Assignment 2

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Problem 1. Consider a simple process H associated with a partition $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ such that $H_t = H_{t_i}$ for $t \in [t_i, t_{i+1})$ and H_{t_i} is $\mathcal{F}_{t_i}$-measurable. Prove that the stochastic integral with respect to a standard Brownian motion B defined as

$$I(T) := \int_0^T H_u \, dB_u = \sum_{i=0}^{n-1} H_{t_i} \big(B_{t_{i+1}} - B_{t_i} \big)$$

satisfies $\mathbb{E}[I(T)] = 0$.

Solution 1.

Proof. The first thing we must do is convince ourselves that everything we have is in fact $\mathcal{F}_t$-measurable. From the definition of a filtration we have

$$\mathcal{F}_0 = \mathcal{F}_{t_0} \subseteq \mathcal{F}_{t_1} \subseteq \cdots \mathcal{F}_{t_i} \subseteq \cdots \quad \text{for } 0 = t_0 < t_1 < \cdots < t_i \cdots$$

thus, we see that

$$\begin{aligned} H_{t_i} \in \mathcal{F}_{t_i} \subseteq \mathcal{F}_t &\quad \text{for } t \in [t_i, t_{i+1}), \text{ and} \\ \big(B_{t_{i+1}} - B_{t_i} \big) &\quad \text{is independent of } \mathcal{F} \end{aligned}$$

So,

$$\begin{aligned} \mathbb{E}[I(T)] &= \mathbb{E}\Big[\sum_{i=0}^{n-1} H_{t_i} \big(B_{t_{i+1}} - B_{t_i} \big) \Big] \\ &= \sum_{i=0}^{n-1} \mathbb{E}\Big[H_{t_i} \big(B_{t_{i+1}} - B_{t_i} \big) \Big] \quad \text{(by linearity of expectation)} \\ &= \sum_{i=0}^{n-1} \mathbb{E}\Big[\mathbb{E}\big[H_{t_i} \big(B_{t_{i+1}} - B_{t_i} \big) | \mathcal{F}_t \big] \Big] \quad \text{(by the tower property)} \\ &= \sum_{i=0}^{n-1} \mathbb{E}\Big[H_{t_i} \cdot \mathbb{E}\big[B_{t_{i+1}} - B_{t_i} | \mathcal{F}_t \big] \Big] \quad \text{(since } H_{t_i} \text{ is } \mathcal{F}_t\text{-measurable)} \\ &= \sum_{i=0}^{n-1} \mathbb{E}\Big[H_{t_i} \cdot 0 \Big] = 0 \end{aligned}$$

Where the final step was achieved by realizing that Brownian motion is defined to have independent increments with mean zero. Thus we conclude with the result

$$\mathbb{E}[I(T)] = \mathbb{E}\left(\int_0^T H_u \, dB_u \right) = \mathbb{E}\left(\sum_{i=0}^{n-1} H_{t_i} \left(B_{t_{i+1}} - B_{t_i} \right) \right) = 0$$

as desired. $\square$

Problem 2. Suppose that on a risk-neutral filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ the price of a risky asset at time t is given by the stochastic differential equation

$$S_t = S_0 + \int_0^t r S_u \, du + \int_0^t \sigma S_u \, dW_u$$

for $0 \leq t \leq T$. Use Itô's formula to give a stochastic differential equation satisfied by $\ln(S_t)$.

Solution 2. Since we are looking for the SDE satisfied by $\ln(S_t)$ we will consider $f(x) = \ln x$. We quickly compute our derivatives for use in Itô's formula,

$$f_x(x) = \frac{1}{x} \quad f_{xx}(x) = -\frac{1}{x^2}$$

and from Itô's formula we have

$$f(x) = f(0) + \int_0^t \frac{1}{x} \, dx - \frac{1}{2} \int_0^t \frac{1}{x^2} \, d\langle x \rangle$$

Evaluating x at S_t gives us

$$\ln(S_t) = \ln(S_0) + \int_0^t \frac{1}{S_u} \, dS_u - \frac{1}{2} \int_0^t \frac{1}{S_u^2} \, d\langle S_{(\cdot)} \rangle_u$$

In differential form our SDE for S_t is

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t$$

For an Itô process $Y_t = Y_0 + \int_0^t H \, du + \int_0^t K \, dB_u$ the quadratic variation of the process can be simplified as

$$\langle Y_{(\cdot)} \rangle_t = \left\langle Y_0 + \int_0^{(\cdot)} H \, du + \int_0^{(\cdot)} K \, dB_{(\cdot)} \right\rangle_t = \int_0^t K^2 \, d\langle B_{(\cdot)} \rangle_u = \int_0^t K^2 \, du$$

Thus,

$$d\langle S_{(\cdot)} \rangle_t = (\sigma S_t)^2 \, d\langle W_{(\cdot)} \rangle_t = \sigma^2 S_t^2 \, dt$$

With this we may continue to hack away at our SDE for $\ln(S_t)$,

$$\begin{aligned}
\ln(S_t) &= \ln(S_0) + \int_0^t \frac{1}{S_u}\, dS_u - \frac{1}{2}\int_0^t \frac{1}{S_u^2}\, d\langle S_{(\cdot)}\rangle_u \\
&= \ln(S_0) + \int_0^t \frac{1}{S_u}\big[rS_u\, du + \sigma S_u\, dW_u\big] - \frac{1}{2}\int_0^t \frac{1}{S_u^2}\sigma^2 S_u^2\, du \\
&= \ln(S_0) + \int_0^t \big[r\, du + \sigma\, dW_u\big] - \frac{1}{2}\int_0^t \sigma^2\, du \\
&= \ln(S_0) + \int_0^t \big[r - \frac{1}{2}\sigma^2\big]\, du + \int_0^t \sigma\, dW_u
\end{aligned}$$

or in differential form

$$d\ln(S_t) = \big[r - \frac{1}{2}\sigma^2\big]\, dt + \sigma\, dW_t$$

with initial condition $\ln(S_0)$, and so our task is now complete.

Problem 3. Let W_t be a standard Brownian motion. Use Itô's formula to prove the following:

Problem 3 (a). For a (deterministic) function $h(t)$ with continuous derivative on $[0,\infty)$:

$$\int_0^t h(s)\, dW_s = h(t)W_t - \int_0^t h'(s)W_s\, ds$$

Solution 3 (a).

Proof. If we rearrange our equation so that we have

$$h(t)W_t = \int_0^t h(s)\, dW_s + \int_0^t h'(s)W_s\, ds$$

We see that we now have a natural function $f(t,x)$ for use in Itô's formula, namely, $f(t,x) = h(t)x$. We compute our derivatives

$$f_t(t,x) = h'(t)x \quad f_x(t,x) = h(t) \quad f_{xx}(t,x) = 0$$

and apply Itô's formula

$$\begin{aligned}
f(t,x) &= f(0,0) + \int_0^t f_s(s,x)\, ds + \int_0^t f_x(s,x)\, dx + \frac{1}{2}\int_0^t f_{xx}(t,x)\, d\langle x\rangle \\
&= f(0,0) + \int_0^t h'(s)x\, ds + \int_0^t h(s)\, dx + \frac{1}{2}\int_0^t (0)\, d\langle x\rangle \\
&= f(0,0) + \int_0^t h'(s)x\, ds + \int_0^t h(s)\, dx
\end{aligned}$$

Evaluating x at W_t we get

$$
\begin{aligned}
h(t)W_t &= h(0)W_0 + \int_0^t h'(s)W_s\,ds + \int_0^t h(s)\,dW_s \\
&= \int_0^t h'(s)W_s\,ds + \int_0^t h(s)\,dW_s
\end{aligned}
$$

We see that this is correct, but for completeness we arrange our equation to obtain

$$
\int_0^t h(s)\,dW_s = h(t)W_t - \int_0^t h(s)W_s\,ds
$$

as desired. □

Problem 3 (b).

$$
Z_t = \exp\left(\int_0^t \theta(s)\,ds - \frac{1}{2}\int_0^t \theta(s)^2\,dW_s\right)
$$

satisfies

$$
dZ_t = \theta(t)Z_t\,dW_t
$$

Solution 3 (b).

Proof. Letting $M_t = \int_0^t \theta(s)\,dW_s - \frac{1}{2}\int_0^t \theta(s)^2\,ds$ we see that $Z_t = e^{M_t}$ is now formulated as a SDE for M_t. With $f(t,x) = e^x$, thus $\frac{\partial}{\partial t}f(t,x) = 0$ and $\frac{\partial^n}{\partial x^n}f(t,x) = e^x$, we use Itô's formula

$$
\begin{aligned}
f(t,x) &= f(0,0) + \int_0^t e^x\,dx + \frac{1}{2}\int_0^t e^x\,d\langle x\rangle \\
\implies Z_t &= Z_0 + \int_0^t e^{M_u}\,dM_u + \frac{1}{2}\int_0^t e^{M_u}\,d\langle M_{(\cdot)}\rangle_u \\
&= 1 + \int_0^t Z_u\,dM_u + \frac{1}{2}\int_0^t Z_u\,d\langle M_{(\cdot)}\rangle_u
\end{aligned}
$$

Computing our differential terms

$$
\begin{aligned}
dM_t &= \theta(t)\,dW_t - \frac{1}{2}\theta(t)^2\,dt \\
\langle M_{(\cdot)}\rangle_t &= \left\langle \int_0^{(\cdot)} \theta(u)\,dW_u - \frac{1}{2}\int_0^{(\cdot)} \theta(u)^2\,du \right\rangle_t = \int_0^t \theta(u)^2\,d\langle W_{(\cdot)}\rangle_t = \int_0^t \theta(u)^2\,du \\
\implies d\langle M_{(\cdot)}\rangle_t &= \theta(t)^2\,dt
\end{aligned}
$$

so

$$\begin{aligned}
Z_t &= 1 + \int_0^t Z_u \, dM_u + \frac{1}{2}\int_0^t Z_u \, d\langle M_{(\cdot)}\rangle_u \\
&= 1 + \int_0^t Z_u \Big[\theta(u)\, dW_u - \frac{1}{2}\theta(t)^2 \, du\Big] + \frac{1}{2}\int_0^t Z_u \theta(u)^2 \, du \\
&= 1 + \int_0^t Z_u \theta(u)\, dW_u - \frac{1}{2}\int_0^t Z_u \theta(u)^2 \, du + \frac{1}{2}\int_0^t Z_u \theta(u)^2 \, du \\
&= 1 + \int_0^t Z_u \theta(u)\, dW_u
\end{aligned}$$

In differential form our SDE becomes

$$dZ_t = Z_t \theta(t)\, dW_t \quad \text{with initial condition } Z_0 = 1$$

as desired. □

Problem 3 (c). For $x > 0$ a constant, the process

$$X_t = (x^{1/3} + \frac{1}{3}W_t)^3$$

satisfies the SDE

$$dX_t = \frac{1}{3}X_t^{1/3}\, dt + X_t^{2/3}\, dW_t$$

Solution 3 (c).

Proof. Consider $f(t, y) = (x + \frac{1}{3}y)^3$, for constant $x > 0$, then our derivatives are

$$f_t(t, y) = 0 \quad f_y(t, y) = (x^{1/3} + \frac{1}{3}y)^2 \quad f_{yy}(t, y) = \frac{1}{3}(x^{1/3} + \frac{1}{3}y)$$

and so by Itô's formula we have

$$f(t, y) = f(0, 0) + \int_0^t (x^{1/3} + \frac{1}{3}y)^2 \, dy + \int_0^t \frac{1}{3}(x^{1/3} + \frac{1}{3}y)\, d\langle y\rangle$$

Substituting y for B_t we get

$$\begin{aligned}
X_t &= X_0 + \int_0^t \big(x^{1/3} + \frac{1}{3}W_u\big)^2 \, dW_u + \int_0^t \frac{1}{3}\big(x^{1/3} + \frac{1}{3}W_u\big)\, d\langle W_{(\cdot)}\rangle_u \\
&= (x^{1/3})^3 + \int_0^t \big(x^{1/3} + \frac{1}{3}W_u\big)^2 \, dW_u + \frac{1}{3}\int_0^t \big(x^{1/3} + \frac{1}{3}W_u\big)\, du
\end{aligned}$$

and in differential form we have the SDE

$$dX_t = (x^{1/3} + \frac{1}{3}W_t)^2\, dW_t + \frac{1}{3}(x^{1/3} + \frac{1}{3}W_t)\, dt$$
$$dX_t = X_t^{2/3}\, dW_t + \frac{1}{3}X_t^{1/3}\, dt \quad \text{with initial condition } X_0 = x$$

as desired. □

Problem 5. Consider the process X_t given by the SDE

$$dX_t = -X_t\, dt + e^{-t}\, dB_t$$

with $X_0 = 0$ and B_t standard Brownian motion. Show that

$$\mathbb{E}[X_t] = 0$$

and

$$\mathrm{Var}[X_t] = te^{-2t}$$

by solving ODEs for $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$.

Solution 5. In integral form our process X_t is given by the SDE

$$X_t = X_0 - \int_0^t X_u\, du + \int_0^t e^{-u}\, dB_u$$

Taking the expectation

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}\Big[0 - \int_0^t X_u\, du + \int_0^t e^{-u}\, dB_u\Big] \\
&= -\mathbb{E}\Big[\int_0^t X_u\, du\Big] + \mathbb{E}\Big[\int_0^t e^{-u}\, dB_u\Big] \quad \text{(by linearity)} \\
&= -\mathbb{E}\Big[\int_0^t X_u\, du\Big]
\end{aligned}$$

Where the final line was achieved by realizing that e^{-u} is $\mathcal{F}_t$-measurable, permitting us to apply the theorem verified in Problem 1. Our goal is to create a differential equation with $\mathbb{E}[X_t]$ as our function. We see that we are remarkably close to doing so if only there was a way to swap the expectation and integration operations. Fortunately we have Fubini's theorem in our toolbox:

$$\begin{aligned}
\mathbb{E}\Big[\int_0^t X_u\, du\Big] &= \int_\Omega \int_0^t X_u(\omega)\, du\, d\mathbb{P}(\omega) \\
&= \int_0^t \int_\Omega X_u(\omega)\, d\mathbb{P}(\omega)\, du \\
&= \int_0^t \mathbb{E}[X_u]\, du
\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}[X_t] &= -\mathbb{E}\Big[\int_0^t X_u\,du\Big]\\ &= -\int_0^t \mathbb{E}[X_u]\,du\end{aligned}$$

And now we see that we have created a natural ODE. Letting $\phi(t) = \mathbb{E}[X_t]$

$$\begin{aligned}\phi(t) &= -\int_0^t \phi(u)\,du\\ \implies d\phi(t) &= -\phi(t)\,dt\\ \implies \frac{d\phi(t)}{\phi(t)} &= -\,dt\\ \implies \ln\phi(t) &= -t + k\\ \implies \phi(t) &= e^{-t+k} = Ce^{-t}\end{aligned}$$

Using our initial condition $X_0 = 0 = \mathbb{E}[X_0] = \phi(0)$

$$\begin{aligned}\phi(0) &= 0 = Ce^{-t}\\ \implies C &= 0\end{aligned}$$

and so we're left with the conclusion

$$\phi(t) = \mathbb{E}[X_t] = 0\cdot e^{-t} = 0$$

as desired. For $\mathbb{E}[X_t^2]$, we lean on Itô's formula noting that since we are interested in X_t^2 we should consider $f(x) = x^2$. With our derivatives $f_x(x) = 2x,\ f_{xx}(x) = 2$ we apply Itô's formula

$$X_t^2 = X_0 + \int_0^t 2X_u\,dX_u + \frac{1}{2}\int_0^t 2\,d\langle X_{(\cdot)}\rangle_u$$

Since we were given dX_t in the question we quickly compute $d\langle X_{(\cdot)}\rangle_t$

$$\begin{aligned}dX_t &= -X_t\,dt + e^{-t}\,dB_t\\ d\langle X_{(\cdot)}\rangle_t &= \left(e^{-t}\right)^2 d\langle B_{(\cdot)}\rangle_t = e^{-2t}dt\end{aligned}$$

Thus,

$$
\begin{aligned}
X_t^2 &= X_0 + \int_0^t 2X_u\, dX_u + \frac{1}{2}\int_0^t 2\, d\langle X_{(\cdot)}\rangle_u \\
&= 0 + \int_0^t 2X_u\Big[-X_u\, du + e^{-u}\, dB_u\Big] + \int_0^t e^{-2u}\, du \\
&= -\int_0^t 2X_u^2\, du + 2\int_0^t e^{-u}X_u\, dB_u + \int_0^t e^{-2u}\, du \\
\implies \mathbb{E}[X_t^2] &= \mathbb{E}\left[-\int_0^t 2X_u^2\, du + 2\int_0^t e^{-u}X_u\, dB_u + \int_0^t e^{-2u}\, du\right] \\
&= -2E\left[\int_0^t X_u^2\, du\right] + 2\mathbb{E}\left[\int_0^t e^{-u}X_u\, dB_u\right] + \mathbb{E}\left[\int_0^t e^{-2u}\, du\right] \quad \text{(by linearity)} \\
&= -2\mathbb{E}\left[\int_0^t X_u^2\, du\right] + 0 + \mathbb{E}\left[\int_0^t e^{-2u}\, du\right] \quad \text{(from Problem 1)} \\
&= -2\mathbb{E}\left[\int_0^t X_u^2\, du\right] + \mathbb{E}\Big[-\frac{1}{2}e^{-2t}\Big] \\
&= -2\mathbb{E}\left[\int_0^t X_u^2\, du\right] - \frac{1}{2}e^{-2t}
\end{aligned}
$$

Where the last line was achieved by realizing that $-\frac{1}{2}e^{-2t}$ is deterministic in t. Once again we apply Fubini's Theorem on the remaining expectation,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t X_u^2\, du\right] &= \int_\Omega \int_0^t X_u^2\, du\, d\mathbb{P}(\omega) \\
&= \int_0^t \int_\Omega X_u^2\, d\mathbb{P}(\omega)\, du \\
&= \int_0^t \mathbb{E}\Big[X_u^2\Big]\, du
\end{aligned}
$$

From this we see a natural ODE for $\mathbb{E}\big[X_t^2\big]$ emerge. Letting $\psi(t) = \mathbb{E}\big[X_t^2\big]$

$$
\begin{aligned}
\psi(t) &= -2\int_0^t \psi(u)\, du - \frac{1}{2}e^{-2t} \\
d\psi(t) &= \big[e^{-2t} - 2\psi(t)\big]\, dt \\
\psi'(t) &= e^{-2t} - 2\psi(t)
\end{aligned}
$$

and so we go about solving our ODE in the typical manner

$$\begin{aligned}
e^{2t}\psi'(t) + 2e^{2t}\psi(t) &= 1 \\
\frac{d}{dt}\Big[e^{2t}\psi(t)\Big] &= 1 \\
\int \frac{d}{dt}\Big[e^{2t}\psi(t)\Big]\,dt &= \int 1\,dt \\
e^{2t}\psi(t) &= t + C \\
\psi(t) &= te^{-2t} + Ce^{-2t}
\end{aligned}$$

Using our initial condition $X_0 = 0 \iff X_0^2 = 0 \implies \mathbb{E}[X_0^2] = \psi(0) = 0$

$$\begin{aligned}
&\psi(0) = 0 = 0 \cdot e^{-2\cdot 0} + Ce^{-2\cdot 0} \\
&\implies C = 0 \\
&\implies \psi(t) = te^{-2t}
\end{aligned}$$

Thus we conclude with

$$\mathbb{E}[X_t^2] = \psi(t) = te^{-2t}$$

as desired.

Problem 6. Recall that stochastic integrals

$$\int_0^T H_u\,dB_u$$

are martingales provided that the integrand H is adapted and satisfies some technical (integrability) conditions. Using Iô's formula find a process X_t such that

$$B_t^3 - X_t$$

is a martingale.

Solution 6. With $f(t, x, y) = x^3 - y$ we take our derivatives

$$\begin{aligned}
&f_t(t,x,y) = 0 \\
&f_x(t,x,y) = 3x^2 \quad f_{xx}(t,x,y) = 6x \\
&f_y(t,x,y) = -1 \quad f_{yy}(t,x,y) = 0 \\
&f_{xy}(t,x,y) = 0
\end{aligned}$$

then, by Itô's formula evaluating $x = B_t$ and $y = X_t$, we have

$$\begin{aligned}
B_t^3 - X_t &= B_0^3 - X_0 + \int_0^t 3B_u^2\,dB_u + \int_0^t (-1)\,dX_u + \frac{1}{2}\int_0^t 6B_u\,d\langle B_{(\cdot)}\rangle_u \\
&= B_0^3 - X_0 + \int_0^t 3B_u^2\,dB_u - \int_0^t dX_u + \int_0^t 3B_u\,du
\end{aligned}$$

where the last line was achieved from recognizing that the quadratic variation of Brownian motion $\langle B_{(\cdot)} \rangle_t = dt$. Notice that without $-X_t$ we would be left with $B_t^3 = B_0^3 + \int_0^t 3B_u^2 \, dB_u + \int_0^t 3B_u \, du$ showing us that we may have reason to believe that B_t^3 is not a martingale due to the appearance of the drift term $3\int_0^t B_u \, du$. We later confirm this hypothesis, but for the time being we propose that $-X_t$ be some process that evaluates in a such a way to annihilate this drift. To achieve this annihilation it's immediately obvious that we must set

$$\int_0^t dX_u = \int_0^t 3B_u \, du \quad \text{and} \quad X_0 = x_0 \in \mathbb{R}$$

Thus

$$\begin{aligned} X_t &= \int_0^t 3B_u \, du \quad \text{and} \quad X_0 = x_0 \\ X_t &= 3tB_t \quad \text{and} \quad X_0 = x_0 \end{aligned}$$

is our process making $B_t^3 - X_t$ a martingale. To verify we take an expectation

$$\begin{aligned} \mathbb{E}\Big[B_t^3 - 3tB_t \big| \mathcal{F}_s\Big] &= \mathbb{E}\Big[B_t^3 \big| \mathcal{F}_s\Big] - 3\mathbb{E}\Big[tB_t \big| \mathcal{F}_s\Big] \quad \text{(by linearity)} \\ &= \mathbb{E}\Big[B_t^3 \big| \mathcal{F}_s\Big] - 3t\mathbb{E}\Big[B_t \big| \mathcal{F}_s\Big] \quad \text{(since } t \text{ is deterministic)} \\ &= \mathbb{E}\Big[\big(B_t - B_s + B_s\big)^3 \big| \mathcal{F}_s\Big] - 3tB_s \quad \text{(since } B_t \text{ is a martingale)} \end{aligned}$$

Our strategy is to expand the cubic term inside the remaining expectation in such a way that we are only left with either independent increments of $B_t - B_s$ or isolated $\mathcal{F}_s$ measurable Brownian motions B_s. So,

$$\begin{aligned} (B_t - B_s + B_s)^3 &= ((B_t - B_s) + B_s)^3 \quad \text{and recalling the binomial expansion} \\ &= (B_t - B_s)^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + B_s^3 \end{aligned}$$

Again applying linearity of expectation and recognizing that B_s, B_s^2, B_s^3 are $\mathcal{F}_s$-measurable we simplify our expectation to

$$\begin{aligned} \mathbb{E}\Big[\big(B_t - B_s + B_s\big)^3 \big| \mathcal{F}_s\Big] = \mathbb{E}\Big[(B_t - B_s)^3 \big| \mathcal{F}_s\Big] + 3B_s\mathbb{E}\Big[(B_t - B_s)^2 \big| \mathcal{F}_s\Big] + 3B_s^2\mathbb{E}\Big[(B_t - B_s) \big| \mathcal{F}_s\Big] \\ + B_s^3 \end{aligned}$$

Using the lemma

$$\mathbb{E}[(B_t - B_s)^m] = \begin{cases} 0 & \text{if } m \text{ odd} \\ 1 \cdot 3 \cdot \dots \cdot (m-3) \cdot (m-1) \cdot (t-s)^{m/2} & \text{if } m \text{ even} \end{cases}$$

we see that

$$\begin{aligned} \mathbb{E}\Big[\big(B_t - B_s + B_s\big)^3 \big| \mathcal{F}_s\Big] &= \mathbb{E}\Big[(B_t - B_s)^3 \big| \mathcal{F}_s\Big] + 3B_s\mathbb{E}\Big[(B_t - B_s)^2 \big| \mathcal{F}_s\Big] + 3B_s^2\mathbb{E}\Big[(B_t - B_s) \big| \mathcal{F}_s\Big] \\ &\quad + B_s^3 \\ &= 0 + 3B_s(t-s) + 0 + B_s^3 \\ &= 3tB_s - 3sB_s + B_s^3 \end{aligned}$$

This convinces us that B_t^3 is indeed not a martingale alone since we have just shown that $\mathbb{E}[B_t^3|\mathcal{F}_s] \neq B_s^3$. Placing our ingredients together we get

$$\begin{aligned}\mathbb{E}\Big[B_t^3 - 3tB_t\big|\mathcal{F}_s\Big] &= \big[3tB_s - 3sB_s + B_s^3\big] - 3tB_s \\ &= B_s^3 - 3sB_s\end{aligned}$$

as desired.

Problem 7. In the continuous time Black-Scholes model prove the put-call parity relationship

$$P(t,T,S,K) = C(t,T,S,K) + e^{-r(T-t)}K - S_t$$

between the price at time t of a European call option, denoted $C(t,T,S,K)$, and the price of a European put option, denoted $P(t,T,S,K)$, with common strike price K and maturity T.

Solution 7.

Proof. Assume not. That is, assume

$$P_t \neq C_t + e^{-r(T-t)}K - S_t$$

We first consider the case $P_t < C_t + e^{-r(T-t)}K - S_t$ and build the strategy

Action at time $= t$	Cash Flow
Long 1 put	$-P_t$
Short 1 call	$+C_t$
Long underlying asset	$-S_t$
Borrow at risk free rate	$P_t - C_t + S_t$
Net	0

Where we have exactly funded our long positions with the proceeds from our short position and borrowing. Note that our borrowing at the riskless rate is $P_t - C_t + S_t < e^{-r(T-t)}K$ by assumption. We see that at maturity either one of two cases will occur

Action at time $= T$	Cash flow if $S_T > K$	Cash flow if $K > S_T$
Put payoff	0	$+K - S_T$
Call payoff	$-(S_T - K)$	0
Sell asset	$+S_T$	$+S_T$
Return funds	$-(P_T - C_T + S_T)e^{r(T-t)}$	$-(P_T - C_T + S_T)e^{r(T-t)}$
Net	$+K - (P_T - C_T + S_T)e^{r(T-t)}$	$+K - (P_T - C_T + S_T)e^{r(T-t)}$

We note that in either case the net cash flow $K - (P_T - C_T + S_T)e^{-r(T-t)} > 0$ since

$$P_T - C_T + S_T < e^{-r(T-t)}K \iff (P_T - C_T + S_T)e^{r(T-t)} < K$$

Thus we have managed to construct a risk-neutral portfolio with returns exceeding the risk free rate. Contradiction! The Black-Scholes model assumes that there may not be arbitrage in the market. Thus by the no-arbitrage assumption we are forced to conclude that our consideration for $P_t < C_t + e^{-r(T-t)}K - S_t$ is false. We now consider the case $P_t > C_t + e^{-r(T-t)}K - S_t$ and build the strategy

Action at time $= t$	Cash Flow
Short 1 put	$+P_t$
Long 1 call	$-C_t$
Short underlying asset	$-S_t$
Invest at risk free rate	$-P_t + C_t - S_t$
Net	0

Where we have exactly funded our long position and investment with the proceeds from our short positions. Note that our investment at the riskless rate is $-P_t+C_t-S_t > e^{-r(T-t)}K$ by assumption. We see that at maturity either one of two cases will occur

Action at time $= T$	Cash flow if $S_T > K$	Cash flow if $K > S_T$
Put payoff	0	$-(K - S_T)$
Call payoff	$+S_T - K$	0
Return asset	$-S_T$	$-S_T$
Receive funds	$(P_T - C_T + S_T)e^{r(T-t)}$	$(P_T - C_T + S_T)e^{r(T-t)}$
Net	$(P_T - C_T + S_T)e^{r(T-t)} - K$	$(P_T - C_T + S_T)e^{r(T-t)} - K$

We note that in either case the net cash flow $(P_T - C_T + S_T)e^{-r(T-t)} - K > 0$ since

$$P_T - C_T + S_T > e^{-r(T-t)}K \iff (P_T - C_T + S_T)e^{r(T-t)} > K$$

Thus we have managed to construct a risk-neutral portfolio with returns exceeding the risk free rate. Contradiction! The Black-Scholes model assumes that there may not be arbitrage in the market. Thus by the no-arbitrage assumption we are forced to conclude that our consideration for $P_t > C_t + e^{-r(T-t)}K - S_t$ is false. Since we have determined that $P_t < C_t + e^{-r(T-t)}K - S_t$ cannot be true and that $P_t > C_t + e^{-r(T-t)}K - S_t$ cannot be true we are forced to reject the initial assumption that $P_t \neq C_t + e^{-r(T-t)}K - S_t$ and conclude that

$$P_t = C_t + e^{-r(T-t)}K - S_t$$

as desired. $\square$

Appendix A The Long & Hard Way of Deriving Put-Call Parity

I only left this section in for my own future reference. Feel free to ignore.

Proof. We will first derive the Black-Scholes price of a European put option on an underlying asset process S, strike K, and expiry at time T. That is, a European put option with payoff $h_T = (K - S_T)^+$ at time T.

By the risk neutral pricing formula we have that

$$\begin{aligned}
P_t(S_t) &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(K - S_T)^+ | \mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(K - S_T) \cdot \mathbb{1}_{\{K>S_T\}} | \mathcal{F}_t] \quad \text{and, by linearity we have} \\
&= Ke^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{K>S_T\}} | \mathcal{F}_t] - \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{K>S_T\}} | \mathcal{F}_t]
\end{aligned}$$

And so it is now our task to determine the expectations $\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t]$ and $E_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t]$. We consider first the expectation $\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t]$, noting that

$$\begin{aligned}
K > S_T &\implies K > S_t \exp\left[(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\right] \\
&\implies \log(K) > \log(S_t) + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t) \\
&\implies \frac{-\log(\frac{S_t}{K}) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma} > W_T - W_t
\end{aligned}$$

For brevity let $-Y_t = \frac{-\log(\frac{S_t}{K}) - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma}$, then we may rewrite our problem as

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{-Y_t>W_T-W_t\}}|\mathcal{F}_t]$$

Since $-Y_t$ is $\mathcal{F}_t$-measurable and the Brownian increment $W_T - W_t$ is independent of our filtration we may use the result that permits us to write

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{-Y_t>W_T-W_t\}}|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{-Y_t>W_T-W_t\}}]$$

and thus we have reduced our problem to a simple problem of integration with the normal distribution function. That is, since the increment $W_T - W_t \sim N(0, T-t)$ we have

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{-Y_t>W_T-W_t\}}] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{-Y_t} e^{-\frac{1}{2(T-t)}z^2}\, dz$$

We use the substitution

$$\begin{aligned}
u = \frac{z}{\sqrt{T-t}} &\implies du = \frac{dz}{\sqrt{T-t}} \\
\therefore -d_2 := u(-Y_t) &= \frac{-Y_t}{\sqrt{T-t}}
\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{-d_2>W_T-W_t\}}] &= \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{-d_2} e^{-\frac{1}{2(T-t)}z^2}\,dz \\ &= \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{-d_2} e^{-\frac{1}{2(T-t)}(u\sqrt{T-t})^2}\sqrt{T-t}\,du \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_2} e^{-\frac{1}{2}u^2}\,du \\ &= \Phi[-d_2]\end{aligned}$$

as expected. We now consider the expectation $\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T\cdot\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t]$. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T\cdot\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}S_0\exp\big[(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)\big]\mathbb{1}_{\{K>S_T\}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[S_0\exp\big[-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)\big]\mathbb{1}_{\{K>S_T\}}\middle|\mathcal{F}_t\right]\end{aligned}$$

Once again letting $-Y_t = \frac{-\log(\frac{S_t}{K})-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma}$ we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T\cdot\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}\left[S_t\exp\big[-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)\big]\mathbb{1}_{\{K>S_T\}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[S_t\exp\big[-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T-W_t)\big]\mathbb{1}_{\{-Y_t>W_T-W_t\}}\middle|\mathcal{F}_t\right] \\ &= S_te^{-\frac{1}{2}\sigma^2(T-t)}\mathbb{E}_{\mathbb{Q}}\left[e^{\sigma(W_T-W_t)}\mathbb{1}_{\{-Y_t>W_T-W_t\}}\middle|\mathcal{F}_t\right]\end{aligned}$$

We again note that our expectation contains W_T-W_t and a function of W_T-W_t both of which are random variables independent of our filtration, and $-Y_t$ which is $\mathcal{F}_t$-measurable. Thus we write

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T\cdot\mathbb{1}_{\{K>S_T\}}|\mathcal{F}_t] &= S_te^{-\frac{1}{2}\sigma^2(T-t)}\mathbb{E}_{\mathbb{Q}}\left[e^{\sigma(W_T-W_t)}\mathbb{1}_{\{-Y_t>W_T-W_t\}}\middle|\mathcal{F}_t\right] \\ &= S_te^{-\frac{1}{2}\sigma^2(T-t)}\mathbb{E}_{\mathbb{Q}}\left[e^{\sigma(W_T-W_t)}\mathbb{1}_{\{-Y_t>W_T-W_t\}}\right] \\ &= \frac{S_te^{-\frac{1}{2}\sigma^2(T-t)}}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{-Y_t} e^{\sigma z}e^{-\frac{1}{2(T-t)}z^2}\,dz\end{aligned}$$

Performing the substitution

$$u = \frac{z}{\sqrt{T-t}} \implies dz = \sqrt{T-t}\,du$$

$$\therefore -d_2 := u(-Y_t) = \frac{-Y_t}{\sqrt{T-t}}$$

Thus

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{K>S_T\}}] &= \frac{S_t e^{-\frac{1}{2}\sigma^2(T-t)}}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{-d_2} e^{\sigma u\sqrt{T-t}} e^{-\frac{1}{2(T-t)}(u\sqrt{T-t})^2} \sqrt{T-t}\, du \\
&= \frac{S_t e^{-\frac{1}{2}\sigma^2(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{\sigma u\sqrt{T-t}} e^{-\frac{1}{2}u^2}\, du \\
&= \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-\frac{1}{2}\sigma^2(T-t)+\sigma u\sqrt{T-t}-\frac{1}{2}u^2}\, du
\end{aligned}$$

We then recognize that the exponentiated term in the integrand is conveniently a perfect square

$$-\frac{1}{2}\sigma^2(T-t) + \sigma u\sqrt{T-t} - \frac{1}{2}u^2 = -\frac{1}{2}(\sigma\sqrt{T-t} - u)^2$$

So we have

$$\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{K>S_T\}}] = \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-\frac{1}{2}(\sigma\sqrt{T-t}-u)^2}\, du$$

Performing another substitution

$$\begin{aligned}
v = u - \sigma\sqrt{T-t} &\implies dv = du \\
\therefore -d_1 := v(-d_2) &= -d_2 - \sigma\sqrt{T-t}
\end{aligned}$$

We are left with

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{K>S_T\}}] &= \frac{S_t}{\sqrt{2\pi}} \int_{-\infty}^{-d_1} e^{-\frac{1}{2}v^2}\, dv \\
&= S_t\Phi[-d_1]
\end{aligned}$$

Finally, we conclude that the Black-Scholes price of a European-style put option on an underlying asset process S, strike K, and expiry at time T is

$$\begin{aligned}
P_t(S_t) &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(K-S_T)^+] \\
&= Ke^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{K>S_T\}}] - \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{K>S_T\}}] \\
&= Ke^{-r(T-t)}\Phi[-d_2] - S_t\Phi[-d_1]
\end{aligned}$$

as expected, with $\Phi(x)$ the normal cumulative distribution function. The second component in proving the put-call parity relationship is to retrieve the Black-Scholes price of a European call option on an underlying asset process S, strike K, and expiry at time T. Fortunately this has already been done by us (in class) and is

$$\begin{aligned}
C_t(S_t) &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S_T-K)^+|\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}S_T \cdot \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] - Ke^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= S_t\Phi[d_1] - Ke^{-r(T-t)}\Phi[d_2]
\end{aligned}$$

With all our ingredients ready we finally tackle the put-call parity equation:

$$\begin{aligned}
P(t,T,S,K) &= C(t,T,S,K) + e^{-r(T-t)}K - S_t \\
Ke^{-r(T-t)}\Phi[-d_2] - S_t\Phi[-d_1] &= S_t\Phi[d_1] - Ke^{-r(T-t)}\Phi[d_2] + e^{-r(T-t)}K - S_t \\
e^{-r(T-t)}K\big(\Phi[-d_2] + \Phi[d_2]\big) &= S_t\big(\Phi[d_1] + \Phi[-d_1]\big) + e^{-r(T-t)}K - S_t
\end{aligned}$$

Recalling the properties of the normal distribution function, $\Phi(-x) = 1 - \Phi(x)$ we get

$$\begin{aligned}
e^{-r(T-t)}K\big(1 - \Phi[d_2] + \Phi[d_2]\big) &= S_t\big(\Phi[d_1] + 1 - \Phi[d_1]\big) + e^{-r(T-t)}K - S_t \\
e^{-r(T-t)}K &= S_t + e^{-r(T-t)}K - S_t \\
e^{-r(T-t)}K - S_t &= e^{-r(T-t)}K - S_t
\end{aligned}$$

and we see that this is now trivially true and get our result as desired. $\square$