Mathematical & Computational Finance II
Lecture Notes

Simulation and Computational Finance

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1 Simulation & Monte-Carlo

We continue our discussion on Monte-Carlo methods.

1.1 Tilted Densities

A special case of the importance sampling estimator to reduce variance that works fairly well is to use the tilted density of the desired pdf. Using tilted densities is a common method of generating a sampling density g from the original density f. We use the moment generating function of $X \sim f$. Denote the MGF of X

$$m_X(t) = \mathbb{E}_f\left[e^{tX}\right]$$

Then we say that the tilted density of f is given by

$$f_t(X) = \frac{e^{tx} f(x)}{m_X(t)}$$

for $t \in (-\infty, \infty)$. The tilted density is useful since a random variable with density f_t tends to be larger than a random variable with density f, when $t > 0$, and smaller than $t < 0$. That is, we may sample more frequently from the region where we expect X to be large by tilting the density with $t > 0$.

Example:

Suppose $X_1, ..., X_n$ are i.i.d. random variables generated from the density f_i. Let $S_n := \sum_{i=1}^{n} X_i$ and suppose we wish to estimate

$$\theta := \mathbb{P}(S_n \geq a)$$

for some constant a. If a is large so that we have some extremely rare event we should use the importance sampling estimator to compute $\hat{\theta}$. Since S_n is large when the X_i's are large it makes sense to sample each X_i from the its tilted density function $f_{i,t}$, for some $t > 0$. That is,

$$\begin{aligned}
\theta &= \mathbb{E}\left[\mathbb{1}_{S_n \geq a}\right] \\
&= \mathbb{E}_f\left[\mathbb{1}_{S_n \geq a} \prod_{i=1}^{n} \frac{f_i(X_i)}{f_{i,t}(X_i)}\right] \\
&= \mathbb{E}_f\left[\mathbb{1}_{S_n \geq a} \left(\prod_{i=1}^{n} m_{X_i}(t)\right) e^{-tS_n}\right]
\end{aligned}$$

where $\mathbb{E}_f$ denotes the expectation with respect to X_i under the tilted density $f_{i,t}$, and $m_{X_I}(t)$ is the MGF of X_i. Writing $m(t) := \prod_{i=1}^{n} m_{X_i}(t)$, we find that the importance sampling estimator (tilted density estimator) $\hat{\theta}$ is

$$\hat{\theta} \leq M(t)e^{-ta}$$

We can show that a good choice for t would be one that minimizes the upper bound $M(t)e^{-ta}$.

1.2 Conditional Monte-Carlo

The idea to conditional Monte-Carlo estimators is to apply to tower property. That is, given a sample we will condition on a simpler/related model.

Example:

Consider the correlated Brownian process (Heston model/Cox-Ingersoll-Ross model for volatility)

$$\begin{aligned}
dS_t &= rS_t\,dt + \sqrt{v_t}S_t\,dB_t^{(1)} \\
dV_t &= \kappa(\alpha - v_t)\,dt + \sigma\sqrt{v_t}\,dB_t^{(2)} \\
d\langle B_{(\cdot)}^{(1)}, B_{(\cdot)}^{(2)}\rangle_t &= \rho\,dt
\end{aligned}$$

Conditioning on the path $\{v_s : 0 \leq s \leq T\}$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[(\,S_T - K\,)^+ | v_s, 0 \leq s \leq T\right]$$

We can show that this turns out to be a Black-Scholes-like model (see assignment 4), which depends on the realized volatility

$$\frac{1}{T}\int_0^T v_s\,ds$$

Thus, our conditional estimator $\hat{\mu}_{cond}$ is

$$\hat{\mu}_{cond} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\left(S_T - K\right)^+ | v_s, 0 \leq s \leq T\right]$$

Under the crude estimator we needed a discretization over the n time steps and simulate a price process by generating $2n$ random variates $B^{(1)}, B^{(2)}$. However, under the conditional estimator we no longer need to simulate the price process and only require the simulation of the volatility process involving only n variates $B^{(1)}$.

1.3 Stratified Sampling

The stratified sampler is similar to condition: The idea is to break the range of the quantity of interest into discrete strata. If we split the values a random variable X into m discrete strata we may then estimate the parameter in each strata separately and meaningfully recombine the stratified estimators together (i.e. weighted average).

In order for stratified sampling to be effective we have to have *a priori* knowledge of how and where to stratify the range of X. We normalize the size of the strata by the contribution of the variance of the strata to the whole range. The problem is that the variance of the strata are usually not knowable in advance (if it was then we likely wouldn't need to use estimation). To solve this we usual run some pilot sample to determine the sample variance s_k^2 for stratum k.