Inverse Multidimensional Scaling

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Introduction

The data in a typical multidimensional scaling situation is an $n \times n$ matrix $\Delta = \{\delta_{ij}\}$ of *dissimilarities* between n objects. The dissimilarities are supposed to give imprecise and/or incomplete information about the *distances* of the n objects in some metric space $< X, d >$. In general terms, the problem is to embed the objects as points in the space in such a way that the distances between the points approximate the dissimilarities between the objects. There are still many variations possible on this theme (cf. [6]). In this paper we restrict our attention to Euclidean scaling, in which $< X, d >$ is a finite-dimensional Euclidean space.

We develop some notation for the Euclidean case. Suppose X are the coordinates of n points in d dimensions. The $n \times d$ matrix X is called a *configuration.* We write $\mathcal{R}^{n\times d}$ for the space of centered configurations (in which the columns of X sum to zero), and we write $d_{ij}(X)$ for the Euclidean distance between points i and j.

The basic problem we discuss in this paper is the *Metric Multidimensional Scaling* or MMS problem. In MMS we want to find $X \in \mathcal{R}^{n\times d}$ in such a way that the loss function

$$\sigma(X, W, \Delta) \triangleq 1/2 \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(\delta_{ij} - d_{ij}(X))^2 \tag{1}$$

is minimized over X. Following Kruskal [11],[12] we call $\sigma(X, W, \Delta)$ the STRESS of a configuration (for given W and Δ).

We can suppose, without loss of generality, that dissimilarities and weights are symmetric and hollow (have zero diagonal). De Leeuw [2] shows how to partition STRESS in such a way that the asymmetric and diagonal parts end up in additive components that do not depend on the configuration. We can also suppose without loss of generality that half the weighted sum of squares of the dissimilarities is equal to one. Moreover, we suppose the weights and dissimilarities are nonnegative. Write $\mathcal{H}^{n\times n}$ for the space of symmetric, nonnegative, and hollow matrices.

The MMS Problem can be made more specific. In order to do this, we have to distinguish between *global minima* and *local minima.*

- A configuration $\hat{X}$ corresponds with a global minimum of STRESS if $\sigma(\hat{X}, W, \Delta) \leq \sigma(X, W, \Delta)$ for all $X \in \mathcal{R}^{n\times d}$.
- A configuration $\hat{X}$ corresponds with a local minimum of STRESS if there is a neighborhood $\mathcal{N} \subseteq \mathcal{R}^{n\times d}$ of $\hat{X}$ such that $\sigma(\hat{X}, W, \Delta) \leq \sigma(X, W, \Delta)$ for all $X \in \mathcal{N}$.

A problem in MMS is that there are multiple local minima. If local minima were unique, there would be no reason to dinstinguish local minima from global minima in the first place, but all indications [5], [9] are that most MMS problems have a host of different local minima. In order to describe this situation mathematically, we define the (set-valued) maps, on $\mathcal{H}^{n\times n} \times \mathcal{H}^{n\times n}$,

$$\mathcal{X}_{local}(W,\Delta) \stackrel{\Delta}{=} \{X \in \mathcal{R}^{n\times d} \mid \sigma(X,W,\Delta) \text{ has a local minimum at } X\}, \tag{2a}$$

$$\mathcal{X}_{global}(W,\Delta) \stackrel{\Delta}{=} \{X \in \mathcal{R}^{n\times d} \mid \sigma(X,W,\Delta) \text{ has a global minimum at } X\}. \tag{2b}$$

The first map, the local minima map, associates with each pair (Δ, W) the configuarions that are local minima, the second map does the same with the global minima. MMS can be defined as the technique that studies these local and global minima maps. Any scaling technique is a configuration-valued function that maps data (W, Δ) into $\mathcal{R}^{n\times d}$, which means that it implements a particular *selection* from the minima-maps. It can be argued that we are really only interested in global minima. Global minimization techniques for MMS are still in their infancy. The problems connected with the global minima map have hardly been touched on [9], except in the special case of one-dimensional scaling [10]. Thus we concentrate here on the local minima map, which has been studied in much greater detail, and is a much simpler object. But it helps to think of the local minima map as an approximation of the global minima map. In fact, global minimum algorithms that use multiple random starts use the representation

$$\mathcal{X}_{global}(W,\Delta) = \{\hat{X} \in \mathcal{R}^{n\times d} \mid \sigma(\hat{X},W,D) \leq \sigma(X,W,\Delta) \text{ for all } X \in \mathcal{X}_{local}(W,\Delta)\} \tag{3}$$

Using Differentiability

To study the local minima map we translate some standard results into our notation. Let

$$\mathcal{X}_{diff}(W,\Delta) \stackrel{\Delta}{=} \{X \in \mathcal{R}^{n\times d} \mid \sigma(X,W,\Delta) \text{ is differentiable at } X\}. \tag{4}$$

De Leeuw [3] has shown that if the weights and dissimilarities are non-negative, then

$$\mathcal{X}_{local}(W,\Delta) \subseteq \mathcal{X}_{diff}(W,\Delta). \tag{5}$$

But this means that if

$$\mathcal{X}_{partial}(W,\Delta) \stackrel{\Delta}{=} \{X \in \mathcal{R}^{n\times d} \mid \frac{\partial\sigma(X,W,\Delta)}{\partial X} = 0\}, \tag{6}$$

then

$$\mathcal{X}_{local}(W,\Delta) \subseteq \mathcal{X}_{partial}(W,\Delta). \tag{7}$$

Most MMS algorithms use gradient or subgradient type methods to find a configuration in $\mathcal{X}_{partial}(W,\Delta)$, and then hope it will also be in $\mathcal{X}_{local}(W,\Delta)$. This is not necessarily true, of course. We can have vanishing partials in saddle points as well (De Leeuw [5] shows that STRESS has no local maxima). Actually we have to be a bit more precise here. The MMS algorithms look for configurations with

$$\left\|\frac{\partial\sigma(X,W,\Delta)}{\partial X}\right\| < \epsilon \tag{8}$$

for some small $\epsilon > 0$. If we are in a region where the STRESS is very flat, we still could be a long way from the nearest local minimum (or saddle-point). This makes it necessary, in practice, to look at the second derivatives of STRESS as well.

The second partials make it possible to make (7) more precise. We define the regions where the Hessian is non-negative definite, and where it is positive definite. We write them as

$$\mathcal{X}_{nne-hes}(W,\Delta) \triangleq \{X \in \mathcal{R}^{n\times d} \mid \frac{\partial^2\sigma(X,W,\Delta)}{\partial X^2} \gtrsim 0\}, \tag{9a}$$

$$\mathcal{X}_{pos-hes}(W,\Delta) \triangleq \{X \in \mathcal{R}^{n\times d} \mid \frac{\partial^2\sigma(X,W,\Delta)}{\partial X^2} > 0\}. \tag{9b}$$

It follows that

$$\mathcal{X}_{pos-hes}(W,\Delta)\cap\mathcal{X}_{partial}(W,\Delta) \subseteq \mathcal{X}_{local}(W,\Delta) \subseteq \mathcal{X}_{nne-hes}(W,\Delta)\cap\mathcal{X}_{partial}(W,\Delta). \tag{10}$$

This is just saying that a necessary condition for a configuration to be a local minimum is that the partials vanish and the Hessian is non-negative definite, a sufficient condition is that the partials vanish and the Hessian is positive definite. Let

$$\mathcal{X}_{l-local}(W,\Delta) \triangleq \mathcal{X}_{pos-hes}(W,\Delta) \cap \mathcal{X}_{partial}(W,\Delta), \tag{11a}$$

$$\mathcal{X}_{u-local}(W,\Delta) \triangleq \mathcal{X}_{nne-hes}(W,\Delta) \cap \mathcal{X}_{partial}(W,\Delta). \tag{11b}$$

Then, instead of studying $\mathcal{X}_{local}$ directly, we can study $\mathcal{X}_{partial}$ or $\mathcal{X}_{l-local}$ and $\mathcal{X}_{u-local}$. These maps are far from simple. De Leeuw [5] has shown that STRESS has local minima, sharp ridges, and other irregularities. There seems to be no obvious relationship between the different local minima, and there are no systematic results on the number of local minima. In order to compute the map, or a selection from the map, we need complicated iterative algorithms, perhaps with multiple random starts. Some results are available for very special cases, such as unidimensional scaling and full-dimensional scaling (cf. below), but for $1 < p < n-1$ almost nothing is known.

Inverse Metric Multidimensional Scaling

In order to understand the mappings $\mathcal{X}_{partial}$, $\mathcal{X}_{l-local}$, and $\mathcal{X}_{u-local}$ a bit better, we look at their inverses. Thus in stead of finding the configurations which are optimal for a given set of weights and dissimilarities, we now look at the weights and dissimilarities for which a given configuration is optimal. There is one obvious reason to do this: it turns out that the inverse maps are comparatively simple. And by studying the inverses in detail, we learn a great deal about the maps themselves. There is a useful analogous situation. In an eigenvalue problem we compute the eigenvectors of a given matrix, in an inverse eigenvalue problem we compute matrices of which a given orthogonal system is a matrix of eigenvectors. MMS is quite close to an eigenvalue problem in various aspects [2], although versions of MMS that use SSTRESS or STRAIN are much more like eigenvalue problems.

The inverse MMS problem for SSTRESS and STRAIN is discussed in Groenen, De Leeuw, and Mathar [8].

For the time being, we restrict ourselves to configurations X which have $d_{ij} > 0$ for all $i \neq j$. Since we are interested in local minima, this causes no real loss of generality [3]. The inverse of $\mathcal{X}_{partial}$, for instance, is defined as

$$\mathcal{X}^{+}_{partial}(X) \triangleq \{W \in \mathcal{H}^{n\times n}, \Delta \in \mathcal{H}^{n\times n} \mid \frac{\partial\sigma(X,W,\Delta)}{\partial X} = 0\}. \tag{12}$$

Inverses for the other maps are defined in the same way, but we will analyze the partial-map in this section. In order to do that efficiently we also define

$$\mathcal{X}^{+}_{partial}(X,W) \triangleq \{\Delta \in \mathcal{H}^{n\times n} \mid \frac{\partial\sigma(X,W,\Delta)}{\partial X} = 0\}. \tag{13}$$

This is just the set of dissimilarity matrices for which X is stationary for given W. For our computations, we also need an orthonormal column-centered matrix K, of dimensions $n \times (n-r-1)$, such that $K'X = 0$. Here $r \triangleq \text{rank}(X)$.

Theorem Inverse:

$$\mathcal{X}^{+}_{partial}(X,W) = \{\Delta \in \mathcal{H}^{n\times n} \mid \delta_{ij} = d_{ij}(1 - \frac{t_{ij}}{w_{ij}})\}, \tag{14}$$

where T is of the form $T = KMK'$, with M an arbitrary real symmetric matrix (of order $n-r-1$), and satisfies $t_{ij} \leq w_{ij}$ for all $i \neq j$.

Proof: The stationary equations have to be brought into a convenient form. We use the notation familiar from earlier papers, such as [6], which gives

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij}X = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}}A_{ij}X, \tag{15}$$

with

$$A_{ij} \triangleq (e_i - e_j)(e_i - e_j)', \tag{16}$$

and with the e_i the unit-vectors of $\mathcal{R}^n$. We have to solve (15) for Δ for given X and W. We make the transformation indicated in the Theorem, i.e. we define

$$t_{ij} \triangleq w_{ij} - w_{ij}\frac{\delta_{ij}}{d_{ij}}, \tag{17}$$

and we solve for t_{ij}. Equation (15) transforms to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}A_{ij}X = 0. \tag{18}$$

To solve (18) we have to realize that the A_{ij} are a basis for the symmetric, doubly centered (SDC) matrices of order n. Thus (18) is solved if we find all SDC matrices T such that $TX = 0$. But that means $T = KMK'$, with M an arbitrary symmetric matrix. Thus there are $1/2(n-r)(n-r-1)$ independent solutions in all.

Of course we must also have $\delta_{ij} \geq 0$, which translates to $t_{ij} \leq w_{ij}$. **Q.E.D.**

A brief comment is in order here. The t_{ij} are defined by (17) only for $i \neq j$. We can define the t_{ii} in a completely arbitrary way, because no matter how we define them (18) will still be true. Thus (18) does not define the t_{ii} and we simply choose them in such a way that T is SDC.

In order to facilitate comparison with other basic MMS papers, such as [4], [6], [7], we define

$$V \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij}, \tag{19a}$$

$$B \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}}A_{ij}. \tag{19b}$$

In this notation $T = V - B$, and Theorem Inverse simply says that $B = V - KMK'$.

Corollary Bounded: $\mathcal{X}^{+}_{partial}(X, W)$ is a closed, bounded, convex polyhedron, containing $D(X)$.

Proof: Closedness, polyhedrality, and convexity follow directly from the representation in the theorem. Obviously $D(X) \in \mathcal{X}^{+}_{partial}(X, W)$. Only boundedness is nontrivial. We have to show [14] that the set cannot contain a ray. From (18) we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}d_{ij}^{2}(X) = 0. \tag{20}$$

This means that not all t_{ij} can have the same sign, at least one t_{ij} has to be negative. For this t_{ij}, and for some $\lambda > 0$ we have that $\lambda t_{ij} < -w_{ij}$, and thus the set of matrices T cannot be unbounded. **Q.E.D.**

Corollary Dominate: If $\Delta_1 \in \mathcal{X}^{+}_{partial}(X, W)$ and $\Delta_2 \in \mathcal{X}^{+}_{partial}(X, W)$ and $\delta_{ij1} \leq \delta_{ij2}$ for all $i < j$, then actually $\Delta_1 = \Delta_2$.

Proof: We have $\delta_{ij1} \leq \delta_{ij2}$ if and only if $t_{ij1} \leq t_{ij2}$. But, from (20),

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (t_{ij1} - t_{ij2})d_{ij}^{2}(X) = 0, \tag{21}$$

which is impossible unless $T_1 = T_2$. **Q.E.D.**

Corollary Only:

$$\mathcal{X}^+_{partial}(X) = \{W \in \mathcal{H}^{n\times n}, \Delta \in \mathcal{H}^{n\times n} \mid \Delta \in \mathcal{X}^+_{partial}(X, W)\}. \tag{22}$$

Proof: Directly from the representation in the Theorem. **Q.E.D.**

From the last corrollary we can choose W arbitrarily in $\mathcal{H}^{n\times n}$, and for each W there is a corresponding set of dissimilarities. Thus weights are not very essential to the formulation of the problem, and we shall largely ignore them further on.

If Δ_1 and Δ_2 are two different elements of $\mathcal{X}^+_{partial}$, then we can measure their distance by using

$$1/2\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(\delta_{1ij} - \delta_{2ij})^2 = 1/2\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2(t_{ij1} - t_{ij2})^2. \tag{23}$$

In particular, the distance between Δ and $D(X)$, which of course is simply STRESS, is equal to

$$\sigma(X, W, \Delta) = 1/2\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 t_{ij}^2. \tag{24}$$

Now suppose we have m configurations $X_1, \cdots, X_m$. We can ask for the set of dissimilarity matrices for which all X_j are stationary points. It is not necessary that all X_j have the same rank. Each of the configurations defines an affine space of dimension $1/2(n - r_j)(n - r_j - 1)$, and if these spaces are "in general position", they have an intersection if $1/2\sum_{j=1}^{m}(n - r_j)(n - r_j - 1) \geq 1/2(m-1)n(n-1)$. If all r_j are equal to p this works out to

$$m \leq \frac{n(n-1)}{n(n-1) - (n-p)(n-p-1)}. \tag{25}$$

It is tempting to use (25) as an upper bound on the number of stationary points of STRESS, but the reasoning here is difficult to make rigourous.

Computing the Inverse Map

We now go into more detail in describing the convex polyhedron defined in Theorem Inverse. From the computational point of view, it is convenient to use a basis P_r for the symmetric matrices of order $n - r - 1$. Define

$$Q_r \triangleq K P_r K', \tag{26}$$

and then write

$$T = \sum_{r=1}^{R} \theta_r Q_r. \tag{27}$$

If we limit ourselves to the case $w_{ij} = 1$ for all $i \neq j$ then we must have

$$\sum_{r=1}^{R} \theta_r q_{rij} \leq 1 \tag{28}$$

for all $i < j$, which are $N \triangleq {}^1\!/\!_2 n(n-1)$ linear inequalities in R unknowns. Obviously, these linear inequalities describe the bounded convex polyhedron of Corollary Bounded.

Bounded convex polyhedra can be described in term of their edges. Compare [13], [15]. We find the edges of the polyhedron by an enumerative procedure which looks at all subsystems of R rows of (28). If the complete system is written as $Q\theta \geq -u$, with u a vector with all elements equal to $+1$, then we can write a subsystem as

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \theta \geq - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{29}$$

with Q_1 of order R. We then check if Q_1 is nonsingular. If it is not, we go to the next subsystem. If it is, we compute $\tilde{\theta} = -Q_1^{-1}u_1$. If $Q_2\tilde{\theta} \geq -u_2$ we add $\tilde{\theta}$ to our list of edges. If not, we go to the next subsystem. This can be done quite efficiently by using pivoting techniques, moving one row into the basis and another one out of the basis in one pivot, and cycling through the candidate subsets lexicographically [1].

We start with a really simple example. Let's call it Example Square. Consider the configuration

$$X = \begin{pmatrix} -1/2 & -1/2 \\ +1/2 & -1/2 \\ +1/2 & +1/2 \\ -1/2 & +1/2 \end{pmatrix},$$

with distances

$$D = \begin{pmatrix} 0 & 1 & \sqrt{2} & 1 \\ 1 & 0 & 1 & \sqrt{2} \\ \sqrt{2} & 1 & 0 & 1 \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix}.$$

We now want to find all dissimilarity matrices Δ for which X gives a stationary value of STRESS. Throughout, we fix W at $w_{ij} = 1$. For K we find

$$K = \begin{pmatrix} -1/2 \\ +1/2 \\ -1/2 \\ +1/2 \end{pmatrix},$$

and thus

$$T = \theta \begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{pmatrix},$$

with $-1 \leq \theta \leq +1$. The two extreme points are

$$\Delta_1 = \begin{pmatrix} 0 & 0 & 2\sqrt{2} & 0 \\ 0 & 0 & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & 0 \end{pmatrix},$$

and

$$\Delta_2 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}.$$

Any convex combination $\lambda\Delta_1 + (1-\lambda)\Delta_2$ of these edges has the four points arranged on a square, with the length of the side equal to $2(1-\lambda)$ and the length of the diagonals equal to $2\lambda\sqrt{2}$. The distance matrix D is exactly in the middle of the edges. In the interval we also have the matrix with all six dissimilarities equal, which is the distance matrix of a regular simplex in three dimensions. For the distances between the edges and their centroid we find

$$\begin{array}{c} \\ D \\ \Delta_1 \\ \Delta_2 \end{array} \begin{array}{c} \begin{array}{ccc} D & \Delta_1 & \Delta_2 \end{array} \\ \begin{pmatrix} 0 & 8 & 8 \\ 8 & 0 & 32 \\ 8 & 32 & 0 \end{pmatrix} \end{array}.$$

Thus the STRESS of each edge is 8.

Improved Approximation

We know that $\mathcal{X}^+_{partial}(X, W)$ is a compact convex set. It is clear that $\mathcal{X}^+_{u-local}(X, W)$ is a more precise approximation of $\mathcal{X}_{local}(X, W)$, and we shall see that it also is convex and compact (although not necessarily polyhedral). Remember that $\mathcal{X}^+_{u-local}(X, W)$ is the set of all disimilarity matrices for which X is stationary, and the Hessian is positive semi-definite.

First, we need a convenient expression for the Hessian of STRESS. This has been discussed earlier in [4]. We start with

$$\frac{\partial \sigma}{\partial x_s} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij}x_s - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}}A_{ij}x_s. \tag{30}$$

Thus

$$\frac{\partial^2 \sigma}{\partial x_s \partial x_t} = \delta^{st}\{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}}A_{ij}\} + \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{\delta_{ij}}{d_{ij}^3}A_{ij}x_s x_t' A_{ij}. \tag{31}$$

Now substitute

$$w_{ij}\frac{\delta_{ij}}{d_{ij}} = w_{ij} - t_{ij}. \tag{32}$$

This gives

$$\frac{\partial^2\sigma}{\partial x_s \partial x_t} = \delta^{st}T - \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}\frac{(x_{is}-x_{js})(x_{it}-x_{jt})}{d_{ij}^2}A_{ij} + \\ + \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{(x_{is}-x_{js})(x_{it}-x_{jt})}{d_{ij}^2}A_{ij}. \tag{33}$$

Here superscripted δ is the Kronecker symbol, it does not have anything to do with the subscripted δ's. It is convenient at this point to define the $np \times np$ supermatricesmatrices $H_0, H_1, \cdots, H_r$ with submatrices

$$H_{rst} \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ijr}\frac{(x_{is}-x_{js})(x_{it}-x_{jt})}{d_{ij}^2}A_{ij}, \tag{34a}$$

$$H_{0st} \triangleq \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\frac{(x_{is}-x_{js})(x_{it}-x_{jt})}{d_{ij}^2}A_{ij}. \tag{34b}$$

Also

$$\overline{Q}_r \triangleq \underbrace{Q_r \oplus \cdots \oplus Q_r}_{\text{p times}}, \tag{35}$$

i.e., $\overline{Q}_r$ is the diagonal supermatrix with the Q_r repeated along the diagonal.

Theorem Improved: $\mathcal{X}^+_{u-local}(X, W)$ is a compact convex set.

Proof: We have $\Delta \in \mathcal{X}^+_{u-local}(X, W)$ if and only if Δ is of the form in Theorem Inverse, with in addition

$$\sum_{r=1}^{R} \theta_r(\overline{Q}_r - H_r) \gtrsim -H_0. \tag{36}$$

But this means that $\mathcal{X}^+_{u-local}(X, W)$ is the intersection of the convex set defined by (36) and the compact convex set from Theorem Inverse, i.e. it is a compact convex set. **Q.E.D.**

Unfortunately, $\mathcal{X}^+_{u-local}(X, W)$ is more difficult to describe than $\mathcal{X}^+_{partial}(X, W)$, because it is not polyhedral. We can approximate it by polyhedral sets, by cutting off the edges that are not in the cone, using the eigenvectors corresponding to the positive eigenvalues. This makes arbitrarily precise approximation possible, but the number of edges will increase very rapidly.

In our Example Square, we can still carry out the necessary computations quite easily. We

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$$\frac{\partial^2\sigma}{\partial x_s \partial x_t} = \delta^{st} T - \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} \frac{(x_{is}-x_{js})(x_{it}-x_{jt})}{d_{ij}^2} A_{ij} + \\ + \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \frac{(x_{is}-x_{js})(x_{it}-x_{jt})}{d_{ij}^2} A_{ij}. \tag{33}$$

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In our Example Square, we can still carry out the necessary computations quite easily. We first give H_0, which is of course the Hessian at $\theta = 0$, i.e. at $\Delta = D$.

$$H_0 = \left(\begin{array}{cccc|cccc}
\frac{3}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
-1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & \frac{3}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{2} & -1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\hline
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -1 \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{3}{2} & -1 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 & \frac{3}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{3}{2}
\end{array}\right)$$

Also

$$\overline{Q}_1 - H_1 = \left(\begin{array}{cccc|cccc}
\frac{3}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
-1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
-\frac{1}{2} & 0 & \frac{3}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{2} & -1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\hline
\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & -1 \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{3}{2} & -1 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 & \frac{3}{2} & 0 \\
0 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{3}{2}
\end{array}\right)$$

We use the results of De Leeuw [4] on the Hessian. It is shown in that paper that the Hessian has at least $^1/_2 p(p+1)$ eigenvalues equal to zero, corresponding with the rotational and translational invariance of the distances, and it has at least one eigenvalue equal to n.

FULL DIMENSIONAL SCALING

In MDS we minimize STRESS over D, on the conditon that $D = D(X)$, i.e. D are the Euclidean distances between the n points of a configuration in p dimensions. Now suppose we drop the constraint of p dimensions, and merely require that the D are Euclidean distances between points of any configuration. This defines *full dimensional scaling,* or FDS. There is no need to emphasize the fact that FDS is metric, because non-metric full-dimensional scaling does not make sense (any dissimilarity matrix can be fitted perfectly in $n-2$ dimensions nonmetrically). The most interesting result on FDS is that all local minima are global. This result is due to De Leeuw [5], but because the proof is difficult to find and simple to reproduce, we give it here for completeness.

Theorem Full: In FDS all local minima are global.

Proof: The FDS problem can be formulated as minimization of

$$\sigma(C) = {}^1/_2 \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(\delta_{ij} - d_{ij}(C))^2 \tag{37}$$

over the convex cone of all positive semi-definite matrices C. Here

$$d_{ij}(C) = \sqrt{c_{ii} + c_{jj} - 2c_{ij}}. \tag{38}$$

Thus $d_{ij}(C)$ is the square root of a linear function of C, which means it is concave in C. Obviously $d^2_{ij}(C)$ is linear in C. It follows that $\sigma(C)$ is convex in C, and thus the FDS problem minimizes a convex function over a convex set. All local mimima are global. **Q.E.D.**

It now makes sense to define *inverse* FDS. Given a configuration X, find the weights and/or dissimilarities for which X is the unique solution to the FDS problem. Thus we define

$$\mathcal{X}_{full}(W, \Delta) \triangleq \{X \in \mathcal{R}^{n\times d} \mid X \text{solves the FDS problem}\}, \tag{39a}$$

$$\mathcal{X}^{+}_{full}(W, \Delta) \triangleq \{\Delta \in \mathcal{H}^{n\times n} \mid X \text{solves the FDS problem}\}. \tag{39b}$$

Theorem Inverse Full: $\mathcal{X}^{+}_{full}(W, \Delta)$ is a compact convex set.

Proof: If we minimize a differentiable convex function $f(\bullet)$ over a convex cone $\mathcal{K}$, then the necessary and sufficient conditions for a minimum [14] are

- $\hat{x} \in \mathcal{K}$,
- $-\nabla f(\hat{x}) \in \mathcal{K}^{\circ}$,
- $< \hat{x}, \nabla f(\hat{x}) >= 0$.

In our case this means that the necessary and sufficient conditions for the FDS problem are case

$$C \gtrsim 0, \tag{40a}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_{ij}\delta_{ij}}{d_{ij}(C)}A_{ij} \gtrsim 0, \tag{40b}$$

$$\text{tr } C\Big\{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}A_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_{ij}\delta_{ij}}{d_{ij}(C)}A_{ij}\Big\} = 0, \tag{40c}$$

which translates to $T \gtrsim 0$ and $TX = 0$. But this is the same as $T = KMK'$, with M a positive semidefinite matrix. So again we have the intersection of a convex cone and the compact convex set of Theorem Inverse. **Q.E.D.**

For our simple example we have $T \gtrsim 0$ if and only if $\theta \geq 0$. Thus the dissimilarity matrices for which X solves the FDS problem are the ones on the line segment between D and Δ_2.

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Appendix

Let us extend the previous configuration by adding a point in the origin, so that

$$X = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \\ 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 1 & \sqrt{2} & 1 & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 1 & \sqrt{2} & \frac{1}{2}\sqrt{2} \\ \sqrt{2} & 1 & 0 & 1 & \frac{1}{2}\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \end{pmatrix}.$$

X has null space

$$K = \begin{pmatrix} +\frac{1}{2\sqrt{5}} & +\frac{1}{2} \\ +\frac{1}{2\sqrt{5}} & -\frac{1}{2} \\ +\frac{1}{2\sqrt{5}} & +\frac{1}{2} \\ +\frac{1}{2\sqrt{5}} & -\frac{1}{2} \\ -\frac{2}{\sqrt{5}} & 0 \end{pmatrix},$$

which implies that there are 3 parameters, i.e., μ_{11}, μ_{12}, and μ_{22} that can be varied to obtain δ_{ij} such that X is stationary point. We have found 5 cornerpoints for the polyhedral convex set with nonnegative δ_{ij} for all ij. The first cornerpoint has

$$M_1 = \begin{pmatrix} -20 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \Delta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ \frac{5}{2}\sqrt{2} & \frac{5}{2}\sqrt{2} & \frac{5}{2}\sqrt{2} & \frac{5}{2}\sqrt{2} & 0 \end{pmatrix}$$

with STRESS 40. The Hessian has four eigenvalues of 5, two of -10, and three eigenvalues due to the rotational invariance of the STRESS function. This shows that X is a saddle point dissimilarity matrix Δ_1.

The second cornerpoint has

$$M_2 = \begin{pmatrix} -\frac{15}{2} & -\frac{5}{2}\sqrt{5} \\ -\frac{5}{2}\sqrt{5} & \frac{5}{2} \end{pmatrix} \text{ and } \Delta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ 0 & 0 & 0 & \frac{5}{2}\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ 0 & \frac{5}{2}\sqrt{2} & 0 & 0 & 0 \\ \frac{5}{2}\sqrt{2} & 0 & \frac{5}{2}\sqrt{2} & 0 & 0 \end{pmatrix}$$

with STRESS 27.5. The corresponding Hessian has five eigenvalues equal to 5, four equal to zero and one equal to -10, which indicates that X is a saddle point.

The third cornerpoint has only the a different sign for μ_{12}, i.e.,

$$M_3 = \begin{pmatrix} -\frac{15}{2} & \frac{5}{2}\sqrt{5} \\ \frac{5}{2}\sqrt{5} & \frac{5}{2} \end{pmatrix} \text{ which gives } \Delta_3 = \begin{pmatrix} 0 & 0 & \frac{5}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ \frac{5}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{2}\sqrt{2} \\ \frac{5}{2}\sqrt{2} & 0 & \frac{5}{2}\sqrt{2} & 0 & 0 \end{pmatrix}$$

also with STRESS 27.5. The corresponding Hessian has exactly the same eigenvalues as the one for Δ_2, so that X is also a saddle point for Δ_3.

The fourth cornerpoint has

$$M_4 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \text{ and } \Delta_4 = \begin{pmatrix} 0 & 0 & \frac{5}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{2}\sqrt{2} & 0 \\ \frac{5}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which gives STRESS 15. The Hessian has six eigenvalues equal to 5, and four equal to 0, so that X is a nonisolated local minimum for point for Δ_4.

The final cornerpoint differs from the previous one in the sign of μ_{22}, i.e.,

$$M_5 = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \text{ which gives } \Delta_5 = \begin{pmatrix} 0 & \frac{5}{2} & 0 & \frac{5}{2} & 0 \\ \frac{5}{2} & 0 & \frac{5}{2} & 0 & 0 \\ 0 & \frac{5}{2} & 0 & \frac{5}{2} & 0 \\ \frac{5}{2} & 0 & \frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

also with STRESS 15. However, the Hessian is different from Δ_4; it has only four eigenvalues equal to 5, and six eigenvalues equal to 0. Thus X is also a nonisolated local minimum for point for Δ_5.

The third example concerns a configuration of four points on a line with coordinates

$$X_u = \begin{pmatrix} -3 \\ -1 \\ +1 \\ +3 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 2 & 4 & 6 \\ 2 & 0 & 2 & 4 \\ 4 & 2 & 0 & 2 \\ 6 & 4 & 2 & 0 \end{pmatrix}.$$

The null space of X_u is spanned by

$$K_u = \begin{pmatrix} +2 & +1 \\ -2 & -3 \\ -2 & +3 \\ +2 & -1 \end{pmatrix}.$$

We have found seven cornerpoints of that produces dissimilarities with X_u as local minima. They are local minima, because the STRESS is a piecewise linear quadratic function, where the pieces depends only on the order of the coordinates of X. The cornerpoints are summarized in Table A.

Table A. The cornerpoints of the polyhedral set that defines dissimilarities Δ for which X_u is a local minimum.

cornerpoint	M	STRESS	Δ
1	$\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$	16	$\begin{pmatrix} 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 4 \\ 4 & 0 & 0 & 0 \\ 8 & 4 & 0 & 0 \end{pmatrix}$
2	$\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$	80	$\begin{pmatrix} 0 & 0 & 0 & 12 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$	144	$\begin{pmatrix} 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 4 \\ 12 & 0 & 0 & 8 \\ 0 & 4 & 8 & 0 \end{pmatrix}$
4	$\begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix}$	144	$\begin{pmatrix} 0 & 8 & 4 & 0 \\ 8 & 0 & 0 & 12 \\ 4 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \end{pmatrix}$
5	$\begin{pmatrix} -\frac{5}{16} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{4} \end{pmatrix}$	224	$\begin{pmatrix} 0 & 0 & 12 & 0 \\ 0 & 0 & 4 & 0 \\ 12 & 4 & 0 & 12 \\ 0 & 0 & 12 & 0 \end{pmatrix}$
6	$\begin{pmatrix} -\frac{5}{16} & -\frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{4} \end{pmatrix}$	224	$\begin{pmatrix} 0 & 12 & 0 & 0 \\ 12 & 0 & 4 & 12 \\ 0 & 4 & 0 & 0 \\ 0 & 12 & 0 & 0 \end{pmatrix}$
7	$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}$	464	$\begin{pmatrix} 0 & 12 & 0 & 0 \\ 12 & 0 & 16 & 0 \\ 0 & 16 & 0 & 12 \\ 0 & 0 & 12 & 0 \end{pmatrix}$

In the next few pages we give some additional examples (without comments).

```
Solution X
 -3.0
 -1.0
  1.0
  3.0

Zero space K:
 -.512 -.195
  .828 -.121
 -.121  .828
 -.195 -.512

Distances:
  .0 2.0 4.0 6.0
 2.0  .0 2.0 4.0
 4.0 2.0  .0 2.0
 6.0 4.0 2.0  .0

STRESS:  80
M:
  -2.0  -2.0
  -2.0  -2.0
Eigenvalues M:
    .0  -4.0
Eigenvalues B-matrix:
   1.0   1.0    .0    .0
Eigenvalues Hessian:
   4.0   4.0   4.0    .0
Eigenvalues Classical Scaling:
  72.0   8.0    .0 -40.0
Dissimilarities:
   .0  .0  .0 12.0
   .0  .0 4.0   .0
   .0 4.0  .0   .0
 12.0  .0  .0   .0

STRESS:  144
M:
  -2.472   2.0
   2.0   6.472
Eigenvalues M:
   6.899  -2.899
Eigenvalues B-matrix:
```

```
   2.725   1.0    .275    .0
Eigenvalues Hessian:
   4.0   4.0   4.0    .0
Eigenvalues Classical Scaling:
  77.617   6.365    .0 -27.982
Dissimilarities:
   .0  .0 12.0  .0
   .0  .0   .0 4.0
 12.0  .0   .0 8.0
   .0 4.0  8.0  .0

STRESS:  224
M:
  -1.708    .0
     .0  11.708
Eigenvalues M:
  11.708  -1.708
Eigenvalues B-matrix:
   3.927   1.0    .573    .0
Eigenvalues Hessian:
   4.0   4.0   4.0    .0
Eigenvalues Classical Scaling:
  97.093    .0    .0 -21.093
Dissimilarities:
   .0  .0 12.0   .0
   .0  .0  4.0   .0
 12.0 4.0   .0 12.0
   .0  .0 12.0   .0

STRESS:  16
M:
  -2.667    .667
    .667  -2.667
Eigenvalues M:
  -2.0  -3.333
Eigenvalues B-matrix:
   1.0    .500    .167    .0
Eigenvalues Hessian:
   4.0   4.0   4.0    .0
Eigenvalues Classical Scaling:
  33.889    .0  -1.889  -8.0
Dissimilarities:
  .0  .0 4.0 8.0
  .0  .0  .0 4.0
```

```
 4.0  .0  .0  .0
 8.0 4.0  .0  .0

STRESS:  464
M:
  14.0  -6.0
  -6.0  14.0
Eigenvalues M:
  20.0   8.0
Eigenvalues B-matrix:
   6.0   3.0   1.0    .0
Eigenvalues Hessian:
   4.0   4.0   4.0    .0
Eigenvalues Classical Scaling:
 160.333   8.0    .0 -32.333
Dissimilarities:
   .0 12.0   .0   .0
 12.0   .0 16.0   .0
   .0 16.0   .0 12.0
   .0   .0 12.0   .0

STRESS:  224
M:
  11.708    .0
    .0  -1.708
Eigenvalues M:
  11.708  -1.708
Eigenvalues B-matrix:
   3.927   1.0   .573    .0
Eigenvalues Hessian:
   4.0   4.0   4.0    .0
Eigenvalues Classical Scaling:
  97.093    .0    .0 -21.093
Dissimilarities:
   .0 12.0  .0   .0
 12.0   .0 4.0 12.0
   .0  4.0  .0   .0
   .0 12.0  .0   .0

STRESS:  144
M:
   6.472   2.0
   2.0  -2.472
Eigenvalues M:
```

```
   6.899  -2.899
Eigenvalues B-matrix:
   2.725    1.0     .275     .0
Eigenvalues Hessian:
   4.0    4.0    4.0     .0
Eigenvalues Classical Scaling:
  77.617    6.365     .0 -27.982
Dissimilarities:
  .0  8.0 4.0    .0
 8.0    .0   .0 12.0
 4.0    .0   .0    .0
  .0 12.0   .0    .0

Squared distances between dissimilarities:
(row 1 are distances of X)
      .0    80.0   144.0   224.0    16.0   464.0   224.0   144.0
    80.0      .0   384.0   432.0    64.0   576.0   432.0   384.0
   144.0   384.0      .0    48.0   192.0   576.0   432.0   256.0
   224.0   432.0    48.0      .0   304.0   432.0   576.0   432.0
    16.0    64.0   192.0   304.0      .0   640.0   304.0   192.0
   464.0   576.0   576.0   432.0   640.0      .0   432.0   576.0
   224.0   432.0   432.0   576.0   304.0   432.0      .0    48.0
   144.0   384.0   256.0   432.0   192.0   576.0    48.0      .0
```

```
Solution X
 -1.0   -.577
  1.0   -.577
   .0   1.155
   .0     .0

Zero space K:
 -.289
 -.289
 -.289
  .866

Distances:
   .0    2.0    2.0    1.155
  2.0     .0    2.0    1.155
  2.0    2.0     .0    1.155
  1.155 1.155 1.155   .0

STRESS:  48
M:
   12.0
Eigenvalues M:
   12.0
Eigenvalues B-matrix:
    4.0      1.0      1.0       .0
Eigenvalues Hessian:
    4.0      4.0      4.0       .0       .0       .0     -6.0     -6.0
Eigenvalues Classical Scaling:
   16.0       .0       .0       .0
Dissimilarities:
   .0     .0     .0    4.619
   .0     .0     .0    4.619
   .0     .0     .0    4.619
  4.619 4.619 4.619   .0

STRESS:  5.333333333
M:
   -4.0
Eigenvalues M:
   -4.0
Eigenvalues B-matrix:
    1.0      1.0       .0       .0
Eigenvalues Hessian:
```

```
   4.0     4.0     4.0     2.0     2.0      .0      .0      .0
Eigenvalues Classical Scaling:
   3.556   3.556    .0    -1.778
Dissimilarities:
  .0    2.667 2.667 .0
 2.667  .0    2.667 .0
 2.667 2.667  .0    .0
  .0     .0    .0   .0

Squared distances between dissimilarities:
(row 1 are distances of X)
      .0       48.0       5.333
    48.0         .0      85.333
     5.333    85.333       .0
```

```
Solution X
 -0.500 -0.500
  0.500 -0.500
  0.500  0.500
 -0.500  0.500

Zero space K:
  0.500
 -0.500
  0.500
 -0.500

Distances:
 0.0   1.0   1.414 1.0
 1.0   0.0   1.0   1.414
 1.414 1.0   0.0   1.0
 1.0   1.414 1.0   0.0

STRESS:  8
M:
  -4.0
Eigenvalues M:
  -4.0
Eigenvalues B-matrix:
   1.0   1.0   0.0   0.0
Eigenvalues Hessian:
   4.0   4.0   4.0   4.0   0.0   0.0   0.0   0.0
Eigenvalues classical scaling:
   4.0   4.0   0.0  -4.0
Dissimilarities:
 0.0   0.0   2.828 0.0
 0.0   0.0   0.0   2.828
 2.828 0.0   0.0   0.0
 0.0   2.828 0.0   0.0

STRESS:  8
M:
   4.0
Eigenvalues M:
   4.0
Eigenvalues B-matrix:
   2.0   1.0   1.0   0.0
Eigenvalues Hessian:
```

```
   4.0    4.0    0.0    0.0    0.0    0.0    0.0    0.0
Eigenvalues classical scaling:
   4.0    0.0    0.0    0.0
Dissimilarities:
 0.0 2.0 0.0 2.0
 2.0 0.0 2.0 0.0
 0.0 2.0 0.0 2.0
 2.0 0.0 2.0 0.0

Squared distances between dissimilarities:
(row 1 are distances of x)
     0.0      8.0      8.0
     8.0      0.0     32.0
     8.0     32.0      0.0
```

```
Solution X
 -.707  .000
  .000  .707
  .707  .000
  .000 -.707
  .000  .000

Zero space K:
  .500 -.224
 -.500 -.224
  .500 -.224
 -.500 -.224
  .000  .894

Distances:
  .000 1.000 1.414 1.000 .707
 1.000  .000 1.000 1.414 .707
 1.414 1.000  .000 1.000 .707
 1.000 1.414 1.000  .000 .707
  .707  .707  .707  .707 .000

STRESS:  40
M:
    .000    .000
    .000  20.000
Eigenvalues M:
  20.000    .000
Eigenvalues B-matrix:
   5.000   1.000   1.000   1.000    .000
Eigenvalues Hessian:
   5.000   5.000   5.000   5.000    .000    .000    .000    .000 -10.000 -10.000
Eigenvalues Classical Scaling:
  10.000    .000    .000    .000    .000
Dissimilarities:
  .000  .000  .000  .000 3.536
  .000  .000  .000  .000 3.536
  .000  .000  .000  .000 3.536
  .000  .000  .000  .000 3.536
 3.536 3.536 3.536 3.536  .000

STRESS:  27.5
M:
  -2.500  -5.590
```

```
  -5.590   7.500
Eigenvalues M:
  10.000  -5.000
Eigenvalues B-matrix:
   3.000   1.000   1.000    .000    .000
Eigenvalues Hessian:
   5.000   5.000   5.000   5.000   5.000    .000    .000    .000    .000 -10.000
Eigenvalues Classical Scaling:
   8.555   6.250    .000    .000  -7.305
Dissimilarities:
   .000   .000   .000   .000 3.536
   .000   .000   .000 3.536   .000
   .000   .000   .000   .000 3.536
   .000 3.536   .000   .000   .000
  3.536   .000 3.536   .000   .000

STRESS:  27.5
M:
  -2.500   5.590
   5.590   7.500
Eigenvalues M:
  10.000  -5.000
Eigenvalues B-matrix:
   3.000   1.000   1.000    .000    .000
Eigenvalues Hessian:
   5.000   5.000   5.000   5.000   5.000    .000    .000    .000    .000 -10.000
Eigenvalues Classical Scaling:
   8.555   6.250    .000    .000  -7.305
Dissimilarities:
   .000   .000 3.536   .000   .000
   .000   .000   .000   .000 3.536
  3.536   .000   .000   .000   .000
   .000   .000   .000   .000 3.536
   .000 3.536   .000 3.536   .000

STRESS:  15
M:
  -5.000    .000
    .000  -5.000
Eigenvalues M:
  -5.000  -5.000
Eigenvalues B-matrix:
   1.000   1.000    .000    .000    .000
Eigenvalues Hessian:
```

```
   5.000    5.000    5.000    5.000    5.000    5.000     .000     .000     .000     .000
Eigenvalues Classical Scaling:
   6.250    6.250     .000   -1.250   -6.250
Dissimilarities:
   .000   .000 3.536   .000 .000
   .000   .000   .000 3.536 .000
  3.536   .000   .000   .000 .000
   .000 3.536   .000   .000 .000
   .000   .000   .000   .000 .000

STRESS:  15
M:
   5.000     .000
    .000   -5.000
Eigenvalues M:
   5.000   -5.000
Eigenvalues B-matrix:
   2.000    1.000    1.000     .000     .000
Eigenvalues Hessian:
   5.000    5.000    5.000    5.000     .000     .000     .000     .000     .000     .000
Eigenvalues Classical Scaling:
   6.250     .000     .000     .000   -1.250
Dissimilarities:
   .000 2.500   .000 2.500 .000
  2.500   .000 2.500   .000 .000
   .000 2.500   .000 2.500 .000
  2.500   .000 2.500   .000 .000
   .000   .000   .000   .000 .000

Squared distances between dissimilarities:
(row 1 are distances of X)
    .000 40.000 27.500 27.500 15.000 15.000
  40.000    .000 37.500 37.500 75.000 75.000
  27.500 37.500    .000 75.000 37.500 62.500
  27.500 37.500 75.000    .000 37.500 62.500
  15.000 75.000 37.500 37.500    .000 50.000
  15.000 75.000 62.500 62.500 50.000    .000
```

```
Solution X
   .0    1.0
  .951   .309
  .588 -.809
 -.588 -.809
 -.951   .309

Zero space K:
 -.178   .607
 -.213 -.596
  .522   .357
 -.632   .018
  .501 -.387

Distances:
   .0    1.176 1.902 1.902 1.176
 1.176   .0    1.176 1.902 1.902
 1.902 1.176   .0    1.176 1.902
 1.902 1.902 1.176   .0    1.176
 1.176 1.902 1.902 1.176   .0

STRESS:  19.09830056
M:
   2.390  -1.959
  -1.959  -2.390
Eigenvalues M:
   3.090  -3.090
Eigenvalues B-matrix:
   1.618   1.0     1.0      .382     .0
Eigenvalues Hessian:
   5.0     5.0     4.045   3.455   1.545    .955    .0     .0     .0     .0
Eigenvalues Classical Scaling:
   9.341   3.488   3.159    .0    -7.168
Dissimilarities:
   .0     .0   4.253   .0   1.625
   .0     .0     .0   2.629 2.629
  4.253   .0     .0   1.625   .0
   .0   2.629 1.625   .0   2.629
  1.625 2.629   .0   2.629   .0

STRESS:  9.549150282
M:
  -3.090     .0
```

```
    .0    -3.090
Eigenvalues M:
  -3.090  -3.090
Eigenvalues B-matrix:
   1.0     1.0      .382    .382    .0
Eigenvalues Hessian:
   5.0     5.0     5.0     4.045   4.045   1.545   1.545    .0     .0     .0
Eigenvalues Classical Scaling:
   5.590   5.590    .0    -2.135  -2.135
Dissimilarities:
  .0     .0    2.629 2.629  .0
  .0     .0     .0   2.629 2.629
 2.629  .0     .0     .0   2.629
 2.629 2.629  .0     .0     .0
  .0   2.629 2.629  .0     .0

STRESS:  19.09830056
M:
   2.602   1.667
   1.667  -2.602
Eigenvalues M:
   3.090  -3.090
Eigenvalues B-matrix:
   1.618   1.0     1.0      .382    .0
Eigenvalues Hessian:
   5.0     5.0     4.045   3.455   1.545    .955    .0     .0     .0     .0
Eigenvalues Classical Scaling:
   9.341   3.488   3.159    .0     -7.168
Dissimilarities:
  .0     .0    2.629 2.629  .0
  .0     .0    1.625  .0   4.253
 2.629 1.625  .0   2.629  .0
 2.629  .0   2.629  .0   1.625
  .0   4.253  .0   1.625  .0

STRESS:  65.45084972
M:
   8.090    .0
    .0     8.090
Eigenvalues M:
   8.090   8.090
Eigenvalues B-matrix:
   2.618   2.618   1.0    1.0     .0
Eigenvalues Hessian:
```

```
  5.0     5.0     5.0     .0     .0     .0     -1.545  -1.545  -4.045  -4.045
Eigenvalues Classical Scaling:
 14.635  14.635    .0    -5.590  -5.590
Dissimilarities:
  .0   4.253  .0    .0   4.253
 4.253  .0   4.253  .0    .0
  .0   4.253  .0   4.253  .0
  .0    .0   4.253  .0   4.253
 4.253  .0    .0   4.253  .0

STRESS:  19.09830056
M:
   -.782   2.990
   2.990    .782
Eigenvalues M:
   3.090  -3.090
Eigenvalues B-matrix:
   1.618   1.0     1.0      .382    .0
Eigenvalues Hessian:
   5.0     5.0     4.045   3.455   1.545    .955    .0     .0     .0     .0
Eigenvalues Classical Scaling:
   9.341   3.488   3.159    .0    -7.168
Dissimilarities:
  .0   1.625  .0   4.253  .0
 1.625  .0   2.629  .0   2.629
  .0   2.629  .0   1.625 2.629
 4.253  .0   1.625  .0    .0
  .0   2.629 2.629  .0    .0

STRESS:  19.09830056
M:
  -3.085    .180
    .180   3.085
Eigenvalues M:
   3.090  -3.090
Eigenvalues B-matrix:
   1.618   1.0     1.0      .382    .0
Eigenvalues Hessian:
   5.0     5.0     4.045   3.455   1.545    .955    .0     .0     .0     .0
Eigenvalues Classical Scaling:
   9.341   3.488   3.159    .0    -7.168
Dissimilarities:
  .0   2.629  .0   2.629 1.625
 2.629  .0   1.625 2.629  .0
```

```
  .0   1.625  .0     .0   4.253
 2.629 2.629  .0     .0    .0
 1.625  .0   4.253  .0    .0

STRESS:  19.09830056
M:
  -1.125  -2.878
  -2.878   1.125
Eigenvalues M:
   3.090  -3.090
Eigenvalues B-matrix:
   1.618   1.0     1.0      .382    .0
Eigenvalues Hessian:
   5.0     5.0     4.045   3.455   1.545    .955    .0     .0     .0     .0
Eigenvalues Classical Scaling:
   9.341   3.488   3.159    .0    -7.168
Dissimilarities:
  .0    1.625 2.629  .0   2.629
 1.625  .0     .0   4.253  .0
 2.629  .0     .0     .0   2.629
  .0   4.253  .0     .0   1.625
 2.629  .0    2.629 1.625  .0

Squared distances between dissimilarities:
(row 1 are distances of X)
  .0     19.098    9.549   19.098   65.451   19.098   19.098   19.098
19.098     .0     28.647   26.393   84.549   69.098   69.098   26.393
 9.549   28.647     .0     28.647  125.0     28.647   28.647   28.647
19.098   26.393   28.647     .0     84.549   26.393   69.098   69.098
65.451   84.549  125.0     84.549     .0     84.549   84.549   84.549
19.098   69.098   28.647   26.393   84.549     .0     26.393   69.098
19.098   69.098   28.647   69.098   84.549   26.393     .0     26.393
19.098   26.393   28.647   69.098   84.549   69.098   26.393     .0
```

Six Points Equally Spaced on a Circle

```
     MSIX2 SHOWCORNERS2 SIX
Solution X
   .866   .500
   .866  -.500
   .0   -1.0
  -.866  -.500
  -.866   .500
   .0    1.0

Zero space K:
   .132  .640  .271
   .260 -.640  .153
  -.647  .213 -.189
   .644  .213 -.200
  -.253 -.213  .625
  -.135 -.213 -.661

Distances:
   .0    1.0    1.732 2.0    1.732 1.0
  1.0     .0    1.0    1.732 2.0    1.732
  1.732 1.0     .0    1.0    1.732 2.0
  2.0    1.732 1.0     .0    1.0    1.732
  1.732 2.0    1.732 1.0     .0    1.0
  1.0    1.732 2.0    1.732 1.0     .0

STRESS:  72
M:
  -3.639  -1.965  -2.174
  -1.965  -4.364   1.810
  -2.174   1.810  -3.997
Eigenvalues M:
     .0    -6.0    -6.0
Eigenvalues B-matrix:
   1.0     1.0     1.0      .0      .0      .0
Eigenvalues Hessian:
   6.0     6.0     6.0     6.0     6.0     6.0     6.0      .0      .0      .0
            .0      .0
Eigenvalues Classical Scaling:
  18.0    18.0    18.0      .0   -18.0   -18.0
Dissimilarities:
  .0     .0     .0    6.0     .0     .0
  .0     .0     .0     .0    6.0     .0
  .0     .0     .0     .0     .0    6.0
```

```
 6.0    .0    .0    .0    .0    .0
  .0   6.0    .0    .0    .0    .0
  .0    .0   6.0    .0    .0    .0

STRESS:  72
M:
  -1.770  -3.985   1.680
  -3.985  -2.182  -2.354
   1.680  -2.354   3.952
Eigenvalues M:
   6.0       .0    -6.0
Eigenvalues B-matrix:
   2.0      1.0      1.0      1.0       .0       .0
Eigenvalues Hessian:
   6.0      6.0      6.0      6.0      6.0       .0       .0       .0       .0       .0
          .0       .0
Eigenvalues Classical Scaling:
  18.0     18.0      9.0       .0     -9.0    -18.0
Dissimilarities:
  .0    .0    .0   6.0    .0    .0
  .0    .0   3.0    .0    .0   5.196
  .0   3.0    .0    .0   5.196  .0
 6.0    .0    .0    .0    .0    .0
  .0    .0   5.196  .0    .0   3.0
  .0   5.196  .0    .0   3.0    .0

STRESS:  72
M:
   3.639   2.020   2.125
   2.020  -2.182   4.164
   2.125   4.164  -1.457
Eigenvalues M:
   6.0       .0    -6.0
Eigenvalues B-matrix:
   2.0      1.0      1.0      1.0       .0       .0
Eigenvalues Hessian:
   6.0      6.0      6.0      6.0      6.0       .0       .0       .0       .0       .0
          .0       .0
Eigenvalues Classical Scaling:
  18.0     18.0      9.0       .0     -9.0    -18.0
Dissimilarities:
  .0    .0   5.196  .0    .0   3.0
  .0    .0    .0    .0   6.0    .0
 5.196  .0    .0   3.0    .0    .0
```

```
  .0    .0   3.0    .0    .0   5.196
  .0   6.0    .0    .0    .0    .0
 3.0    .0    .0   5.196  .0    .0
```

STRESS: 65.25

```
M:
   5.361   3.473  -2.137
   3.473   -.955    .566
  -2.137    .566   9.094
Eigenvalues M:
  10.158   6.0    -2.658
Eigenvalues B-matrix:
   2.693   2.0     1.0     1.0      .557    .0
Eigenvalues Hessian:
   6.0     6.0     5.358   4.552   3.255    .0      .0      .0      .0     -.190
        -2.613  -5.862
Eigenvalues Classical Scaling:
  15.151  14.862    .0      -.328  -1.362 -11.448
Dissimilarities:
  .0    .0   5.196  .0    .0   3.0
  .0    .0    .0   2.598 1.500 2.598
 5.196  .0    .0   3.0    .0    .0
  .0   2.598 3.0    .0   4.500  .0
  .0   1.500  .0   4.500  .0   4.500
 3.0   2.598  .0    .0   4.500  .0
```

STRESS: 36

```
M:
  -2.361   1.965   2.174
   1.965  -1.636  -1.810
   2.174  -1.810  -2.003
Eigenvalues M:
    .0      .0    -6.0
Eigenvalues B-matrix:
   1.0     1.0     1.0     1.0      .0      .0
Eigenvalues Hessian:
   6.0     6.0     6.0     6.0     3.0     3.0     3.0     3.0      .0      .0
          .0      .0
Eigenvalues Classical Scaling:
   6.0     6.0     6.0     6.0      .0   -12.0
Dissimilarities:
  .0    .0   3.464  .0   3.464  .0
  .0    .0    .0   3.464  .0   3.464
 3.464  .0    .0    .0   3.464  .0
```

```
 .0    3.464  .0     .0     .0    3.464
3.464  .0    3.464  .0     .0     .0
 .0    3.464  .0    3.464  .0     .0
```

STRESS: 40

```
M:
   2.951   2.948  -2.718
   2.948  -1.455  -2.716
  -2.718  -2.716   2.504
Eigenvalues M:
   7.292    .0     -3.292
Eigenvalues B-matrix:
   2.215   1.0     1.0     1.0      .451     .0
Eigenvalues Hessian:
   6.0     6.0     5.612   4.0     3.503   2.0     1.852    .0      .0      .0
            .0    -2.967
Eigenvalues Classical Scaling:
  13.292   6.0     2.708    .0    -3.333  -6.0
Dissimilarities:
   .0     .0    3.464  .0    3.464  .0
   .0     .0     .0   3.464  .0    3.464
  3.464  .0     .0    2.0     .0    2.0
   .0   3.464  2.0     .0    4.0     .0
  3.464  .0     .0    4.0     .0    2.0
   .0   3.464  2.0     .0    2.0     .0
```

STRESS: 20

```
M:
   2.185   2.302   -.824
   2.302  -2.0    -1.116
   -.824  -1.116  -2.185
Eigenvalues M:
   3.464  -2.0    -3.464
Eigenvalues B-matrix:
   1.577   1.0     1.0      .667    .423     .0
Eigenvalues Hessian:
   6.0     5.752   5.473   4.463   3.841   3.0     2.0     1.065    .407    .0
            .0      .0
Eigenvalues Classical Scaling:
   8.969   3.737   2.592    .0    -1.070  -4.894
Dissimilarities:
   .0     .0    3.464  .0    3.464  .0
   .0     .0     .0   2.309 2.0   2.309
  3.464  .0     .0    2.0     .0    2.0
```

```
  .0    2.309 2.0     .0    2.0    2.309
 3.464 2.0     .0    2.0     .0     .0
  .0    2.309 2.0    2.309   .0     .0

STRESS:  40
M:
   7.212    .646    .642
    .646  -1.455  -1.599
    .642  -1.599  -1.758
Eigenvalues M:
   7.292    .0    -3.292
Eigenvalues B-matrix:
   2.215   1.0     1.0     1.0      .451    .0
Eigenvalues Hessian:
   6.0     6.0     5.612   4.0     3.503   2.0     1.852    .0      .0      .0
          .0     -2.967
Eigenvalues Classical Scaling:
  13.292   6.0     2.708    .0    -3.333  -6.0
Dissimilarities:
  .0     .0    3.464  .0    3.464  .0
  .0     .0    2.0     .0    2.0    3.464
 3.464 2.0     .0    4.0     .0     .0
  .0     .0    4.0     .0    2.0    3.464
 3.464 2.0     .0    2.0     .0     .0
  .0    3.464  .0    3.464  .0     .0

STRESS:  52
M:
   7.978   1.292  -1.252
   1.292   -.909  -3.199
  -1.252  -3.199   2.931
Eigenvalues M:
   8.745   4.0    -2.745
Eigenvalues B-matrix:
   2.457   1.667   1.0     1.0      .543    .0
Eigenvalues Hessian:
   6.0     5.784   5.588   4.0     3.0      .695    .602    .0      .0      .0
       -1.190  -4.479
Eigenvalues Classical Scaling:
  14.167  11.029    .0     -.138   -.363 -10.030
Dissimilarities:
  .0     .0    3.464  .0    3.464  .0
  .0     .0    2.0    1.155  .0    4.619
 3.464 2.0     .0    4.0     .0     .0
```

```
 .0    1.155 4.0     .0    4.0    1.155
3.464   .0     .0    4.0    .0    2.0
 .0    4.619  .0    1.155 2.0     .0

STRESS:  52
M:
   3.170   3.294  -1.647
   3.294   -.909  -1.026
  -1.647  -1.026   7.739
Eigenvalues M:
   8.745   4.0    -2.745
Eigenvalues B-matrix:
   2.457   1.667   1.0     1.0      .543     .0
Eigenvalues Hessian:
   6.0      5.784   5.588   4.0     3.0      .695     .602     .0       .0       .0
         -1.190  -4.479
Eigenvalues Classical Scaling:
  14.167  11.029    .0      -.138    -.363 -10.030
Dissimilarities:
  .0     .0    4.619  .0    1.155 2.0
  .0     .0     .0    3.464  .0   3.464
 4.619  .0     .0    2.0    1.155  .0
  .0    3.464 2.0     .0    4.0     .0
 1.155  .0    1.155 4.0     .0    4.0
 2.0    3.464  .0     .0    4.0     .0

STRESS:  59.04
M:
   7.161   2.358  -2.174
   2.358   -.764  -2.173
  -2.174  -2.173   6.803
Eigenvalues M:
  10.109   4.800  -1.709
Eigenvalues B-matrix:
   2.685   1.800   1.0     1.0      .715     .0
Eigenvalues Hessian:
   6.0      5.799   5.400   3.709   2.751    .600     .204     .0       .0       .0
         -1.950  -5.713
Eigenvalues Classical Scaling:
  16.338   9.780   3.629    .0     -5.460  -8.448
Dissimilarities:
  .0     .0    4.157  .0    2.078 1.200
  .0     .0    1.200 2.078  .0    4.157
 4.157 1.200  .0    3.600  .0     .0
```

```
  .0    2.078 3.600  .0    4.800  .0
 2.078  .0     .0    4.800  .0    3.600
 1.200 4.157  .0     .0    3.600  .0

STRESS:  40
M:
  -1.803   1.647   -.099
   1.647  -1.455   -.513
   -.099   -.513   7.257
Eigenvalues M:
   7.292    .0    -3.292
Eigenvalues B-matrix:
   2.215   1.0     1.0     1.0      .451    .0
Eigenvalues Hessian:
   6.0     6.0     5.612   4.0     3.503   2.0     1.852    .0      .0      .0
          .0     -2.967
Eigenvalues Classical Scaling:
  13.292   6.0     2.708    .0    -3.333  -6.0
Dissimilarities:
  .0     .0    3.464 2.0     .0    2.0
  .0     .0     .0   3.464  .0    3.464
 3.464  .0     .0     .0   3.464  .0
 2.0   3.464  .0     .0    2.0    .0
  .0     .0    3.464 2.0     .0    4.0
 2.0   3.464  .0     .0    4.0    .0

STRESS:  20
M:
  -2.568   1.001   1.796
   1.001  -2.0     1.086
   1.796   1.086   2.568
Eigenvalues M:
   3.464  -2.0    -3.464
Eigenvalues B-matrix:
   1.577   1.0     1.0      .667    .423    .0
Eigenvalues Hessian:
   6.0     5.752   5.473   4.463   3.841   3.0     2.0     1.065    .407    .0
          .0      .0
Eigenvalues Classical Scaling:
   8.969   3.737   2.592    .0    -1.070  -4.894
Dissimilarities:
  .0     .0    3.464 2.0     .0    2.0
  .0     .0     .0   2.309 2.0    2.309
 3.464  .0     .0     .0   3.464  .0
```

```
 2.0    2.309  .0     .0     .0    2.309
  .0    2.0    3.464  .0     .0    2.0
 2.0    2.309  .0    2.309  2.0     .0

STRESS:  52
M:
   7.770    .328  -1.631
    .328  -1.273   -.302
  -1.631   -.302   7.502
Eigenvalues M:
   9.292   6.0    -1.292
Eigenvalues B-matrix:
   2.549   2.0     1.0     1.0      .785    .0
Eigenvalues Hessian:
   6.0     5.256   5.222   3.956   3.0      .426    .0      .0      .0     -.174
       -2.682  -5.005
Eigenvalues Classical Scaling:
  12.958   9.145   2.610    .0    -2.479  -7.568
Dissimilarities:
  .0     .0    3.464 2.0     .0    2.0
  .0     .0    2.0     .0    2.0   3.464
 3.464 2.0     .0    4.0     .0     .0
 2.0     .0    4.0     .0    4.0     .0
  .0    2.0     .0    4.0     .0    4.0
 2.0    3.464  .0     .0    4.0     .0

STRESS:  20
M:
  -2.022   1.301  -1.170
   1.301  -2.0    -2.203
  -1.170  -2.203   2.022
Eigenvalues M:
   3.464  -2.0    -3.464
Eigenvalues B-matrix:
   1.577   1.0     1.0      .667    .423    .0
Eigenvalues Hessian:
   6.0     5.752   5.473   4.463   3.841   3.0     2.0     1.065    .407    .0
         .0      .0
Eigenvalues Classical Scaling:
   8.969   3.737   2.592    .0    -1.070  -4.894
Dissimilarities:
  .0     .0    2.309 2.0    2.309  .0
  .0     .0     .0   3.464   .0   3.464
 2.309  .0     .0     .0    2.309 2.0
```

```
 2.0    3.464  .0     .0    2.0    .0
 2.309  .0    2.309 2.0     .0    2.0
  .0    3.464 2.0     .0    2.0    .0

STRESS:  65.25
M:
   9.418   -.280  -1.804
   -.280   -.955  -3.508
  -1.804  -3.508   5.037
Eigenvalues M:
  10.158   6.0    -2.658
Eigenvalues B-matrix:
   2.693   2.0     1.0     1.0      .557    .0
Eigenvalues Hessian:
   6.0     6.0     5.358   4.552   3.255    .0      .0      .0      .0     -.190
        -2.613  -5.862
Eigenvalues Classical Scaling:
  15.151  14.862    .0     -.328   -1.362 -11.448
Dissimilarities:
  .0     .0    2.598 1.500 2.598  .0
  .0     .0    3.0    .0     .0   5.196
 2.598 3.0     .0    4.500  .0     .0
 1.500  .0    4.500  .0    4.500  .0
 2.598  .0     .0    4.500  .0    3.0
  .0    5.196  .0     .0    3.0    .0

STRESS:  20
M:
   2.240  -1.001   2.191
  -1.001  -2.0    -1.086
   2.191  -1.086  -2.240
Eigenvalues M:
   3.464  -2.0    -3.464
Eigenvalues B-matrix:
   1.577   1.0     1.0      .667    .423    .0
Eigenvalues Hessian:
   6.0     5.752   5.473   4.463   3.841   3.0     2.0     1.065    .407    .0
           .0      .0
Eigenvalues Classical Scaling:
   8.969   3.737   2.592    .0     -1.070  -4.894
Dissimilarities:
  .0     .0    2.309 2.0    2.309  .0
  .0     .0    2.0     .0   2.0   3.464
 2.309 2.0     .0    2.0    2.309  .0
```

```
 2.0    .0    2.0    .0     .0    3.464
 2.309 2.0    2.309  .0     .0     .0
  .0   3.464  .0    3.464  .0     .0

STRESS:  180
M:
  14.361  -1.965  -2.174
  -1.965  13.636   1.810
  -2.174   1.810  14.003
Eigenvalues M:
  18.0     12.0     12.0
Eigenvalues B-matrix:
   4.0      3.0      3.0      1.0      1.0       .0
Eigenvalues Hessian:
   6.0      6.0      6.0      6.0       .0       .0       .0    -3.0     -3.0     -9.0
        -9.0   -12.0
Eigenvalues Classical Scaling:
  36.0     18.0     18.0       .0    -18.0    -18.0
Dissimilarities:
  .0   6.0     .0     .0     .0    6.0
 6.0    .0    6.0     .0     .0     .0
  .0   6.0     .0    6.0     .0     .0
  .0    .0    6.0     .0    6.0     .0
  .0    .0     .0    6.0     .0    6.0
 6.0    .0     .0     .0    6.0     .0

STRESS:  72
M:
  -1.869   1.965  -3.805
   1.965   4.364  -1.810
  -3.805  -1.810  -2.495
Eigenvalues M:
   6.0       .0     -6.0
Eigenvalues B-matrix:
   2.0      1.0      1.0      1.0       .0       .0
Eigenvalues Hessian:
   6.0      6.0      6.0      6.0      6.0       .0       .0       .0       .0       .0
           .0       .0
Eigenvalues Classical Scaling:
  18.0     18.0      9.0       .0     -9.0    -18.0
Dissimilarities:
  .0   3.0     .0     .0    5.196  .0
 3.0    .0     .0    5.196  .0     .0
  .0    .0     .0     .0     .0    6.0
```

```
  .0   5.196  .0     .0   3.0    .0
 5.196  .0     .0   3.0    .0    .0
  .0     .0   6.0    .0    .0    .0

STRESS:  65.25
M:
   9.442  -1.760   -.507
  -1.760   5.591  -2.897
   -.507  -2.897  -1.533
Eigenvalues M:
  10.158   6.0    -2.658
Eigenvalues B-matrix:
   2.693   2.0     1.0     1.0      .557    .0
Eigenvalues Hessian:
   6.0     6.0     5.358   4.552   3.255    .0      .0      .0      .0     -.190
        -2.613  -5.862
Eigenvalues Classical Scaling:
  15.151  14.862    .0     -.328  -1.362 -11.448
Dissimilarities:
  .0   3.0     .0     .0   5.196  .0
 3.0    .0   4.500  .0     .0   2.598
  .0   4.500  .0   4.500  .0   1.500
  .0     .0   4.500  .0   3.0   2.598
 5.196  .0     .0   3.0    .0    .0
  .0   2.598 1.500 2.598  .0    .0

STRESS:  59.04
M:
   7.239  -2.391   2.095
  -2.391   7.091  -1.412
   2.095  -1.412  -1.130
Eigenvalues M:
  10.109   4.800  -1.709
Eigenvalues B-matrix:
   2.685   1.800   1.0     1.0      .715    .0
Eigenvalues Hessian:
   6.0     5.799   5.400   3.709   2.751    .600    .204    .0      .0      .0
        -1.950  -5.713
Eigenvalues Classical Scaling:
  16.338   9.780   3.629    .0    -5.460  -8.448
Dissimilarities:
  .0   3.600  .0     .0   4.157 1.200
 3.600  .0   4.800  .0     .0   2.078
  .0   4.800  .0   3.600 2.078  .0
```

```
  .0     .0    3.600  .0    1.200 4.157
 4.157  .0    2.078 1.200  .0     .0
 1.200 2.078   .0   4.157  .0     .0

STRESS:  52
M:
   7.803  -1.647    .099
  -1.647   7.455    .513
    .099    .513  -1.257
Eigenvalues M:
   9.292   6.0    -1.292
Eigenvalues B-matrix:
   2.549   2.0     1.0     1.0      .785    .0
Eigenvalues Hessian:
   6.0     5.256   5.222   3.956   3.0      .426    .0      .0      .0     -.174
        -2.682  -5.005
Eigenvalues Classical Scaling:
  12.958   9.145   2.610    .0     -2.479  -7.568
Dissimilarities:
  .0    4.0     .0     .0    3.464 2.0
 4.0     .0    4.0     .0    2.0     .0
  .0    4.0     .0    4.0     .0    2.0
  .0     .0    4.0     .0    2.0    3.464
 3.464 2.0     .0    2.0     .0     .0
 2.0     .0    2.0   3.464  .0     .0

STRESS:  65.25
M:
   5.459  -2.464   3.200
  -2.464   8.864   1.516
   3.200   1.516   -.822
Eigenvalues M:
  10.158   6.0    -2.658
Eigenvalues B-matrix:
   2.693   2.0     1.0     1.0      .557    .0
Eigenvalues Hessian:
   6.0     6.0     5.358   4.552   3.255    .0      .0      .0      .0     -.190
        -2.613  -5.862
Eigenvalues Classical Scaling:
  15.151  14.862    .0     -.328  -1.362 -11.448
Dissimilarities:
  .0    4.500  .0     .0    2.598 3.0
 4.500  .0    4.500  .0    1.500  .0
  .0    4.500  .0    3.0   2.598  .0
```

```
  .0    .0   3.0    .0    .0   5.196
 2.598 1.500 2.598  .0    .0    .0
 3.0    .0    .0   5.196  .0    .0

STRESS:  40
M:
  -1.770   -.328   1.631
   -.328   7.273    .302
   1.631    .302  -1.502
Eigenvalues M:
   7.292    .0    -3.292
Eigenvalues B-matrix:
   2.215   1.0     1.0     1.0      .451    .0
Eigenvalues Hessian:
   6.0     6.0     5.612   4.0     3.503   2.0     1.852    .0      .0      .0
          .0    -2.967
Eigenvalues Classical Scaling:
  13.292   6.0     2.708    .0    -3.333  -6.0
Dissimilarities:
  .0   4.0    .0    .0   3.464 2.0
 4.0    .0   2.0   3.464  .0    .0
  .0   2.0    .0    .0   3.464 2.0
  .0   3.464  .0    .0    .0   3.464
 3.464  .0   3.464  .0    .0    .0
 2.0    .0   2.0   3.464  .0    .0

STRESS:  59.04
M:
   1.495  -1.572   3.044
  -1.572   9.709   1.448
   3.044   1.448   1.996
Eigenvalues M:
  10.109   4.800  -1.709
Eigenvalues B-matrix:
   2.685   1.800   1.0     1.0      .715    .0
Eigenvalues Hessian:
   6.0     5.799   5.400   3.709   2.751    .600    .204    .0      .0      .0
        -1.950  -5.713
Eigenvalues Classical Scaling:
  16.338   9.780   3.629    .0    -5.460  -8.448
Dissimilarities:
  .0   4.800  .0    .0   2.078 3.600
 4.800  .0   3.600 2.078  .0    .0
  .0   3.600  .0   1.200 4.157  .0
```

```
  .0    2.078 1.200  .0     .0   4.157
 2.078  .0    4.157  .0     .0   1.200
 3.600  .0     .0    4.157 1.200  .0

STRESS:  52
M:
   3.256  -1.983   3.097
  -1.983   7.818   -.181
   3.097   -.181  -1.075
Eigenvalues M:
   8.745   4.0    -2.745
Eigenvalues B-matrix:
   2.457   1.667   1.0     1.0     .543    .0
Eigenvalues Hessian:
   6.0     5.784   5.588   4.0     3.0     .695    .602    .0      .0      .0
        -1.190  -4.479
Eigenvalues Classical Scaling:
  14.167  11.029    .0     -.138   -.363 -10.030
Dissimilarities:
  .0    4.0    .0     .0    3.464 2.0
 4.0    .0    4.0    1.155  .0   1.155
  .0    4.0    .0    2.0    3.464  .0
  .0    1.155 2.0     .0     .0   4.619
 3.464  .0    3.464  .0     .0    .0
 2.0    1.155  .0    4.619  .0    .0

STRESS:  52
M:
   3.049  -2.948   2.718
  -2.948   7.455   2.716
   2.718   2.716   3.496
Eigenvalues M:
   9.292   6.0    -1.292
Eigenvalues B-matrix:
   2.549   2.0     1.0     1.0     .785    .0
Eigenvalues Hessian:
   6.0     5.256   5.222   3.956   3.0     .426    .0      .0      .0     -.174
        -2.682  -5.005
Eigenvalues Classical Scaling:
  12.958   9.145   2.610    .0    -2.479  -7.568
Dissimilarities:
  .0    4.0    .0    2.0     .0   4.0
 4.0    .0    4.0     .0    2.0    .0
  .0    4.0    .0    2.0    3.464  .0
```

```
2.0    .0   2.0    .0    .0   3.464
 .0   2.0   3.464  .0    .0   2.0
4.0    .0    .0   3.464 2.0    .0

STRESS:  65.25
M:
  -1.303  -1.713   2.644
  -1.713   8.864   2.331
   2.644   2.331   5.939
Eigenvalues M:
  10.158   6.0    -2.658
Eigenvalues B-matrix:
   2.693   2.0     1.0     1.0     .557    .0
Eigenvalues Hessian:
   6.0     6.0     5.358   4.552   3.255   .0     .0     .0     .0    -.190
        -2.613  -5.862
Eigenvalues Classical Scaling:
  15.151  14.862    .0     -.328  -1.362 -11.448
Dissimilarities:
  .0   4.500  .0   1.500  .0   4.500
 4.500  .0   3.0   2.598  .0    .0
  .0   3.0    .0    .0   5.196  .0
 1.500 2.598  .0    .0    .0   2.598
  .0    .0   5.196  .0    .0   3.0
 4.500  .0    .0   2.598 3.0    .0

STRESS:  52
M:
  -1.212   -.646   -.642
   -.646   7.455   1.599
   -.642   1.599   7.758
Eigenvalues M:
   9.292   6.0    -1.292
Eigenvalues B-matrix:
   2.549   2.0     1.0     1.0     .785    .0
Eigenvalues Hessian:
   6.0     5.256   5.222   3.956   3.0     .426   .0     .0     .0    -.174
        -2.682  -5.005
Eigenvalues Classical Scaling:
  12.958   9.145   2.610    .0    -2.479  -7.568
Dissimilarities:
  .0   4.0    .0   2.0    .0   4.0
 4.0    .0   2.0   3.464  .0    .0
  .0   2.0    .0    .0   3.464 2.0
```

```
 2.0   3.464  .0     .0   2.0    .0
  .0    .0   3.464 2.0    .0   4.0
 4.0    .0   2.0    .0   4.0    .0

STRESS:  52
M:
   8.360  -2.011  -2.224
  -2.011   3.091  -3.169
  -2.224  -3.169   2.549
Eigenvalues M:
   9.292   6.0    -1.292
Eigenvalues B-matrix:
   2.549   2.0     1.0     1.0      .785    .0
Eigenvalues Hessian:
   6.0     5.256   5.222   3.956   3.0      .426    .0      .0      .0     -.174
        -2.682  -5.005
Eigenvalues Classical Scaling:
  12.958   9.145   2.610    .0    -2.479  -7.568
Dissimilarities:
  .0   2.0    .0   2.0   3.464  .0
 2.0    .0   4.0    .0    .0   3.464
  .0   4.0    .0   4.0    .0   2.0
 2.0    .0   4.0    .0   4.0    .0
 3.464  .0    .0   4.0    .0   2.0
  .0   3.464 2.0    .0   2.0    .0

STRESS:  40
M:
   3.049  -2.993   2.669
  -2.993   2.909  -2.263
   2.669  -2.263  -1.958
Eigenvalues M:
   7.292    .0    -3.292
Eigenvalues B-matrix:
   2.215   1.0     1.0     1.0      .451    .0
Eigenvalues Hessian:
   6.0     6.0     5.612   4.0     3.503   2.0     1.852    .0      .0      .0
          .0    -2.967
Eigenvalues Classical Scaling:
  13.292   6.0     2.708    .0    -3.333  -6.0
Dissimilarities:
  .0   2.0    .0   2.0   3.464  .0
 2.0    .0   4.0    .0    .0   3.464
  .0   4.0    .0   2.0   3.464  .0
```

```
 2.0    .0   2.0    .0    .0   3.464
 3.464  .0   3.464  .0    .0    .0
  .0   3.464  .0   3.464  .0    .0

STRESS:  20
M:
  -1.978  -1.337   1.203
  -1.337   2.364  -1.781
   1.203  -1.781  -2.385
Eigenvalues M:
   3.464  -2.0    -3.464
Eigenvalues B-matrix:
   1.577   1.0     1.0     .667    .423    .0
Eigenvalues Hessian:
   6.0     5.752   5.473   4.463   3.841   3.0     2.0     1.065    .407    .0
           .0      .0
Eigenvalues Classical Scaling:
   8.969   3.737   2.592    .0    -1.070  -4.894
Dissimilarities:
   .0   2.0    .0   2.0   3.464  .0
  2.0    .0   2.0   2.309  .0   2.309
   .0   2.0    .0    .0   3.464 2.0
  2.0   2.309  .0    .0    .0   2.309
  3.464  .0   3.464  .0    .0    .0
   .0   2.309 2.0   2.309  .0    .0

STRESS:  65.25
M:
  -1.376   2.743  -1.396
   2.743   5.591   1.992
  -1.396   1.992   9.285
Eigenvalues M:
  10.158   6.0    -2.658
Eigenvalues B-matrix:
   2.693   2.0     1.0     1.0     .557    .0
Eigenvalues Hessian:
   6.0     6.0     5.358   4.552   3.255    .0      .0      .0      .0     -.190
         -2.613  -5.862
Eigenvalues Classical Scaling:
  15.151  14.862    .0     -.328  -1.362 -11.448
Dissimilarities:
   .0   3.0   2.598  .0    .0   4.500
  3.0    .0    .0   5.196  .0    .0
  2.598  .0    .0    .0   2.598 1.500
```

```
 .0    5.196  .0     .0    3.0     .0
 .0     .0    2.598 3.0     .0    4.500
4.500   .0    1.500  .0    4.500   .0

STRESS:  52
M:
  -1.595   2.657    .329
   2.657   3.455   1.569
    .329   1.569   8.140
Eigenvalues M:
   8.745   4.0     -2.745
Eigenvalues B-matrix:
   2.457   1.667   1.0      1.0      .543     .0
Eigenvalues Hessian:
   6.0      5.784   5.588   4.0      3.0      .695     .602     .0       .0       .0
        -1.190  -4.479
Eigenvalues Classical Scaling:
  14.167  11.029    .0     -.138    -.363 -10.030
Dissimilarities:
  .0    2.0    3.464  .0     .0    4.0
 2.0     .0     .0   4.619  .0    1.155
 3.464  .0     .0     .0   3.464  .0
  .0   4.619  .0     .0    2.0   1.155
  .0     .0   3.464 2.0     .0    4.0
 4.0   1.155  .0    1.155 4.0     .0

STRESS:  59.04
M:
   1.417   3.177  -1.226
   3.177   1.855    .688
  -1.226    .688   9.928
Eigenvalues M:
  10.109   4.800  -1.709
Eigenvalues B-matrix:
   2.685   1.800   1.0      1.0      .715     .0
Eigenvalues Hessian:
   6.0      5.799   5.400   3.709   2.751    .600     .204     .0       .0       .0
        -1.950  -5.713
Eigenvalues Classical Scaling:
  16.338   9.780   3.629    .0     -5.460  -8.448
Dissimilarities:
  .0    1.200 4.157  .0     .0    3.600
 1.200  .0     .0   4.157  .0    2.078
 4.157  .0     .0   1.200 2.078  .0
```

```
  .0    4.157 1.200  .0    3.600  .0
  .0     .0   2.078 3.600  .0    4.800
 3.600 2.078  .0     .0    4.800  .0

STRESS:  40
M:
  -2.360   2.011   2.224
   2.011   2.909   3.169
   2.224   3.169   3.451
Eigenvalues M:
   7.292    .0    -3.292
Eigenvalues B-matrix:
   2.215   1.0     1.0     1.0      .451    .0
Eigenvalues Hessian:
   6.0     6.0     5.612   4.0     3.503   2.0     1.852    .0      .0      .0
          .0    -2.967
Eigenvalues Classical Scaling:
  13.292   6.0     2.708    .0    -3.333  -6.0
Dissimilarities:
  .0    2.0   3.464  .0    .0    4.0
 2.0    .0    .0   3.464 2.0    .0
 3.464  .0    .0    .0   3.464  .0
  .0   3.464  .0    .0    .0   3.464
  .0    2.0   3.464  .0    .0    2.0
 4.0    .0    .0   3.464 2.0    .0

STRESS:  52
M:
   2.951   2.993  -2.669
   2.993   3.091   2.263
  -2.669   2.263   7.958
Eigenvalues M:
   9.292   6.0    -1.292
Eigenvalues B-matrix:
   2.549   2.0     1.0     1.0      .785    .0
Eigenvalues Hessian:
   6.0     5.256   5.222   3.956   3.0      .426    .0      .0      .0     -.174
        -2.682  -5.005
Eigenvalues Classical Scaling:
  12.958   9.145   2.610    .0    -2.479  -7.568
Dissimilarities:
  .0    2.0   3.464  .0    .0    4.0
 2.0    .0    .0   3.464 2.0    .0
 3.464  .0    .0   2.0    .0    2.0
```

```
 .0   3.464 2.0    .0   4.0    .0
 .0   2.0    .0   4.0    .0   4.0
4.0    .0   2.0    .0   4.0    .0

STRESS:  59.04
M:
  -1.416   1.212   1.384
   1.212   7.091   2.499
   1.384   2.499   7.525
Eigenvalues M:
  10.109   4.800  -1.709
Eigenvalues B-matrix:
   2.685   1.800   1.0     1.0      .715    .0
Eigenvalues Hessian:
   6.0     5.799   5.400   3.709   2.751    .600    .204    .0      .0      .0
        -1.950  -5.713
Eigenvalues Classical Scaling:
  16.338   9.780   3.629    .0    -5.460  -8.448
Dissimilarities:
  .0   3.600 2.078  .0    .0   4.800
 3.600  .0   1.200 4.157  .0    .0
 2.078 1.200  .0    .0   4.157  .0
  .0   4.157  .0    .0   1.200 2.078
  .0    .0   4.157 1.200  .0   3.600
 4.800  .0    .0   2.078 3.600  .0

STRESS:  59.04
M:
  10.071   -.426   -.514
   -.426   1.855  -3.223
   -.514  -3.223   1.274
Eigenvalues M:
  10.109   4.800  -1.709
Eigenvalues B-matrix:
   2.685   1.800   1.0     1.0      .715    .0
Eigenvalues Hessian:
   6.0     5.799   5.400   3.709   2.751    .600    .204    .0      .0      .0
        -1.950  -5.713
Eigenvalues Classical Scaling:
  16.338   9.780   3.629    .0    -5.460  -8.448
Dissimilarities:
  .0   1.200 2.078  .0   4.157  .0
 1.200  .0   3.600  .0    .0   4.157
 2.078 3.600  .0   4.800  .0    .0
```

```
  .0     .0    4.800  .0    3.600 2.078
 4.157  .0     .0    3.600  .0    1.200
  .0    4.157  .0    2.078 1.200  .0

STRESS:  52
M:
   8.021  -1.346   1.120
  -1.346   3.455  -2.776
   1.120  -2.776  -1.476
Eigenvalues M:
   8.745   4.0    -2.745
Eigenvalues B-matrix:
   2.457   1.667   1.0     1.0      .543     .0
Eigenvalues Hessian:
   6.0     5.784   5.588   4.0     3.0      .695     .602     .0       .0       .0
         -1.190  -4.479
Eigenvalues Classical Scaling:
  14.167  11.029    .0     -.138   -.363 -10.030
Dissimilarities:
  .0    2.0    1.155  .0    4.619  .0
 2.0     .0    4.0     .0     .0   3.464
 1.155 4.0     .0    4.0    1.155  .0
  .0     .0    4.0     .0   2.0    3.464
 4.619  .0    1.155 2.0     .0     .0
  .0    3.464  .0    3.464  .0     .0

STRESS:  20
M:
  -2.579   1.665   1.153
   1.665   2.364   1.479
   1.153   1.479  -1.784
Eigenvalues M:
   3.464  -2.0    -3.464
Eigenvalues B-matrix:
   1.577   1.0     1.0      .667    .423     .0
Eigenvalues Hessian:
   6.0     5.752   5.473   4.463   3.841   3.0     2.0      1.065    .407     .0
          .0       .0
Eigenvalues Classical Scaling:
   8.969   3.737   2.592    .0    -1.070  -4.894
Dissimilarities:
  .0    2.0    2.309  .0    2.309 2.0
 2.0     .0     .0   3.464 2.0     .0
 2.309  .0     .0     .0   2.309 2.0
```

```
 .0    3.464  .0     .0     .0    3.464
2.309  2.0    2.309  .0     .0    .0
2.0    .0     2.0    3.464  .0    .0

STRESS:  52
M:
  -1.551    .018   2.702
    .018   7.818   1.992
   2.702   1.992   3.733
Eigenvalues M:
   8.745   4.0    -2.745
Eigenvalues B-matrix:
   2.457   1.667   1.0     1.0      .543    .0
Eigenvalues Hessian:
   6.0     5.784   5.588   4.0     3.0      .695    .602    .0      .0      .0
      -1.190  -4.479
Eigenvalues Classical Scaling:
  14.167  11.029    .0     -.138   -.363 -10.030
Dissimilarities:
  .0    4.0    1.155  .0    1.155  4.0
 4.0    .0     2.0    3.464 .0     .0
 1.155  2.0    .0     .0    4.619  .0
  .0    3.464  .0     .0    .0     3.464
 1.155  .0     4.619  .0    .0     2.0
 4.0    .0     .0     3.464 2.0    .0
```