Fourier Transforms: Background theory

Signal Processing

Discrete-time signals

Discrete-time signals are represented as sequences of numbers x, in which the n^{th} number in the sequence is denoted $x[n]$ is formally written as:

`Eq. 2.1`:

$$x = \{x[n]\} \quad -\infty < n < \infty$$

If such sequence arises from periodic sampling of an analog signal $x_a(t)$, the numeric value of the nth number in the sequence is equal to the value of the analog signal, $x_a(t)$, at time nT.

`Eq. 2.2`:

$$x = x_a(nT) \quad -\infty < n < \infty$$

The quantity T is the *sampling period*, and its reciprical is the *sampling frequency*.

The *unit sample sequence* is defined as the sequence:

`Eq. 2.3`:

$$\partial[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

Note: the unit sample sequence plays the same role for discrete-time signals and systems that the unit impulse function (Dirac delta function) does for continuous-time ssignals and systens. The unit samples sequence is often referred as an *impulse*.

One important aspects of the impulse sequence is that and arbitrary sequence can be represented as a sum of scaled, delayed impulses.

`Eq. 2.5`:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\partial[n-k]$$

The *unit step sequence* is defined as:

`Eq. 2.6`:

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Discrete-time systems

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$.

`Eq. 2.19`

$$y[n] = Tx[n]$$

Linear Systems

The class of linear systems is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear iff it holds for the *additive property* and the *homogeneity / scaling property*.

The principle of superposition can be defined as:

Eq. 2.24

$$Tax_1[n] + bx_2[n] = aTx_1[n] + bTx_2[n]$$

Time-Invariant systems

A time-invariant system (also, shift-invariant system) is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence.

This means that for all n_0, the input sequence with values $x_1[n] = x_2[n]$ produces the output sequence with values $y_1[n] = y[n - n_0]$.

LTI systems (linear and time invariant)

- Must abide linearity property (additivity and homogeneity principles)
- If the linearity property is combined with the representation of a general sequence a s a linear combination of delayed impulses (`Eq. 2.5`), it follows that a linear system can be completely characterized by its impulse response.
- Specifically, let $h_k[n]$ be the response of the system to the input $\partial[n - k]$, an impulse ocurring at $n = k$. Then, using `Eq. 2.5` to represent the input it follows that:

Eq. 2.47

$$y[n] = T \sum_{k=-\infty}^{\infty} x[k]\partial[n - k]$$

and the principle of superposition in `Eq. 2.24`, we can write

Eq. 2.48

$$y[n] = \sum_{k=-\infty}^{\infty} Tx[k]\partial[n - k] = \sum_{k=-\infty}^{\infty} Tx[k]h_k[n]$$

Applying the constraint of time invariance:

Eq. 2.49

$$y[n] = \sum_{k=-\infty}^{\infty} Tx[k]\partial[n - k] = \sum_{k=-\infty}^{\infty} Tx[k]h_k[n]$$