Mathematical & Computational Finance II
Lecture Notes

Introduction to Stochastic Calculus

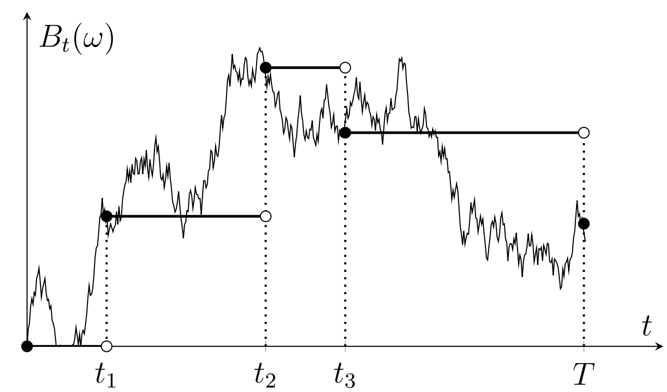
October 1 2015
Last update: December 4, 2017

1 Itô Calculus

Last time we claimed

$$\int_0^t B_u \, dB_u = \frac{1}{2} B_t^2 - \frac{t}{2}$$

Proof. To show this we will set up a sequence of simple functions $H^n \longrightarrow B \in \mathcal{H}_T$ and split our Brownian Motion into n intervals.



Define

$$H^n(u) = \begin{cases} B(0) & \text{if } 0 \leq u < \frac{T}{n} \\ B(\frac{T}{n}) & \text{if } \frac{T}{n} \leq u < 2\frac{T}{n} \\ \vdots & \vdots \\ B(k\frac{T}{n}) & \text{if } k\frac{T}{n} \leq u < (k+1)\frac{T}{n} \\ \vdots & \vdots \\ B((n-1)\frac{T}{n}) & \text{if } (n-1)\frac{T}{n} \leq u < T \end{cases}$$

Then, H^n is a sequence of simple processes in H_T such that $H^n \longrightarrow B$. Using the definition and letting $B_k = B(k\frac{T}{n})$

$$\implies \int_0^T H^n(u)\,dB(u) = \sum_{k=0}^{n-1} B_k(B_{k+1} - B_k)$$

Doing some algebra we note that

$$\begin{aligned}
\frac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2 &= \frac{1}{2}\sum_{k=0}^{n-1}(B_{k+1}^2 - 2B_kB_{k+1} + B_k^2) \\
&= \frac{1}{2}B_n^2 + \frac{1}{2}\sum_{k=0}^{n-1}B_k^2 - \sum_{k=0}^{n-1}B_kB_{k+1} + \frac{1}{2}\sum_{k=0}^{n-1}B_k^2 \\
&= \frac{1}{2}B_n^2 - \sum_{k=0}^{n-1}B_k(B_{k+1} - B_k) \\
\implies \sum_{k=0}^{n-1}B_k(B_{k+1} - B_k) &= \frac{1}{2}B_n^2 - \sum_{k=0}^{n-1}(B_{k+1} - B_k)^2 \\
\iff \int_0^T H^n(u)\,dB(u) &= \frac{1}{2}B(T)^2 - \sum_{k=0}^{n-1}(B_{k+1} - B_k)^2
\end{aligned}$$

Then, to show that

$$\int_0^T B(u)\,dB(u) = \frac{1}{2}B(T)^2 - \frac{T}{2}$$

as we have claimed we will consider the limit in L^2.

$$\begin{aligned}
D &= \left\| \int_0^T H^n(u)\,dB(u) - \Big(\frac{1}{2}B(T)^2 - \frac{1}{2}T\Big) \right\|_2^2 \\
&= \left\| \left(\Big[\frac{1}{2}B(T)^2 - \frac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2\Big] - \Big[\frac{1}{2}B(T)^2 - \frac{1}{2}T\Big] \right) \right\|_2^2 \quad \text{as } n \longrightarrow \infty
\end{aligned}$$

We can simplify this a little by noting

$$\begin{aligned}
D &= \left\| \left(\frac{1}{2}\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2 - \frac{1}{2}T \right) \right\|_2^2 \\
&= \mathbb{E}\left[\frac{1}{4}\left(\sum_{k=0}^{n-1}(B_{k+1} - B_k)^2 - T \right)^2 \right]
\end{aligned}$$

which resembles the form we saw when looking at quadratic variation of Brownian motion. Before showing that $D \longrightarrow 0$ we will need the assistance of a few preliminary results.

Lemma 1. If $s \leq t$

$$\mathbb{E}[(B_t - B_s)^m] = \begin{cases} 0 & \text{if } m \text{ odd} \\ 1 \cdot 3 \cdots (m-3) \cdot (m-1)(t-s)^{m/2} & \text{if } m \text{ even} \end{cases}$$

Proof. Note that $(B_t - B_s) \sim N(0, (t-s))$, therefore the claim is clearly true for $m = 1, 2$ (first and second moments of a normal random variable). If X is a normal random variable with mean 0 and variance σ^2 then

$$\mathbb{E}[X^m] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^m e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx$$

If m is odd then we have

$$\begin{aligned}
\mathbb{E}[X^m] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^m e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \lim_{k\to\infty} \left(\int_{-k}^{0} x^m e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx + \int_{0}^{k} x^m e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx \right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \lim_{k\to\infty} \left(-\int_{0}^{k} v^m e^{-\frac{1}{2}\frac{v^2}{\sigma^2}}\, dv + \int_{0}^{k} x^m e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx \right) \quad (\text{let } v = -x \text{ in the first integral}) \\
&= 0 \quad (\text{realising that } v \text{ and } x \text{ in the integrands are dummy variables})
\end{aligned}$$

If m is even then let $m = 2n$ for $n = 1, 2, \cdots$, and we have

$$\begin{aligned}
\mathbb{E}[X^{2n}] &= \frac{1}{\sigma\sqrt{2\pi}} \lim_{k\to\infty} \left(\int_{-k}^{0} x^{2n} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx + \int_{0}^{k} x^{2n} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx \right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \lim_{k\to\infty} \left(\int_{0}^{k} v^{2n} e^{-\frac{1}{2}\frac{v^2}{\sigma^2}}\, dx + \int_{0}^{k} x^{2n} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx \right) \\
&= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \lim_{k\to\infty} \left(\int_{0}^{k} v^{2n} e^{-\frac{1}{2}\frac{v^2}{\sigma^2}}\, dv \right) \\
&= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \lim_{k\to\infty} \left(\int_{0}^{k} (\sigma u)^{2n} e^{-\frac{1}{2}u^2} \sigma\, du \right) \quad (\text{letting } u(v) = v/\sigma) \\
&= \sigma^{2n}\sqrt{\frac{2}{\pi}} \lim_{k\to\infty} \left(\int_{0}^{k/\sigma} u^{2n} e^{-\frac{1}{2}u^2}\, du \right) \\
&= \sigma^{2n}\sqrt{\frac{2}{\pi}} \lim_{k\to\infty} \left(2^{n-\frac{1}{2}} \int_{0}^{k^2/2\sigma^2} w^{n-\frac{1}{2}} e^{-w}\, dw \right) \quad (\text{letting } w(u) = \frac{u^2}{2}) \\
&= \sigma^{2n}\sqrt{\frac{2}{\pi}} 2^{n-\frac{1}{2}} \Gamma\Big(n + \frac{1}{2}\Big) \quad \Big(\text{where } \Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x}\, dx, \alpha > 0\Big) \\
&= \sigma^{2n} \frac{2^n}{\sqrt{\pi}} \Gamma\Big(n + \frac{1}{2}\Big)
\end{aligned}$$

Note that

$$
\begin{aligned}
\Gamma\Big(n+\frac{1}{2}\Big) &= \Big(n-\frac{1}{2}\Big)\cdot\Big(n-\frac{3}{2}\Big)\cdots\Big(\frac{3}{2}\Big)\cdot\Big(\frac{1}{2}\Big)\cdot\Gamma\Big(\frac{1}{2}\Big)\\
&= \Big(n-\frac{1}{2}\Big)\cdot\Big(n-\frac{3}{2}\Big)\cdots\Big(\frac{3}{2}\Big)\cdot\Big(\frac{1}{2}\Big)\cdot\sqrt{\pi}\\
&= \Big[\prod_{i=1}^{n}\Big(n-\frac{2i-1}{2}\Big)\Big]\sqrt{\pi}
\end{aligned}
$$

Substituting this result back into the expression for $\mathbb{E}[X^{2n}]$ yields

$$
\begin{aligned}
\mathbb{E}[X^{2n}] &= \sigma^{2n}\frac{2^n}{\sqrt{\pi}}\Big[\prod_{i=1}^{n}\Big(n-\frac{2i-1}{2}\Big)\Big]\sqrt{\pi}\\
&= \sigma^{2n}\Big[\prod_{i=1}^{n}\Big(2n-2i+1\Big)\Big]\\
&= [\sigma^2]^n\cdot 1\cdot 3\cdots(2n-3)\cdot(2n-1)
\end{aligned}
$$

Substituting $X = B_t - B_s$ we have $\sigma^2 = (t-s)$ and obtain the result as desired. □

Lemma 2. Suppose $\pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ is a partition of $[0, T]$. Then,

$$
\mathbb{E}\Bigg[\bigg(\Big[\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2\Big] - T\bigg)^2\Bigg] = 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2
$$

Proof. Note that

$$
\begin{aligned}
\bigg(\Big[\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2\Big] - T\bigg)^2 &= \sum_{j=0}^{n-1}(B_{t_{j+1}-B_{t_j}})^4 - 2T\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2\\
&\quad + \sum_{j=0}^{n-1}\sum_{\substack{i=0\\ i\neq j}}^{n=1}(B_{t_{j+1}-B_{t_j}})^2(B_{t_{i+1}-B_{t_i}})^2 + T^2
\end{aligned}
$$

But by Lemma 1 we have

$$
\begin{aligned}
\mathbb{E}[(B_{t_{j+1}-B_{t_j}})^4] &= 3(t_{j+1}-t_j)^2\\
\mathbb{E}[(B_{t_{j+1}-B_{t_j}})^2] &= t_{j+1}-t_j
\end{aligned}
$$

and by independence, for $i \neq j$

$$
\begin{aligned}
\mathbb{E}[(B_{t_{j+1}-B_{t_j}})^2(B_{t_{i+1}-B_{t_i}})^2] &= \mathbb{E}[(B_{t_{j+1}-B_{t_j}})^2]\mathbb{E}[(B_{t_{u+1}-B_{t_i}})^2]\\
&= (t_{j+1}-t_j)(t_{i+1}-t_i)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\left(\Big[\sum_{j=0}^{n-1}(B_{t_{j+1}}-B_{t_j})^2\Big]-T\right)^2\right] &= 3\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 - 2T\sum_{j=0}^{n-1}(t_{j+1}-t_j) \\
&\qquad + \sum_{j=0}^{n-1}\sum_{\substack{i=0 \\ i\neq j}}^{n=1}(t_{j+1}-t_j)(t_{i+1}-t_i) + T^2 \\
&= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 - 2T\sum_{j=0}^{n-1}(t_{j+1}-t_j) \\
&\qquad + \sum_{j=0}^{n-1}\sum_{i=0}^{n=1}(t_{j+1}-t_j)(t_{i+1}-t_i) + T^2 \\
&= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 - 2T^2 + \left(\sum_{j=0}^{n-1}(t_{j+1}-t_j)\right)^2 + T^2 \\
&= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2 - 2T^2 + T^2 + T^2 \\
&= 2\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2
\end{aligned}
$$

as desired. $\square$

Finally, going back to our original integral we have, with $t_k = k\frac{T}{n}$,

$$
\begin{aligned}
D &= \mathbb{E}\left[\frac{1}{4}\left(\sum_{k=0}^{n-1}(B_{k+1}-B_k)^2 - T\right)^2\right] \quad \text{(from Lemma 2)} \\
&= \frac{1}{2}\sum_{k=0}^{n-1}(t_{k+1}-t_k)^2 = \frac{1}{2}\sum_{k=0}^{n-1}((k+1)\frac{T}{n} - k\frac{T}{n})^2 \\
&= \frac{1}{2}\sum_{k=0}^{n-1}\left(\frac{T}{n}\right)^2 = \frac{1}{2}\frac{T^2}{n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty
\end{aligned}
$$

Since we have shown that, as $n \longrightarrow \infty$,

$$
\begin{aligned}
D &= \mathbb{E}\left[\frac{1}{4}\left(\sum_{k=0}^{n-1}(B_{k+1}-B_k)^2 - T\right)^2\right] \longrightarrow 0 \quad \text{which is equivalent to} \\
&= \left\| \int_0^T H^n(u)\,dB(u) - \left(\frac{1}{2}B(T)^2 - \frac{1}{2}T\right)\right\|_2^2 \longrightarrow 0
\end{aligned}
$$

We may conclude that

$$
\int_0^T B_u\,dB_u = \frac{1}{2}B_T^2 - \frac{T}{2}
$$

$\square$

After having gone through all these steps we realise that we never actually want to go through the definitions again to actually be able to compute these integrals. We will soon discuss better ways of finding solutions.

If we define $I(t) = \int_0^t B_u \, dB_u = \frac{1}{2}B_t^2 - \frac{t}{2}$ then we can see that $I(t)$ is a martingale (from the fact that Brownian increments are martingales). In general, any stochastic integral $\int_0^t H_u \, dB_u$ is a martingale for "nice" integrands H. So, the martingale property gives us

$$\mathbb{E}[I(t)|\mathcal{F}_0] = \mathbb{E}[I(t)] = I(0) = 0$$

So we see another reason why $\int_0^t B_u \, dB_u$ cannot be $\frac{1}{2}B_t^2$ since the martingale property requires $\mathbb{E}[\int_0^t B_u \, dB_u] = 0 = \mathbb{E}[\frac{1}{2}B_t^2 - \frac{t}{2}]$. Thus, $\mathbb{E}[\frac{1}{2}B_t^2] = \frac{1}{2}t \neq 0$ violating the martingale property.

Theorem: For an Itô $I_t(H) = \int_0^t H_u \, dB_u$, the quadratic variation is $\langle I_{(\cdot)}(H) \rangle_t = \int_0^t H_u^2 \, du$.

Proof. (Very brief points). For a suitable H, Itô isometry (Pythagoras' Principle) gives us

$$\mathbb{E}\left[\left(\int_0^t H_u \, dB_u\right)^2\right] = \mathbb{E}\left[\int_0^t H_u^2 \, du\right]$$

$\vdots$

and we obtain $\langle I_{(\cdot)}(H) \rangle_t = \int_0^t H_u^2 \, du$ as desired. $\square$

For example, if we set $H = 1$ we have

$$I_t(1) = \int_0^t 1 \, dB_u = B_t \quad \text{and}$$
$$\langle I_{(\cdot)}(1) \rangle_t = \int_0^t 1^2 \, du = t$$

Showing us again that the quadratic variation of Brownian motion is equal to t.

1.1 Itô's Lemma/Differentiation Rule

The usual chain rule from Calculus gives us

$$\frac{d}{dt} f(g(t)) = \frac{d}{dg} f(g(t)) \frac{d}{dt} g(t) \equiv f'(g(t)) \cdot g'(t)$$

However, if we substitute our Brownian motion $g(t) = B(t)$ then we are faced with the dilemma that $\frac{d}{dt}(B(t))$ doesn't exist. In integral form the chain rule is formulated as

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(u)) \, dB(u)$$

But we have seen that, taking $f(x) = x^2$,

$$x^2 = 0 + \int_0^x (u^2)'\,du = 2\int_0^t u\,du \quad \text{but,}$$
$$B(t)^2 \neq B(0)^2 + \int_0^x [B(u)^2]'\,dB(u) = 2\int_0^2 B(u)\,dB(u)$$

Since we proved that

$$\int_0^t B(u)\,dB(u) = \frac{1}{2}B(u)^2 + \frac{t}{2}$$
$$\iff B(t)^2 = t + \int_0^t 2B(u)\,dB(u)$$

So clearly we require some correction term in our stochastic generalization to the chain rule.

Theorem: Itô's Rule. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and that f' and f'' are continuous. Then, for Brownian Motion $B(t)$ we have

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(u))\,dB(u) + \frac{1}{2}\int_0^t f''(B(u))du$$

Example: Consider $f(x) = x^2$. Computing our derivatives,

$$f(x) = x^2 \quad f'(x) = 2x \quad f''(x) = 2$$

So,

$$\begin{aligned} B(t)^2 &= 0 + \int_0^t 2B(u)\,dB(u) + \frac{1}{2}\int_0^t 2\,du \\ &= 2\int_0^t 2B(u)\,dB(u) + t \\ &= 2(\frac{1}{2}B(t)^2 - \frac{t}{2}) + t \\ &= B(t)^2 \end{aligned}$$

Proof. (Sketch of proof). By Taylor's theorem we have, for f twice differentiable,

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(\gamma)(b-a)^2$$

for some $\gamma \in (a, b)$, and if our time partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, then

$$f(B_{t_{i+1}}) - f(B_{t_i}) = f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}f''(\gamma)(B_{t_{i+1}} - B_{t_i})^2$$

for some $\gamma \in (B_{t_i}, B_{t_{i+1}})$. So,

$$f(B_t) - f(B_{t_0}) = \sum_{i=0}^{n-1} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}\sum_{i=0}^{n-1} f''(\gamma)(B_{t_{i+1}} - B_{t_i})^2$$

Note that the second sum would be 0 if the quadratic variation was 0. However, since we are dealing with Brownian Motion this not necessarily so. We consider the limit of $|\pi| \longrightarrow 0$, so

$$\sum_{i=0}^{n-1} f'(\gamma)(B_{t_{i+1}} - B_{t_i}) \longrightarrow \int_0^t f'(B_u)\, dB_u$$
$$\frac{1}{2}\sum_{i=0}^{n-1} f''(\gamma)(B_{t_{i+1}} - B_{t_i})^2 \longrightarrow \frac{1}{2}\int_0^t f''(B_u)\, d\langle B\rangle_u = \frac{1}{2}\int_0^t f''(B_u)\, du$$

Where we have claimed earlier that $\langle B\rangle_t = t$ so $d\langle B\rangle_t = dt$, hence

$$f(B_t) - f(B_{t_0}) = \int_0^t f'(B_0)\, dB_u + \frac{1}{2}\int_0^t f''(B_u)\, du$$
$$\iff f(B_t) = f(B_{t_0}) + \int_0^t f'(B_0)\, dB_u + \frac{1}{2}\int_0^t f''(B_u)\, du$$

as desired. $\square$

Theorem: Itô's Rule in Two Variables. If $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ (we may think of t, x as time and space variables) with continuous partial derivatives

$$\frac{\partial}{\partial t} f(t,x) \quad \frac{\partial}{\partial x} f(t,x) \quad \frac{\partial^2}{\partial x^2} f(t,x)$$

(i.e $f \in \mathcal{C}^{1,2}$) then

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial}{\partial u} f(u, B_u) du + \int_0^t \frac{\partial}{\partial x} f(u, B_u)\, dB_u + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(u, B_u)\, du$$

Example: Take $f(t,x) = \frac{1}{2}x^2 - \frac{t}{2}$. Then,

$$f_t = -\frac{1}{2} \quad f_x = x \quad f_{xx} = 1$$

By Itô's Rule we have

$$\begin{aligned}
\frac{1}{2}B_t^2 - \frac{t}{2} &= 0 + \int_0^t \Big(-\frac{1}{2}\Big)\, du + \int_0^t B_u\, dB_u + \frac{1}{2}\int_0^t \Big(1\Big)\, du \\
&= -\frac{t}{2} + \int_0^t B_u\, dB_u + \frac{t}{2} \\
&= \int_0^t B_u\, dB_u \\
&= \frac{1}{2}B_t^2 - \frac{t}{2}
\end{aligned}$$

1.2 Some Finance Example

Say we have

$$f(t,x) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma x}$$

Apply Itô's Rule,

$$S_t = f(t, B_t) = f(0, B_0) + \int_0^t f_u(u, B_u)\,du + \int_0^t f_x(u, B_u)\,dB_u + \frac{1}{2}\int_0^t f_{xx}(u, B_u)\,du$$

and noting that

$$f_t(t,x) = f(t,x)(\mu - \frac{1}{2}\sigma^2) \quad f_x(t,x) = f(t,x)\sigma \quad f_{xx}(t,x) = f(t,x)\sigma^2$$

we get

$$\begin{aligned}
S_t &= S_0 + \int_0^t f(u, B_u)(\mu - \frac{1}{2}\sigma^2)\,du + \int_0^t f(u, B_u)\sigma\,dB_u + \frac{1}{2}\int_0^t f(u, B_u)\sigma^2\,du \\
&= S_0 + \int_0^t f(u, B_u)\mu\,du - \int_0^t f(u, B_u)\frac{1}{2}\sigma^2\,du + \int_0^t f(u, B_u)\sigma\,dB_u + \frac{1}{2}\int_0^t f(u, B_u)\sigma^2\,du \\
&= S_0 + \int_0^t f(u, B_u)\mu\,du + \int_0^t f(u, B_u)\sigma\,dB_u
\end{aligned}$$

but $f(u, B_u)$ is just S_u so

$$\begin{aligned}
S_t &= S_0 + \int_0^t S_u\mu\,du + \int_0^t S_u\sigma\,dB_u \\
&= S_0 + \mu\int_0^t S_u\,du + \sigma\int_0^t S_u\,dB_u
\end{aligned}$$

We sometimes choose to put it into differential form

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$$

or equivalently the differential equation

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dB_t$$

1.3 Stochastic Differential Equations

The previous equation is an example of a stochastic differential equation (SDE). The (a?) solution to this SDE is

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

We say that

$$S_t = S_0 + \mu\int_0^t S_u\,du + \sigma\int_0^t S_u\,dB_u$$

is the "Black-Scholes" SDE. Recall that in an ODE we have

$$\frac{dy(t)}{dt} = f(t, y(t))$$

which has integral form

$$y(t) = y(0) + \int_0^t f(u, y(u))\, du$$

1.3.1 Existence & Uniqueness

We will show the conditions for a solution to exist given a SDE.

Definition 1. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with filtration $(B_t)_{t\geq 0}$ (standard Brownian Motion) with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then, an Itô process X_t is a stochastic process of the form

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dB_s$$

where

1. $X_0 \in \mathcal{F}_0$
2. K, H are adapted to $\mathcal{F}_t$
3. $\int_0^T |K_s|\, ds < \infty$ and $\int_0^T |H_s|\, dB_s < \infty$ a.s.

We want to show a general result for some arbitrary X_t satisfying these properties, not just a Brownian Motion.

Proposition 1. Suppose an Itô process X can be written as

$$X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dB_s$$
$$X_t^* = X_0^* + \int_0^t K_s^*\, ds + \int_0^t H_s^*\, dB_s$$

Then

$$\begin{aligned} X_0 &= X_0^* \quad \mathbb{P} \text{ a.s.} \\ H_s &= H_s^* \quad d\lambda \times d\mathbb{P} \text{ a.s.}^1 \\ K_u &= K_s^* \quad d\lambda \times d\mathbb{P} \text{ a.s.} \end{aligned}$$

In particular if X is a martingale then $K = 0$.

Proof. Proof omitted ☐

[1] Where λ is the Lebesgue measure.

Proposition 2. Consider a partition π, then we can show that

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 = \int_0^t |H_s|^2 \, ds$$

That is, the quadratic variation of an Itô process, $\langle X_t \rangle = \langle \int_0^{(\cdot)} H_s \, dB_s \rangle_t = \int_0^t H_s^2 \, du$.

Theorem: Existence. Suppose X_t is an Itô process satisfying

$$dX_t = K_t \, dt + H_t \, dB_t$$

and that $f \in \mathcal{C}^{1,2}$. Then,

$$f(t, X_t) = f(0, X_0) + \int_0^t f_u(u, X_u) \, du + \int_0^t f_x(u, X_u) \, dX_u + \frac{1}{2} \int_0^t f_{xx}(u, X_u) \, d\langle X \rangle_u$$

Note that

$$\begin{aligned} d\langle X \rangle_t &= H_t^2 \, dt \quad \text{since we have} \\ \langle X_t \rangle &= \int_0^t H_s^2 \, du \end{aligned}$$

So

$$f(t, X_t) = f(0, X_0) + \int_0^t f_u(u, X_u) \, du + \int_0^t f_x(u, X_u) \big[K_u \, du + H_u \, dB_u\big] + \frac{1}{2} \int_0^t f_{xx}(u, X_u) \big[H_u^2 \, du\big]$$

More generally, a SDE in differential form

$$dX_t = f(t, X_t) \, dt + \sigma(t, X_t) \, dB_t$$

with initial condition $x_0 = \xi$, is defined if the integrals

$$\int_0^t f(u, X_u) \, du \quad \text{and} \quad \int_0^t \sigma(u, X_u) \, dB_u$$

"make sense". We write[2]

$$X_t = \xi + \int_0^t f(u, X_u) du + \int_0^t \sigma(u, X_u) \, dB_u$$

Theorem: Uniqueness. If

$$|f(t, x) - f(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq k|x - x'|$$

and

$$|f(t, x)|^2 + |\sigma(t, x)|^2 \leq k_0^2 (1 + |x|)^2$$

[2]Note that the integral form of a differential equation "actually means something" unlike the differential form.

then there exists a unique solution to the SDE

$$X_t = \xi + \int_0^t f(u, X_u) du + \int_0^t \sigma(u, X_u)\, dB_u$$

with initial condition $x_0 = \xi$, such that $\exists\, c \in \mathbb{R}$

$$\mathbb{E}\Big[\sup_{0 \le t \le T} |X_t|^2\Big] < c\Big(1 + \mathbb{E}\big[|\xi|^2\big]\Big)$$

Proof. "The proof is similar to Picard approximation done for existence and uniqueness in ODEs" $\square$