Using Fractional Calculus to Find the Roots of Systems of Polynomial Equations

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Abstract

Many problems in the physical and social sciences can be solved by finding the zeros of systems of polynomial equations. This is a challenging problem when the polynomials are multi-dimensional. I present a procedure for finding the roots of a system of polynomial equations using fractional calculus and Newton's method. I demonstrate how this works for the simple problem of finding the roots of one-dimensional polynomials of high degree. I conclude by discussing how the method can be extended to general algebraic equations in multiple dimensions and evaluating its relative strengths and shortcomings.

1 Introduction

Constrained optimization is a fundamental problem in science and engineering. This usually involves simultaneously solving for the roots of the derivatives of the constraints. In certain situations, the properties of the constraints (such as being linear or convex for example) make the solution to these optimization procedures much easier. Good algorithms already exist for these kinds of systems. However, in the case of nonlinear or nonconvex constraints, finding a solution to the system is nontrivial. While difficult, these systems are not impossible to solve and many disciplines would benefit from the ability to model complex phenomena without being restricted by limited theoretical advances. See [1] for an overview including applications in economics and statistics.

This paper shows that such systems can be solved generally for any function that can be continuously differentiated in the fractional sense. The fractional derivative of a function is a continuous extension of derivatives of integer order to derivatives whose order can take on values in $\mathbb{R}$. Allowing derivatives to be continuous has helped solve many problems in science. Applications of fractional calculus have been used in biology, viscoelastic systems, electrochemistry, and other dynamical systems [2]. Others have applied fractional methods in economics and finance (see for example [3] and [4]).

2 Fractional Root Finding

To begin, I give a brief review of fractional calculus. The main idea began with L'Hospital and Leibniz who considered derivatives of non-integer order. Mathematically speaking, if $\frac{d^n}{dx^n}f(x)$ is the nth derivative of a function $f(x)$, then the fractional derivative is the result of allowing $n \in \mathbb{R}$ instead of $n \in \mathbb{N}$. A consistent theory evaded Leibniz and L'Hospital who left its resolution to be developed by future mathematicians.

Eventually, several mathematicians developed a consistent theory of fractional calculus by combining differentiation and integration into one operator—the differintegral. A natural candidate for such an operator can be derived from Cauchy's Repeated Integration Formula

$$\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f(t)dt.$$

The gamma function $\Gamma(x)$ can be interpreted as the continuous extension of $(n-1)!$ since $\Gamma(n) = (n-1)!$. Substitution in the above equation gives

$$\frac{1}{\Gamma(n)}\int_0^x (x-t)^{n-1}f(t)dt$$

which allows n to take on values in $\mathbb{R}$. Common notation usually replaces n with α to bring attention to this change in the order of the differintegral operator. Thus the final operator is typically written

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt.$$

The above is only a heuristic. Rigorous definitions for such operators are not unique, so fractional derivatives or integrals must be taken with respect to a specified differintegral operator. The two most common are the Riemann-Liouville and Caputo forms. The former is most similar to the above derivation and suits the purposes of this paper.

2.1 Fractional Derivative of a Polynomial

Here I present a simple example of a fractional derivative of a single variable polynomial. Let $p(x) = c_2x^2 + c_1x + c_0$ denote an arbitrary quadratic polynomial and let D^α denote the fractional derivative operator where $\alpha \in (0, 1)$. For polynomials, the fractional derivative is additive but this is not the case in general. Thus,

$$D^\alpha p(x) = D^\alpha c_2x^2 + D^\alpha c_1x + D^\alpha c_0.$$

For any monomial x^n, the kth derivative is expressed $\frac{n!}{(n-k)!}x^{n-k}$. Similar to the example in the previous section, the gamma function can replace the factorials to get

$$D^\alpha x^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{k-\alpha}.$$

In the case of polynomials, this is a consistent and correct fractional derivative. Thus the αth fractional derivative of the above polynomial becomes

$$D^\alpha p(x) = \frac{\Gamma(3)c_2}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{\Gamma(2)c_1}{\Gamma(2-\alpha)}x^{1-\alpha} + \frac{\Gamma(1)c_0}{\Gamma(1-\alpha)}x^{-\alpha}.$$

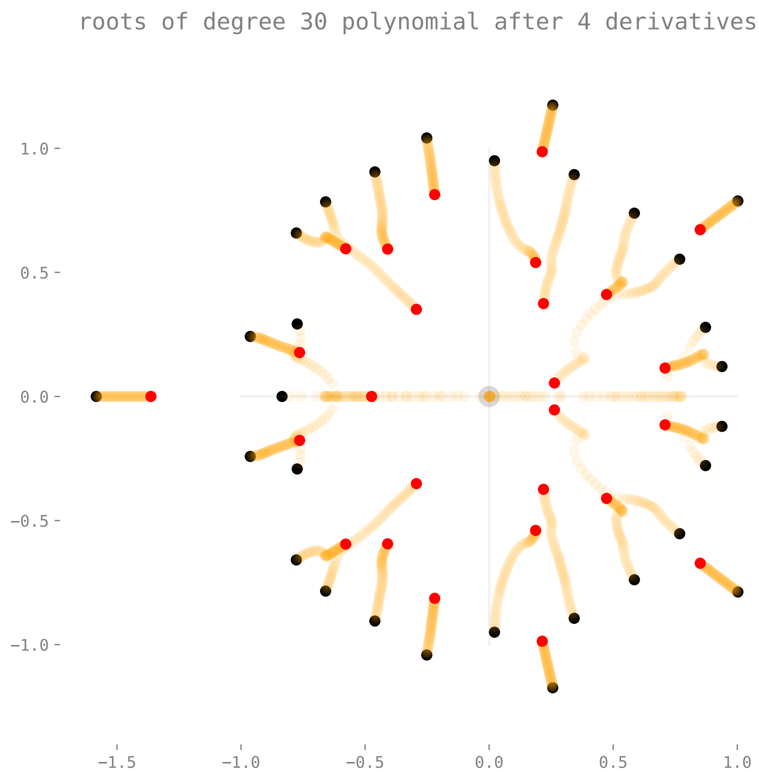
2.2 A New Root Finding Method

Here I present a root finding method that uses fractional calculus and Newton's method. The idea behind the algorithm is that many functions have derivatives (or antiderivatives) whose roots are much easier to compute. These derivative roots can be helpful in computing the roots of the original function when the order of derivatives is made to be continuous. In essence, one can compute the roots of the original function by tracking the roots as they continuously move through differintegration space. At the first iteration, the roots of the more manageable function are computed. At each successive iteration, an order of differintegration α_i is chosen and the roots of the previous iteration are used as the initial guesses for Newton's method, which computes each of the roots one at a time before proceeding to the next iteration.

This algorithm is consistent with already developed theory. In fact, differintegrals are equivalent to a choice of homotopy for the methods used by the Bertini method of finding roots [5].

3 Example: Polynomial Root-Finding

The original motivation for this paper was exploring solutions to systems of random utility functions which can be represented as polynomials with random variable coefficients. To better understand the problem, I looked at the roots of random polynomials where the coefficients were independent draws from a standard normal distribution. It is well known that such roots of single variable polynomials tend to lie on the complex unit circle in $\mathbb{C}$ as the degree of the polynomial tends to infinity. It is also the case that the Gauss-Lucas Theorem guarantees that the roots of a derivative of a polynomial will lie in the convex hull of the roots of the original polynomial. Visualizations of the roots for integer-order derivatives of high degree polynomials led to an investigation of fractional derivatives and their roots and the development of the algorithm described above.



Code and results for the implementation of fractional root finding can be found at `https://github.com/dgmiller/polyrand`.

4 Conclusion

The Weierstrauss Approximation Theorem states that any continuous function defined on a fixed interval can be approximated arbitrarily by a polynomial function. Therefore, the above method can be extended to apply to systems of equations for any system of algebraic functions that are continuous and defined on a fixed interval in $\mathbb{R}$.

The method can also be extended to multidimensional polynomials and hence to general continuous algebraic equations. The continuity of fractional derivatives also applies in the multidimensional case and so constitute a homotopy for multidimensional functions. In general, it is easier to find the roots of lower dimensional polynomials. Thus the roots of an arbitrary, multidimensional polynomial can be found by differentiating down to a lower dimensional sub-polynomial and stepping back through antiderivatives up to the dimension of the original polynomial. This is left for future research.

The fractional root finding method does have some shortcomings in its current form. Solutions to single variable polynomials are straightforward to compute without the use of fractional calculus and the current proposed method is slow compared to existing solutions. Its promise lies in its potential to be simple and effective in multiple dimensions. However, it may suffer from the curse of dimensionality. The initial guesses will likely be far from the actual solution as the dimension of the equation increases, no matter how small the change in the order of the derivative. It may be possible to use the properties of polynomials (such as Boucher's theorem) to overcome this danger. Even so, the algorithm may not be very fast. Studying the properties of multidimensional polynomials with random variable coefficients and the roots of their fractional derivatives may give insight into how to extend this method.

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