Assignment 4

David Fleischer – 27101449
MACF 401 - Mathematical & Computational Finance I

Due: April 4 2016
Last update: April 2, 2016

Part I

Solution 5.2:

(i) We have

$$\begin{aligned} f(\sigma) &= pe^{\sigma} + q^{-\sigma} \\ &= pe^{\sigma} + (1-p)e^{-\sigma} \\ &= pe^{\sigma} + e^{-\sigma} - pe^{-\sigma} \\ &= p\left(e^{\sigma} - e^{-\sigma}\right) + e^{-\sigma} \end{aligned}$$

However, note that $\forall\, x \in \mathbb{R}$ we have $(x-1)^2 > 0$, hence

$$\begin{aligned} (x-1)^2 = x^2 - 2x + 1 &\geq 0 \\ \implies x^2 + 1 &\geq 2 \\ \implies \frac{x^2+1}{x} &\geq 2 \\ \implies x + x^{-1} &\geq 2 \end{aligned}$$

Thus $(e^{\sigma} - e^{-\sigma}) \geq 2$ and since $p > \frac{1}{2}$ we find

$$p\left(e^{\sigma} - e^{-\sigma}\right) \geq 1$$

This, together with the final term $e^{-\sigma} > 0$ gives us that

$$f(\sigma) = p\left(e^{\sigma} - e^{-\sigma}\right) + e^{-\sigma} > 1$$

as desired.
(ii) By definition we have that $M_n = \sum_{j=1}^{n} X_j$ depends on only the first n coin tosses $\omega_1 \cdots \omega_n$. Note that $\left(\frac{1}{f(\sigma)}\right)^n$ is deterministic we find that $S_n = e^{\sigma M_n}\left(\frac{1}{f(\sigma)}\right)^n$ is adapted since

a measurable function of an adapted process is itself adapted.

Now, to confirm the martingale property:

$$\begin{aligned}
\mathbb{E}_n\left[S_{n+1}\right] &= \mathbb{E}_n\left[e^{\sigma M_{n+1}}\left(\frac{1}{f(\sigma)}\right)^{n+1}\right] \\
&= \mathbb{E}_n\left[e^{\sigma(M_n+X_{n+1})}\left(\frac{1}{f(\sigma)}\right)^{n+1}\right] \quad \text{(by definition of } M_{n+1}\text{)} \\
&= \mathbb{E}_n\left[e^{\sigma M_n}\left(\frac{1}{f(\sigma)}\right)^{n}\left(\frac{1}{f(\sigma)}\right)e^{\sigma X_{n+1}}\right] \\
&= \mathbb{E}_n\left[\frac{S_n}{f(\sigma)}e^{\sigma X_{n+1}}\right] \quad \text{(by definition of } S_n\text{)} \\
&= \frac{S_n}{f(\sigma)}\mathbb{E}_n\left[e^{\sigma X_{n+1}}\right] \quad \text{(adaptedness of } \frac{S_n}{f(\sigma)} \text{ to the first } n \text{ coin tosses)} \\
&= \frac{S_n}{f(\sigma)}\left(pe^{\sigma\cdot(1)}+qe^{\sigma\cdot(-1)}\right) \quad \text{(by the independence lemma)} \\
&= \frac{S_n}{f(\sigma)}f(\sigma) \\
&= S_n
\end{aligned}$$

Therefore, since S_n is both adapted and satisfies the martingale property we have that S_n is indeed a martingale, as desired.

(iii) Applying the Optional Sampling Theorem we find that the stopped process $S_{n\wedge\tau_1}$ must be a martingale. So

$$\begin{aligned}
\mathbb{E}_0\left[S_{n\wedge\tau_1}\right] &= \mathbb{E}_0\left[e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1}\right] \\
&= S_{0\wedge\tau_1} \quad \text{(by the martingale property)} \\
&= S_0 \quad \text{(since } \tau_1 > 0\text{)} \\
&= e^{\sigma M_0}\left(\frac{1}{f(\sigma)}\right)^{0} \\
&= 1
\end{aligned}$$

Note

$$\lim_{n\to\infty}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} = \begin{cases}\left(\frac{1}{f(\sigma)}\right)^{\tau_1} & \text{if } \tau_1 < \infty \\ 0 & \text{if } \tau_1 = \infty\end{cases}$$

By the definition of $M_{n\wedge\tau_1}$ we note that

$$M_{n\wedge\tau_1} \leq 1$$

Hence

$$0 \leq e^{\sigma M_{n\wedge\tau_1}} \leq e^{\sigma}$$

Now, considering first $\tau_1 < \infty$

$$\begin{aligned}\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} &= \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \lim_{n\to\infty} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} \\ &= e^{\sigma} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\end{aligned}$$

and for the case of $\tau_1 = \infty$ we use our result that $e^{\sigma M_n}$ is bound above and below, so

$$\begin{aligned}\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} &= \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \lim_{n\to\infty} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} \\ &= \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \cdot 0 \\ &= 0\end{aligned}$$

Therefore, we may combine both cases as

$$\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} = \mathbb{1}_{\{\tau_1<\infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}$$

Now we wish to take the limit of the expectation of the stopped process $S_{n\wedge\tau_1}$ as $n \to \infty$. However, we have already shown that $\mathbb{E}_0[S_{n\wedge\tau_1}] = S_{0\wedge\tau_1} = S_0 = 1$. So

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}_0[S_{n\wedge\tau_1}] &= \lim_{n\to\infty} 1 \\ &= 1\end{aligned}$$

Thus

$$\begin{aligned}1 &= \lim_{n\to\infty} \mathbb{E}_0\left[S_{n\wedge\tau_1}\right] \\ &= \mathbb{E}_0\left[\lim_{n\to\infty} S_{n\wedge\tau_1}\right] \quad \text{(Dominated Convergence)} \\ &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \\ \implies e^{-\sigma} &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right]\end{aligned}$$

and taking the limit as $\sigma \downarrow 0$

$$\begin{aligned}\lim_{\sigma\downarrow 0} e^{-\sigma} &= \lim_{\sigma\downarrow 0} \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \\ \implies 1 &= \lim_{\sigma\downarrow 0} \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \\ &= \mathbb{E}_0\left[\lim_{\sigma\downarrow 0} \mathbb{1}_{\{\tau_1<\infty\}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \quad \text{(Dominated Convergence)} \\ &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} \lim_{\sigma\downarrow 0} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \\ &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}}\right] \\ &= \mathbb{P}\left(\{\tau_1 < \infty\}\right)\end{aligned}$$

as desired.

(iv) Let $\alpha \in (0,1)$. We will first solve for the σ satisfying

$$\begin{aligned} \alpha &= \frac{1}{f(\sigma)} \\ &= \frac{1}{pe^{\sigma} + qe^{-\sigma}} \\ \implies \alpha\left(pe^{\sigma} + qe^{-\sigma}\right) &= 1 \\ \implies \alpha pe^{\sigma} + \alpha qe^{-\sigma} &= 1 \\ \implies \alpha p + \alpha q\left(e^{-\sigma}\right)^2 &= e^{-\sigma} \\ \implies \alpha q\left(e^{-\sigma}\right)^2 - e^{-\sigma} + \alpha p &= 0 \\ \implies e^{-\sigma} &= \frac{1 \pm \sqrt{1 - 4\alpha^2 q}}{2\alpha pq} \end{aligned}$$

We require $\sigma > 0$ so then $0 < e^{-\sigma} < 1$. For this purpose we take the negative root

$$e^{-\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

and from **(iii)** we find

$$\begin{aligned} e^{-\sigma} &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1 < \infty\}}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \\ \implies \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1 < \infty\}}\left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] &= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q} \end{aligned}$$

Since we initially performed the substitution $\alpha = \frac{1}{f(\sigma)}$ we may write

$$\mathbb{E}\left[\mathbb{1}_{\{\tau_1 < \infty\}}\alpha^{\tau_1}\right] = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

Noting that

$$\mathbb{E}\left[\alpha^{\tau_1}\right] = \mathbb{E}\left[\mathbb{1}_{\{\tau_1 = \infty\}}\alpha^{\tau_1}\right] + \mathbb{E}\left[\mathbb{1}_{\{\tau_1 < \infty\}}\alpha^{\tau_1}\right]$$

and since $\alpha \in (0,1)$ we find $\mathbb{1}_{\{\tau_1 = \infty\}}\alpha^{\tau_1} = 0$. Therefore, we may conclude with

$$\mathbb{E}\left[\alpha^{\tau_1}\right] = \mathbb{E}\left[\mathbb{1}_{\{\tau_1 < \infty\}}\alpha^{\tau_1}\right] = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

as desired.

(v) By the Dominated Convergence Theorem write

$$\begin{aligned}
\mathbb{E}\left[\tau_1 \alpha^{\tau_1 - 1}\right] &= \frac{\partial}{\partial \alpha} \mathbb{E}\left[\alpha^{\tau_1}\right] \\
&= \frac{\partial}{\partial \alpha} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q} \qquad \text{(from } \textbf{(iv)}\text{)} \\
&= \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}}
\end{aligned}$$

Since we had defined $\alpha \in (0,1)$ we must take the limit as $\alpha \uparrow 1$

$$\begin{aligned}
\lim_{\alpha \uparrow 1} \mathbb{E}\left[\tau_1 \alpha^{\tau_1 - 1}\right] &= \lim_{\alpha \uparrow 1} \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha^2 q \sqrt{1 - 4\alpha^2 pq}} \\
&= \frac{1 - \sqrt{1 - 4pq}}{2q\sqrt{1 - 4pq}} \\
&= \frac{1 - \sqrt{1 - 4p(1-p)}}{2(1-p)\sqrt{1 - 4p(1-p)}} \\
&= \frac{1 - \sqrt{1 - 4p + 4p^2}}{2(1-p)\sqrt{1 - 4p + 4p^2}} \\
&= \frac{1 - \sqrt{(1-2p)^2}}{2(1-p)\sqrt{(1-2p)^2}} \\
&= \frac{1 - |1-2p|}{2(1-p)|1-2p|} \\
&= \frac{1 - (2p-1)}{2(1-p)(2p-1)} \qquad \text{since } 1 - 2p < 0 \text{ for } \frac{1}{2} < p < 1 \\
&= \frac{2 - 2p}{2(1-p)(2p-1)} \\
&= \frac{1}{2p-1} = \frac{1}{p+p-1} = \frac{1}{p-q} \qquad \text{(I think the final expression is the nicest)}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{p-q} &= \lim_{\alpha \uparrow 1} \mathbb{E}\left[\tau_1 \alpha^{\tau_1 - 1}\right] \\
&= \mathbb{E}\left[\lim_{\alpha \uparrow 1} \tau_1 \alpha^{\tau_1 - 1}\right] \qquad \text{(Dominated Convergence)} \\
&= \mathbb{E}\left[\tau_1\right]
\end{aligned}$$

as desired.

Solution 5.3:
(i) With the substitution $x = e^{\sigma_0}$ we solve

$$\begin{aligned} 1 &= px + \frac{q}{x} \\ &= \frac{px^2 + q}{x} \\ \implies x &= px^2 + q \\ \implies px^2 - x + q &= 0 \\ \implies x &= \frac{1 \pm \sqrt{1-4pq}}{2p} \\ &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} \\ &= \frac{1 \pm \sqrt{(1-2p)^2}}{2p} \\ &= \frac{1 \pm (1-2p)}{2p} \end{aligned}$$

Clearly we must have $x = e^{\sigma} > 0$ for all $\sigma \in \mathbb{R}$, so we are required to take the positive term. This yields

$$\begin{aligned} x &= \frac{1+1-2p}{2p} \\ &= \frac{1-p}{p} \\ &= \frac{q}{p} \\ \implies e^{\sigma_0} &= \frac{q}{p} \\ \implies \sigma_0 &= \log \frac{q}{p} \end{aligned}$$

Since $q > p$ we have that $\frac{q}{p} > 1 \implies \sigma_0 = \log \frac{q}{p} > 1 > 0$, which satisfies our positivity criteria. That is, we have found $\sigma_0 > 0$ such that $f(\sigma_0) = 1$ since

$$\begin{aligned} f(\sigma_0) &= pe^{\sigma_0} + qe^{-\sigma_0} \\ &= pe^{log \frac{q}{p}} + qe^{-\log \frac{q}{p}} \\ &= p \cdot \frac{q}{p} + q \cdot \frac{p}{q} \\ &= q + p = 1 \end{aligned}$$

We wish now to confirm that $f(\sigma) > 1$ for all $\sigma > \sigma_0 = \log \frac{q}{p}$. To this end, again let

$x = e^\sigma$ and calculate

$$\begin{aligned}
1 &\le px + \frac{q}{x} \\
1 &\le \frac{px^2 + q}{x} \\
x &\le px^2 + q \\
0 &\le px^2 - x + q \\
0 &\le (x-1)(px-q)
\end{aligned}$$

which is satisfied when $x \ge \frac{q}{p} \implies e^\sigma \ge \frac{q}{p} \implies \sigma \ge \log \frac{q}{p} = \sigma_0$, as desired.

(ii) We go through the same process as in **(5.2)**. Defining the process $S_n = e^{\sigma M_n}\left(\frac{1}{f(\sigma)}\right)^n$ we see by the same argument that S_n is an adapted process since it is a measurable function of M_n, which is itself a function of the first n coin tosses. To confirm the martingale property we have

Now, to confirm the martingale property:

$$\begin{aligned}
\mathbb{E}_n\left[S_{n+1}\right] &= \mathbb{E}_n\left[e^{\sigma M_{n+1}}\left(\frac{1}{f(\sigma)}\right)^{n+1}\right] \\
&= \mathbb{E}_n\left[e^{\sigma(M_n + X_{n+1})}\left(\frac{1}{f(\sigma)}\right)^{n+1}\right] \quad \text{(by definition of } M_{n+1}\text{)} \\
&= \mathbb{E}_n\left[e^{\sigma M_n}\left(\frac{1}{f(\sigma)}\right)^{n}\left(\frac{1}{f(\sigma)}\right)e^{\sigma X_{n+1}}\right] \\
&= \mathbb{E}_n\left[\frac{S_n}{f(\sigma)}e^{\sigma X_{n+1}}\right] \quad \text{(by definition of } S_n\text{)} \\
&= \frac{S_n}{f(\sigma)}\mathbb{E}_n\left[e^{\sigma X_{n+1}}\right] \quad \text{(adaptedness of } \frac{S_n}{f(\sigma)} \text{ to the first } n \text{ coin tosses)} \\
&= \frac{S_n}{f(\sigma)}\left(pe^{\sigma\cdot(1)} + qe^{\sigma\cdot(-1)}\right) \quad \text{(by the independence lemma)} \\
&= \frac{S_n}{f(\sigma)}f(\sigma) \\
&= S_n
\end{aligned}$$

So S_n remains a martingale for the case of $0 < p < \frac{1}{2}$. Now, again applying the Optional Sampling Theorem we have that the stopped process $S_{n\tau_1}$ must be a martingale. Hence

$$\begin{aligned}
\mathbb{E}_0\left[S_{n\wedge\tau_1}\right] &= \mathbb{E}_0\left[e^{\sigma M_{n\wedge\tau_1}}\left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1}\right] \\
&= S_{0\wedge\tau_1} \quad \text{(by the martingale property)} \\
&= S_0 \quad (\text{since } \tau_1 > 0) \\
&= e^{\sigma M_0}\left(\frac{1}{f(\sigma)}\right)^0 \\
&= 1
\end{aligned}$$

Suppose now that $\sigma > \sigma_0 = \log \frac{q}{p}$ from part **(i)**. Then we have that $f(\sigma) > 1$. Therefore, $0 < \frac{1}{f(\sigma)} < 1$. From this fact we find

$$\lim_{n\to\infty} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} = \begin{cases} \left(\frac{1}{f(\sigma)}\right)^{\tau_1} & \text{if } \tau_1 < \infty \\ 0 & \text{if } \tau_1 = \infty \end{cases}$$

By the definition of $M_{n\wedge\tau_1}$ we note that

$$M_{n\wedge\tau_1} \leq 1$$

Hence

$$0 \leq e^{\sigma M_{n\wedge\tau_1}} \leq e^{\sigma}$$

Now, considering first $\tau_1 < \infty$

$$\begin{aligned} \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} &= \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \lim_{n\to\infty} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} \\ &= e^{\sigma} \left(\frac{1}{f(\sigma)}\right)^{\tau_1} \end{aligned}$$

and for the case of $\tau_1 = \infty$ we use our result that $e^{\sigma M_n}$ is bound above and below, so

$$\begin{aligned} \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} &= \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \lim_{n\to\infty} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} \\ &= \lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, we may combine both cases as

$$\lim_{n\to\infty} e^{\sigma M_{n\wedge\tau_1}} \left(\frac{1}{f(\sigma)}\right)^{n\wedge\tau_1} = \mathbb{1}_{\{\tau_1<\infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}$$

Now we wish to take the limit of the expectation of the stopped process $S_{n\wedge\tau_1}$ as $n \to \infty$. However, we have already shown that $\mathbb{E}_0[S_{n\wedge\tau_1}] = S_{0\wedge\tau_1} = S_0 = 1$. So

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}_0[S_{n\wedge\tau_1}] &= \lim_{n\to\infty} 1 \\ &= 1 \end{aligned}$$

Thus

$$\begin{aligned} 1 &= \lim_{n\to\infty} \mathbb{E}_0\left[S_{n\wedge\tau_1}\right] \\ &= \mathbb{E}_0\left[\lim_{n\to\infty} S_{n\wedge\tau_1}\right] \quad \text{(Dominated Convergence)} \\ &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} e^{\sigma} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \\ \implies e^{-\sigma} &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}} \left(\frac{1}{f(\sigma)}\right)^{\tau_1}\right] \end{aligned}$$

Instead of taking taking the limit as $\sigma \downarrow 0$ as we did in **(5.2)** we now consider the limit as $\sigma \downarrow \sigma_0$ since we have restricted σ such that $\sigma > \sigma_0$. Taking this limit

$$\begin{aligned}
\lim_{\sigma \downarrow \sigma_0} e^{-\sigma} &= \lim_{\sigma \downarrow \sigma_0} \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
\implies e^{-\sigma_0} &= \lim_{\sigma \downarrow \sigma_0} \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
&= \mathbb{E}_0 \left[\lim_{\sigma \downarrow \sigma_0} \mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \qquad \text{(Dominated Convergence)} \\
&= \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \lim_{\sigma \downarrow \sigma_0} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right] \\
&= \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \right] \\
&= \mathbb{P} \left(\{\tau_1 < \infty\} \right)
\end{aligned}$$

We may simplify this as

$$\begin{aligned}
\mathbb{P} \left(\{\tau_1 < \infty\} \right) &= e^{-\sigma_0} \\
&= e^{-\log \frac{q}{p}} \\
&= \frac{p}{q}
\end{aligned}$$

which is a permissible probability for this event since our definition of p and q satisfying $0 < p < \frac{1}{2}$ and $q = 1 - p$ implies that $0 < \frac{p}{q} < 1$.

(iii) Let $\alpha \in (0, 1)$. Repeating the arguments from **(5.2.iv)** (since the preliminary steps are not affected by the requirement $0 < p < \frac{1}{2}$) we find that the σ satisfying

$$\alpha = \frac{1}{f(\sigma)}$$

yielding

$$e^{-\sigma} = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

From **(5.3.ii)** we have that

$$e^{-\sigma} = \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \left(\frac{1}{f(\sigma)} \right)^{\tau_1} \right]$$

Therefore, with our substitution of $\alpha = \frac{1}{f(\sigma)}$, we may write

$$\mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1} \right] = \frac{1 - \sqrt{1 - 4\alpha^2 pq}}{2\alpha q}$$

Now, write

$$\mathbb{E}_0 \left[\alpha^{\tau_1} \right] = \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 = \infty\}} \alpha^{\tau_1} \right] + \mathbb{E}_0 \left[\mathbb{1}_{\{\tau_1 < \infty\}} \alpha^{\tau_1} \right]$$

but if $\tau_1 = \infty$ then, recalling $\alpha \in (0,1)$, we have $\mathbb{1}_{\{\tau_1=\infty\}}\alpha^{\tau_1} = 0$. Hence

$$\begin{aligned}
\mathbb{E}_0\left[\alpha^{\tau_1}\right] &= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1=\infty\}}\alpha^{\tau_1}\right] + \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}}\alpha^{\tau_1}\right] \\
&= \mathbb{E}_0\left[\mathbb{1}_{\{\tau_1<\infty\}}\alpha^{\tau_1}\right] \\
&= \frac{1-\sqrt{1-4\alpha^2 pq}}{2\alpha q}
\end{aligned}$$

as desired.

(iv) Applying the Dominated Convergence Theorem we write

$$\begin{aligned}
\mathbb{E}\left[\tau_1\alpha^{\tau_1-1}\right] &= \frac{\partial}{\partial\alpha}\mathbb{E}\left[\alpha^{\tau_1}\right] \\
&= \frac{\partial}{\partial\alpha}\frac{1-\sqrt{1-4\alpha^2 pq}}{2\alpha q} \quad \text{(from } \mathbf{(5.3.iii)}\text{)} \\
&= \frac{1-\sqrt{1-4\alpha^2 pq}}{2\alpha^2 q\sqrt{1-4\alpha^2 pq}}
\end{aligned}$$

Since we had defined $\alpha \in (0,1)$ we must take the limit as $\alpha \uparrow 1$

$$\begin{aligned}
\lim_{\alpha\uparrow 1}\mathbb{E}\left[\tau_1\alpha^{\tau_1-1}\right] &= \lim_{\alpha\uparrow 1}\frac{1-\sqrt{1-4\alpha^2 pq}}{2\alpha^2 q\sqrt{1-4\alpha^2 pq}} \\
&= \frac{1-\sqrt{1-4pq}}{2q\sqrt{1-4pq}} \\
&= \frac{1-\sqrt{1-4p(1-p)}}{2(1-p)\sqrt{1-4p(1-p)}} \\
&= \frac{1-\sqrt{1-4p+4p^2}}{2(1-p)\sqrt{1-4p+4p^2}} \\
&= \frac{1-\sqrt{(1-2p)^2}}{2(1-p)\sqrt{(1-2p)^2}} \\
&= \frac{1-|1-2p|}{2(1-p)|1-2p|}
\end{aligned}$$

Now, since $0 < p < \frac{1}{2}$ we have that $1-2p > 0$, thus

$$\begin{aligned}
\lim_{\alpha\uparrow 1}\mathbb{E}\left[\tau_1\alpha^{\tau_1-1}\right] &= \frac{1-(1-2p)}{2(1-p)(1-2p)} \\
&= \frac{p}{(1-p)(1-2p)} \\
&= \frac{p}{q(1-p-p)} = \frac{p}{q(q-p)}
\end{aligned}$$

as desired.

Solution 5.6: $S_0 = 4, K = 4, u = 2, d = \frac{1}{2}, r = \frac{1}{4}$. Show that an American put expiring at time $N = 1$ has price $V_0 = 0.8$, expiring at time $N = 3$ has price $V_0 = 0.928$, and expiring at time $N = 5$ has price $V_0 = 0.96896$.

Solution 5.7: **(i)** Recall that we had found

$$v(s) = \begin{cases} 4 - s & \text{if } s \leq 2 \\ \frac{4}{s} & \text{if } s \geq 4 \end{cases}$$

so with $s = 2^j$ we have

$$v(2^j) = \begin{cases} 4 - 2^j & \text{if } j \leq 1 \\ \frac{4}{2^j} & \text{if } j \geq 2 \end{cases}$$

Therefore, for $j \leq 0$, we find

$$\begin{aligned} c(2^j) &= v(2^j) - \frac{4}{5}\left[\frac{1}{2}v(2 \cdot 2^j) + \frac{1}{2}v(\frac{1}{2}2^j)\right] \\ &= (4 - 2^j) - \frac{2}{5}\left[(4 - 2^{j+1}) + (4 - 2^{j-1})\right] \\ &= 4 - 2^j - \frac{2}{5}\left[8 - 2^{j-1}(4 + 1)\right] \\ &= 4 - 2^j - \frac{2}{5}\left[8 - 5 \cdot 2^{j-1}\right] \\ &= 4 - 2^j - \frac{16}{5} + 2^j \\ &= \frac{4}{5} \end{aligned}$$

For $j = 1$ we find

$$\begin{aligned} c(2^j) &= v(2^j) - \frac{4}{5}\left[\frac{1}{2}v(2 \cdot 2^j) + \frac{1}{2}v(\frac{1}{2}2^j)\right] \\ &= v(2) - \frac{2}{5}\left[v(4) + v(1)\right] \\ &= (4 - 2) - \frac{2}{5}\left[\frac{4}{4} + (4 - 1)\right] \\ &= 2 - \frac{2}{5}\left[1 + 3\right] \\ &= 2 - \frac{8}{5} \\ &= \frac{2}{5} \end{aligned}$$

and for $j \geq 2$

$$\begin{aligned}
c(2^j) &= v(2^j) - \frac{4}{5}\left[\frac{1}{2}v(2\cdot 2^j) + \frac{1}{2}v(\frac{1}{2}2^j)\right] \\
&= \frac{4}{2^j} - \frac{2}{5}\left[\frac{4}{2^{j+1}} + \frac{4}{2^{j-1}}\right] \\
&= \frac{4}{2^j} - \frac{2}{5}\frac{4}{2^{j-1}}\left[\frac{1}{4} + 1\right] \\
&= \frac{4}{2^j} - \frac{8}{5\cdot 2^{j-1}}\frac{5}{4} \\
&= \frac{4}{2^j} - \frac{2}{2^{j-1}} \\
&= \frac{4}{2^j} - \frac{4}{2^j} \\
&= 0
\end{aligned}$$

(ii) For $j \leq 0$ we find

$$\begin{aligned}
\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{(4 - 2^{j+1}) - (4 - 2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{-(2^{j+1} - 2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= -1
\end{aligned}$$

for $j = 1$

$$\begin{aligned}
\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{v(4) - v(1)}{4 - 1} \\
&= \frac{\frac{4}{4} - (4 - 1)}{3} \\
&= -\frac{2}{3}
\end{aligned}$$

and for $j \geq 2$

$$\begin{aligned}
\delta(2^j) &= \frac{v(2^{j+1}) - v(2^{j-1})}{2^{j+1} - 2^{j-1}} \\
&= \frac{\frac{4}{2^{j+1}} - \frac{4}{2^{j-1}}}{2^{j+1} - 2^{j-1}} \\
&= \frac{4}{2^{j-1}} \cdot \frac{\frac{1}{4} - 1}{2^{j+1} - 2^{j-1}} \\
&= -\frac{4}{2^{j-1}} \cdot \frac{\frac{3}{4}}{3 \cdot 2^{j-1}} \\
&= -\frac{1}{2^{j-1} \cdot 2^{j-1}} \\
&= -\frac{1}{2^{2j-2}} \\
&= -\frac{1}{2^{2j}2^{-2}} \\
&= -\frac{4}{2^{2j}}
\end{aligned}$$

(iii)

Solution 5.8:
(i)
(ii)
(iii)
(iv)