Assignment guidlines:

- You should not hand in a first draft. Rewrite your solutions carefully and neatly. Use complete sentences. Make sure your arguments are clear.

Problems:

1. Consider a simple process H with associated partition $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ such that $H_t = H_{t_i}$ for $t \in [t_i, t_{i+1})$ and H_{t_i} is $\mathcal{F}_{t_i}$-measurable. Prove that the stochastic integral with respect to a standard Brownian motion B defined as
$$I(T) := \int_0^T H_u dB_u = \sum_{i=0}^{n-1} H_{t_i}(B_{t_{i+1}} - B_{t_i})$$
satisfies $E[I(T)] = 0$. *[Use the definition and be explicit and rigorous in your proof.]*
Solution:
See the solution to Question 2 provided in the "2014 Midterm: Solutions" file on Moodle

2. Suppose that on the risk-neutral filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Q})$ the price of the risky asset at time t is given by the stochastic differential equation
$$S_t = S_0 + \int_0^t rS_u du + \int_0^t \sigma S_u dW_u$$
for $0 \leq t \leq T$ where W_t is a standard Brownian motion with respect to $(\mathcal{F}_t, \mathbf{Q})$. Use Itô's formula to give a stochastic differential equation satisfied by $\ln(S_t)$.
Solution:
See the solution to Question 3(b) provided in the "2014 Midterm: Solutions" file on Moodle

3. Let W_t be a standard Brownian motion. Use Itô's formula to prove the following:

   (a) For a (deterministic) function $h(t)$ with continuous derivative on $[0, \infty)$:
$$\int_0^t h(s)dW_s = h(t)W_t - \int_0^t h'(s)W_s ds.$$
   Solution:
   Consider the function $f(t,x) = h(t)x$ and apply Itô's formula to find that
$$\begin{aligned} h(t)W_t = f(t,W_t) &= f(0,W_0) + \int_0^t \frac{\partial}{\partial s} f(s,W_s)ds + \int_0^t \frac{\partial}{\partial x} f(s,W_s)dW_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(s,W_s)d\langle W.\rangle_s \\ &= h(0)W_0 + \int_0^t h'(s)W_s ds + \int_0^t h(s)dW_s + \frac{1}{2}\int_0^t 0\,ds \\ &= 0 + \int_0^t h'(s)W_s ds + \int_0^t h(s)dW_s + 0. \end{aligned}$$
   Therefore,
$$\int_0^t h(s)dW_s = h(t)W_t - \int_0^t h'(s)W_s ds$$
   as desired.

(b) The process

$$Z_t = \exp\left(\int_0^t \theta(s)dW_s - \frac{1}{2}\int_0^t \theta^2(s)ds\right)$$

satisfies

$$dZ_t = \theta(t)Z_t dW_t$$

where θ is a (deterministic) integrable function.

Solution:

Let

$$Y_t = \int_0^t \theta(s)dW_s - \frac{1}{2}\int_0^t \theta^2(s)ds$$

and note that

$$\langle Y_\cdot\rangle_t = \langle \int_0^\cdot \theta(s)dW_s\rangle_t = \int_0^t \theta^2(s)ds.$$

Apply Itô's formulat to the function $f(x) = e^x$ to find that

$$\begin{aligned}
Z_t = f(Y_t) &= f(Y_0) + \int_0^t f'(Y_s)dY_s + \frac{1}{2}\int_0^t f''(Y_s)d\langle Y_\cdot\rangle_s \\
&= f(0) + \int_0^t e^{Y_s}[\theta(s)dW_s - \frac{1}{2}\theta^2(s)ds] + \frac{1}{2}\int_0^t e^{Y_s}[\theta^2(s)ds] \\
&= 1 + \int_0^t Z_s\theta(s)dW_s - \frac{1}{2}\int_0^t Z_s\theta^2(s)ds + \frac{1}{2}\int_0^t Z_s\theta^2(s)ds \\
&= 1 + \int_0^t Z_s\theta(s)dW_s
\end{aligned}$$

which is the integral form of the given SDE.

(c) For $x > 0$ a constant the process

$$X_t = (x^{1/3} + \frac{1}{3}W_t)^3$$

satisfies the SDE

$$dX_t = \frac{1}{3}X_t^{1/3}dt + X_t^{2/3}dW_t.$$

Solution:

Let $f(y) = (x^{1/3} + \frac{1}{3}y)^3$ and apply Itô's formula to find

$$\begin{aligned}
X_t = f(W_t) &= f(W_0) + \int_0^t f'(W_u)dW_u + \frac{1}{2}\int_0^t f''(W_u)\langle W_\cdot\rangle_u \\
&= (x^{1/3} + \frac{1}{3}0)^3 + \int_0^t 3(x^{1/3} + \frac{1}{3}W_u)^2\left(\frac{1}{3}\right)dW_u + \frac{1}{2}\int_0^t 2(x^{1/3} + \frac{1}{3}W_u)\left(\frac{1}{3}\right)du \\
&= x + \int_0^t \left[(x^{1/3} + \frac{1}{3}W_u)^3\right]^{2/3} dW_u + \frac{1}{3}\int_0^t \left[(x^{1/3} + \frac{1}{3}W_u)^3\right]^{1/3} du \\
&= x + \int_0^t X_u^{2/3}dW_u + \frac{1}{3}\int_0^t X_u^{1/3}du
\end{aligned}$$

which is the integral form of the given SDE.

4. **[MAST 729/881 Only]** Consider the vector-valued stochastic process $X_t = \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}$ where $X_t^{(1)} = a\cos(B_t)$ and $X_t^{(2)} = b\sin(B_t)$. Show that X_t is a solution of

$$dX_t = -\frac{1}{2}X_t dt + MX_t dB_t$$

for some (2×2)-matrix M. What is M? *[Hint: Consider the components separately.]*
Solution:
See the solution to Question 5 provided in the "2014 Midterm: Solutions" file on Moodle

5. Consider the process X given by the SDE

$$dX_t = -X_t dt + e^{-t} dB_t$$

with $X_0 = 0$ and B_t a standard Brownian motion. Show that

$$E[X_t] = 0$$

and

$$Var[X_t] = te^{-2t}$$

by solving ODEs for $E[X_t]$ and $E[X_t^2]$.
Solution:
Write the integral form of the SDE

$$X_t = -\int_0^t X_u du + \int_0^t e^{-u} dB_u$$

and take the expectation of both sides to find

$$\begin{aligned} E[X_t] &= E\left[-\int_0^t X_u du\right] + E\left[\int_0^t e^{-u} dB_u\right] \\ &= -E\left[\int_0^t X_u du\right] + 0 \\ &= -E\left[\int_0^t X_u du\right] \end{aligned} \tag{1}$$

where we have used the fact that stochastic integrals are martingales so that

$$E\left[\int_0^t e^{-u} dB_u\right] = E\left[\int_0^t e^{-u} dB_u \middle| \mathcal{F}_0\right] = \int_0^0 e^{-u} dB_u = 0.$$

Therefore, applying Fubini's Theorem to exchange the expectation and integral in equation (1), we have

$$E[X_t] = -\int_0^t E[X_u] du.$$

Define $f(t) = E[X_t]$ and note that the above equation is equivalent to the ordinary differential equation (ODE)

$$\frac{d}{dt} f(t) = -f(t)$$

with initial condition $f(0) = 0$. The unique solution to this ODE is $f(t) = 0$ so we have $E[X_t] = 0$. Next, define $Y_t = X_t^2$ and apply Itô's formula to find that

$$\begin{aligned} Y_t &= \int_0^t 2X_u dX_u + \frac{1}{2}\int_0^t 2d\langle X.\rangle_u \\ &= \int_0^t 2X_u[-X_u du + e^{-u}dB_u] + \int_0^t e^{-2u}du \\ &= -2\int_0^t X_u^2 du + 2\int_0^t X_u e^{-u}dB_u + \int_0^t e^{-2u}du \\ &= -2\int_0^t Y_u du + 2\int_0^t X_u e^{-u}dB_u + \frac{1}{2}(1-e^{-2t}) \qquad (2) \end{aligned}$$

where we have used the fact that

$$\langle X.\rangle_t = \langle \int_0^\cdot e^{-u}dB_u\rangle_t = \int_0^t \left[e^{-u}\right]^2 du = \int_0^t e^{-2u}du.$$

Taking the expectation of both sides of equation (2) and using the fact that stochastic integrals are martingales, and hence there expectations are equal to zero, we have

$$\begin{aligned} E[Y_t] &= E\left[-2\int_0^t Y_u du\right] + \frac{1}{2}(1-e^{-2t}) \\ &= -2\int_0^t E[Y_u]du + \frac{1}{2}(1-e^{-2t}) \qquad (3) \end{aligned}$$

using Fubini's Theorem to exchange the order of expectation and integration in the first integral. Define $g(t) = E[Y_t]$ and note that equation (3) is equivalent to the ODE

$$\frac{dg}{dt}(t) = -2g(t) + e^{-2t}$$

with initial condition $g(t) = 0$. Note this is a nonhomogenous linear ODE and it can be shown, using elementary methods (MATH 370), that the unique solution is

$$g(t) = te^{-2t}.$$

Therefore, we have that

$$Var[X_t] = E[X_t^2] - (E[X_t])^2 = E[Y_t] - 0 = te^{-2t}.$$

<u>Alternative Solution:</u>
We can actually solve this simple SDE by applying Itô's formula to the product $f(t,x) = e^t x$ to find that

$$\begin{aligned} e^t X_t &= f(0, X_0) + \int_0^t e^u X_u du + \int_0^t e^u dX_u + \frac{1}{2}\int_0^2 0d\langle X.\rangle_u \\ &= 0 + \int_0^t e^u X_u du + \int_0^t e^u[-X_u du + e^{-u}dB_u] \\ &= \int_0^t dB_u \\ &= B_t. \end{aligned}$$

Therefore, multiplying both sides by e^{-t}, we have

$$X_t = e^{-t} B_t.$$

Then

$$E[X_t] = E[e^{-t} B_t] = e^{-t} E[B_t] = 0$$

and

$$Var[X_t] = E[X_t^2] = E[e^{-2t} B_t^2] = e^{-2t} E[B_t^2] = e^{-2t} t.$$

Note that solving SDEs is not always that easy!

6. Recall that stochastic integrals

$$\int_0^T H_u dB_u$$

are martingales provided that the integrand H is adapted and satisfies some technical (integrability) conditions. Using Itô's formula find a process X_t such that

$$B_t^3 - X_t$$

is a martingale.
Solution:
Apply Itô's formula to the function $f(x) = x^3$ to find that

$$\begin{aligned} B_t^3 = f(B_t) &= f(B_0) + \int_0^t f'(B_u) dB_u + \frac{1}{2} \int_0^t f''(B_u) du \\ &= 0 + \int_0^t 3B_u^2 dB_u + \frac{1}{2} \int_0^t 6B_u du \\ &= 3 \int_0^t B_u^2 dB_u + 3 \int_0^t B_u du. \end{aligned}$$

Therefore, if we define

$$X_t = 3 \int_0^t B_u du$$

we have that

$$B_t^3 - X_t = 3 \int_0^t B_u^2 dB_u$$

is a martingale since the stochastic integral $\int_0^t B_u^2 dB_u$ is a martingale (and a martingale multiplied by a constant is also a martingale).

7. In the continous-time Black-Scholes model prove the put-call parity relationship

$$P(t, T, S, K) = C(t, T, S, K) + e^{-r(T-t)} K - S_t$$

between the price at time t of a European call option, denoted $C(t, T, S, K)$, and the price of a European put option, denoted by $P(t, T, S, K)$, with common strike price K and maturity T.
Solution:
See the solution to Question 4(a) provided in the "2014 Midterm: Solutions" file on Moodle