Mathematical & Computational Finance II
Lecture Notes

Introduction to Stochastic Calculus

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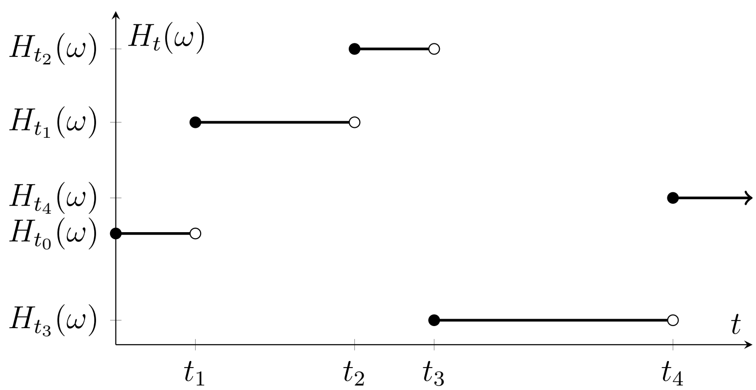
1 Stochastic Integrals

Last time we discussed the quadratic variation of Brownian Motion and why that means its interesting. Now, we elaborate and move onto stochastic integration.

Definition 1. A simple process H from $[0, T] \times \Omega \to \mathbb{R}$ is such that[1]

$$H_t(\omega) = \begin{cases} H_0(\omega) & t \in (t_0, t_1] \\ H_{t_i}(\omega) & t \in (t_i, t_{i+1}] \end{cases}$$

with partition $0 = t_0 \leq \cdots \leq t_N = T$ of $[0, T]$ where H_{t_i} is $\mathcal{F}_{t_i}$-measurable (i.e. it's a random variable).



Definition 2. Let $t \in [0, T]$ and consider the partition $0 = t_0 < t_1 < \cdots < t_m = T$. Suppose that $H \in \mathcal{H}_T$ is a simple process with representation

$$H_t(\omega) = \begin{cases} H_0(\omega) & t \in (t_0, t_1] \\ H_{t_i}(\omega) & \text{if } t \in (t_i, t_{i+1}] \end{cases}$$

[1] For Itô integrals we require that our processes be "right continuous", i.e. we have $t \in (t_i, t_{i+1}]$.

with respect to Brownian Motion B_t on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Consider some random variables $\{H_{t_i}\}_{i=0}^m$ such that each H_{t_i} is $\mathcal{F}_{t_i}$-measurable and bounded. We define the stochastic integral (Itô integral) with respect to the Brownian motion B_t as

$$W(T) = \int_0^T H_u \, dB_u$$

1.1 Properties

For simple processes H and K and constants α, β

1. Linearity: $\int_0^t (\alpha H_u + \beta K_u) \, dB_u = \alpha \int_0^t H_u \, dB_u + \beta \int_0^t K_u \, dB_u$
2. $W(t) = \int_0^t H_u \, dB_u$ is a martingale.
3. $\mathbb{E}[(W(t))^2] = \mathbb{E}[(\int_0^t H_u \, dB_u)^2] = \mathbb{E}[\int_0^t H_u^2 \, du]$

For some visual intuition of the "simple proof" that integrals of simple processes are linear, let $X_t = H_t + K_t$ then we can see the following

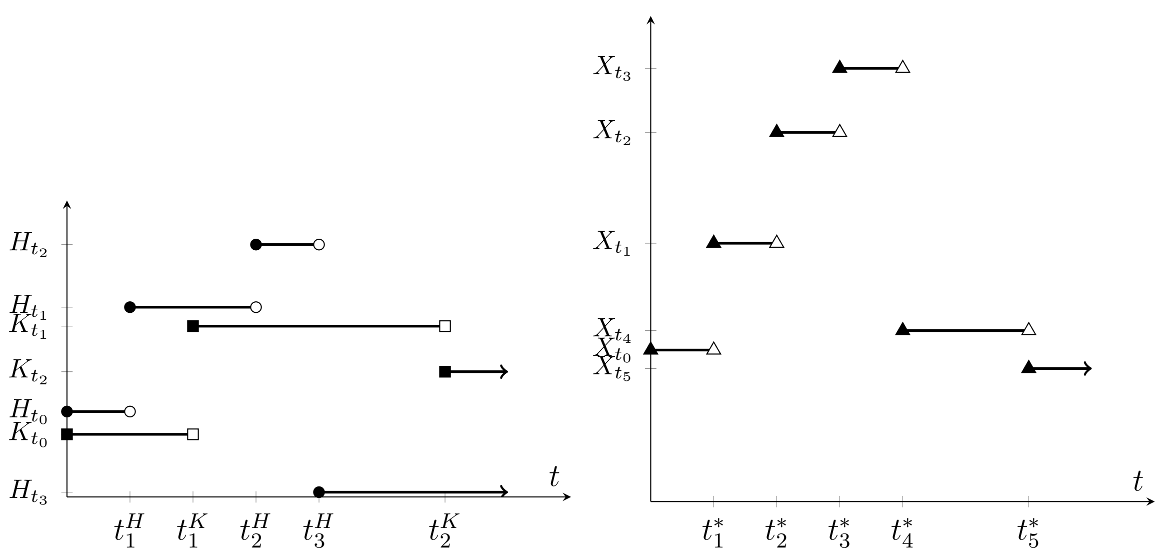


Figure 1: By combining the existing partitions of H and K we can produce a new simple process which has the value of $X_t = H_t + K_t$ at every point t defined for H and K. Clearly we see that scaling H and K by constants α, β produces an analoguous result.

Proof. Proof that $W(t) = \int_0^t H_u \, dB_u$ is a martingale. By assumption there exists a partition $0 = t_0 \leq \cdots \leq t_N = T$ such that

$$H_t(\omega) = \begin{cases} H_0(\omega) & t \in (t_0, t_1] \\ H_{t_i}(\omega) & t \in (t_i, t_{i+1}] \end{cases}$$

where $H_{t_i} \in \mathcal{F}_{t_i}$. Suppose $t \in (t_k, t_{k+1}]$ then,

$$W(t) = H_{t_k}(B_t - B_{t_k}) + \sum_{i=1}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

To complete the proof we must show that $W(t)$ is $\mathcal{F}_t$-measurable, integrable, and satisfies the martingale property: $\mathbb{E}[W(t)|\mathcal{F}_s] = W(s)$. Clearly $W(t)$ is $\mathcal{F}_t$-measurable since its components

$$\begin{aligned} H_{t_{i-1}} &\in \mathcal{F}_{t_{i-1}} \subset \mathcal{F}_t \quad \text{and} \\ (B_{t_i} - B_{t_{i-1}}) &\in \mathcal{F}_{t_i} \subset \mathcal{F}_t \quad \text{and} \\ H_{t_k} &\in \mathcal{F}_{t_k} \subset \mathcal{F}_t \quad \text{and} \\ (B_t - B_{t_k}) &\in \mathcal{F}_{t_k} \subset \mathcal{F}_t \end{aligned}$$

So every component of $W(t)$ is $\mathcal{F}_t$-measuable and a sum/product of $\mathcal{F}_t$-measurable elements is itself $\mathcal{F}_t$-measurable.[2] [3] For $0 \leq s < t$ and $s \in (t_j, t_{j+1}]$ for $j < k$, thus $s < t$, we have

$$\begin{aligned} \mathbb{E}[W(t)|\mathcal{F}_s] &= \mathbb{E}[H_{t_k}(B_t - B_{t_k}) + \sum_{i=1}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \\ &= \mathbb{E}[H_{t_k}(B_t - B_{t_k})|\mathcal{F}_s] + \mathbb{E}[\sum_{i=1}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \end{aligned}$$

But notice

$$\begin{aligned} \mathbb{E}[H_{t_k}(B_t - B_{t_k})|\mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[H_{t_k}(B_t - B_{t_k})|\mathcal{F}_{t_k}]|\mathcal{F}_s] \quad \text{(by the tower property)} \\ &= \mathbb{E}[H_{t_k}\mathbb{E}[B_t - B_{t_k}|\mathcal{F}_{t_k}]|\mathcal{F}_s] \quad \text{(taking out what's known)} \\ &= \mathbb{E}[H_{t_k} \cdot 0|\mathcal{F}_s] = 0 \quad \text{(by independent increments with mean 0)} \end{aligned}$$

[2] Is this statement obvious?

[3] I think we skip the integrability condition and move on directly to the martingale property.

So we're left with

$$
\begin{aligned}
\mathbb{E}[W(t)|\mathcal{F}_s] &= 0 + \mathbb{E}[\sum_{i=1}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \\
&= \mathbb{E}[\sum_{i=1}^{j} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) + H_{t_j}(B_s - B_{t_j}) + H_s(B_{t_{j+1}} - B_{t_j}) + \\
&\qquad \sum_{i=j+2}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \\
&= \mathbb{E}[W(s) + H_s(B_{t_{j+1}} - B_{t_j}) + \sum_{i=j+2}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \\
&= W(s) + H_s\mathbb{E}[B_{t_{j+1}} - B_{t_j}|\mathcal{F}_s] + \mathbb{E}[\sum_{i=j+2}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \\
&= W(s) + H_s \cdot 0 + \mathbb{E}[\sum_{i=j+2}^{k} H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_s] \\
&= W(s) + \sum_{i=j+2}^{k} \mathbb{E}[\mathbb{E}[H_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_{t_{i-1}}]|\mathcal{F}_s] \\
&= W(s) + \sum_{i=j+2}^{k} \mathbb{E}[H_{t_{i-1}}\mathbb{E}[B_{t_i} - B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}]|\mathcal{F}_s] \\
&= W(s) + \sum_{i=j+2}^{k} \mathbb{E}[H_{t_{i-1}} \cdot 0|\mathcal{F}_s] \\
&= W(s)
\end{aligned}
$$

□

We want to extend this definition of the Itô integral to a wider class of integrands. Define a family $\mathcal{H}_T$ of processes H_T on $[0,T] \times \Omega$ such that if $H_T \in \mathcal{H}_T$ then

1. $H_T \in \mathcal{F}_T \quad \forall\, t \in [0,T]$ (adapted process)
2. $\mathbb{E}[\int_0^T (H_u)^2\, du] < \infty$ (square integrable)

Theorem: We may approximate H_T by a family of simple processes. This is similar to how a Reimann can be approximated with piecewise constant functions.

Proof. If $H \in \mathcal{H}_T$ then there exists a sequence of simple process $\{H^m\} \in \mathcal{H}_T$ (by definition all simple process are elements of $\mathcal{H}_T$) such that

$$\lim_{m\to\infty} \|H - H^m\|_2 = 0 \quad \text{where}$$

$$\|H\|_2 = \left(\mathbb{E}\Big[\int_0^T (H_u)^2\, du\Big]\right)^{1/2}$$

So $L^2([0,T] \times \Omega, \mathcal{B}(\mathbb{R}) \times \mathcal{F}, dx \times \mathbb{P})$ is a complete[4] space. Letting $\pi_n = \{0 = t_0 \leq \cdots \leq t_n = T\}$ and $\{\pi\}_{n=0}^{\infty}$ be a sequence of refinements[5] on our partition of $[0,T]$ (i.e. if $p \in \pi_n \implies p \in \pi_{n+1}$) with $|\pi_n| \longrightarrow 0$ associated with a sequence of simple processes $\{H_n\}_{n=1}^{\infty}$ with limit $H^n \longrightarrow H$ a.s. Then,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T H_u^m \, dB_u - \int_0^T H_u^n \, dB_u\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T \left(H_u^m - H_u^n\right) dB_u\right)^2\right] \quad \text{(linearity)} \\
&= \mathbb{E}\left[\int_0^T \left(H_u^m - H_u^n\right)^2 du\right] \quad \text{(Itô isometry)} \\
&= \mathbb{E}\left[\int_0^T \left(H_u^m + H_u - H_u - H_u^n\right)^2 du\right] \\
&\leq 2\mathbb{E}\left[\int_0^T \left(H_u^m - H_u\right)^2 du\right] + 2\mathbb{E}\left[\int_0^T \left(H_u^n - H_u\right)^2 du\right] \\
&\qquad \text{(by } (a+b)^2 \leq 2a^2 + 2b^2\text{)}
\end{aligned}$$

And so we should see

$$2\mathbb{E}\left[\int_0^T \left(H_u^m - H_u\right)^2 du\right] + 2\mathbb{E}\left[\int_0^T \left(H_u^n - H_u\right)^2 du\right] \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty$$

That is, $\{\int_0^T H_u^n \, dB_u\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence, $\lim_{n\to\infty} \int_0^T H_u^m \, dB_u$ exists and we define $\int_0^T H_u \, dB_u = \lim_{n\to\infty} \int_0^T H_u^n \, dB_u$. "This converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$." $\square$

If for $t \in [0,T]$ we wish to define $\int_0^t H_u^n \, dB_u$, note that if $H \in \mathcal{H}_T$ then $H_{(\cdot)} \mathbb{1}_{[0,t]}(\cdot) \in \mathcal{H}_T$, so

1. Take $H^n \longrightarrow H$
2. $H_{(\cdot)} \mathbb{1}_{[0,t]}(\cdot) \in \mathcal{H}_T$

and proceed the same way to define your integral $\int_0^t H_u \, dB_u$

1.2 Some Properties for $H^1, H^2 \in \mathcal{H}_T$

1. $I(t) = \int_0^t H_u \, dB_u$ is a continuous process.
2. Linearity: $\int_0^t (\alpha H_u^1 + \beta H_u^2) \, dB_u = \alpha \int_0^t H_u^1 \, dB_u + \beta \int_0^t H_u^2 \, dB_u$
3. $I(t)$ is $\mathcal{F}_t$-measurable (adapted).

[4] 1. Any Cauchy sequence has a limit in the space, 2. Any convergent sequence is a Cauchy sequence.

[5] That is, we increase the granularity of our partition, keeping all the previous partitions from the previous steps when adding new partitions.

4. $I(t)$ is a $(\mathcal{F}_t, \mathbb{P})$-martingale:

$$\mathbb{E}\left[\int_0^t H_u \, dB_u | \mathcal{F}_s\right] = \int_0^s H_u \, dB_u \quad 0 \le s \le t \le T$$

5. Itô isometry holds:

$$\mathbb{E}\left[\left(\int_0^t H_u \, dB_u\right)^2\right] = \mathbb{E}\left[\int_0^t H_u^2 \, du\right]$$

As an example we can consider H to be a Brownian Motion. Let $H_u = B_u$ and use the definitions to calculate $\int_0^t H_u \, dB_u$.

$$\begin{aligned} \int_0^t B_u \, dB_u &= \cdots \\ &\vdots \quad \text{(steps discussed at a later date)} \\ &= \frac{1}{2} B_t^2 - \frac{t}{2} \end{aligned}$$

To show this is so we need a convergent process and apply the definitions above. The $-\frac{t}{2}$ term comes from the nonzero quadratic variation of B_t.