Extended Kalman Filters

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Kalman Filter Algorithm

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$$
$$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\, \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$$

return μ_t, Σ_t

Kalman Filter vs Bayes Filter

Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$

$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$

$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\, \bar{\mu}_t)$

$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$

return μ_t, Σ_t

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$**):**

for all x_t do

$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$

$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$

endfor

return $bel(x_t)$

Note: Predicted belief is Gaussian and posterior belief is also Gaussian

Assumptions in Kalman Filter

- Initial belief $Bel(x_0)$ is a Gaussian distribution
- State at time *t+1* is a linear function of state at time *t*

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

- Observations are linear in the state

$$z_t = C_t x_t + \delta_t$$

- Error terms are zero-mean random variables which are normally distributed

$$\varepsilon_t = N(0, R_t) \qquad \delta_t = N(0, Q_t)$$

Linear Gaussian System

- State Transition Probability is Gaussian

$$
\begin{aligned}
& p(x_t \mid u_t, x_{t-1}) \\
&\quad = \quad \det\left(2\pi R_t\right)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\}
\end{aligned}
$$

- Measurement Probability is also Gaussian

$$
p(z_t \mid x_t) \quad = \quad \det\left(2\pi Q_t\right)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}(z_t - C_t\ x_t)^T\ Q_t^{-1}\ (z_t - C_t\ x_t)\right\}
$$

Prediction

- Predicted mean

$$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$$

- Predicted Covariance

$$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$$

Measurement Update

- Posterior Belief

$$bel(x_t) \quad = \quad \eta \quad \underbrace{p(z_t \mid x_t)}_{\sim\mathcal{N}(z_t;C_t x_t,Q_t)} \quad \underbrace{\overline{bel}(x_t)}_{\sim\mathcal{N}(x_t;\bar{\mu}_t,\bar{\Sigma}_t)}$$

$$bel(x_t) \quad = \quad \eta \ \exp\left\{-J_t\right\}$$

$$J_t \quad = \quad \tfrac{1}{2}\ (z_t - C_t x_t)^T\ Q_t^{-1}\ (z_t - C_t x_t) + \tfrac{1}{2}\ (x_t - \bar{\mu}_t)^T\ \bar{\Sigma}_t^{-1}\ (x_t - \bar{\mu}_t)$$

Derivative of Quadratic Form in Gaussian Distribution

- First derivative of the quadratic expression inside exponential gives us the mean of the distribution.
- Second derivative gives us the covariance.

$$C = \frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu) = \frac{1}{2} x \Sigma^{-1} x - 2x^\top \Sigma^{-1}\mu + \mu^\top \mu$$

$$\frac{\partial C}{\partial x} = \Sigma^{-1}x - \Sigma^{-1}\mu$$

$$\frac{\partial^2 C}{\partial x^2} = \Sigma^{-1}$$

Measurement Update

$$
\begin{aligned}
J_t &= \tfrac{1}{2}\,(z_t - C_t x_t)^T\, Q_t^{-1}\,(z_t - C_t x_t) + \tfrac{1}{2}\,(x_t - \bar{\mu}_t)^T\, \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t) \\
\frac{\partial J}{\partial x_t} &= -C_t^T\, Q_t^{-1}\,(z_t - C_t x_t) + \bar{\Sigma}_t^{-1}\,(x_t - \bar{\mu}_t) \\
\frac{\partial^2 J}{\partial x_t^2} &= C_t^T\, Q_t^{-1}\, C_t + \bar{\Sigma}_t^{-1} \\
\Sigma_t &= (C_t^T\, Q_t^{-1}\, C_t + \bar{\Sigma}_t^{-1})^{-1}
\end{aligned}
$$

The mean of $bel(x_t)$ is the minimum of this quadratic function, which we now calculate by setting the first derivative of J_t to zero (and substituting μ_t for x_t):

$$
C_t^T\, Q_t^{-1}\,(z_t - C_t \mu_t) = \bar{\Sigma}_t^{-1}\,(\mu_t - \bar{\mu}_t)
$$

$$
\boxed{\mu_t = \bar{\mu}_t + K_t\,(z_t - C_t\,\bar{\mu}_t)}
\qquad
\boxed{K_t = \Sigma_t\, C_t^T\, Q_t^{-1}}
$$

Kalman Gain

- Kalman gain is a function of state covariance and the state covariance is obtained after twice inversion of the predicted state covariance matrix. In case of large state space (robot pose and landmarks !) this is computationally very expensive

$$K_t = \Sigma_t \, C_t^T \, Q_t^{-1}$$

$$\Sigma_t = \underbrace{(C_t^T \, Q_t^{-1} \, C_t + \bar{\Sigma}_t^{-1})^{-1}}_{\text{[n x n] Matrix}}$$

Kalman Filter Algorithm

Algorithm Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$$\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$$
$$\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$$

$$K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\, \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t\, C_t)\, \bar{\Sigma}_t$$

return μ_t, Σ_t

Extended Kalman Filter

- In mobile robotics the state transition model and the measurement model are generally <u>**non-linear**</u>

$$\begin{aligned} x_t &= g(u_t, x_{t-1}) + \varepsilon_t \\ z_t &= h(x_t) + \delta_t \ . \end{aligned}$$

- EKF linearizes the process and measurement models and then estimate the belief for this linearized system
- Note: In Kalman Filter the belief is <u>**exactly Gaussian**</u>

Linearization via Taylor Series Expansion

- Linearized process model:

$$
\begin{aligned}
g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=:\, G_t} (x_{t-1} - \mu_{t-1}) \\
&= g(u_t, \mu_{t-1}) + G_t\, (x_{t-1} - \mu_{t-1})
\end{aligned}
$$

$$
g'(u_t, x_{t-1}) := \frac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}}
$$

$$
\begin{aligned}
& p(x_t \mid u_t, x_{t-1}) \\
&\quad \approx \det\left(2\pi R_t\right)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\, [x_t - g(u_t, \mu_{t-1}) - G_t\, (x_{t-1} - \mu_{t-1})]^T \right. \\
&\qquad \left. R_t^{-1}\, [x_t - g(u_t, \mu_{t-1}) - G_t\, (x_{t-1} - \mu_{t-1})]\right\}
\end{aligned}
$$

Linearization via Taylor Series Expansion

- Linearized measurement model:

$$
\begin{aligned}
h(x_t) &\approx h(\bar{\mu}_t) + \underbrace{h'(\bar{\mu}_t)}_{=:\, H_t} (x_t - \bar{\mu}_t) \\
&= h(\bar{\mu}_t) + H_t\, (x_t - \bar{\mu}_t)
\end{aligned}
$$

$$
\begin{aligned}
p(z_t \mid x_t) &= \det\left(2\pi Q_t\right)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\left[z_t - h(\bar{\mu}_t) - H_t\, (x_t - \bar{\mu}_t)\right]^T \right. \\
&\quad \left. Q_t^{-1} \left[z_t - h(\bar{\mu}_t) - H_t\, (x_t - \bar{\mu}_t)\right]\right\}
\end{aligned}
$$

EKF Algorithm

- Prediction:

$$\overline{bel}(x_t) \quad = \quad \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\sim\mathcal{N}(x_t; g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1}), R_t)} \quad \underbrace{bel(x_{t-1})}_{\sim\mathcal{N}(x_{t-1}; \mu_{t-1}, \Sigma_{t-1})} \quad dx_{t-1}$$

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$
$$\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$$

- Correction:

$$bel(x_t) \quad = \quad \eta \quad \underbrace{p(z_t \mid x_t)}_{\sim\mathcal{N}(z_t; h(\bar{\mu}_t) + H_t \; (x_t - \bar{\mu}_t), Q_t)} \quad \underbrace{\overline{bel}(x_t)}_{\sim\mathcal{N}(x_t; \bar{\mu}_t, \bar{\Sigma}_t)}$$

$$\begin{aligned} K_t &= \bar{\Sigma}_t \, H_t^T \, (H_t \, \bar{\Sigma}_{t-1} \, H_t^T + Q_t)^{-1} \\ \mu_t &= \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t)) \\ \Sigma_t &= (I - K_t \, H_t) \, \bar{\Sigma}_t \end{aligned}$$

EKF Algorithm

Algorithm Extended_Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)**:**

$\bar{\mu}_t = g(u_t, \mu_{t-1})$
$\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t^T + R_t$
$K_t = \bar{\Sigma}_t\ H_t^T (H_t\ \bar{\Sigma}_t\ H_t^T + Q_t)^{-1}$
$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$
$\Sigma_t = (I - K_t\ H_t)\ \bar{\Sigma}_t$
return μ_t, Σ_t

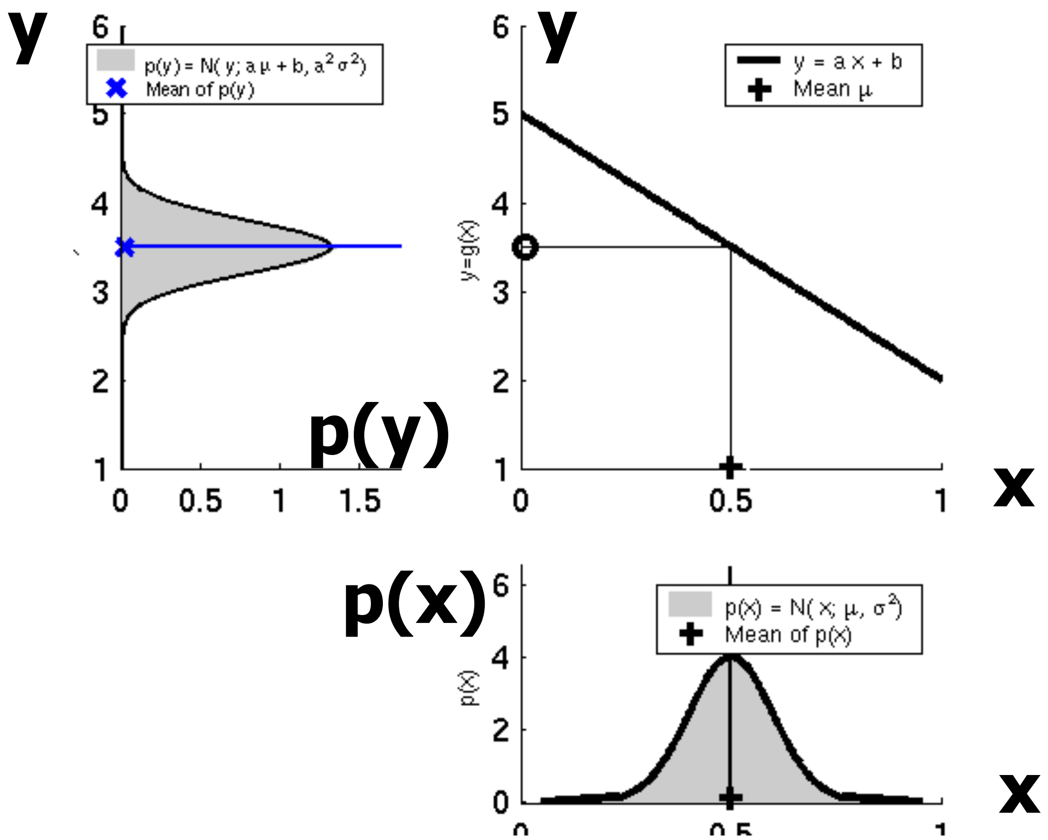
Algorithm Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)**:**

$\bar{\mu}_t = A_t\ \mu_{t-1} + B_t\ u_t$
$\bar{\Sigma}_t = A_t\ \Sigma_{t-1}\ A_t^T + R_t$
$K_t = \bar{\Sigma}_t\ C_t^T (C_t\ \bar{\Sigma}_t\ C_t^T + Q_t)^{-1}$
$\mu_t = \bar{\mu}_t + K_t (z_t - C_t\ \bar{\mu}_t)$
$\Sigma_t = (I - K_t\ C_t)\ \bar{\Sigma}_t$
return μ_t, Σ_t

	Kalman filter	EKF
state prediction	$A_t\ \mu_{t-1} + B_t\ u_t$	$g(u_t, \mu_{t-1})$
measurement prediction	$C_t\ \bar{\mu}_t$	$h(\bar{\mu}_t)$

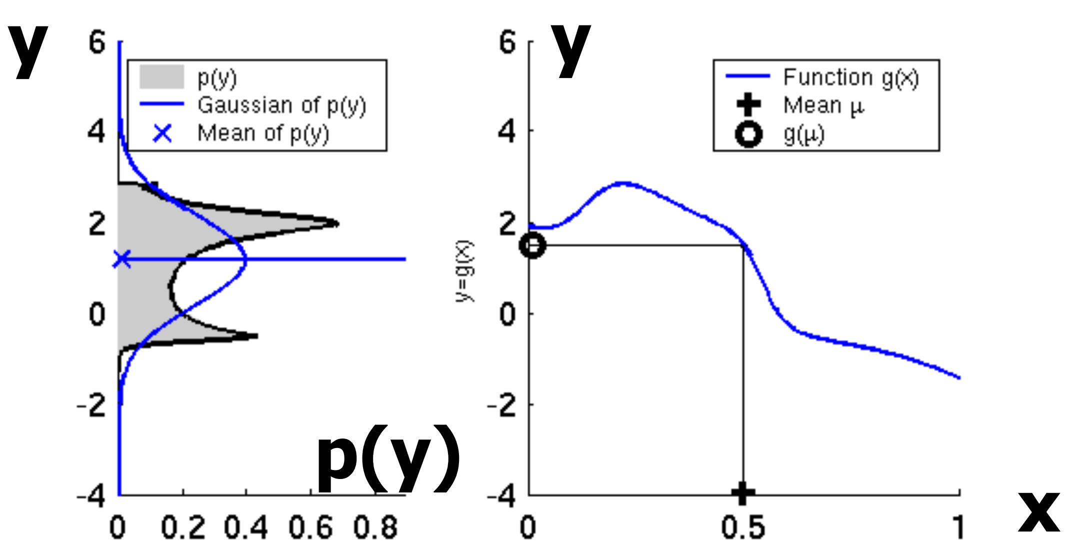
Linear System

- Gaussian remains Gaussian

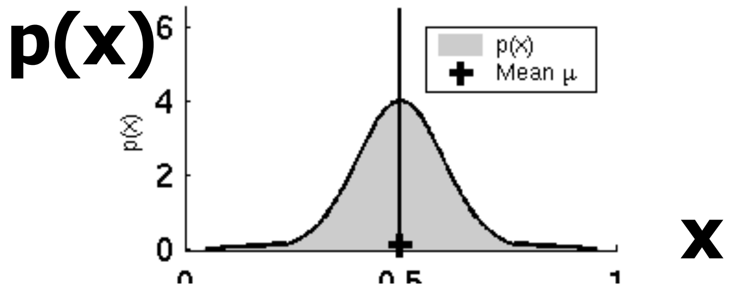


Non-Linear System

- Gaussian assumptions becomes invalid

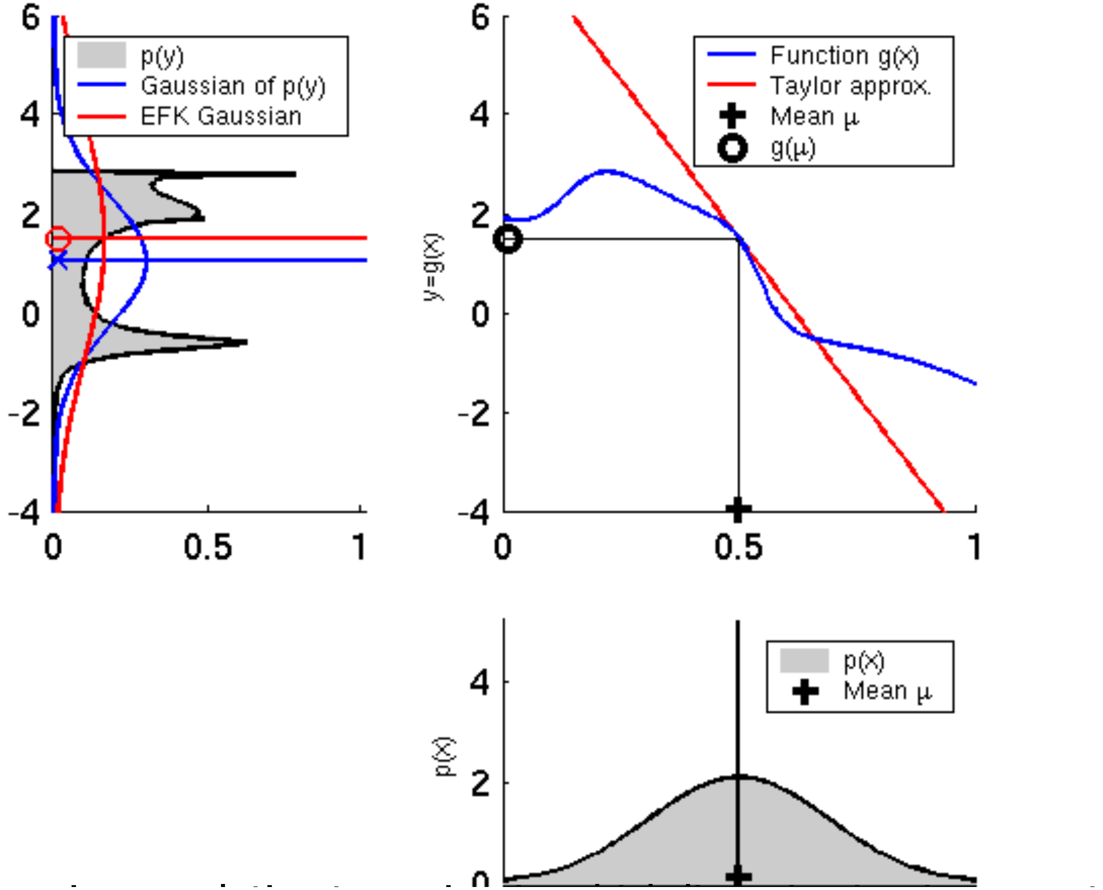


“Gaussian of p(y)” has mean and variance of y under p(y)



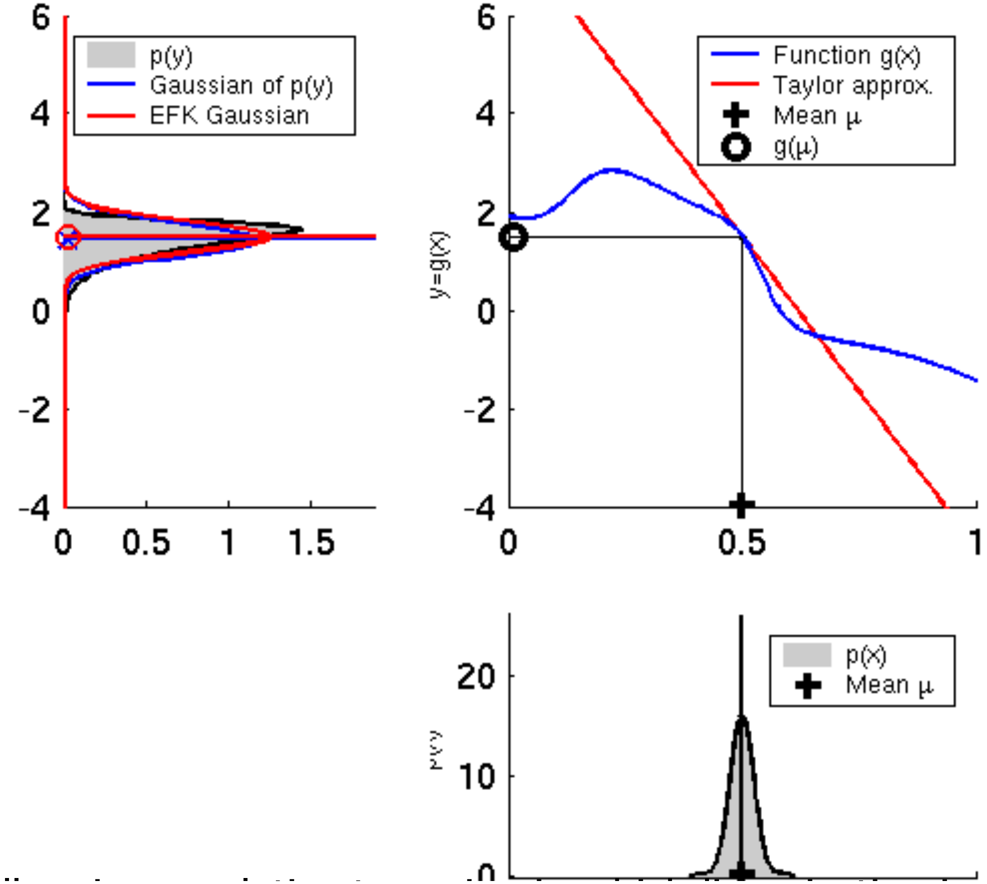
EKF Linearization

- Bel(x) has high variance

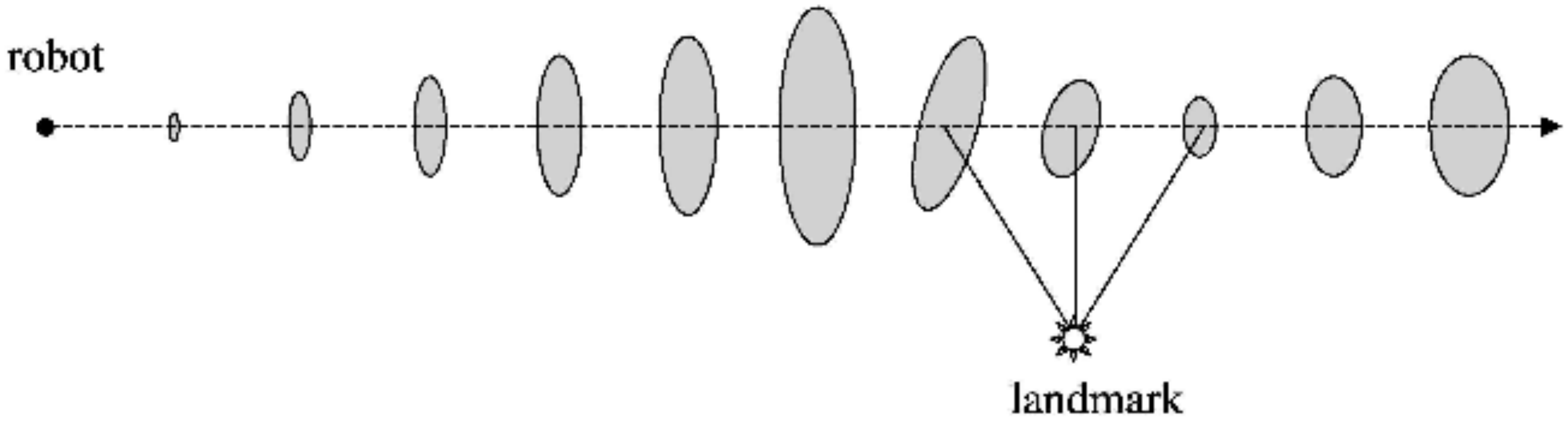


EKF Linearization

- Bel(x) has small variance



EKF Localization Example

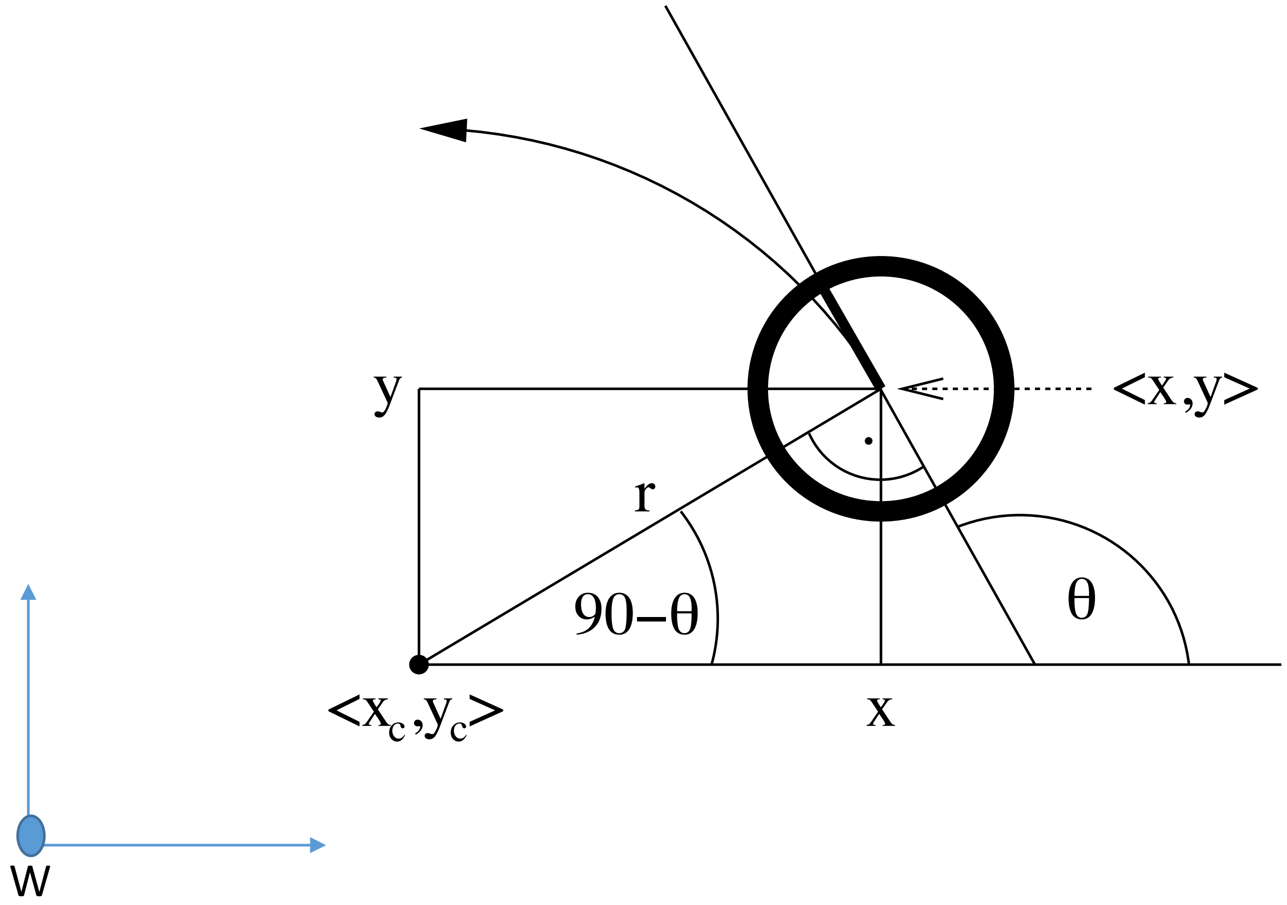


Robot Motion Models

- **Velocity Model:** Assumes that the control is given as a linear and angular velocity commands to the motors. These velocities remain constant over the sampling period.
- **Acceleration Model:** Slightly more complicated as it assumes a constant acceleration i.e. a linearly increasing velocities.
- **Odometry Model:** Assumes the accessibility of odometry information either from wheel encoders or other means like visual / laser odometry.

Robot Motion Model

- Velocity Motion Model:



$$x_c = x - \frac{v}{\omega}\sin\theta$$
$$y_c = y + \frac{v}{\omega}\cos\theta$$

Velocity Motion Model: Exact

Let $\boldsymbol{x}_{t-1} = \begin{pmatrix} x_{t-1} & y_{t-1} & \theta_{t-1} \end{pmatrix}^{\mathsf{T}}$ the initial pose, $\boldsymbol{u}_t = (v_t \quad \omega_t)^{\mathsf{T}}$ the constant control in the interval between $(t-1)\Delta t$ and $t\Delta t$, knowing the rotation center $\boldsymbol{c} = (x_c \quad y_c)$, we have the differential motion model

$$\begin{aligned}
\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} &= \begin{pmatrix} x_c + \frac{v_t}{\omega_t}\sin(\theta_{t-1} + \omega_t\Delta t) \\ y_c - \frac{v_t}{\omega_t}\cos(\theta_{t-1} + \omega_t\Delta t) \\ \theta_{t-1} + \omega_t\Delta t \end{pmatrix} \\
&= \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin(\theta_{t-1}) + \frac{v_t}{\omega_t}\sin(\theta_{t-1} + \omega_t\Delta t) \\ \frac{v_t}{\omega_t}\cos(\theta_{t-1}) - \frac{v_t}{\omega_t}\cos(\theta_{t-1} + \omega_t\Delta t) \\ \omega_t\Delta t \end{pmatrix}
\end{aligned}$$

Velocity Motion Model: Noisy

$$\underbrace{\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}}_{x_t} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\ \sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\ \cos(\theta + \omega_t \Delta t) \\ \omega \Delta t \end{pmatrix}}_{g(u_t, x_{t-1})} + \mathcal{N}(0, R_t)$$

Linearization of Velocity Model

$$g(u_t, x_{t-1}) \quad \approx \quad g(u_t, \mu_{t-1}) \; + \; G_t \; (x_{t-1} - \mu_{t-1})$$

$$G_t \quad = \quad g'(u_t, \mu_{t-1}) \quad = \quad \left(\begin{array}{ccc} 1 & 0 & \frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{array} \right)$$

EKF Prediction:

$$\bar{\mu}_t = \mu_{t-1} + \left(\begin{array}{c} -\frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} + \frac{v_t}{\omega_t} \ \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \ \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{array} \right)$$

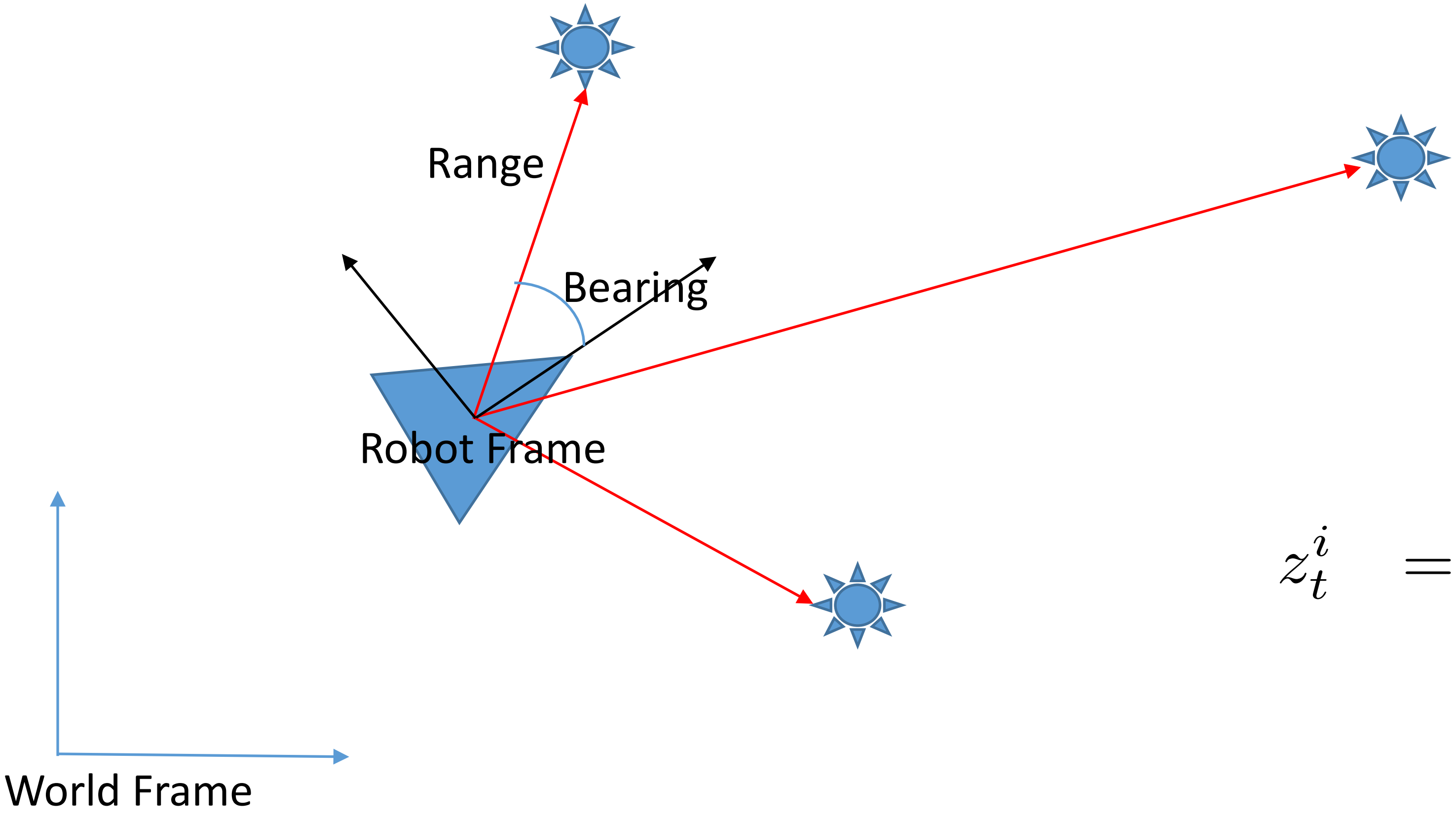
$$G_t = \left(\begin{array}{ccc} 1 & 0 & \frac{v_t}{\omega_t} \cos \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t} \sin \mu_{t-1,\theta} - \frac{v_t}{\omega_t} \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{array} \right)$$

$$\bar{\Sigma}_t = G_t \ \Sigma_{t-1} \ G_t^T + R_t$$

Measurement Models

- Depends upon the sensor:
  - Range and Bearing Sensor (Lidar)
  - Range only sensor (RF devices)
  - Sonar
  - Camera

Range and Bearing Sensor Model



$$z_t^i = \begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix}$$

Measurement Model

$$
\begin{aligned}
z_t^i &= \begin{pmatrix} r_t^i \\ \phi_t^i \\ s_t^i \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ m_{j,s} \end{pmatrix} + \begin{pmatrix} \mathcal{N}(0, \sigma_r) \\ \mathcal{N}(0, \sigma_\phi) \\ \mathcal{N}(0, \sigma_s) \end{pmatrix}
\end{aligned}
$$

$$
z_t^i = h(x_t, j, m) + \mathcal{N}(0, Q_t)
$$

$$
h(x_t, j, m) = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \\ m_{j,s} \end{pmatrix} \qquad Q_t = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\phi & 0 \\ 0 & 0 & \sigma_s \end{pmatrix}
$$

Linearized Measurement Model

$$h(x_t, j, m) \quad \approx \quad h(\bar{\mu}_t, j, m) \; + \; H_t^i \; (x_t - \bar{\mu}_t)$$

$$H_t^t \quad = \quad h'(\bar{\mu}_t, j, m) \quad = \quad \left(\begin{array}{ccc} \frac{\partial r_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial \phi_t^i}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial s_t^i}{\partial \bar{\mu}_{t,x}} & \frac{\partial s_t^i}{\partial \bar{\mu}_{t,y}} & \frac{\partial s_t^i}{\partial \bar{\mu}_{t,\theta}} \end{array} \right)$$

Linearized Measurement Model

$$H_t^i \quad = \quad \begin{pmatrix} \frac{m_{j,x} - \bar{\mu}_{t,x}}{\sqrt{q}_t} & \frac{y_t - \bar{\mu}_{t,y}}{\sqrt{q}_t} & 0 \\ \frac{\bar{\mu}_{t,y} - y_t}{q_t} & \frac{m_{j,x} - \bar{\mu}_{t,x}}{q_t} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

with $q_t = (m_{j,x} - \bar{\mu}_{t,x})^2 + (m_{j,y} - \bar{\mu}_{t,y})^2$

Algorithm EKF_localization_known_correspondences$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, c_t, m)$**:**

$$\bar{\mu}_t = \mu_{t-1} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\mu_{t-1,\theta} + \frac{v_t}{\omega_t}\ \sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \frac{v_t}{\omega_t}\cos\mu_{t-1,\theta} - \frac{v_t}{\omega_t}\ \cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ \omega_t \Delta t \end{pmatrix}$$

$$G_t = \begin{pmatrix} 1 & 0 & \frac{v_t}{\omega_t}\cos\mu_{t-1,\theta} - \frac{v_t}{\omega_t}\cos(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 1 & \frac{v_t}{\omega_t}\sin\mu_{t-1,\theta} - \frac{v_t}{\omega_t}\sin(\mu_{t-1,\theta} + \omega_t \Delta t) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{\Sigma}_t = G_t\ \Sigma_{t-1}\ G_t^T + R_t$$

$$Q_t = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_\phi & 0 \\ 0 & 0 & \sigma_s \end{pmatrix}$$

for all observed features $z_t^i = (r_t^i\ \ \phi_t^i\ \ s_t^i)^T$ *do*

$$j = c_t^i$$

$$\delta = \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} m_{j,x} - \bar{\mu}_{t,x} \\ m_{j,y} - \bar{\mu}_{t,y} \end{pmatrix}$$

$$q = \delta^T \delta$$

$$\hat{z}_t^i = \begin{pmatrix} \sqrt{q} \\ \operatorname{atan2}(\delta_y, \delta_x) - \bar{\mu}_{t,\theta} \\ m_{j,s} \end{pmatrix}$$

$$H_t^i = \frac{1}{q}\begin{pmatrix} \sqrt{q}\delta_x & -\sqrt{q}\delta_y & 0 \\ \delta_y & \delta_x & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_t^i = \bar{\Sigma}_t\ H_t^{i,T}(H_t^i\ \bar{\Sigma}_t\ H_t^{i,T} + Q_t)^{-1}$$

$$\bar{\mu}_t = \bar{\mu}_t + K_t^i(z_t^i - h(\bar{\mu}_t))$$
$$\bar{\Sigma}_t = (I - K_t^i H_t^i)\bar{\Sigma}_t$$

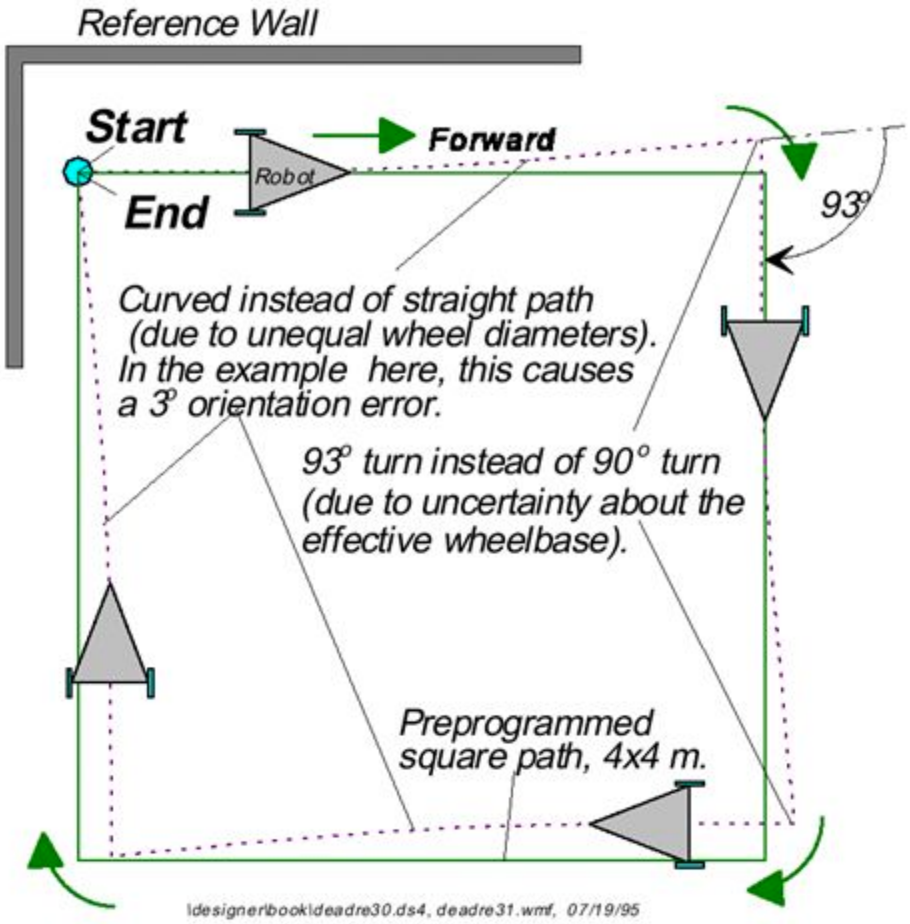
endfor

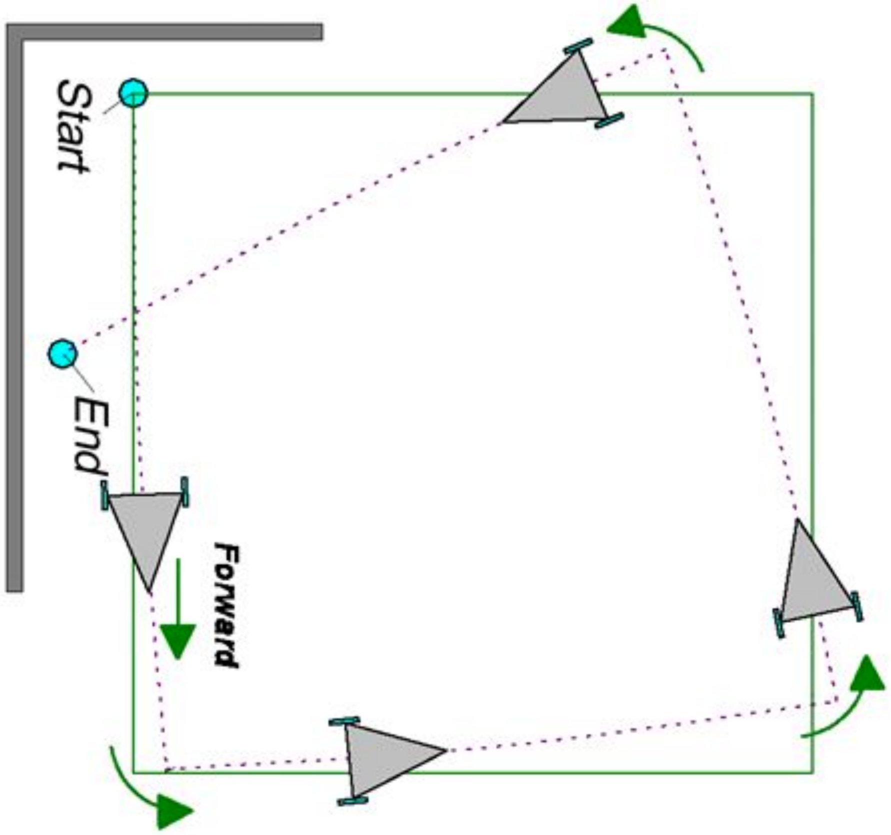
$$\mu_t = \bar{\mu}_t$$

$$\Sigma_t = \bar{\Sigma}_t$$

return μ_t, Σ_t

Typical Motion Error



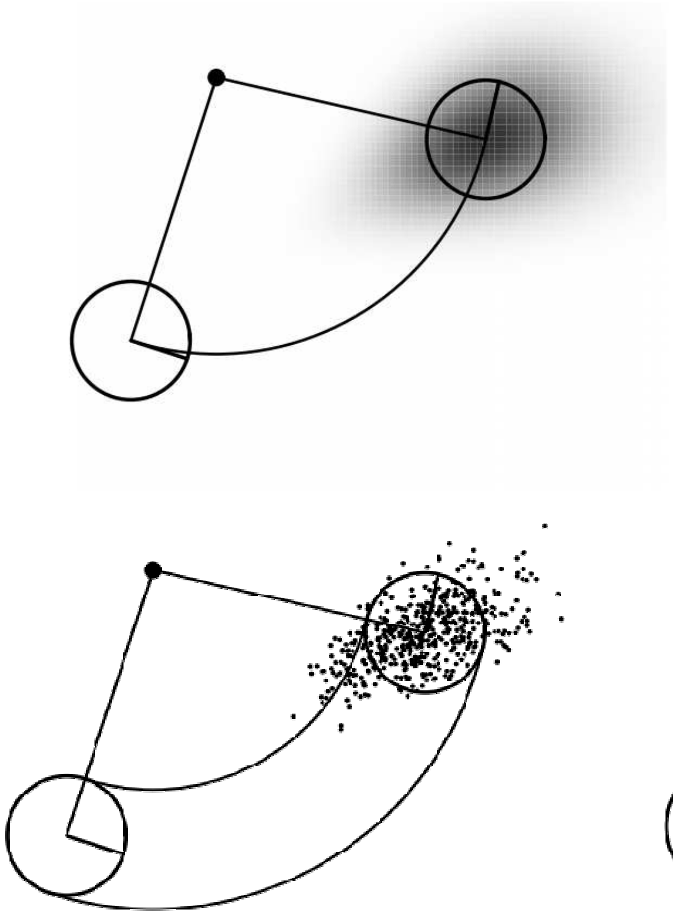


Slide Credit: R. Seigwart

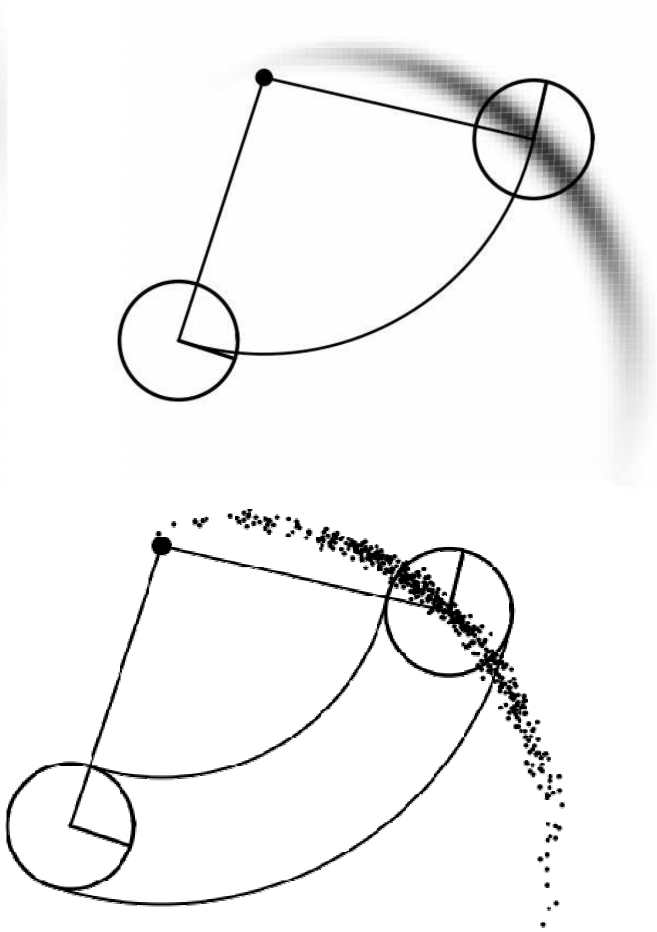
Velocity Motion Model: Additive Noise

$$\underbrace{\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}}_{x_t} = \underbrace{\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v_t}{\omega_t}\sin\theta + \frac{v_t}{\omega_t}\sin(\theta + \omega_t \Delta t) \\ \frac{v_t}{\omega_t}\cos\theta - \frac{v_t}{\omega_t}\cos(\theta + \omega_t \Delta t) \\ \omega \Delta t \end{pmatrix}}_{g(u_t, x_{t-1})} + \mathcal{N}(0, R_t)$$

Observed Noise in Motion Model



Small error in angular velocity



Large error in angular velocity

Actual Noise in Motion Model

$$\begin{pmatrix} \hat{v} \\ \hat{\omega} \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} + \begin{pmatrix} \varepsilon_{\alpha_1|v|+\alpha_2|\omega|} \\ \varepsilon_{\alpha_3|v|+\alpha_4|\omega|} \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}} \sin(\theta + \hat{\omega}\Delta t) \\ \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}} \cos(\theta + \hat{\omega}\Delta t) \\ \hat{\omega}\Delta t \end{pmatrix}$$

EKF Localization

Prediction:

$$G_t = \frac{\partial g(u_t,\mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$$

Jacobian of *g* w.r.t location

$$B_t = \frac{\partial g(u_t,\mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$$

Jacobian of *g* w.r.t control

$$Q_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$$

Motion noise

$$\bar{\mu}_t = g(u_t, \mu_{t-1})$$

Predicted mean

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + B_t Q_t B_t^T$$

Predicted covariance

EKF Localization

Correction:

$$\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$$

Predicted measurement mean

$$H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$$

Jacobian of *h* w.r.t locatior

$$R_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$$

$$S_t = H_t \bar{\Sigma}_t H_t^T + R_t$$

Innovation covariance

$$K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$$

Kalman gain

$$\mu_t = \bar{\mu}_t + K_t (z_t - \hat{z}_t)$$

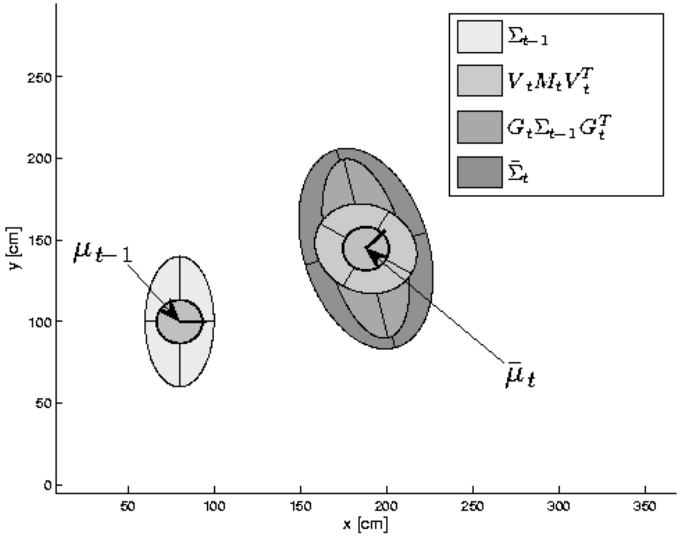
Updated mean

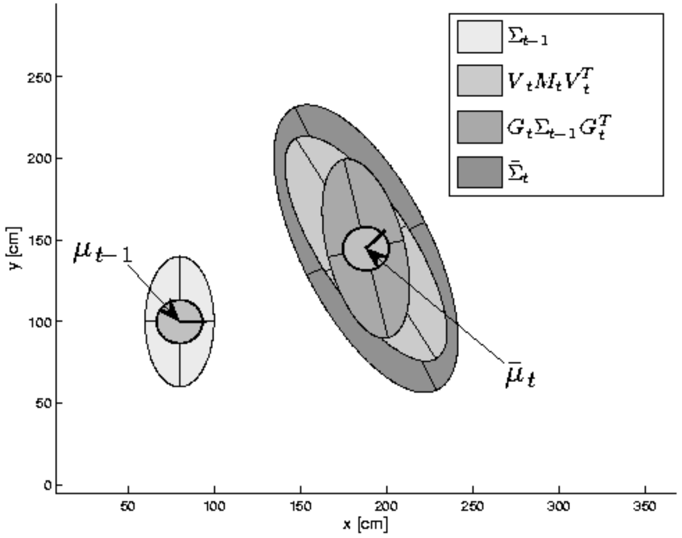
$$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$$

Updated covariance

EKF Prediction

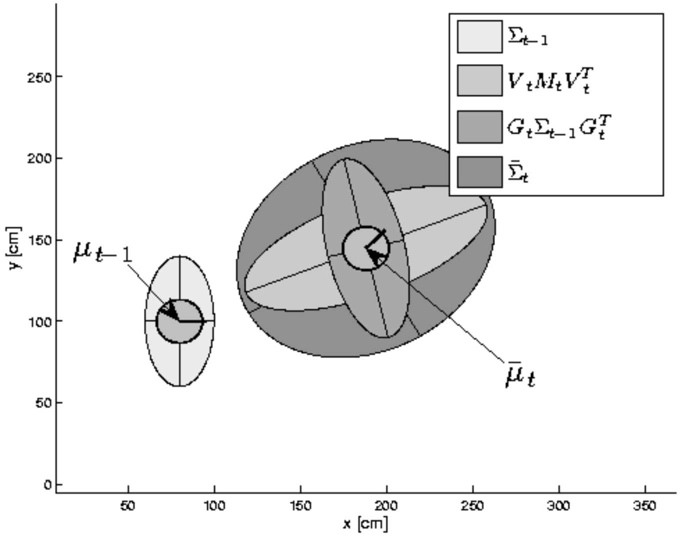
Small Motion Noise

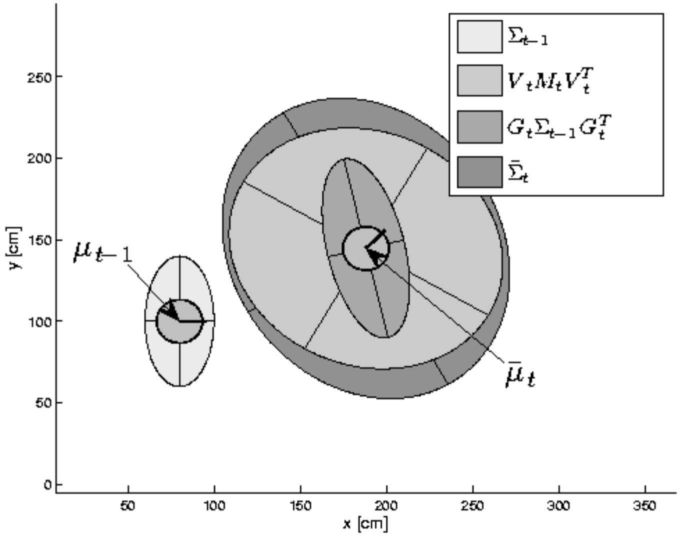




High Rotation Noise

High Translation Noise

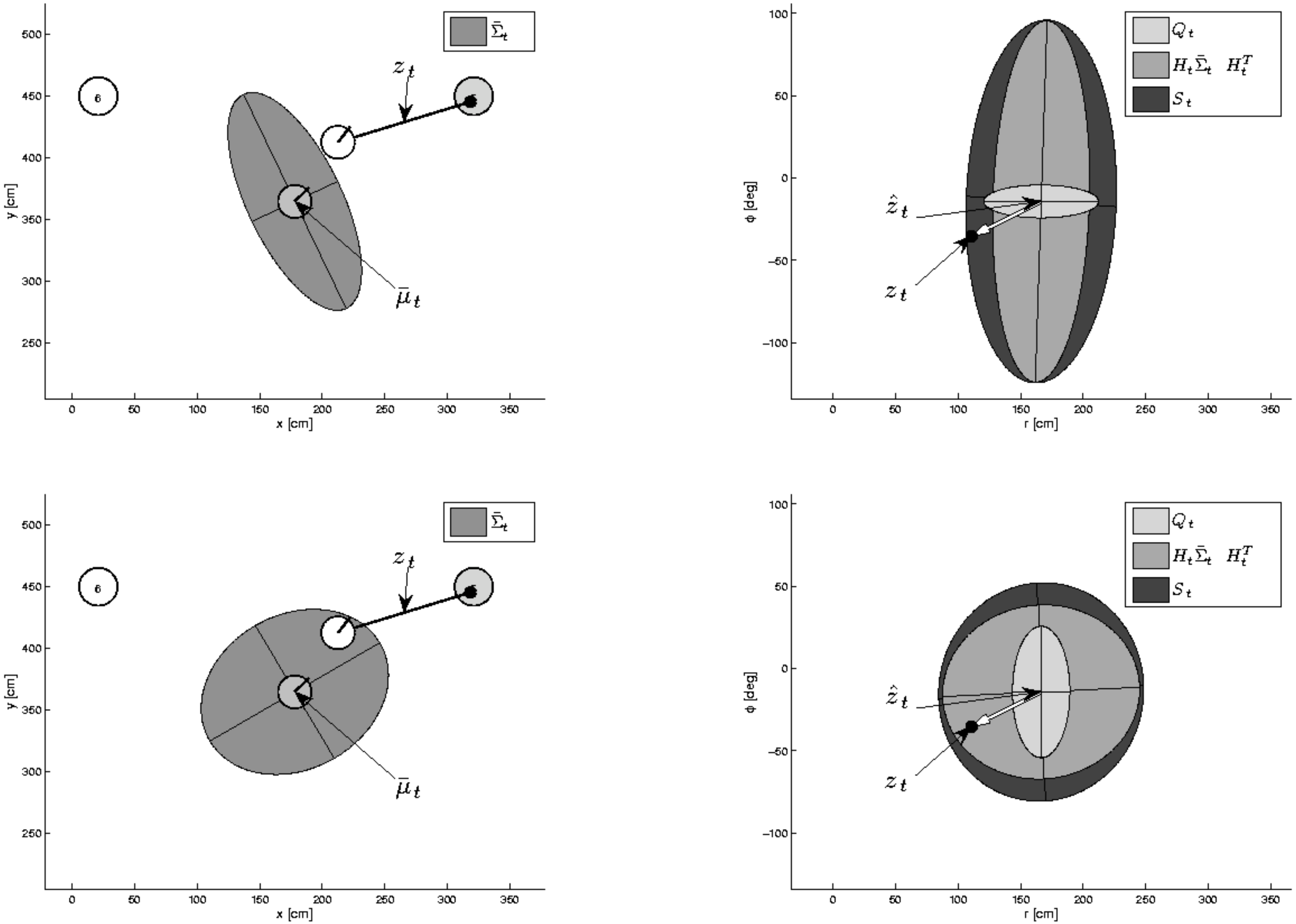




High Motion Noise

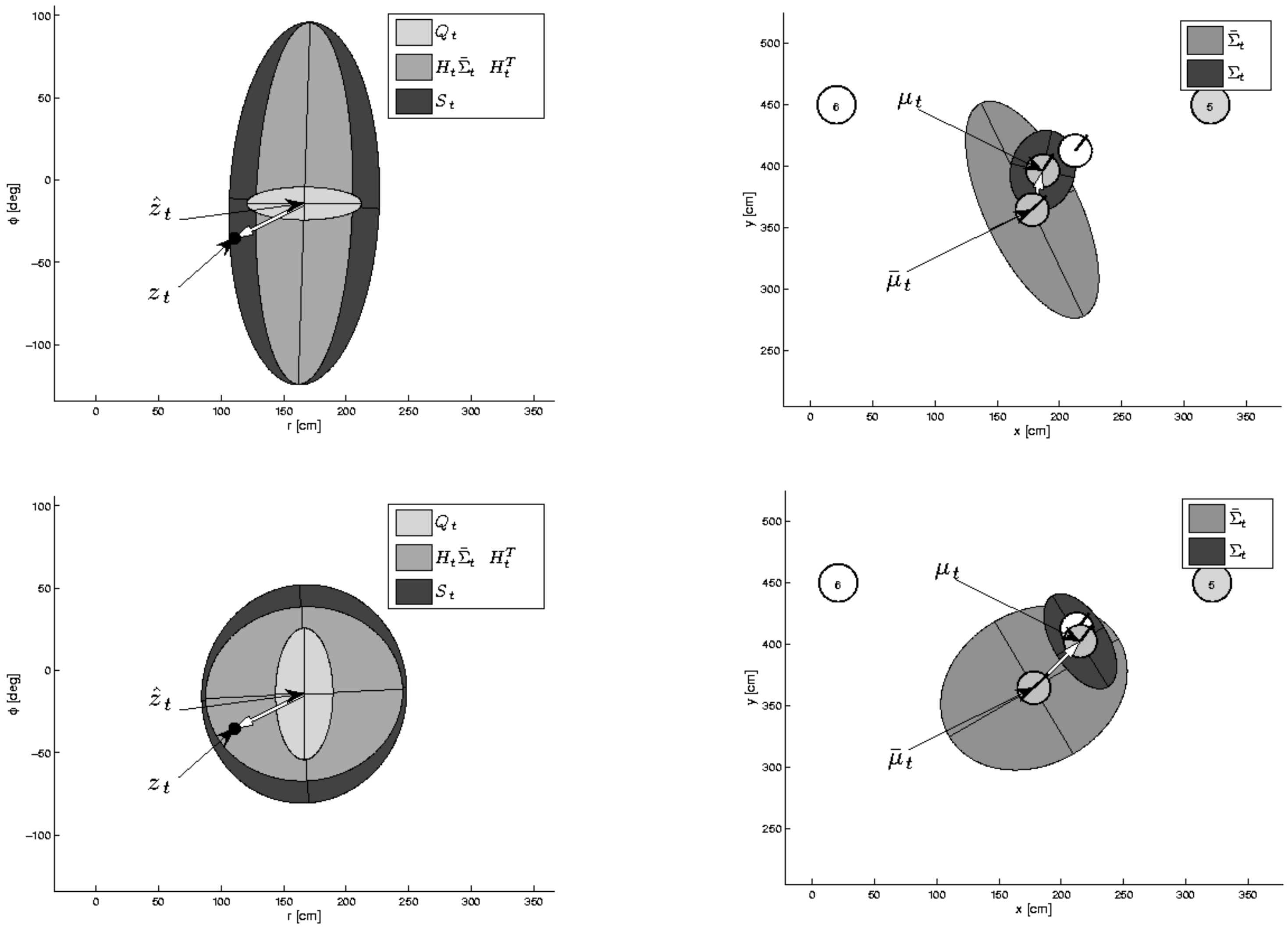
EKF Measurement

Two predicted robot location along with their uncertainty.
True robot = White



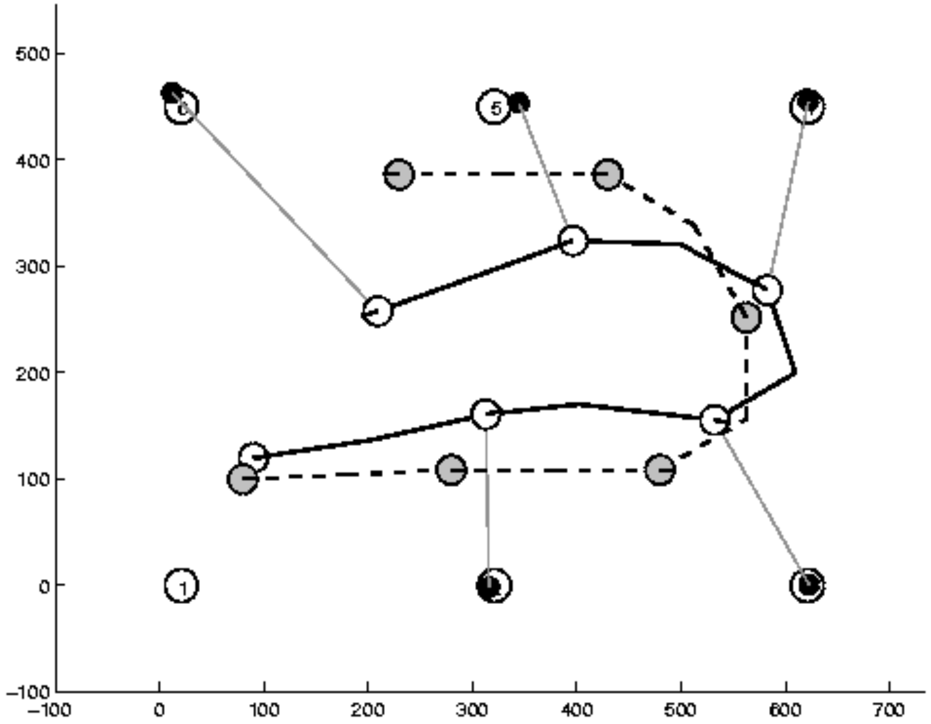
Resulting measurement predictions. White arrow indicates the innovation

EKF Correction

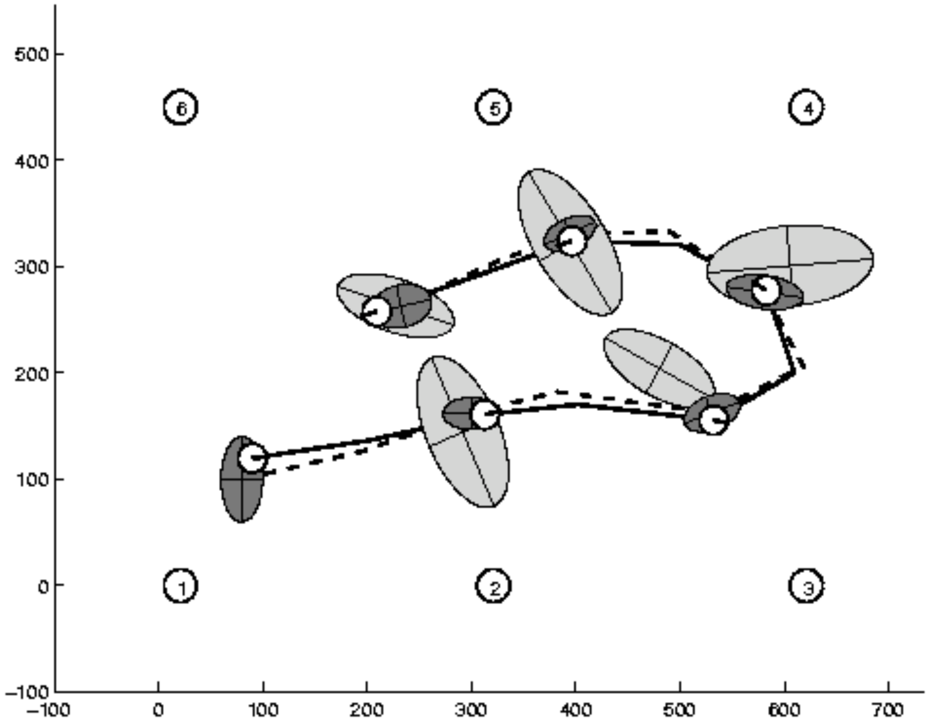


Position uncertainty is reduced !!

Estimation sequence with a good range sensor

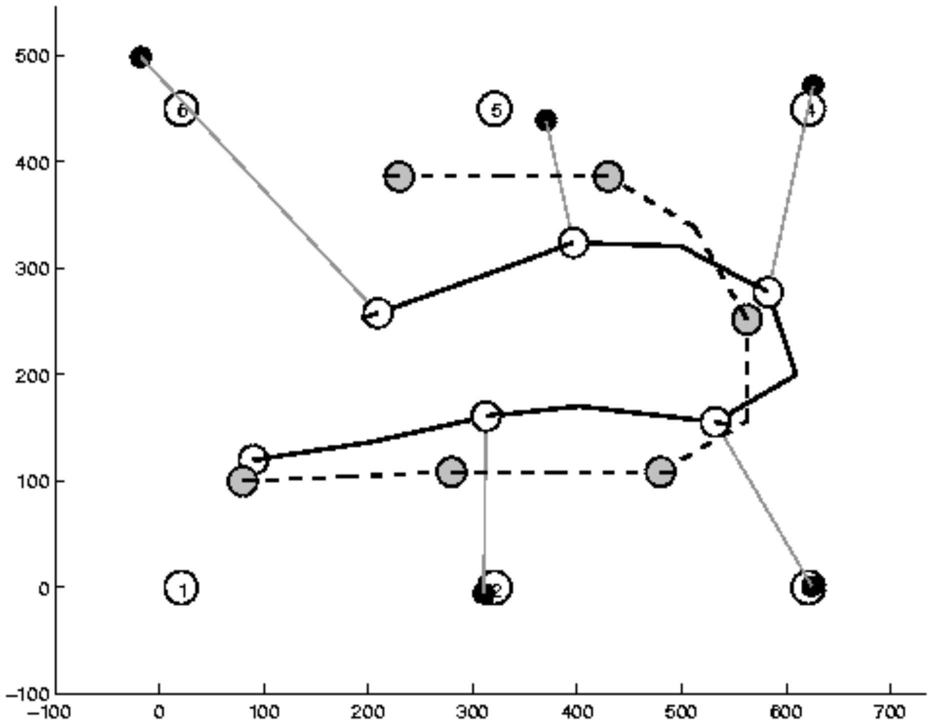


Dashed line is trajectory estimated from motion control. Solid line is true motion of the robot

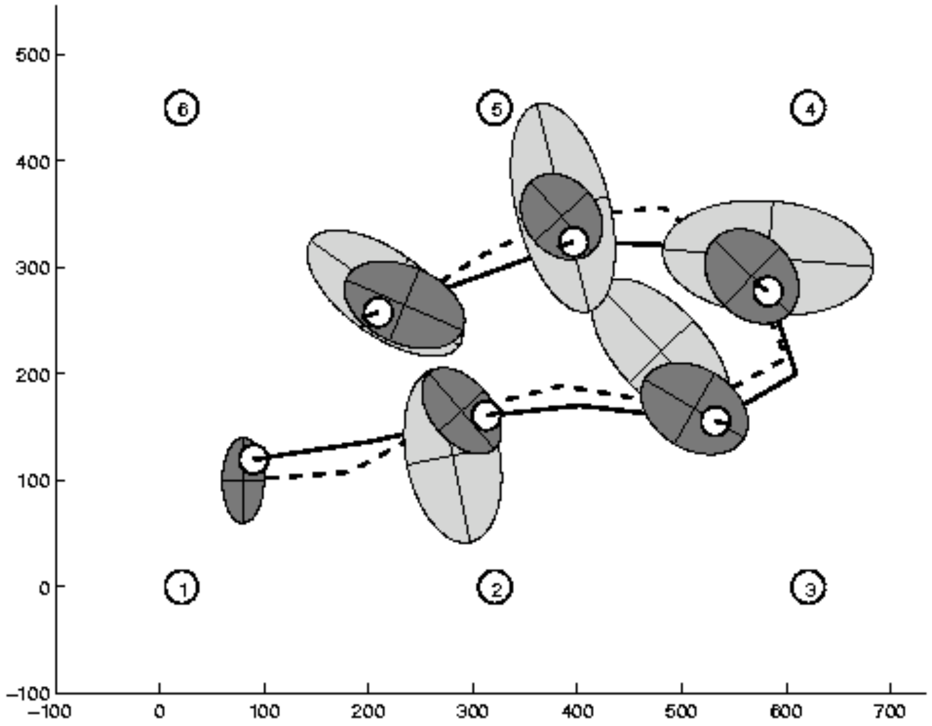


Dashed line is corrected robot trajectory along with predicted and corrected uncertainty. Solid line is true motion of the robot

Estimation sequence with a poor range sensor



Dashed line is trajectory estimated from motion control. Solid line is true motion of the robot



Dashed line is corrected robot trajectory along with predicted and corrected uncertainty. Solid line is true motion of the robot

Limitations of EKF

- If the non-linearity about the mean is small then the EKF approximation can be good.
- If the uncertainty in the state is more then the robot is affected more by the non-linearities in state transition.