CENTERING IN MULTILEVEL MODELS

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Consider the situation in which we have m groups of individuals, where group j has n_j members. We consider a general multilevel model, i.e. a random coefficient model for each group of the form

$$\underline{y}_{ij} = \underline{\beta}_{0j} + \sum_{s=1}^{p} x_{is}\underline{\beta}_{sj} + \underline{\epsilon}_{ij},$$

where the coefficients are the outcomes of a second regression model

$$\underline{\beta}_{sj} = \gamma_{s0} + \sum_{r=1}^{q} z_{jr}\gamma_{rs} + \underline{\delta}_{js},$$

where $s = 0, \cdots, p$. Here we have used the "Dutch convention" [1] of underlining random variables. The disturbance vectors $\underline{\epsilon}_j$ and $\underline{\delta}_j$ are supposed to have zero expectations. They are uncorrelated with each other and have

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dispersions $\mathbf{V}(\underline{\epsilon}_j) = \sigma^2 I$ and $\mathbf{V}(\underline{\delta}_j) = \Omega$. It follows that

$$\mathbf{E}(\underline{y}_{ij}) = \gamma_{00} + \sum_{r=1}^{q} z_{jr}\gamma_{r0} + \sum_{s=1}^{p} x_{is}\gamma_{s0} + \sum_{s=1}^{p}\sum_{r=1}^{q} x_{is}z_{jr}\gamma_{rs},$$

and

$$\mathbf{C}(\underline{y}_{ij}, \underline{y}_{kj}) = \omega_{00} + \sum_{s=1}^{p}(x_{is} + x_{ks})\omega_{0s} + \sum_{s=1}^{p}\sum_{t=1}^{p} x_{is}x_{kt}\omega_{st} + \delta^{ij}\sigma^2$$

Typically, we define more restrictive models for the same data by requiring that some of the regression coefficients γ_{rs} and some of the variance and covariance components ω_{st} are zero.

In multilevel analysis the scaling and centering of the predictors is often arbitrary. Also, there are sometimes theoretical reasons to choose a particular form of centering. See Raudenbush and Bryk [3, p. 31-34] or Kreft et al. [2]. In this article we consider the effect on the model of translations. Suppose we replace x_{is} by $\tilde{x}_{is} = x_{is} - a_s$. Thus we subtract a constant from each individual-level predictor, and we use the same constant for all groups. If the a_s are the predictor means, this means *grant mean centering* of all predictor variables.

After some algebra we see that

$$\gamma_{00} + \sum_{r=1}^{q} z_{jr}\gamma_{r0} + \sum_{s=1}^{p} x_{is}\gamma_{s0} + \sum_{s=1}^{p}\sum_{r=1}^{q} x_{is}z_{jr}\gamma_{rs} =$$
$$= \tilde{\gamma}_{00} + \sum_{r=1}^{q} z_{jr}\tilde{\gamma}_{r0} + \sum_{s=1}^{p} \tilde{x}_{is}\gamma_{s0} + \sum_{s=1}^{p}\sum_{r=1}^{q} \tilde{x}_{is}z_{jr}\gamma_{rs},$$

with

$$\tilde{\gamma}_{r0} = \gamma_{r0} + \sum_{s=1}^{p} \gamma_{rs}a_s$$

for all $r = 0, \cdots, q$. Thus the translation of the predictor can be compensated by a linear transformation of the regression coefficients, and any vector of expected values generated by the untranslated model can also be generated by the translated model. This is a useful type of invariance. But it is important to observe that if we restrict our untranslated model, for instance by requiring one or more γ_{r0} to be zero, then those same $gamma_{r0}$ will no longer be zero in the corresponding translated model. We have invariance of the expected values under translation if the regression coefficients of the group-level predictors are non-zero.

In the same way we can see that

$$\omega_{00} + \sum_{s=1}^{p}(x_{is} + x_{ks})\omega_{0s} + \sum_{s=1}^{p}\sum_{t=1}^{p} x_{is}x_{kt}\omega_{st} =$$
$$= \tilde{\omega}_{00} + \sum_{s=1}^{p}(\tilde{x}_{is} + \tilde{x}_{ks})\tilde{\omega}_{0s} + \sum_{s=1}^{p}\sum_{t=1}^{p} \tilde{x}_{is}\tilde{x}_{kt}\tilde{\omega}_{st}$$

if

$$
\begin{aligned}
\tilde{\omega}_{00} &= \omega_{00} + 2\sum_{s=1}^{p}\omega_{0s}a_s + \sum_{s=1}^{p}\sum_{t=1}^{p}\omega_{st}a_sa_t,\\
\tilde{\omega}_{0s} &= \omega_{0s} + \sum_{t=1}^{p}\omega_{st}a_t.
\end{aligned}
$$

Thus we have invariance under translation of the variance and covariance components as well, but, again, only if we do not require the ω_{0s}, i.e. the covariances of the slopes and the intercepts, to be zero. If we center by using the grant mean of the predictors we stiil fit the same model, at least in the case in which we do not restrict the γ_{r0} or the ω_{s0} to be zero.

If we translate by $\tilde{x}_{is} = x_{is} - a_{js}$ and thus subtract a different constant for each group, the situation becomes more complicated. If the a_{js} are the group means of the predictors, this is *within-group centering*. The relevant formulas are derived in [2], and we will not repeat them here. The conclusion is that separate translations for each group cannot be compensated for by adjusting the regression coefficients and the variance components. In this case, there is no invariance, and we are fitting a truly different model. or, in other words, choosing between a translated and a non-translated model becomes a matter of either theoretical or statistical (goodness-of-fit) considerations.

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