Least Squares Estimation, PCA, RANSAC and Rigid Body Transformation

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Least Squares Estimation

- Various Applications !!
- Sensor Data Estimation
  - Range sensor
  - GPS
- Robot location
- Feature Estimation
  - Line
  - Plane

Least Squares as Optimization

$$e = \|Y - A\theta\|^2 .$$
$$= \langle Y - A\theta, Y - A\theta \rangle$$
$$= (Y - A\theta)^T (Y - A\theta)$$
$$= (Y^T - \theta^T A^T)(Y - A\theta)$$
$$= Y^T Y - \underbrace{Y^T A \theta}_{\substack{\text{a number} \\ \text{can be written as } \theta^T A^T Y}} - \theta^T A^T Y + \theta^T A^T A \theta .$$

$$= Y^T Y - 2\theta^T A^T Y + \theta^T A^T A \theta .$$

$$\frac{\partial e}{\partial \theta} = -2 A^T Y + 2 A^T A \theta .$$

$$\frac{\partial e}{\partial \theta} = 0$$
$$\Rightarrow \cancel{2} A^T A \theta = \cancel{2} A^T Y$$
$$\Rightarrow \boxed{\theta = (A^T A)^{-1} A^T Y}$$

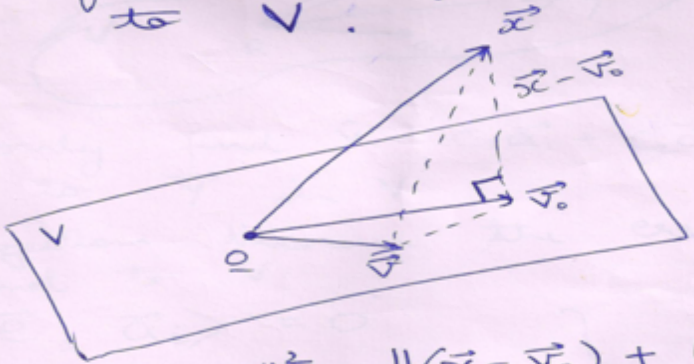
Least Squares from Projection Theorem

★ In order to solve the least squares problem we need to first see the Projection Theorem

Statement: Let S be a inner product space and let V be a subspace of S. For any vector $x \in S$ there exist a unique vector $v_0 \in V$ closest to $\vec{x}$

i.e. $\|\vec{x} - \vec{v}_0\| \leq \|\vec{x} - \vec{v}\|$ for every $v \in V$.

Conversely $\vec{v}_0$ is the minimizer of $\|\vec{x} - \vec{v}_0\|$ if and only if $\vec{x} - \vec{v}_0$ is orthogonal to V.



Proof: Consider $\|\vec{x} - \vec{v}\|^2 = \|(\vec{x} - \vec{v}_0) + (v_0 - \vec{v})\|^2$

$\because (\vec{x} - \vec{v}_0) \perp (\vec{v}_0 - \vec{v})$

$$\Rightarrow \|\vec{x} - \vec{v}\|^2 = \|\vec{x} - \vec{v}_0\|^2 + \|\vec{v}_0 - \vec{v}\|^2$$
$$\geq \|\vec{x} - \vec{v}_0\|^2$$

Hence Proved

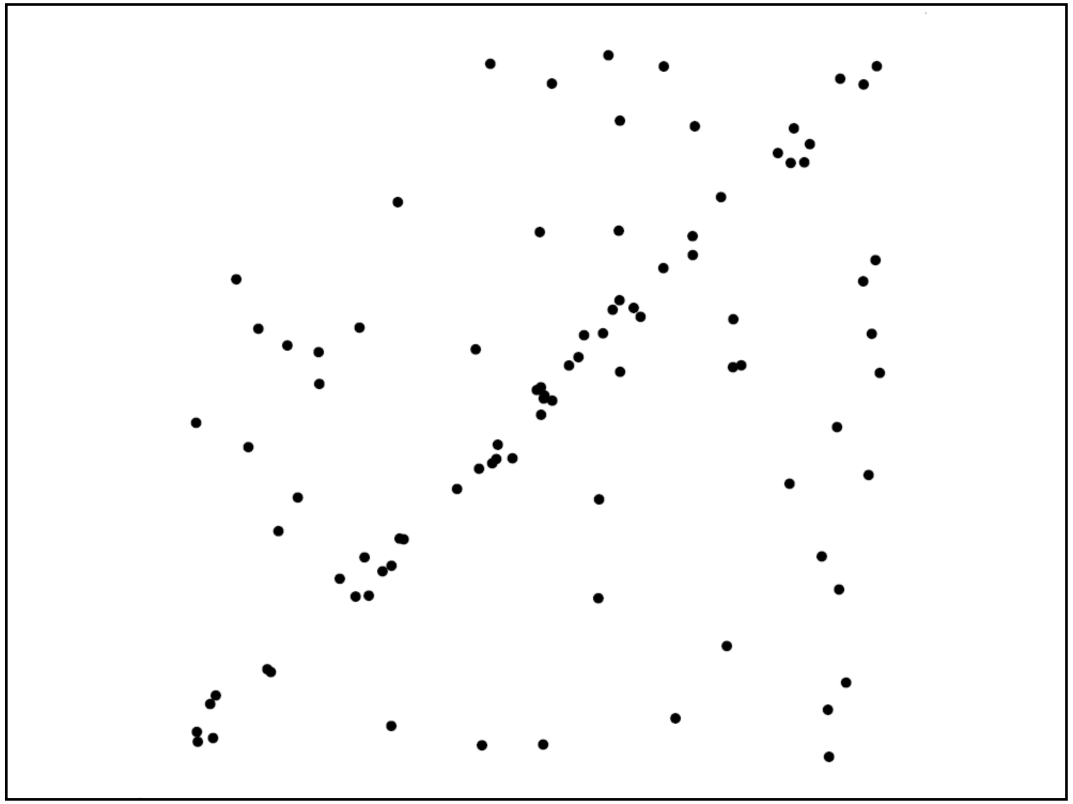
Principal Component Analysis

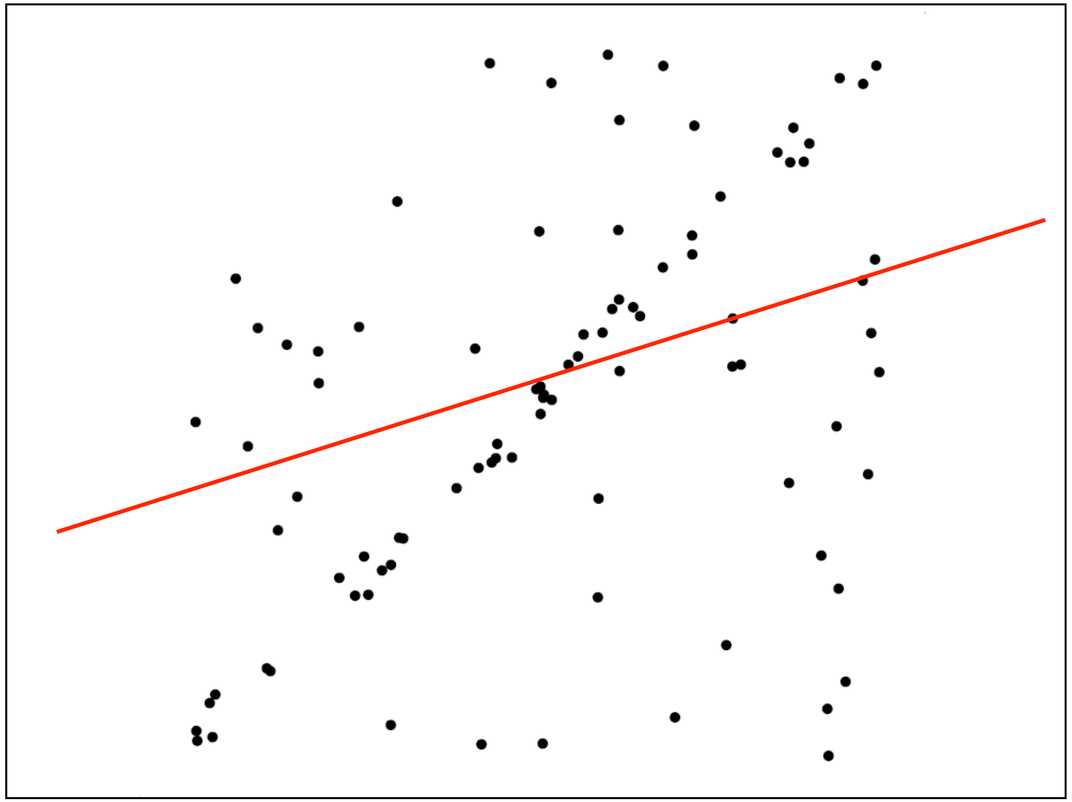
- It is a very important mathematical tool from linear algebra
- Allows us to describe the covariance structure in the sample dataset (which is often confusing)
- The most important dimensions in a sample dataset are those along which the variance is maximum
  - Data about students in class: dimensions can be [Gate score, Height, Weight, Color of Hair, Home Town]. Which one is least informative ?
- PCA helps us identify important dimensions (i.e. Max-variance subspace)
- Low rank approximation / Dimensionality reduction

PCA

- The max-variance subspace is given by the eigenvector (corresponding to max eigenvalue) of the sample covariance matrix of data.
- Process:
  - Find sample mean
  - Make the data zero mean
  - Compute sample covariance matrix ('R')
  - Compute Eigenvectors and Eigenvalues of 'R'
  - Eigenvector corresponding to the largest eigenvalue is called the principal component.
  - The kth principal component of data is given by projecting the data point onto the kth eigenvector.

What if we have outliers ?





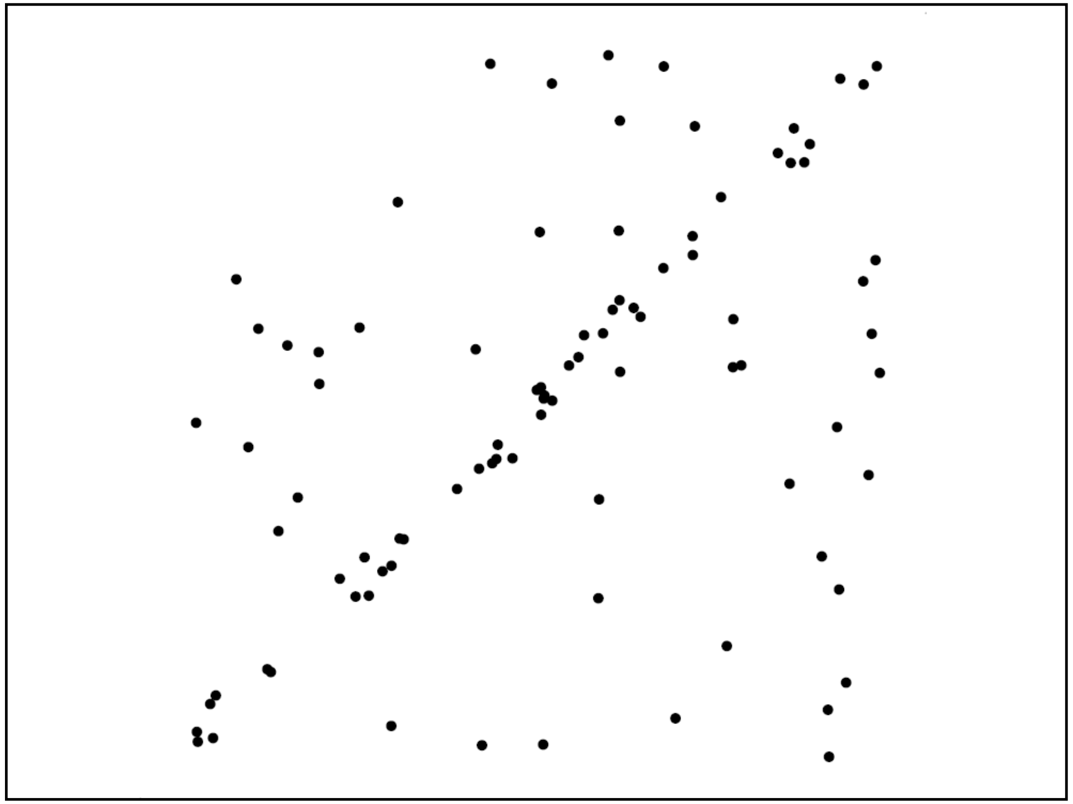
Least Squares Fit

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

Random Sampling and Consensus (RANSAC)

- Robust Estimation
- Outlier Rejection

RANSAC



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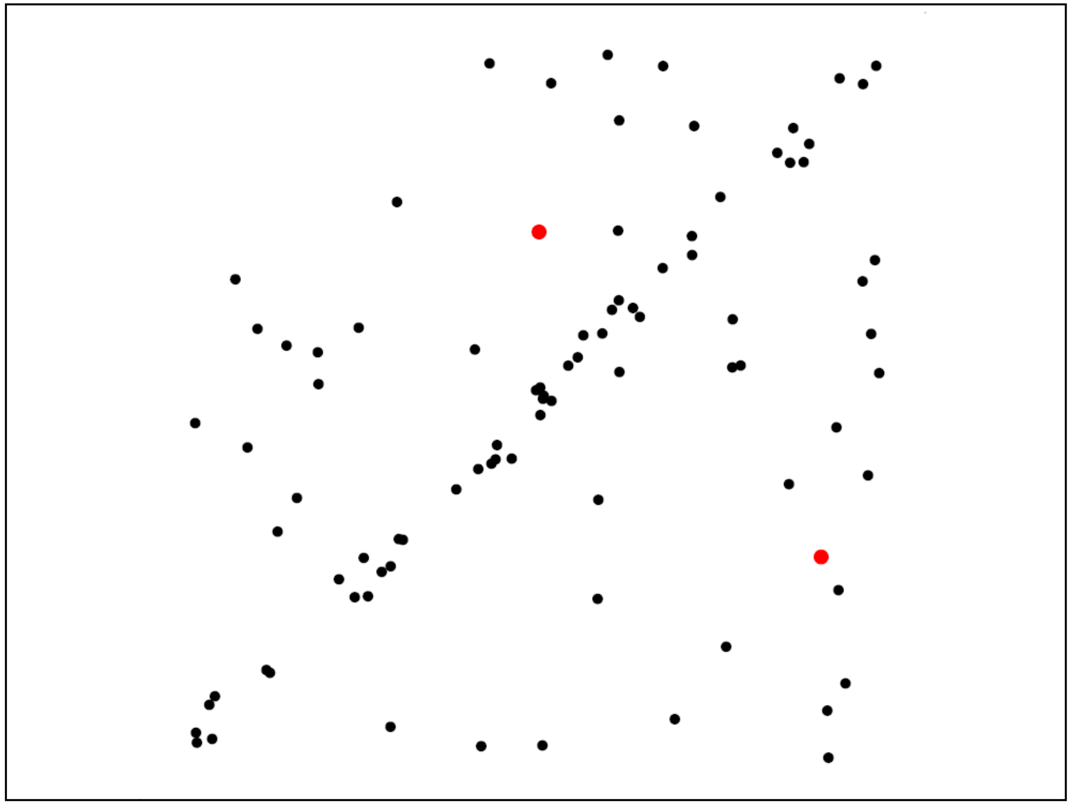
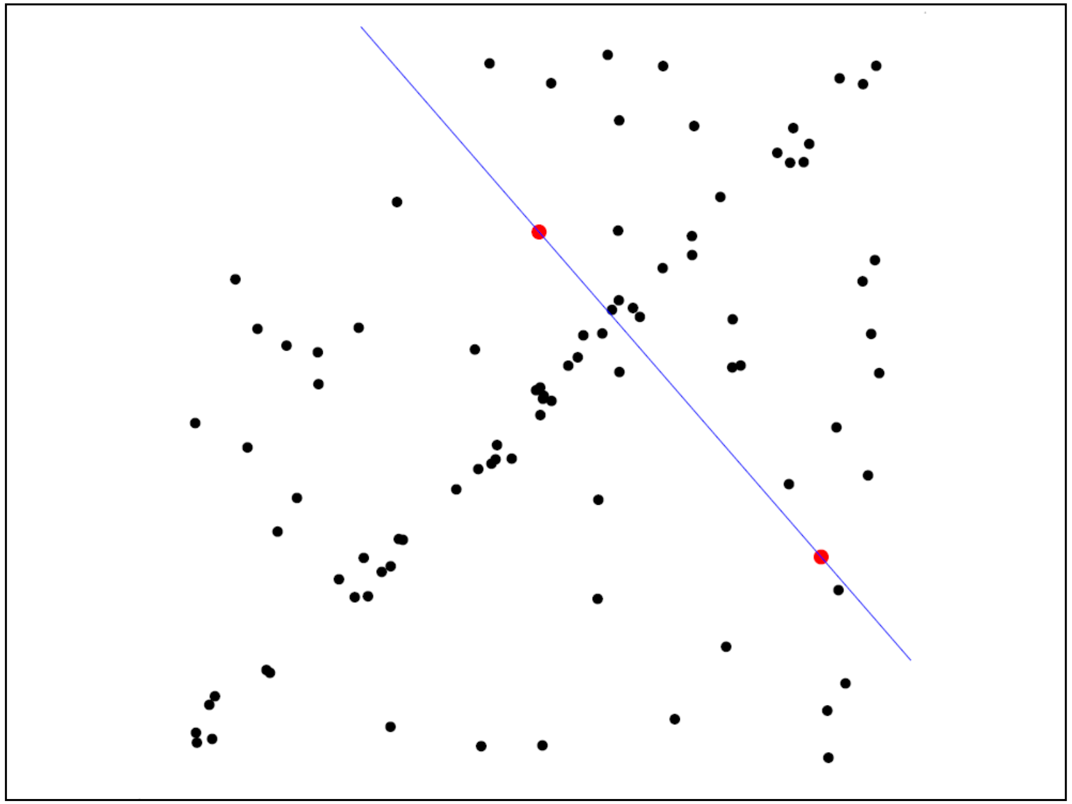
RANSAC



1. Randomly select minimal subset of points

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

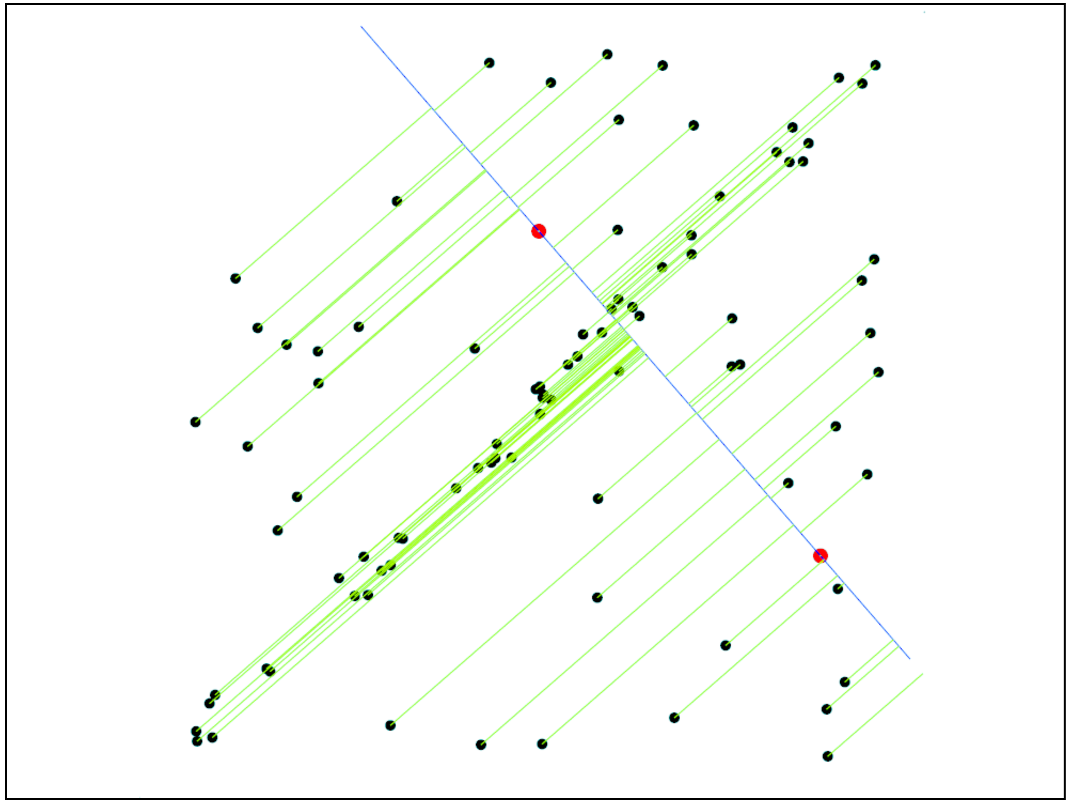
RANSAC



1. Randomly select minimal subset of points
2. Hypothesize a model

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

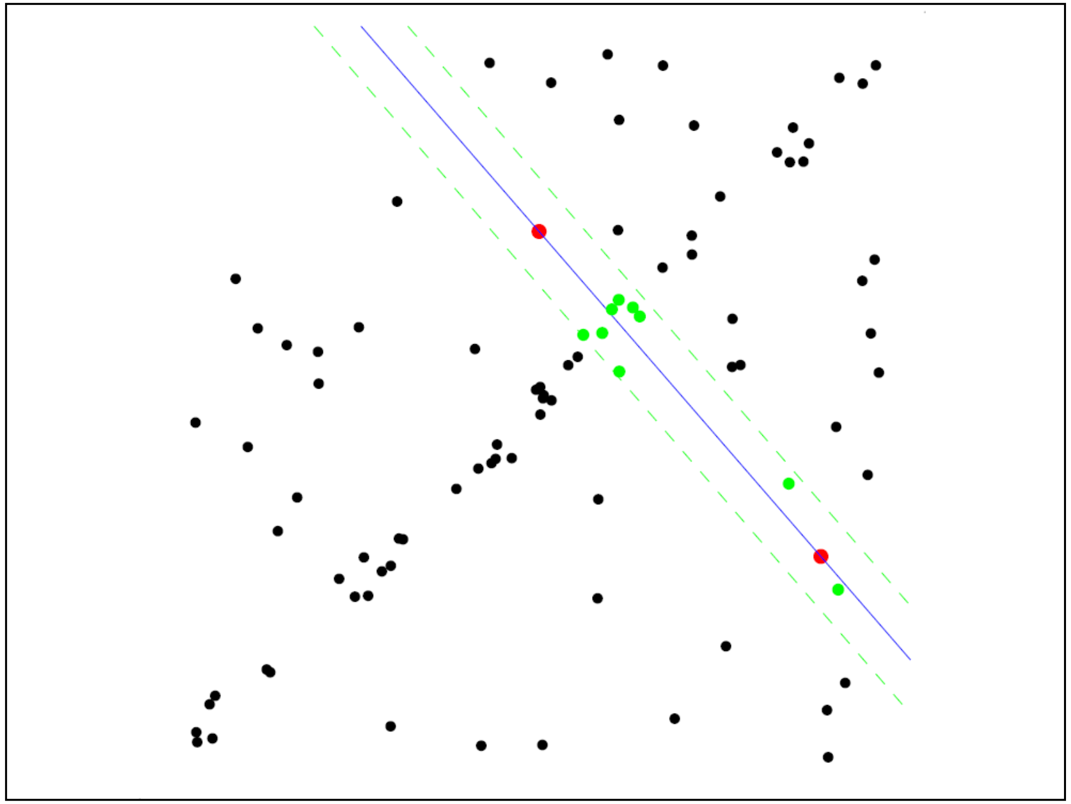
RANSAC



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

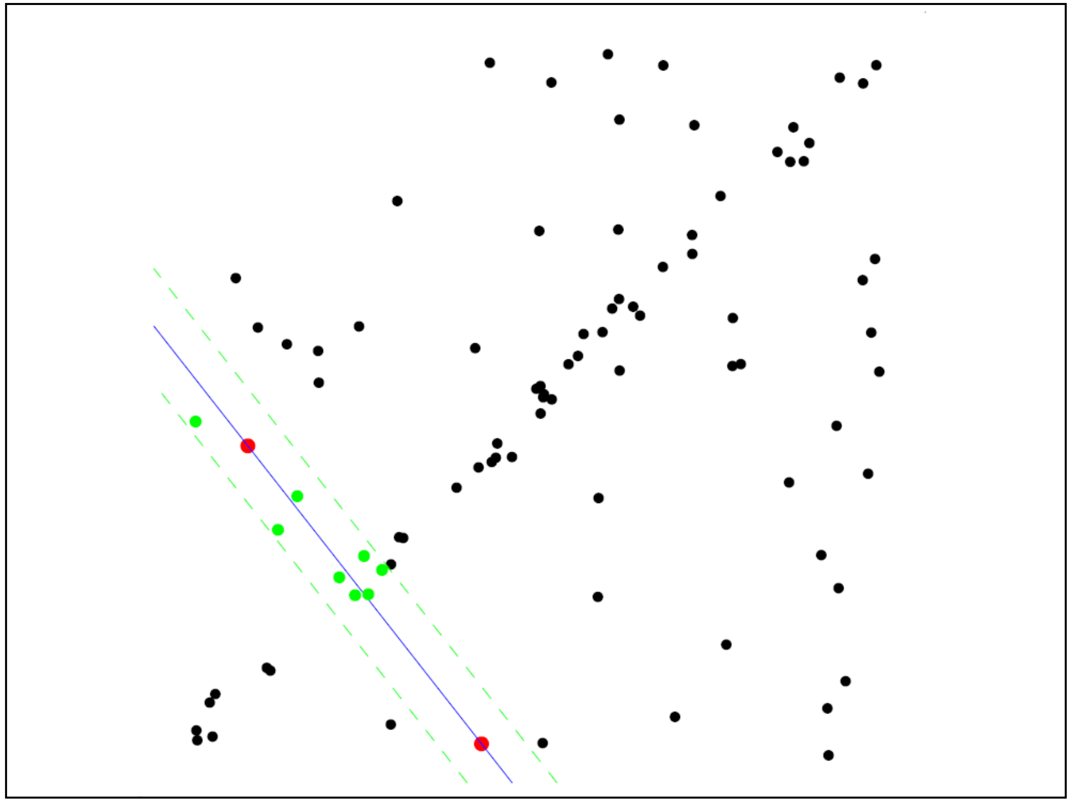
RANSAC



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
4. Select points consistent with model

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

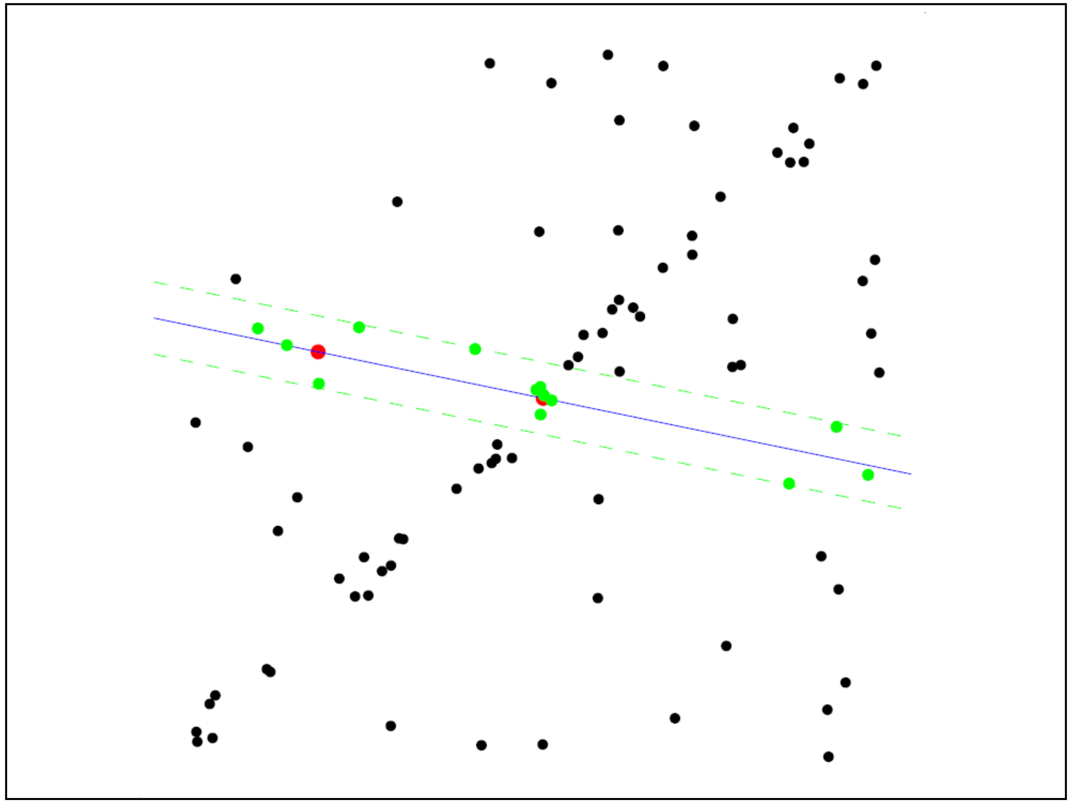
RANSAC



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
4. Select points consistent with model
5. Repeat *hypothesize-and-verify* loop

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

RANSAC



1. Randomly select minimal subset of points
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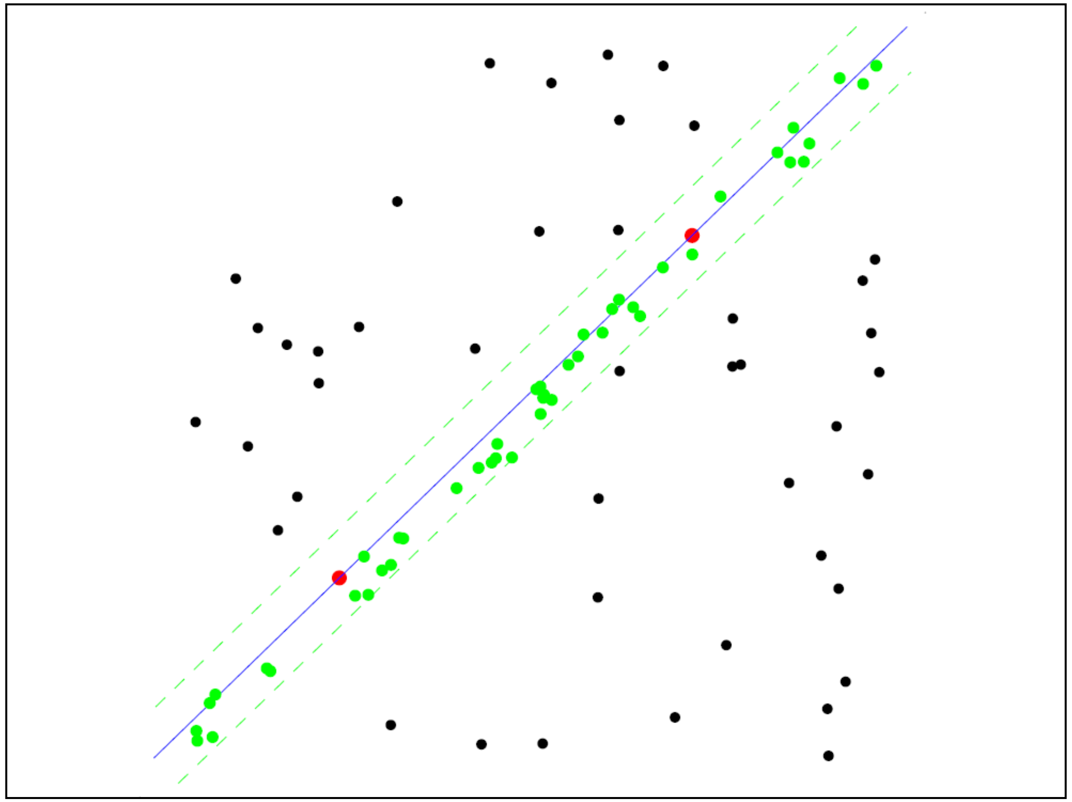
Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

RANSAC



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
4. Select points consistent with model
5. Repeat *hypothesize-and-verify* loop

Slide Source: http://cs.gmu.edu/~kosecka/cs682/lect-fitting.pdf

RANSAC

- Number of Iterations required ?

$$N = \frac{\log(1-p)}{\log(1-w^s)}$$

Covariance Propagation

- *Propagating Covariance in Computer Vision*. Robert M. Haralick

$$\Sigma_\Theta = \left[\frac{\partial^2 F}{\partial \Theta^2}(X, \Theta) \right]^{-1} \frac{\partial^2 F^T}{\partial X \partial \Theta}(X, \Theta) \Sigma_X \frac{\partial^2 F}{\partial X \partial \Theta}(X, \Theta) \left[\frac{\partial^2 F}{\partial \Theta^2}(X, \Theta) \right]^{-1}$$

Multivariate Gaussian Distribution

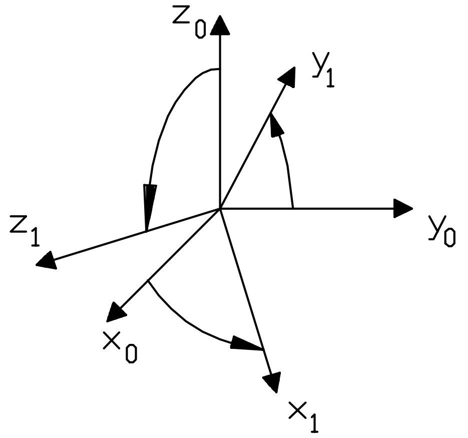
- Linear Transformation of Gaussian Random Vector

$$Y = AX + b$$
$$\Sigma_Y = A\Sigma_X A^\top$$

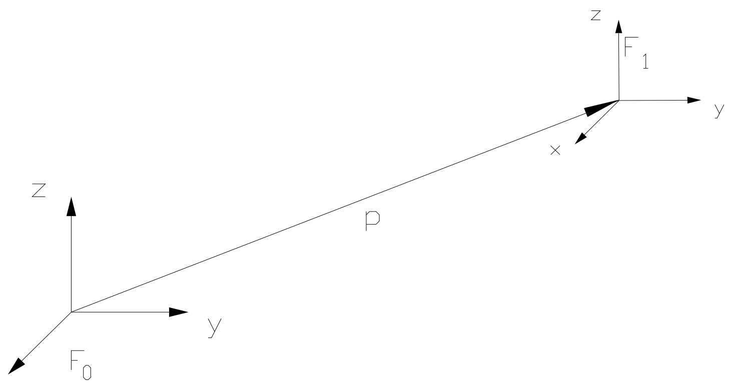
Multivariate Gaussian Distribution

- Decorrelation Transform

Rigid Body Transformation



Rotation



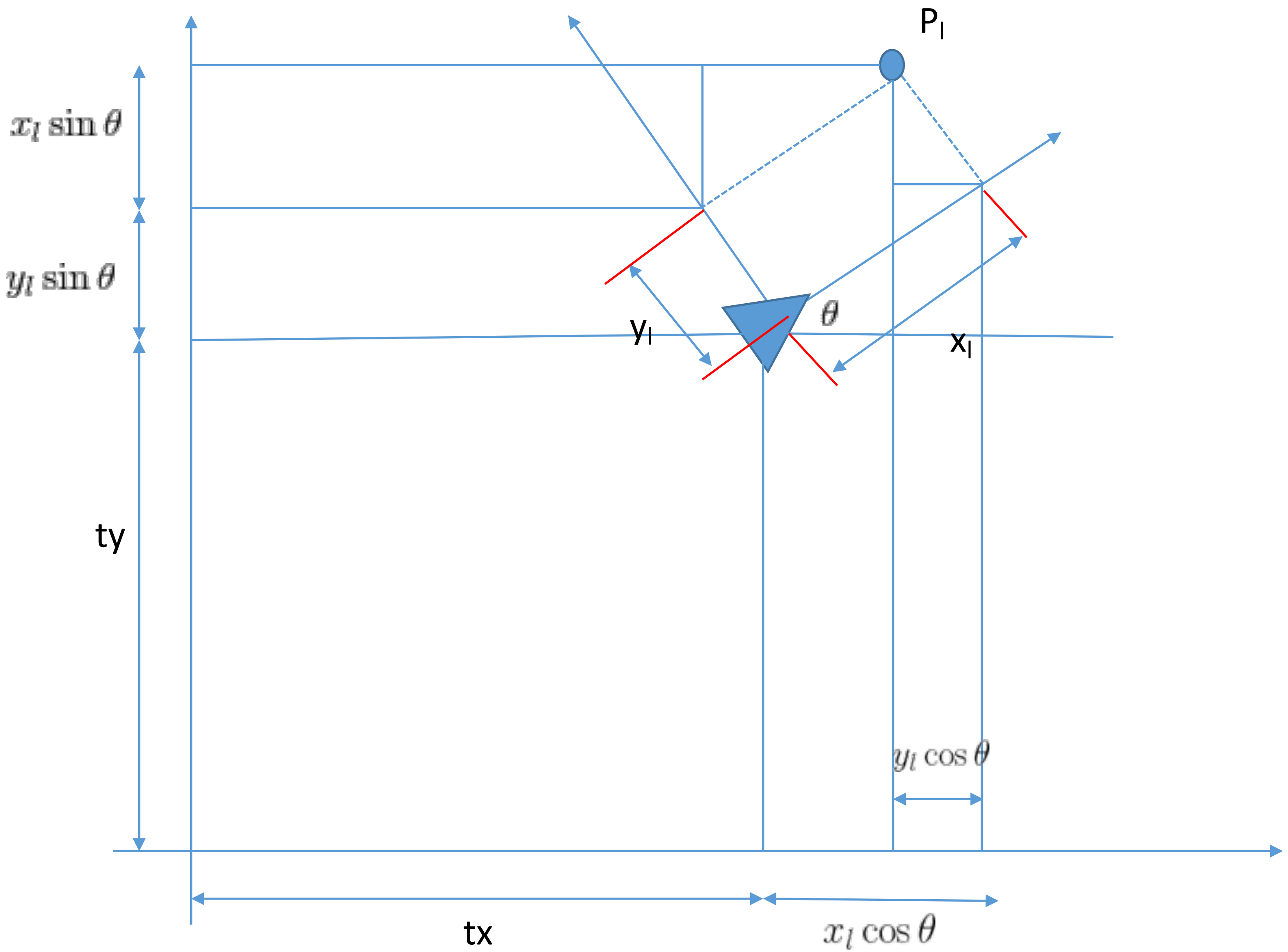
Rotation and Translation

<u>REFERENCES</u>

→ Estimating Uncertain Spatial Relationships in Robotics. Smith, Self and Cheesman

→Large Area Visually Augmented Navigation for Autonomous Underwater Vehicles. Ryan Eustice.

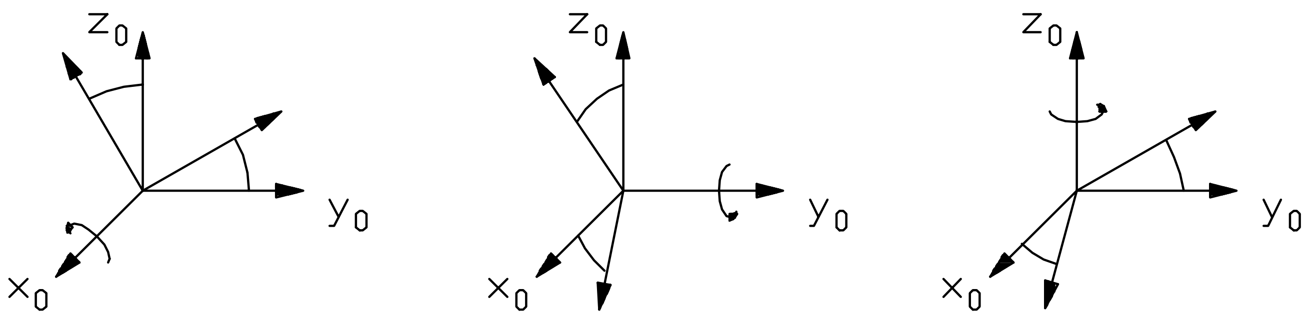
2D Rigid Body Transformation



2D Rigid Body Transformation

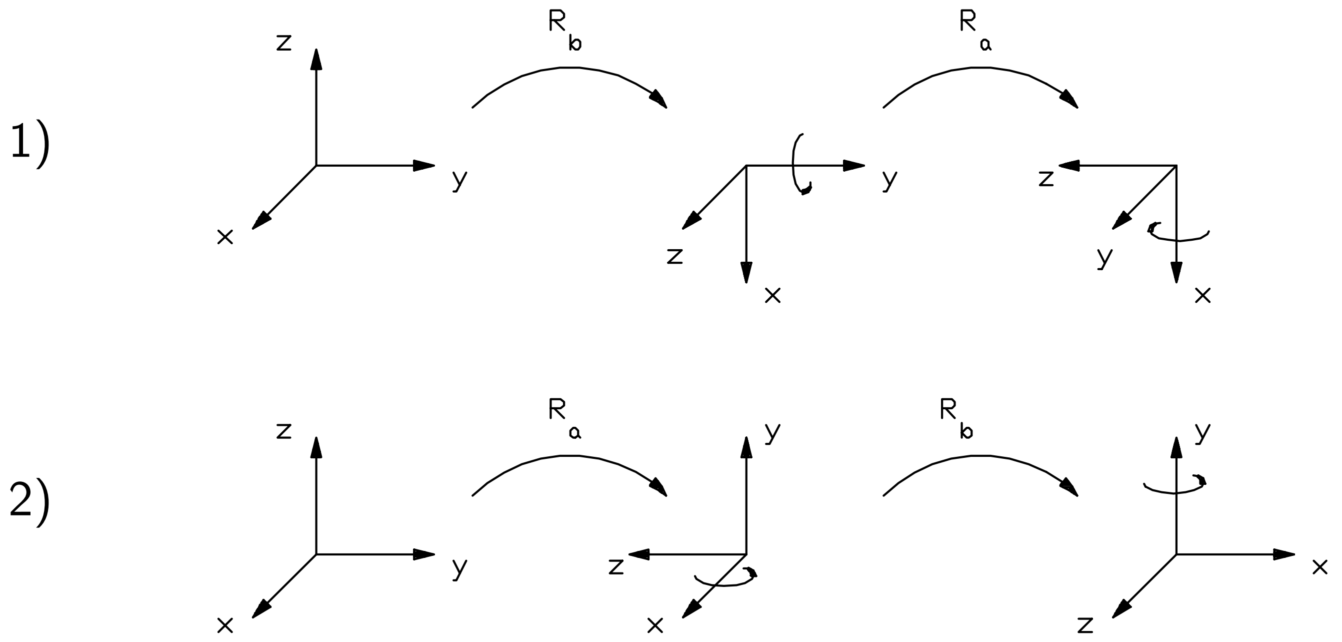
$$P_w = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_l \\ y_l \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

3D Rotation



$$
\begin{aligned}
{}^{i}_{j}\mathrm{R} &= \mathrm{rotxyz}(\boldsymbol{\Theta}_{ij}) \\
&= \mathrm{rotz}(\psi_{ij})^{\top}\,\mathrm{roty}(\theta_{ij})^{\top}\,\mathrm{rotx}(\phi_{ij})^{\top} \quad \text{(principle rotation sequence)} \\
&= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\top} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}^{\top} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}^{\top} \\
&= \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}.
\end{aligned}
$$

Sequence of Rotation Matters !



Case 1) $\mathbf{R}_b$ followed by $\mathbf{R}_a$:

$$\mathbf{R} = \mathbf{R}_b\mathbf{R}_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

Case 2) $\mathbf{R}_a$ followed by $\mathbf{R}_b$:

$$\mathbf{R} = \mathbf{R}_a\mathbf{R}_b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Angles from Rotation Matrix

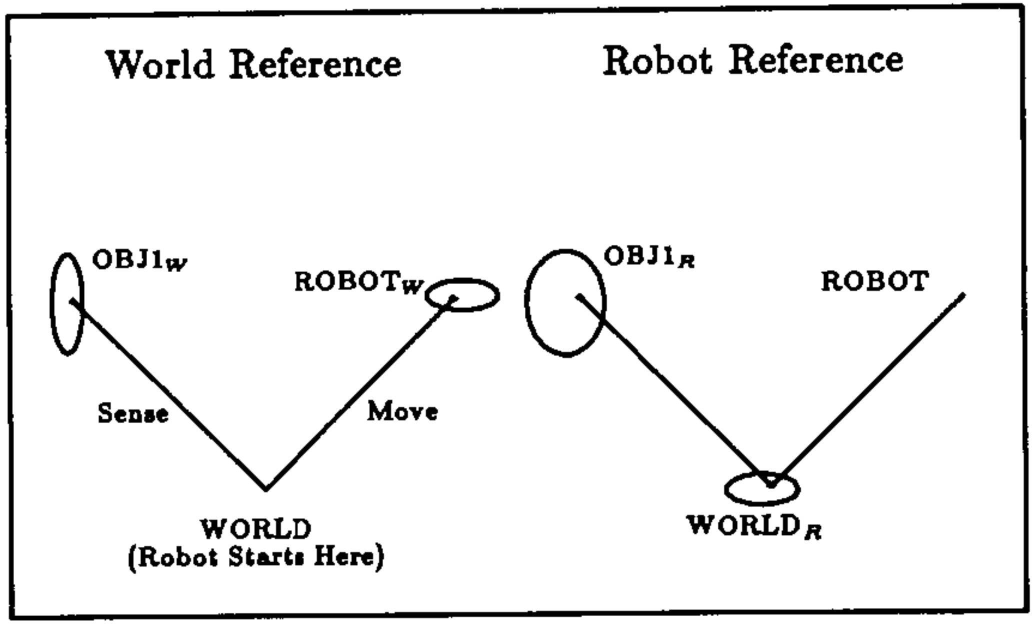
$$\begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi+\cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi+\cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi+\sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi+\sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

$$\begin{aligned} \phi_{ij} &= \operatorname{atan2}\left({}^{i}_{j}\mathrm{R}_{1,3}\sin\psi_{ij} - {}^{i}_{j}\mathrm{R}_{2,3}\cos\psi_{ij}, \quad -{}^{i}_{j}\mathrm{R}_{1,2}\sin\psi_{ij} + {}^{i}_{j}\mathrm{R}_{2,2}\cos\psi_{ij}\right) \\ \theta_{ij} &= \operatorname{atan2}\left(-{}^{i}_{j}\mathrm{R}_{3,1}, \quad {}^{i}_{j}\mathrm{R}_{1,1}\cos\psi_{ij} + {}^{i}_{j}\mathrm{R}_{2,1}\sin\psi_{ij}\right) \\ \psi_{ij} &= \operatorname{atan2}\left({}^{i}_{j}\mathrm{R}_{2,1}, \quad {}^{i}_{j}\mathrm{R}_{1,1}\right) \end{aligned}$$

$$\mathsf{atan2}(y,x) = \begin{cases} \arctan(y/x) & x > 0 \\ \pi + \arctan(y/x) & y \geq 0, x < 0 \\ -\pi + \arctan(y/x) & y < 0, x < 0 \\ \pi/2 & y > 0, x = 0 \\ -\pi/2 & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

Robot state

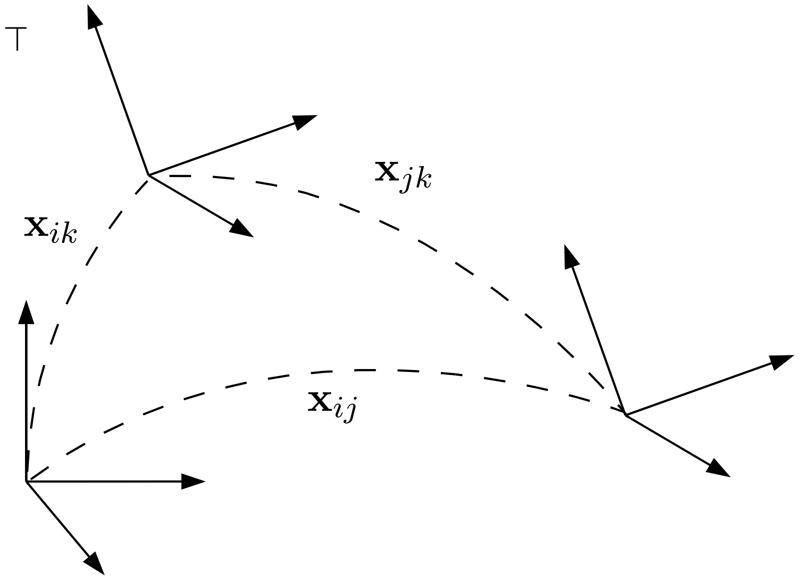
$$\mathbf{x} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix} \qquad \hat{\mathbf{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{\phi} \end{bmatrix}, \qquad \mathbf{C}(\mathbf{x}) = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\phi} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{y\phi} \\ \sigma_{x\phi} & \sigma_{y\phi} & \sigma_\phi^2 \end{bmatrix}$$



Reference: Estimating Uncertain Spatial Relationships in Robotics. Smith, Self and Cheesman

Arbitrary Transformation

$$\mathbf{x}_{ij} = \left[{}^{i}\mathbf{t}_{ij}{}^{\top}, \mathbf{\Theta}_{ij}^{\top}\right]^{\top} = [x_{ij}, y_{ij}, z_{ij}, \phi_{ij}, \theta_{ij}, \psi_{ij}]^{\top}$$



$${}_{j}^{i}\mathrm{H} = \begin{bmatrix} {}_{j}^{i}\mathrm{R} & {}^{i}\mathbf{t}_{ij} \\ \mathbf{0} & 1 \end{bmatrix}$$

$${}_{k}^{i}\mathrm{H} = {}_{j}^{i}\mathrm{H}\ {}_{k}^{j}\mathrm{H}$$

$$\begin{aligned} \mathbf{x}_{ik} &= \mathbf{x}_{ij} \oplus \mathbf{x}_{jk} \\ &= [x_{ik}, y_{ik}, z_{ik}, \phi_{ik}, \theta_{ik}, \psi_{ik}]^{\top} \\ &= \begin{bmatrix} {}_{j}^{i}\mathrm{R} \begin{bmatrix} x_{jk} \\ y_{jk} \\ z_{jk} \end{bmatrix} + \begin{bmatrix} x_{ij} \\ y_{ij} \\ z_{ij} \end{bmatrix} \\ \operatorname{atan2}\left({}_{k}^{i}\mathrm{R}_{1,3} \sin\psi_{ik} - {}_{k}^{i}\mathrm{R}_{2,3} \cos\psi_{ik}, \quad -{}_{k}^{i}\mathrm{R}_{1,2} \sin\psi_{ik} + {}_{k}^{i}\mathrm{R}_{2,2} \cos\psi_{ik}\right) \\ \operatorname{atan2}\left(-{}_{k}^{i}\mathrm{R}_{3,1}, \quad {}_{k}^{i}\mathrm{R}_{1,1} \cos\psi_{ik} + {}_{k}^{i}\mathrm{R}_{2,1} \sin\psi_{ik}\right) \\ \operatorname{atan2}\left({}_{k}^{i}\mathrm{R}_{2,1}, \quad {}_{k}^{i}\mathrm{R}_{1,1}\right) \end{bmatrix} \end{aligned}$$

Let us look at 2D for simplicity

$$\mathbf{x}_{ik} \triangleq \mathbf{x}_{ij} \oplus \mathbf{x}_{jk} = \begin{bmatrix} x_{jk}\cos\phi_{ij} - y_{jk}\sin\phi_{ij} + x_{ij} \\ x_{jk}\sin\phi_{ij} + y_{jk}\cos\phi_{ij} + y_{ij} \\ \phi_{ij} + \phi_{jk} \end{bmatrix}$$

$$\hat{\mathbf{x}}_{ik} \approx \hat{\mathbf{x}}_{ij} \oplus \hat{\mathbf{x}}_{jk}$$

Mean

$$\mathbf{C}(\mathbf{x}_{ik}) \approx \mathbf{J}_{\oplus} \begin{bmatrix} \mathbf{C}(\mathbf{x}_{ij}) & \mathbf{C}(\mathbf{x}_{ij}, \mathbf{x}_{jk}) \\ \mathbf{C}(\mathbf{x}_{jk}, \mathbf{x}_{ij}) & \mathbf{C}(\mathbf{x}_{jk}) \end{bmatrix} \mathbf{J}_{\oplus}^{T}$$

First Order Estimate of Covariance

Reference: Estimating Uncertain Spatial Relationships in Robotics. Smith, Self and Cheesman

Jacobian

$$\mathbf{J}_{\oplus} \triangleq \frac{\partial(\mathbf{x}_{ij} \oplus \mathbf{x}_{jk})}{\partial(\mathbf{x}_{ij}, \mathbf{x}_{jk})} = \frac{\partial \mathbf{x}_{ik}}{\partial(\mathbf{x}_{ij}, \mathbf{x}_{jk})} = \begin{bmatrix} 1 & 0 & -(y_{ik} - y_{ij}) & \cos\phi_{ij} & -\sin\phi_{ij} & 0 \\ 0 & 1 & (x_{ik} - x_{ij}) & \sin\phi_{ij} & \cos\phi_{ij} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{x}_{ik} \triangleq \mathbf{x}_{ij} \oplus \mathbf{x}_{jk} = \begin{bmatrix} x_{jk}\cos\phi_{ij} - y_{jk}\sin\phi_{ij} + x_{ij} \\ x_{jk}\sin\phi_{ij} + y_{jk}\cos\phi_{ij} + y_{ij} \\ \phi_{ij} + \phi_{jk} \end{bmatrix}$$

If the two spatial relationships are independent then:

$$\mathbf{C}(\mathbf{x}_{ik}) \approx \mathbf{J}_{1\oplus}\mathbf{C}(\mathbf{x}_{ij})\mathbf{J}_{1\oplus}^T + \mathbf{J}_{2\oplus}\mathbf{C}(\mathbf{x}_{jk})\mathbf{J}_{2\oplus}^T$$

$$\mathbf{J}_{\oplus} = \begin{bmatrix} \mathbf{J}_{1\oplus} & \mathbf{J}_{2\oplus} \end{bmatrix}$$

Reference: Estimating Uncertain Spatial Relationships in Robotics. Smith, Self and Cheesman

Example

Example:
Given vehicle pose in the local-level frame, $\mathbf{x}_{\ell v}$, we can compute the corresponding sensor pose in the local-level frame as

$$\mathbf{x}_{\ell s} = \mathbf{x}_{\ell v} \oplus \mathbf{x}_{vs}$$

where $\mathbf{x}_{vs}$ is the static sensor-to-vehicle pose. Assuming that vehicle pose $\mathbf{x}_{\ell v}$ is a random variable with covariance $\Sigma_{\mathbf{x}_{\ell v}}$, then to first-order the covariance of the sensor pose in the local-level frame is

$$\Sigma_{\mathbf{x}_{\ell s}} = \mathrm{J}_{\oplus 1} \Sigma_{\mathbf{x}_{\ell v}} \mathrm{J}_{\oplus 1}^{\top}$$

since $\mathbf{x}_{vs}$ is static and assumed *known* (i.e., $\mathbf{x}_{vs}$ is not a random variable).

Inverse Operation

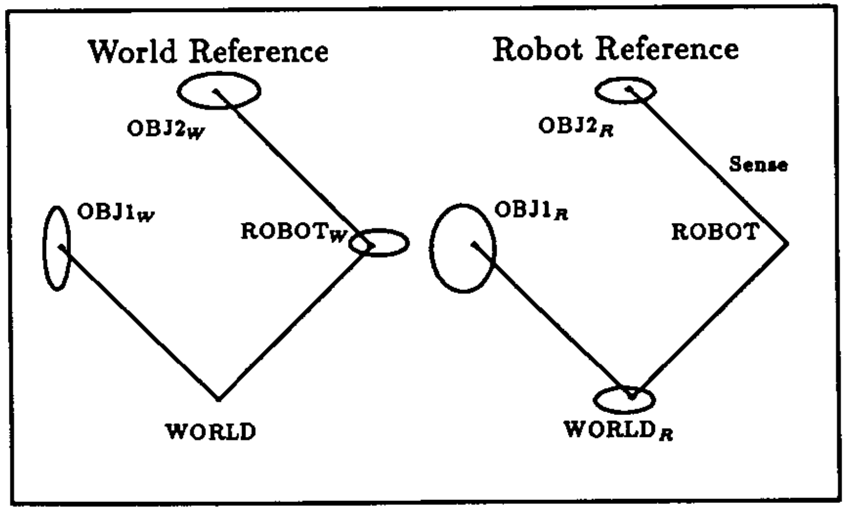
$$ {}^{j}_{i}\mathrm{H} = {}^{i}_{j}\mathrm{H}^{-1} $$

$$
\begin{aligned}
\mathbf{x}_{ji} &= \ominus \mathbf{x}_{ij} \\
&= [x_{ji}, y_{ji}, z_{ji}, \phi_{ji}, \theta_{ji}, \psi_{ji}]^{\top} \\
&= \begin{bmatrix}
-{}^{i}_{j}\mathrm{R}^{\top} \begin{bmatrix} x_{ij} \\ y_{ij} \\ z_{ij} \end{bmatrix} \\
\mathrm{atan2}\big({}^{i}_{j}\mathrm{R}_{3,1} \sin\psi_{ji} - {}^{i}_{j}\mathrm{R}_{3,2} \cos\psi_{ji}, \quad -{}^{i}_{j}\mathrm{R}_{2,1} \sin\psi_{ji} + {}^{i}_{j}\mathrm{R}_{2,2} \cos\psi_{ji}\big) \\
\mathrm{atan2}\big(-{}^{i}_{j}\mathrm{R}_{1,3}, \quad {}^{i}_{j}\mathrm{R}_{1,1} \cos\psi_{ji} + {}^{i}_{j}\mathrm{R}_{1,2} \sin\psi_{ji}\big) \\
\mathrm{atan2}\big({}^{i}_{j}\mathrm{R}_{1,2}, \quad {}^{i}_{j}\mathrm{R}_{1,1}\big)
\end{bmatrix}
\end{aligned}
$$

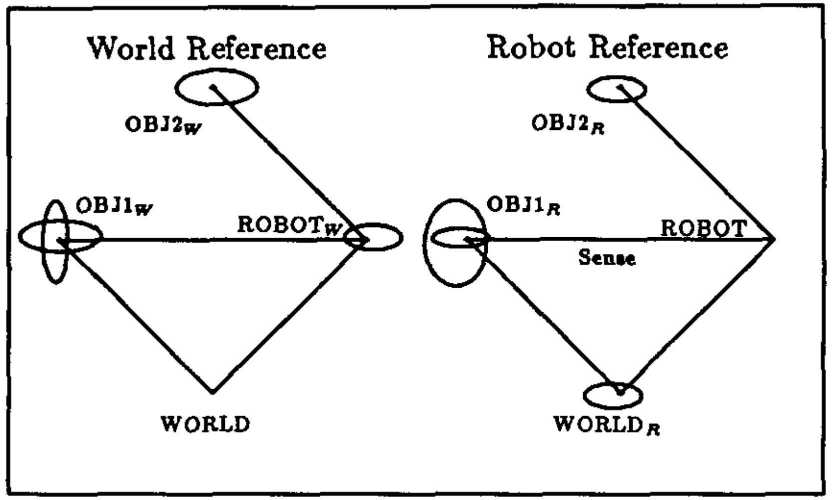
Recall:

$$
\begin{aligned}
\phi_{ij} &= \mathrm{atan2}\big({}^{i}_{j}\mathrm{R}_{1,3} \sin\psi_{ij} - {}^{i}_{j}\mathrm{R}_{2,3} \cos\psi_{ij}, \quad -{}^{i}_{j}\mathrm{R}_{1,2} \sin\psi_{ij} + {}^{i}_{j}\mathrm{R}_{2,2} \cos\psi_{ij}\big) \\
\theta_{ij} &= \mathrm{atan2}\big(-{}^{i}_{j}\mathrm{R}_{3,1}, \quad {}^{i}_{j}\mathrm{R}_{1,1} \cos\psi_{ij} + {}^{i}_{j}\mathrm{R}_{2,1} \sin\psi_{ij}\big) \\
\psi_{ij} &= \mathrm{atan2}\big({}^{i}_{j}\mathrm{R}_{2,1}, \quad {}^{i}_{j}\mathrm{R}_{1,1}\big)
\end{aligned}
$$

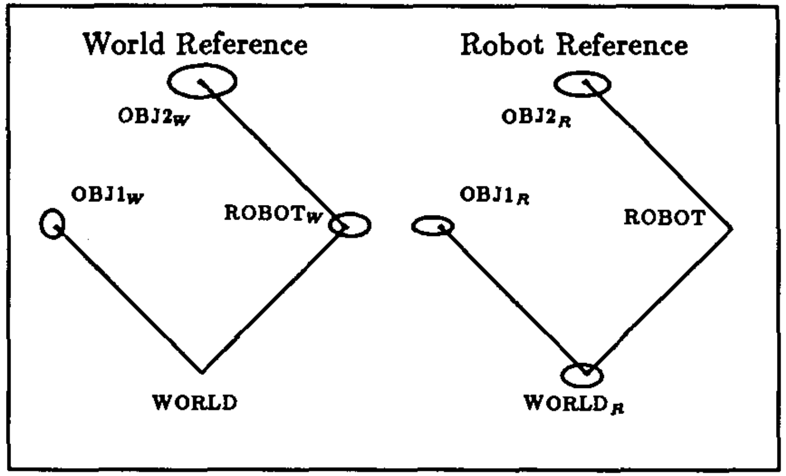
Uncertainity of Spatial Relations



Robot sees object 2



Robot sees object 1 again



Updated Estimates