Mathematical & Computational Finance II
Lecture Notes

Introduction to Stochastic Calculus

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1 Examples with Itô's Rule

"If you're clever with Itô's formula there's a lot that you can accomplish."

Before we get started it's useful to remember that we may simplify the quadratic variation for standard Brownian motion as $\langle B_{(\cdot)}\rangle_t = t$, thus the differential $d\langle B_{(\cdot)}\rangle_t = dt$.

For the following examples let B_t be a standard Brownian motion with filtration $(\{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$.

1.1 Example 1

Use Itô's formula to find the SDE satisfied by the following function of Brownian motion:

$$f(t,x) = \cos(tx)$$

Solution

Recall from Itô's formula we get

$$f(t,B_t) = f(0,B_0) + \int_0^t f_t(u,B_u)\,du + \int_0^t f_x(u,B_u)\,dB_u + \frac{1}{2}\int_0^t f_{xx}(u,B_u)\,d\langle B_{(\cdot)}\rangle_u$$

Computing our first and second order derivatives we have

$$\begin{aligned} f_t(t,x) &= -x\sin(tx) \\ f_x(t,x) &= -t\sin(tx) \\ f_{xx}(t,x) &= -t^2\cos(tx) \end{aligned}$$

So

$$\begin{aligned} f(t,B_t) &= cos(0) + \int_0^t -B_u\sin(uB_u)\,du + \int_0^t -u\sin(u,B_u)\,dB_u + \frac{1}{2}\int_0^t -u^2\cos(uB_u)\,du \\ &\qquad\qquad \text{(from } \langle B_{(\cdot)}\rangle_t = t,\ \text{so } d\langle B_{(\cdot)}\rangle_t = dt) \\ &= 1 - \int_0^t \big(B_u\sin(uB_u) + u^2\cos(uB_u)\big)\,du - \int_0^t u\sin(uB_u)\,dB_u \end{aligned}$$

1.2 Example 2

Use Itô's formula to find the SDE satisfied by the following function of Brownian motion:

$$f(t,x) = e^{x^2} = f(x)$$

Solution

Note we have only a function of x so

$$\begin{aligned}
f_t(x) &= 0 \\
f_x(x) &= e^{x^2} 2x \\
f_{xx}(x) &= e^{x^2}(2x)^2 + e^{x^2} 2
\end{aligned}$$

So

$$\begin{aligned}
f(B_t) &= 1 + \int_0^t 0\, du + \int_0^t e^{B_u^2} 2B_u\, dB_u + \frac{1}{2}\int_0^t \left(e^{B_u^2}(2B_u)^2 + e^{B_u^2} 2\right) du \\
&= 1 + 2\int_0^t B_u e^{B_u^2}\, dB_u + \frac{1}{2} 2 \int_0^t (1 + 2B_u^2) e^{B_u^2}\, du \\
&= 1 + \int_0^t (1 + 2B_u^2) e^{B_u^2}\, du + 2\int_0^t B_u e^{B_u^2}\, dB_u
\end{aligned}$$

1.3 Example 3

Use Itô's formula to find the SDE satisfied by the following function of Brownian motion:

$$f(t,x) = \arctan(t+x)$$

Solution

Taking our derivatives we get

$$\begin{aligned}
f_t(t,x) &= \frac{1}{1 + (t+x)^2} \\
f_x(t,x) &= \frac{1}{1 + (t+x)^2} \\
f_{xx}(t,x) &= \frac{-1}{\left(1 + (t+x)^2\right)^2} 2(t+x)
\end{aligned}$$

So

$$
\begin{aligned}
f(t,B_t) &= \arctan(0+0) + \int_0^t \frac{1}{1+(u+B_u)^2}\,du + \int_0^t \frac{1}{1+(u+B_u)^2}\,dB_u \\
&\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}\int_0^t \frac{-2(u+B_u)}{\left(1+(u+B_u)^2\right)^2}\,du \\
&= \int_0^t \frac{1}{1+(u+B_u)^2}\,du + \int_0^t \frac{1}{1+(u+B_u)^2}\,dB_u - \int_0^t \frac{u+B_u}{\left(1+(u+B_u)^2\right)^2}\,du \\
&= \int_0^t \frac{1}{1+(u+B_u)^2} - \frac{u+B_u}{\left(1+(u+B_u)^2\right)^2}\,du + \int_0^t \frac{1}{1+(u+B_u)^2}\,dB_u \\
&= \int_0^t \frac{1+(u+B_u)^2-(u+B_u)}{\left(1+(u+B_u)^2\right)^2}\,du + \int_0^t \frac{1}{1+(u+B_u)^2}\,dB_u
\end{aligned}
$$

1.4 Example 4

Suppose S_t satisfies to SDE

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t, \quad S_0 = s_0, \quad 0 \le t \le T$$

In integral form[1] we have

$$f(t,S_t) = f(0,S_0) + \int_0^t \mu f(u,B_u)\,du + \int_0^t \sigma f(u,B_u)\,dB_u$$

Find the SDE satisfied by $G(t) = f(t,S_t) = e^{\alpha(T-t)}S_t$ for $\alpha \in \mathbb{R}$. Apply Itô's formula to $G(t) = f(t,S_t)$

$$G(t) = f(0,S_0) + \int_0^t f_t(u,S_u)\,du + \int_0^t f_x(u,S_u)\,dS_u + \frac{1}{2}\int_0^t f_{xx}(u,S_u)\,d\langle S_{(\cdot)}\rangle_u$$

Computing our derivatives for $G(t) = e^{\alpha(T-t)}S_t$ we get

$$
\begin{aligned}
f_t(t,S_t) &= e^{\alpha(T-t)}(-\alpha)S_t \\
f_{S_t}(t,S_t) &= e^{\alpha(T-t)} \\
f_{S_tS_t}(t,S_t) &= 0
\end{aligned}
$$

Plugging in we get

$$G(t) = e^{\alpha T}s_0 + \int_0^t (-\alpha)e^{\alpha(T-u)}S_u\,du + \int_0^t e^{\alpha(T-u)}\,dS_u + \frac{1}{2}\int_0^t 0\,d\langle S_{(\cdot)}\rangle_u$$

[1] "The integral form of a differential equation actually means something"

Note that we are considering the quadratic variation of S_t in the final differential term $d\langle S_{(\cdot)}\rangle_u$. The quadratic variation of S_t is a little more involved than the quadratic variation of B_t. We know that $\langle B_{(\cdot)}\rangle_t = t$ but we cannot say the same for $\langle S_{(\cdot)}\rangle_t$ since we have $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$. As a result we can't wave our hands as before and say $d\langle S_{(\cdot)}\rangle_t = dt$. Fortunately we have the integrand in this case to be 0 so we don't have to think about the consequences, but "you could figure out what this thing is from the definitions & formulas". Moving on,

$$G(t) = e^{\alpha T}s_0 + \int_0^t (-\alpha)e^{\alpha(T-u)}S_u\,du + \int_0^t e^{\alpha(T-u)}\,dS_u + 0$$

From our initial assumption of S_t satisfying the SDE $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$ we can break down our SDE to

$$\begin{aligned}
G(t) &= e^{\alpha T}s_0 + \int_0^t (-\alpha)e^{\alpha(T-u)}S_u\,du + \int_0^t e^{\alpha(T_u)}\left(\mu S_u\,du + \sigma S_u\,dB_u\right) \\
&= e^{\alpha T}s_0 + \int_0^t (-\alpha)e^{\alpha(T-u)}S_u\,du + \int_0^t e^{\alpha(T-u)}\mu S_u\,du + e^{\alpha(T-u)}\sigma S_u\,dB_u \\
&= e^{\alpha T}s_0 + \int_0^t (\mu-\alpha)e^{\alpha(T-u)}S_u\,du + \int_0^t e^{\alpha(T-u)}\sigma S_u\,dB_u
\end{aligned}$$

But $e^{\alpha(T-u)}S_u \equiv G(u)$, so

$$G(t) = e^{\alpha T}s_0 + \int_0^t (\mu-\alpha)G(u)\,du + \int_0^t \sigma G(u)\,dB_u$$

2 Ticks with Itô's Isometry (Pythagoras Principle)

Use the Itô isometry to calculate the variances of the following Itô integrals and explain why the stochastic integrals are well defined.

2.1 Example 1

Find the variance of

$$\int_0^t |B_s|^{1/2}\,dB_s$$

Solution:

Let $Y_t = \int_0^t |B_s|^{1/2}\,dB_s$. Then Y_t is a martingale[2] so $\mathbb{E}[Y_t] = 0$ and $\mathbb{E}^2[Y_t] = 0$.

[2] This is because, for a simple process H, stochastic integrals with respect to Brownian motion of the form $\int_0^t H_u\,dB_u$ are martingales. See Sept. 29 notes.

Now, computing $\mathbb{E}[Y_t^2]$

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \mathbb{E}\left[\int_0^t \left(|B_s|^{1/2}\right)^2 ds\right] \quad \text{(by Itô's Isometry)} \\
&= \mathbb{E}\left[\int_0^t |B_s|\, ds\right] \\
&= \int_0^t \mathbb{E}[|B_s|]\, ds \quad \text{(by Fubini's Theorem}^3\text{)}
\end{aligned}$$

We know B_s is normal with mean zero and variance s, so $|B_s|$ has distribution[4]

$$f(x) = \frac{2}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}, \quad x > 0$$

Computing $\mathbb{E}[|B_s|]$, for brevity denoting $X := |B_s|$,

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x \frac{2}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}\, dx \\
&= \frac{2}{\sqrt{2\pi s}} \int_0^\infty x e^{-\frac{x^2}{2s}}\, dx \\
&= \frac{2}{\sqrt{2\pi s}} \int_0^\infty x e^{-\frac{x^2}{2s}}\, dx
\end{aligned}$$

[3] The liberal use of Fubini's Theorem in these contexts is not trivial and is something that ought to be verified.

[4] There's some methods you can do to derive this (that I don't currently know) but the intuitive way of of seeing this is the following: Since we've restricted the domain of our distribution to only half of $\mathbb{R}$ then every $f(x)$ should take on twice the value. This intuition obviously relies on the fact that our distribution has mean zero and is symmetric about the mean, i.e. symmetric about the point we're folding over.

Let $u = \frac{-x^2}{2s}$, so $du = \frac{-2x}{2s}\,dx \iff -s\,du = x\,dx$, so

$$\begin{aligned}
\mathbb{E}[X] &= \frac{2}{\sqrt{2\pi s}}\int_0^{-\infty} -se^u\,du \\
&= \frac{2}{\sqrt{2\pi s}}\int_{-\infty}^0 se^u\,du \\
&= \frac{2s}{\sqrt{2\pi s}}\int_{-\infty}^0 e^u\,du \\
&= \frac{2s}{\sqrt{2\pi s}}\left[e^x\right]_{x=-\infty}^{x=0} \\
&= \frac{2s}{\sqrt{2\pi s}}\left[1-0\right] \\
&= \frac{2s}{\sqrt{2\pi s}} \\
&= 2s\frac{\sqrt{2s}}{2\sqrt{\pi s}} \\
\mathbb{E}[X] = \mathbb{E}[|B_s|] &= \frac{\sqrt{2s}}{\sqrt{\pi}}
\end{aligned}$$

So we have $\mathbb{E}[|B_s|] = \frac{\sqrt{2s}}{\sqrt{\pi}}$, which is well defined since $s \geq 0$, then

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \int_0^t \mathbb{E}[|B_s|]\,ds = \int_0^t \frac{\sqrt{2s}}{\sqrt{\pi}}\,ds \\
&= \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^t \sqrt{s}\,ds \\
&= \frac{\sqrt{2}}{\sqrt{\pi}}\frac{2}{3}t^{3/2}
\end{aligned}$$

So $\mathrm{Var}\left[\int_0^t |B_s|^{1/2}\right] \equiv \mathrm{Var}[Y_t] = \mathbb{E}[Y_t^2] - \mathbb{E}^2[Y_t] = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{2}{3}t^{3/2} - 0$, then clearly, for some $t \in \mathbb{R}$, we have $\mathrm{Var}[Y_t^2] < \infty$. Since our variance is finite over some finite interval and "the integrand $\mathbb{E}[|B_s|] = \frac{\sqrt{2s}}{\sqrt{\pi}}$ is adapted"[5] then we have that our variance is well defined, as desired.

2.2 Example 2

Find the variance of

$$\int_0^t |B_s + s|^2\,dB_s$$

Solution:

[5]That is, the integrand is $\mathcal{F}$-measurable (i.e. it's a random variable in our probability space).

Let $Y_t = \int_0^t |B_s + s|^2 \, dB_s$. Then Y_t is a martingale[6] so $\mathbb{E}[Y_t] = \mathbb{E}^2[Y_t] = 0$ and

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \mathbb{E}\Big[\Big(\int_0^t |B_s + s|^2 \, dB_s\Big)\Big] \\
&= \mathbb{E}\Big[\int_0^t |B_s + s|^4 \, ds\Big] \quad \text{(by Itô's Isometry)} \\
&= \int_0^t \mathbb{E}\big[|B_s + s|^4\big] \, ds \quad \text{(by Fubini's Theorem)} \\
&= \int_0^t \mathbb{E}\big[(B_s^4 + 4sB_s^3 + 6s^2B_s^2 + 4s^3B_s + s^4)\big] \, ds \\
&\qquad\qquad\qquad\qquad \text{(removing } |\cdot| \text{ due to exponentiation by an even power)} \\
&= \int_0^t \big(\mathbb{E}[B_s^4] + 4s\mathbb{E}[B_s^3] + 6s^2\mathbb{E}[B_s^2] + 4s^3\mathbb{E}[B_s] + s^4\big) \, ds
\end{aligned}$$

Recall our (useful!) lemma stating, for $s \leq t$,

$$\mathbb{E}\big[(B_t - B_s)^m\big] = \begin{cases} 0 & m \text{ odd} \\ 1 \cdot 3 \cdots (m-3) \cdot (m-1)(t-s)^{m/2} & m \text{ even} \end{cases}$$

Hence

$$\begin{aligned}
\mathbb{E}[Y_t^2] &= \int_0^t \big(\mathbb{E}[B_s^4] + 4s\mathbb{E}[B_s^3] + 6s^2\mathbb{E}[B_s^2] + 4s^3\mathbb{E}[B_s] + s^4\big) \, ds \\
&= \int_0^t \big(3s^2 + 0 + 6s^2 s + 0 + s^4\big) \, ds \\
&= \int_0^t \big(3s^2 + 6s^3 + s^4\big) \, ds \\
&= \frac{1}{3}3t^3 + \frac{1}{4}6t^4 + \frac{1}{5}t^5 \\
\mathbb{E}[Y_t^2] &= t^3 + \frac{3}{2}t^4 + \frac{1}{5}t^5
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Var}\left[\int_0^t |B_s + s|^2 \, dB_s\right] = \mathrm{Var}[Y_t] &= \mathbb{E}[Y_t^2] - \mathbb{E}^2[Y_t] \\
&= t^3 + \frac{3}{2}t^4 + \frac{1}{5}t^5 + 0 \\
&= t^3 + \frac{3}{2}t^4 + \frac{1}{5}t^5
\end{aligned}$$

By the same logic as above, we have an adapted integrand and finite variance therefore our variance is well defined $\mathrm{Var}\left[\int_0^t |B_s + s|^2 \, dB_s\right] = t^3 + \frac{3}{2}t^4 + \frac{1}{5}t^5$, as desired.

[6] Can we write $H_s := |B_s + s|^2$? Is the integrand a simple process? If so then it's clear (to me) why this is a martingale.

3 Expectations of SDEs as ODEs

The oldest trick in the book of mathematics is to transform a hard problem into an easier one. In our case we will sometimes try calculate to the expectation or variance of a stochastic differential equation by solving a related ODE.

3.1 Example 1

Let B_t be standard Brownian motion and consider

$$Y_t = e^{\lambda B_t}$$

where λ is some nonzero constant.

Step 1: Write the appropriate SDE

We will use Itô's formula on $f(x) = e^{\lambda x}$. Notice that we have no time parameter so $f_t(x) = 0$ and

$$f_x(x) = \lambda e^{\lambda x} \quad f_{xx}(x) = \lambda^2 e^{\lambda x}$$

From Itô's formula we find

$$\begin{aligned} f(B_t) &= f(B_0) + \int_0^t f'(B_u)\, dB_u + \frac{1}{2}\int_0^t f''(B_u)\, d\langle B_{(\cdot)}\rangle_u \\ &= f(0) + \lambda \int_0^t f(B_u)\, dB_u + \frac{1}{2}\lambda^2 \int_0^t f(B_u)\, du \quad \text{(once again seeing } d\langle B_{(\cdot)}\rangle = dt) \end{aligned}$$

Since $f(B_t) \equiv Y_t$ we have that Y_t satisfies the SDE

$$Y_t = 1 + \frac{1}{2}\lambda^2 \int_0^t Y_u\, du + \lambda \int_0^t Y_u\, dB_u$$

Or in differential form

$$dY_t = \frac{1}{2}\lambda^2 Y_t\, dt + \lambda Y_t\, dB_t, \quad Y_0 = 1$$

Step 2: Take the expectation of the SDE to yield an ODE

Define $\phi(t) := \mathbb{E}[Y_t]$ and notice that this is now a deterministic function of t (time, in our case). We take the expected value of the integral form of our SDE

$$\mathbb{E}[Y_t] = 1 + \frac{1}{2}\lambda^2 \mathbb{E}\left[\int_0^t Y_u\, du\right] + \lambda \mathbb{E}\left[\int_0^t Y_u\, dB_u\right]$$

But expectations of stochastic integrals are zero,[7] so $\mathbb{E}\left[\int_0^t Y_u\, dB_u\right] = 0$, hence

$$\mathbb{E}[Y_t] = 1 + \frac{1}{2}\lambda^2 \mathbb{E}\left[\int_0^t Y_u\, du\right]$$

[7] Since stochastic integrals of simple processes are martingales.

Now, explicitly working through Fubini's Theorem to swap the integral operation with the expectation we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^t Y_u\,du\Big] &= \int_\Omega \int_0^t Y_u(\omega)\,du\,d\mathbb{P}(\omega)\\ &= \int_0^t \int_\Omega Y_u(\omega)\,d\mathbb{P}(\omega)\,du\\ &= \int_0^t \mathbb{E}[Y_u]\,du\end{aligned}$$

So our expectation of Y_t is equivalent to

$$\mathbb{E}[Y_t] = 1 + \frac{1}{2}\lambda^2 \int_0^t \mathbb{E}[Y_u]\,du$$

That is, we have just constructed the following ODE for $\phi(t) := \mathbb{E}[Y_t]$

$$\phi(t) = 1 + \frac{1}{2}\lambda^2 \int_0^t \phi(u)\,du$$

In differential form

$$d\phi(t) = \frac{1}{2}\lambda^2\phi(t)$$

Step 3: Solving the ODE

We now go through the steps to solve for $\phi(t)$

$$\begin{aligned}\frac{d\phi(t)}{\phi(t)} &= \frac{1}{2}\lambda^2\\ \implies \log(\phi(t)) &= \frac{1}{2}t\lambda^2 + c\\ \implies \phi(t) &= e^c e^{\frac{1}{2}t\lambda^2}\\ &= Ce^{\frac{1}{2}t\lambda^2}\end{aligned}$$

and using the initial condition $\phi(0) = \mathbb{E}[Y_0] = e^{\lambda B_0} = e^{\lambda(0)} = 1$ we see that

$$1 = Ce^{\frac{1}{2}\lambda^2(0)} \implies C = 1$$

Thus

$$\phi(t) = e^{\frac{1}{2}t\lambda^2}$$

We should realize that this result was perhaps anticipated by us since taking the expectation $\mathbb{E}[Y_t] = \mathbb{E}[e^{\lambda B_t}]$ is equivalent to computing the moment generating function of a normal random variable with mean zero and variance t which has the form $e^{\frac{\sigma^2}{2}t}$.

4 Itô's Formula in Multiple Dimensions

Definition 1. Let $B_t^{(j)}$ be a standard Brownian motion for $j = 1, \cdots, m$ and suppose that $B_t^{(i)}$ is independent of $B_t^{(j)}$ for $i \neq j$. Then, a <u>n-dimensional[8] Itô process</u> is a vector valued

[8] I'm still trying to figure out why it's n-dimensional and not m-dimensional.

process $X_t = (X_t^{(1)}, \cdots, X_t^{(n)})$ where components $X_t^{(i)}$ are given by

$$X_t^{(i)} = X_0^{(i)} + \int_0^t K_s^{(i)}\, ds + \sum_{j=1}^{m} \int_0^t H_s^{(i,j)}\, dB^{(j)}\, ds$$

We require that, for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$

1. $K^{(i)}$ and $H^{(i,j)}$ be adapted to $\{\mathcal{F}_t\}$
2. $\int_0^T |K_s^{(i)}|\, ds < \infty \quad \mathbb{P}-$a.s.
3. $\int_0^T |H_s^{(i,j)}|^2\, ds < \infty \quad \mathbb{P}-$a.s.

Theorem: Itô's Formula in n-dimensions. Suppose $X_t = (X_t^{(1)}, \cdots, X_t^{(n)})$ is an n-dimensional Itô process as defined above and suppose $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is in $\mathcal{C}^{1,2}$. Then

$$\begin{aligned} f(t, X_t^{(1)}, \cdots, X_t^{(n)}) = f(0, X_0^{(1)}, \cdots, X_0^{(n)}) &+ \int_0^t \frac{\partial}{\partial s} f(s, X_s^{(1)}, \cdots, X_s^{(n)})\, ds \\ &+ \sum_{i=1}^{n} \int_0^t \frac{\partial}{\partial x_i} f(s, X_s^{(1)}, \cdots, X_s^{(n)})\, dX_s^{(i)} \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s^{(1)}, \cdots, X_s^{(n)})\, d\langle X_{(\cdot)}^{(i)}, X_{(\cdot)}^{(j)} \rangle_s \end{aligned}$$

where

$$\begin{aligned} dX_s^{(i)} &= K_s^{(i)}\, ds + \sum_{j=1}^{m} H_s^{(i,j)}\, dB_s^{(i)} \\ d\langle X_{(\cdot)}^{(i)}, X_{(\cdot)}^{(j)} \rangle_s &= \sum_{r=1}^{m} H_s^{(i,r)} H_s^{(j,r)}\, ds \end{aligned}$$

4.1 Example 1

Consider the processes X, Y with lognormal dynamics

$$\begin{aligned} dX_t &= \mu_t^X X_t\, dt + \sigma_t^X X_t\, dB_t \\ dY_t &= \mu_t^Y Y_t\, dt + \sigma_t^Y Y_t\, dB_t \end{aligned}$$

Show that the process $Z_t = \frac{X_t}{Y_t}$ is also lognormally distributed with dynamics[9]

$$dZ_t = \mu_t^Z Z_t\, dt + \sigma_t^Z Z_t\, dB_t$$

Solution:

[9]We may think of this quotient as the relative performance of two assets.

Let $f(x,y) = \frac{x}{y}$. We first compute our derivative terms

$$\begin{aligned}
f_x(x,y) &= \frac{1}{y} \\
f_y(x,y) &= -\frac{x}{y^2} \\
f_{xx}(x,y) &= 0 \\
f_{yy}(x,y) &= \frac{2x}{y^3} \\
f_{xy}(x,y) = f_{yx}(x,y) &= -\frac{1}{y^2}
\end{aligned}$$

Before getting started with Itô's Formula notice that we require $Y_t \neq 0$ $\mathbb{P}$-a.s. However we can show that a solution[10] to the SDE for Y_t is

$$Y_t = Y_0 e^{\int_0^t (\mu_u^Y - \frac{1}{2}[\sigma_u^Y]^2)\, du + \int_0^t \sigma_u \, dB_u}$$

and provided that $Y_0 \neq 0$ (and since this isn't standard Brownian motion we can assume this) we have $Y_t \neq 0$ $\mathbb{P}-$a.s.

Moving on, we apply the multidimensional Itô formula (for $Z_t = f(X_t, Y_t)$)

$$\begin{aligned}
Z_t = Z_0 &+ \int_0^t f_x(X_u, Y_u)\, dX_u + \int_0^t f_y(X_u, Y_u)\, dY_u \\
&+ \frac{1}{2}\int_0^t f_{xx}(X_u, Y_u)\, d\langle X_{(\cdot)}\rangle_u + \frac{1}{2}\int_0^t f_{yy}(X_u, Y_u)\, d\langle X_{(\cdot)}\rangle_u \\
&+ \frac{1}{2}\int_0^t f_{xy}(X_u, Y_u)\, d\langle X_{(\cdot)}, Y_{(\cdot)}\rangle_u + \frac{1}{2}\int_0^t f_{yx}(X_u, Y_u)\, d\langle Y_{(\cdot)}, X_{(\cdot)}\rangle_u \\
= Z_0 &+ \int_0^t f_x(X_u, Y_u)\, dX_u + \int_0^t f_y(X_u, Y_u)\, dY_u \\
&+ \frac{1}{2}\int_0^t f_{xx}(X_u, Y_u)\, d\langle X_{(\cdot)}\rangle_u + \frac{1}{2}\int_0^t f_{yy}(X_u, Y_u)\, d\langle X_{(\cdot)}\rangle_u \\
&+ \int_0^t f_{xy}(X_u, Y_u)\, d\langle X_{(\cdot)}, Y_{(\cdot)}\rangle_u
\end{aligned}$$

Notice that since second order cross-term partial derivatives are symmetric ($f_{xy} = f_{yx}$) and since the quadratic covariation operation is also symmetric[11] we were able to combine the last two integrals. Furthermore, because we are considering independent processes, we have that the quadratic covariance of X_t, Y_t is zero[12], so

$$\begin{aligned}
Z_t = Z_0 &+ \int_0^t f_x(X_u, Y_u)\, dX_u + \int_0^t f_y(X_u, Y_u)\, dY_u \\
&+ \frac{1}{2}\int_0^t f_{xx}(X_u, Y_u)\, d\langle X_{(\cdot)}\rangle_u + \frac{1}{2}\int_0^t f_{yy}(X_u, Y_u)\, d\langle X_{(\cdot)}\rangle_u
\end{aligned}$$

[10]Did we already do this? Are we just pulling this out of a hat? Proof left as an exercise for the reader...
[11]$d\langle X, Y\rangle_t = d\langle Y, X\rangle_t$: Is this something to prove?
[12]I think we prove this later on in the notes.

Substituting our partial derivatives of $f(x,y)$ and differential equations for $dX_t,\ dY_t$

$$\begin{aligned}
Z_t &= Z_0 + \int_0^t \frac{1}{Y_u}[\mu_u^X X_u\,du + \sigma_u^X X_u\,dB_u] - \int_0^t \frac{X_u}{Y_u^2}[\mu_u^Y Y_u\,du + \sigma_u^Y Y_u\,dB_u] \\
&\quad - \int_0^t \frac{X_u}{Y_u}[\sigma_u^X - \sigma_u^Y]\,dB_u + \frac{1}{2}\int_0^t \frac{2X_u}{Y_u^3}[\sigma_u^Y]^2 Y_u^2\,du \\
&= Z_0 + \int_0^t \frac{X_u}{Y_u}\big(\mu_u^X - \mu_u^Y + [\sigma_u^Y]^2 - \sigma_u^X\sigma_u^Y\big)\,du + \int_0^t \frac{X_u}{Y_u}\big(\sigma_u^X - \sigma_u^Y\big)\,dB_u
\end{aligned}$$

But notice $\frac{X_u}{Y_u} \equiv Z_u$ and so Z_t satisfies the differential equation

$$Z_t = Z_0 + \int_0^t \mu_u^Z Z_u\,du + \int_0^t \sigma_u^Z Z_u\,dB_u$$

where $\mu_u^Z = \mu_u^X - \mu_u^Y + [\sigma_u^Y]^2 - \sigma_u^X\sigma_u^Y$ and $\sigma_u^Z = \sigma_u^X - \sigma_u^Y$

4.2 Example 2

For constants $c, \alpha_1, \cdots, \alpha_n$ give the SDE satisfied by the process

$$X_t = \exp\Big(ct + \sum_{j=1}^m \alpha_j B_t^{(j)}\Big)$$

Solution:

Let $f(t, x_1, \cdots, x_m) = \exp\Big(ct + \sum_{j=1}^M \alpha_j B_t^{(j)}\Big)$. Then,

$$\begin{aligned}
f_t(t, x_1, \cdots, x_m) &= cf(t, x_1, \cdots, x_m) \\
f_{x_i}(t, x_1, \cdots, x_m) &= \alpha_i f(t, x_1, \cdots, x_m) \\
f_{x_i,x_j}(t, x_1, \cdots, x_m) &= \begin{cases} \alpha_i^2 f(t, x_1, \cdots, x_m) & \text{if } i = j \\ \alpha_i\alpha_j f(t, x_1, \cdots, x_m) & \text{if } i \neq j \end{cases}
\end{aligned}$$

Then, by Itô's formula, we have

$$\begin{aligned}
X_t = f(t, x_1, \cdots, x_m) = f(0, B_0^{(1)}, \cdots, B_0^{(m)}) &+ \int_0^t cf(u, B_u^{(1)}, \cdots, B_u^{(m)})\,du \\
&+ \sum_{i=1}^m \int_0^t \alpha_i f(t, B_u^{(1)}, \cdots, B_u^{(m)})\,dB_u^{(i)} \\
&+ \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \int_0^t \alpha_i\alpha_j f(u, B_u^{(1)}, \cdots, B_u^{(m)})\,d\langle B_{(\cdot)}^{(i)}, B_{(\cdot)}^{(j)}\rangle_u
\end{aligned}$$

and note that[13]

$$\langle B_{(\cdot)}^{(i)}, B_{(\cdot)}^{(j)}\rangle_t = \begin{cases} t & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

[13] This is the part about quadratic covariation being zero of independent processes that we had invoked earlier.

That is, the quadratic covariance of a process with itself is just its variance and the quadratic covariation of independent processes is zero. So we may write $d\langle B^{(i)}_{(\cdot)}, B^{(i)}_{(\cdot)}\rangle_t = d\langle B^{(i)}_{(\cdot)}\rangle_t = dt$. Therefore, we simplify our equation to

$$\begin{aligned} X_t = f(t, x_1, \cdots, x_m) = f(0, B^{(1)}_0, \cdots, B^{(m)}_0) + \int_0^t cf(u, B^{(1)}_u, \cdots, B^{(m)}_u)\, du \\ + \sum_{i=1}^{m} \int_0^t \alpha_i f(t, B^{(1)}_u, \cdots, B^{(m)}_u)\, dB^{(i)}_u \\ + \frac{1}{2}\sum_{i=1}^{m} \int_0^t \alpha_i^2 f(u, B^{(1)}_u, \cdots, B^{(m)}_u)\, du \end{aligned}$$

But $f(u, B^{(1)}_u, \cdots, B^{(m)}_u) \equiv X_u$ so we see that X_t satisfies the SDE

$$\begin{aligned} X_t &= 1 + \int_0^t cX_u\, du + \sum_{n=1}^{m} \int_0^t \alpha_i X_u\, dB^{(i)}_u + \frac{1}{2}\sum_{i=1}^{m} \int_0^t \alpha_i^2 X_u\, du \\ &= 1 + \int_0^t \Big(c + \frac{1}{2}\sum_{i=1}^{m} \alpha_i^2\Big) X_u\, du + \sum_{i=1}^{m} \int_0^t \alpha_i X_u\, dB^{(i)}_u \end{aligned}$$

5 A Discussion on Itô's Formula in Multi-/2-Dimensions

Earlier we were faced with differential quadratic covariation terms $d\langle X^{(i)}_{(\cdot)}, X^{(j)}_{(\cdot)}\rangle_t$. Write the quadratic covariation as

$$\langle X_{(\cdot)}, Y_{(\cdot)}\rangle_t = \lim_{|\pi|\to 0} \sum_{i=1}^{n} \big(X_{t_i} - X_{t_{i-1}}\big)\big(Y_{t_i} - Y_{t_{i-1}}\big)$$

Supposedly in the limit this sum converges almost surely. But we were able to write[14]

$$\langle B^{(i)}_{(\cdot)}, B^{(j)}_{(\cdot)}\rangle_t = \begin{cases} t & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

From this can we say anything useful about $\langle X_{(\cdot)}, Y_{(\cdot)}\rangle_t$ and its differential?[15]

[14] It's obvious that two independent normally distributed processes have covariance zero, but is it obvious that their quadratic covariation is zero?

[15] I don't really know what we're about to show, so I didn't really know how to preface what we're about to do...

5.1 Consider Itô's Formula in 2-Dimensions

Omitting the steps showing $d\langle X^{(i)}_{(\cdot)}, X^{(i)}_{(\cdot)}\rangle_t = d\langle X^{(i)}_{(\cdot)}\rangle_t = dt$, and $f_{xy} = f_{yx}$, and the symmetry of quadratic covariation, we have

$$\begin{aligned}
f(t, X^{(1)}_t, X^{(2)}_t) = f(t, X^{(1)}_0, X^{(2)}_0) &+ \int_0^t f_u(u, X^{(1)}_u, X^{(2)}_u)\, du \\
&+ \int_0^t f_{x_1}(u, X^{(1)}_u, X^{(2)}_u)\, dX^{(1)}_u + \int_0^t f_{x_2}(u, X^{(1)}_u, X^{(2)}_u)\, dX^{(2)}_u \\
&+ \frac{1}{2}\int_0^t f_{x_1x_1}(u, X^{(1)}_u, X^{(2)}_u)\, d\langle X^{(1)}_{(\cdot)}\rangle_u + \frac{1}{2}\int_0^t f_{x_2x_2}(u, X^{(1)}_u, X^{(2)}_u)\, d\langle X^{(1)}_{(\cdot)}\rangle_u \\
&+ \int_0^t f_{x_1x_2}(u, X^{(1)}_u, X^{(2)}_u)\, d\langle X^{(1)}_{(\cdot)}, X^{(2)}_{(\cdot)}\rangle_t
\end{aligned}$$

So, for some 2-dimensional process (where does this come from?)

$$\begin{aligned}
X^{(1)}_t &= X^{(1)}_0 + \int_0^t (???)\, du + \int_0^t H^{(1,1)}_u\, dB^{(1)}_u + \int_0^t H^{(1,2)}_u\, dB^{(1)}_u \\
X^{(2)}_t &= X^{(2)}_0 + \int_0^t (???)\, du + \int_0^t H^{(2,1)}_u\, dB^{(2)}_u + \int_0^t H^{(2,2)}_u\, dB^{(2)}_u
\end{aligned}$$

So[16]

$$\begin{aligned}
d\langle X^{(1)}_{(\cdot)}\rangle_t &= \left[H^{(1,1)}_t\right]^2 dt + \left[H^{(1,2)}_t\right]^2 dt \\
d\langle X^{(2)}_{(\cdot)}\rangle_t &= \left[H^{(2,1)}_t\right]^2 dt + \left[H^{(2,2)}_t\right]^2 dt
\end{aligned}$$

If we start thinking in "silly calculus" we can see

$$\vdots$$

$$???$$

$$\vdots$$

so finally

$$d\langle X^{(1)}_{(\cdot)}, X^{(2)}_{(\cdot)}\rangle_t = H^{(1,1)}_t H^{(2,1)}_t\, dt + H^{(1,2)}_t H^{(2,2)}_t\, dt$$

6 Some Financial Modelling in Continuous Time

"Our model in continuous time markets will use the mathematics that we've been building up and eventually lead to us implementing the mathematics in a computer environment."

[16] Not obvious to me.

The modelling framework is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a standard Brownian motion with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Suppose we have a money market account with price

$$\begin{aligned} S^0(t) &\text{ at time t} \\ \text{with initial condition } S^0(0) &= 1 \end{aligned}$$

which pay interest at some continuously compounded rate r. Therefore

$$\begin{aligned} dS^0(t) &= rS^0(t)\,dt \\ \implies S^0(t) &= S^0(0)e^{rt} \end{aligned}$$

Suppose we also are interesting in a risky asset with price $S^1(t)$ at time t and that

$$\begin{aligned} dS^1(t) &= \mu S^1(t)\,dt + \sigma S^1(t)\,dB_t \\ \text{with initial condition } S^1(0) &= s_0 \end{aligned}$$

Then the solution to this SDE[17] is

$$S^1(t) = s_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Our space $(\Omega, \mathcal{F}, \mathbb{P})$ has "real world" probability measure $\mathbb{P}$ (the physical measure) which governs the movement of the asset price.

We decide at time $= t$ how much to invest in both S^0 and S^1 and we require that the decision to invest only on information in the past (this is similar to the efficient market hypothesis). How do we model this? We do this from our filtration $\{\mathcal{F}_t\}_{t\geq 0}$

Definition 2. A trading strategy is a process $H = (H^0, H^1)$ where H^i_t is the number of units of S^i held at time $= t$.

We require that $H = (H^0, H^1)$ be adapted to our filtration $\mathcal{F}_t$ (i.e. $H^i \in \mathcal{F}_t$).

Definition 3. For a given trading strategy, the wealth process is a function of our strategy.

$$V_t(H) = H^0_t S^0_t + H^1_t S^1_t$$

6.1 An Illustrative Example in Discrete Time

In discrete time once (S^0_{n-1}, S^1_{n-1}) are known at $t = n - 1$ the components H^0_n, H^1_n are chosen given wealth

$$V_n(H) = H^0_n S^0_{n-1} + H^1_n S^1_{n-1}$$

[17] Proof left as an exercise to the reader?

That is, you can only base your information of how much to invest now based on the price at the previous time step.

If we assume our portfolio is self-financed (no net injections of new capital) then we have

$$V_n(H) = H_n^0 S_n^0 + H_n^1 S_n^1 = H_{n+1}^0 S_n^0 + H_{n+1}^1 S_{n+1}^1$$

Then

$$\begin{aligned} V_n - V_{n-1} &= H_n^0\big(S_n^0 - S_{n-1}^0\big) + H_n^1\big(S_n^1 - S_{n-1}^1\big) \\ \Delta V &= H_n^0 \Delta S^0 + H_n^1 \Delta S^1 \end{aligned}$$

So, the self-financing condition implies that ΔV comes only from the ΔS^i.

6.2 Returning to Continuous Time

We gave the previous formula in discrete time to show more transparently what will be occurring in continuous time since it becomes a bit more technical.

Definition 4. The portfolio process H is self-financing if

$$dV_t = H_t^0 \, dS_t^0 + H_t^1 \, dS_t^1$$

If we use Itô's formula, formally, we get

$$dV_t = H_t^0 \, dS_t^0 + H_t^1 \, dS_t^1 + (dH_t^0) S_t^0 + (dH_t^1) S_t^1 + \text{QV terms}$$

So the self-financing condition implies that $dH_t^i = 0$. So any increase in units of one can only come from a corresponding decrease in another.

If H is self-financing then we have

$$\begin{aligned} V_t &= V_0 + \int_0^t H_u^0 \, dS_u^0 + \int_0^t H_u^1 \, dS_u^1 \\ &= V_0 + \int_0^t H_u^0 r S_u^0 \, du + \int_0^t H_u^1 \big(\mu S_u^1 \, du + \sigma S_u^1 \, dB_u\big) \\ &= V_0 + r \int_0^t H_u^0 S_u^0 \, du + \mu \int_0^t H_u^1 S_u^1 \, du + \sigma \int_0^t H_u^1 S_u^1 \, dB_u \end{aligned}$$

We have some technical requirements on $[0, T]$ including

1. $\int_0^t |H_u^0| \, du < \infty$ a.s.
2. $\int_0^t (H_u^1)^2 \, du < \infty$ a.s
3. etc...[18]

[18] "We require a few other things but I don't want to say."

Definition 5. If X is any price process then the discounted price process is defined to be

$$\begin{aligned}
\overline{X}_t &= \left(S_t^0\right)^{-1} X_t \\
&= (e^{rt})^{-1} X_t \\
&= e^{-rt} X_t \\
\Longrightarrow \quad \overline{S}_t^1 &= e^{-rt} S_t^1 \\
\Longrightarrow \quad \overline{V}_t &= e^{-rt} V_t
\end{aligned}$$

Lemma 1. The portfolio process $H = (H^0, H^1)$ is self-financing if and only if we can write

$$\overline{V}_t = \overline{V}_0 + \int_0^t H_u^1 \, d\overline{S}_u^1$$

and notice that $\overline{V}_0 = V_0$.

Being able to write our discount value process like this permits us to discard a lot of the terms in the non-discounted price process V_t.

Exercise: Prove if self-financing holds then $\overline{V}_t$ satisfies the above and vice-versa.[19]

Definition 6. A European contingent claim is a positive $\mathcal{F}$-measurable random variable h.

For example, if $h_T = (S_t^1 - K)^+$ then h is a European call option with strike K and maturity T.

Our future goal will be to find the price at time $= 0$ of a European contingent claim.

[19] Strategy: Apply Itô's formula to show that all extraneous terms $= 0$.