CSE 546 HW1

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Contents

Question 1

1.1. Since

$$E[X] = \int max(X_1, X_2) f(x) dx_1 dx_2 \tag{1}$$

where $f(x) = 1$ is the PDF for the uniform random variates. We consider two regions, one below the line defined by $X = X_1 = X_2$ and one above said line for our integrations. This gives us the following integral:

$$\begin{aligned} E[X] &= \int_0^1 \int_{x_2}^1 x_1 dx_1 dx_2 + \int_0^1 \int_{x_1}^1 x_2 dx_2 dx_1 \\ &= \int_0^1 (\frac{1}{2} - \frac{1}{2} x_2^2) dx_2 + \int_0^1 (\frac{1}{2} - \frac{1}{2} x_1^2) dx_1 \\ &= (\frac{1}{2} x_2 - \frac{1}{6} x_2^2)|_0^1 + (\frac{1}{2} x_1 - \frac{1}{6} x_1^2)|_0^1 \\ &= \frac{2}{3} \end{aligned} \tag{2}$$

1.2. Since

$$Var[X] = E[X^2] - E[X]^2 \tag{3}$$

and we now know $E[X] = 2/3$, we solve the integral presented above but with the integrand squared as follows

$$
\begin{aligned}
E[X^2] &= \int_0^1 \int_{x_2}^1 x_1^2 dx_1 dx_2 + \int_0^1 \int_{x_1}^1 x_2^2 dx_2 dx_1 \\
&= \int_0^1 (\frac{1}{3} - \frac{1}{3}x_2^3) dx_2 + \int_0^1 (\frac{1}{3} - \frac{1}{3}x_1^3) dx_1 \\
&= (\frac{1}{3}x_2 - \frac{1}{12}x_2^4)|_0^1 + (\frac{1}{3}x_1 - \frac{1}{12}x_1^4)|_0^1 \\
&= \frac{1}{2}
\end{aligned} \tag{4}
$$

which when combined with the definition for $Var[X]$ gives

$$Var[X] = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} \tag{5}$$

1.3. Since

$$Cov[X, X_1] = E[XX_1] - E[X]E[X_1] \tag{6}$$

and we know $E[X]$ from 1.1 and $E[X_1] = 1/2$ is the trivial result for the uniform distribution from $[0, 1]$, we need only calculate the first term as follows:

$$
\begin{aligned}
E[XX_1] &= \int_0^1 \int_{x_2}^1 x_1^2 dx_1 dx_2 + \int_0^1 \int_{x_1}^1 x_2 x_1 dx_2 dx_1 \\
&= \int_0^1 (\frac{1}{3} - \frac{1}{3}x_2^3) dx_2 + \int_0^1 x_1(\frac{1}{2} - \frac{1}{2}x_1^2) dx_1 \\
&= (\frac{1}{3}x_2 - \frac{1}{12}x_2^4)|_0^1 + (\frac{1}{4}x_1^2 - \frac{1}{8}x_1^4)|_0^1 \\
&= \frac{3}{8}
\end{aligned} \tag{7}
$$

giving us

$$Cov[X, X_1] = E[XX_1] - E[X]E[X_1] = \frac{3}{8} - \frac{2}{3}\frac{1}{2} = \frac{1}{24} \tag{8}$$

Note: For this question, I collaborated with Matt Wilde.

Question 2

2.1. For the log-likelihood of G given λ and i.i.d. samples, we have

$$
\begin{aligned}
LL &= \log\left(P(G|\theta)\right) \\
&= \Pi_i^n P(G_i|\theta) \\
&= \log\left(\frac{\lambda^{\Sigma_i^n k_i} e^{-\lambda n}}{\Pi_i^n k_i!}\right) \\
&= \Sigma_i^n k_i \log\lambda - \lambda n - \log \Pi_i^n k_i!
\end{aligned} \tag{9}
$$

2.2. To compute the MLE for λ in general,

$$\frac{\partial}{\partial\lambda}\left[LL\right] = 0 \tag{10}$$

which yields

$$
\begin{aligned}
0 &= \frac{\partial}{\partial\lambda}\left(\Sigma_i^n k_i \log\lambda - \lambda n - \log \Pi_i^n k_i!\right) \\
&= \frac{\Sigma_i^n k_i}{\lambda} - n \\
\hat{\lambda}_{MLE} &= \frac{\Sigma_i^n k_i}{n}
\end{aligned} \tag{11}
$$

2.3. For the observed set G, I use Eqn. 11 to compute λ_{MLE} as

$$\hat{\lambda}_{MLE} = \frac{4+1+3+5+5+1+3+8}{8} = 3.75 \tag{12}$$

Note: For this question, I collaborated with Matt Wilde.