Mathematical & Computational Finance II
Lecture Notes

Welcome to Measure Theory

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1 Measure Theoretic Prerequisites

Definition 1. For a sample space Ω we say that a collection of subsets $\mathcal{F}$ is a σ-algebra if $\mathcal{F}$ satisfies three conditions:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$ then $\overline{A} \in \mathcal{F}$
3. For any countable set of subsets in $\mathcal{F}$, the union of these subsets is in $\mathcal{F}$. Symbolically, if $(A_n)_{n\geq 1} \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

A set $A \in \mathcal{F}$ is called a measurable set[1].

Lemma 1. If $A_1, A_2, ..., A_N \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_N \in \mathcal{F}$.

Proof. From Definition **??** Conditions **??** & **??** we have $\overline{\Omega} \in \mathcal{F}$. But Ω is our sample space, so, $\overline{\Omega} = \emptyset \in \mathcal{F}$. By Condition **??** we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Construct $(A_n)_{n\geq 1}$ where for $n \geq N+1, A_n = \emptyset$. So,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \qquad \text{but,}$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{N} A_n \cup \emptyset \cup \emptyset \cup \emptyset \cup \cdots = \bigcup_{n=1}^{N} A_n$$

$$\implies \bigcup_{n=1}^{N} A_n \in \mathcal{F}$$

□

Lemma 2. If $(A_n)_{n\geq 1} \in \mathcal{F}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

[1] This is something worthy of definition itself but is omitted in this course.

Proof. By De Morgan's laws we have,

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots} = \overline{A_1} \cap \overline{A_2} \cap \cdots \iff (\cap_{n=1}^{\infty} A_n) = \overline{(\cup_{n=1}^{\infty} \overline{A_n})} &\quad \text{but,} \\
\cup_{n=1}^{\infty} \overline{A_n} \in \mathcal{F} &\quad \text{From Definition ?? Condition ?? and} \\
\overline{(\cup_{n=1}^{\infty} \overline{A_n})} \in \mathcal{F} &\quad \text{From Condition ??} \\
\implies \overline{(\cup_{n=1}^{\infty} \overline{A_n})} = (\cap_{n=1}^{\infty} A_n) \in \mathcal{F} &
\end{aligned}$$

□

Lemma 3. If $(A_n)_{n\geq 1} \in \mathcal{F}$ then $\bigcap_{n=1}^{N} A_n \in \mathcal{F}$. That is, Lemma 2 holds for finite intersections.

Proof. From Definition **??** Condition **??** we have $\Omega \in F$. From Lemma **??** we have $\bigcap_{n=1}^{\infty} \in \mathcal{F}$. Construct $(A_n)_{n\geq 1}$ where for $n \geq N+1, A_n = \Omega$. So,

$$\begin{aligned}
\bigcap_{n=1}^{\infty} A_n \in \mathcal{F} &\quad \text{but,} \\
\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{N} A_n \cap \Omega \cap \Omega \cap \Omega \cap \cdots = \bigcap_{n=1}^{N} A_n & \\
\implies \bigcap_{n=1}^{N} A_n \in \mathcal{F} &
\end{aligned}$$

□

<u>Examples</u>: Consider two extreme cases,

1. $\mathcal{F} = \{\emptyset, \Omega\}$
2. $\mathcal{F} = \mathcal{P}(\Omega) = 2^{\Omega}$

Both satisfy Definition **??** Condition **??** by construction.The first satisfies Condition **??** since $\overline{\emptyset} = \Omega \in \mathcal{F}$ and $\overline{\Omega} = \emptyset \in \mathcal{F}$. The first also satisfies Condition **??** since a countable union of $\emptyset$ and/or Ω will be either $\emptyset$ (in the case of unions of strictly $\emptyset$) or Ω (all other cases). Thus, the first example is a σ-algebra. The second satisfies Condition **??** since if $A \in \Omega$ then $\{A\} \in \mathcal{P}(\Omega)$ and $\overline{\{A\}} = (\mathcal{P}(\Omega) \setminus \{A\}) \in \mathcal{P}(\Omega)$ (by definition of the power set). Finally, the second example satisfies Condition **??** since, by construction, unions of elements in the power set is already an element in the power set. Thus, the second example is a σ-algebra.

Definition 2. If $\mathcal{C}$ is any collection of subsets of a sample space Ω (not necessarily a σ-algebra), we let $\sigma(\mathcal{C})$ denote the <u>smallest σ-algebra</u> containing $\mathcal{C}$. That is, $\sigma(\mathcal{C})$ must contain Ω and become closed under intersection and union (with respect to Ω). We say $\mathcal{C}$ generates $\sigma(\mathcal{C})$.

Example
Let $A \subsetneqq \Omega$ and $\mathcal{C} = \{A\}$ then it is clear that $\{A, \overline{A}\}$ is not closed under unions, but

$$\sigma(\mathcal{C}) = \{A, \overline{A}, \emptyset, \Omega\}$$

Definition 3. If $\mathcal{C} = \{A \subseteq \mathbb{R}^n : A \text{ is open}^2 \text{ in } \mathbb{R}^n\}$ then $\sigma(\mathcal{C})$ is called the family of Borel sets on $\mathbb{R}^n$. We write in this case $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^n)$.

Proposition 1. $\mathcal{B}(\mathbb{R}) = \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$
To do: Think about this.

1.1 Probability Measures

Definition 4. A function $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is called a probability measure if

1. $\mathbb{P}(\Omega) = 1$
2. $0 \leq \mathbb{P}(A) \leq 1$
3. If $A_1, A_2, ...$ are disjoint in $\mathcal{F}$ then $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

Some consequences of Definiton **??** (stated without proof... to do: state with proof):

1. $\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A), A \in \mathcal{F}$
2. $\mathbb{P}(\emptyset) = 0$
3. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
4. If $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$
5. $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ (for not necessarily disjoint A_n)
6. etc... (there's more but we didn't elaborate – these are usual results you would except from a basic probability course)

Definition 5. A function $F : \mathbb{R} \to \mathbb{R}$ is a distribution function if

1. $\forall x, y \in \mathbb{R}, x \leq y \implies F(x) \leq F(y)$
2. $\lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) = 1$
3. F is right continuous, that is, $\forall a \in \mathbb{R}^+, \lim_{x \to a^+} F(x) = F(a)$

Proposition 2. If $\mathbb{P}$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))^3$ then $F(x) = \mathbb{P}((-\infty, x])$ is a distribution function.

To do: Figure out proof ... "This is easy to prove, but hard to prove the converse".

[2] Worthy of definition.
[3] To be a probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ means to be a function $\mathbb{P} : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$.

2 Integration

Definition 6. A function $s : \Omega \to \mathbb{R}$ is simple if we can write

$$s = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n}$$

where $a_n \geq 0$ and $A_1, A_2, ..., A_N$ are disjoint sets in our σ-algebra $\mathcal{F}$.

Definition 7. Let s be a simple function. Then the expectation (the integral) of s is

$$\mathbb{E}[s] = \int s \, d\mathbb{P} = \int_{\Omega} s(\omega) \, \mathbb{P}(d\omega) = \sum_{n=1}^{N} a_n \mathbb{P}(A_n)$$

Proposition 3. If $(s_n)_{n \geq 1}$ is a sequence of increasing simple functions bound by some function s,

$$s_n \leq s_{n+1} \leq s$$

and $s_n(\omega) \longrightarrow s(\omega)$ as $n \longrightarrow \infty, \forall \omega \in \mathbb{R}$ then,

$$\int s_n \, d\mathbb{P} \longrightarrow \int s \, d\mathbb{P}$$

Definition 8. A function $X : \Omega \to \mathbb{R}$ is called a random variable, or $\mathcal{F}$-measurable, if $\{\omega \in \Omega : X(\omega) < \lambda\} \in \mathcal{F}$ for all $\lambda \in \mathbb{R}$. We write $X \in \mathcal{F}$ (i.e. X is $\mathcal{F}$-measurable).

From Definition **??** we can prove a bunch of facts like the sum of two random variables is a random variable, etc... Some consequences of our definitions:

1. $\cdots$
2. $\cdots$
3. If you have a sequence of random variables $X_n \in \mathcal{F}, n \in \mathbb{N}$ then the $\inf X_n \in \mathcal{F}$ and the $\sup X_n \in \mathcal{F}$ (i.e. these are random variables).

We're doing this so that we can show that it's possible to approximate any random variable with simple functions or a sequence of simple functions.

Proposition 4. If $X \geq 0$ is a random variable then there exists (a sequence?) s_n such that s_n is increasing and converges to X, denoted $s_n \uparrow X$.

Proposition 5. If $X \in \mathcal{F}$ and $X \geq 0$ and $(s_n), (s_m)$ are simple, with $s_n \uparrow X$ and $s_m \uparrow X$, then

$$\lim_n \int s_n \, d\mathbb{P} = \lim_m \int s_m \, d\mathbb{P}$$

Definition 9. If $X \in \mathcal{F}$ and $X \geq 0$ then

$$\mathbb{E}[X] = \lim \int s_n \, d\mathbb{P}$$

when $s_n \uparrow X$. For a general $X \in \mathcal{F}$ write

$$X = X^+ - X^-$$

and define

$$\mathbb{E}[X] = \mathbb{E}[X^+] - E[X^-] \quad \text{(although you can't have } \infty - \infty\text{, etc...)}$$

This has all the basic properties of expectation

1. $\mathbb{E}[\alpha X_1 + \beta X_2] = \alpha\mathbb{E}[X_1] + \beta\mathbb{E}[X_2]$
2. $\cdots$
3. Monotone convergences: If $X_n \uparrow X$ then $\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X]$
4. If $X_n \geq 0$ then $\mathbb{E}[\liminf_n X_n] \leq \liminf_n \mathbb{E}[X_n]$ (Fatou's Lemma)

2.1 Integrability

Definition 10. A nonnegative $X \in \mathcal{F}$ is <u>integrable</u> if $\mathbb{E}[X] < \infty$. To show this we write $X \in L^1(\mathcal{F})$.

Proposition 6.

1. $X, Y \in L^1 \implies X + Y \in L^1$
   $\lambda X \in L^1, \forall \lambda \in \mathbb{R}$
   $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$
2. $X \in L^1 \iff |X| \subset L^1$
   $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$
3. $|X| \leq Y \in L^1 \iff |X| \in L^1$

Theorem: (Lebesgue) Dominated Convergence Theorem.
Let $(X_n)_{n \in \mathbb{N}} \in \mathcal{F}$. If $\exists Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $|X_n| \leq Y$ for all n, if

$$\begin{aligned} X_n(\omega) &\longrightarrow X(\omega) \quad \text{a.s.}^4 \quad \text{then,} \\ \mathbb{E}[X_n] &\longrightarrow \mathbb{E}[X] \quad \text{a.s} \end{aligned}$$

Proof omitted.

Lemma 4. Suppose $Z \in \mathcal{F}$ and $\mathbb{E}[\mathbb{1}_A Z] = \int_A Z \, d\mathbb{P} \leq 0$ for all $A \in \mathcal{F}$. Then $Z \leq 0$ a.s.[5]

[4] A property holds "almost surely" (a.s.) if it holds everywhere except on a set of measure 0.
[5] $\mathbb{P}(\{\omega \in \Omega : Z(\omega) > 0\}) = 0$

Theorem: (Another) Dominated Convergence Theorem.
If $X, Y, (X_n)_{n\geq 1} \in \mathcal{F}$ with $Y \in L^1$ and

$$\begin{aligned}
&|X_n| \leq Y && \forall\, n, \text{ a.s and} \\
&X_n \longrightarrow X && \text{a.s. then,} \\
&X_n \in L^1 && \text{and} \\
&\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X] && \text{a.s}
\end{aligned}$$

Proof omitted.

Definition 11. For $1 \leq p < \infty$ let $\underline{L^p}$ consist of all random variables $X \in \mathcal{F}$ such that

$$\mathbb{E}[|X|^p] < \infty$$

We can prove that, for $X, Y \in L^p$,

$$\mathbb{E}[|X+Y|^p] \leq 2^{p-1}(E[|X|^p] + E[|Y|^p]) < \infty$$

So L^p is a linear space[6]

[6]From some theorem we have, for $a, b \geq 0$ and $1 \leq p < \infty, (a+b)^p \leq 2^{p-1}(a^p + b^p)$