

# Fourier Transforms: Background theory

## Signal Processing

### Discrete-time signals

Discrete-time signals are represented as sequences of numbers  $x$ , in which the  $n^{th}$  number in the sequence is denoted  $x[n]$  is formally written as:

Eq. 2.1:

$$x = \{x[n]\} \quad -\infty < n < \infty$$

If such sequence arises from periodic sampling of an analog signal  $x_a(t)$ , the numeric value of the  $n$ th number in the sequence is equal to the value of the analog signal,  $x_a(t)$ , at time  $nT$ .

Eq. 2.2:

$$x = x_a(nT) \quad -\infty < n < \infty$$

The quantity  $T$  is the *sampling period*, and its reciprocal is the *sampling frequency*.

The *unit sample sequence* is defined as the sequence:

Eq. 2.3:

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases}$$

*Note:* the unit sample sequence plays the same role for discrete-time signals and systems that the unit impulse function (Dirac delta function) does for continuous-time signals and systems. The unit sample sequence is often referred as an *impulse*.

One important aspect of the impulse sequence is that any arbitrary sequence can be represented as a sum of scaled, delayed impulses.

Eq. 2.5:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The *unit step sequence* is defined as:

Eq. 2.6:

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

## Discrete-time systems

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values  $x[n]$  into an output sequence with values  $y[n]$ .

Eq. 2.19

$$y[n] = Tx[n]$$

## Linear Systems

The class of linear systems is defined by the principle of superposition. If  $y_1[n]$  and  $y_2[n]$  are the responses of a system when  $x_1[n]$  and  $x_2[n]$  are the respective inputs, then the system is linear iff it holds for the *additive property* and the *homogeneity / scaling property*.

The principle of superposition can be defined as:

Eq. 2.24

$$T(ax_1[n] + bx_2[n]) = aTx_1[n] + bTx_2[n]$$

## Time-Invariant systems

A time-invariant system (also, shift-invariant system) is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence. ## LTI systems (linear and time invariant)

- Must abide linearity property (additivity and homogeneity principles)
- If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses (Eq. 2.5), it follows that a linear system can be completely characterized by its impulse response.
- Specifically, let  $h_k[n]$  be the response of the system to the input  $\delta[n - k]$ , an impulse occurring at  $n = k$ . Then, using Eq. 2.5 to represent the input it follows that:

Eq. 2.47

$$y[n] = T \sum_{k=-\infty}^{\infty} x[k]\delta[n - k]$$

and the principle of superposition in Eq. 2.24, we can write

Eq. 2.48

$$y[n] = \sum_{k=-\infty}^{\infty} Tx[k]\delta[n - k] = \sum_{k=-\infty}^{\infty} Tx[k]h_k[n]$$