Mathematical & Computational Finance II
Lecture Notes

The Black Scholes World

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1 Hedging in the Black-Scholes World

Last time we went through the hedging portion fairly hurriedly. We will go through the material a little more thoroughly here now.

By the risk neutral pricing formula we know that if we have the price of the underlying asset S_t^1 then the price of a derivative security with payoff $f_T = f(S_T^1)$ will be

$$V_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f(S_T^1)|\mathcal{F}_t]$$

at time $t \in [0, T]$. Since

$$S_T^1 = S_t^1 e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T - W_t)}$$

and that $S_t^1 \in \mathcal{F}_t$ and $(W_T - W_t)$ is independent of the filtration we have that

$$V_t = e^{-r(T-t)}F(T-t, S_t^1)$$

where[1]

$$\begin{aligned} F(T-t,x) &= \mathbb{E}_{\mathbb{Q}}[f(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{(T-t)}z}) \cdot e^{-\frac{1}{2}z^2}\,dz \end{aligned}$$

because we're dealing an expectation of a function of normally distributed random variable[2]. Note that $F(T-t, x)$ is simply us restating the expectation of the payoff of a derivative security for an underlying asset S^1 with price x. The key is that we're permitted to formulate our payoff as such from results introduced previously. We can show that F is differentiable with respect to t and x (but we omit this?). If we write

$$G(t,x) = F(T-t, e^{rt}x)$$

[1]From our lemma/result introduced in the previous lecture?
[2]Should this have variance $(T-t)$?

then we have created a function giving us the expectation of the payoff for an asset whose value is compounded up to t. Thus,[3]

$$\begin{aligned} V_t &= e^{-rT}G(t,S_t^1) = e^{-rT}F(T-t,e^{rt}S_t^1) \\ \overline{V}_t &= e^{-rT}G(t,\overline{S}_t^1) = e^{-rT}F(T-s,e^{rt}\overline{S}_t^1) = e^{-rT}F(T-t,S_t^1) \end{aligned}$$

We apply Itô's formula to $e^{-rT}G(t,\overline{S}_t^1)$ to find

$$\overline{V}_t = e^{-rT}\Big(G(0,\overline{S}_0^1) + \int_0^t \frac{\partial}{\partial u}G(u,\overline{S}_u^1)\,du + \int_0^t \frac{\partial}{\partial x}G(u,\overline{S}_u^1)\,d\overline{S}_u^1 + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}G(u,\overline{S}_u^1)\,d\langle \overline{S}_{(\cdot)}^1\rangle_u\Big)$$

Under $\mathbb{Q}$ we know that $\overline{S}^1$ and $\overline{V}$ are martingales, thus

$$\begin{aligned} d\overline{S}_t^1 &= \overline{S}_t^1\sigma\,dW_t \quad \text{and} \\ d\langle \overline{S}_{(\cdot)}^1\rangle_t &= (\sigma\overline{S}_t^1)^2\,du \end{aligned}$$

so we see that the Itô expansion becomes

$$\overline{V}_t = e^{-rT}\Big(G(0,\overline{S}_0^1) + \int_0^t \frac{\partial}{\partial u}G(u,\overline{S}_u^1)\,du + \int_0^t \frac{\partial}{\partial x}G(u,\overline{S}_u^1)\overline{S}_t^1\sigma\,dW_t + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}G(u,\overline{S}_u^1)\sigma^2(\overline{S}_t^1)^2\,du\Big)$$

Since $\overline{V}$ is a martingale we must have that all bounded variation terms du be equal to zero.[4] That is,

$$\frac{\partial}{\partial t}G(t,\overline{S}_t^1) + \frac{1}{2}\frac{\partial^2}{\partial x^2}G(t,\overline{S}_t^1)\sigma^2(\overline{S}_t^1)^2 = 0 \quad \forall\ t \in [0,T]$$

When $t = T$ we must also have the terminal condition

$$\overline{V}_t = f_T = f(S_T^1)$$

But note that if $t = T$ we have[5]

$$F(0,e^{rT}x) = f(e^{rT}x)$$

Applying the multivariate chain rule, making the substitution $u = T - t$ and $v = e^{rT}x$ such that

$$G(t,x) = F(u,v)$$

we get

$$\begin{aligned} \frac{\partial G(t,x)}{\partial t} &= \frac{\partial F(u,v)}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F(u,v)}{\partial v}\frac{\partial v}{\partial t} \\ &= \frac{\partial F}{\partial u}\cdot(-1) + \frac{\partial F}{\partial v}re^{rt}x \\ &= -\frac{\partial F}{\partial u} + rv\frac{\partial F}{\partial v} \\ \frac{\partial G(t,x)}{\partial x} &= e^{rt}\frac{\partial F}{\partial v} \\ \frac{\partial^2 G(t,x)}{\partial x^2} &= e^{2rt}\frac{\partial^2 F}{\partial v^2} \end{aligned}$$

[3] Should this be $e^{-r(T-t)}$?

[4] This is something worthy of proof but we omit this step.

[5] Where does the $f(e^{rT}x)$ come from?

If we set $x = \overline{S}_t^1$ then we see $v = e^{rt}x = S_t^1$, hence

$$\begin{aligned} G(t, \overline{S}_t^1) &= F(T-t, e^{rt}\overline{S}_t^1) \\ &= F(T-t, S_t^1) \end{aligned}$$

Substituting these partial derivatives into the the equation we set to zero above we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}G(t, \overline{S}_t^1) + \frac{1}{2}\frac{\partial^2}{\partial x^2}G(t, \overline{S}_t^1)\sigma^2(\overline{S}_t^1)^2 \\ \implies 0 &= -\frac{\partial F(u, S_t^1)}{\partial u} + rS_t^1\frac{\partial F(u, S_t^1)}{\partial S_t^1} + \frac{1}{2}e^{2rt}\frac{\partial F(u, S_t^1)}{\partial (S_t^1)^2}\sigma^2(\overline{S}_t^1)^2 \\ &= -\frac{\partial F(u, S_t^1)}{\partial u} + rS_t^1\frac{\partial F(u, S_t^1)}{\partial S_t^1} + \frac{1}{2}e^{2rt}\frac{\partial F(u, S_t^1)}{\partial (S_t^1)^2}\sigma^2 e^{-2rt}(S_t^1)^2 \\ &= -\frac{\partial F(u, S_t^1)}{\partial u} + rS_t^1\frac{\partial F(u, S_t^1)}{\partial S_t^1} + \frac{\sigma^2}{2}(S_t^1)^2\frac{\partial F(u, S_t^1)}{\partial (S_t^1)^2} \end{aligned}$$

However, note that

$$\begin{aligned} \overline{V}_t &= e^{-rT}G(t, \overline{S}_t^1) = F(T-t, S_t^1) \\ \implies e^{-rt}V_t &= F(T-t, S_t^1) \end{aligned}$$

so

$$\begin{aligned} \frac{\partial F}{\partial t} &= e^{-rt}\frac{\partial V}{\partial t} - re^{-rt}V \\ \implies \frac{\partial F}{\partial (T-t)} &= -e^{-rt}\frac{\partial V}{\partial t} + re^{-rt}V \end{aligned}$$

and noting that

$$\frac{\partial F}{\partial S} = e^{-rt}\frac{\partial V}{\partial S} \quad \text{and} \quad \frac{\partial^2 F}{\partial S^2} = e^{-rt}\frac{\partial^2 V}{\partial S^2}$$

We have

$$\begin{aligned} 0 &= -\frac{\partial F(u, S_t^1)}{\partial u} + rS_t^1\frac{\partial F(u, S_t^1)}{\partial S_t^1} + \frac{\sigma^2}{2}(S_t^1)^2\frac{\partial F(u, S_t^1)}{\partial (S_t^1)^2} \\ &= -\left[-e^{-rt}\frac{\partial V}{\partial t} + re^{-rt}V\right] + rSe^{-rt}\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2e^{-rt}\frac{\partial^2 V}{\partial S^2} \\ &= \frac{\partial V}{\partial t} - rV + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial S^2} \end{aligned}$$

with terminal condition $V(T, S) = f(S_T^1)$. This is the Black-Scholes PDE, in particular we call it a backwards parabolic partial differential equation and it is related to the heat equation. It turns out that there is a deep relationship between solutions to PDEs and to those of SDEs. In this case

$$V(t, S) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f(S_T^1)|S_t^1 = s]$$

is called the Feynman-Kac solution. We see that (Vt, S) is a solution to the Black-Scholes PDE but the conditional expectation is a SDE under $\mathbb{Q}$ such that

$$dS_t^1 = rS_t^1\,dt + \sigma S_t^1\,dW_t$$

If we are permitted to assume that the Black-Scholes model and its PDE hold then an implication is that the non-zero terms in the original Itô expansion above

$$\overline{V}_t = e^{-rT}G(0, \overline{S}_0^1) + e^{-rT}\int_0^t \frac{\partial}{\partial x}G(u, \overline{S}_u^1)\,d\overline{S}_u^1$$

can be rewritten as

$$\overline{V}_t = e^{-rT}F(T, S_0^1) + e^{-rT}\int_0^t e^{ru}\frac{\partial F(T-u, S_u^1)}{\partial x}\,d\overline{S}_u^1$$

However, recall that we had as our hedge strategy H^* such that

$$\begin{aligned}\overline{V}_t(H^*) &= \overline{V}_0 + \int_0^t \sigma H_u^1 \overline{S}_u^1\,dW_u\\ &= \overline{V}_0 + \int_0^t H_u^1\,d\overline{S}_u^1\end{aligned}$$

So we see that the integrand $e^{ru}\frac{\partial F(T-u,S_u^1)}{\partial u}$ above is equal to the H^1 component of our portfolio process when we came up with the minimal hedge. Recall that from the Martingale Representation Theorem we had

$$\begin{aligned}N_t &= \mathbb{E}_{\mathbb{Q}}[e^{-rT}f_T|\mathcal{F}_t] \quad \text{(martingale by the tower property)}\\ V_t &= e^{rt}N_t\\ &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}f(S_t^1)|\mathcal{F}_t]\\ \Longrightarrow\ V_t &= e^{-r(T-t)}F(T-t, S_t^1)\\ \Longrightarrow\ N_t &= e^{-rT}F(T-t, S_t^1)\\ &\begin{cases} H_t^1 &= \frac{\gamma_t e^{rt}}{\sigma S_t^1}\\ H_t^0 &= N_t - \frac{\gamma_t}{\sigma} = N_t - e^{-rt}S_t^1 H_t^1\\ &= e^{-rT}\left[F(T-t, S_t^1) - S_t^1\frac{\partial F(T-t,S_t^1)}{\partial x}\right]\end{cases}\end{aligned}$$

and using the partial derivative $\frac{\partial F}{\partial x} = e^{-rt}\frac{\partial V}{\partial S}$ computed above we get

$$H_t^1 = e^{rt}\frac{\partial F}{\partial x} = e^{rt}e^{-rt}\frac{\partial V}{\partial S} = \frac{\partial V}{\partial S}$$

We interpret this partial derivative as the hedge ratio. That is, the number of shares of S^1 to be held at time t. For puts and calls we can derive an explicit formula for $\frac{\partial V}{\partial S}$.[6] We call this quantity the option "delta" and the hedging strategy $H^* = (H^0, H^1)$ is called "delta hedging".

[6] For a later date?

1.1 Greeks

We may consider a variety of option price sensitivities, called "Greeks". Namely,

$$\begin{aligned}
\Delta &= \frac{\partial V}{\partial S} \quad \Gamma = \frac{\partial^2 V}{\partial S^2} \\
\Theta &= \frac{\partial V}{\partial t} \\
\rho &= \frac{\partial V}{\partial r} \\
\nu &= \frac{\partial V}{\partial \sigma} \quad \text{“vega”}
\end{aligned}$$

For a European call option we can prove that its just a calculus exercise to show that

$$H_t^1 = \frac{\partial V}{\partial S} = \Delta = \Phi(d_1)$$

We will elaborate on the Greeks more in the future but not we will look into the Black-Scholes implied volatility.

2 Black-Scholes Implied Volatility

Implied volatility is the σ which matches the observed/quoted price to the Black-Scholes price from the formula. We have some map $\sigma \mapsto V(t, S_t, T, r, \sigma, K)$ and our problem is to somehow meaningfully invert it given all other parameters. Letting C be the price of a call option, we want to solve for σ the equation

$$C^{obs} = C^{BS}(\sigma)$$

using some numerical method (bisection, Newton, Newton-Raphson, ...). Newton's method is particularly applicable since it relies on the use of derivatives of the function (i.e. $\frac{\partial V}{\partial \sigma}$) to estimate the function output.

3 A Discussion on Exotic Options

The idea will be to price more interesting options (that are currently not available in the market) using the σ^{obs} from the Black-Scholes model.

3.1 Barrier Options

These options become either cancelled or activated when the underlying asset passes some threshold (i.e. passes a barrier).

3.1.1 Down & Out Call Option

We consider a barrier option on asset S^1 that gives us the right to buy the asset for strike K at time T as long as $S^1_t \geq H \ \forall \ t \in [0, T]$. Mathematically, we write the payoff as

$$f_T = (S^1_T - K)^+ \mathbb{1}_{S_t \geq H \ \forall \ t \in [0,T]}$$

We see that this option must be cheaper than a vanilla call option since we reduce the chance that it will be exercised in the money. The question is, of course, by how much? Clearly it's related to H, but it's not immediately obvious by how much.

3.1.2 Up & In Call Option

Similar to down & out option but instead of being cancelled at barrier H the option is instead activated at barrier H. That is, the buyer may only exercise the contract if the underlying asset passes the threshold H before maturity. If we define

$$\overline{S}^1_t = \sup_{t \in [0,T]} S^1_t$$
$$\underline{S}^1_t = \inf_{t \in [0,T]} S^1_t$$

then we see we may reformulate the payoff of a down & out call as

$$f_T = (S^1_T - K)^+ \mathbb{1}_{\underline{S}^1_t \geq H}$$

and the payoff of an up & in call as

$$f_T = (S^1_T - K)^+ \mathbb{1}_{\overline{S}^1_t \geq H}$$

3.2 Lookback Type Options

The strike price of a lookback option is based on either the max or the min of the underlying asset price over the term of the contract. We write the payoff of a lookback call with strike based on the lower bound of the asset path as

$$f_T = S^1_T - \underline{S}^1_T$$

3.3 Asian Options

The payoff of an Asian option is defined by some "average" price over the term of the contract.