Mathematical & Computational Finance II
Lecture Notes

Continuous Time Finance

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1 Market Processes in Continuous Time

Recall that we were discussing finance & option pricing in continuous time. Our modelling framework was a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by Brownian motion B_t where $\mathbb{P}$ was the "real world"[1] probability space.

We had a money market account S_t^0 such that

$$dS_t^0 = rS_t^0 dt, \quad S_0^0 = 1$$

with solution

$$S_t^0 = e^{rt}$$

and a risky asset S_t^1 whose SDE

$$dS_t^1 = \mu S_t^1 \, dt + \sigma S_t^1 \, dB_t, \quad S_0^1 = s_0$$

has solution[2]

$$S_t^1 = s_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Our wealth process V_t associated with a portfolio $H = (H_t^0, H_t^1)$ of S_t^0, S_t^1, with H_t^i denoting the quantity of asset S^i at time t, was

$$V_t(H) = H_t^0 S_t^0 + H_t^1 S_t^1$$

and we say V_t is self financing if

$$\begin{aligned} dV_t &= H_t^0 \, dS_t^0 + H_t^1 \, dS_t^1 \\ &= H_t^0 r S_t^0 \, dt + H_t^1 \left[\mu S_t^1 \, dt + \sigma S_t^1 \, dB_t\right] \\ &= H_t^0 \left(r S_t^0\right) dt + H_t^1 \left(\mu S_t^1\right) dt + H_t^1 (\sigma S_t^1) \, dB_t \end{aligned}$$

[1] I want to elaborate more on what this means.

[2] See October 1. The easiest way to see this is by starting with the solution & applying Itô's formula to get the SDE.

That is, movements in wealth come strictly from movements in the asset prices (i.e. no wealth movements from net injections/withdrawals of capital). Our discounted price processes are

$$\begin{aligned}\overline{S}_t^1 &= e^{-rt} S_t^1 \\ \overline{V}_t &= e^{-rt} V_t\end{aligned}$$

We can apply Itô's formula on $f(t,x) = e^{-rt}x$ to find the SDEs that these processes satisfy. First taking our derivatives,

$$\begin{aligned}f_t(t,x) &= -re^{-rt}x \\ f_x(t,x) &= e^{-rt} \\ f_{xx}(t,x) &= 0\end{aligned}$$

So for $\overline{S}_t^1$ we have

$$\begin{aligned}\overline{S}_t^1 = f(t, S_t^1) &= f(0, S_0^1) + \int_0^t f_u(u, S_u^1)\, du + \int_0^t f_x(u, S_u^1)\, dS_u^1 + \frac{1}{2}\int_0^t f_{xx}(u, S_u^1)\, d\langle S_{(\cdot)}^1\rangle_u \\ &= e^{-r(0)} s_0 + \int_0^t -re^{-ru} S_u^1\, du + \int_0^t e^{-ru}\, dS_u^1 + 0 \\ &= 1 - r\int_0^t e^{-ru} S_u^1\, du + \int_0^t e^{-ru}\, dS_u^1 \\ &= 1 - r\int_0^t e^{-ru} S_u^1\, du + \int_0^t e^{-ru}\left[\mu S_u^1\, du + \sigma S_u^1\, dB_u\right] \\ &= 1 - r\int_0^t e^{-ru} S_u^1\, du + \mu\int_0^t e^{-ru} S_u^1\, du + \sigma\int_0^t e^{-ru} S_u^1\, dB_u \\ &= 1 + (\mu - r)\int_u^t e^{-ru} S_u^1\, du + \sigma\int_0^t e^{-ru} S_u^1\, dB_u \\ \implies d\overline{S}_t^1 &= (\mu - r)\overline{S}_u^1\, dt + \sigma \overline{S}_t^1\, dB_t\end{aligned}$$

and for $\overline{V}_t$ we have

$$
\begin{aligned}
\overline{V}_t = f(t, V_t) &= f(0, V_0) + \int_0^t f_u(u, V_u)\,du + \int_0^t f_x(u, V_u)\,dV_u + \frac{1}{2}\int_0^t f_{xx}(u, V_u)\,d\langle V_{(\cdot)}\rangle_u \\
&= e^{-r(0)}v_0 + \int_0^t -re^{-ru}V_u\,du \int_0^t e^{-ru}\,dV_u + 0 \\
&= v_0 - r\int_0^t e^{-ru}V_u\,du + \int_0^t e^{-ru}\big[H_u^0\big(rS_u^0\big)\,du + H_u^1\big(\mu S_u^1\big)\,du + H_u^1(\sigma S_u^1)\,dB_u\big] \\
&= v_0 - r\int_0^t e^{-ru}V_u\,du + \int_0^t e^{-ru}H_u^0\big(rS_u^0\big)\,du + \int_0^t e^{-ru}H_u^1\big(\mu S_u^1\big)\,du \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_0^t e^{-ru}H_u^1\big(\sigma S_u^1\big)\,dB_u \\
&= v_0 - r\int_0^t e^{-ru}\big(H_u^0S_u^0 + H_u^1S_u^1\big)\,du + r\int_0^t e^{-ru}H_u^0S_u^0\,du \\
&\qquad\qquad + \Big[\mu\int_0^t e^{-ru}H_u^1S_u^1\,du + \sigma\int_0^t e^{-ru}H_u^1S_u^1\,dB_u\Big] \\
&= v_0 + r\Big[\int_0^t e^{-ru}\Big(H_u^0S_u^0 - H_u^0S_u^0 - H_u^1S_u^1\Big)\,du\Big] \\
&\qquad\qquad + \Big[\mu\int_0^t e^{-ru}H_u^1S_u^1\,du + \sigma\int_0^t e^{-ru}H_u^1S_u^1\,dB_u\Big] \\
&= v_0 - r\int_0^t e^{-ru}H_u^1S_u^1\,du + \mu\int_0^t e^{-ru}H_u^1S_u^1\,du + \sigma\int_0^t e^{-ru}H_u^1S_u^1\,dB_u \\
&= v_0 + (\mu - r)\int_0^t e^{-ru}H_u^1S_u^1\,du + \sigma\int_0^t e^{-ru}H_u^1S_u^1\,dB_u \\
\implies d\overline{V}_t &= (\mu - r)H_t^1\overline{S}_t^1\,dt + \sigma H_t^1\overline{S}_t^1\,dB_t \\
\iff d\overline{V}_t &= H_t^1\big[(\mu - r)\overline{S}_t^1\,dt + \sigma\overline{S}_t^1\,dB_t\big]
\end{aligned}
$$

Note that our discounted risky asset and wealth processes have the term $(\mu - r)$ appearing. Intuitively we can think of this as the risk premium for the risky asset.

2 Probability Measures

Our goal is to be able to select a portfolio process whose payoff is equal to that of European contingent claim with payoff $h_T \in \mathcal{F}_T$ at time T. We should note that our "real world" measure $\mathbb{P}$ assigns probabilities to different states of the world (as do all measures) – and these states in turn affect the value process V_t (i.e. not necessarily a martingale). We say that these states and corresponding probabilities are a reflection of investors' beliefs. However, under $\mathbb{P}$ it's not usually possible to value V_t as a discounted sum of independent cash flows since V_t is not a martingale.

So, we want to be able to construct a different probability measure $\mathbb{Q}$ (i.e. it assigns probabilities in a manner different than $\mathbb{P}$) under which our price process *is* a martingale. We call this measure $\mathbb{Q}$ the "risk neutral" measure.[3] The key insight is that with the right choice of $\mathbb{Q}$ we not only have a price process that is now a martingale, but also expectations with respect to $\mathbb{Q}$ that are identical to those under $\mathbb{P}$ (i.e. the real world prices).

Using risk neutral measure $\mathbb{Q}$ we are able to price things in the following way: Suppose that we have an easy no-arbitrage argument allowing us to pin down the value of our contingent claim. For example, if V_t is the price process for a European call on asset with price S_t^1 with exercise date T and strike price K, then $h_T = (S_T^1 - K)^+$. In this case, the martingale property with respect to $\mathbb{Q}$ buys us

$$\begin{aligned} V_t &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}h_T] \quad \text{(by the martingale property)} \\ &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T^1 - K)^+] \end{aligned}$$

Now that we've seen the advantages of moving into a risk neutral framework we are faced with the obvious two questions

1. How do we construct $\mathbb{Q}$?
2. What does the path of our risky asset S_t^1 look like now that we've gone from $\mathbb{P}$ to $\mathbb{Q}$?

Girsanov's Theorem provides an answer to these, but itself relies on other results from stochastic calculus.

Theorem: Lévy's Theorem. *(Used to prove Girsanov's Theorem)* Suppose $(W_t)_{t\geq 0}$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by W. If

1. W_t is continuous ($\mathbb{P}$ a.s.)
2. W is a $(\mathcal{F}_t, \mathbb{P})$-martingale
3. $W_t^2 - t$ is a $(\mathcal{F}_t, \mathbb{P})$-martingale

then W_t is a standard Brownian motion.

Proof. Proof largely omitted, but here's a few notes.

Note that we've seen that the converse of this theorem is true since we define Brownian motion to have these properties, but it is not immediately obvious that this should hold going the other direction.

[3] "The idea behind the name "risk neutral" is that we may price securities as if we are indifferent to any volatility in the dividend stream or price process.". That is, we are guaranteed that a replicating portfolio will always be the same value as our contingent claim.

The way that you prove Lévy's Theorem is by considering increments of W and showing that the conditional[4] characteristic function is equal to a standard normal random variable. That is

$$\begin{aligned}\mathbb{E}[e^{iu(W_t-W_s)}|\mathcal{F}_s] &= \cdots \\ &\vdots \\ &= e^{-\frac{1}{2}u^2(t-s)} \\ &= \mathbb{E}[e^{iu(W_t-W_s)}]\end{aligned}$$

$\square$

The reason we need Lévy's Theorem is because we want to do something called a "change of measure" which relies on Girsanov's Theorem (discussed later) which itself relies on Lévy's Theorem. Eventually we will go into a risk neutral measure, but it's not trivial how to get there from our real world measure $\mathbb{P}$.

2.1 Switching Probability Measures

The goal is to be able to enter a measure such that the discounted European contingent claim price process *is* a martingale. It turns out that (with some technical requirements) a price process V_t for a derivative on S_t avoids arbitrage opportunities only if a risk-neutral measure for the price process of the underlying S is also a risk-neutral measure for V_t. So we see that we want to construct a risk-neutral measure for S. However, to do so we need to be careful which drift process, say Θ_t, we select.

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $(B_t)_{t\geq 0}$ Brownian motion being a $(\mathcal{F}, \mathbb{P})$-martingale, we consider the adapted process $(\Theta_t)_{t\geq 0}$ and define[5]

$$\Lambda_t = e^{-\int_0^t \Theta_u \, dB_u - \frac{1}{2}\int_0^t \Theta_u^2 \, du}$$

Letting $Z_t = -\int_0^t \Theta_u \, dB_u - \frac{1}{2}\int_0^t \Theta_u^2 \, du$ we get $\Lambda_t = e^{Z_t}$ and note that Λ_t is a SDE for Z_t with $f(t,x) = e^x$. From Itô's formula we know

$$\begin{aligned}\Lambda_t = f(t, Z_t) &= f(0,0) + \int_0^t f_u(u)\, du + \int_0^t f_x(Z_u)\, dZ_u + \frac{1}{2}\int_0^t f_{xx}(Z_u)\, d\langle Z_{(\cdot)}\rangle_u \\ &= Z_0 + \int_0^t e^{Z_u}\, dZ_u + \frac{1}{2}\int_0^t e^{Z_u}\, d\langle Z_{(\cdot)}\rangle_u \\ &= 1 + \int_0^t e^{Z_u}\big[-\Theta_u\, dB_u - \frac{1}{2}\Theta_u^2\, du\big] + \frac{1}{2}\int_0^t e^{Z_u}\, d\langle Z_{(\cdot)}\rangle_u \\ &= 1 - \int_0^t e^{Z_u}\Theta_u\, dB_u - \frac{1}{2}\int_0^t e^{Z_u}\Theta_u^2\, du + \frac{1}{2}\int_0^t e^{Z_u}\, d\langle Z_{(\cdot)}\rangle_u\end{aligned}$$

[4] I think it's a good bet that we've got to use the tower property of conditional expectation: Condition and take out what's known.

[5] I think we call Λ_t an "exponential martingale".

Fortunately we know what the quadratic variation of an Itô process is[6] ,

$$\begin{aligned}
\langle Z_{(\cdot)} \rangle_t &= \left\langle \int_0^{(\cdot)} -\Theta_u \, dB_u + \frac{1}{2} \int_0^{(\cdot)} -\Theta_u^2 \, du \right\rangle_t \\
&= \left\langle \int_0^{(\cdot)} -\Theta_u \, dB_u \right\rangle_t \\
&= \int_0^t \Theta_u^2 \, du \\
\implies d\langle Z_{(\cdot)} \rangle_t &= \Theta_t^2 \, dt
\end{aligned}$$

So

$$\begin{aligned}
\Lambda_t &= 1 - \int_0^t e^{Z_u} \Theta_u \, dB_u - \frac{1}{2} \int_0^t e^{Z_u} \Theta_u^2 \, du + \frac{1}{2} \int_0^t e^{Z_u} \, d\langle Z_{(\cdot)} \rangle_u \\
&= 1 - \int_0^t e^{Z_u} \Theta_u \, dB_u - \frac{1}{2} \int_0^t e^{Z_u} \Theta_u \, du + \frac{1}{2} \int_0^t e^{Z_u} \Theta_u^2 \, du \\
&= 1 - \int_0^t e^{Z_u} \Theta_u \, dB_u
\end{aligned}$$

And so we end up with the SDE

$$\begin{aligned}
\Lambda_t &= 1 - \int_0^t \Lambda_u \Theta_u \, dB_u \\
\implies d\Lambda_t &= -\Lambda_t \Theta_t \, dB_t
\end{aligned}$$

We will propose Λ_t as a candidate density but we should first be careful:

Definition 1. A probability density function on $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{F}_t$-measurable random variable ϕ such that

1. $\phi(\omega) > 0$ a.s.
2. $\int_\Omega \phi(\omega) \, d\mathbb{P}(\omega) = \mathbb{E}[\phi] = 1$

Suppose Λ_t is a martingale on $[0, T]$, so

$$\mathbb{E}[\Lambda_t] = \mathbb{E}[\Lambda_t | \mathcal{F}_0] = \Lambda_0 = 1$$

So Λ_t is indeed a density and we will use it as a candidate as a probability density function for our proposed probability measure. We define a new probability measure $\mathbb{P}^\Theta$ on $(\Omega, \mathcal{F}_T)$ by

$$\begin{aligned}
\mathbb{P}^\Theta(A) &= \int_A \Lambda_T(\omega) \, d\mathbb{P}(\omega) \\
&= \mathbb{E}[\mathbb{1}_A \Lambda_T] \quad \forall\ A \in \mathcal{F}_T
\end{aligned}$$

[6] See October 1, Proposition 2.

We can write (see Radon-Nikodym density/derivative)

$$\Lambda_T = \frac{d\mathbb{P}^\Theta}{d\mathbb{P}}\bigg|_{\mathcal{F}_T}$$

This is telling us how much probability weight $\mathbb{P}^\Theta$ is assigning to the states of the world relative to $\mathbb{P}$. It is capturing the adjustment we ought to be making to the probabilities given by $\mathbb{P}$. As an analogue, the density of the normal distribution tells you how much weight is assigned under the normal distribution to a given small interval of the real line, while the Radon-Nikodym density/derivative is telling us the weight assigned under $\mathbb{P}^\Theta$ to a small portion of the state space under $\mathbb{P}$.

The key is that Λ_T is a (true) martingale if

$$\mathbb{E}_\mathbb{P}\left[e^{-\frac{1}{2}\int_0^T \Theta_u^2\,du}\right] < \infty$$

But under our new measure $\mathbb{P}^\Theta$ the Brownian motion $(B_t)_{t\geq 0}$ is no longer a Brownian motion.[7]

Lemma 1. For a process $(X_t)_{t\geq 0}$, $X_t\Lambda_t$ is a martingale under $\mathbb{P}$ if and only if X_t is a martingale under $\mathbb{P}^\Theta$.

We will prove this in one direction.

Proof. Suppose $X_t\Lambda_t$ is a $\mathbb{P}$ martingale. We want to show that

$$\mathbb{E}_{\mathbb{P}^\Theta}[X_t|\mathcal{F}_s] = X_s \quad \forall\ 0 \leq s \leq t$$

That is, X_t is a martingale in our new measure. Essentially, the whole proof relies on a deep understanding of the definition of conditional expectation. From the definition,

$$\int_A \mathbb{E}_{\mathbb{P}^\Theta}[X_t|\mathcal{F}_s]\,d\mathbb{P} = \int_A X_s\,d\mathbb{P}^\Theta \quad \forall\ A \in \mathcal{F}_s$$

Using the formalism[8] $\Lambda_T\,d\mathbb{P} = d\mathbb{P}^\Theta$ we get

$$\int_A X_s\,d\mathbb{P}^\Theta = \int_A X_s\frac{d\mathbb{P}^\Theta}{d\mathbb{P}}\,d\mathbb{P} = \int_A X_s\Lambda_T\,d\mathbb{P}$$

[7] I'm not exactly sure what part we introduced the drift parameters.
[8] That is, this isn't fundamentally true but we use it as notation.

Leaning on the definition of conditional expectation

$$\begin{aligned}
\int_A X_s \Lambda_T \, d\mathbb{P} &= \int_A \mathbb{E}_{\mathbb{P}}[X_s \Lambda_T | \mathcal{F}_s] \, d\mathbb{P} \\
&= \int_A X_s \mathbb{E}_{\mathbb{P}}[\Lambda_T | \mathcal{F}_s] \, d\mathbb{P} \quad \text{(taking out what is known)} \\
&= \int_A X_s \Lambda_s \, d\mathbb{P}_s \quad \text{(since } \Lambda_T \text{ is a } \mathbb{P} \text{ martingale)} \\
&= \int_A \mathbb{E}_{\mathbb{P}}[X_t \Lambda_t | \mathcal{F}_s] \, d\mathbb{P} \quad \text{(by definition of conditional expectation)} \\
&= \int_A X_t \Lambda_t \, d\mathbb{P} \quad \text{(by definition of conditional expectation)}
\end{aligned}$$

"A lot of these steps may seem useless, but they're really not."

$$\begin{aligned}
&= \int_A X_t \mathbb{E}_{\mathbb{P}}[\Lambda_T | \mathcal{F}_t] \, d\mathbb{P} \\
&= \int_A X_t \Lambda_T \, d\mathbb{P} \\
&= \int_A X_t \frac{d\mathbb{P}^\Theta}{d\mathbb{P}} \, d\mathbb{P} \\
&= \int_A X_t \, d\mathbb{P}^\Theta \\
&= \int_A \mathbb{E}_{\mathbb{P}^\Theta}[X_t | \mathcal{F}_s] \, d\mathbb{P}^\Theta = \mathbb{E}_{\mathbb{P}^\Theta}[X_t | \mathcal{F}_s] = X_s
\end{aligned}$$

□

The point is that from this we get Girsanov's Theorem.

Theorem: Girsanov's Theorem. If Θ is adapted and

$$\int_0^T \Theta_u^2 \, du < \infty$$

and

$$\Lambda_t = e^{-\int_0^t \Theta_u \, dB_u - \frac{1}{2} \int_0^t \Theta_u^2 \, du}$$

is a (true) martingale then the process W defined by the SDE

$$\begin{aligned}
W_t &= B_t + \int_0^t \Theta_u \, du \\
dW_t &= dB_t + \Theta_t \, dt
\end{aligned}$$

is a standard Brownian motion $(\mathcal{F}_t, \mathbb{P}^\Theta)$

Proof. (*Stated without proof, but done using Lévy's Theorem*) $\square$

So, Girsanov's Theorem tells us that W_t progresses as the sum of a Brownian motion under $\mathbb{P}$ and some process Θ_t (related to the Radon-Nikodym derivative characterizing $\mathbb{P}^{\Theta}$). We therefore want to choose a process Θ_t so that the path of W_t with respect to $\mathbb{P}^{\Theta}$ cancels out the real world drift of the discounted process $\overline{S}^1_t$, leaving us with a pure Brownian motion, with respect to our new measure, to model the underlying asset.

2.2 An Important Example

Consider Θ_t is constant $\Theta \in \mathbb{R}$, so

$$W_t = B_t + \int_0^t \Theta \, du = B_t + \Theta t$$

Under $\mathbb{P}$ we have

$$\begin{aligned} B_T &\sim N(0, T) \\ W_T &\sim N(\Theta t, T) \end{aligned}$$

We introduce $\mathbb{P}^{\Theta}$ to remove the drift Θt from the process W_t. This is a very important concept in mathematical finance since when we remove drift our process becomes a martingale (provided it otherwise satisfied the other criteria).

2.3 Constructing the Risk Neutral Measure

Definition 2. If we set $\Theta = \frac{\mu - r}{\sigma}$ then the measure $\mathbb{P}^{\Theta}$ (from now on denoted $\mathbb{Q}$) is called the <u>risk neutral</u> or <u>martingale</u> measure.

Setting $\Theta = \frac{\mu - r}{\sigma}$ we have

$$\begin{aligned} W_t &= B_t + \left(\frac{\mu - r}{\sigma}\right)t \\ \implies B_t &= W_t - \left(\frac{\mu - r}{\sigma}\right)t \\ \implies dB_t &= dW_t - \left(\frac{\mu - r}{\sigma}\right)dt \end{aligned}$$

Using S^1_t to solve for dS^1_t under $\mathbb{Q}$ we have

$$\begin{aligned} dS^1_t &= \mu S^1_t \, dt + \sigma S^1_t \, dB_t \\ &= \mu S^1_t \, dt + \sigma S^1_t \left[dW_t - \left(\frac{\mu - r}{\sigma}\right)dt\right] \\ &= \mu S^1_t \, dt + \sigma S^1_t \, dW_t - (\mu - r) S^1_t \, dt \\ &= r S^1_t \, dt + \sigma S^1_t \, dW_t \quad \text{(remember that } W_t \text{ is Brownian under } \mathbb{Q}) \end{aligned}$$

Notice that our drift is the risk-free rate. This aligns with the binomial model since the expected returns in this model is precisely the risk-free rate. Furthermore,

$$\begin{aligned}
d\overline{S}_t^1 &= (\mu - r)\overline{S}_t^1\, dt + \sigma \overline{S}_t^1\, dB_t \\
&= (\mu - r)\overline{S}_t^1\, dt + \sigma \overline{S}_t^1 \Big[dW_t - \Big(\frac{\mu - r}{\sigma}\Big) dt\Big] \\
&= (\mu - r)\overline{S}_t^1\, dt + \sigma \overline{S}_t^1\, dW_t - (\mu - r)\overline{S}_t^1\, dt \\
&= \sigma \overline{S}_t^1\, dW_t
\end{aligned}$$

Notice that we have confirmed that we have now achieved our goal set above: Select a process Θ_t such that the discounted asset process $\overline{S}_t^1$ becomes Brownian. Additionally,

$$\begin{aligned}
d\overline{V}_t &= H_t^1 \big[(\mu - r)\overline{S}_t^1\, dt + \sigma \overline{S}_t^1\, dB_t\big] \\
&= H_t^1 \Big[(\mu - r)\overline{S}_t^1\, dt + \sigma \overline{S}_t^1 \Big(dW_t - \Big[\frac{\mu - r}{\sigma}\Big] dt\Big)\Big] \\
&= H_t^1 \big[(\mu - r)\overline{S}_t^1\, dt + \sigma \overline{S}_t^1\, dW_t - (\mu - r)\overline{S}_t^1\, dt\big] \\
&= H_t^1 \sigma \overline{S}_t^1\, dW_t
\end{aligned}$$

We say $(\mu - r)$ is the risk premium and $\frac{\mu - r}{\sigma}$ the market price of risk.

Definition 3. A martingale measure is a probability measure $\mathbb{Q}$ that makes all discounted price processes martingales.

Under $\mathbb{Q}$ the expected return is[9]

$$\mathbb{E}_{\mathbb{Q}}\Big[\frac{dS_t^1}{S_t}\Big] = r\, dt$$

This means that the expected return is precisely the risk free rate, or the rate in our money market account. So, under $\mathbb{Q}$ we've essentially removed risk. How does this help us? Consider hedging & replication.

3 Option Pricing: The Replicating Portfolio

Consider

$$\begin{aligned}
M_t &= \int_0^t \sigma H_u^1 \overline{S}_u^1\, dW_u \quad \text{and} \\
\overline{V}_t(H) &= \overline{V}_0 + M_t
\end{aligned}$$

In general these are local martingales.[10,11] Because these are martingales we have something called a martingale representation.

[9]Note that this is just notation since differentials are pretty handwavy.

[10]"I'm not going to tell you what this means since it's unimportant for our purposes."

[11]"There's actually a bunch of mathematics behind this but we'll just assume that it's a typical martingale."

Theorem: Martingale Representation Theorem/Itô Representation Theorem[12]**.** Suppose $(M_t, \mathcal{F}_t)_{0\leq t\leq T}$ is a square integrable martingale and $\mathcal{F}_t$ is the filtration generated by the Brownian motion W_t. Then, there exists an adapted process $(H_t)_{0\leq t\leq T}$ such that

1. $\mathbb{E}[\int_0^T H_u^2\, du] < \infty$
2. $M_t = M_0 + \int_0^t H_u\, dB_u$ (this item is the key point)

Proof. Stated without proof □

Suppose that for a contingent claim h_T we define the stochastic process $(N_t)_{t\geq 0}$ by

$$N_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}h_T|\mathcal{F}_t] \quad 0 \leq t \leq T$$

Then N_t is a martingale with respect to $(\mathcal{F}_t, \mathbb{Q})$. Why? Use the tower property of conditional expectation to find out!

So, by the Martingale Representation Theorem there exists a process γ_t such that

$$\begin{aligned} N_t &= N_0 + \int_0^t \gamma_s\, dW_s \\ dN_t &= \gamma_t\, dW_t \end{aligned}$$

That is, γ_t is the process by which N_t's movement follows[13], where N_t is the discounted value of the contingent claim at time t. Note however that while we are guaranteed that γ_t exists we aren't given what γ_t is. Regardless, the MRT delivered us the existence of γ_t to build our portfolio process. Take

$$\begin{aligned} H_t^1 &= \frac{\gamma_t e^{rt}}{\sigma S_t^1} \\ H_t^0 &= N_t - \frac{\gamma_t}{\sigma} \end{aligned}$$

and consider the strategy $H^* = (H^0, H^1)$. Given this strategy we have to prove that

Lemma 2. H^* is self financing and $N_t = \overline{V}_t(H^*) = e^{-rt}V_t(H^*)$.

To check that $N_t = \overline{V}_t = e^{-rt}V_t(H^*)$ we can just plug in H^* into our wealth process

$$\begin{aligned} \overline{V}_t(H^*) = e^{-rt}V_t(H^*) &= e^{-rt}\big[H_t^0 S_t^0 + H_t^1 S_t^1\big] \\ &= e^{-rt}\big[\big(N_t - \frac{\gamma_t}{\sigma}\big)S_t^0 + \frac{\gamma_t e^{rt}}{\sigma S_t^1}S_t^1\big] \\ &= \big(N_t - \frac{\gamma_t}{\sigma}\big)e^{-rt}S_t^0 + \frac{\gamma_t}{\sigma} \\ &= N_t - \frac{\gamma_t}{\sigma} + \frac{\gamma_t}{\sigma} \quad (\text{since } e^{-rt}S_t^0 = 1) \\ &= N_t \end{aligned}$$

[12]The Martingale Representation Theorem is a generalized result and is sometimes referred to as the Itô Representation Theorem when discussion Brownian Motion.

[13]This is my interpretation.

So, we have $N_t = \overline{V}_t \iff N_0 + \int_0^t \gamma_u \, dW_u = \overline{V}_0 + \int_0^t \sigma H_u^1 \overline{S}_u^1 \, dW_u$. It should be clear that the integrands γ_t and $\sigma H_t^1 \overline{S}_t^1$ are the same, but lets check using our proposal for H_t^1

$$H_t^1 = \frac{\gamma_t e^{rt}}{\sigma S_t^1} = \frac{\sigma H_t^1 \overline{S}_t^1 e^{rt}}{\sigma S_t^1} = \frac{\sigma H_t^1 S_t^1}{\sigma S_t^1} = H_t^1 \quad \text{as desired}$$

To check whether H^* is self financing we must see that it satisfies the self financing condition

$$\begin{aligned} dV_t(H) &= H_t^0 \, dS_t^0 + H_t^1 \, dS_t^1 \\ &= H_t^0 r S_t^0 \, dt + H_t^1 \big[\mu S_t^1 \, dt + \sigma S_t^1 \, dB_t\big] \end{aligned}$$

That is, movements in wealth come strictly from movements in asset prices. So,

$$\begin{aligned} V_t(H^*) = e^{rt}\overline{V}_t(H^*) &= e^{rt}\big(\overline{V}_0 + \int_0^t \sigma H_u^1 \overline{S}_u^1 \, dW_u\big) \\ &= \overline{V_0} + e^{rt}\int_0^t \sigma H_u^1 \overline{S}_u^1 \, dW_u \end{aligned}$$

Applying Itô's formula with $V_t(H^*) = f(t, \overline{V}_t(H^*)) = e^{rt}\overline{V}_t(H^*) \equiv e^{rt}x$

$$\begin{aligned} f(t,x) = V_t(H^*) &= V_0 + \int_0^t r e^{ru}\overline{V}_u(H^*) \, du + \int_0^t e^{ru} \, d\overline{V}_u \\ \implies dV_t(H^*) &= r e^{rt}\overline{V}_t(H^*) \, dt + e^{rt} \, d\overline{V}_t \\ &= r e^{rt}\Big(e^{-rt}\big[H_t^0 S_t^0 + H_t^1 S_t^1\big]\Big) \, dt + e^{rt}\big[\gamma_t \, dW_t\big] \\ &= r\big[H_t^0 S_t^0 + H_t^1 S_t^1\big] \, dt + e^{rt}\big[\sigma H_t^1 S_t^1 e^{-rt}\big] \, dW_t \\ &\qquad \text{(since the integrands of } \overline{V}_t \text{ and } N_t \text{ are identical)} \\ &= H_t^0 r S_t^0 \, dt + H_t^1 r S_t^1 \, dt + H_t^1 \sigma S_t^1 , dW_t \\ &= H_t^0 r S_t^0 \, dt + H_t^1 \big[r S_t^1 \, dt + \sigma S_t^1 \, dW_t\big] \quad \text{as desired} \end{aligned}$$

We see that we get a SDE identical to the self financing portfolio and by existence & uniqueness we know its the right one to satisfy our needs. Additionally, notice that we have $\mu = r$ in our self financing portfolio using strategy H^*, making the expected return precisely the riskless rate which: In line with our expected return under the martingale measure $\mathbb{Q}$.

Finally, note

$$\begin{aligned} N_t = \overline{V}_t(H^*) &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} h_T | \mathcal{F}_t] \quad 0 \le t \le T \\ \implies \overline{V}_0(H^*) = V_0(H^*) &= \mathbb{E}_{\mathbb{Q}}[e^{-rT} h_T | \mathcal{F}_0] \\ &= \mathbb{E}_{\mathbb{Q}}[e^{-rT} h_T | \mathcal{F}_0] \\ &= e^{-rT} h_T \quad \text{(I think this is intuitive, but I'd like more)} \\ \implies V_T(H^*) = e^{rT}\overline{V}_0(H^*) &= e^{rT} e^{-rT} h_T = h_T \end{aligned}$$

That is, H^* replicates the payoff of the European contingent claim and we require

$$V_0(H^*) = \mathbb{E}_{\mathbb{Q}}[e^{-rT} h_T]$$

initial capital to hedge.

3.1 The Minimal Hedge

By absence of arbitrage type arguments[14] we have that

$$V_0(H^*) = \mathbb{E}_{\mathbb{Q}}[e^{-rT} h_T]$$

is the rational price at time 0 for the European contingent claim h_T. Thus if Φ is any other self financing portfolio process with initial capital x then

$$V_T(\Phi) \geq V_T(H^*) = h_T$$

Thus H^* is the minimal hedge.

3.2 Determining the Process γ_t

Recall that we are guaranteed that the process γ_t exists but we aren't given a recipe how to find out what it is. Thankfully we have ways for doing so. With our future model (Black-Scholes) we can figure out γ_t which the MRT does not explicitly deliver but which our hedging depends heavily on.

[14] Left as an exercise, but maybe it's straightforward since you can consider cases where its $\leq$ and $\geq$.