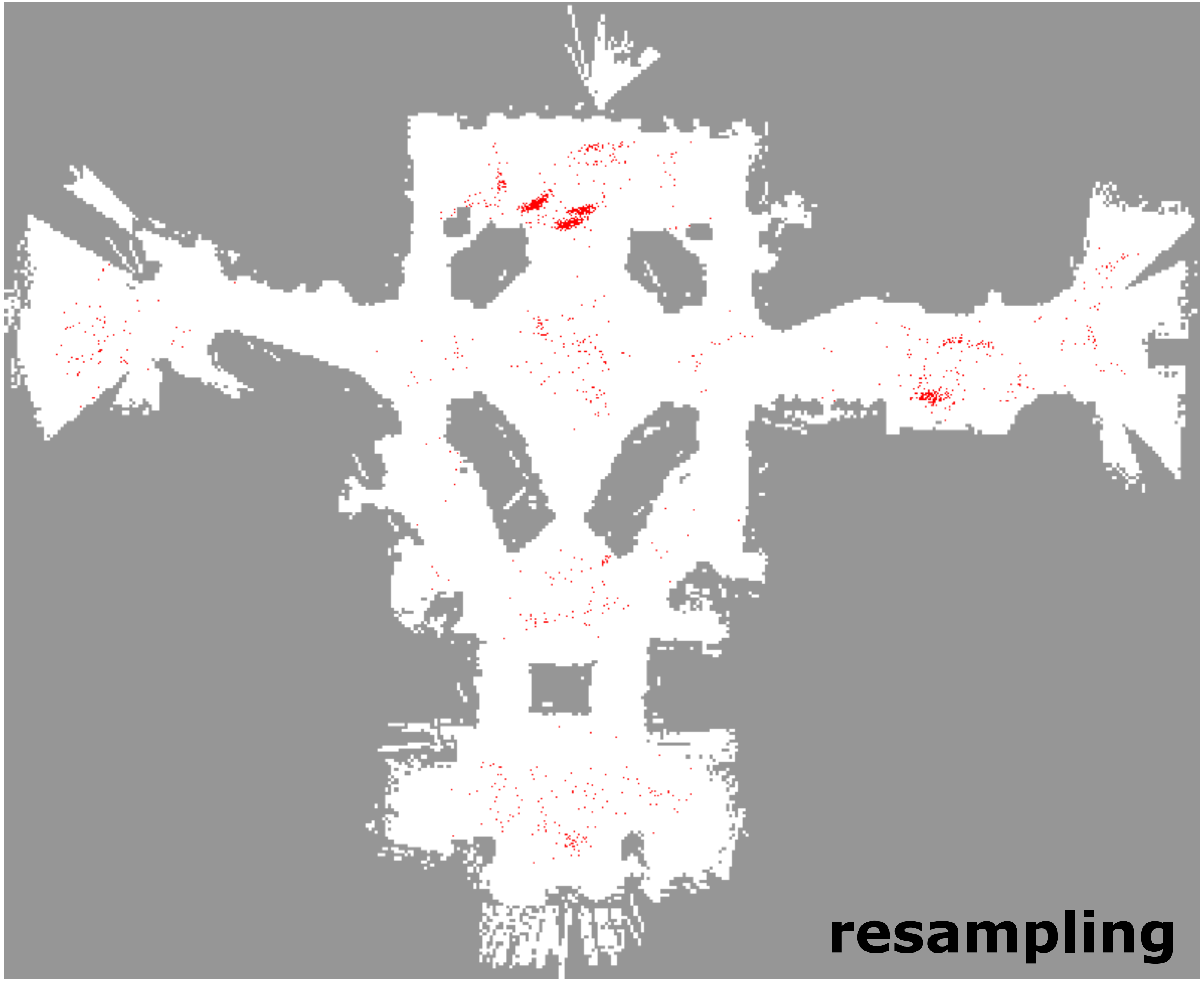
Slide Credit: Cyrill Stachnis



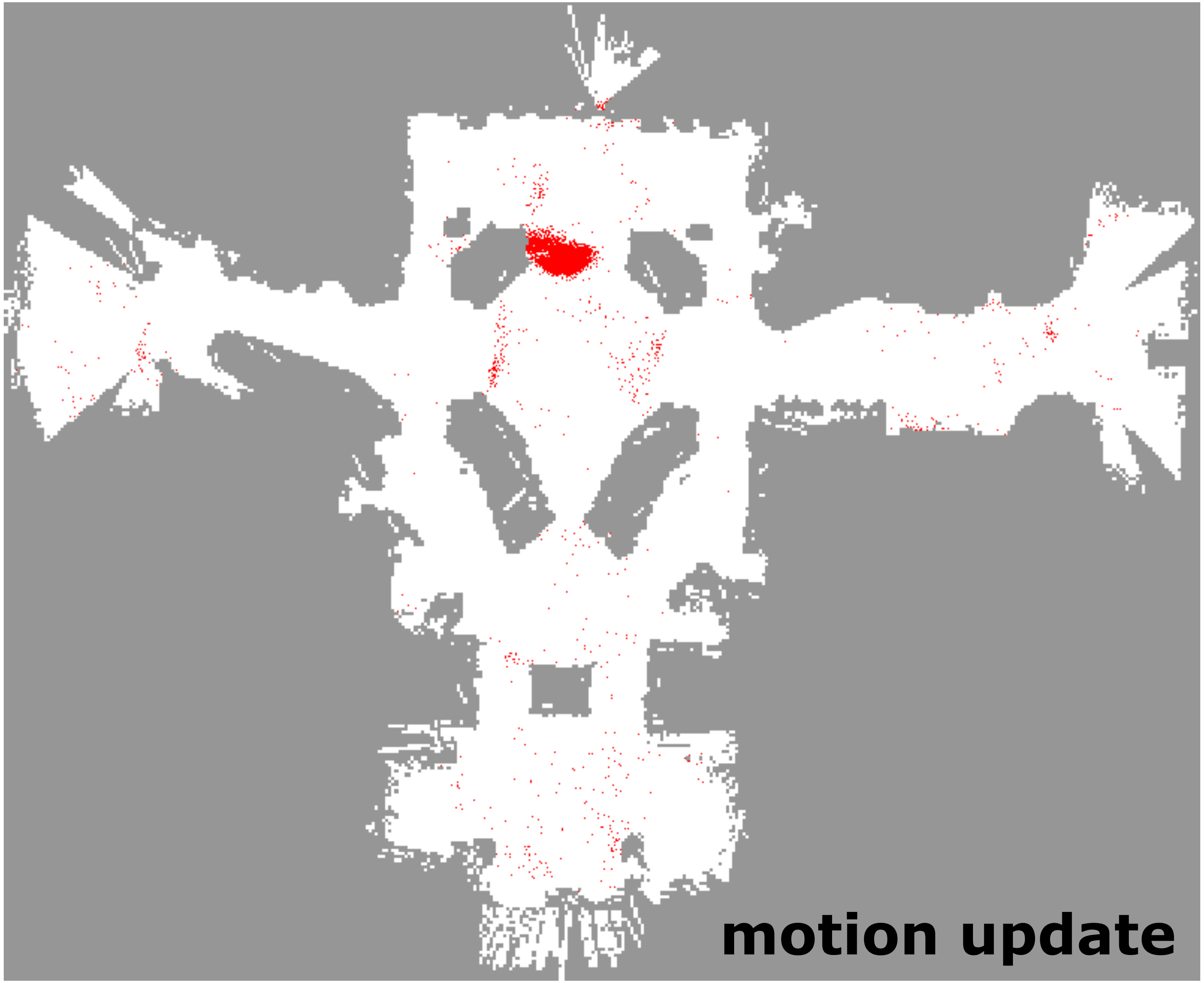
Slide Credit: Cyrill Stachnis



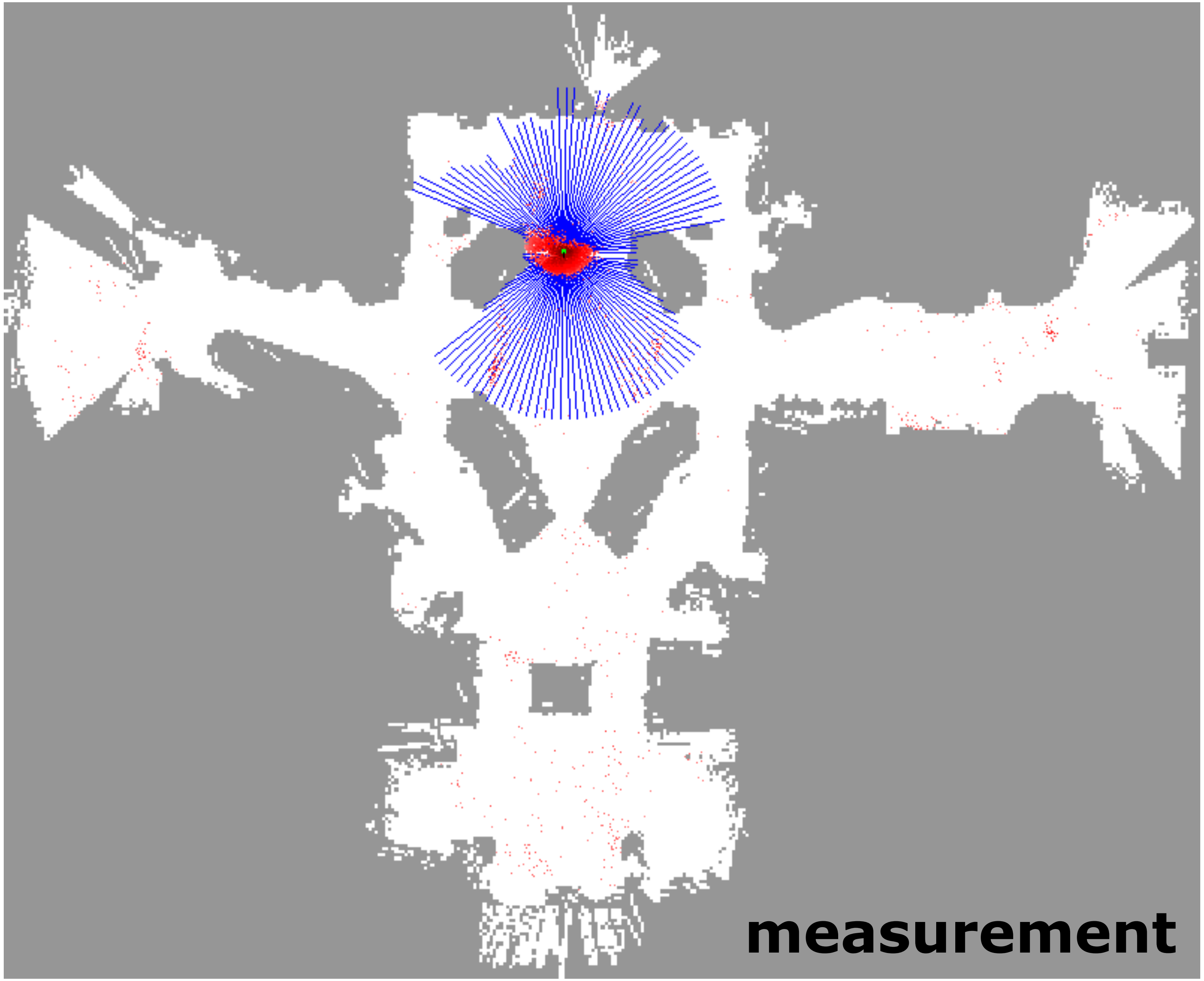
Slide Credit: Cyrill Stachnis



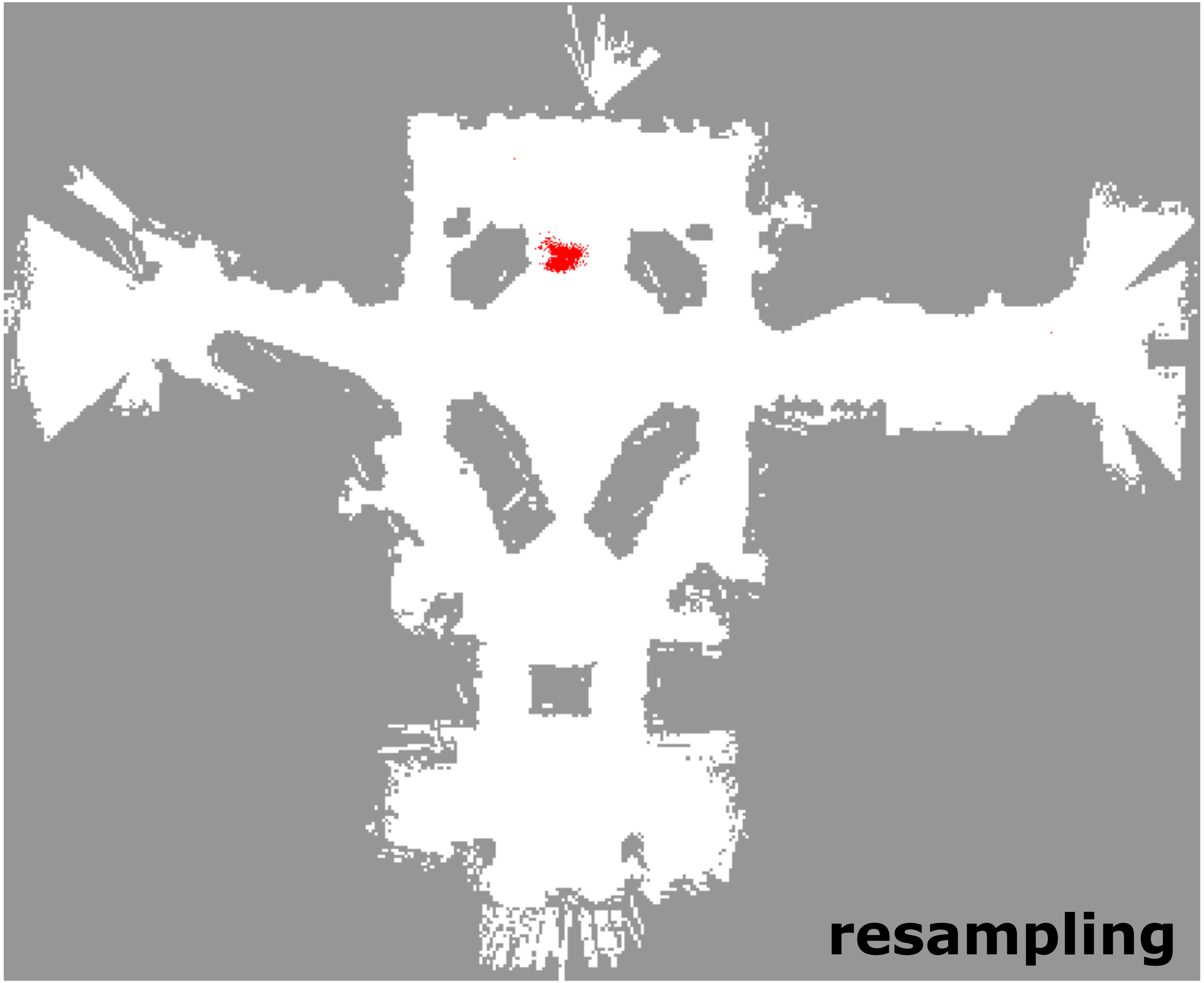
Slide Credit: Cyrill Stachnis



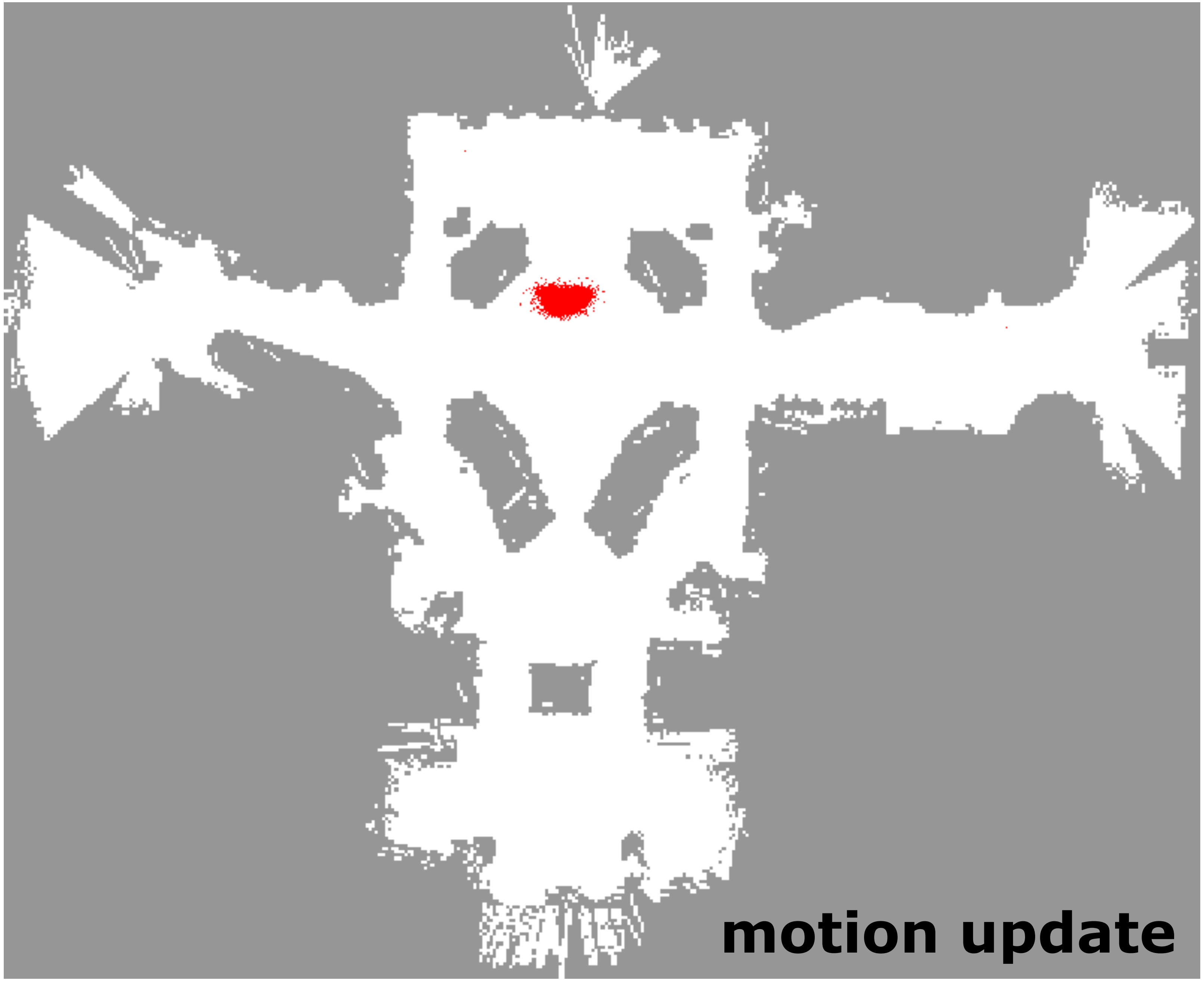
Slide Credit: Cyrill Stachnis



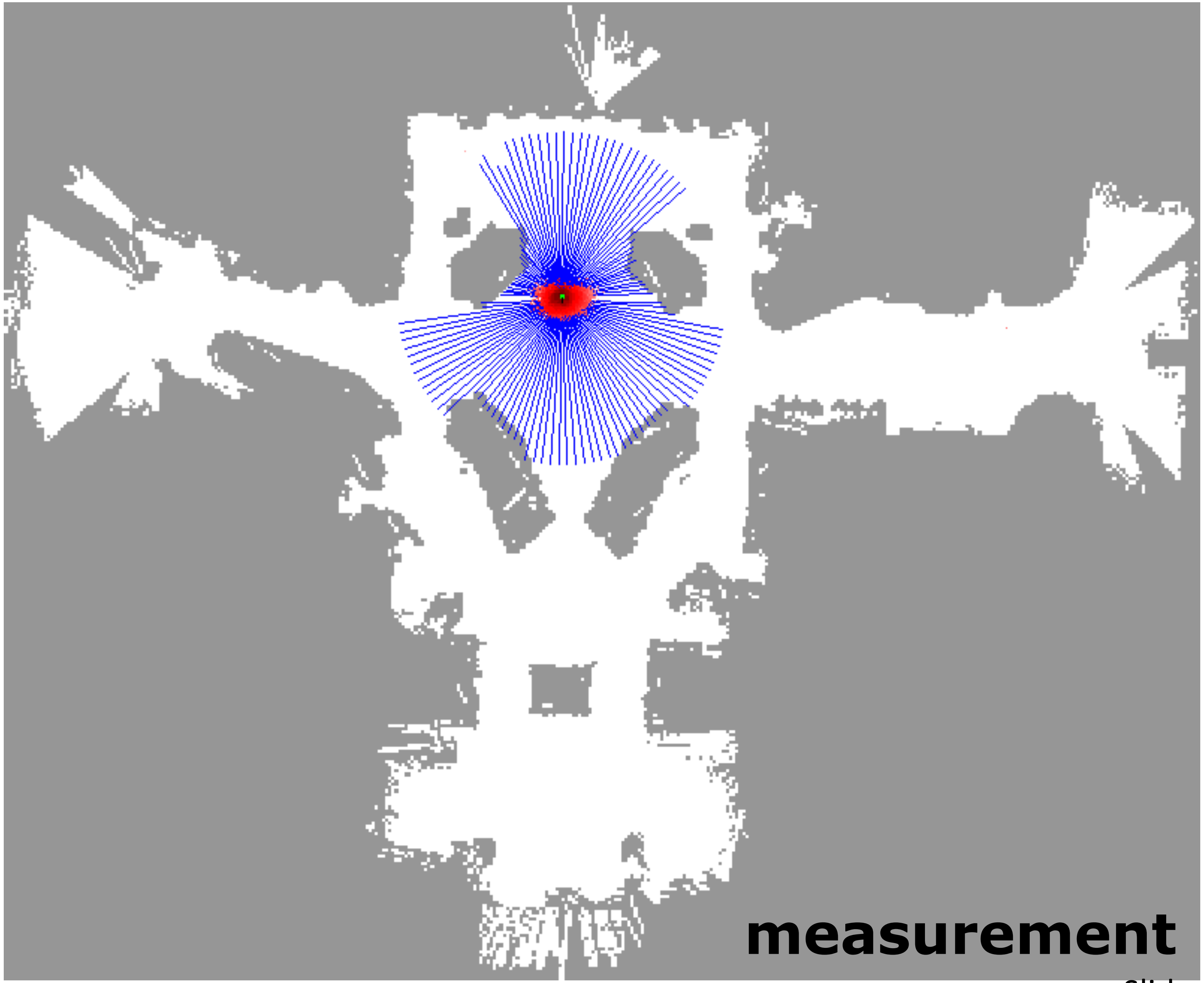
Slide Credit: Cyrill Stachnis



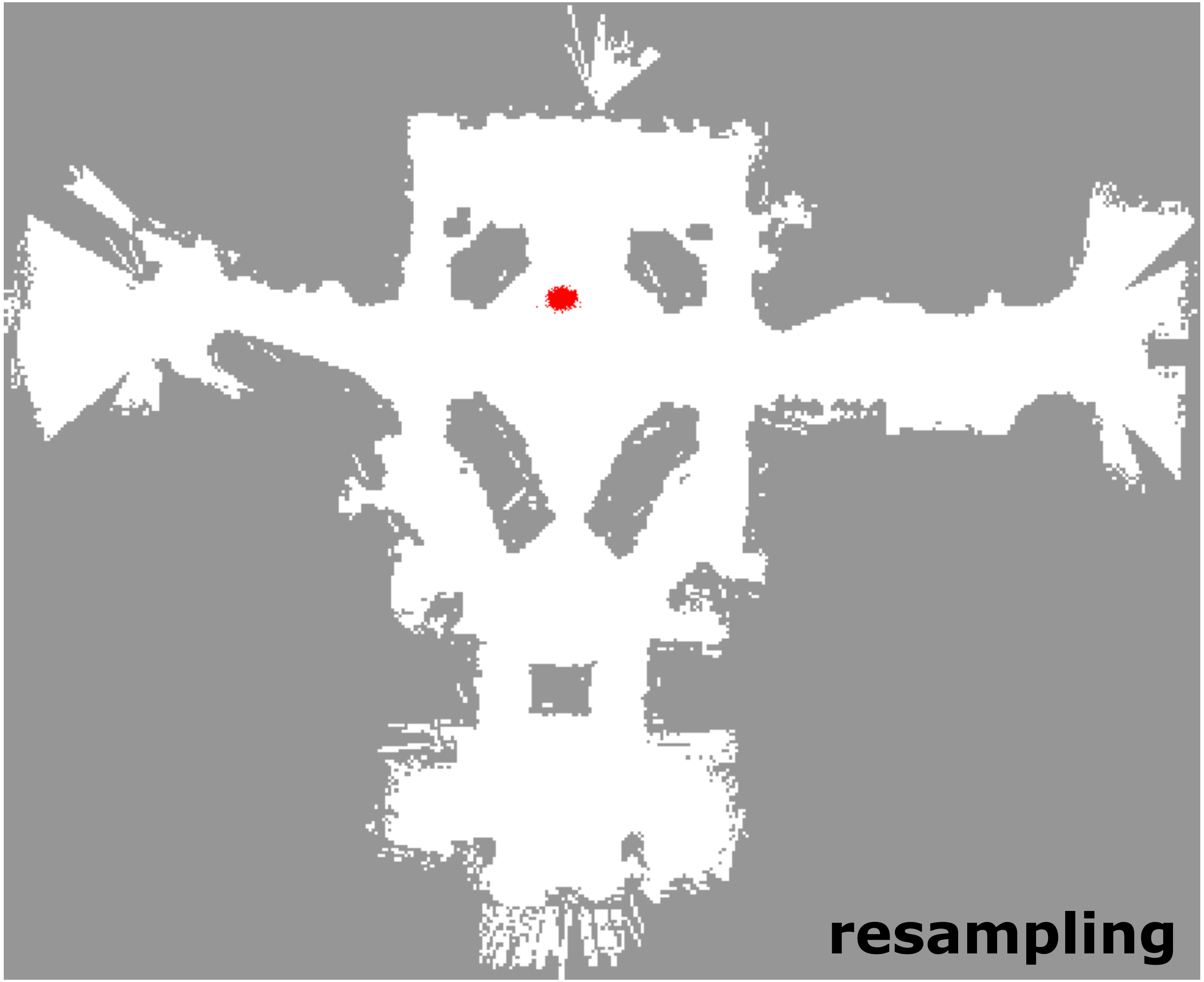
Slide Credit: Cyrill Stachnis



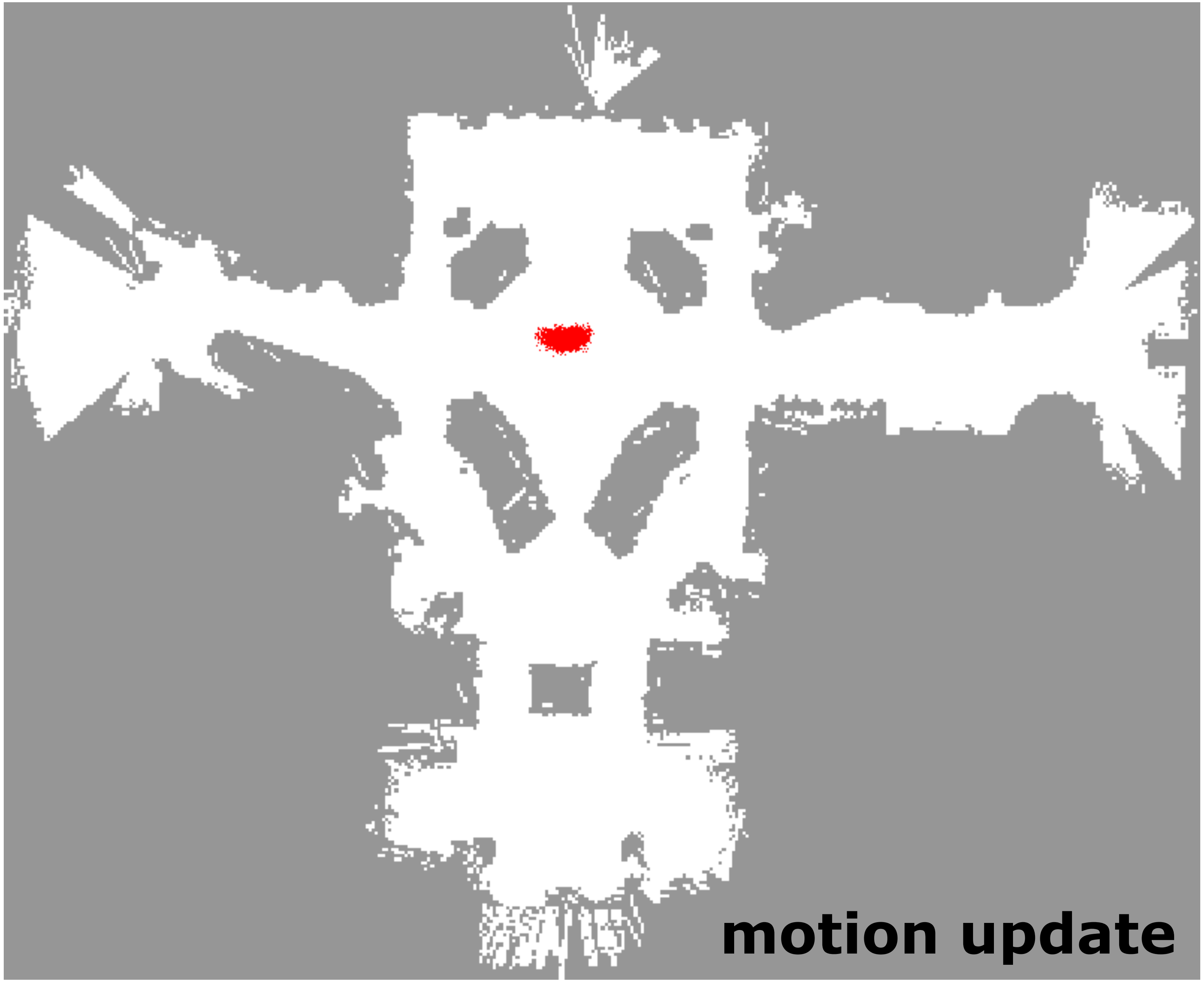
Slide Credit: Cyrill Stachnis



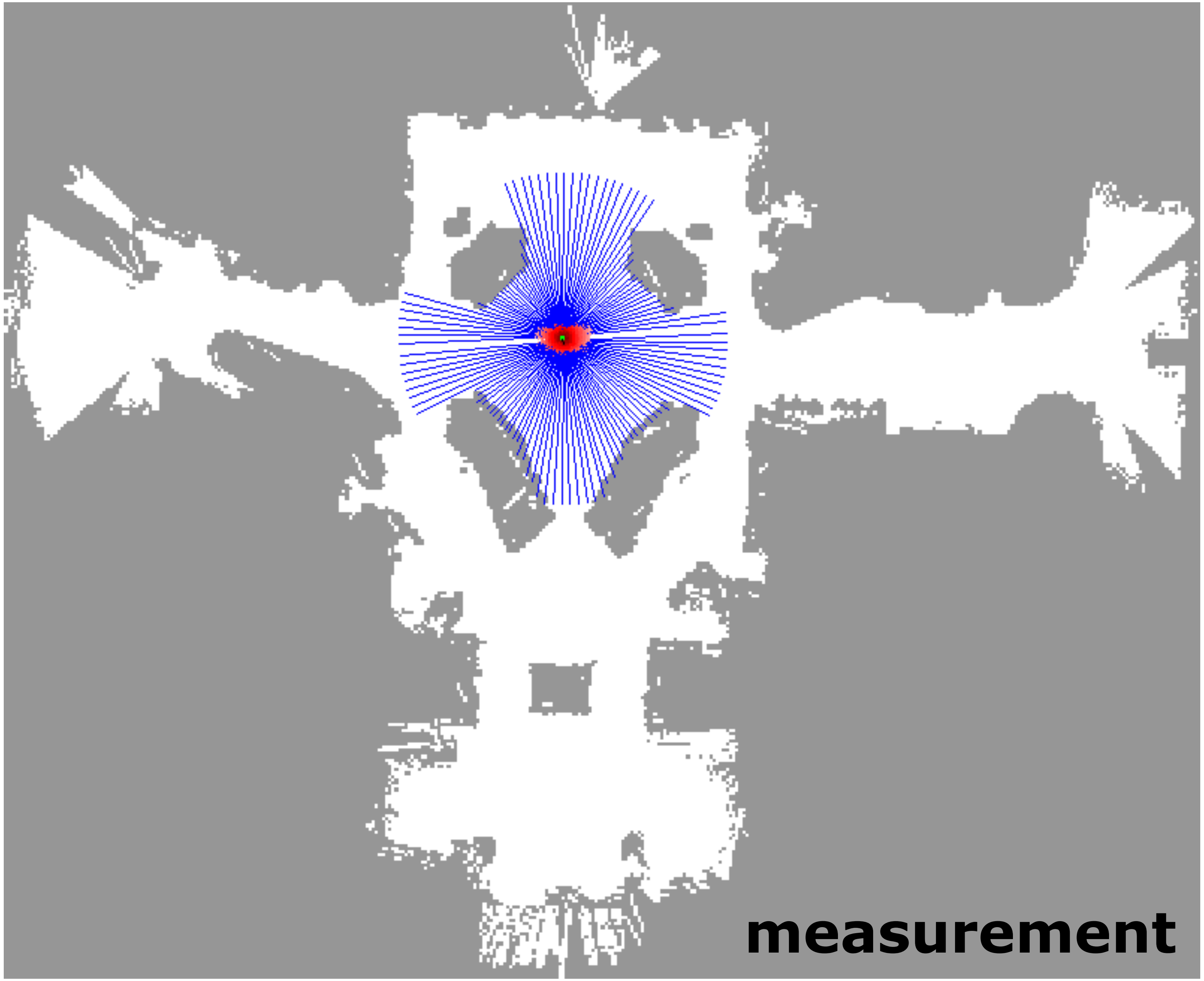
Slide Credit: Cyrill Stachnis



Slide Credit: Cyrill Stachnis

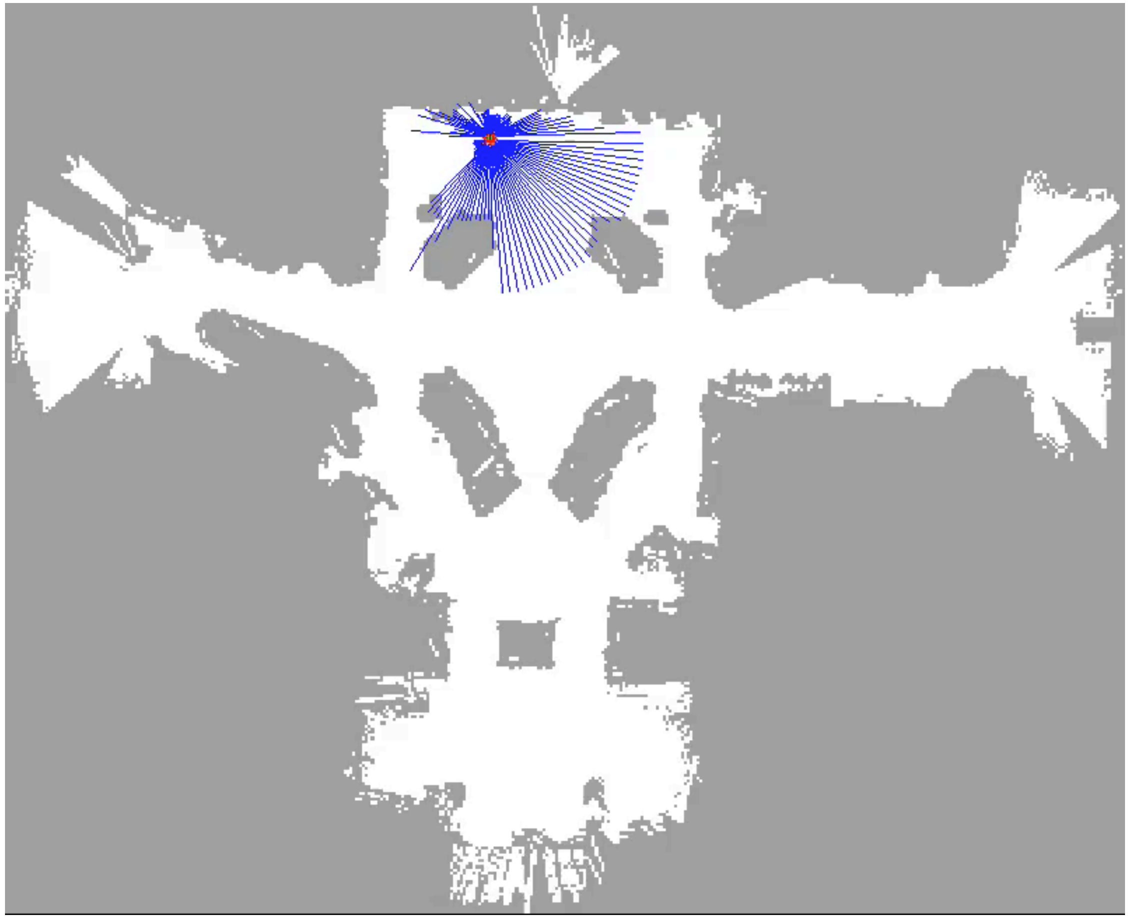


Slide Credit: Cyrill Stachnis



Slide Credit: Cyrill Stachnis

Particle Filter in Action (Kidnapped Robot)



Video Credit: Sebastian Thrun

Recall Bayes Filter

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$**):**

for all x_t do

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\ dx$$

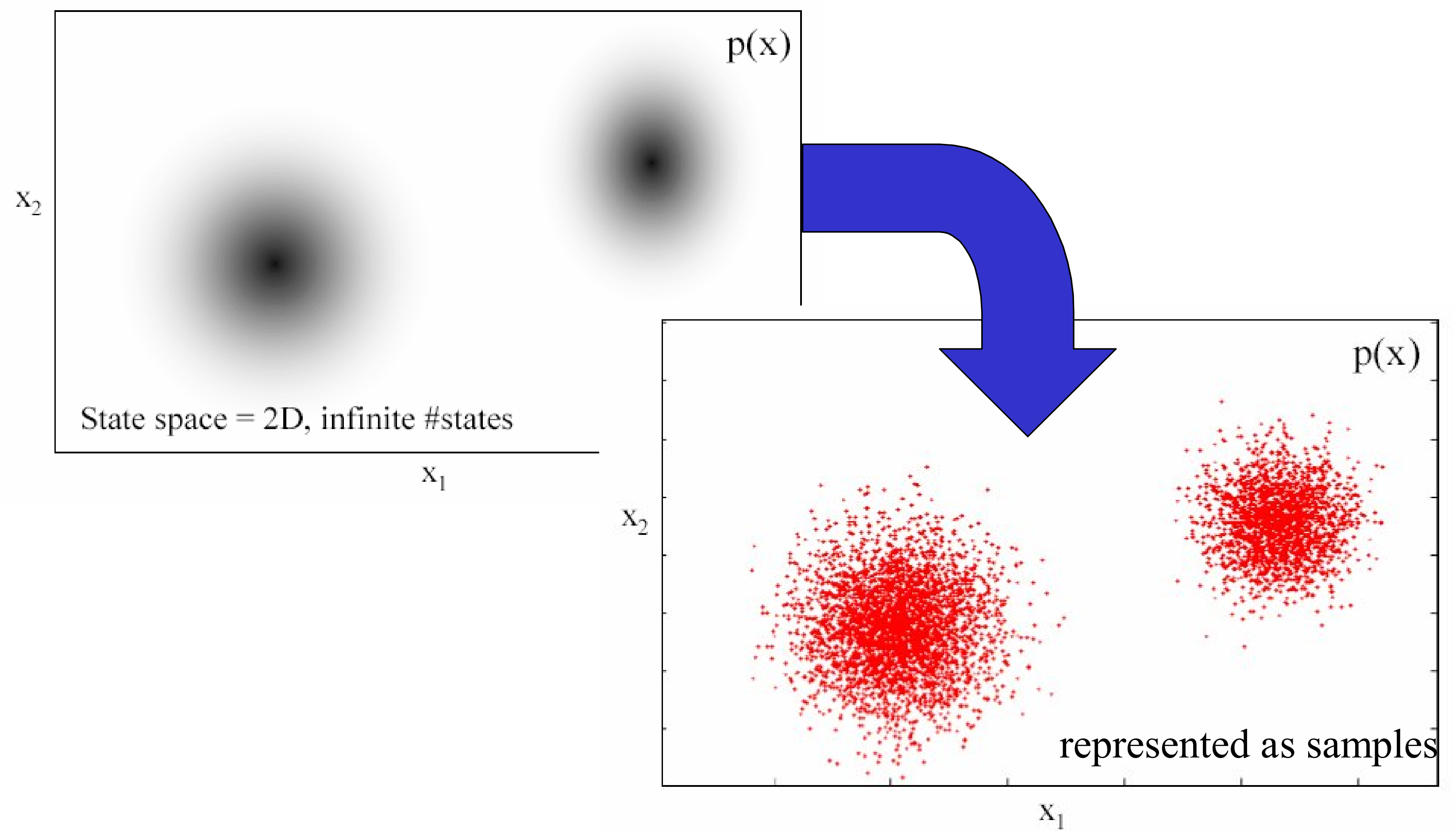
$$bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$$

endfor

return $bel(x_t)$

- In Particle filter the probabilities are represented by random observations of the actual distribution (called particles) !
- More number of particles → better estimate of distributions [Law of Large Numbers !]

Represent distributions with particles



Slide Credit: Robert Collins, Penn State Univ

Particle Filter Algorithm

Algorithm Particle_filter($\mathcal{X}_{t-1}, u_t, z_t$):

$\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$

for $m = 1$ to M do

sample $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$

$w_t^{[m]} = p(z_t \mid x_t^{[m]})$

$\bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t + \langle x_t^{[m]}, w_t^{[m]} \rangle$

endfor

for $m = 1$ to M do

draw i with probability $\propto w_t^{[i]}$

add $x_t^{[i]}$ to $\mathcal{X}_t$

endfor

return $\mathcal{X}_t$

Importance sampling

- In the re-sampling step we need samples from the distribution $Bel(\mathrm{x_t})$ but we have samples from $\overline{Bel}\ (\mathrm{x_t})$.
- Importance sampling allows us to approximate the target distribution "f" (here $Bel(\mathrm{x_t})$) from a proposal distribution "g" (here $\overline{Bel}\ (\mathrm{x_t})$).
- Random samples of proposal distribution "g" weighted with

w(x) = f(x) / g(x) ------ {Importance Weight}

are equivalent to samples from target distribution "f"

Importance Sampling: Why it works ?

- Let X be a random variable with pdf given by f(x)

$$P(x \in A) = \int_{x \in A} f(x)dx = \int f(x)I(x \in \mathrm{A})dx = E_f[\,I(x \in A)]$$

where $I(x \in \mathrm{A})$ is the indicator function

$$\begin{aligned} E_f[\,I(x \in A)] &= \int f(x)I(x \in \mathrm{A})dx \\ &= \int \frac{f(x)}{g(x)} g(x)I(x \in \mathrm{A})dx \\ &= \int w(x)g(x)I(x \in \mathrm{A})dx \\ &= E_g[\,w(x)I(x \in A)] \end{aligned}$$

Particle Filter Algorithm

Algorithm Particle_filter($\mathcal{X}_{t-1}, u_t, z_t$):

$\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$

for $m = 1$ to M do

sample $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$

$w_t^{[m]} = p(z_t \mid x_t^{[m]})$

$\bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t + \langle x_t^{[m]}, w_t^{[m]} \rangle$

endfor

for $m = 1$ to M do

draw i with probability $\propto w_t^{[i]}$

add $x_t^{[i]}$ to $\mathcal{X}_t$

endfor

return $\mathcal{X}_t$

Derivation of Particle Filter

- Consider the proposal distribution

$$\mathrm{g}(\mathrm{x_t}) = \overline{Bel}\,(\mathrm{x_t})$$

- Consider the target distribution

$$\mathrm{f}(\mathrm{x_t}) = Bel\,(\mathrm{x_t})$$

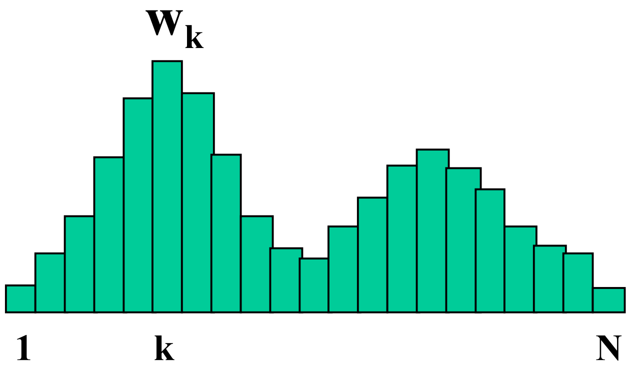
- For any particle $\mathrm{x}_\mathrm{t}^{[m]}$

$$\mathrm{w}_\mathrm{t}^{[m]} = \frac{target\ distribution}{proposal\ distribution}$$

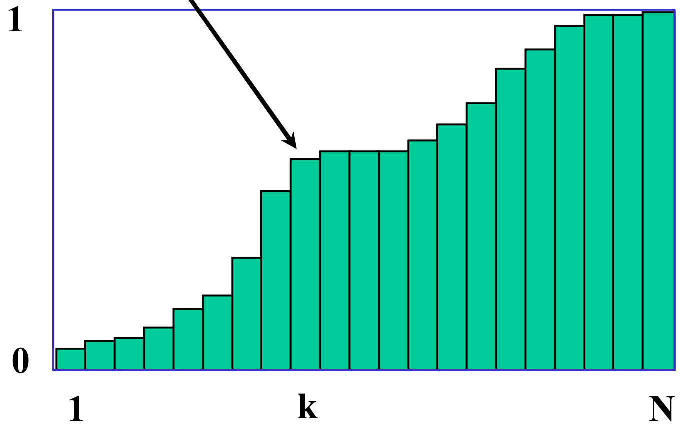
$$= \frac{Bel\,(x_t^{[m]})}{\overline{Bel}(x_t^{[m]})} = \frac{\eta P(z_t|x_t^{[m]})\,\overline{Bel}(x_t^{[m]})}{\overline{Bel}(x_t^{[m]})} = \eta P(z_t|x_t^{[m]})$$

Re-sampling using CDF

- Cumulative Distribution Function (CDF)



$$c(k) = \sum_{i=1}^{k} w_i / \sum_{i=1}^{N} w_i$$



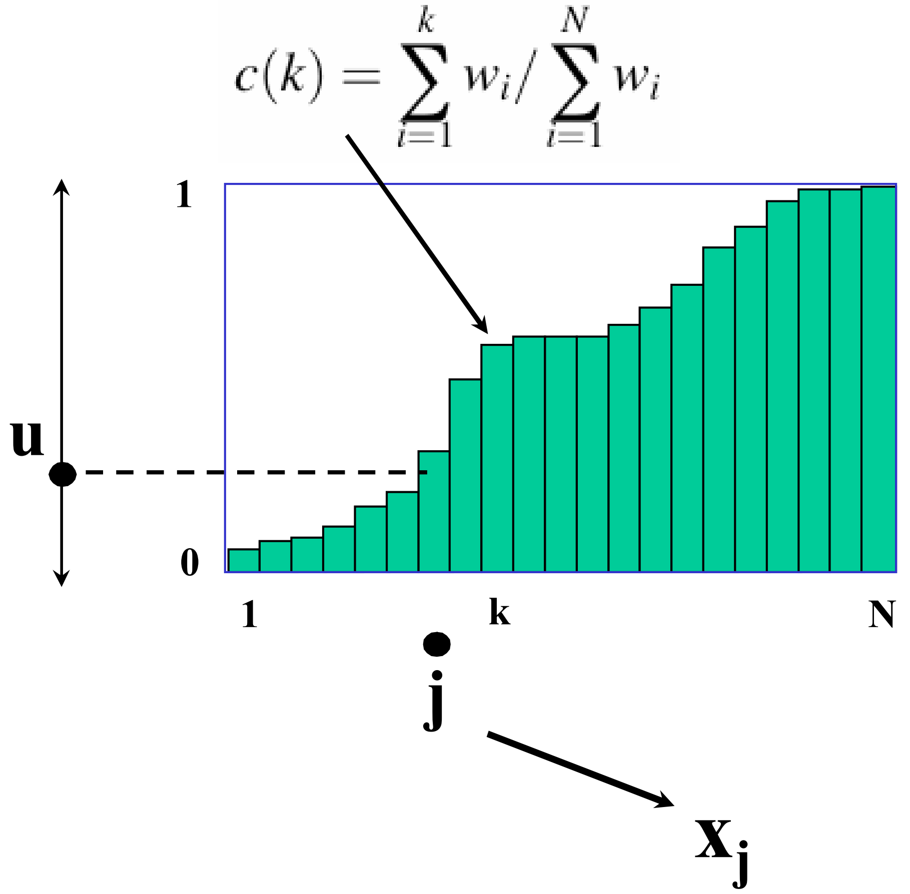
cumulative distribution function

$$F(x) = P(X \leq x)$$

Slide Credit: Robert Collins, Penn State Univ

Re-sampling using CDF

- Generate uniform random variable $u \in [0,1]$
- Draw a line intersecting c(k)
- If the index of intersection is j, then return sample x_j
- Complexity : $O(N^2)$



Slide Credit: Robert Collins, Penn State Univ

Efficient Re-sampling

```
Algorithm 2: Resampling Algorithm
```
$[\{\mathbf{x}_k^{j*}, w_k^j, i^j\}_{j=1}^{N_s}] = \text{RESAMPLE } [\{\mathbf{x}_k^i, w_k^i\}_{i=1}^{N_s}]$

- Initialize the CDF: $c_1 = 0$
- FOR $i = 2 : N_s$
  - Construct CDF: $c_i = c_{i-1} + w_k^i$
- END FOR
- Start at the bottom of the CDF: $i = 1$
- Draw a starting point: $u_1 \sim \mathbb{U}[0, N_s^{-1}]$
- FOR $j = 1 : N_s$
  - Move along the CDF: $u_j = u_1 + N_s^{-1}(j-1)$
  - WHILE $u_j > c_i$
  - $*$ $i = i + 1$
  - END WHILE
  - Assign sample: $\mathbf{x}_k^{j*} = \mathbf{x}_k^i$
  - Assign weight: $w_k^j = N_s^{-1}$
  - Assign parent: $i^j = i$
- END FOR

Basic idea: choose one initial small random number; deterministically sample the rest by "crawling" up the cdf function. This is O(N).

Reference: A Tutorial on particle filter for online non-linear non-gaussian bayesian tracking by Sanjeev Arunapalam et al

Monte Carlo Localization (Particle Filter)

Algorithm Particle_filter($\mathcal{X}_{t-1}, u_t, z_t$):

$\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$

for $m = 1$ to M do

  sample $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$ → Prediction Step: Motion Model

  $w_t^{[m]} = p(z_t \mid x_t^{[m]})$ → Correction step: Observation Model

  $\bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t + \langle x_t^{[m]}, w_t^{[m]} \rangle$

endfor

for $m = 1$ to M do

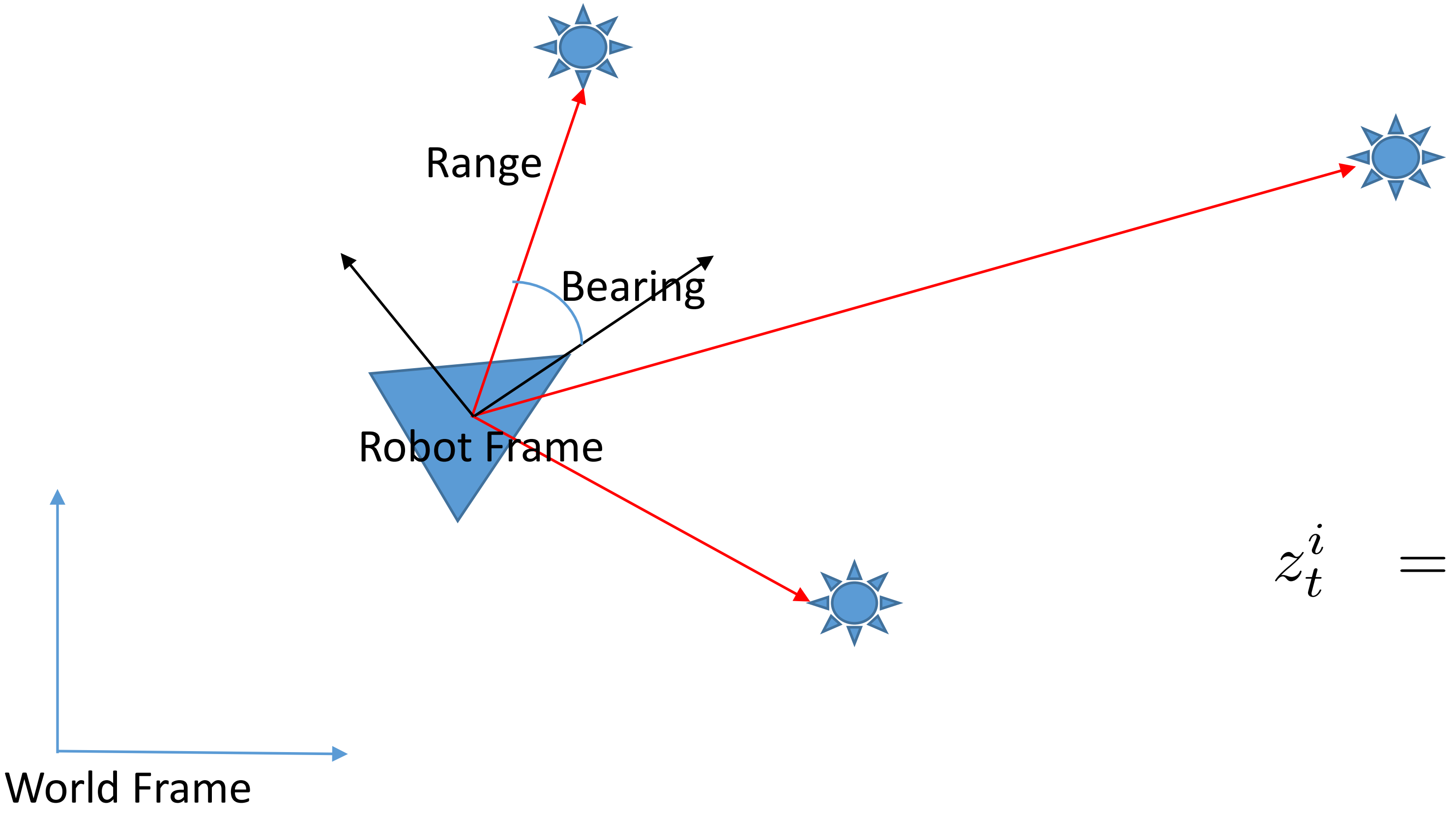
  draw i with probability $\propto w_t^{[i]}$

  add $x_t^{[i]}$ to $\mathcal{X}_t$

endfor

return $\mathcal{X}_t$

Range and Bearing Sensor Model



$$z_t^i = \begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix}$$

Importance weight from measurement

- For ‘n’ measurements

$$p(z_t \mid c_t, x_t, m) \quad = \quad \prod_i p(z_t^i \mid c_t, x_t, m)$$

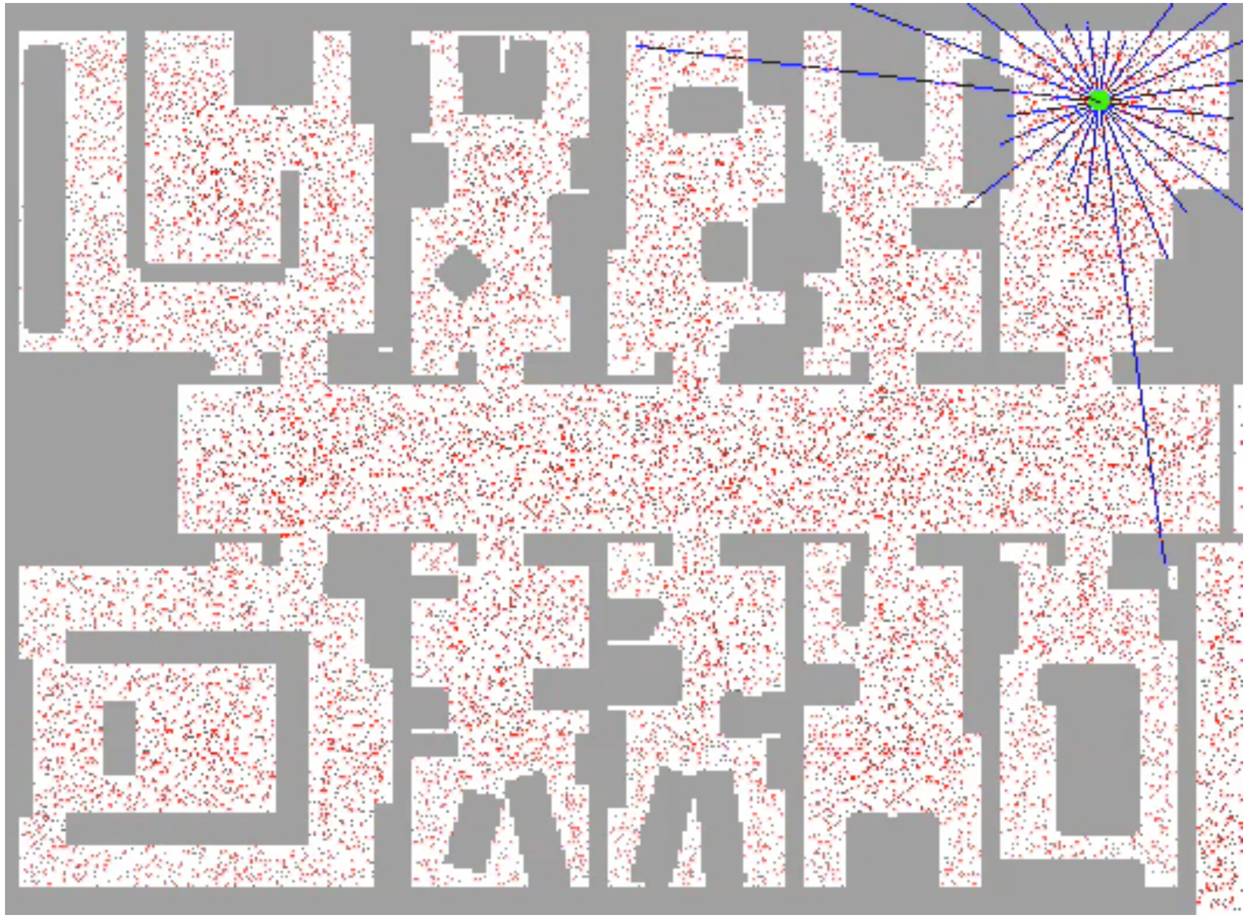
$$\begin{aligned} & p(z_t^i \mid c_t^i, x_t, m) \\ & \quad = \quad \eta \, \exp\left\{-\tfrac{1}{2}(z_t^i - h(x_t, c_t^i, m))^T \, Q_t^{-1} \, (z_t^i - h(x_t, c_t^i, m))\right\} \end{aligned}$$

Where $x_t = x_t^{[m]}$, corresponding to each particle.

PF Localization summary

- In the context of localization, the particles are propagated according to the motion model.
- They are then weighted according to the likelihood of the observations.
- In a re-sampling step, new particles are drawn with a probability proportional to the likelihood of the observation.

Particle Filter in Action (Global Localization)



Importance Sampling