Mathematical & Computational Finance II
Lecture Notes

Mathematical Finance

November 24 2015
Last update: December 4, 2017

1 Option Pricing with Different Borrowing & Lending Rates

Suppose we have our bank account S^0 and some risky asset S^1. We now impose the restriction on the bank account that we may not short sell. That is, we are no longer able to borrow at the risk free rate. How does this change our ability to price options?

Suppose we are on the real world probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the bank account S^0 with rate r and risky assets S^i such that

$$dS_t^0 = r_t S_t^0 \, dt \quad S_0^0 = 1$$

$$dS_t^i = S_t^i \left[b_i(t) \, dt + \sum_{j=1}^d \sigma_{i,j} \, dW_j(t) \right] \quad j = 1, ..., d$$

where $W(t) = (W_1(t), ..., W_d(t))^T$ is a d-dimensional Brownian motion. Our investor is not permitted to short S^0. That is, the investor may only borrow at some rate $R(t)$ such that

$$R(t, \omega) \geq r(t) \, \forall_{(t,\omega) \, \in \, [0,T] \times \Omega}$$

We introduce a bank account

$$dP^-(t) = P^-(t)R(t) \, dt \quad P^-(0) = 1$$

Assume that we cannot hold a positive number of units of P^- (i.e. we may only borrow P^- and cannot invest in P^-). Let $X(t)$ be the wealth at time t and x be the initial wealth. Let

$$\vec{\Pi}(t) = (\Pi_1(t), ..., \Pi_d(t))^T$$

be the vector of <u>proportions</u> of wealth invested in the asset processes S^i. We have some technical requirements for $\overline{\Pi}(t)$: It must be adapted to our filtration $\mathcal{F}_t$ and

$$\int_0^T \left\| \vec{\Pi}(t)X(t) \right\|^2 \, dt < \infty \quad \text{a.s.}$$

We impose these requirements so that we cannot have our wealth process explode in a finite amount of time (i.e. no infinite wealth in some time interval). We have the fraction of wealth invested in the bank account as

$$1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}$$

which should be positive when lending and negative when borrowing. Under self-financing strategies we have

$$dX(t) = X(t)\left[\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)r(t)\,dt + \vec{\Pi}^T(t)\Big(b(t)\,dt + \sigma(t)\,dW(t)\Big)\right] \quad X(0) = 0$$

where the dimensions of the vectors are

$$\vec{\Pi}^T(t) : d \times 1$$
$$b(t) : d \times 1$$
$$\sigma(t) : d \times d$$
$$dW(t) : d \times 1$$

This gives us an <u>admissible</u> wealth process if $X(t) \geq 0 \; \forall_{t\in[0,T]}$ a.s. If we enforce borrowing at the higher rate $R(t)$ from account P^- we have

$$\text{invest } \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ \text{ in } S^0(t)$$
$$\text{borrow } \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^- \text{ in } P^-(t)$$

where $(x)^+$ and $(x)^-$ mean the max/min of x and 0, respectively. For a self-financing strategy the wealth equation becomes

$$dX(t) = X(t)\left[\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ r(t)\,dt + \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^- R(t)\,dt + \vec{\Pi}^T(t)\Big(b(t)\,dt + \sigma(t)\,dW(t)\Big)\right]$$

For some invertible covariance matrix $\sigma(t)$, let the market price of risk be

$$\theta_0(t) = \sigma^{-1}(t)\left[b(t) - r(t)\vec{\mathbb{1}}\right]$$

By Girsanov's Theorem we have that

$$W_0(t) = W(t) + \int_0^t \theta_0^T(s)\,ds$$

is a d-dimensional Brownian motion with respect to a new measure $\mathbb{P}_0$ defined by

$$\mathbb{P}_0(A) = \mathbb{E}_{\mathbb{P}}\left[Z_0(T)\mathbb{1}_A\right] \quad \forall_{A\in\mathcal{F}_T}$$

where Z is the stochastic exponential martingale of θ, as defined in the notes before (in the change of measure notes).

Theorem: . Let B be a non-negative $\mathcal{F}_T$-measurable random variable. The fair price of B at time $t = 0$ is given by

$$p^0 = \mathbb{E}_{\mathbb{P}_0}\left[\gamma_0(T)B\right]$$

and there exists a unique corresponding hedging portfolio/process (replicating portfolio) Π^0 with a wealth process satisfying

$$X(0) = p^0$$

In our problem we have

$$\gamma_0(t) = e^{-\int_0^t r(s)\,ds}$$

We say that the "fair price" is fair to both buyers and sellers of the contingent claim (i.e. they both may replicate the payoff). If $R(t) > r(t)$ then we should consider the perspective of both the buy and seller of the derivative.

1.1 The Buyer's Perspective

Denote the the initial cost of the cheapest hedging strategy that replicates the claim by p^+. We should note that this market is no longer necessarily "complete". That is, such a hedging strategy is no longer guaranteed to exist!

We will consider the spread between rates. Let $V(t)$ be a (progressively measurable) process such that

$$V(t) \in [0, R(t) - r(t)] \quad \forall_t \ \mathbb{P} \text{ a.s.}$$

We define an auxiliary market M_V with the same assets but with a single bank account with interest rate $r_V(t) = r(t) + V(t)$ (essentially this interest rate is $R(t)$). The use of this auxiliary market is that M_V is now a complete market for every V satisfying

$$V(t) \in [0, R(t) - r(t)]$$

Define $\theta_V(t)$, $Z_V(t)$, $W_V(t)$, $\mathbb{P}_V$ and $\gamma_V(t)$ as before but with the replacement of $r(t)$ by $r_v(t)$. In this market M_V the fair price is

$$p^V = \mathbb{E}_{\mathbb{P}_V}\left[\gamma_V(t)B\right]$$

To compute the fair buying price, suppose we know p^+ and the corresponding Π^+ in advance and define

$$r^+(t) = r(t) + V^+(t) \quad \forall_{t \in [0,T]} \ \mathbb{P} \text{ a.s.}$$

with

$$V^+(t) = \begin{cases} 0 & \text{if } \left(1 - \vec{\Pi}^+(t)^T \vec{\mathbb{1}}\right) > 0 \\ R(t) - r(t) & \text{if } \left(1 - \vec{\Pi}^+(t)^T \vec{\mathbb{1}}\right) \leq 0 \end{cases}$$

where if $\left(1 - \vec{\pi}^+(t)^T \vec{\mathbb{1}}\right) > 0$ implies that "you don't need to short the asset". This adjusts the interest we're dealing with so that Π^+ hedges B in M_{V^+}. However, we assumed that we knew p^+ and Π^+ in advance, which is likely not the case in the real world.

The cheapest strategy in M_{V+} should correspond to some strategy in M_V. That is, every hedge for B in M_{V+} should be a hedge for B in some M_V, and the investor has at least as good terms for borrowing and lending in every market M_V as compared to M_V^+. If initial wealth $x > 0$ then every portfolio with an admissible wealth process in M_V has admissible wealth in M_V^+. That is, the terminal wealth in M_V is $\geq$ the terminal wealth in the corresponding M_{V+}. The idea is that the replication of B is cheaper in every market M_V than M_{V+}.

If $p^+ = p^V$ then this strategy must be the smallest in M_{V+}.

Lemma 1. *Formalizes the above.* Let Π be a portfolio process and $X_V(t)$, $X_{V+}(t)$ be the corresponding wealth processes in M_V and M_{V+}. Let $X_V(0) = X_{V+}(0)$ then

$$X_V(t) = X_{V+}(t) \quad \forall_{t \in [0,T]} \text{ a.s.}$$

if and only if

$$\left[R(t) - r(t) - V(t)\right] \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^- + V(t) \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ = 0 \quad \forall_{t \in [0,t]} \text{ a.s.}$$

Proof. Compare the dynamics of

$$dX_V(t) = X_V(t) \left[\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right) r_V(t)\, dt + \vec{\Pi}^T(t)\left(b(t)\, dt + \sigma(t)\, dW(t)\right)\right]$$

with that of

$$dX(t) = X(t) \left[\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ r(t)\, dt + \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^- R(t)\, dt + \vec{\Pi}^T(t)\left(b(t)\, dt + \sigma(t)\, dW(t)\right)\right]$$

We have the terms $\vec{\Pi}^T(t)\left(b(t)\, dt + \sigma(t)\, dW(t)\right)$ in both processes (i.e. the stock investment process) and using the assumption that $X_V(0) = X_{V+}(0)$ we have that

$$X_V(t) = X_{V+}(t)$$

if and only if the bounded variation terms dt are equal

$$\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ r(t) = \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ r(t) + \left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^- R(t)$$

That is, starting from the same initial condition (and with the same asset process), the only way to end up with the same process throughout t is for the bounded variation terms to be identical.

$\square$

Theorem: . Let B be a European contingent claim and $V(t)$ be a (progressively measurable) adapted process such that

$$V(t) \in [0, R(t) - r(t)] \quad \forall_{t \in [0,T]} \text{ a.s.}$$

If the hedging strategy Π^V replicates B in M_V and satisfies the above

$$\left[R(t) - r(t) - V(t)\right]\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^- + V(t)\left(1 - \vec{\Pi}^T(t)\vec{\mathbb{1}}\right)^+ = 0$$

then Π^V is a hedging strategy for B in M_{V+} and

$$p^+ = p^V$$

The idea is that we will always use the worst-case interest rate to hedge our contingent claim.

Corollary 1. If the portfolio process Π^{R-r} corresponding to the contingent claim valuation problem in M_{R-r}

$$\Pi^{R-r}(t)^T\vec{\mathbb{1}} \geq 1 \quad \forall_{t \in [0,T]} \quad \text{(no lending condition)}$$

then

$$p^+ = p^{R-r}$$

Corollary 2. If Π^0, which hedges B in M_0 satisfies

$$\Pi^0(t)^T\vec{\mathbb{1}} \leq 1 \quad \forall_{t \in [0,T]} \quad \text{(no borrowing condition)}$$

that is, we do not need to buy more of the stock, then

$$p^+ = p^0$$

Example (a): European call option

Say $d = 1$, $r(t) = r$, $b(t) = b$, $\sigma(t) = \sigma$, and $R(t) = R > r$ (i.e. the Black-Scholes model). Then we have

$$B = (S_1(T) - K)^+$$

The Black-Scholes formula gives us that we must always borrow to replicate the call and corollary 1 gives us

$$\begin{aligned}
p^+_{Call} &= p^{R-r} \\
&= \mathbb{E}_{R-r}\left[\exp\left\{(-RT)(S(T) - K)^+\right\}\right] \\
&= BS_{Call}(S, T, R, \sigma)
\end{aligned}$$

We have that the fair buying price is obtained by substituting R for r in the Black-Scholes call formula. Note that this result is not immediately obvious or trivial. It's plausible that we could have had

$$p^- < p^+_{Call} < p^{R-r}$$

Example (b): European put option

For the corresponding put option we can find that the fair buying price will be

$$\begin{aligned}
p^+_{Put} &= p^0 \quad \text{(Black-Scholes price without changes!)} \\
&= BS_{Put}(S, T, r, \sigma)
\end{aligned}$$

1.2 The Seller's Perspective

The seller of a contingent claim will want to hedge $-B$ with a replicating portfolio process such that

$$Y(0) = y < 0$$
$$Y(T) = -B$$

We get similar theorems, lemmas, corollaries as in the buyer's perspective, but now with the addition of

Theorem: . Let B be a European contingent claim.[1] Let $V(t)$ be an adapted process satisfying

$$V(t) = [0, R(t) - r(t)] \quad \forall_{t \in [0,T]} \quad \mathbb{P} \text{ a.s.}$$

If Π^V hedges B in M_V then Π^V is a hedging strategy against $-B$ in M_{V^+}. Moreover, if

$$p^V = \inf \left\{ p_\mu : \mu(t) \in [0, R(t) - r(t)] \ \forall_{t \in [0,T]} \text{ a.s.} \right\}$$

then

$$p^- = p^V$$

Example (a): European call option

We find that $p^V \geq p^0$, so the fair selling price is the same as the Black-Scholes price using interest rate r.

$$p^-_{Call} = BS_{Call}(S, T, r, \sigma)$$

Example (b): European put option

We have $p^V \geq p^{R-r}$, so the fair selling price is the same as the Black-Scholes price with interest rate r.

$$p^-_{Put} = BS_{Put}(S, T, R, \sigma)$$

1.3 Domain of Negotiation

These fair prices with respect to the buyer and seller are price bounds at which these options are traded. Thus, we have the market price $\in [p^-, p^+]$ and we say that this domain is the "domain of negotiation" between buyers and sellers. Note that p^- is the lowest price a seller is willing to accept and p^+ the highest price a buyer is willing to pay. It may end up that

$$[p^-, p^+] = \emptyset \quad \text{(no price is accepted between parties)}$$
$$[p^-, p^+] = p \quad \text{(a singleton price is accepted)}$$

In summary, for a European option we have

$$\text{European call} : p \in \left[BS_{Call}(S, T, r, \sigma), BS_{Call}(S, T, R, \sigma) \right]$$

$$\text{European put} : p \in \left[BS_{Put}(S, T, R, \sigma), BS_{Put}(S, T, r, \sigma) \right]$$

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[1]This holds for American options as well.