Assignment guidlines:

- You should not hand in a first draft. Rewrite your solutions carefully and neatly. Use complete sentences. Make sure your arguments are clear.

Problems:

1. Consider a simple process H with associated partition $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ such that $H_t = H_{t_i}$ for $t \in [t_i, t_{i+1})$ and H_{t_i} is $\mathcal{F}_{t_i}$-measurable. Prove that the stochastic integral with respect to a standard Brownian motion B defined as
$$I(T) := \int_0^T H_u dB_u = \sum_{i=0}^{n-1} H_{t_i}(B_{t_{i+1}} - B_{t_i})$$
satisfies $E[I(T)] = 0$. *[Use the definition and be explicit and rigorous in your proof.]*

2. Suppose that on the risk-neutral filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{Q})$ the price of the risky asset at time t is given by the stochastic differential equation
$$S_t = S_0 + \int_0^t rS_u du + \int_0^t \sigma S_u dW_u$$
for $0 \leq t \leq T$. Use Itô's formula to give a stochastic differential equation satisfied by $\ln(S_t)$.

3. Let W_t be a standard Brownian motion. Use Itô's formula to prove the following:
   (a) For a (deterministic) function $h(t)$ with continuous derivative on $[0, \infty)$:
   $$\int_0^t h(s)dW_s = h(t)W_t - \int_0^t h'(s)W_s ds.$$
   (b) The process
   $$Z_t = \exp\left(\int_0^t \theta(s)dW_s - \frac{1}{2}\int_0^t \theta^2(s)ds\right)$$
   satisfies
   $$dZ_t = \theta(t)Z_t dW_t$$
   where θ is a (deterministic) integrable function.
   (c) For $x > 0$ a constant the process
   $$X_t = (x^{1/3} + \frac{1}{3}B_t)^3$$
   satisfies the SDE
   $$dX_t = \frac{1}{3}X_t^{1/3}dt + X_t^{2/3}dW_t.$$

4. **[MAST 729/881 Only]** Consider the vector-valued stochastic process $X_t = \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}$ where $X_t^{(1)} = a\cos(B_t)$ and $X_t^{(2)} = b\sin(B_t)$. Show that X_t is a solution of
$$dX_t = -\frac{1}{2}X_t dt + MX_t dB_t$$
for some (2×2)-matrix M. What is M? *[Hint: Consider the components separately.]*

5. Consider the process X given by the SDE

$$dX_t = -X_t dt + e^{-t} dB_t$$

with $X_0 = 0$ and B_t a standard Brownian motion. Show that

$$E[X_t] = 0$$

and

$$Var[X_t] = te^{-2t}$$

by solving ODEs for $E[X_t]$ and $E[X_t^2]$.

6. Recall that stochastic integrals

$$\int_0^T H_u dB_u$$

are martingales provided that the integrand H is adapted and satisfies some technical (integrability) conditions. Using Itô's formula find a process X_t such that

$$B_t^3 - X_t$$

is a martingale.

7. In the continous-time Black-Scholes model prove the put-call parity relationship

$$P(t, T, S, K) = C(t, T, S, K) + e^{-r(T-t)} K - S_t$$

between the price at time t of a European call option, denoted $C(t, T, S, K)$, and the price of a European put option, denoted by $P(t, T, S, K)$, with common strike price K and maturity T.