MACF 402 Assignment #4
Question #2

a) Note that

$$S_{t_i} = S_0 \exp\{(r - 1/2\sigma^2) i \Delta t + \sum_{j=1}^{i} \sigma (B_{t_j} - B_{t_{j-1}})\}$$

$$= S_0 \exp\{(r - 1/2\sigma^2) i \Delta t + \sum_{j=1}^{i} \sigma \sqrt{\Delta t}\, Z_j\}$$

Hence.

$$X = \log\left(\left[\prod_{i=1}^{n} S_{t_i}\right]^{1/n}\right) - \log(S_0)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \log S_{t_i} - \log(S_0)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \{\log S_0 + (r - 1/2\sigma^2) i \Delta t + \sum_{j=1}^{i} \sigma \sqrt{\Delta t}\, Z_j\} - \log(S_0)$$

$$= \frac{1}{n} (r - 1/2\sigma^2) \Delta t \sum_{i=1}^{n} i + \frac{\sigma \sqrt{\Delta t}}{n} \sum_{i=1}^{n} \sum_{j=1}^{i} Z_j$$

Note $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and

$$\sum_{i=1}^{n} \sum_{j=1}^{i} a_j = \sum_{i=1}^{n} [a_1 + a_2 + \cdots + a_{i-1} + a_i]$$

$$= n a_1 + (n-1) a_2 + \cdots + 2 a_{n-1} + 1 a_n$$

$$= \sum_{i=1}^{n} i\, a_{n-i+1}$$

$$\Leftrightarrow \frac{(r - 1/2\sigma^2) \Delta t (n+1)}{2} + \frac{\sigma \sqrt{\Delta t}}{n} \sum_{i=1}^{n} i\, Z_{n-i+1}$$

$$= \frac{(n+1)}{2} (r - 1/2\sigma^2) \Delta t + \sum_{i=1}^{n} \sigma i Z_i$$ (index doesn't matter)

(b) $E[X] = \frac{(n+1)}{2}(r - \frac{1}{2}\sigma^2)\Delta t + \sum_{i=1}^{n} \sigma_i E[Z_i]$

$$= \frac{(n+1)}{2}(r - \tfrac{1}{2}\sigma^2)\frac{n\Delta t}{n} \qquad (n\Delta t = T)$$

$$= \frac{(n+1)}{2n} T (r - \tfrac{1}{2}\sigma^2)$$

$$VAR[X] = \sum_{i=1}^{n} \sigma_i^2$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^2}\sigma^2 \Delta t$$

$$= \frac{1}{n^2}\sigma^2\Delta t \sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^2}\sigma^2\Delta t \, \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{\sigma^2 (n+1)(2n+1)}{6n^2} T \qquad (n\Delta t = T)$$

Since X is a linear comb of indep r.v.s it is also normal and hence

$$X \sim N\left((r - \tfrac{1}{2}\sigma^2)\frac{(n+1)}{2n}T, \; \frac{\sigma^2 (n+1)(2n+1)T}{6n^2} \right)$$

(c) $h_T = \left(\left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} - K \right)^+$

$$= \begin{cases} \left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} - K & \left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} > K \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} - K & \log\left(\left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} \right) > \log K \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} - K & X > \log(K/S_0) \\ 0 & \text{else} \end{cases}$$

where $X = \log\left(\left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} \right) - \log S_0$

$$C_0^{GA} = E_Q\left[e^{-rT} h_T \right]$$

$$= E_Q\left[e^{-rT} \left(\left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} - K \right) 1_{\{X > \log(K/S_0)\}} \right]$$

$$= E_Q\left[e^{-rT} \quad S_0 \exp\left\{ \log\left(\left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n} \right) - \log S_0 \right\} 1_{\{X > \log(K/S_0)\}} \right]$$

$$- K e^{-rT} Q\left(X > \log(K/S_0) \right)$$

$$= e^{-rT} S_0 \, E_Q\left[e^{X} 1_{\{X > \log(K/S_0)\}} \right]$$

$$- K e^{-rT} Q\left(X > \log(K/S_0) \right)$$

where $X \sim N(\hat{\mu}, \hat{\sigma}^2)$

$$\hat{\mu} = (r - 1/2\,\sigma^2) \frac{(n+1)}{2n} T \qquad \hat{\sigma}^2 = \frac{\sigma^2 (n+1)(2n+1)}{6n^2} T$$

Note, if $X \sim N(\mu, \sigma^2)$ we have

$$E_Q[e^X 1_{\{X>c\}}]$$

$$= \int_c^\infty e^x \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}\,dx \;=$$

let $z = \frac{x-\mu}{\sigma}$ $\quad dz = \frac{dx}{\sigma}$ $\quad x = \sigma z + \mu$

$z(\infty) = \infty \quad z(c) = \frac{c-\mu}{\sigma}$

$$= \int_{\frac{c-\mu}{\sigma}}^\infty \frac{e^{\sigma z+\mu}\, e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz$$

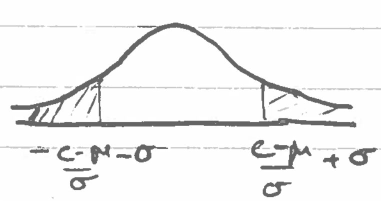
$$= e^\mu \int_{\frac{c-\mu}{\sigma}}^\infty \frac{e^{-\frac{1}{2}z^2+\sigma z}}{\sqrt{2\pi}}\,dz$$

$$= e^\mu \int_{\frac{c-\mu}{\sigma}}^\infty \frac{e^{-\frac{1}{2}(z-\sigma)^2+\frac{1}{2}\sigma^2}}{\sqrt{2\pi}}\,dz$$

$$= e^{\mu+\frac{1}{2}\sigma^2} \int_{\frac{c-\mu}{\sigma}}^\infty \frac{e^{-1/2(z-\sigma)^2}}{\sqrt{2\pi}}\,dz \;=$$

let $v = z-\sigma$ $\quad dv = dz$ $\quad v(\infty) = \infty$ $\quad v\left(\frac{c-\mu}{\sigma}\right) = \frac{c-\mu}{\sigma} - \sigma$

$$= e^{\mu+1/2\sigma^2} \int_{\frac{c-\mu}{\sigma}-\sigma}^\infty \frac{e^{-1/2 v^2}}{\sqrt{2\pi}}\,dz$$



$$= e^{\mu + 1/2\sigma^2} \left[1 - \Phi\left(\frac{c-\mu}{\sigma} - \sigma \right) \right]$$

$$= e^{\mu + 1/2\sigma^2} \Phi\left(-\left(\frac{c-\mu}{\sigma}\right) + \sigma \right)$$

$$= e^{\mu + 1/2\sigma^2} \Phi\left(\frac{-c+\mu}{\sigma} + \sigma \right)$$

$$= e^{\mu + 1/2\sigma^2} \Phi\left(\frac{-c+\mu+\sigma^2}{\sigma} \right)$$

Hence, with $\mu = \breve{\mu}$, $\sigma = \breve{\sigma}$, and

$c = \log(K/S_0)$ we have.

$$E_Q\left[e^X 1_{\{X > \log(K/S_0)\}} \right]$$

$$= e^{\breve{\mu} + 1/2\breve{\sigma}^2} \Phi\left(\frac{\log(S_0/K) + \breve{\mu} + \breve{\sigma}^2}{\breve{\sigma}} \right)$$

Note $\breve{\sigma}^2 = \hat{\sigma}^2 T$

$$\breve{\mu}^2 = (r - 1/2\sigma^2)\frac{(n+1)}{2n} T$$

$$= (r - 1/2\sigma^2)\frac{(n+1)}{2n} T + \frac{1}{2}\hat{\sigma}^2 T - \frac{1}{2}\hat{\sigma}^2 T$$

$$= \hat{\mu} \cdot T - 1/2\,\hat{\sigma}^2 T \qquad \left(\Rightarrow \breve{\mu}^2 + \frac{1}{2}\breve{\sigma}^2 = \hat{\mu} T \right)$$

$$= \hat{\mu} T - \frac{1}{2}\breve{\sigma}^2$$

So that

$$\frac{\log(S_0/K) + \tilde{\mu} + \tilde{\sigma}^2}{\tilde{\sigma}} = \frac{\log(S_0/K) + \hat{\mu}T + \frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}} = \hat{d}_1$$

Similarly, for $X \sim N(\mu, \sigma^2)$

$$Q(X > c) = \int_c^\infty \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}\,dx$$

$$= \int_{\frac{c-\mu}{\sigma}}^\infty \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz = 1 - \Phi\left(\frac{c-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{-c+\mu}{\sigma}\right)$$

so that, with $\mu = \tilde{\mu}$, $\sigma^2 = \tilde{\sigma}^2$, and $c = \log\left(\frac{K}{S_0}\right)$ we have

$$\frac{\log\left(\frac{S_0}{K}\right) + \tilde{\mu}}{\tilde{\sigma}} = \frac{\log\left(\frac{S_0}{K}\right) + \hat{\mu}T - \frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}T}$$

$$= \hat{d}_1 - \hat{\sigma}\sqrt{T} = d_2$$

Then

$$C_0^{GA} = e^{-rT}\left[S_0 e^{\hat{\mu}T}\,\Phi(\hat{d}_1) - K\,\Phi(\hat{d}_2)\right]$$

as desired