1 Inroduction

Orthogonal packing in two and more dimensions includes a family of problems that are relevant in production and scheduling [ESI, WHS07, Har00], reconfigurable computing [Wik09], transport and warehouse logistics. We consider d-dimensional rectangular objects, items (boxes) and container(s) with integer sizes w_i^k and W_k, respectively, for $k = \overline{1,d}$ and $i = \overline{1,n}$. The items should be packed into the container(s) with their sides parallel to those of the container, without rotations.

The basic problem is the *d-Dimensional Orthogonal Packing Problem* (OPP). This is a *decision problem* which asks whether all given items fit into the given container. OPP belongs to the class of *strongly $\mathcal{NP}$-complete* problems [GJ79, FS04a]. By defining an objective function and further constraints we can distinguish other types of d-dimensional orthogonal packing problems, e.g.,

1. the *Strip-Packing Problem* (SPP) consists in minimizing the height W_d of a container that can hold all boxes, where the sizes W_1, W_2, ..., W_{d-1} of the other dimensions of the container are fixed;

2. the *Orthogonal Bin-Packing Problem* (BPP) consists in determining the minimal number of identical containers that are required to pack all the items;

3. the *Orthogonal Knapsack Problem* (OKP) consists in maximizing the total value of a subset of the boxes packable into the given container with the assumption that every box has a certain value.

The problems 1.–3. are *optimization problems* because they have an objective function. They are *strongly $\mathcal{NP}$-hard* [GJ79, FS04a]. The corresponding solution methods typically rely on those for OPP, that is why it is important to study this basic decision problem.

2 Relaxations and bounds

Many solution methods use *bounds on the solution value*. For example, the simplest bound for OPP is the *volume bound*: if the total volume of the items exceeds that of the container, then the instance is infeasible. Bounds are obtained from *relaxations* of the main problem, i.e., somewhat simpler problems. Bounds should be preferably quickly computable. However, some bounds are so strong that it pays off to spend more time for their computation. In many methods, the following relaxations of orthogonal packing are used to obtain bounds:

1. *Conservative scales* (CS) [FS04b] are modified item sizes such that if a packing exists, it is also feasible with the modified sizes. Thus, the volume bound for the modified instance is valid for the original instance. Often it is stronger, which is heavily used in algorithms. CS can be obtained by combinatorial heuristics called *dual-feasible functions* (DFF) [FS04b, CAdC08] and as dual solutions [CLM05] of the set-partitioning formulation of the simple 1D relaxation (see below).

2. The *simple one-dimensional relaxation* of 2D SPP: instead of n items sized $w_i \times h_i$, we consider n *types* of items with sizes $w_i \times 1$; for each type $i = \overline{1,n}$, h_i copies should be packed. This leads to a special *One-Dimensional Cutting-Stock Problem* (1D CSP) [Wik08, GG61, AV08]: given stock bars of size W and n item types of size w_i and order demands h_i, $i = \overline{1,n}$, minimize the number of bars needed to obtain all the items so that each bar has at most one item of each type.

1D CSP itself is $\mathcal{NP}$-hard, that is why its LP bounds are used. The LP relaxation of the *set-partitioning model* of 1D CSP is very tight [GG61, RST02]. It proved very effective for 2D packing [LA01, CLM05, BB07, BKRS09, Roh08].

3. The *Cumulative-Resource Non-Preemptive Scheduling Problem*, cf. [CJCM08] or, equivalently, the *One-Dimensional Contiguous Cutting-Stock Problem* (CCSP), cf. [MMV03, AVPT09]: in terms of scheduling, we have n jobs and a single resource, job i consumes w_i units of the resource for h_i units of time. Each job should be processed without interruption (*non-preemptiveness*). The resource can be consumed cumulatively, but in any moment of time up to W units. The goal is to minimize the total processing time. This is a relaxation of the two-dimensional SPP: the geometry of the items in one dimension is relaxed, see Figure 1. Thus, any feasible SPP layout is feasible for CCSP. Note that 1D CSP is a relaxation of CCSP.

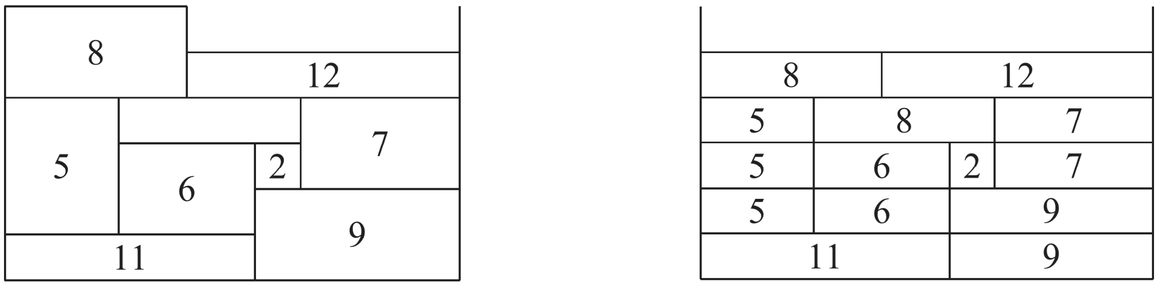

Figure 1: An instance of 2D SPP and a corresponding non-preemptive cumulative-resource schedule of CCSP [Sch08]. In this case, the latter coincides with an optimum of the corresponding 1D CSP. The numbers denote item widths w_i, $i = \overline{1,n}$.

Thus, the set-partitioning formulation of 1D CSP provides strong bounds for OPP, in particular it delivers good conservative scales. *The LP relaxations of 1D CSP and of CCSP would be the main instruments in the project.*

Further bounds are obtained, e.g., by preprocessing the combinatorial properties of packings: non-overlapping [CJCM08], transitivity of item orderings [PS07], transitive orientability of the overlapping relations and containment [FSvdV07] and others.

3 Models and exact algorithms for OPP and SPP

Exact methods for strongly $\mathcal{NP}$-hard problems are typically based on *branch-and-bound* which is the most efficient technique for such problems nowadays [NW88]. Branch-and-bound is a principle to separate the solution space and prune unpromising parts using bounds. For the computational success, of crucial importance are both the *branching strategy* and the employed bounds, which typically depend on the chosen *model* of the problem [NW88].

Thus, to begin our classification of exact methods, we start with the known models of OPP. The "natural" OPP model defines item coordinates as variables and just states the containment and non-overlapping conditions:

Find a set of coordinates x_i^k, $k = \overline{1,d}$, $i = \overline{1,n}$, satisfying

$$0 \le x_i^k \le W_k - w_i^k \quad \forall k, i, \tag{1a}$$

$$x_i^k + w_i^k \le x_j^k \quad \text{or} \quad x_j^k + w_j^k \le x_i^k \quad \text{for at least one } k, \quad \forall i < j, \tag{1b}$$

or prove that none exist.

Today's most successful approaches to OPP are based on this model implemented using Constraint Programming (CP) [CJCM08, SO08, BCP08, PS07]. The method in [CJCM08] uses branching and bounding techniques from scheduling problems.

Another powerful model is the graph-theoretical model of Fekete and Schepers [FS04a]. It considers the intersection relations of the item projections on each axis, which are described by the so-called *intersection graphs*. Given a system of d such graphs (one for each dimension), called *packing class*, we can restore the item orderings along each axis and, finally, the coordinates. Thus, the goal of a corresponding enumeration scheme is to obtain a packing class. Note that item orderings and coordinates are handled implicitly. Branching is performed on the edges of the graphs. The bounds use, e.g., the combinatorial properties of the graphs (intervalness). The corresponding exact algorithm [FSvdV07] proved very efficient both for OPP as well as for OKP in two and three dimensions. Belov and Rohling [BR09] improved this algorithm by introducing bounds and branching strategy based on the 1D cutting-stock LP relaxation. However, this algorithm is not as strong as that based on CP [CJCM08].

Other methods are much less efficient. They use, e.g., the direct layout coding [MMV03, AVPT09] or Integer Linear Programming (ILP) formulations [Pad00, BB07].

4 Today's exact algorithms for OKP and BPP

For the 2D knapsack problem, one of the latest algorithms is [BB07]. It has a two-level enumeration scheme: at the first level, all candidate subsets of items that could fit into the knapsack are enumerated. For each such subset, the corresponding OPP is solved as an ILP by commercial software. At the first level, strong LP bounds based on 1D CSP are applied, which lead to very strong decrease of the number of candidate subsets. The branching strategy in the first level is *LP-based*, i.e., the solutions of the LP give hints for branching variable selection.

For bin packing, the exact algorithm in [PS07] uses LP in the upper level (to combine whole bins), and Constraint Programming to pack the bins by solving OKPs. The algorithm can be seen as a merge of CP and LP techniques. Another exact method [CCM07] is purely combinatorial.

5 Today's heuristic algorithms

Heuristics are non-exact algorithms mostly based on *intuitive principles* aimed at quickly finding good solutions. They allow the efficient handling of complicated constraints, e.g., industrial ones. For orthogonal packing, all published heuristics are, to the best of our knowledge, purely combinatorial. For literature reviews, see, e.g., [EP09, BSM08, PAVOT08].

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