

## Chapter 1

# MAMMA: a 1-D steady state model of volcanic conduits

MAMMA (Magma Ascent Mathematical Modeling and Analysis) is a FORTRAN90 code designed to solve a conservative model for steady magma ascent in a volcanic conduit, described as a compressible multicomponent two-phase flow. It is an open-source code (<https://github.com/demichie/MAMMA>) mainly developed by Mattia de'Michieli Vitturi. This document has been redacted by Alvaro Aravena Ponce.

### 1.1 System of equations

The system of conservation equations is derived from the theory of thermodynamically compatible systems [1], considering the effects of the main processes that magmas experience during ascent, such as crystallization, rheological changes, fragmentation, injection of external water, physical interaction with conduit walls, outgassing and degassing. The system is described as a mixture of two phases ( $i = 1, 2$ ), each one characterized by a volumetric fraction ( $\alpha_i$ ), density ( $\rho_i$ ), velocity ( $u_i$ ) and specific entropy ( $s_i$ ). Below the fragmentation level, phase 1 is a mixture of crystals, dissolved water and melt (continuous phase); while phase 2 is composed by the exsolved gas bubbles (dispersed phase). Above magma fragmentation, phase 1 is constituted by magma fragments (dispersed phase) and phase 2 is the exsolved gas mixture (continuous phase). Magma

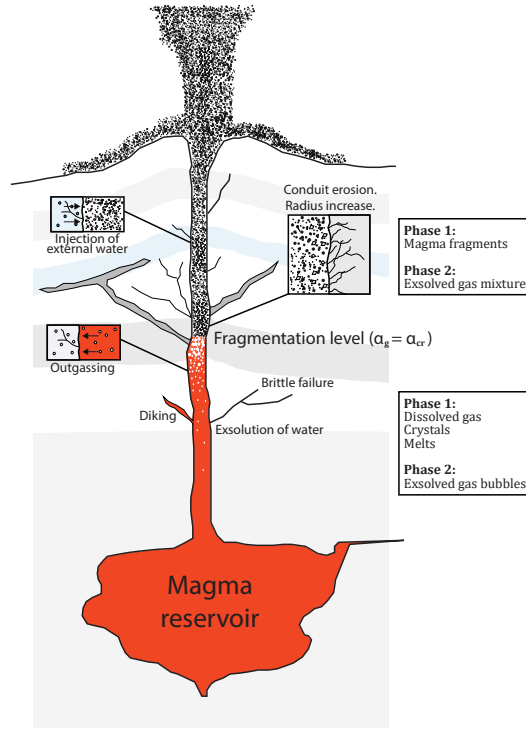


FIGURE 1.1: Schematic illustration of volcanic conduits.

fragmentation occurs when the exsolved gas volumetric fraction reaches a critical value ( $\alpha_g = \alpha_{cr}$ ) [2] (Fig. 1.1).

The components of the system are characterized by an equation of state; while pressure ( $p_i$ ) and temperature ( $T_i$ ) of both phases are derived from the internal energy ( $e_i$ ):

$$p_i = \rho_i^2 \frac{\partial e_i}{\partial \rho_i} \quad (1.1)$$

$$T_i = \frac{\partial e_i}{\partial s_i} \quad (1.2)$$

The model is capable of describing conduits with elliptical cross sections, and it includes the conservation laws of total mass (Eq. 1.3), momentum (Eq. 1.4) and energy (Eq. 1.5).

$$\frac{\partial}{\partial z} \left( \rho u R_{eq}^2 \right) = 2J_{ex} f_{\epsilon_1} R_{eq} - 2J_{lat} f_{\epsilon_1} R_{eq} \quad (1.3)$$

$$\begin{aligned} \frac{\partial}{\partial z} \left( \left( \alpha_1(\rho_1 u_1^2 + p_1) + \alpha_2(\rho_2 u_2^2 + p_2) \right) R_{eq}^2 \right) \\ = -\rho g R_{eq}^2 - 2J_{lat} f_{\epsilon_1} u_2 R_{eq} - \frac{8\chi_1 \mu u_1}{f_{\epsilon_2}^2} - \frac{\chi_2 \lambda_w \rho_2 u_2^2 R_{eq}}{4f_{\epsilon_2}^2} \end{aligned} \quad (1.4)$$

$$\begin{aligned} \frac{\partial}{\partial z} \left( \left( \alpha_1 \rho_1 u_1 \left( e_1 + \frac{p_1}{\rho_1} + \frac{u_1^2}{2} \right) + \alpha_2 \rho_2 u_2 \left( e_2 + \frac{p_2}{\rho_2} + \frac{u_2^2}{2} \right) \right. \right. \\ \left. \left. - \rho x_1 x_2 (u_1 - u_2)(s_1 - s_2) T \right) R_{eq}^2 \right) = -\rho g u R_{eq}^2 - \frac{8\chi_1 \mu u_1^2}{f_{\epsilon_2}^2} \\ - \frac{\chi_2 \lambda_w \rho_2 u_2^3 R_{eq}}{4f_{\epsilon_2}^2} + 2J_{ex} f_{\epsilon_1} c_w T_w R_{eq} - 2J_{lat} f_{\epsilon_1} \left( c_g T + \frac{u_2^2}{2} \right) R_{eq} \end{aligned} \quad (1.5)$$

where  $z$  is the vertical coordinate,  $\rho$  is mixture density,  $u$  is mixture velocity,  $R_{eq}$  is the equivalent conduit radius (Eq. 1.6),  $J_{ex}$  is the mass flux of external water,  $f_{\epsilon_1}$  is a conduit eccentricity-derived factor (Eq. 1.7),  $J_{lat}$  is the lateral gas flux through conduit walls,  $g$  is the acceleration of gravity,  $\chi_i$  controls the inclusion of wall friction contribution (1 or 0, function of the continuous phase index),  $\mu$  is mixture viscosity,  $f_{\epsilon_2}$  is an additional conduit eccentricity-derived factor (Eq. 1.8),  $\lambda_w$  is a drag coefficient [3],  $x_i$  is the mass fraction of phase  $i$ ,  $T$  is mixture temperature,  $c_w$  is the specific heat capacity of external water,  $T_w$  is the external water temperature and  $c_g$  is the specific heat capacity of exsolved water.

$$R_{eq} = \sqrt{R_a \cdot R_b} \quad (1.6)$$

$$f_{\epsilon_1} = \frac{3(1 + \sqrt{1 - \epsilon^2}) - \sqrt{(3 + \sqrt{1 - \epsilon^2}) \cdot (1 + 3\sqrt{1 - \epsilon^2})}}{2 \cdot \sqrt[4]{1 - \epsilon^2}} \quad (1.7)$$

$$f_{\epsilon_2} = \sqrt{\frac{2\sqrt{1 - \epsilon^2}}{2 - \epsilon^2}} \quad (1.8)$$

where  $\epsilon$  is conduit eccentricity (Eq. 1.9),  $R_a$  is the maximum semi-axis and  $R_b$  is the minimum semi-axis.

$$\epsilon = \sqrt{1 - \frac{R_b^2}{R_a^2}} \quad (1.9)$$

Phase 1 volume fraction is governed by the following equation:

$$\frac{\partial}{\partial z} \left( \rho u \alpha_1 R_{eq}^2 \right) = -\frac{1}{\tau(p)} (p_2 - p_1) R_{eq}^2 \quad (1.10)$$

where  $\tau^{(p)}$  is the relaxation parameter which controls the pressure difference between both phases.

Furthermore, the model includes an additional equation for controlling the relative velocity between phases:

$$\begin{aligned} \frac{\partial}{\partial z} \left( \left( \frac{u_1^2}{2} - \frac{u_2^2}{2} + e_1 - e_2 + \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} - (s_1 - s_2)T \right) R_{eq}^2 \right) \\ = -\frac{8\chi_1\mu u_1}{\alpha_1\rho_1 f_{\epsilon_2}^2} + \frac{\chi_2\lambda_w u_2^2 R_{eq}}{4\alpha_2 f_{\epsilon_2}^2} - \frac{\rho}{\rho_1\rho_2} \delta_f (u_1 - u_2) R_{eq}^2 \end{aligned} \quad (1.11)$$

where  $\delta_f$  is the drag factor.

Finally, the system of equations presents the mass conservation laws of crystals (Eq. 1.12), dissolved water (Eq. 1.13) and exsolved gas (Eq. 1.14).

$$\frac{\partial}{\partial z} \left( \alpha_1 \rho_c \alpha_c u_1 R_{eq}^2 \right) = -\frac{1}{\tau^{(c)}} \alpha_1 \rho_c (\alpha_c - \alpha_c^{eq}) R_{eq}^2 \quad (1.12)$$

$$\begin{aligned} \frac{\partial}{\partial z} \left( x_d \alpha_1 (\rho_1 - \alpha_c \rho_c) u_1 R_{eq}^2 \right) \\ = 2J_{ex} f_{\epsilon_1} R_{eq} - \frac{1}{\tau^{(d)}} (x_d - x_d^{eq}) \alpha_1 (\rho_1 - \alpha_c \rho_c) R_{eq}^2 \end{aligned} \quad (1.13)$$

$$\frac{\partial}{\partial z} \left( \alpha_2 \rho_2 u_2 R_{eq}^2 \right) = -2J_{lat} f_{\epsilon_1} R_{eq} + \frac{1}{\tau^{(d)}} (x_d - x_d^{eq}) \alpha_1 (\rho_1 - \alpha_c \rho_c) R_{eq}^2 \quad (1.14)$$

where  $\rho_c$  is density of crystals,  $\alpha_c$  is the volume fraction of crystals in phase 1,  $\tau^{(c)}$  is the crystallization relaxation parameter,  $\alpha_c^{eq}$  is the equilibrium value of crystals volume fraction in phase 1,  $x_d$  is the mass fraction of dissolved water in the phase composed by melt and dissolved water,  $\tau^{(d)}$  is the relaxation parameter which controls the gas exsolution rate, and  $x_d^{eq}$  is the equilibrium value of the mass fraction of dissolved water in the phase composed by melt and dissolved water.

For the model solution, it employs a numerical shooting technique: for a given inlet pressure, the model adjusts the inlet flow rate until the appropriate boundary condition (choked flow or atmospheric pressure) is reached. For the spatial integration of the equations, a well-established PI step-size control technique is adopted, with the relaxation terms treated implicitly to guarantee the stability of the numerical scheme.

TABLE 1.1: Available models for calculating  $\theta_c(\alpha_c)$  in MAMMA.

Model	Equation	Auxiliary variables
None	$\theta_c = 1.0$	
Costa [4]	$\theta_c = \left(1 - c_1 \cdot \operatorname{erf}\left(\frac{\sqrt{\pi}}{2} \alpha_c \left(1 + \frac{c_2}{(1-\alpha_c)^{c_3}}\right)\right)\right)^{\frac{c_4}{c_1}}$	$c_1 = 0.9995$ . $c_2 = 1.0$ . $c_3 = 1.0$ . $c_4 = -2.5$ .
Dingwell [8]	$\theta_c = \left(1 + 0.75 \cdot \frac{\alpha_c}{c - \alpha_c}\right)^2$	$c = 0.84$
Lejeune-Richet [9]	$\theta_c = \left(1 - \frac{\alpha_c}{c_1}\right)^{-c_2}$	$c_1 = 0.7$ . $c_2 = 3.4$ .
Melnik-Sparks v1 [10]	$\log_{10}\left(\frac{\theta_c}{c_1}\right) = \operatorname{atan}(c_2 \cdot (\alpha_c - c_3)) + \frac{\pi}{2}$	$c_1 = 1.6$ . $c_2 = 20.6$ . $c_3 = 0.62$ .
Melnik-Sparks v2 [11]	$\log_{10}\left(\frac{\theta_c}{c_1}\right) = \operatorname{atan}(c_2 \cdot (\alpha_c - c_3)) + \frac{\pi}{2}$	$c_1 = 1.4$ . $c_2 = 8.6$ . $c_3 = 0.69$ .
Vona v1 [12]	$\theta_c = \frac{1 + \phi^{c_2}}{\left(1 - (1 - c_3) \operatorname{erf}\left(\frac{\sqrt{\pi}}{2(1-c_3)} \phi(1 + \phi^{c_4})\right)\right)^{c_1 c_5}}$	$\phi = \alpha_c / c_1$ . $c_1 = 0.27$ . $c_2 = 12.16$ . $c_3 = 0.032$ . $c_4 = 0.84$ . $c_5 = 2.8$ .
Vona v2 [12]	$\theta_c = \frac{1 + \phi^{c_2}}{\left(1 - (1 - c_3) \operatorname{erf}\left(\frac{\sqrt{\pi}}{2(1-c_3)} \phi(1 + \phi^{c_4})\right)\right)^{c_1 c_5}}$	$\phi = \alpha_c / c_1$ . $c_1 = 0.39$ . $c_2 = 1.16$ . $c_3 = 0.03$ . $c_4 = 0.84$ . $c_5 = 2.8$ .

## 1.2 Constitutive equations

In order to offer the possibility of describing the behaviour of a wide range of magma compositions and volcanic phenomena, a complete set of constitutive equations has been implemented in the code.

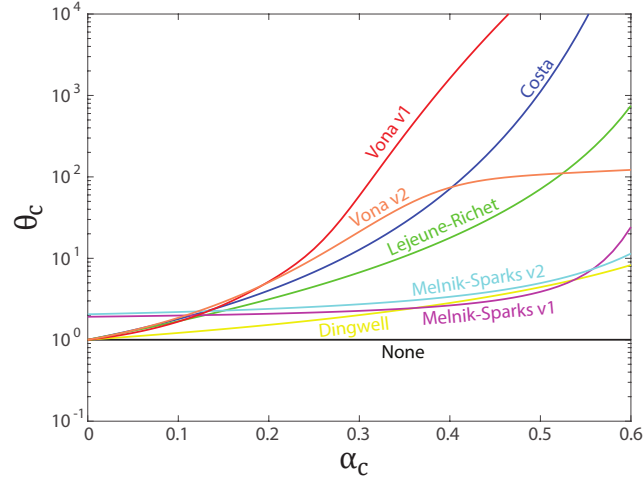
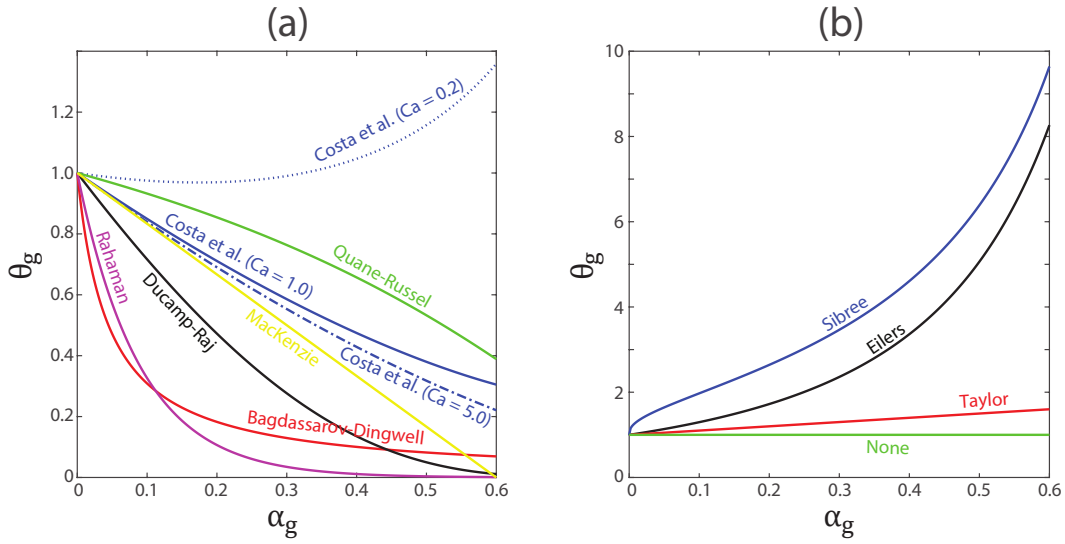
### 1.2.1 Viscosity models

Since it has been suggested a strong effect of crystal content [4, 5, 6] and exsolved gas bubbles [6, 7] on the resulting mixture rheology, magma viscosity ( $\mu$ ) is evaluated using the following expression:

$$\mu = \mu_{melt} \cdot \theta_c(\alpha_c) \cdot \theta_g(\alpha_g) \quad (1.15)$$

where  $\mu_{melt}$  is the crystals and bubbles-free viscosity; whereas  $\theta_c(\alpha_c)$  and  $\theta_g(\alpha_g)$  account for the effect of crystals and bubbles on the resulting viscosity, respectively.

The following models are implemented for calculating  $\mu_{melt}$ , while the available expressions for calculating  $\theta_c$  and  $\theta_g$  are shown in Tables 1.1 and 1.2 and Figs. 1.3 and 1.2.

FIGURE 1.2: Available models for calculating  $\theta_c(\alpha_c)$  in MAMMA.FIGURE 1.3: Available models for calculating  $\theta_g(\alpha_g)$  in MAMMA.

### Hess and Dingwell [24]

This model is based on a multiple non-linear regression of 111 measurements of viscosity, and is adapted for studying rhyolitic magmas:

$$\log_{10}(\mu_{melt}) = -3.545 + 0.833 \cdot \ln(w) + \frac{9601 - 2368 \cdot \ln(w)}{T - (195.7 + 32.25 \cdot \ln(w))} \quad (1.16)$$

where  $\mu_{melt}$  is expressed in  $\text{Pa} \cdot \text{s}$ ,  $w$  is  $\text{H}_2\text{O}$  concentration in wt.% and  $T$  is temperature in K.

TABLE 1.2: Available models for calculating  $\theta_g(\alpha_g)$  in MAMMA.

Model	Equation	Auxiliary variables
None	$\theta_g = 1.0$	
Bagdassarov-Dingwell [13]	$\theta_g = \frac{1}{1+b \cdot \alpha_g}$	$b = 22.4$
Costa et al. [14]	$\theta_g = \frac{1+25 \cdot \text{Ca}^2(1-\alpha_g)^{8/3}}{(1-\alpha_g) \cdot (1+25 \cdot \text{Ca}^2)}$	Ca <sup>(1)</sup>
Ducamp-Raj [15]	$\theta_g = \exp\left(\frac{b \cdot \alpha_g}{1-\alpha_g}\right)$	$b = -3$
Eilers [16, 17]	$\theta_g = \left(1 + \frac{1.25 \alpha_g}{1-b \cdot \alpha_g}\right)^2$	$b = 1.29$
Mackenzie [18]	$\theta_g = 1 - \frac{5}{3} \alpha_g$	
Quane-Russel [19]	$\theta_g = \exp\left(\frac{b \cdot \alpha_g}{1-\alpha_g}\right)$	$b = -0.63$ <sup>(2)</sup>
Rahaman [20]	$\theta_g = \exp(-b \cdot \alpha_g)$	$b = 11.2$
Sibree [21]	$\theta_g = \frac{1}{1-(b \cdot \alpha_g)^{1/3}}$	$b = 1.2$
Taylor [22]	$\theta_g = 1 + \alpha_g$	

<sup>(1)</sup> Capillarity number. Calculated following Llewellyn and Manga [23].

<sup>(2)</sup> Adapted for Phlegrean Fields.

### Romano et al. [25]

This model has been calibrated using samples from Vesuvius and Phlegrean Fields.

For trachytic magmas, melt viscosity is calculated using:

$$\log_{10}(\mu_{melt}) = -3.5405 + 0.14467 \cdot \ln(w) + \frac{9618.9 - 498.79 \cdot \ln(w)}{T - (191.78 - 35.518 \cdot \ln(w))} \quad (1.17)$$

On the other hand, the following equation is adapted for studying phonolitic magmas:

$$\log_{10}(\mu_{melt}) = -5.8996 - 0.2857 \cdot \ln(w) + \frac{10775 - 394.83 \cdot \ln(w)}{T - (148.71 - 21.65 \cdot \ln(w))} \quad (1.18)$$

### Giordano et al. [26]

This model predicts the non-Arrhenian Newtonian viscosity of silicate melts as a function of  $T$  and melt composition (major elements). Melt viscosity ( $\mu_{melt}$ ) is calculated using:

$$\log_{10}(\mu_{melt}) = -4.55 + \frac{B_G}{T - C_G} \quad (1.19)$$

where  $B_G$  and  $C_G$  are composition-dependent constants (Eq. 1.20 and Eq. 1.21, respectively).

TABLE 1.3: Coefficients for calculation of  $B_G$  and  $C_G$  from melt composition (mol% oxide) [26].

Coefficient	Value	Oxides
$b_1$	159.6	$M_1 = \text{SiO}_2 + \text{TiO}_2$
$b_2$	-173.3	$M_2 = \text{Al}_2\text{O}_3$
$b_3$	72.1	$M_3 = \text{FeO(T)} + \text{MnO} + \text{P}_2\text{O}_5$
$b_4$	75.7	$M_4 = \text{MgO}$
$b_5$	-39.0	$M_5 = \text{CaO}$
$b_6$	-84.1	$M_6 = \text{Na}_2\text{O} + \text{V}^{(1)}$
$b_7$	141.5	$M_7 = \text{V} + \ln(1 + \text{H}_2\text{O})$
$b_{11}$	-2.43	$M_{11} = (\text{SiO}_2 + \text{TiO}_2) \cdot (\text{FM}^{(2)})$
$b_{12}$	-0.91	$M_{12} = (\text{SiO}_2 + \text{TA}^{(3)} + \text{P}_2\text{O}_5) \cdot (\text{NK}^{(4)} + \text{H}_2\text{O})$
$b_{13}$	17.6	$M_{13} = (\text{Al}_2\text{O}_3) \cdot (\text{NK})$
$c_1$	2.75	$N_1 = \text{SiO}_2$
$c_2$	15.7	$N_2 = \text{TA}$
$c_3$	8.3	$N_3 = \text{FM}$
$c_4$	10.2	$N_4 = \text{CaO}$
$c_5$	-12.3	$N_5 = \text{NK}$
$c_6$	-99.5	$N_6 = \ln(1 + \text{V})$
$c_{11}$	0.30	$N_{11} = (\text{Al}_2\text{O}_3 + \text{FM} + \text{CaO} - \text{P}_2\text{O}_5) \cdot (\text{NK} + \text{V})$

<sup>(1)</sup>  $\text{V} = \text{H}_2\text{O} + \text{F}_2\text{O}_{-1}$ .<sup>(2)</sup>  $\text{FM} = \text{FeO(T)} + \text{MnO} + \text{MgO}$ .<sup>(3)</sup>  $\text{TA} = \text{TiO}_2 + \text{Al}_2\text{O}_3$ .<sup>(4)</sup>  $\text{NK} = \text{Na}_2\text{O} + \text{K}_2\text{O}$ .

$$B_G = \sum_{i=1}^7 (b_i M_i) + \sum_{j=1}^3 b_{1j} M_{1j} \quad (1.20)$$

$$C_G = \sum_{i=1}^6 (c_i N_i) + c_{11} N_{11} \quad (1.21)$$

where  $M_i$ ,  $M_{1j}$ ,  $N_i$  and  $N_{11}$  refer to the combinations of mol% oxides reported in Table 1.3, and  $b_i$ ,  $b_{1j}$ ,  $c_i$  and  $c_{11}$  are constant values (Table 1.3).

### Giordano et al. [27]

This model was calibrated using data derived from Stromboli samples, thus it is adapted for studying basaltic magmas. Melt viscosity ( $\mu_{melt}$ ) is determined using the following equation:



$$\log_{10}(\mu_{melt}) = -4.55 + \frac{6101 - 63.66 \cdot w^*}{T - (567 - 160.3 \cdot \log_{10}(1 + w^*))} \quad (1.22)$$

where  $w^*$  is  $H_2O$  concentration in mol%.

### Whittington et al. [28]

In this case, the viscosity model is adapted to dacitic magmas and uses the following formulation:

$$\log_{10}(\mu_{melt}) = -4.43 + \frac{7618.3 - 17.25 \cdot \log_{10}(w + 0.26)}{T - (406.1 - 292.6 \cdot \log_{10}(w + 0.26))} \quad (1.23)$$

### Di Genova et al. [29]

This work includes two viscosity models, adapted for studying the pantelleritic melts from the Khaggiar lava flow. The first formulation (Di Genova v1) uses the parametrization proposed by Giordano et al. [27]:

$$\log_{10}(\mu_{melt}) = -4.55 + \frac{4278.17 + 8.6 \cdot w^*}{T - (513 - 245.3 \cdot \log_{10}(1 + w^*))} \quad (1.24)$$

The second model (Di Genova v2) uses the following parametrization:

$$\log_{10}(\mu_{melt}) = -4.55 + \frac{10528.64 - 4672.21 \cdot \log_{10}(1 + w^*)}{T - (172.27 + 89.75 \cdot \log_{10}(1 + w^*))} \quad (1.25)$$

## 1.2.2 Solubility models

### Henry's law

The equilibrium value of dissolved water is calculated using the following expression:

$$x_d^{eq} = \sigma \left( \frac{p_g}{1[\text{Pa}]} \right)^\epsilon \quad (1.26)$$

where  $p_g$  is pressure of the gas component,  $\sigma$  is the solubility coefficient and  $\epsilon$  is the solubility exponent.

### Polynomial fit

When the polynomial fit is employed,  $x_d^{eq}$  is computed with the following expression:

$$x_d^{eq} = c_1 \cdot \left( \frac{p_g}{1[\text{Pa}]} \right)^2 + c_2 \cdot \left( \frac{p_g}{1[\text{Pa}]} \right) \quad (1.27)$$

where  $c_1$  and  $c_2$  are fitting parameters.

### Zhang model [30]

In this case,  $x_d^{eq}$  is calculated using the following equations:

$$x_d^{eq} = 0.01 \cdot \left( d_1(T) \cdot \sqrt{p_1} + d_2(T) \cdot p_1 + d_3(T) \cdot \sqrt{p_1^3} \right) \quad (1.28)$$

$$d_1(T) = 0.4874 - \frac{608}{T} + \frac{489530}{T^2} \quad (1.29)$$

$$d_2(T) = -0.060602 + \frac{135.6}{T} - \frac{69200}{T^2} \quad (1.30)$$

$$d_3(T) = 0.00253 - \frac{4.154}{T} + \frac{1509}{T^2} \quad (1.31)$$

where  $p_1$  and  $T$  are expressed in MPa and K, respectively.

### 1.2.3 Crystallization models

#### de'Michieli Vitturi et al. [31]

The equilibrium volume fraction of crystals ( $\alpha_c^{eq}$ ) is calculated using Eq. 1.32.

$$\alpha_c^{eq} = \min \left( \alpha_{c,max}, \alpha_{c,0} + 0.55 \cdot (0.58815 \cdot p^{-0.5226}) \right) \quad (1.32)$$

where  $\alpha_{c,max}$  is the maximum crystallinity,  $\alpha_{c,0}$  is the initial volume fraction of crystals and  $\min()$  is the minimum function.

### 1.2.4 Outgassing models

#### Forchheimer's law [3]

The model is dependent on the relative position of the fragmentation level. Below magma fragmentation, since a non-linear relationship between pressure gradient and gas flow rate has been recognized, Degruyter et al. [3] describe the outgassing process using the Forchheimer's law, which includes the influence of viscous (linear term) and inertial forces (quadratic term) (Eq. 1.33). Above magma fragmentation, the model presented by Yoshida and Koyaguchi [32] is considered; and the presence of a transitional domain is also assumed (Eq. 1.33). Please note that  $|dp/dz| = \delta_f \cdot \Delta u$ .

$$\left| \frac{dp}{dz} \right| = \begin{cases} \frac{\mu_g}{k_D} (\Delta u) + \frac{\rho_g}{k_I} (\Delta u)^2 & \text{if } \alpha_g \leq \alpha_{cr} \\ \left( \frac{\mu_g}{k_D} (\Delta u) + \frac{\rho_g}{k_I} (\Delta u)^2 \right)^{1-t} \cdot \left( \frac{3C_D}{8r_a} \rho_g (\Delta u)^2 \right)^t & \text{if } \alpha_{cr} < \alpha_g < \alpha_t \\ \frac{3C_D}{8r_a} \rho_g (\Delta u)^2 & \text{if } \alpha_g \geq \alpha_t \end{cases} \quad (1.33)$$

where  $\Delta u$  is the velocity difference between both phases, subscript  $g$  refers to the exsolved gas phase,  $C_D$  is a drag coefficient,  $r_a$  is the average size of the fragmented magma particles,  $t = (\alpha_g - \alpha_{cr})/(\alpha_t - \alpha_{cr})$ ,  $\alpha_t$  controls the range of the transitional domain, while  $k_D$  and  $k_I$  are the Darcian and inertial permeabilities, respectively (Eq. 1.34 and Eq. 1.35).

$$k_D = \frac{(f_{rb} r_b)^2}{8} \alpha_g^m \quad (1.34)$$

$$k_I = \frac{f_{rb} r_b}{f} \alpha_g^{(1+3m)/2} \quad (1.35)$$

$$r_b = \left( \frac{\alpha_g}{\frac{4\pi}{3} N_{bd} \alpha_l} \right)^{1/3} \quad (1.36)$$

where  $f_{rb}$  is the throat-bubble size ratio (0.1 - 1),  $r_b$  is the average bubble size,  $N_{bd}$  is the bubble density number ( $10^8 - 10^{16} \text{ m}^{-3}$ ) and  $f$  and  $m$  are fitting parameters.

#### Darcy's law

In this case, the inertial forces below magma fragmentation (quadratic term) and the transitional domain are not considered, and thus the resulting model is described by

the following expression:

$$\left| \frac{dp}{dz} \right| = \begin{cases} \frac{\mu_g}{k_D} (\Delta u) & \text{if } \alpha_g \leq \alpha_{cr} \\ \frac{3C_D}{8r_a} \rho_g (\Delta u)^2 & \text{if } \alpha_g > \alpha_{cr} \end{cases} \quad (1.37)$$

### 1.2.5 Degassing model

If lateral degassing is considered, it follows Eq. 1.38.

$$J_{lat} = \frac{\rho_2 \alpha_2 k}{\mu_2 f_{\epsilon_2}} \frac{\partial p}{\partial r} \Big|_{r=R_{eq}} \quad (1.38)$$

where  $k$  is country rocks permeability.

### 1.2.6 Injection of external water

When the injection of external water is considered, it is modelled using the following equation [2]:

$$J_{ex} = \frac{\rho_w k_a}{\mu_w f_{\epsilon_2}} \frac{\partial p}{\partial r} \Big|_{r=R_{eq}} \quad (1.39)$$

where  $\rho_w$  is external water density,  $k_a$  is the aquifer permeability and  $\mu_w$  is external water viscosity.

### 1.2.7 Equations of state

In order to define the specific internal energy and entropy of melt, crystals and dissolved water, a linearized version of the Mie-Gruneisen equation of state [33] was adopted:

$$e_k(\rho_k, T) = \bar{e}_k + c_{v,k} T + \frac{\rho_{0,k} C_{0,k}^2 - \gamma_k p_{0,k}}{\gamma_k \rho_k} \quad (1.40)$$

$$s_k(\rho_k, T) = s_{0,k} + c_{v,k} \cdot \ln \left( \frac{T}{T_{0,k}} \left( \frac{\rho_{0,k}}{\rho_k} \right)^{\gamma_k - 1} \right) \quad (1.41)$$

$$p_k(\rho_k, T) = c_{v,k} (\gamma_k - 1) \rho_k T - \frac{\rho_{0,k} C_{0,k}^2 - \gamma_k p_{0,k}}{\gamma_k} \quad (1.42)$$

where  $\bar{e}_k$  is formation energy,  $c_{v,k}$  is the specific heat capacity at constant volume,  $\rho_{0,k}$  and  $C_{0,k}$  are density and sound speed at a reference state,  $\gamma_k$  is the adiabatic

exponent and  $p_{0,k}$ ,  $s_{0,k}$  and  $T_{0,k}$  are pressure, specific entropy and temperature at a reference state. Subscript  $k$  refers to melt, dissolved water or crystals.

For the equation of state of exsolved water, two models are available:

### Ideal gas

The internal energy, specific entropy and pressure are calculated using equations 1.43, 1.44 and 1.45, respectively.

$$e_g(\rho_g, T) = c_{v,g}T + \bar{e}_g \quad (1.43)$$

$$s_g(\rho_g, T) = c_{v,g} \cdot \ln \left( \frac{T}{T_{0,g}} \left( \frac{\rho_{0,g}}{\rho_g} \right)^{\gamma_g - 1} \right) \quad (1.44)$$

$$p_g(\rho_g, T) = c_{v,g}(\gamma_g - 1) \cdot T \cdot \rho_g \quad (1.45)$$

where  $c_{v,g}$  is the specific heat capacity at constant volume,  $\bar{e}_g$  is the formation energy,  $T_{0,g}$  and  $\rho_{0,g}$  are temperature and density at a reference state and  $\gamma_g$  is the adiabatic exponent. Subscript  $g$  refers to exsolved gas.

### Van der Waals

In this case, we use the following equations:

$$e_g(\rho_g, T) = c_{v,g}T - a \cdot \rho + \bar{e}_g \quad (1.46)$$

$$s_g(\rho_g, T) = c_{v,g} \cdot \ln \left( \frac{T}{T_{0,g}} \left( \frac{\rho_{0,g}}{\rho_g} \cdot (1 - b \cdot \rho_g) \right)^{\gamma_g - 1} \right) \quad (1.47)$$

$$p_g(\rho_g, T) = c_{v,g}(\gamma_g - 1) \cdot T \cdot \frac{\rho_g}{1 - b\rho_g} - a\rho_g^2 \quad (1.48)$$

where:

$$a = \frac{27}{64} \cdot \frac{c_{v,g}^2(\gamma_g - 1)^2 T_{cr,g}^2}{p_{cr,g}} \quad (1.49)$$

$$b = \frac{1}{8} \cdot \frac{c_{v,g}(\gamma_g - 1) T_{cr,g}}{p_{cr,g}} \quad (1.50)$$

### 1.3 Outputs of the model

MAMMA provides the profiles along the conduit of the following parameters:

- (a) Velocity of both phases.
- (b) Density of both phases.
- (c) Pressure of both phases.
- (d) Temperature.
- (e) Dissolved gas mass fraction and the equilibrium value.
- (f) Exsolved gas volume fraction.
- (g) Crystals volume fraction.
- (h) Mass discharge rate.
- (i) Mixture viscosity.

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