Mathematical & Computational Finance II
Lecture Notes

The Black-Scholes World

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1 The Minimal Hedge

Last time we had that if f_T (previously denoted h_T) is the payoff of a European contingent claim which may be exercised at time T we need that

$$f_T \in \mathcal{F}_T \quad \text{(i.e. } f_T \text{ is measurable at time } T\text{)}$$
$$\mathbb{E}_{\mathbb{P}}[e^{-rT} f_T] < \infty \quad \text{(i.e. the discounted payoff is integrable wrt } \mathbb{P}\text{)}$$

Then we can say that the rational/no-arbitrage price is

$$C(T, f_T) = \mathbb{E}_{\mathbb{Q}}[e^{-rT} f_T]$$

where $\mathbb{Q}$ is our risk neutral measure. To construct $\mathbb{Q}$ we take $\Theta = \frac{\mu - r}{\sigma}$ (under the Black-Scholes model) and have $\mathbb{Q}$ be defined by, for $0 \leq t \leq T$,

$$\Lambda_t = e^{-\int_0^t \Theta_u \, dB_u - \frac{1}{2}\int_0^t \Theta_u^2 \, du} \quad \text{and}$$
$$\mathbb{Q}(A) = \int_A \Lambda \, d\mathbb{P} \quad \forall\, A \in \mathcal{F}_T \quad \text{and}$$
$$W_t = B_t + \int_0^t \Theta_u \, du$$

and we have the result where W_t is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$. We also had found a minimal hedge/portfolio process $H^* = (H^0, H^1)$ given by

$$H_t^1 = \frac{\gamma_t}{\sigma} \frac{e^{rt}}{S_t^1}$$
$$H_t^0 = N_t - e^{rt} S_t^1 H_t^1$$

where $N_t = \mathbb{E}_{\mathbb{Q}}[e^{-rt} f_T | \mathcal{F}_t]$ and γ_t is known to exist by the Martingale Representation Theorem such that $N_t = N_0 + \int_0^t \gamma_s \, dW_s$. We had determined that the value $C(T, f_T)$ is the amount of initial capital needed to replicate the option payoff using the portfolio process H^*.

2 Derivation of the Black-Scholes Price in the Risk Neutral Framework

On $(\Omega, \mathcal{F}, \mathbb{P})$ (the real world space) we have

$$\begin{aligned} dS_t^0 &= rS_t^0\,dt \\ dS_t^1 &= \mu S_t^0\,dt + \sigma S_t^1\,dB_t \end{aligned}$$

and on the risk neutral space $(\Omega, \mathcal{F}, \mathbb{Q})$ we have

$$\begin{aligned} dS_t^0 &= rS^0\,dt \\ dS_t^1 &= rS_t^1\,dt + \sigma S_t^1\,dW_t \end{aligned}$$

Using Itô's formula we can show that the solution to the SDE for S_t^1, in the risk neutral space, is

$$S_t^1 = S_0^1 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

Theorem: Risk Neutral Pricing Theorem. In the Black-Scholes model any option defined by a nonnegative $\mathcal{F}_T$-measurable random variable, say f_T which is square integrable under $\mathbb{Q}$ (and thus $\mathbb{P}$), is replicable. The value at time $t \in [0, T]$ of any replicating portfolio is

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f_T|\mathcal{F}_t]$$

Proof. We could sketch this proof but we've basically already done it using the Martingale Representation Theorem, etc... The only modification is that now we have to deal with $(T-t)$ appearing. □

If we assume $f_T = f(S_T)$ then we have

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f(S_T)|\mathcal{F}_T]$$

Note that

$$\begin{aligned} \frac{S_T^1}{S_t^1} &= \frac{S_0^1 \exp\left[(r-\frac{1}{2}\sigma^2)T - \sigma W_T\right]}{S_0^1 \exp\left[(r-\frac{1}{2}\sigma^2)t - \sigma W_t)\right]} \\ \iff S_T^1 &= S_t^1 e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} \\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}f(S_t^1 e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)})\middle|\mathcal{F}_t\right] \end{aligned}$$

Noting that $W_T - W_t$ is independent of our filtration and S_t^1 is $\mathcal{F}_t$ measurable.

Theorem: "FACT". On a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ let X and Y be random variables and $\mathcal{A}$ a sub-σ-algebra of $\mathcal{G}$. Suppose X is $\mathcal{A}$-measurable and Y is independent of $\mathcal{A}$, then for any bounded measurable function $f(X, Y)$ define some other function $\phi(X)$ as

$$\phi(X) = \mathbb{E}[f(x, Y)] \quad \forall\ x \in \mathbb{R}$$

then we have

$$\mathbb{E}[f(x, Y)|\mathcal{A}] = \phi(X) = \mathbb{E}[f(x, Y)]$$

Basically, this is telling us that if we take a point x (since X is $\mathcal{A}$ measurable we have that x is known) and we can just compute the ordinary expectation as desired.

Moving on, with

$$F(t,x) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} f\big(x,^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}\big)]$$

we have that

$$V_t = F(t, S_t^1)$$

and $\sigma(W_T - W_t)$ is just normally distributed with mean 0 and variance $\sigma^2(T-t)$ so this is easily computable, and using the standard normal density we have

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f\big(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}\big) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

From this we have successfully reduced the Black-Scholes option pricing problem to a relatively simple integration problem.

Lemma 1. For constant $k > 0$ we have that the conditional probability under the measure $\mathbb{Q}$

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{S_T>k}|\mathcal{F}_t] = \Phi\Big[\frac{\log(\frac{S_t}{K}) + (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}\Big]$$

where $\Phi(x)$ is the normal CDF and the conditional $\mathcal{F}_t$ means that we are given all the information available to about the system up to time t (i.e. the price). We prove this using basic calculus & some integration tricks:

Proof. Let

$$d_2 = \frac{\log(\frac{S_t}{K}) + (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}$$

so that

$$V_0 = e^{-rt}\Phi[d_2]$$

Note that since $S_T \equiv S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}$ we have

$$\begin{aligned} S_T > K &\iff S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)} > K \\ \implies \log(S_t) + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t) &> \log(K) \\ \implies W_T - W_t &> \frac{-\log(\frac{S_t}{K}) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma} \end{aligned}$$

For brevity let $Y_t = \frac{\log(\frac{S_t}{K})+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma}$ so that we have $W_T - W_t > -Y_t$, hence

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{S_T>K}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{W_T-W_t>-Y_t}|\mathcal{F}_t]$$

Since we have that Y_t is $\mathcal{F}_t$-measurable and $(W_T - W_t)$ is independent of our filtration we may use our theorem above ("FACT") so that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{W_T - W_t > -Y_t} | \mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{W_T - W_t > -Y_t}] \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-Y_t}^{\infty} e^{-\frac{1}{2(T-t)} z^2} \, dz \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{Y_t} e^{-\frac{1}{2(T-t)} z^2} \, dz
\end{aligned}$$

With the substitution

$$u(z) = \frac{z}{\sqrt{T-t}} \implies d[u(z)] = \frac{dz}{\sqrt{T-t}}$$

we have

$$\begin{aligned}
u(Y_t) &= \frac{\log(\frac{S_t}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \equiv d_2 \\
\implies \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{Y_t} e^{-\frac{1}{2(T-t)} z^2} \, dz &= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{d_2} e^{-\frac{1}{2(T-t)}[u\sqrt{T-t}]^2} \left[\sqrt{T-t}\, du\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{1}{2}u^2} \, du \\
&= \Phi[d_2]
\end{aligned}$$

as desired. $\square$

2.1 The Binary Option

Consider the example of a binary option with payoff

$$h(S_T) = \begin{cases} 1 & \text{if } S_T > K \\ 0 & \text{else} \end{cases}$$

We say that this European-style contingent is a "cash-or-nothing" call option. We can find the correct price for this option using the lemma above. In this example we consider time $t = 0$,

$$\begin{aligned}
V_0 &= \mathbb{E}_{\mathbb{Q}}[e^{-rT} h(S_T)] \\
&= e^{-rT} \mathbb{E}_{\mathbb{Q}}[h(S_T)] \quad \text{(since } e^{-rt} \text{ is known)} \\
&= e^{-rT} \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{S_T > K}] \\
&= e^{-rT} \Phi[d_2]
\end{aligned}$$

with d_2 to be defined as earlier.

2.2 The "Asset-or-Nothing" Option

Consider the example of an option with payoff

$$h(S_T) = \begin{cases} S_T & \text{if } S_T > K \\ 0 & \text{else} \end{cases}$$

We call this the "asset-or-nothing" call option. By our risk neutral pricing formula we have that, at time $t = 0$,

$$V_0^{AON} = \mathbb{E}_{\mathbb{Q}}[e^{-rT} S_T \cdot \mathbb{1}_{S_T > K}]$$

This is similar to the "all-or-nothing" option but we now have the included S_T term. We proceed by recall our substitution for $S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W_t)}$ and evaluating at $t = 0$ to get

$$\begin{aligned} V_0^{AON} &= \mathbb{E}_{\mathbb{Q}}[e^{-rT} S_0 e^{(r-\frac{1}{2}\sigma^2)(T-0)+\sigma(W_T - W_0)} \cdot \mathbb{1}_{S_T > K}] \\ &= \mathbb{E}_{\mathbb{Q}}[S_0 e^{\frac{1}{2}\sigma^2 T + \sigma W_T} \cdot \mathbb{1}_{S_T > K}] \end{aligned}$$

Again recalling our lemma we have

$$\begin{aligned} V_0^{AON} &= \mathbb{E}_{\mathbb{Q}}[S_0 e^{\frac{1}{2}\sigma^2 T + \sigma W_T} \cdot \mathbb{1}_{W_T > -Y_0}] \\ &= \mathbb{E}_{\mathbb{Q}}[S_0 e^{\frac{1}{2}\sigma^2 T + \sigma W_T} \cdot \mathbb{1}_{-W_T < Y_0}] \quad \text{(by symmetry of } W_T \sim N(0,T)) \\ &= \mathbb{E}_{\mathbb{Q}}[S_0 e^{\frac{1}{2}\sigma^2 T - \sigma(-W_T)} \cdot \mathbb{1}_{-W_T < Y_0}] \quad \text{(to make all our } W_T \text{ terms the same sign)} \\ &= S_0 e^{\frac{1}{2}\sigma^2 T} \mathbb{E}_{\mathbb{Q}}[e^{-\sigma(-W_T)} \cdot \mathbb{1}_{-W_T < Y_0}] \quad \text{(taking out what is known)} \\ &= S_0 e^{\frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi T}} \int_{\infty}^{Y_0} e^{-\sigma z} e^{-\frac{1}{2T} z^2}\, dz \end{aligned}$$

From here we use the substitution

$$\begin{aligned} u = \frac{z}{\sqrt{T}} &\iff z = u\sqrt{T} \implies dz = \sqrt{T}\, du \\ \therefore\ u(Y_0) &= d_2 \end{aligned}$$

So we simplify our integral to

$$\begin{aligned} V_0^{AON} &= S_0 e^{\frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi T}} \int_{\infty}^{Y_0} e^{-\sigma z} e^{-\frac{1}{2T} z^2}\, dz \\ &= S_0 e^{\frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi T}} \int_{\infty}^{d_2} e^{-\sigma u\sqrt{T}} e^{-\frac{1}{2T}(u\sqrt{T})^2} \left[\sqrt{T}\, du\right] \\ &= S_0 e^{\frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{d_2} e^{-\sigma u\sqrt{T}} e^{-\frac{1}{2}u^2}\, du \\ &= S_0 \frac{1}{\sqrt{2\pi}} \int_{\infty}^{d_2} e^{-\frac{1}{2}u^2 - \sigma u\sqrt{T} - \frac{1}{2}\sigma^2 T}\, du \end{aligned}$$

We realize that our integrand is a perfect square (or complete the square to see this),

$$
\begin{aligned}
-\frac{1}{2}u^2 - \sigma\sqrt{T}u - \frac{1}{2}\sigma^2 T &= -\frac{1}{2}\left(u^2 + 2\sigma\sqrt{T}u + \sigma^2 T\right) \\
&= -\frac{1}{2}\left(u + \sigma\sqrt{T}\right)^2
\end{aligned}
$$

So our integral becomes

$$
V_0^{AON} = S_0 e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{\infty}^{d_2} e^{-\frac{1}{2}(u+\sigma\sqrt{T})^2}\, du
$$

Substituting again

$$
\begin{aligned}
v = u + \sigma\sqrt{T} &\implies dv = du \\
\therefore\ v(d_2) &= d_2 + \sigma\sqrt{T}
\end{aligned}
$$

Letting $d_1 = d_2 + \sigma\sqrt{T}$ we have

$$
\begin{aligned}
V_0^{AON} &= S_0 \int_{\infty}^{d_2} e^{-\frac{1}{2}(u+\sigma\sqrt{T})^2}\, du \\
&= S_0 \int_{\infty}^{d_1} e^{-\frac{1}{2}v^2}\, dv \\
&= S_0\Phi[d_1]
\end{aligned}
$$

Therefore we conclude that the correct price at time $t = 0$ for a European-style "asset-or-nothing" call option is

$$
V_O^{AON} = S_0\Phi[d_1]
$$

with d_1 defined as above.

3 The European Call Option

We claim that the Black-Scholes time $t = 0$ price of a European call option with payoff

$$
h(S_T) = (S_T - K)^+
$$

is

$$
C_0 = S_0\Phi[d_1] - Ke^{-rT}\Phi[d_2]
$$

That is, we claim that the correct price is identical to a portfolio of 1 long "asset-or-nothing" call options and and K short "cash-or-nothing" call options. Where

$$
\begin{aligned}
d_1 &= \frac{\log(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T} = \frac{\log(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}
\end{aligned}
$$

Proof. By the risk neutral pricing argument we have that

$$\begin{aligned}
C_0 &= \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - K)^+] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - K)\mathbb{1}_{S_T > K}] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-rT}S_T\mathbb{1}_{S_T > K}] - \mathbb{E}_{\mathbb{Q}}[e^{-rT}K\mathbb{1}_{S_T > K}] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-rT}S_T\mathbb{1}_{S_T > K}] - K\mathbb{E}_{\mathbb{Q}}[e^{-rT}\mathbb{1}_{S_T > K}] \\
&= S_0\Phi[d_1] - Ke^{-rT}\Phi[d_2]
\end{aligned}$$

□

In generality if we're at time t, for $0 \leq t < T$, we have

$$C_t = S_t\Phi[d_1(t)] - Ke^{-r(T-t)}\Phi[d_2(t)]$$

where

$$\begin{aligned}
d_1(t) &= \frac{\log(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\
d_2(t) &= d_1(t) - \sigma\sqrt{T-t} = \frac{\log(\frac{S_t}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

3.1 The European Put Option

We could go through these steps again to derive a price for a European-style put option, but if we already have the price of a call option then its instead quicker to do so using put-call parity:

$$C - P = S_t - e^{-r(T-t)}K$$

So,

$$\begin{aligned}
P_t &= e^{-r(T-t)}K - S_t + \Big(S_t\Phi[d_1(t)] - Ke^{-r(T-t)}\Phi[d_2(t)]\Big) \\
&= Ke^{-r(T-t)}\big(1 - \Phi[d_2(t)]\big) - S_t\big(1 - \Phi[d_1(t)]\big)
\end{aligned}$$

By symmetry of the normal distribution, $1 - \Phi(x) = \Phi(-x)$, we have

$$P_t = Ke^{-r(T-t)}\Phi[-d_2(t)] - S_t\Phi[-d_1(t)]$$

To have done this rigorously we would need to prove put-call parity, but we'll call it sufficient for now.

4 Hedging

When we were constructing the hedging process $H^* = (H^0, H^1)$ we figured out the ratios for each component

$$\begin{aligned}
H_t^1 &= \frac{\gamma_t e^{rt}}{\sigma S_t^1} \\
H_t^0 &= N_t - \frac{\gamma_t}{\sigma}
\end{aligned}$$

but were left without the γ_t guaranteed to exist by the Martingale Representation theorem. If we're actually going to do anything with this process we'll need to make this γ_t component explicit. Fortunately we have that

$$V_t = \mathbb{E}_{\mathbb{Q}}\Big[e^{-r(T-t)} f(S_t^1 \cdot \exp\big[(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\big]\Big|\mathcal{F}_t\Big]$$

By our "FACT" we know this is just a function of the asset price and time

$$V_t = e^{-r(T-t)} F(T-t, S_t^1)$$

We can prove (it's not easy) that in this model F is differentiable with respect to both t and x, so, we may write

$$G(t,x) = F(T-t, e^{rt}x)$$

then we have

$$\begin{aligned} V_t e^{-rt} &= e^{-rT} G(t, e^{-rt} S_t^1) \\ \overline{V}_t &= e^{-rT} G(t, \overline{S}_t^1) \end{aligned}$$

Applying Itô's formula gives us

$$\overline{V}_t = e^{-rT}\left[G(0,\overline{S}_0^1) + \int_0^t \frac{\partial}{\partial u} G(u,\overline{S}_u^1)\,du + \int_0^t \frac{\partial}{\partial x} G(u,\overline{S}_u^1)\,d\overline{S}_u^1 + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} G(u,\overline{S}_u^1)\,d\langle \overline{S}_{(\cdot)}^1\rangle_u\right]$$

Under the probability measure $\mathbb{Q}$ we know that $\overline{S}^1$ and $\overline{V}$ are martingales thus no drift term should appear in their SDEs. Therefore, since we know that

$$\begin{aligned} d\overline{S}_t^1 &= \overline{S}_t^1 \sigma\, dW_t \quad \text{(notice no drift in } dt) \\ d\langle \overline{S}_{(\cdot)}^1\rangle_t &= \big(\overline{S}_t^1\sigma\big)^2\, dt \end{aligned}$$

we must have that

$$\begin{aligned} \overline{V}_t &= e^{-rT}\left[G(0,\overline{S}_0^1) + \int_0^t \frac{\partial}{\partial u} G(u,\overline{S}_u^1)\,du + \int_0^t \frac{\partial}{\partial x} G(u,\overline{S}_u^1)\,d\overline{S}_u^1 + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} G(u,\overline{S}_u^1)\,d\langle \overline{S}_{(\cdot)}^1\rangle_u\right] \\ &= e^{-rT}\left[G(0,\overline{S}_0^1) + \int_0^t \frac{\partial}{\partial u} G(u,\overline{S}_u^1)\,du + \int_0^t \frac{\partial}{\partial x} G(u,\overline{S}_u^1)\overline{S}_u^1\sigma\,dW_u + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} G(u,\overline{S}_u^1)\big(\overline{S}_u^1\sigma\big)^2\,du\right] \\ &= e^{-rT}\left[G(0,\overline{S}_0^1) + \int_0^t \left(\frac{\partial}{\partial u} G(u,\overline{S}_u^1) + \frac{1}{2}\frac{\partial^2}{\partial x^2} G(u,\overline{S}_u^1)(\overline{S}_u^1)^2\sigma^2\right)du + \int_0^t \frac{\partial}{\partial x} G(u,\overline{S}_u^1)\overline{S}_u^1\sigma\,dW_u\right] \end{aligned}$$

has as its drift term du be equal to zero. That is, we must satisfy

$$\frac{\partial}{\partial u} G(u,\overline{S}_u^1) + \frac{1}{2}\frac{\partial^2}{\partial x^2} G(u,\overline{S}_u^1)(\overline{S}_u^1)^2\sigma^2 = 0$$

Using

$$\begin{aligned} G(T,x) &= F(0, e^{rT}x) = f(e^{rT}, x) \\ G(t,x) &= F(T-t, e^{rt}x) \end{aligned}$$

we apply the chain rule for partial derivatives on our equation above to obtain the partial differential equation

$$\frac{\partial F}{\partial u}(u, S_u^1) + rS_t^1 \frac{\partial F}{\partial x}(u, S_u^1) + \frac{1}{2}\sigma^2 (S_t^1)^2 \frac{\partial^2 F}{\partial x^2} F(u, S_u^1) = 0$$