Fractional Fourier Transform:A Survey

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ABSTRACT
The Fractional Fourier transform (FRFT), which provides generalization of conventional Fourier Transform was introduced many years ago in mathematics literature by Namias. In this paper, definition, properties of fractional Fourier transform and its relationship with other transforms is discussed. Various definitions of discrete version of FRFT and their comparison is presented. FRFT falls under the category of Linear time frequency representations. Some of the applications of FRFT such as detection of signals in noise, image compression, reduction of side lobe levels using convolutional windows, and time-frequency analysis are illustrated with examples. It has been observed that FRFT can be used in more effective manner compared to Fourier transform with additional degrees of freedom.

Categories and Subject Descriptors
C.3 [**Special-Purpose And Application-Based Systems**]: Signal processing systems; I.5 [**Pattern Recognition**]: Signal processing

Keywords
Fractional order, Signal detection, Time-Frequency representation, Rotation, Image compression, Window-function.

1. INTRODUCTION
Fourier analysis is one of the most frequently used tools in signal processing and many other scientific fields[25]. The Fourier transform operator can be visualized as a change in representation of the signal corresponding to a counter clockwise rotation of the axis by an angle $\frac{\pi}{2}$[25]. The fractional Fourier transform, is a generalized Fourier transform, can be considered as a rotation by an angle α in the time-frequency plane and is also called as *rotational Fourier transform* or *angular Fourier transform*[30]. Like Discrete Fourier Transform(DFT), a discrete version of Fourier Transform, Discrete Fractional Fourier Transform (DFRFT) is a discrete version of FRFT[11, 15, 17]. There is no single definition for DFRFT till now. All the possible definitions of different types of DFRFT are presented and their specific applications are also discussed in this paper. The FRFT has been found to have many applications in the areas of signal processing such as filtering , restoration, enhancement, correlation, convolution, multiplexing , pattern recognition, beam forming, perspective projections , system synthesis, speaker recognition, neural networks, pattern recognition, array signal processing, radar, sonar, communication, information security, etc[26, 16, 7, 4, 3, 6, 8, 14, 10, 1]. In this paper an attempt is made to illustrate the efficieny of FRFT in the area of Signal and Image Processing.

The paper is organized as follows. Section 2 deals with the definition of Fractional Fourier transform,it's Properties and relationship with other transforms. Concept of Discrete Fractional Fourier Transform (DFRFT) and its various definitions is presented in Section 3. Application of Fractional Fourier transform for the detection of signal in the presence of noise, Image Compression, time-frequency analysis, analysis of convolutional windows etc. are illustrated in section 4. Finally, conclusions are drawn in Section 5.

2. FRACTIONAL FOURIER TRANSFORM
The Continuous time fractional Fourier transform (CFRFT) with an angle α of a signal $f(t)$ is defined as[21, 30],

$$F_\alpha\left(u\right) \ = \int_{-\infty}^{\infty} f\left(t\right) K_\alpha\left(u,t\right) dt \qquad (1)$$

Where the transform kernel $K_\alpha\left(u,t\right)$ is given by[18, 19],

$$K_\alpha\left(u,t\right) = \begin{cases} \sqrt{\frac{1-jcot\alpha}{2\pi}} e^{\frac{j\left(u^2\ +\ t^2\right)}{2} cot\alpha - jutcosec\alpha} & \alpha \neq k\pi \\ \partial\left(u-t\right) & \alpha = 2k\pi \\ \partial\left(u+t\right) & \alpha = \left(2k+1\right)\pi \end{cases} \qquad (2)$$

where $\alpha = a\pi/2$ is rotation angle in the phase plane and a is a constant ranging from 0 to 4. The Trans-

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form Kernel is also defined as[21, 23, 28],

$$K_\alpha(u,t) = \sum_{n=0}^{\infty} e^{-jn\alpha}\psi_n(t)\psi_n(u) \qquad (3)$$

where $\psi_n(t)$ is the n^{th} order continuous Hermite$-$Gaussian function. The FRFT of a given signal $f(t)$(From Eqn.2)can be computed as follows,

1. Multiplication by a chirp signal
2. A Fourier transform with its argument scaled by $cosec\alpha$.
3. Multiplication by another chirp signal
4. Multiplication by another constant

The inverse fractional Fourier transform is defined as[2, 5, 12, 22, 23, 24, 27],

$$f\,(t) = \int_{-\infty}^{\infty} F_\alpha\,(u)\,K_{-\alpha}\,(u,t)\,du \qquad (4)$$

The relationship between fractional domain with the traditional time-frequency plane can be expressed in matrix form as,

$$\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad (5)$$

FRFT will have an advantage of providing rotation in the time$-$frequency plane. FRFT of Some of the commonly used signals is shown in Fig.1. From Fig it is observed that FRFT of a Gaussian is also Gaussian. Fig.1 illustrates how the FRFT of a rectangular signal evolves into a sinc function as α is varied from 0 to $\pi/2$. Some of the useful properties of FRFT are listed

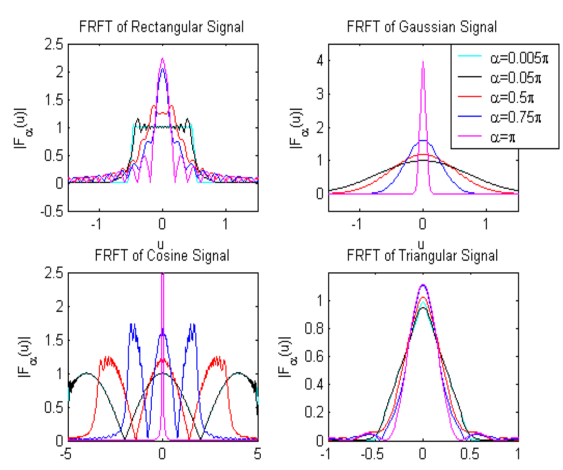

Figure 1: FRFT of some common signals

below.

1. When the value of $\alpha = 0$, the FRFT of the signal is the signal itself. When $\alpha = 0$, $cot\alpha = 1/\alpha$, $\;sin\alpha = \alpha$

$$K_\alpha\,(u,t) = \sqrt{\frac{1}{j2\pi\alpha}}e^{-j\left(\frac{u-t}{2\alpha}\right)^2} \qquad (6)$$

$$F_0\,(u) = \int_{-\infty}^{\infty} f\,(t)\sqrt{\frac{1}{j2\pi\alpha}}e^{-j\left(\frac{u-t}{2\alpha}\right)^2}dt = f(t) \qquad (7)$$

we have an identity that, $\delta(t) = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{j\pi\varepsilon}}e^{\left(\frac{-t^2}{j\varepsilon}\right)}$

2. When $\alpha = 2\pi$ FRFT of the signal is same as the identity operator.
3. When $\alpha = \pi/2$ FRFT gives the Fourier transform of the signal. When $\alpha = \pi/2$,$cot\alpha = 0, cosec\alpha = 1$ and

$$K_\alpha\,(u,t) = \sqrt{\frac{1}{2\pi}}e^{-jut} \qquad (8)$$

$$F_{\frac{\pi}{2}}\,(u) = \int_{-\infty}^{\infty} f\,(t)\sqrt{\frac{1}{2\pi}}e^{-jut}dt = F(u) \qquad (9)$$

4. The FRFT of order $-\alpha$ is the inverse of the FRFT of order α .
5. Successive applications of FRFT are equivalent to a single transform whose order is equal to the sum of the individual orders.

$$F_\alpha\;F_\beta = F_{\alpha+\beta} \qquad (10)$$

6. According to Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2\;dt = \int_{-\infty}^{\infty} |F_\alpha(u)|^2\;du \qquad (11)$$

7. The symmetry properties of the Fourier transform for even and odd real sequences do not extend to the FRFT. So, for a real signal,

$$F_\alpha^*\,(u) = F_{-\alpha}(u) \qquad (12)$$

FRFT of some of the commonly used signals and some of its properties is shown in Table.1.

2.1 Relationship with other Transforms

To derive the *relationship between FRFT and Fourier Transform*, let $a = \frac{cot\alpha}{2}$, $b = sec\alpha$, $c = \sqrt{1-jcot\alpha}$. The FRFT can be written as,

$$e^{-jau^2}F_\alpha(u) = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ja(t^2-2but)}dt \qquad (13)$$

Let $g(t) = c\,f(t)e^{jat^2}$, The above expression reduces to,

$$e^{-jau^2}F_\alpha(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-j2abut}dt = G(2abu) \qquad (14)$$

$$F_\alpha(u) = e^{jau^2}G(2abu) \qquad (15)$$

Table 1: FRFT of some common signals

Signal	FRFT
1	$\sqrt{1+j\tan\alpha}\ e^{-\frac{ju^2\tan\alpha}{2}}$
$e^{\frac{-t^2}{2}}$	$e^{\frac{-u^2}{2}}$
e^{jvt}	$\sqrt{1+j\tan\alpha}\ e^{juv\sec\alpha-\frac{j(v^2+u^2)\tan\alpha}{2}}$
$\delta\left(t-\tau\right)$	$\sqrt{\frac{1-j\cot\alpha}{2\pi}}e^{\ j\frac{u^2+\tau^2}{2}\cot\alpha-ju\tau\ \ \mathrm{cosec}\alpha}$
$f(-t)$	$F_\alpha(-u)$
$f(t)e^{jvt}$	$e^{juv\cos\alpha-j\frac{v^2\cos\alpha\sin\alpha}{2}}F_\alpha(u-v\sin\alpha)$
$f\left(t-\tau\right)$	$e^{-ju\tau\cos\alpha+j\frac{\tau^2\cos\alpha\sin\alpha}{2}}F_\alpha(u-\tau\cos\alpha)$
$tf(t)$	$ucos\alpha F_\alpha(u)+jsin\alpha F'_\alpha(u)$
$\frac{df}{dt}$	$cos\alpha F'_\alpha(u)+jusin\alpha F_\alpha(u)$
$\int\limits_a^t f(t')dt'$	$\sec\alpha e^{-\frac{ju^2\tan\alpha}{2}}\int\limits_a^u F_\alpha(z)e^{\frac{jz^2\tan\alpha}{2}}dz$
$f(t)*g(t)$	$e^{-\frac{ju^2}{2}\cot\left(\frac{\alpha\pi}{2}\right)}F^\alpha(u)G^\alpha(u)$

where $g(t)$ and $G(u)$ are Fourier Transform pair.

The *Wigner Distribution(WD)* of a signal $f(t)$ is defined as,

$$W_f(t,u)=\int_{-\infty}^{\infty}f\left(t+\frac{\tau}{2}\right)f^*\left(t-\frac{\tau}{2}\right)e^{-ju\tau}d\tau \qquad (16)$$

The Wigner distribution depicts the distribution of signals energy over the entire time-frequency plane. For a given time domain signal $f(t)$ the Wigner distribution of its Fourier Transform can be obtained by rotating the Wigner Distribution of $f(t)$ by $\frac{\pi}{2}$ in Time$-$Frequency plane in clockwise direction. Similarly the Wigner distribution of α^{th} order FRFT of $f(t)$ is obtained by rotating the Wigner distribution of $f(t)$ by $\frac{\alpha\pi}{2}$ in Time$-$Frequency plane[3, 6, 19, 12]. In mathematical form it is represented as,

$$W_{F_\alpha}(t,u)=W_f\left(t\cos\left(\phi\right)-u\sin\left(\phi\right),t\sin\left(\phi\right)+u\cos\left(\phi\right)\right) \qquad (17)$$

where F_α is the α^{th} order FRFT of function $f(t)$.

The *Ambiguity Function(AF)* for $f(t)$ is defined as,

$$A\left(\tau,f\right)=\int_{-\infty}^{\infty}f(t)f^*(t-\tau)\ e^{-j2\pi ft}\ dt \qquad (18)$$

The ambiguity function and wigner distribution are related by two-dimensional fourier transform. Wigner distribution gives and idea about the distribution of the energy of a signal, whereas, the ambiguity function gives a correlative interpretation. So, It can be easily said that FRFT and ambiguity function are related by a rotation angle[13, 29].

The *continuous wavelet transform* (CWT) of the signal $f(t)$ is defined as,

$$L_\psi f\left(a,t\right)=\frac{1}{\sqrt{c_\psi}}\ \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty}\bar{\psi}\left(\frac{u-t}{a}\right)f(u)du \qquad (19)$$

where $a\neq 0$. The kernels of FRFT for different values of α can be regarded as wavelet family.

3. DISCRETE FRFT

The evaluation of continuous time fractional fourier transform is done by it's discrete version named Discrete Fractional Fourier Transform(DFRFT)[5, 24, 28, 9, 1]. Although, the properties of CFRFT have been well studied, a universally agreed upon discrete counterpart is yet to be found. The DFRFT should satisfy the following properties

1. Unitary property for the Transform to be reversible .
2. Additivity of orders needed for rotation.
3. Generalization of discrete Fourier transform (DFT) is needed to complete theory.
4. Similarity with CFRFT is needed from the point of view of Theory.

Additionally, it is required to have a closed form and be efficient to implement. Pei and Deng has classified some of the possible definitions of DFRFT in 2000[17, 20]. The straight forward way to obtain discrete version is to sample the continuous time fractional fourier transform (*Direct form type DFRFT*) and is given by,

$$F_\alpha(m)=\sum_{n=-N}^{N}K_\alpha(m,n)f(n) \qquad (20)$$

Kernel $K_\alpha(m,n)$ is defined as,

$$K_\alpha(m,n)=C_\alpha\,e^{\frac{jcot\alpha}{2}\left(m^2\Delta u^2+n^2\Delta t^2\right)}e^{\frac{-j2\pi mnS}{2M+1}} \qquad (21)$$

where,

$$C_\alpha=\sqrt{\frac{|\sin\alpha|-j\mathrm{sgn}(\sin\alpha)\cos\alpha}{2M+1}} \qquad (22)$$

$f(n)$ denotes samples of input with $n=(-N,N)$, $F_\alpha(m)$ is the samples of output with $m=(-M,M)$, S is integer prime to $2M+1$, and $\Delta u\Delta t=\frac{2\pi S\sin\alpha}{2M+1}$. The disadvantage of direct form DFRFT is that it does not obey unitary, additivity, reversibility and closed form definitions. However an improved version of direct type DFRFT is possible. The second kind of DFRFT is *Linear combination type DFRFT*, which is derived by using the linear combination of DFT, linearity,inverse operation and IDFT. It is given by,

$$F_\alpha\left[f(n)\right]=\sum_{n=0}^{N}K_\alpha(n,k)f(n) \qquad (23)$$

where

$$\begin{aligned}K_\alpha(n,k)&=a_0(\alpha)\delta(n-k)+\frac{a_1(\alpha)}{\sqrt{N}}e^{\frac{-j2\pi nk}{N}}\\&+a_2(\alpha)\delta\left[(n+k)_N\right]+\frac{a_3(\alpha)}{\sqrt{N}}e^{\frac{j2\pi nk}{N}}\end{aligned} \qquad (24)$$

Eigen vector decomposition type DFRFT is obtained by by finding the eigen vectors and eigen values of DFT matrix and taking the fractional power. This was invented by Pei and Deng, and the results obtained will match to FRFT.

$$F_\alpha [f(n)] = \sum_{k=0}^{N} u_k [n] u_k^T [n] e^{-j\alpha k} \tag{25}$$

Where $u_k[n]$ is k^{th} discrete Hermite-Gaussian function and $k \neq N$, for N odd $k \neq N-1$, for N even. In matrix form,

$$F_\alpha [f(n)] = UEU^T \tag{26}$$

Where U is discrete Hermite-Gaussian matrix consists of discrete Hermite-Gaussian functions, E is a diagonal matrix that consist of the eigen values $e^{-jk\alpha}, k = 0, 1, 2, ...N$ of DFrFT matrix F_α as diagonal elements. *Group theory type DFRFT* is obtained by multiplying DFT with periodic chirps. This is used only when the fractional order equals some specified angles. Comparison of Linear weighted type, Sampling type and eigen decomposition type DFRFT is presented in Table.3. From the Table.3 it is evident that Linear weighted type does not approximate to the continuous FRFT, and hence finds less application. Sampling type DFRFT is very useful when we just want to use the discrete transform to compute the continuous transform, and hence finds many applications in Signal processing like chirp signal detection and estimation, chirp signal removal, fractional filter design, pattern recognition etc. Out of all the above four types of DFRFT's the eigen decomposition type DFRFT works very similar to FRFT. The drawbacks of this type of DFRFT is that it is slow compared to others and finds applications in adaptive filtering, MIMO Systems etc.

Table 2: Comparison of three main types of DFRFT

Property	Linear Weighted type	Sampling type	Eigen decomposition type
Unitarity	√	X	√
Additivity	√	X	√
Approximation	X	√	√
Complexity	O(NlogN)	O(NlogN)	O(N^2)
Close-form	√	√	X

4. APPLICATIONS

FRFT can be used in all applications where Fourier Transform is presently used. Because of the additional degrees of freedom offered by FRFT some gains can be expected compared to Fourier Transform. Some of the applications of FRFT are illustrated with examples in the following subsections.

4.1 Detection of Signal in Noise

FRFT will be used in special cases to filter noise when the signal and noise overlap each other in frequency domain. Many of the signal detection techniques fails in this aspect. Under such circumstances, FRFT of order α is used to rotate both signal and noise in Frequency domains. Then a filter is designed to allow the passage of a signal,which ensures removal of noise completely. Finally, FRFT of order $-\alpha$ is applied to get back the signal. In order to illustrate the noise removal operation, a gaussian signal,$f(t) = e^{-\pi(t-4)^2}$, is added by the noise, $n(t) = e^{-j\pi t^2} rect(\frac{t}{16})$, and fractional order $\alpha = 0.5$ is chosen. The original Signal, Noise, FRFT of Noise and Signal, Recovered signal are shown in Fig.2.

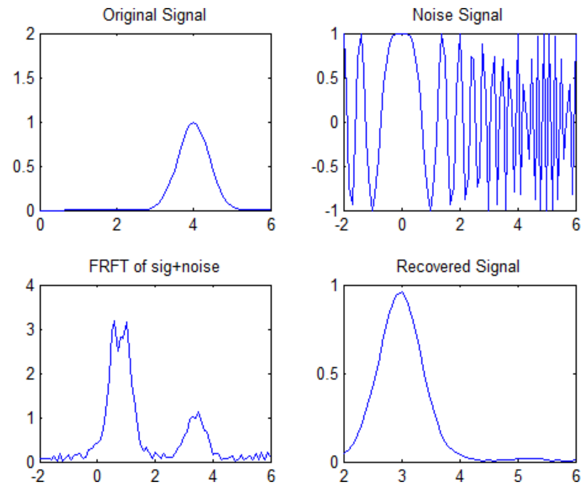

Figure 2: Signal and noise components.

4.2 Image Compression

Image Compression can be divided into two categories as Lossless and Lossy compression.The lossy compression technique will make use of well known transform coding in which any kind of transformation can be chosen, such as Discrete Cosine, Walsh-Hadamard, Fourier etc.The criterion to compare Image Compression Algorithms is to Find RMSE(Root Mean Square Error) and Peak Signal to Noise Ratio(PSNR) which are defined as,

$$RMSE = \sqrt{\frac{1}{MN}\left[\left(\sum_{x=1}^{M}\sum_{y=1}^{N} I(x,y) - I^1(x,y)\right)^2\right]} \tag{27}$$

$$PSNR = 20 \log_{10}\left(\frac{255}{RMSE}\right) \tag{28}$$

Where $I(x, y)$ and $I^1(x, y)$ are original and compressed images. The value of RMSE should be less and PSNR should be high for a good compression scheme. In this paper, Discrete Cosine Transfor(DCT), Discrete Wavelet Transform(DWT), and FRFT are considered. The corresponding outputs are as shown in Fig. 3. PSNR Values for DWT, DCT and FRFT are tabulated in Table.3. From the table it is observed that the performance of DWT is good compared to DCT and FRFT. For values of $\alpha \geq 0.975$, the performance of FRFT is closer to DCT.

4.3 Time-Frequency Analysis

FRFT falls under the category of Linear time frequency transforms. In order to illustrate the efficacy

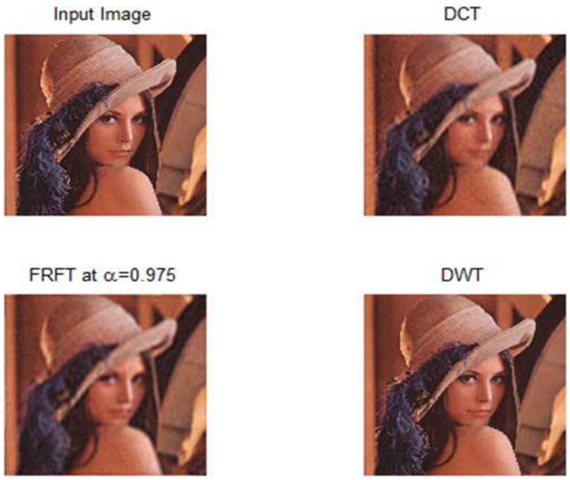

Figure 3: Comparison of DCT, DWT and FRFT.

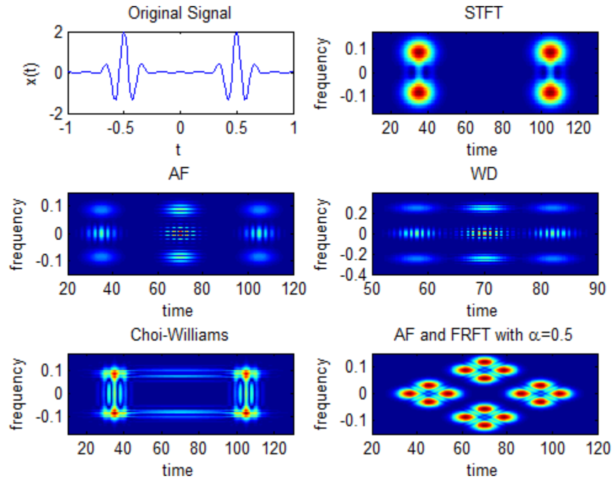

Figure 4: Time-Frequency Representations.

Table 3: PSNR Values of different Transforms

Transform	RMSE	PSNR, dB
DCT	10.6375	25.6520
DWT	5.2966	35.4714
FRFT, α=0	27.5512	7.5933
FRFT, α=0.25	27.6335	7.8735
FRFT, α=0.75	27.4295	8.8873
FRFT, α=0.95	19.3807	19.7591
FRFT, α=0.975	10.9577	25.1555

of FRFT in combination with other TFR's, the following example is considered.

$$x(t) = \left(e^{(t-0.5)^2} + e^{(t+0.5)^2}\right)\left(e^{j12\pi t} + e^{-j12\pi t}\right) \tag{29}$$

The corresponding time frequency plots are shown in Fig.4.It can be observed from the graph that all four auto-terms are located in the middle of the graph for Ambiguity function.Each of four horizontal and vertical objects with respect to the origin represents two cross-terms, while the corner objects represent single cross-terms. In the case of the Wigner Distribution, the corner objects are auto-terms, while the central object represents the four cross-terms. The remaining objects contain two cross terms each. However, all 16 components (4 auto terms and 12 cross-terms) are clearly separated as shown in Fig.4 when the FRFT operator with $\alpha = 0.5$ is used.

4.4 FRFT of Convolutional windows

A window function is a mathematical function that exists for a chosen time interval and is of zero valued outside the interval. In general, windows are divided into two main categories being Fixed and Variable windows.Fixed windows have all the parameters constant or fixed.Rectangular, Triangular, Cosine, Hanning, Hamming, Blackman window are some commonly used fixed windows.Variable windows include Kaiser-Bessel, Dolph-Chebyshev, Poisson, Gaussian, Tseng window etc. Convolutional Windows are obtained by the convolution of the window with itself. If $w(t)$ is time domain window function, then the m^{th} order convolution window is given by,

$$w_m(t) = w(t)*w(t)*........*w(t) \tag{30}$$

By applying the Fractional Fourier Transform to the above equation, and considering $m = 2$,

$$W_2^\alpha(u) = W_\alpha(u)^2 e^{-\frac{ju^2 cot\frac{\alpha\pi}{2}}{2}} \tag{31}$$

In this paper, Rectangular and Triangular windows is considered. *A Rectangular Window* is defined as,

$$\begin{aligned} w(t) &= 1 \qquad |t| \leq 1/2 \\ &= 0 \qquad elsewhere \end{aligned} \tag{32}$$

Calculating FRFT for the convolutional windows and simplifying,

$$W_\alpha(u) = -\sqrt{\frac{1-j\cot\alpha}{8}} \exp\left(-\frac{u^2\tan\alpha}{2}\right)\{A-B\} \tag{33}$$

Where,

$$A = erfi\left[\frac{1+j}{2}\sqrt{\cot\alpha}\left(\frac{1}{2}-u\sec\alpha\right)\right] \tag{34}$$

$$B = erfi\left[\frac{1+j}{2}\sqrt{\cot\alpha}\left(-\frac{1}{2}-u\sec\alpha\right)\right] \tag{35}$$

*A Triangular Window*is defined by the following expression,

$$w(t) = 1-t \tag{36}$$

its corresponding FRFT is,

$$FRFT(1-t) = \sqrt{1+j\tan\alpha}\quad e^{\frac{-ju^2\tan\alpha}{2}}(1+usec\alpha) \tag{37}$$

On applying the FRFT to the Convolutional Window,

$$W_2^\alpha(u) = (1+j\tan\alpha)\quad e^{\frac{-ju^2}{2}(2tan\alpha+cot\alpha)}(1+usec\alpha)^2 \tag{38}$$

The outputs obtained when rectangular and triangular windows is taken into consideration are shown in Figs.5-8.

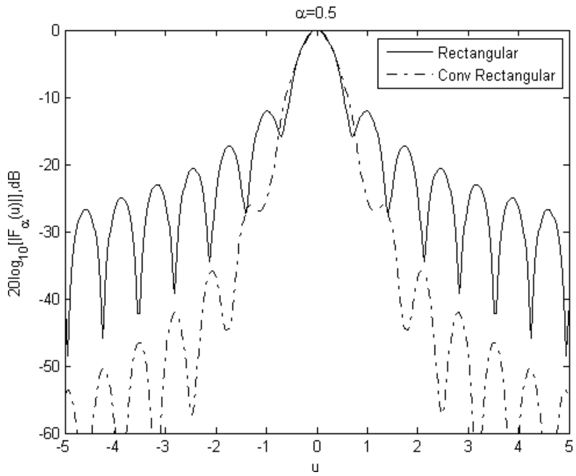

Figure 5: Rectangular and Convoluted Rectangular windows at $\alpha = 0.5$

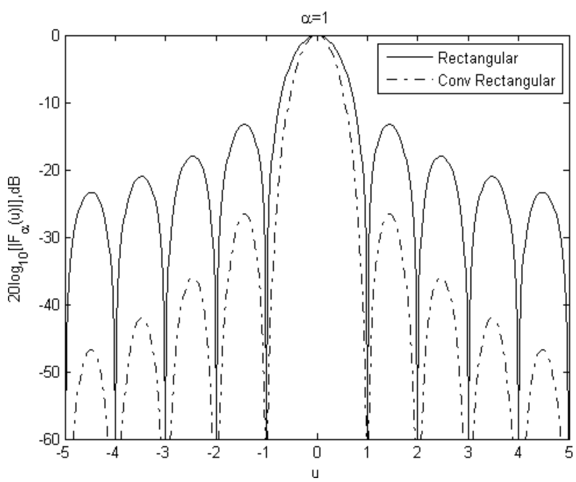

Figure 6: Rectangular and Convoluted Rectangular windows at $\alpha = 1$

5. RESULTS AND CONCLUSIONS

This paper deals with the theory and applications of FRFT in signal processing. Basics , properties of FRFT, theory of DFRFT and different methods of calculation is studied. Later, applications like signal detection, image compression, time-frequency analysis etc are explained by taking an example. In signal detection, 0.5th order FRFT is considered and explained by taking gaussian signal into consideration. The FRFT can also be used effectively in comparison with other transforms for the compression of an image. In image compression, it is observed that for $\alpha \geq 0.975$, the FRFT performs in comparison with DCT. FRFT when combined with AF outper-

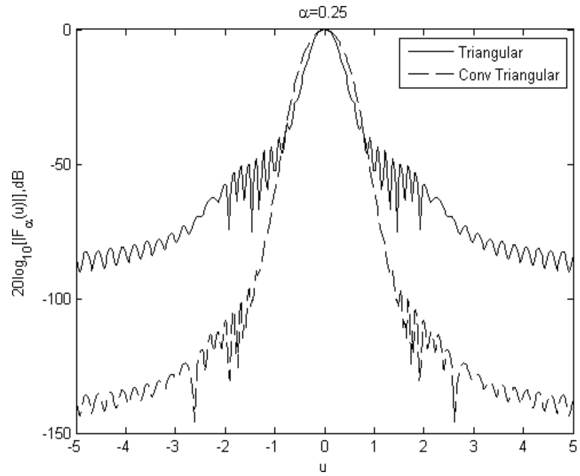

Figure 7: Triangular and Convoluted Triangular windows at $\alpha = 0.25$

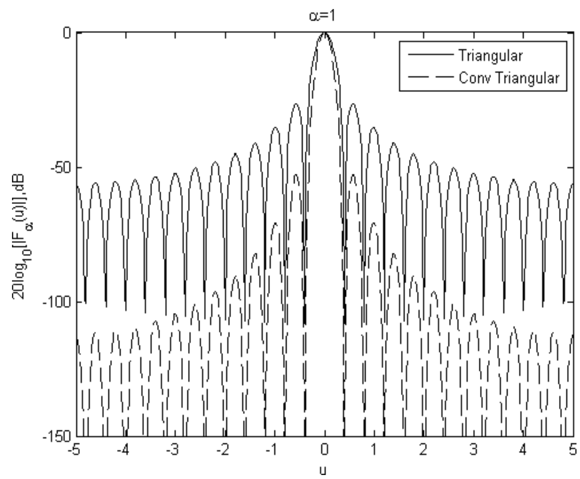

Figure 8: Triangular and Convoluted Triangular windows at $\alpha = 1$**.**

forms with other transforms in time-frequency analysis. Next, the use of FRFT to reduce the side lobe levels is illustrated. It has been demonstrated that the FRFT based convolutional windows reduce the side lobe level compared to FFT based one. From Results it is observed that the sidelobe level is reduced from 13.27 to26.53 in case of convolutional rectangular windows. Similarly there is a reduction of sidelobe level from 45.53 to 90.98 when Triangular window is taken into consideration. It is observed that the plots at $\alpha = 1$ is same as the conventional fourier Transform plots. At other values of $\alpha = 1$ the main lobe width is increased compared to single window,but the sidelobe level is reduced drastically.

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