Mathematical & Computational Finance II
Lecture Notes

Basics of Stochastic Processes

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1 Brownian Motion

Given a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, a d-dimensional Brownian motion starting at the origin (i.e. a standard Brownian motion, $B_0 = \hat{0}$) is a stochastic process such that

1. B_t has independent increments. That is, for $0 = t_0 \leq t_1 \leq \cdots \leq t_n$, we have
$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$$
are independent random variables. So, any partition has independent intervals.
2. For $0 \leq s \leq t, B_t - B_s \sim N(0, (t-s)\mathbb{I}^d)$, where $\mathbb{I}^d$ is the d-dimensional identity matrix.
3. $\mathbb{P}(\{\omega : B_0(\omega) = 0 \text{ and } t \mapsto B_t(\omega) \text{ is continuous}\}) = 1$ (i.e. our process B_t is almost surely continuous).

1.1 Implications

We have
$$\mathbb{P}(B_t - B_s \leq x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{x} e^{-\frac{1}{2}\frac{z^2}{t-s}}\, dz$$

This gives us that $B_t - B_s$ is independent of the "natural filtration" generated by the Brownian motion up to s. That is, $B_t - B_s$ is independent of $\mathcal{F}_s^B = \sigma(B_v - B_0 : 0 \leq v \leq s) = \sigma(B(v) : 0 \leq v \leq s)$. For f is sufficiently "nice"[1] then we have
$$\mathbb{E}[f(B_t - B_s)|\mathcal{F}_s^B] = \mathbb{E}[f(B_t - B_s)]$$
In particular, for $0 \leq s \leq t$,
$$\begin{aligned}\mathbb{E}[B_t|\mathcal{F}_s^B] &= \mathbb{E}[B_s + (B_t - B_s)|\mathcal{F}_s^B] \\ &= B_s + \mathbb{E}[B_t - B_s|\mathcal{F}_s^B] \quad \text{(taking out what is known)} \\ &= B_s + \mathbb{E}[B_t - B_s] \quad \text{(independent increments)}\end{aligned}$$

[1] Lebesgue measurable

But $B_t - B_s \sim N(0, (t-s)\mathbb{I})$ so

$$\begin{aligned}\mathbb{E}[B_t|\mathcal{F}_s^B] &= B_s + 0 \\ &= B_s\end{aligned}$$

$\therefore$ we say that our process $(B_t)_{t\geq 0}$ is a $(\mathcal{F}_t^B, \mathbb{P})$-martingale. We can interpret a $(\mathcal{F}_t^B, \mathbb{P})$-martingale as "our best estimate for the future state of this process is what it is currently at".

1.2 Martingales

Definition 1. Let $(F_t)_{t\geq 0}$ be a filtration[2]. A stochastic process $(M_t)_{t\geq 0}$ is an $(\mathcal{F}_t, \mathbb{P})$-martingale if

1. M_t is $\mathcal{F}_t$-measurable $\forall\, t \geq 0$. "$M_t \in \mathcal{F}_t$" $\iff$ "$\mathcal{F}_t$-adapted".
2. $\mathbb{E}[|M_t|] < \infty, \forall\, t \geq 0$ (i.e. the process is integrable).
3. "The Martingale Property" $\mathbb{E}[M_t|\mathcal{F}_s] = M_s, \forall\, s \leq t$

1.2.1 "We can create other martingales from Brownian motion"

Lemma 1. Suppose $(B_t)_{t\geq 0}$ is a Brownian motion.

1. $(B_t)_{t\geq 0}$ is a $(\mathcal{F}_t^B, \mathbb{P})$-martingale.
2. "Important property for later"[3] $B_t^2 - t = M_t$ is a martingale.
3. For any $\sigma > 0$, $N_t = e^{\sigma B_t - \frac{\sigma^2}{2}t}$ is a $(\mathcal{F}_t, \mathbb{P})$-martingale.

Proof. (Partial) Proof that $B_t^2 - t = M_t$ is a martingale. Note

$$\begin{aligned}\mathbb{E}[B_t^2 - B_s^2|\mathcal{F}_s^B] &= \mathbb{E}[(B_t - B_s)^2 + 2B_s(B_t - B_s)]\mathcal{F}_s^B] \\ &= \mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s^B] + 2\mathbb{E}[B_s(B_t - B_s)]\mathcal{F}_s^B] \quad \text{(by linearity)} \\ &= \mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s^B] + 2B_s\mathbb{E}[(B_t - B_s)]\mathcal{F}_s^B] \quad \text{(taking out what is known)} \\ &= \mathbb{E}[(B_t - B_s)^2] + 2B_s\mathbb{E}[(B_t - B_s)] \quad \text{(independent intervals)} \\ &= \mathbb{E}[(B_t - B_s)^2] + 0 \quad \text{(intervals are normally distributed with mean 0)}\end{aligned}$$

Recall

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}^2[X] \\ \implies \mathbb{E}[X^2] &= \mathrm{Var}[X] + \mathbb{E}^2[X]\end{aligned}$$

[2] Note that each $\mathcal{F}_i$ is a σ-algebra and that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

[3] Maybe when we introduce Itô Calculus?

So substituting X with $B_t - B_s$ we have

$$\begin{aligned}
\mathbb{E}[B_t^2 - B_s^2 | \mathcal{F}_s^B] &= \mathbb{E}[(B_t - B_s)^2] \\
&= \text{Var}[B_t - B_s] + \mathbb{E}^2[B_t - B_s] \\
&= (t - s) + 0^2 \quad \text{(intervals are normally distributed with variance (t - s))} \\
&= t - s
\end{aligned}$$

However, note that $B_s^2 \in \mathcal{F}_s^B$ so

$$\mathbb{E}[B_t^2 - B_s^2 | \mathcal{F}_s^B] = \mathbb{E}[B_t^2 | \mathcal{F}_s^B] - B_s^2 \quad \text{(taking out what is known)}$$

Hence

$$\begin{aligned}
&\mathbb{E}[B_t^2 | \mathcal{F}_s^B] - B_s^2 = t - s \\
\implies &\mathbb{E}[B_t^2 | \mathcal{F}_s^B] - t = B_s^2 - s \\
\implies &\mathbb{E}[B_t^2 - t | \mathcal{F}_s^B] = B_s^2 - s
\end{aligned}$$

or equivalently

$$\mathbb{E}[M_t | \mathcal{F}_s^B] = M_s$$

□

Prove that any $\sigma > 0$, $N_t = e^{\sigma B_t - \frac{\sigma^2}{2} t}$ is a $(\mathcal{F}_t, \mathbb{P})$-martingale.

Proof. Recall that if $Z \sim N(0, 1)$ then for $\lambda > 0$

$$\mathbb{E}[e^{\lambda Z}] = m_Z(\lambda) = e^{\frac{\lambda^2}{2}}$$

So

$$\begin{aligned}
\mathbb{E}[N_t | \mathcal{F}_s^B] &= \mathbb{E}[e^{\sigma B_t - \frac{\sigma^2}{2} t} | \mathcal{F}_s^B] \\
&= \mathbb{E}[e^{\sigma B_t - \sigma B_s + \sigma B_s - \frac{\sigma^2}{2} t} | \mathcal{F}_s^B] = \mathbb{E}[e^{\sigma(B_t - B_s)} e^{\sigma B_s - \frac{\sigma^2}{2} t} | \mathcal{F}_s^B] \\
&= \mathbb{E}[e^{\sigma(B_t - B_s)} | \mathcal{F}_s^B] e^{\sigma B_s - \frac{\sigma^2}{2} t} \quad \text{(taking out what is known)} \\
&= \mathbb{E}[e^{\sigma(B_t - B_s)}] e^{\sigma B_s - \frac{\sigma^2}{2} t} \quad \text{(independent increments)}
\end{aligned}$$

But $\mathbb{E}[e^{\sigma(B_t - B_s)}]$ is the moment generating function[4] of $N(0, t - s)$. So,

$$\begin{aligned}
\mathbb{E}[N_t | \mathcal{F}_s^B] &= e^{\frac{\sigma^2}{2}(t-s)} e^{\sigma B_s - \frac{\sigma^2}{2} t} \\
&= e^{\frac{\sigma^2}{2}(t-s) + \sigma B_s - \frac{\sigma^2}{2} t} \\
&= e^{\frac{\sigma^2}{2} t - \frac{\sigma^2}{2} s + \sigma B_s - \frac{\sigma^2}{2} t} \\
&= e^{\sigma B_s - \frac{\sigma^2}{2} s} \\
&= N_s
\end{aligned}$$

□

[4] If $X \sim N(\mu, \sigma^2)$ then the MGF of X, denoted $M_x(t) = \mathbb{E}[e^{tX}] = e^{\mu t + \frac{\sigma^2}{2} t^2}$.

1.3 Geometric Brownian Motion

Our first model for a stock price will be

$$S_t = S_0 e^{\mu t \sigma B_t - \frac{1}{2}\sigma^2 t}$$

S_t is Geometric Brownian motion (GBM) where μ is our drift parameter and σ is our volatility parameter. Why is this so? Note

$$\begin{aligned}
\ln\left(\frac{S_t}{S_0}\right) &= \mu t + \sigma B_t - \frac{1}{2}\sigma^2 t \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t
\end{aligned}$$

We see that the log returns have a normal distribution, $\ln\left(\frac{S_t}{S_0}\right) \sim N(\mu - \frac{1}{2}\sigma^2), \sigma^2)$, and so returns (and thereby asset prices S_t) have a log-normal distribution.[5]

Recall that for $Z \sim N(0,1)$ then $cZ \sim N(0, c^2)$. So to generate a GBM with variance $t - s = \Delta t$ first generate $Z \sim N(0,1)$ and set $B_{t_n} = B_{t_{n-1}} + \sqrt{\Delta t} Z$.

Theorem: Brownian motion is continuous everywhere but differentiable nowhere.

Proof. (Proof outline) Consider a differentiable function f. Then for some partition of some interval $\pi = \{0 = t_0 \leq t_1 \leq \cdots \leq t_n = T\}$ of $[0,T]$ define the total variation of f over $[0,T]$ to be,

$$V(f) = \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1} - f(t_i)|$$

Where $|\pi| = \max(|t_{i+1} - t_i|)$ and can be interpreted as "the width of the mesh of the partition". If f is differentiable over $[0,T]$ then by the mean value theorem we have

$$f(t_{i+1}) - f(t_i) = f'(t_i^*)(t_{i+1} - t_i)$$

For some $t_i^* \in (t_i, t_{i+1})$. So,

$$\begin{aligned}
V(f) &= \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1} - f(t_i)| \\
&= \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f'(t_i^*)|(t_{i+1} - t_i) \\
&= \int_0^T |f'(u)|\, du \quad \text{(not necessarily finite)}
\end{aligned}$$

[5] A random variable X is lognormally distributed if, for $Y \sim N(\mu, \sigma^2)$, we have $X = e^Y$.

We say that f is of <u>bounded variation</u> on $[0,T]$ if $V(f) < \infty$. Now, define <u>quadratic variation</u> to be

$$QV(f) = \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^2$$

Why do we care about this? If f is twice differentiable we can use the mean value theorem again

$$\begin{aligned}
QV(f) &= \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|^2 \\
&= \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f'(t_i^*)|^2 (t_{i+1} - t_i)^2 \\
&= \lim_{|\pi| \to 0} |\pi| \sum_{i=0}^{n-1} |f'(t_i^*)|^2 (t_{i+1} - t_i) \\
&= (0) \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} |f'(t_i^*)|^2 (t_{i+1} - t_i) \quad \text{(I'm pretty sure this is a handwavy part)} \\
&= 0
\end{aligned}$$

So if f is differentiable on $[0,T]$ then $QV(f) = 0$ and so any function with bounded variation must have 0 quadratic variation.

Theorem: The quadratic variation of a Brownian motion is nonzero. That is, if B_t is a standard one-dimensional Brownian motion then $QV(B_t) = T$ over the interval $[0,T]$. Therefore, the sample paths $t \mapsto B_t(\omega)$ for fixed ω of a Brownian motion have infinite total variation (since $QV(B_t) \neq 0$).

Proof. (We will only begin the proof). Write $D_k(\omega) = B_{t_{k+1}}(\omega) - B_{t_k}(\omega)$ for fixed ω. Then, the quadratic variation of $t \mapsto B_t(\omega)$ is

$$Q_\pi(\omega) = \sum_{k=0}^{n-1} (D_k(\omega))^2$$

Note we can write T as the telescopic sum (with $t_n \equiv T$ and $t_0 \equiv 0$)

$$T = \sum_{k=0}^{n-1} t_{k+1} - t_k$$

so

$$\begin{aligned}
Q_\pi - T &= \sum_{k=0}^{n-1} (D_k)^2 \sum_{k=0}^{n-1} t_{k+1} - t_k \\
&= \sum_{k=0}^{n-1} (D_k)^2 - (t_{k+1} - t_k)
\end{aligned}$$

But we have $D_k = B_{t_{k+1}} - B_{t_k} \sim N(0, t_{k+1} - t_k)$, so

$$\mathbb{E}\left[(D_k)^2\right] = t_{k+1} - t_k$$

hence

$$\mathbb{E}[(D_k)^2 - (t_{k+1} - t_k)] = 0$$

Therefore

$$\mathbb{E}\left[Q_\pi - T\right] = \mathbb{E}\left[\sum_{k=0}^{n-1} (D_k)^2 - (t_{k+1} - t_k)\right] = 0$$

Now, for $j \neq k$, we note that $(D_j)^2 - (t_{j+1} - t_j)$ and $(D_k)^2 - (t_{k+1} - t_k)$ are independent increments of Brownian motion. We wish to show that $(Q_\pi - T)^2$ has expectation 0. However, have just shown that

$$\mathbb{E}[Q_\pi - T] = 0$$

and note that $\mathbb{E}[(Q_\pi - T)^2]$ has some terms that look like

$$\begin{aligned} &\mathbb{E}[(D_j^2 - (t_{j+1} - t_j)) \cdot (D_k^2 - (t_{k+1} - t_k))] \quad \text{but by independence,} \\ &= \mathbb{E}[0] \cdot \mathbb{E}[0] = 0 \end{aligned}$$

So we may discard these terms for $j \neq k$ in our sum and consider the expectation of terms such that $j = k$

$$\mathbb{E}[(Q_\pi - T)^2] = \mathbb{E}[\sum_{k=0}^{n-1}(D_k^4 - 2D_k^2(t_{k+1} - t_k) + (t_{k+1} - t_k)^2)]$$

We now rely on a result[6]

1. $\mathbb{E}[B_t B_s] = \min(t, s)$
2. $\mathbb{E}[(B_t - B_s)^m] = \begin{cases} 0 & \text{if m odd} \\ 1 \cdot 3 \cdot 5 \cdots (m-3) \cdot (m-1)(t-s)^{m/2} & \text{for m even} \end{cases}$

then

$$\begin{aligned} \mathbb{E}[(Q_\pi - T)^2] &= \sum_{k=0}^{n-1} \left(\mathbb{E}[D_k^4] - 2\mathbb{E}[D_k^2(t_{k+1} - t_k)] + \mathbb{E}[(t_{k+1} - t_k)^2]\right) \\ &= \sum_{k=0}^{n-1} \left(\mathbb{E}[D_k^4] - 2(t_{k+1} - t_k)\mathbb{E}[D_k^2] + (t_{k+1} - t_k)^2\right) \\ &= \sum_{k=0}^{n-1} \left(\mathbb{E}[(B_{t_{k+1}} - B_{t_k})^4] - 2(t_{k+1} - t_k)\mathbb{E}[(B_{t_{k+1}} - B_{t_k})^2] + (t_{k+1} - t_k)^2\right) \\ &= \sum_{k=0}^{n-1} \left(3(t_{k+1} - t_k)^2 - 2(t_{k+1} - t_k)^2 + (t_{k+1} - t_k)^2\right) \\ &= 2\sum_{k=0}^{n-1}(t_{k+1} - t_k)^2 \leq 2|\pi|T \end{aligned}$$

[6] Proven in a later lecture (I don't think (1) is ever proven?).

So,

$$\begin{aligned}
\lim_{|\pi|\to 0} \operatorname{Var}(Q_\pi - T) &= 0 \quad \text{and} \\
\mathbb{E}[Q_\pi - T] &= 0 \\
\Longrightarrow \lim_{|\pi|\to 0} (Q_\pi - T) &= 0 \quad \text{a.s., so} \\
\lim_{|\pi|\to 0} Q_\pi &= T \quad \text{a.s.}
\end{aligned}$$

□

We have just shown that the quadratic variation of is nonzero, thus we may state that Brownian motion has unbounded total variation. Therefore, Brownian does not meet the criteria for being differentiable and so we say that Brownian motion is continuous everywhere but differentiable nowhere.

□