Rendering Hasse Diagrams for Lattices using LaTeX

Daniel J. Greenhoe

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This text was typeset using XƎLaTeX, which is part of the TeXfamily of typesetting engines, which is arguably the greatest development since the Gutenberg Press. Graphics were rendered using the *pstricks* and related packages, and LaTeX graphics support.

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The ship appearing throughout this text is loosely based on the *Golden Hind*, a sixteenth century English galleon famous for circumnavigating the globe.[2]

[1] pinyin: Wáng Hàn Zōng Zhōng Mińg Tǐ Fán; translation: Hàn Zōng Wáng's Medium-weight Mińg-style Traditional Characters; literal: 王漢宗~font designer's name; 中~medium; 明~Mińg (a dynasty); 體~style; 繁~traditional

[2] Paine (2000) page 63 ⟨Golden Hind⟩

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CHAPTER 1

INTRODUCTION

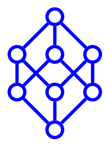
1.1 Ordered sets

This document demonstrates how to render *Hasse diagrams* for finite *lattice*s using LaTeX or XeLaTeX. Both of these software packages are available for free download from `https://ctan.org/`.

A *lattice* is a special case of an *ordered set* (Definition A.2 page 14). An *ordered set* is a set together with an *ordering relation*. However, this amount of structure tends to be insufficient to ensure "well-behaved" mathematical systems. The situation is greatly remedied if every pair of elements in an ordered set (partially or linearly ordered) has both a *least upper bound* and a *greatest lower bound* (Definition A.10 page 20) in the ordered set; in this case, that ordered set is a *lattice* (Definition B.3 page 23). Gian-Carlo Rota (1932–1999) illustrates the advantage of lattices over simple ordered sets by pointing out that the *ordered set* of partitions of an integer "is fraught with pathological properties", while the *lattice* of partitions of a set "remains to this day rich in pleasant surprises".[1]

1.2 Hasse diagrams

A *Hasse diagram* (Definition A.7 page 16) of an *ordered set* is a diagram in which each element of the ordered set is represented by a dot or small circle, and if one element x is less than another element y, then the circle for x is drawn lower than the one for y, and a line connects them. It is not always necessary to label the elements in such a diagram.



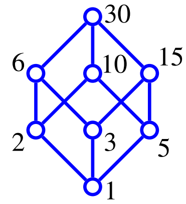
The *Hasse diagram* for a finite *unlabeled* lattice can be plotted within a LaTeX environment using the pst-node package along with a LaTeX source file such as the following:[2]

```
%==============================================================================
% Daniel J. Greenhoe
% LaTeX file
% recommended unit = 5mm
```

[1] Rota (1997) page 1440 ⟨Introduction⟩, Rota (1964) page 498 ⟨partitions of a set⟩

```
%=====================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %-------------------------------
  % nodes
  %-------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){xy}  \Cnode(0,2){xz}  \Cnode(1,2){yz}%
  \Cnode(-1,1){x}   \Cnode(0,1){y}   \Cnode(1,1){z}%
  \Cnode(0,0){b}%
  %-------------------------------
  % node connections
  %-------------------------------
  \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}%
  \ncline{x}{xy}\ncline{x}{xz}%
  \ncline{y}{xy}\ncline{y}{yz}%
  \ncline{z}{xz}\ncline{z}{yz}%
  \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}%
\end{pspicture}%
```

Moreover, one can append labels to a Hasse diagram (e.g. for a *labeled lattice*) as illustrated to the right and as coded below.[3] Such a lattice is over the 8 element set $\{1,2,3,5,6,10,15,30\}$ and the *ordering relation* being the *divides relation* $|$ (x is less than y if x divides y).

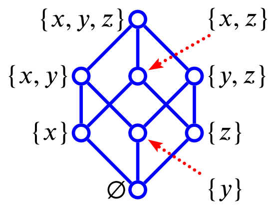



```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
% recommended unit = 7.5mm
%=====================================================================
\begin{pspicture}(-1.6,-\latbot)(1.6,3.3)%
  %-------------------------------
  % nodes
  %-------------------------------
                    \Cnode(0,3){t}%
  \Cnode(-1,2){xy}  \Cnode(0,2){xz}  \Cnode(1,2){yz}%
  \Cnode(-1,1){x}   \Cnode(0,1){y}   \Cnode(1,1){z}%
                    \Cnode(0,0){b}%
  %-------------------------------
  % node connections
  %-------------------------------
  \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}%
  \ncline{x}{xy}\ncline{x}{xz}%
  \ncline{y}{xy}\ncline{y}{yz}%
  \ncline{z}{xz}\ncline{z}{yz}%
  \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}%
  %-------------------------------
  % node labels
  %-------------------------------
  \uput      [ 10](t)  {$30$}%
  \uput      [150](xy) {$6$}%
  \uput{3pt}[ 30](xz) {$10$}%
  \uput      [ 45](yz) {$15$}%
  \uput      [210](x)  {$2$}%
  \uput      [-30](y)  {$3$}%
  \uput      [-45](z)  {$5$}%
  \uput      [-10](b)  {$1$}%
\end{pspicture}%
```

[2] pst-node package: `https://ctan.org/pkg/pst-node`
source file in http://www.github.com/dgreenhoe/hasse: `latu/lat8_2e3.tex`

[3] source file in http://www.github.com/dgreenhoe/hasse: `lat/lat8_2e3_set235.tex`

One important reason why one may *not* want to include labels in a *Hasse diagram* is that many lattices are *isomorphic* to each other, and the labels tend to distract from the nature and fundamental structure of those lattices. For example, the *power set* (Definition C.3 page 33) $2^{\{x,y,z\}}$ of the set $\{x, y, z\}$ with *ordering relation* $\subseteq$, as illustrated to the right,[4] is *isomorphic* to the previous "divides" lattice.



The LaTeX source files as listed previously can be made to output a pdf file with tight borders using the preview package and a "shell" file such as the following:[5]

```
%==========================================================================
% Daniel J. Greenhoe
% XeLaTeX file
% for graphics .tex file to pdf file conversion
% generate tight pdf graphics file for inclusion in a document
%==========================================================================
\input{shelltop.tex}%
\begin{document}%
  \psset{unit=7.5mm}%
  \gsize%
  \input{../../common/math/graphics/lat/lat8_2e3_set235.tex}%
\end{document}%
```

where "shelltop.tex" may be as follows:[6]

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% preamble packages for shell files to
% generate tight pdf graphics file for inclusion in a document
%==========================================================================
\documentclass{article}%
%--------------------------------------
% Style Packages
%--------------------------------------
\usepackage{../../common/sty/packages}%
\usepackage{../../common/sty/fonts}%
\usepackage{../../common/sty/dan}%
\usepackage{../../common/sty/colors_rgb}%
%\usepackage{../../common/sty/colors_cmyk}%
%\usepackage{../../common/sty/colors_gray}%
\usepackage{../../common/sty/math}%
\usepackage{../../common/sty/wavelets}%
\usepackage{../../common/sty/defaults}  % default values
\usepackage{../../common/sty/switches}  %
%--------------------------------------
% color mode
%--------------------------------------
%\selectcolormodel{cmyk}% use cmyk color model
\selectcolormodel{rgb}% use rgb color model
%--------------------------------------
% preview mode
% reference:
          http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput
%--------------------------------------
\usepackage[active,tightpage]{preview}%
\PreviewBorder=0pt%
\PreviewEnvironment{pspicture}%
```

[4] source file in http://www.github.com/dgreenhoe/hasse: `lat/lat8_2e3_setxyz.tex`

[5] http://tex.stackexchange.com/questions/25400/ps2pdf-depscrop-stops-short-with-pstricks-uput

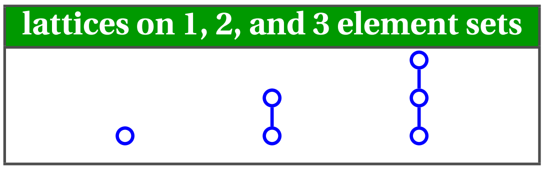
[6] source file in http://www.github.com/dgreenhoe/hasse: `graphics/shelltop.tex`

1.3 Examples

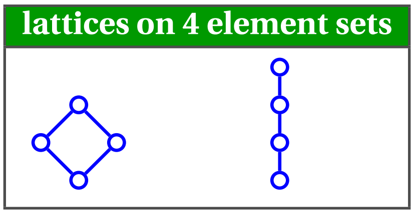
1.3.1 Unlabeled lattices

This subsection demonstrates Hasse diagrams for all the unlabeled lattices of 1–7 elements.

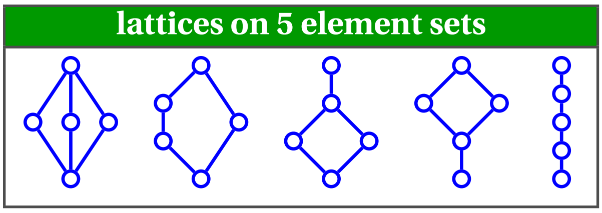
Example 1.1 (lattices on 1–3 element sets). [7]There is only one unlabeled lattice for finite sets with 3 or fewer elements (Proposition B.2 page 29). Thus, these lattices are all *linearly ordered* (Definition A.4 page 15). These 3 lattices are illustrated to the right.



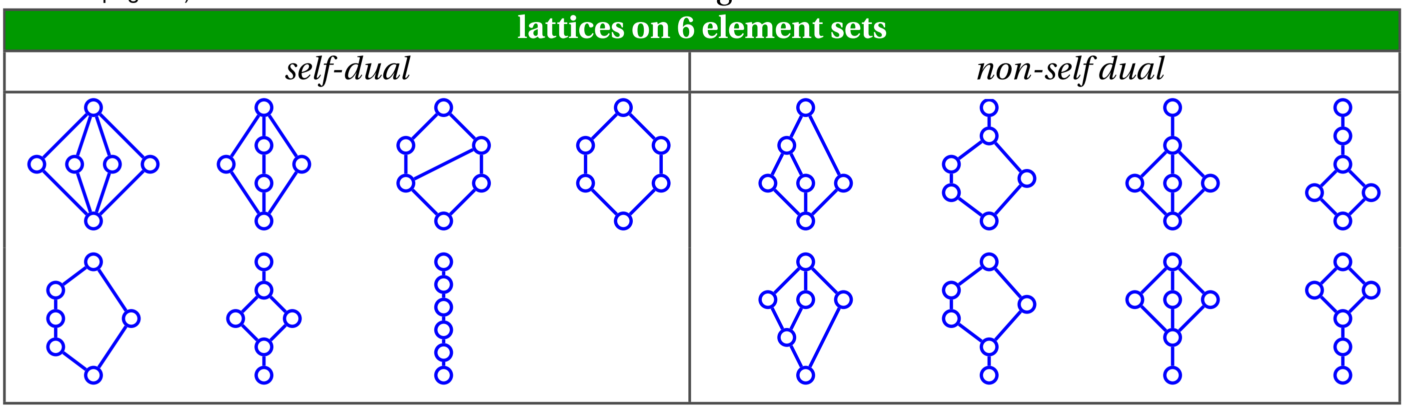
Example 1.2 (lattices on a 4 element set). [8]There are 2 unlabeled lattices on a 4 element set (Proposition B.2 page 29). These are illustrated to the right.



Example 1.3 (lattices on a 5 element set). [9]There are 5 unlabeled lattices on a 5 element set (Proposition B.2 page 29). These are illustrated to the right.



Example 1.4 (lattices on a 6 element set). [10] There are 15 *unlabeled lattices* on a 6 element set (Proposition B.2 page 29). These are illustrated in the following table.



Example 1.5 (lattices on a 7 element set). [11] There are 53 unlabeled lattices on a 7 element set (Proposition B.2 page 29). These are illustrated in Figure 1.1 (page 5).

[7] Kyuno (1979), page 412, Stanley (1997), page 102. For source listings, see Section D.1.1 (page 53). Source files in http://www.github.com/dgreenhoe/hasse: `latu/lat1.tex`, `latu/lat2_l2.tex`, `latu/lat3_l3.tex`

[8] Kyuno (1979), page 412, Stanley (1997), page 102. For source listings, see Section D.1.2 (page 54). Source files in http://www.github.com/dgreenhoe/hasse: `latu/lat4_l4.tex`, `latu/lat4_m2.tex`.

[9] Kyuno (1979), page 413, Stanley (1997), page 102. For source listings, see Section D.1.3 (page 55). Source files in http://www.github.com/dgreenhoe/hasse: `latu/lat5_l2onm2.tex`, `latu/lat5_l5.tex`, `latu/lat5_m2onl2.tex`, `latu/lat5_m3.tex`, `latu/lat5_n5.tex`.

[10] Kyuno (1979), page 413, Stanley (1997), page 102. For source listings, see Section D.1.4 (page 57). Source files in http://www.github.com/dgreenhoe/hasse: see files in the form with `latu/lat6_*.tex`.

[11] Kyuno (1979), pages 413–414. For source listings, see Section D.1.5 (page 64). Source files in http://www.github.com/dgreenhoe/hasse: see files in the form with `latu/lat7_*.tex`.

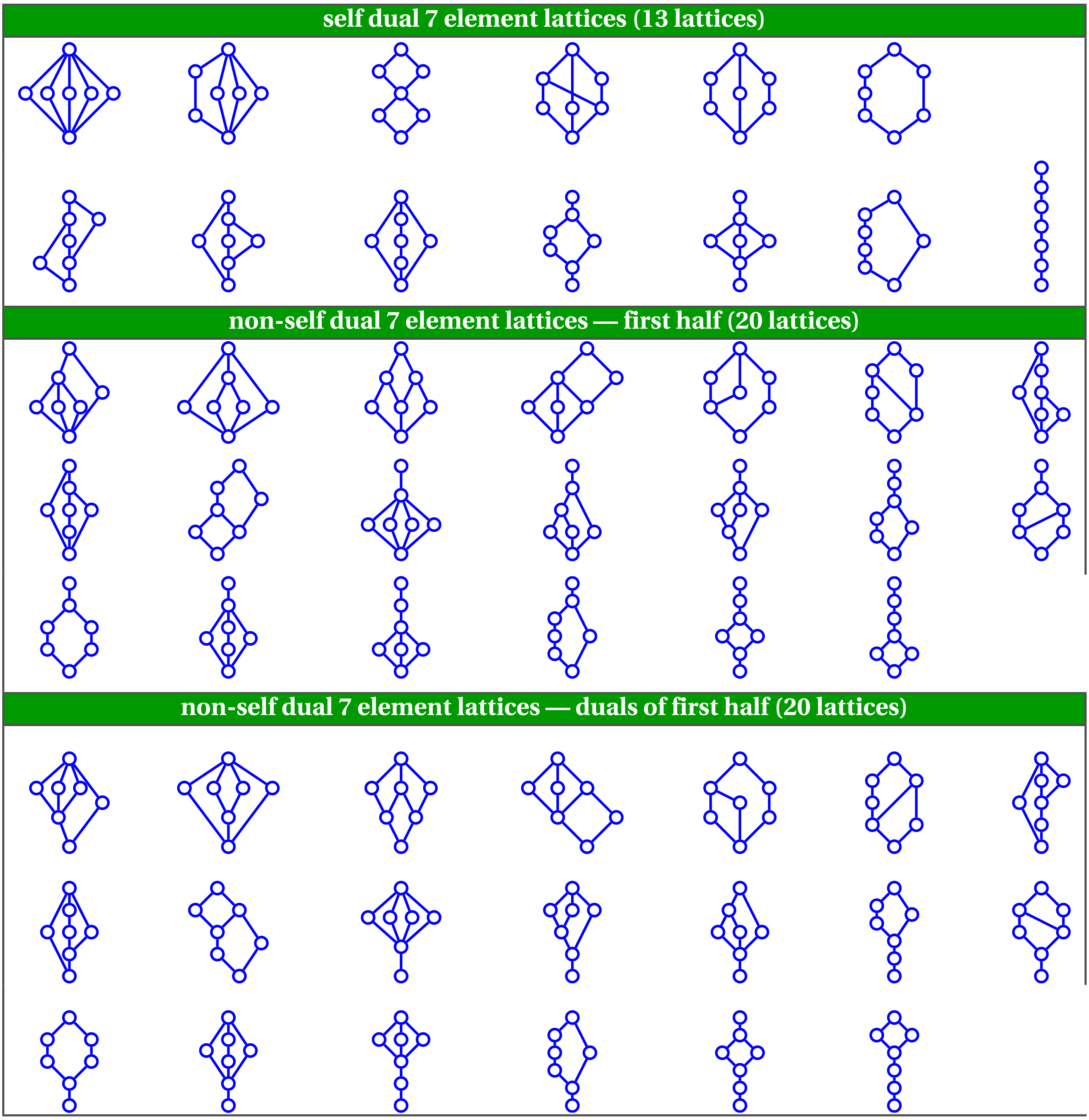
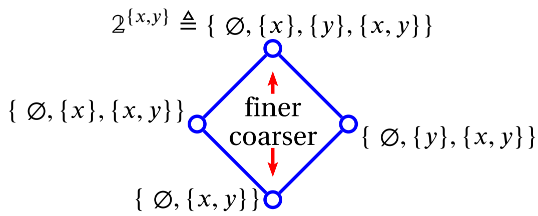


Figure 1.1: The 53 unlabeled lattices on a 7 element set (Example 1.5 page 4)

Example 1.6 (lattices on 8 element sets). There are 222 unlabeled lattices on a 8 element set (Proposition B.2 page 29). See Kyuno's paper[12] for Hasse diagrams of all 222 lattices. Alternatively, one might try using the software package *Mathematica®* with something like[13] `ShowGraph[HasseDiagram[....`

1.3.2 Labeled lattices

Example 1.7. [14]Example C.2 (page 37) lists the four topologies on the set $X \triangleq \{x, y\}$. The lattice of these topologies $\left(\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{T}_4\}, \cup, \cap; \subseteq\right)$ is illustrated by the *Hasse diagram* to the right. For LaTeX source code to produce such a Hasse diagram, see the following:




```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 10mm
%============================================================================
\begin{pspicture}(-3.5,-\latbot)(3.5,2.6)%
  %---------------------------------------
  % settings
  %---------------------------------------
  %\psset{labelsep=1.5mm}%
  %---------------------------------------
  % nodes
  %---------------------------------------
  \Cnode( 0,2){top}%
  \Cnode(-1,1){left}%
  \Cnode( 1,1){right}%
  \Cnode( 0,0){bottom}%
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{top}{left} \ncline{top}{right}%
  \ncline{bottom}{left} \ncline{bottom}{right}%
  %\ncline{finer}  {top}
  %\ncline{coarser}{bottom}
  %---------------------------------------
  % node labels
  %---------------------------------------
  \uput[ 90](top){$\ssP{\setn{x,y}}\eqd\setn{\;\szero, \setn{x}, \setn{y}, \setn{x,y}}$}%
  \uput[150](left){$\setn{\;\szero, \setn{x}, \setn{x,y}}$}%
  \uput[-30](right){$\setn{\;\szero, \setn{y}, \setn{x,y}}$}%
  \uput[-180](bottom){$\setn{\;\szero, \setn{x,y}}$}%
  %---------------------------------------
  % additional labeling
  %---------------------------------------
  \rput[b](0,1.1){\rnode{finer}  {finer}}%
  \rput[t](0,0.9){\rnode{coarser}{coarser}}%
  \ncline[linecolor=red,nodesepA=1pt,nodesepB=5pt]{->}{finer}{top}%
  \ncline[linecolor=red,nodesepA=1pt,nodesepB=5pt]{->}{coarser}{bottom}%
\end{pspicture}%
```

Example 1.8. [15] Example C.3 (page 37) lists the 29 topologies $\mathcal{T}(\{x, y, z\})$. The lattice of these 29 topologies $(\mathcal{T}(\{x, y, z\}), \cup, \cap; \subseteq)$ is illustrated in Figure 1.2 (page 7). For LaTeX source code to produce such a Hasse diagram, see the following:

---

[12] Kyuno (1979), pages 415–421

[13] by Sriram Pemmaraju and Skiena (2003) page 30 ⟨In[96]⟩

[14] Isham (1999), page 44, Isham (1989), page 1515. Source file in http://www.github.com/dgreenhoe/hasse: `setstr/lattopxy.tex`.

[15] Isham (1999), page 44, Isham (1989), page 1516, Steiner (1966), page 386, Greenhoe (2017), pages 10–12 ⟨Example 1.13⟩. Source files in http://www.github.com/dgreenhoe/hasse: `setstr/lattopxyz.tex`.

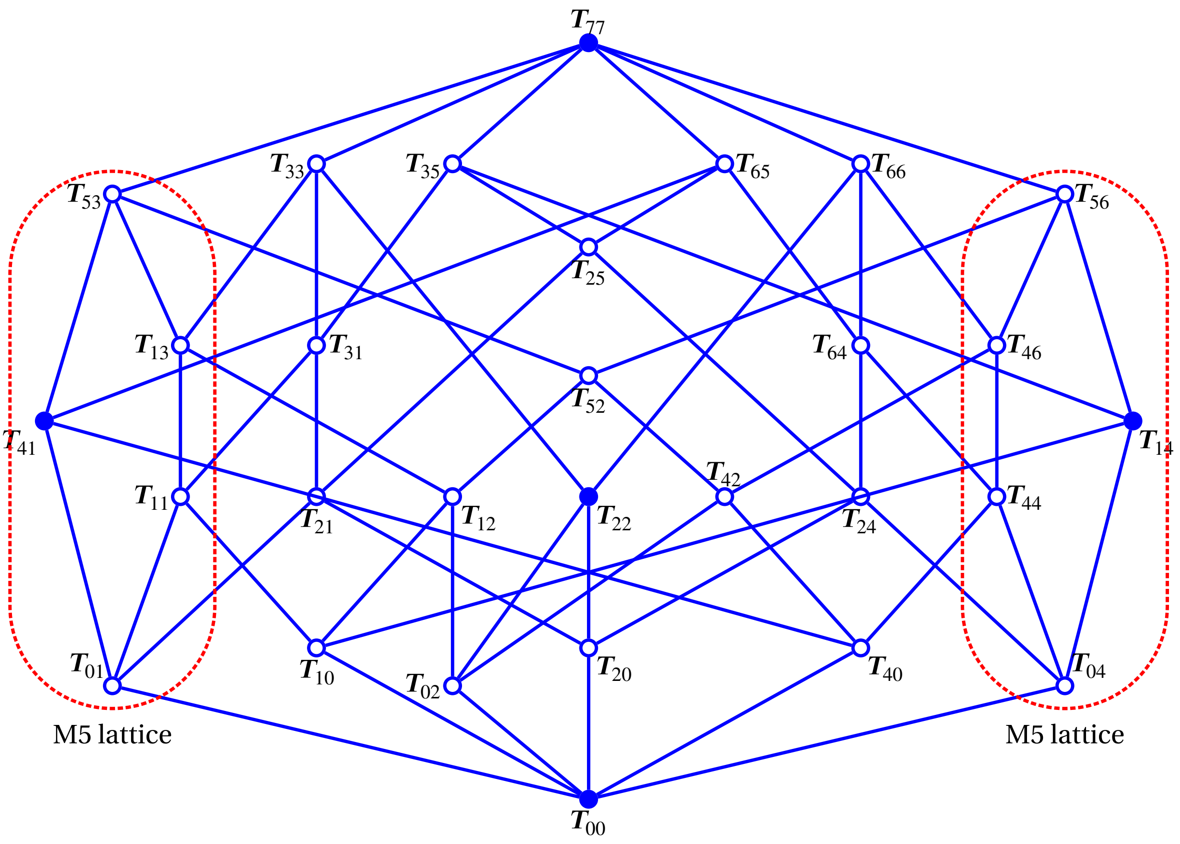


Figure 1.2: Lattice of *topologies* on $X \triangleq \{x, y, z\}$ (see Example 1.8 page 6)

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX File
% lattice of topologies over the set {x,y,z}
% nominal unit=10mm
%============================================================================
{\psset{xunit=1.8\psunit,yunit=2.0\psunit}%
\begin{pspicture}(-4.4,-0.40)(4.40,5.40)%
%---------------------------------
% settings
%---------------------------------
\psset{
  labelsep=1.5mm,
  }
%---------------------------------
% nodes
%---------------------------------
\Cnode*(0,5)          {T77}%
%
\Cnode ( 2.0,4.2)   {T66}%
\Cnode ( 3.5,4.0)   {T56}%
\Cnode ( 1.0,4.2)   {T65}%
\Cnode (-1.0,4.2)   {T35}%
\Cnode (-3.5,4.0)   {T53}%
\Cnode (-2.0,4.2)   {T33}%
%
\Cnode ( 3,3)       {T46}%
\Cnode ( 0,3.65)    {T25}%
\Cnode (-3,3)       {T13}%
\Cnode ( 2,3)       {T64}%
\Cnode ( 0,2.8)     {T52}%
\Cnode (-2,3)       {T31}%
%
\Cnode ( 3,2)       {T44}%
\Cnode ( 2,2)       {T24}%
\Cnode*( 4,2.5)     {T14}%
\Cnode ( 1,2)       {T42}%
```

```
\Cnode*( 0,2)       {T22}%
\Cnode (-1,2)       {T12}%
\Cnode*(-4,2.5)     {T41}%
\Cnode (-2,2)       {T21}%
\Cnode (-3,2)       {T11}%
%
\Cnode ( 2,   1)    {T40}%
\Cnode ( 0,   1)    {T20}%
\Cnode (-2,   1)    {T10}%
\Cnode ( 3.5, 0.75) {T04}%
\Cnode (-1,   0.75) {T02}%
    \Cnode (-3.5, 0.75) {T01}%
\Cnode*( 0,   0)    {T00}%
%-------------------------------
% node connections
%-------------------------------
\ncline{T77}{T33}%
\ncline{T77}{T53}%
\ncline{T77}{T35}%
\ncline{T77}{T65}%
\ncline{T77}{T56}%
\ncline{T77}{T66}%
%
\ncline{T33}{T31}%
\ncline{T33}{T22}%
\ncline{T33}{T13}%
\ncline{T53}{T41}%
\ncline{T53}{T52}%
\ncline{T53}{T13}%
\ncline{T35}{T31}%
\ncline{T35}{T14}%
\ncline{T35}{T25}%
\ncline{T65}{T64}%
\ncline{T65}{T41}%
\ncline{T65}{T25}%
\ncline{T56}{T52}%
\ncline{T56}{T14}%
\ncline{T56}{T46}%
\ncline{T66}{T64}%
\ncline{T66}{T22}%
\ncline{T66}{T46}%
%
\ncline{T31}{T11}%
\ncline{T31}{T21}%
\ncline{T52}{T12}%
\ncline{T52}{T42}%
\ncline{T64}{T24}%
\ncline{T64}{T44}%
\ncline{T13}{T11}%
\ncline{T13}{T12}%
\ncline{T25}{T21}%
\ncline{T25}{T24}%
\ncline{T46}{T42}%
\ncline{T46}{T44}%
%
\ncline{T01}{T11}%
\ncline{T01}{T21}%
\ncline{T01}{T41}%
\ncline{T02}{T12}%
\ncline{T02}{T22}%
\ncline{T02}{T42}%
\ncline{T04}{T14}%
\ncline{T04}{T24}%
\ncline{T04}{T44}%
\ncline{T10}{T11}%
\ncline{T10}{T12}%
\ncline{T10}{T14}%
\ncline{T20}{T21}%
\ncline{T20}{T22}%
\ncline{T20}{T24}%
\ncline{T40}{T41}%
\ncline{T40}{T42}%
\ncline{T40}{T44}%
%
\ncline{T00}{T01}%
\ncline{T00}{T02}%
\ncline{T00}{T04}%
```

```
\ncline{T00}{T10}%
\ncline{T00}{T20}%
\ncline{T00}{T40}%
%---------------------------------
% node labels
%---------------------------------
\uput[ 90](T77){$\topT_{77}$}%
\uput[  0](T66){$\topT_{66}$}%
\uput[  0](T56){$\topT_{56}$}%
\uput[  0](T65){$\topT_{65}$}%
\uput[180](T35){$\topT_{35}$}%
\uput[180](T53){$\topT_{53}$}%
\uput[180](T33){$\topT_{33}$}%
\uput[  0](T46){$\topT_{46}$}%
\uput[-90](T25){$\topT_{25}$}%
\uput[180](T13){$\topT_{13}$}%
\uput[180](T64){$\topT_{64}$}%
\uput[-90](T52){$\topT_{52}$}%
\uput[  0](T31){$\topT_{31}$}%
\uput[  0](T44){$\topT_{44}$}%
\uput[-90](T24){$\topT_{24}$}%
\uput[-45](T14){$\topT_{14}$}%
\uput[ 90](T42){$\topT_{42}$}%
\uput[-45](T22){$\topT_{22}$}%
\uput[-45](T12){$\topT_{12}$}%
\uput[225](T41){$\topT_{41}$}%
\uput[-90](T21){$\topT_{21}$}%
\uput[180](T11){$\topT_{11}$}%
\uput[-45](T40){$\topT_{40}$}%
\uput[-45](T20){$\topT_{20}$}%
\uput[-90](T10){$\topT_{10}$}%
\uput[ 45](T04){$\topT_{04}$}%
\uput[180](T02){$\topT_{02}$}%
\uput[135](T01){$\topT_{01}$}%
\uput[-90](T00){$\topT_{00}$}%
%---------------------------------
% discriptions
%---------------------------------
\psset{
   nodesep=5pt,
   boxsize=0.75\psxunit,
   linestyle=dashed,
   linecolor=red,
   %cornersize=relative,% doesn't seem to work,
   %framearc=1,          % at least with XeLaTeX
   cornersize=absolute,
   linearc=0.75\psxunit,
   }%
\ncbox{T01}{T53}% left  M5 sublattice
\ncbox{T04}{T56}% right M5 sublattice
\rput[t](-3.5,0.5){M5 lattice}
\rput[t]( 3.5,0.5){M5 lattice}
\end{pspicture}}%
```

1.3.3 Lattices of lattices

It is even possible to draw lattices within lattices (draw Hasse diagrams within Hasse diagrams). A Hasse diagram for the *lattice of topologies* on a 3 element set is described in Example 1.9 (page 10) and illustrated in Figure 1.3 (page 10). LaTeX source code for rendering such a diagram is listed in Section D.3 (page 86).

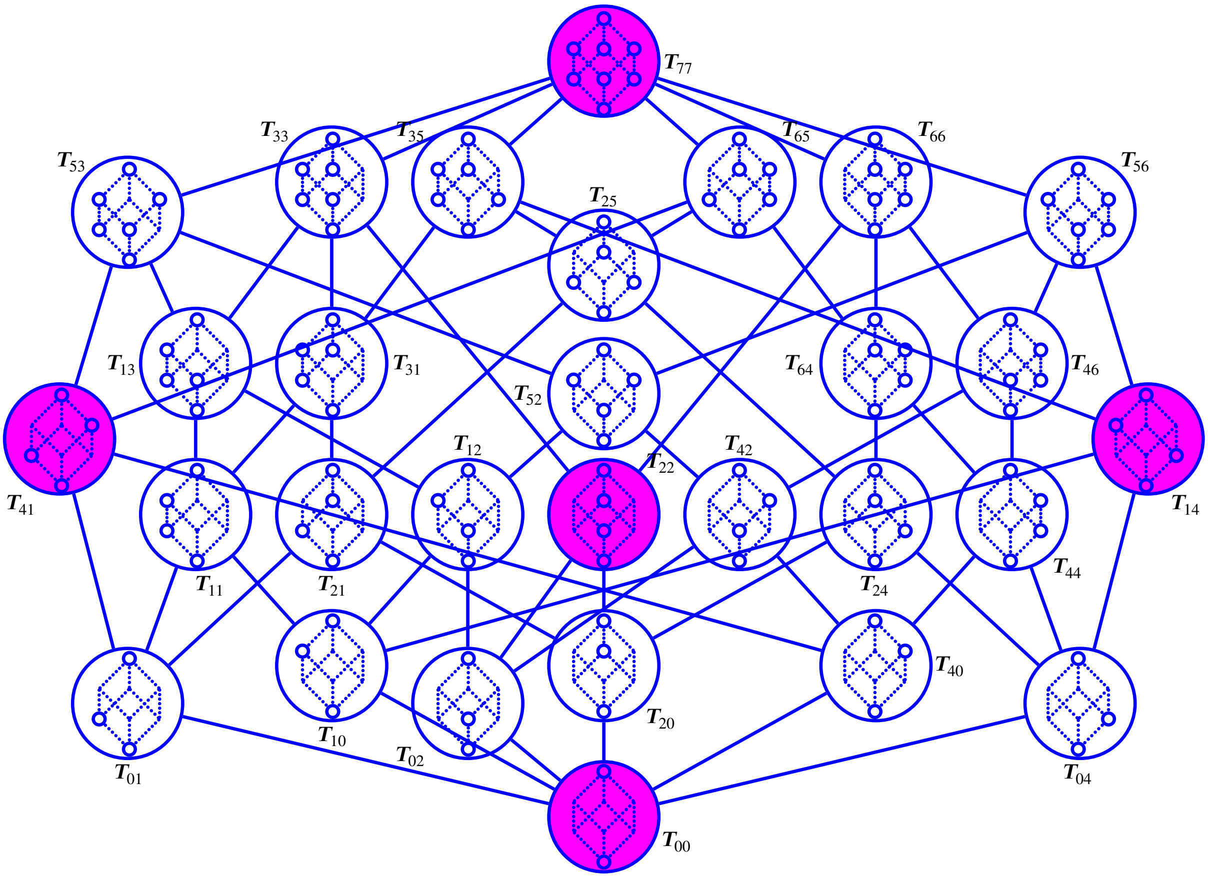
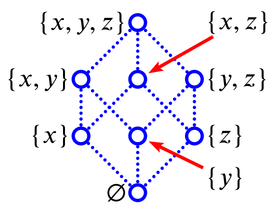


Figure 1.3: Lattice of *topologies* on $X \triangleq \{x, y, z\}$ (see Example 1.9 page 10)

Example 1.9. [16]Let a given topology in $\mathcal{T}(\{x, y, z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the topology. Example C.3 (page 37) lists the 29 topologies $\mathcal{T}(\{x, y, z\})$. The lattice of these 29 topologies $(\mathcal{T}(\{x, y, z\}), \cup, \cap; \subseteq)$ is illustrated in Figure 1.3 (page 10). The five topologies $\boldsymbol{T}_1$, $\boldsymbol{T}_{41}$, $\boldsymbol{T}_{22}$, $\boldsymbol{T}_{14}$, and $\boldsymbol{T}_{77}$ are also *algebras of sets*; these five sets are shaded in Figure C.5 and represented as solid dots in Figure 1.2. For LaTeX source code to produce such a Hasse diagram, see Section D.3 (page 86).



[16] Greenhoe (2016), page 226 ⟨Example 14.14⟩, Greenhoe (2017), pages 10–11 ⟨Example 1.13⟩, Isham (1999), page 44, Isham (1989), page 1516, Steiner (1966), page 386. Source files in http://www.github.com/dgreenhoe/hasse: `setstr/lat2xyzdotted.tex`, `setstr/latlattopxyz.tex`.

Part I

Appendices

APPENDIX A

ORDER

Equivalence relations require *symmetry* ($x \bowtie y \iff y \bowtie x$). However another very important type of relation, the *order relation*, actually requires *anti-symmetry*. This chapter presents some useful structures regarding order relations. Ordering relations on a set allow us to *compare* some pairs of elements in a set and determine whether or not one element is *less than* another. In this case, we say that those two elements are *comparable*; otherwise, they are *incomparable*. A set together with an order relation is called an *ordered set*, a *partially ordered set*, or a *poset* (Definition A.2 page 14).

A.1 Preordered sets

Definition A.1. [1] *Let X be a set.*

DEF

*A relation $\sqsubseteq$ is a **preorder relation** on X if*

1. $x \sqsubseteq x$ $\quad \forall x \in X$ $\quad$ (REFLEXIVE) $\quad$ *and*
2. $x \sqsubseteq y$ *and* $y \sqsubseteq z \implies x \sqsubseteq z$ $\quad \forall x,y,z \in X$ $\quad$ (TRANSITIVE)

*A **preordered set** is the pair* $(X, \sqsubseteq)$.

Example A.1. [2]

EX

$\sqsubseteq$ is a *preorder relation* on the set of *positive integers* $\mathbb{N}$ if

$$n \sqsubseteq m \iff (p \text{ is a prime factor of } n \implies p \text{ is a prime factor of } m)$$

[1] Schröder (2003) page 115, Brown and Watson (1991), page 317

[2] Shen and Vereshchagin (2002) page 43

A.2 Order relations

Definition A.2. [3] *Let X be a set. Let 2^{XX} be the set of all relations on X.*

DEF

*A relation $\le$ is an **order relation** in 2^{XX} if*

1. $x \le x$ $\quad \forall x \in X$ *(reflexive)* and
2. $x \le y$ and $y \le z \implies x \le z$ $\quad \forall x,y,z \in X$ *(transitive)* and
3. $x \le y$ and $y \le x \implies x = y$ $\quad \forall x,y \in X$ *(anti-symmetric)*

(1 and 2: preorder)

*An **ordered set** is the pair $(X, \le)$. The set X is called the **base set** of $(X, \le)$. If $x \le y$ or $y \le x$, then elements x and y are said to be **comparable**, denoted $x \sim y$. Otherwise they are **incomparable**, denoted $x \| y$. The relation $\lneq$ is the relation $\le \setminus =$ ("less than but not equal to"), where $\setminus$ is the* SET DIFFERENCE *operator, and $=$ is the equality relation. An order relation is also called a **partial order relation**. An ordered set is also called a **partially ordered set** or **poset**.*

The familiar relations $\ge$, $<$, and $>$ (next) can be defined in terms of the order relation $\le$ (Definition A.2—previous).

Definition A.3. [4] *Let $(X, \le)$ be an ordered set.*

DEF

The relations $\ge, <, > \in 2^{XX}$ are defined as follows:

$$x \ge y \overset{\text{def}}{\iff} y \le x \qquad \forall x,y \in X$$

$$x \lneq y \overset{\text{def}}{\iff} x \le y \text{ and } x \ne y \qquad \forall x,y \in X$$

$$x \gneq y \overset{\text{def}}{\iff} x \ge y \text{ and } x \ne y \qquad \forall x,y \in X$$

*The relation $\ge$ is called the **dual** of $\le$.*

Theorem A.1. [5] *Let X be a set.*

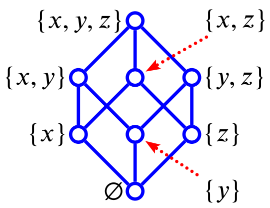
THM

$$(X, \le) \text{ is an ordered set} \iff (X, \ge) \text{ is an ordered set}$$

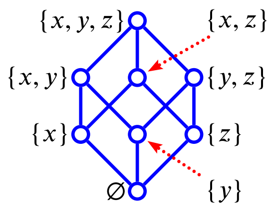
Example A.2.

EX

order relation		dual order relation	
$\le$	(integer less than or equal to)	$\ge$	(integer greater than or equal to)
$\subseteq$	(subset)	$\supseteq$	(super set)
$\mid$	(divides)		(divided by)
$\implies$	(implies)	$\impliedby$	(implied by)



Example A.3. The Hasse diagram to the left illustrates the ordered set $\left(2^{\{x,y,z\}}, \subseteq\right)$ and the Hasse diagram to the right illustrates its dual $\left(2^{\{x,y,z\}}, \supseteq\right)$.



[3] MacLane and Birkhoff (1999) page 470, Beran (1985) page 1, Korselt (1894) page 156 ⟨I, II, (1)⟩, Dedekind (1900) page 373 ⟨I–III⟩

[4] Peirce (1880) page 2

[5] Grätzer (1998), page 3

A.3 Linearly ordered sets

In an ordered set we can say that some element is less than or equal to some other element. That is, we can say that these two elements are *comparable*—we can *compare* them to see which one is lesser or equal to the other. But it is very possible that there are two elements that are not comparable, or *incomparable*. That is, we cannot say that one element is less than the other—it is simply not possible to compare them because their ordered pair is not an element of the order relation.

For example, in the ordered set $(2^{\{x,y,z\}}, \subseteq)$ of Example A.9, we can say that $\{x\} \subseteq \{x,z\}$ (we can compare these two sets with respect to the order relation $\subseteq$), but we cannot say $\{y\} \subseteq \{x,z\}$, nor can we say $\{x,z\} \subseteq \{y\}$. Rather, these two elements $\{y\}$ and $\{x,z\}$ are simply *incomparable*.

However, there are some ordered sets in which every element is comparable with every other element; and in this special case we say that this ordered set is a *totally ordered* set or is *linearly ordered* (next definition).

Definition A.4. [6]

DEF

*A relation $\le$ is a **linear order relation** on X if*

1. *$\le$ is an* ORDER RELATION *(Definition A.2 page 14) and*
2. *$x \le y$ or $y \le x \quad \forall x,y \in X$ (*COMPARABLE*).*

*A **linearly ordered set** is the pair $(X, \le)$.*
*A linearly ordered set is also called a **totally ordered set**, a **fully ordered set**, and a **chain**.*

Definition A.5 (poset product). [7]

DEF

*The **product** $P \times Q$ of ordered pairs $P \triangleq (X, \precsim)$ and $Q \triangleq (Y, \trianglelefteq)$ is the ordered pair $(X \times Y, \le)$ where*

$$(x_1, y_1) \le (x_2, y_2) \quad \overset{\text{def}}{\iff} \quad x_1 \precsim x_2 \text{ and } y_1 \trianglelefteq y_2 \qquad \forall x_1,x_2 \in X;\ y_1,y_2 \in Y$$

A.4 Representation

Definition A.6. [8]

DEF

*y **covers** x in the ordered set $(X, \le)$ if*

1. *$x \le y$ (y is greater than x) and*
2. *$(x \le z \le y) \implies (z = x \text{ or } z = y)$ (there is no element between x and y).*

The case in which y covers x is denoted
$x \prec y$.

Example A.4. Let $(\{x,y,z\}, \le)$ be an ordered set with cover relation $\prec$.

EX

$$\{x < y < z\} \implies \left\{\begin{array}{lll} y & \text{covers} & x \\ z & \text{covers} & y \\ z & \text{does } \textbf{not} \text{ cover} & x \end{array}\right\}$$

An ordered set can be represented in four ways:

1. Hasse diagram
2. tables

[6] MacLane and Birkhoff (1999) page 470, Ore (1935) page 410
[7] Birkhoff (1948) page 7, MacLane and Birkhoff (1967), page 489
[8] Birkhoff (1933) page 445

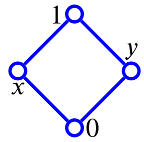
3. set of ordered pairs of order relations
4. set of ordered pairs of cover relations

Definition A.7. *Let $(X, \leq)$ be an ordered pair.*

DEF *A diagram is a **Hasse diagram** of $(X, \leq)$ if it satisfies the following criteria:*
- *Each element in X is represented by a dot or small circle.*
- *For each $x, y \in X$, if $x \prec y$, then y appears at a higher position than x and a line connects x and y.*

Example A.5. Here are three ways of representing the ordered set $\left(2^{\{x,y\}}, \subseteq\right)$;

1. **Hasse diagrams**: If two elements are comparable, then the lesser of the two is drawn lower on the page than the other with a line connecting them.

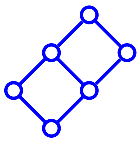


2. Sets of ordered pairs specifying *order relations* (Definition A.2 page 14):

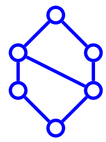
$$\subseteq = \left\{ \begin{array}{ccccc} (\varnothing, \varnothing), & (\{x\},\{x\}), & (\{y\},\{y\}), & (\{x,y\},\{x,y\}), & \\ (\varnothing,\{x\}), & (\varnothing,\{y\}), & (\varnothing,\{x,y\}), & (\{x\},\{x,y\}), & (\{y\},\{x,y\}) \end{array} \right\}$$

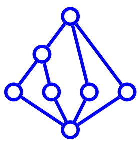
3. Sets of ordered pairs specifying *covering relations*:

$$\prec = \left\{ (\varnothing,\{x\}), \; (\varnothing,\{y\}), \; (\{x\},\{x,y\}), (\{y\},\{x,y\}) \right\}$$

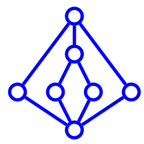


Example A.6. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

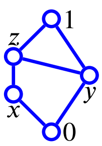




Example A.7. The Hasse diagrams to the left and right represent *equivalent* ordered sets. They are simply drawn differently.

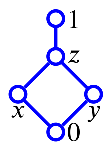


Example A.8. The Hasse diagrams to the left and right represent *equivalent* ordered sets.



In particular, the line extending from 1 to y in the diagram to the left is redundant because other lines already indicate that $z \leq 1$ and $y \leq z$; and thus by the *transitive* property (Definition A.2 page 14), these two relations imply $1 \leq y$. A more concise explanation is that both have the same convering relation:

$$\prec = \{(z,1), (x,z), (0,x), (y,z), (0,y)\}$$



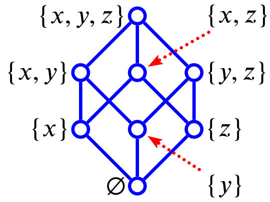
A.5 Examples

Examples of order relations include the following:

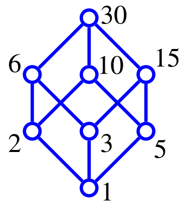
- set inclusion order relation: Example A.9 page 17
- integer divides order relation: Example A.10 page 17

- linear operator order relation: Example A.11 page 17
- projection operator order relation: Example A.12 page 17
- integer order relation: Example A.13 page 17
- coordinatewise order relation Example A.14 page 17
- lexicographical order relation Example A.15 page 18

Example A.9 (Set inclusion order relation). [9]Let X be a set, 2^X the power set of X, and $\subseteq$ the set inclusion relation. Then, $\subseteq$ is an *order relation* on the set 2^X and the pair $(2^X, \subseteq)$ is an ordered set. The ordered set $(2^{\{x,y,z\}}, \subseteq)$ is illustrated to the right by its *Hasse diagram*.



Example A.10 (Integer divides order relation). [10]Let $|$ be the "divides" relation on the set $\mathbb{N}$ of positive integers such that $n|m$ represents m divides n. Then $|$ is an *order relation* on $\mathbb{N}$ and the pair $(\mathbb{N}, |)$ is an *ordered set*. The ordered set $(\{n \in \mathbb{N} | n|2 \text{ or } n|3 \text{ or } n|5\}, |)$ is illustrated by a *Hasse diagram* to the right.



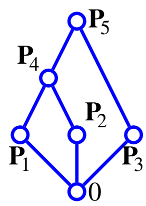
Example A.11 (Operator order relation). [11] Let $\boldsymbol{X}$ be an inner-product space. We can define the order relation $\lesssim$ on the linear operators $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3 \ldots \in X^X$ as follows:

$$\mathbf{L}_1 \lesssim \mathbf{L}_2 \quad \overset{\text{def}}{\iff} \quad \langle \mathbf{L}_2 \boldsymbol{x} - \mathbf{L}_1 \boldsymbol{x} \,|\, \boldsymbol{x} \rangle \geq 0 \qquad \forall \boldsymbol{x} \in \boldsymbol{X}$$

Example A.12 (Projection operator order relation). [12]Let $(\!(\boldsymbol{V}_n)\!)$ be a sequence of subspaces in a Hilbert space $\boldsymbol{X}$. We can define a projection operator $\mathbf{P}_n$ for every subspace $\boldsymbol{V}_n \subseteq \boldsymbol{X}$ in a subspace lattice such that

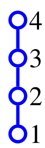
$$\boldsymbol{V}_n = \mathbf{P}_n \boldsymbol{X} \qquad \forall n \in \mathbb{Z}.$$

Each projection operator $\mathbf{P}_n$ in the lattice "projects" the range space $\boldsymbol{X}$ onto a subspace $\boldsymbol{V}_n$. We can define an order relation on the projection operators as follows:



$$\mathbf{P}_1 \leq \mathbf{P}_2 \quad \overset{\text{def}}{\iff} \quad \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1$$

Example A.13 (Integer order relation). Let $\leq$ be the standard order relation on the set of integers $\mathbb{Z}$. Then the ordered pair $(\mathbb{Z}, \leq)$ is a totally ordered set. The totally ordered set $(\{1, 2, 3, 4\}, \leq)$ is illustrated to the right. Other familiar examples of totally ordered sets include the pair $(\mathbb{Q}, \leq)$ (where $\mathbb{Q}$ is the set of rational numbers) and $(\mathbb{R}, \leq)$ (where $\mathbb{R}$ is the set of real numbers).



Example A.14 (Coordinatewise order relation). [13] Let $(X, \leq)$ be an ordered set. Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

The **coordinatewise order relation** $\lesssim$ on the Cartesian product X^n is defined for all $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \lesssim \boldsymbol{y} \quad \overset{\text{def}}{\iff} \quad \{x_1 \leq y_1 \text{ and } x_2 \leq y_2 \text{ and } \ldots \text{ and } x_n \leq y_n\}$$

[9] Menini and Oystaeyen (2004) pages 56–57
[10] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩
[11] Michel and Herget (1993) page 429, Pedersen (2000) page 87
[12] Isham (1999) pages 21–22, Dunford and Schwartz (1957), page 481, Svozil (1994) page 72
[13] Shen and Vereshchagin (2002) page 43

Example A.15 (Lexicographical order relation). [14] Let $(X, \leq)$ be an ordered set.
Let $\boldsymbol{x} \triangleq (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} \triangleq (y_1, y_2, \ldots, y_n)$.

EX

The **lexicographical order relation** $\precsim$ on the Cartesian product X^n is defined forall $\boldsymbol{x}, \boldsymbol{y} \in X^n$ as

$$\boldsymbol{x} \precsim \boldsymbol{y} \stackrel{\text{def}}{\iff} \left\{\begin{array}{llll} (\; x_1 < y_1 & & &) \text{ or} \\ (\; x_2 < y_2 & \text{and} & x_1 = y_1 &) \text{ or} \\ (\; x_3 < y_3 & \text{and} & (x_1, x_2) = (y_1, y_2) &) \text{ or} \\ (\; \ldots & \ldots & \ldots &) \text{ or} \\ (\; x_{n-1} < y_{n-1} & \text{and} & (x_1, x_2, \ldots, x_{n-2}) = (y_1, y_2, \ldots, y_{n-2}) &) \text{ or} \\ (\; x_n \leq y_n & \text{and} & (x_1, x_2, \ldots, x_{n-1}) = (y_1, y_2, \ldots, y_{n-1}) &) \end{array}\right\}$$

The lexicographical order relation is also called the **dictionary order relation** or **alphabetic order relation**.

Definition A.8.

DEF

*An ordered set is **labeled** if the labels on the elements are significant.*
*An ordered set is **unlabeled** if the labels on the elements are not significant.*

Proposition A.1. [15] *Let X_n be a finite set with order $n = |X_n|$. Let P_n be the number of labeled ordered sets on X_n and p_n the number of unlabeled ordered sets.*

PRP

n	0	1	2	3	4	5	6	7	8	9
P_n	1	1	3	19	219	4231	130,023	6,129,859	431,723,379	44,511,042,511
p_n	1	1	2	5	16	63	318	2045	16,999	183,231

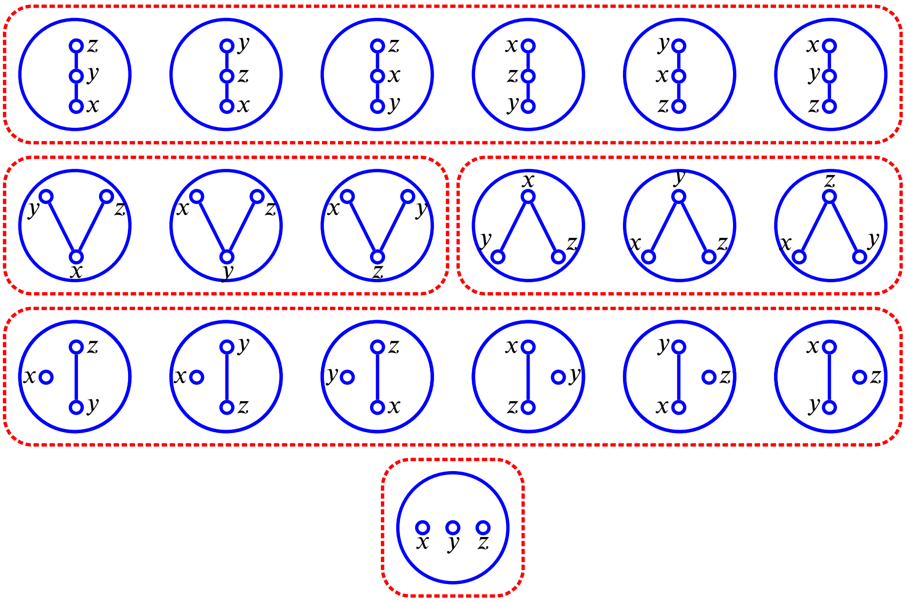


Figure A.1: All possible orderings of the set $\{x, y, z\}$ (Example A.16 page 18).

Example A.16. Proposition A.1 (page 18) indicates that there are exactly 19 labeled order relations on the set $\{x, y, z\}$ and 5 unlabeled order relations.

EX

The 19 labeled order relations on $\{x, y, z,\}$ are represented here using three methods:

1. Hasse diagrams: Figure A.1 page 18
2. order relations: Table A.2 page 19
3. covering relations: Table A.3 page 19

[14] Shen and Vereshchagin (2002) page 44, Halmos (1960) page 58, Hausdorff (1937) page 54

[15] Sloane (2014) ⟨http://oeis.org/A001035⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Comtet (1974) page 60, Brinkmann and McKay (2002)

labeled order relations on $\{x, y, z\}$		
$\le_1$	$=$	$\{\ (x,x),\ (y,y),\ (z,z)\ \}$
$\le_2$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (y,z)\ \}$
$\le_3$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (z,y)\ \}$
$\le_4$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,z)\ \}$
$\le_5$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (z,x)\ \}$
$\le_6$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,y)\ \}$
$\le_7$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (y,x)\ \}$
$\le_8$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,y),\ (x,z)\ \}$
$\le_9$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,y),\ (y,z)\ \}$
$\le_{10}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (z,x),\ (z,y)\ \}$
$\le_{11}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (y,x),\ (z,x)\ \}$
$\le_{12}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,y),\ (z,y)\ \}$
$\le_{13}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,z),\ (y,z)\ \}$
$\le_{14}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,y),\ (y,z),\ (x,z)\ \}$
$\le_{15}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (x,z),\ (x,y),\ (z,y)\ \}$
$\le_{16}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (y,x),\ (y,z),\ (x,z)\ \}$
$\le_{17}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (y,z),\ (y,x),\ (z,x)\ \}$
$\le_{18}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (z,x),\ (z,y),\ (x,y)\ \}$
$\le_{19}$	$=$	$\{\ (x,x),\ (y,y),\ (z,z),\ (z,y),\ (z,x),\ (y,x)\ \}$

Table A.2: labeled order relations on $\{x, y, z\}$

In each of these three methods, the 19 *labeled* order relations are arranged into 5 groups, each group representing one of the 5 *unlabeled* order relations.

A.6 Bounds on ordered sets

In an *ordered set* (Definition A.2 page 14), a pair of elements $\{x, y\}$ may not be *comparable*. Despite this, we may still be able to find elements that *are* comparable to both x and y and are "*greater*" than both of them. Such a greater element is called an *upper bound* of x and y. There may be many elements that are upper bounds of x and y. But if one of these upper bounds is comparable with and is smaller than all the other upper bounds, than this "smallest" of the "greater" elements is called the

labeled cover relations on $\{x, y, z\}$			
$\prec_1$	$= \varnothing$	$\prec_{11}$	$= \{\ (y,x),\ (z,x)\ \}$
$\prec_2$	$= \{\ (y,z)\ \}$	$\prec_{12}$	$= \{\ (x,y),\ (z,y)\ \}$
$\prec_3$	$= \{\ (z,y)\ \}$	$\prec_{13}$	$= \{\ (x,z),\ (y,z)\ \}$
$\prec_4$	$= \{\ (x,z)\ \}$	$\prec_{14}$	$= \{\ (x,y),\ (y,z)\ \}$
$\prec_5$	$= \{\ (z,x)\ \}$	$\prec_{15}$	$= \{\ (x,z),\ (x,y)\ \}$
$\prec_6$	$= \{\ (x,y)\ \}$	$\prec_{16}$	$= \{\ (y,x),\ (y,z)\ \}$
$\prec_7$	$= \{\ (y,x)\ \}$	$\prec_{17}$	$= \{\ (y,z),\ (y,x)\ \}$
$\prec_8$	$= \{\ (x,y),\ (x,z)\ \}$	$\prec_{18}$	$= \{\ (z,x),\ (z,y)\ \}$
$\prec_9$	$= \{\ (x,y),\ (y,z)\ \}$	$\prec_{19}$	$= \{\ (z,y),\ (z,x)\ \}$
$\prec_{10}$	$= \{\ (z,x),\ (z,y)\ \}$		

Table A.3: labeled cover relations on $\{x, y, z\}$

least upper bound (*lub*) of x and y, and is denoted $x \vee y$ (Definition A.9 page 20). Likewise, we may also be able to find elements that are comparable to $\{x, y\}$ and are "*lesser*" than both of them. Such a lesser element is called a *lower bound* of x and y. If one of these lower bounds is comparable with and is larger than all the other lower bounds, than this "largest" of the "lesser" elements is called the *greatest lower bound* (*glb*) of $\{x, y\}$ and is denoted $x \wedge y$ (Definition A.10 page 20). If every pair of elements in an ordered set has both a least upper bound and a greatest lower bound in the ordered set, then that ordered set is a *lattice* (Definition B.3 page 23).

Definition A.9. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

For any set $A \in 2^X$, c is an ***upper bound*** *of A in $(X, \leq)$ if*

1. $x \leq c \quad \forall x \in A.$

An element b is the ***least upper bound***, *or* **lub**, *of A in $(X, \leq)$ if*

2. *b and c are* UPPER BOUNDS *of A* $\implies b \leq c.$

The least upper bound of the set A is denoted $\bigvee A$. It is also called the ***supremum*** *of A, which is denoted $\sup A$. The* ***join*** *$x \vee y$ of x and y is defined as $x \vee y \triangleq \bigvee \{x, y\}$.*

Definition A.10. *Let $(X, \leq)$ be an ordered set and 2^X the power set of X.*

DEF

For any set $A \in 2^X$, p is a ***lower bound*** *of A in $(X, \leq)$ if*

1. $p \leq x \quad \forall x \in A.$

An element a is the ***greatest lower bound***, *or* **glb**, *of A in $(X, \leq)$ if*

2. *a and p are* LOWER BOUNDS *of A* $\implies p \leq a.$

The greatest lower bound of the set A is denoted $\bigwedge A$. It is also called the ***infimum*** *of A, which is denoted $\inf A$. The* ***meet*** *$x \wedge y$ of x and y is defined as $x \wedge y \triangleq \bigwedge \{x, y\}$.*

Definition A.11 (least upper bound property). [16] *Let X be a set. Let $\sup A$ be the supremum (least upper bound) of a set A.*

DEF

A set X satisfies the ***least upper bound property*** *if*

$$\left.\begin{array}{lll} 1. & A \subseteq X & \text{and} \\ 2. & A \neq \varnothing & \text{and} \\ 3. & \exists b \in X \text{ such that } \forall a \in A,\ a \leq b & (A \text{ is bounded above in } X) \end{array}\right\} \implies \exists \sup A \in X$$

A set X that satisfies the least upper bound property is also said to be ***complete***.

Proposition A.2. *Let $(X, \vee, \wedge; \leq)$ be an* ORDERED SET *(Definition A.2 page 14).*

PRP

$$x \leq y \iff \left\{\begin{array}{llll} 1. & x \wedge y = x & \text{and} \\ 2. & x \vee y = y & \end{array}\right\} \quad \forall x, y \in X$$

Proposition A.3. *Let 2^X be the* POWER SET *of a set X.*

PRP

$$A \subseteq B \implies \left\{\begin{array}{llll} 1. & \bigvee A \leq \bigvee B & \text{and} \\ 2. & \bigwedge A \leq \bigwedge B & \end{array}\right\} \quad \forall A, B \in 2^X$$

[16] Pugh (2002) page 13, Rudin (1976) page 4

APPENDIX B

LATTICES

B.1 Semi-lattices

Definition A.9 (page 20) defined the least upper bound $\vee$ of pairs of elements in terms of an ordering relation $\le$. However, the converse development is also possible— we can first define a binary operation $\varovee$ with a handful of "least upper bound like properties", and then define an ordering relation $\lesssim$ in terms of $\varovee$ (Definition B.1 page 21). In fact, Theorem B.1 (page 21) shows that under Definition B.1, $(X, \lesssim)$ *is* a partially ordered set and $\varovee$ is a least upper bound on that ordered set.

The same development is performed with regards to a greatest lower bound $\varowedge$ with the result that $(X, \lesssim)$ *is* a partially ordered set and $\varowedge$ is a greatest lower bound on that ordered set (Theorem B.2 page 22).

Definition B.1. [1] *Let* $\varovee, \lesssim\colon X^2 \to X$ *be binary operators on a set* X.

DEF: *The algebraic structure* $(X, \lesssim, \varovee)$ *is a* ***join semilattice*** *if*

1. $x \varovee x = x \quad \forall x \in X$ (IDEMPOTENT) and
2. $x \varovee y = y \varovee x \quad \forall x,y \in X$ (COMMUTATIVE) and
3. $(x \varovee y) \varovee z = x \varovee (y \varovee z) \quad \forall x,y,z \in X$ (ASSOCIATIVE).

Definition B.2. [2] *Let* $\varowedge, \lesssim\colon X^2 \to X$ *be binary operators on a set* X.

DEF: *The algebraic structure* $(X, \lesssim, \varowedge)$ *is a* ***meet semilattice*** *if*

1. $x \varowedge x = x \quad \forall x \in X$ (IDEMPOTENT) and
2. $x \varowedge y = y \varowedge x \quad \forall x,y \in X$ (COMMUTATIVE) and
3. $(x \varowedge y) \varowedge z = x \varowedge (y \varowedge z) \quad \forall x,y,z \in X$ (ASSOCIATIVE).

Theorem B.1. [3] *Let* $\varovee, \lesssim\colon X^2 \to X$ *be binary operators over a set* X.

THM: $\left\{ (X, \lesssim, \varovee) \text{ is a JOIN SEMILATTICE} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \lesssim) \text{ is a PARTIALLY ORDERED SET} & \text{and} \\ 2. & x \varovee y \text{ is a LEAST UPPER BOUND of } x \text{ and } y & \forall x, y \in X. \end{array} \right\}$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition A.2 (page 14), *reflexive*, *antisymmetric*, and *transitive*;

[1] MacLane and Birkhoff (1999) page 475, Birkhoff (1967) page 22
[2] MacLane and Birkhoff (1999) page 475
[3] MacLane and Birkhoff (1999) page 475

Proof that $\le$ is reflexive:

$$\begin{aligned} & x = x \ovee x && \text{by idempotent hypothesis} \\ \iff & x \le x && \text{by definition of } \le \\ \implies & \le \text{ is reflexive} \end{aligned}$$

Proof that $\le$ is antisymmetric:

$$\begin{aligned} x \le y \text{ and } y \le x \iff & x \ovee y = y \text{ and } y \ovee x = x && \text{by definition of } \le \\ \implies & x \ovee y = y \text{ and } x \ovee y = x && \text{by commutative hypothesis} \\ \implies & x = y \\ \implies & \le \text{ is antisymmetric} \end{aligned}$$

Proof that $\le$ is transitive:

$$\begin{aligned} x \le y \text{ and } y \le z \iff & x \ovee y = y \text{ and } y \ovee z = z && \text{by definition of } \le \\ \implies & (x \ovee y) \ovee z = z \\ \iff & x \ovee (y \ovee z) = z && \text{by associative hypothesis} \\ \implies & x \ovee z = z \\ \iff & x \le z \\ \iff & \le \text{ is transitive} \end{aligned}$$

Proof that $x \ovee y$ is a lub of x and y:

$$\begin{aligned} x \ovee y = y \iff & x \le y && \text{by definition of } \le \\ \iff & x \vee y = y && \text{by definition of } \vee \\ \implies & x \ovee y = x \vee y \\ \implies & x \ovee y \text{ is the lub of } x \text{ and } y \end{aligned}$$

✏

Theorem B.2. [4] *Let $\owedge, \precsim : X^2 \to X$ be binary operators over a set X.*

THM

$$\left\{ \begin{array}{l} (X, \precsim, \owedge) \textit{ is a} \\ \text{MEET SEMILATTICE} \end{array} \right\} \implies \left\{ \begin{array}{lll} 1. & (X, \precsim) \textit{ is a } \text{PARTIALLY ORDERED SET} & \textit{and} \\ 2. & x \owedge y \textit{ is a } \text{GREATEST LOWER BOUND } \textit{of } x \textit{ and } y & \forall x, y \in X. \end{array} \right\}$$

✎PROOF: In order for $(X, \le)$ to be an ordered set, $\le$ must be, according to Definition A.2 (page 14), *reflexive*, *antisymmetric*, and *transitive*;

Proof that $\le$ is reflexive:

$$\begin{aligned} & x = x \owedge x && \text{by idempotent hypothesis} \\ \iff & x \le x && \text{by definition of } \le \\ \implies & \le \text{ is reflexive} \end{aligned}$$

Proof that $\le$ is antisymmetric:

$$\begin{aligned} x \le y \text{ and } y \le x \iff & x \owedge y = x \text{ and } y \owedge x = y && \text{by definition of } \le \\ \implies & x \owedge y = x \text{ and } x \owedge y = y && \text{by commutative hypothesis} \\ \implies & x = y \\ \implies & \le \text{ is antisymmetric} \end{aligned}$$

[4] MacLane and Birkhoff (1999) page 475

Proof that $\le$ is transitive:

$$
\begin{aligned}
x \le y \text{ and } y \le z &\iff x \owedge y = x \text{ and } y \owedge z = y && \text{by definition of } \le \\
&\implies x \owedge (y \owedge z) = x \\
&\iff (x \owedge y) \owedge z = x && \text{by associative hypothesis} \\
&\implies x \owedge z = x \\
&\iff x \le z \\
&\iff \le \text{ is transitive}
\end{aligned}
$$

Proof that $x \owedge y$ is a glb of x and y:

$$
\begin{aligned}
x \owedge y = x &\iff x \le y && \text{by definition of } \le \\
&\iff x \wedge y = x && \text{by definition of } \wedge \\
&\implies x \owedge y = x \wedge y \\
&\implies x \owedge y \text{ is the glb of } x \text{ and } y
\end{aligned}
$$

B.2 Lattices

An *ordered set* is a set together with the additional structure of an ordering relation (Definition A.2 page 14). However, this amount of structure tends to be insufficient to ensure "well-behaved" mathematical systems. This situation is greatly remedied if every pair of elements in an ordered set (partially or linearly ordered) has both a *least upper bound* and a *greatest lower bound* (Definition A.10 page 20) in the ordered set; in this case, that ordered set is a *lattice* (next definition). Gian-Carlo Rota (1932–1999) illustrates the advantage of lattices over simple ordered sets by pointing out that the *ordered set* of partitions of an integer "is fraught with pathological properties", while the *lattice* of partitions of a set "remains to this day rich in pleasant surprises".[5] Further examples of lattices follow in Section B.3 (page 28).

Definition B.3. [6]

DEF

An algebraic structure $\boldsymbol{L} \triangleq (X, \vee, \wedge ; \le)$ *is a* ***lattice*** *if*

1. $(X, \le)$ *is an ordered set* *and*
2. $x, y \in X \implies x \vee y \in X$ *and*
3. $x, y \in X \implies x \wedge y \in X$

The algebraic structure $\boldsymbol{L}^* \triangleq (X, \ovee, \owedge ; \ge)$ *is the* ***dual*** *lattice of* $\boldsymbol{L}$, *where* $\ovee$ *and* $\owedge$ *are determined by* $\ge$. *The* LATTICE $\boldsymbol{L}$ *is* LINEAR *if* $(X, \le)$ *is a* CHAIN *(Definition A.4 page 15)*.

Definition B.3 (previous) characterizes lattices in terms of *order properties*. Under this definition, lattices have an equivalent characterization in terms of *algebraic properties*. In particular, all lattices have four basic algebraic properties: all lattices are *idempotent*, *commutative*, *associative*, and *absorptive*. Conversely, any structure that possesses these four properties *is* a lattice. These results are demonstrated by Theorem B.3 (next). However, note that the four properties are not *independent*, as it is possible to prove that any structure $\boldsymbol{L} \triangleq (X, \vee, \wedge ; \le)$ that is *commutative*, *associative*, and *absorptive*, is also *idempotent*[7] Thus, when proving that $\boldsymbol{L}$ is a lattice, it is only necessary to prove that it is *commutative*, *associative*, and *absorptive*.

[5] Rota (1997) page 1440 ⟨Introduction⟩, Rota (1964) page 498 ⟨partitions of a set⟩

[6] MacLane and Birkhoff (1999) page 473, Birkhoff (1948) page 16, Ore (1935), Birkhoff (1933) page 442, Maeda and Maeda (1970), page 1

[7] Padmanabhan and Rudeanu (2008) page 8, Beran (1985) page 5, McKenzie (1970) page 24

Theorem B.3. [8]

THM

$(X, \vee, \wedge; \leq)$ *is a* LATTICE $\iff$

$$\left\{\begin{array}{rcl|rcl|lll} x \vee x &=& x & x \wedge x &=& x & \forall x \in X & \text{(IDEMPOTENT)} & \textit{and} \\ x \vee y &=& y \vee x & x \wedge y &=& y \wedge x & \forall x,y \in X & \text{(COMMUTATIVE)} & \textit{and} \\ (x \vee y) \vee z &=& x \vee (y \vee z) & (x \wedge y) \wedge z &=& x \wedge (y \wedge z) & \forall x,y,z \in X & \text{(ASSOCIATIVE)} & \textit{and} \\ x \vee (x \wedge y) &=& x & x \wedge (x \vee y) &=& x & \forall x,y \in X & \text{(ABSORPTIVE).} & \end{array}\right\}$$

✎PROOF:

1. Proof that $(X, \vee, \wedge; \leq)$ is a lattice $\implies$ 4 properties:
   These follow directly from the definitions of least upper bound $\vee$ and greatest lower bound $\wedge$. For the absorptive property,

$$\begin{aligned} x \leq y &\implies x \vee (x \wedge y) = x \vee x = x \\ y \leq x &\implies x \vee (x \wedge y) = x \vee y = x \\ x \leq y &\implies x \wedge (x \vee y) = x \wedge y = x \\ y \leq x &\implies x \wedge (x \vee y) = x \wedge x = x \end{aligned}$$

2. Proof that $(X, \vee, \wedge; \leq)$ is a lattice $\impliedby$ 4 properties:

   According to Definition B.3 (page 23), in order for $(X, \vee, \wedge; \leq)$ to be a lattice, $(X, \vee, \wedge; \leq)$ must be an ordered set, $x \vee y$ must be the least upper bound for any $x, y \in X$ and $x \wedge y$ must be the greatest lower bound for any $x, y \in X$.

   (a) By Theorem B.1 (page 21), $(X, \vee, \wedge; \leq)$ is an ordered set.
   (b) By Theorem B.1 (page 21), $x \vee y$ is the least upper bound for any $x, y \in X$.
   (c) Proof that $x \wedge y$ is the greatest lower bound for any $x, y \in X$: To prove this, we must show that $x \leq y \iff x \wedge y = x$.

      Proof that $x \leq y \implies x \wedge y = x$:

$$\begin{aligned} x &= x \wedge (x \vee y) && \text{by absorptive hypothesis} \\ &= x \wedge y && \text{by } x \leq y \text{ hypothesis and definition of } \leq \end{aligned}$$

      Proof that $x \leq y \impliedby x \wedge y = x$:

$$\begin{aligned} y &= y \vee (y \wedge x) && \text{by absorptive hypothesis} \\ &= y \vee (x \wedge y) && \text{by commutative hypothesis} \\ &= y \vee x && \text{by } x \wedge y = x \text{ hypothesis} \\ &= x \vee y && \text{by commutative hypothesis} \\ \implies \quad & x \leq y && \text{by definition of } \leq \end{aligned}$$

✏

[8] MacLane and Birkhoff (1999) pages 473–475 ⟨LEMMA 1, THEOREM 4⟩, Burris and Sankappanavar (1981) pages 4–7, Birkhoff (1938), pages 795–796, Ore (1935) page 409 ⟨(α)⟩, Birkhoff (1933) page 442, Dedekind (1900) pages 371–372 ⟨(1)–(4)⟩. Peirce (1880) credits Boole and Jevons with the *commutative* property: Peirce (1880), page 33 ⟨"(5)"⟩. Peirce (1880) credits Boole and Jevons with the *associative* property. Peirce (1880) credits Jevons (1864) with the *idempotent* property: Jevons (1864), page 41

$$\begin{array}{rcll} A + A &=& A & \text{“Law of Unity”} \\ AA &=& A & \text{“Law of Simplicity”} \end{array}$$

Lemma B.1. [9] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition B.3 page 23)*.

LEM
$$x \leq y \iff x = x \wedge y \qquad \forall x,y \in \mathbf{L}$$

✎PROOF:

1. Proof for $\Longrightarrow$ case: by left hypothesis and definition of $\wedge$ (Definition A.10 page 20).

2. Proof for $\Longleftarrow$ case: by right hypothesis and definition of $\wedge$ (Definition A.10 page 20).

✏

The identities of Theorem B.3 (page 24) occur in pairs that are *duals* of each other. That is, for each identity, if you swap the join and meet operations, you will have the other identity in the pair. Thus, the characterization of lattices provided by Theorem B.3 (page 24) is called *self-dual*. And because of this, lattices support the *principle of duality* (next theorem).

Theorem B.4 (Principle of duality). [10] *Let* $\mathbf{L} \triangleq (X, \vee, \wedge; \leq)$ *be a lattice.*

THM
$$\left\{\begin{array}{l}\phi \text{ is an identity on } \mathbf{L} \text{ in terms}\\ \text{of the operations} \vee \text{ and } \wedge\end{array}\right\} \Longrightarrow \mathbf{T}\phi \text{ is also an identity on } \mathbf{L}$$
where the operator $\mathbf{T}$ *performs the following mapping on the operations of* ϕ*:*
$$\vee \to \wedge, \qquad \wedge \to \vee$$

✎PROOF: For each of the identities in Theorem B.3 (page 24), the operator $\mathbf{T}$ produces another identity that is also in the set of identities:

$$\begin{array}{lclclclclcl}
\mathbf{T}(1a) &=& \mathbf{T}[x \vee y &=& y \vee x] &=& [x \wedge y &=& y \wedge x] &=& (1b)\\
\mathbf{T}(1b) &=& \mathbf{T}[x \wedge y &=& y \wedge x] &=& [x \vee y &=& y \vee x] &=& (1a)\\
\mathbf{T}(2a) &=& \mathbf{T}[x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& [x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& (2b)\\
\mathbf{T}(2b) &=& \mathbf{T}[x \wedge (y \vee z) &=& (x \wedge y) \vee (x \wedge z)] &=& [x \vee (y \wedge z) &=& (x \vee y) \wedge (x \vee z)] &=& (2a)
\end{array}$$

Therefore, if the statement ϕ is consistent with regards to the lattice $\mathbf{L}$, then $\mathbf{T}\phi$ is also consistent with regards to the lattice $\mathbf{L}$. ✏

Proposition B.1 (Monotony laws). [11] *Let* $(X, \vee, \wedge; \leq)$ *be a lattice.*

PRP
$$\left.\begin{array}{lcll} a &\leq& b & \text{and}\\ x &\leq& y. & \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcll} a \wedge x &\leq& b \wedge y & \text{and}\\ a \vee x &\leq& b \vee y. & \end{array}\right.$$

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[9] Holland (1970), page ???

[10] Padmanabhan and Rudeanu (2008) pages 7–8, Beran (1985) pages 29–30

[11] Givant and Halmos (2009) page 39, Doner and Tarski (1969) pages 97–99

✎PROOF:

$$\begin{array}{lll}
1. & (a \wedge x) \le a & \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition A.10 page 20} \\
 & \le b & \text{by left hypothesis} \\
2. & (a \wedge x) \le x & \text{by definition of } \textit{meet} \text{ operation } \wedge \text{ Definition A.10 page 20} \\
 & \le y & \text{by left hypothesis} \\
3. & (a \wedge x) = \underbrace{(a \wedge x)}_{\le b} \wedge \underbrace{(a \wedge x)}_{\le y} & \text{by } \textit{idempotent} \text{ property Theorem B.3 page 24} \\
 & \le b \wedge y & \text{by 1 and 2} \\
4. & (a \vee x) = \underbrace{(a \vee x)}_{\le b} \vee \underbrace{(a \vee x)}_{\le y} & \text{by } \textit{idempotent} \text{ property Theorem B.3 page 24} \\
 & \le b \vee y & \text{by 1 and 2}
\end{array}$$

✏

Minimax inequality. Suppose we arrange a finite sequence of values into m groups of n elements per group. This could be represented as an $m \times n$ matrix. Suppose now we find the minimum value in each row, and the maximum value in each column. We can call the maximum of all the minimum row values the *maximin*, and the minimum of all the maximum column values the *minimax*. Now, which is greater, the maximin or the minimax? The *minimax inequality* demonstrates that the maximin is always less than or equal to the minimax. The minimax inequality is illustrated below and stated formerly in Theorem B.5 (page 26).

$$\underbrace{\bigvee_{1}^{m}\left\{\begin{array}{c}
\bigwedge_{1}^{n}\left\{\; x_{11} \quad x_{12} \quad \cdots \quad x_{1n} \;\right\} \\ \hline
\bigwedge_{1}^{n}\left\{\; x_{21} \quad x_{22} \quad \cdots \quad x_{2n} \;\right\} \\ \hline
\bigwedge_{1}^{n}\left\{\; \vdots \quad \ddots \quad \ddots \quad \vdots \;\right\} \\ \hline
\bigwedge_{1}^{n}\left\{\; x_{m1} \quad x_{m2} \quad \cdots \quad x_{mn} \;\right\}
\end{array}\right\}}_{\text{maximin}}
\quad \le \quad
\underbrace{\bigwedge_{1}^{n}\left\{\begin{array}{c|c|c|c}
\bigvee_{1}^{m} & \bigvee_{1}^{m} & \bigvee_{1}^{m} & \bigvee_{1}^{m} \\
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{array}\right\}}_{\text{minimax}}$$

Theorem B.5 (Minimax inequality). [12] *Let $(X, \vee, \wedge; \le)$ be a lattice.*

THM

$$\underbrace{\bigvee_{i=1}^{m}\bigwedge_{j=1}^{n} x_{ij}}_{\textit{maxmini: largest of the smallest}} \le \underbrace{\bigwedge_{j=1}^{n}\bigvee_{i=1}^{m} x_{ij}}_{\textit{minimax: smallest of the largest}} \qquad \forall x_{ij} \in X$$

[12] Birkhoff (1948) pages 19–20

✎PROOF:

$$\underbrace{\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{smallest for any given } i} \le x_{ij} \le \underbrace{\left(\bigvee_{k=1}^{n} x_{kj}\right)}_{\text{largest for any given } j} \qquad \forall i,j$$

$$\implies \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{k=1}^{n} x_{ik}\right)}_{\text{largest amoung all } i\text{s of the smallest values}} \le \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{k=1}^{m} x_{kj}\right)}_{\text{smallest amoung all } j\text{s of the largest values}}$$

$$\implies \underbrace{\bigvee_{i=1}^{m}\left(\bigwedge_{j=1}^{n} x_{ij}\right)}_{\text{maxmini}} \le \underbrace{\bigwedge_{j=1}^{n}\left(\bigvee_{i=1}^{m} x_{ij}\right)}_{\text{minimax}} \qquad \text{(change of variables)}$$

✎

Distributive inequalities. Special cases of the minimax inequality include three distributive *inequalities* (next theorem). If for some lattice any *one* of these inequalities is an *equality*, then *all three* are *equalities*; and in this case, the lattice is a called a *distributive* lattice.

Theorem B.6 (distributive inequalities). [13]

THM

$(X, \vee, \wedge; \le)$ *is a lattice* $\implies$ *for all* $x,y,z\in X$

$$\begin{array}{rcll}
x \wedge (y \vee z) & \ge & (x \wedge y) \vee (x \wedge z) & \text{(JOIN SUPER-DISTRIBUTIVE) and}\\
x \vee (y \wedge z) & \le & (x \vee y) \wedge (x \vee z) & \text{(MEET SUB-DISTRIBUTIVE) and}\\
(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) & \le & (x \vee y) \wedge (x \vee z) \wedge (y \vee z) & \text{(MEDIAN INEQUALITY).}
\end{array}$$

✎PROOF:

1. Proof that $\wedge$ sub-distributes over $\vee$:

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &\le (x \vee x) \wedge (y \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem B.5 page 26)}\\
&= x \wedge (y \vee z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem B.3 page 24)}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c}\bigwedge\left\{\begin{array}{cc} x & y\end{array}\right\}\\ \hline \bigwedge\left\{\begin{array}{cc} x & z\end{array}\right\}\end{array}\right\} \le \bigwedge\left\{\begin{array}{c|c}\bigvee & \bigvee\\ x & y\\ x & z\end{array}\right\}$$

2. Proof that $\vee$ super-distributes over $\wedge$:

$$\begin{aligned}
x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{idempotent} \text{ property of lattices (Theorem B.3 page 24)}\\
&\le (x \vee y) \wedge (x \vee z) && \text{by } \textit{minimax inequality} \text{ (Theorem B.5 page 26)}
\end{aligned}$$

$$\bigvee\left\{\begin{array}{c}\bigwedge\left\{\begin{array}{cc} x & x\end{array}\right\}\\ \hline \bigwedge\left\{\begin{array}{cc} y & z\end{array}\right\}\end{array}\right\} \le \bigwedge\left\{\begin{array}{c|c}\bigvee & \bigvee\\ x & x\\ y & z\end{array}\right\}$$

3. Proof that of median inequality: by *minimax inequality* (Theorem B.5 page 26)

✎

[13] Davey and Priestley (2002) page 85, Grätzer (2003) page 38, Birkhoff (1933) page 444, Korselt (1894) page 157, Müller-Olm (1997) page 13 ⟨terminology⟩

Modular inequalities. Besides the distributive property, another consequence of the minimax inequality is the *modularity inequality* (next theorem). A lattice in which this inequality becomes equality is said to be *modular*.

Theorem B.7 (Modular inequality). [14] *Let* $(X, \vee, \wedge; \leq)$ *be a* LATTICE *(Definition B.3 page 23)*.

THM $x \leq y \implies x \vee (y \wedge z) \leq y \wedge (x \vee z)$

✎PROOF:

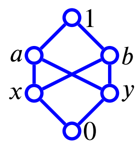
$$\begin{aligned} x \vee (y \wedge z) &= (x \wedge x) \vee (y \wedge z) && \text{by } \textit{absorptive} \text{ property (Theorem B.3 page 24)} \\ &\leq (x \vee y) \wedge (x \vee z) && \text{by the } \textit{minimax inequality} \text{ (Theorem B.5 page 26)} \\ &= y \wedge (x \vee z) && \text{by left hypothesis} \end{aligned}$$

$$\bigvee \left\{ \begin{array}{c} \bigwedge \left\{ \begin{array}{cc} x & x \end{array} \right\} \\ \hline \bigwedge \left\{ \begin{array}{cc} y & z \end{array} \right\} \end{array} \right\} \leq \bigwedge \left\{ \begin{array}{c|c} \bigvee & \bigvee \\ x & x \\ y & z \end{array} \right\}$$

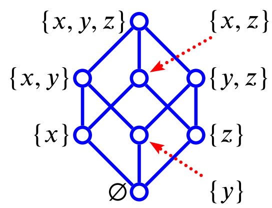
✏

B.3 Examples

Example B.1. [15]the ordered set illustrated to the right is **not** a lattice because, for example, while x and y have *upper bounds* a, b, and 1, x and y have no *least upper bound*. Obviously 1 is not the least upper bound because $a \leq 1$ and $b \leq 1$. And neither a nor b is a least upper bound because $a \not\leq b$ and $b \not\leq a$; rather, a and b are incomparable $(a \| b)$. Note that if we remove either or both of the two lines crossing the center, the ordered set becomes a lattice.



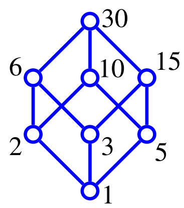
Example B.2 (Discrete lattice). Let 2^A be the power set of a set A, $\subseteq$ the set inclusion relation, $\cup$ the set union operation, and $\cap$ the set intersection operation. Then the tupple $\left(2^{\{x,y,z\}}, \cup, \cap; \subseteq\right)$ is a lattice.



Examples of least upper bounds	Examples of greatest lower bounds
$\{x\} \cup \{z\} = \{x, z\}$	$\{x\} \cap \{z\} = \emptyset$
$\{x, y\} \cup \{y\} = \{x, y\}$	$\{x, y\} \cap \{y\} = \{y\}$
$\{x, z\} \cup \{y, z\} = \{x, y, z\}$	$\{x, z\} \cap \{y, z\} = \{z\}$

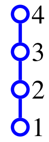
Example B.3 (Integer factor lattice). [16]For any pair of natural numbers $n, m \in \mathbb{N}$, let $n|m$ represent the relation "m divides n", $\operatorname{lcm}(n, m)$ the *least common multiple* of n and m, and $\gcd(n, m)$ the *greatest common divisor* of n and m.

EX $(\{1, 2, 3, 5, 6, 10, 15, 30\}, \gcd, \operatorname{lcm}; |)$ is a lattice.

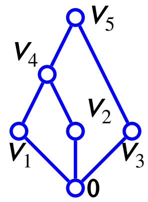


[14] Birkhoff (1948) page 19, Burris and Sankappanavar (1981) page 11, Dedekind (1900) page 374
[15] Oxley (2006) page 54, Farley (1997), page 3, Farley (1996), page 5, Birkhoff (1967) pages 15–16
[16] MacLane and Birkhoff (1999) page 484, Sheffer (1920) page 310 ⟨footnote 1⟩

Example B.4 (Linear lattice). Let $\le$ be the standard counting ordering relation on the set of integers; and for any pair of integers $n, m \in \mathbb{N}$, let $\max(n, m)$ be the maximum of n and m, and $\min(n, m)$ be the minimum of n and m. Then the tupple $(\{1, 2, 3, 4\},\ \max,\ \min\ ;\ \le)$ is a lattice.



Example B.5 (Subspace lattices). [17]Let $(\!(\boldsymbol{V}_n)\!)$ be a sequence of subspaces, $\subseteq$ be the set inclusion relation, $+$ the subspace addition operator, and $\cap$ the set intersection operator. Then the tuple $\left(\left\{\boldsymbol{V}_n\right\},\ +,\ \cap\ ;\ \subseteq\right)$ is a lattice.

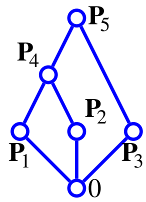


Example B.6 (Projection operator lattices). [18]Let $(\!(\mathbf{P}_n)\!)$ be a sequence of projection operators in a Hilbert space X.

EX

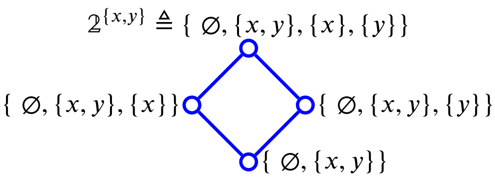
$$\left(\left\{\mathbf{P}_n\right\},\ \vee,\ \wedge\ ;\ \le\right) \text{ is a lattice}$$

$$\begin{array}{llcl} \text{where} & \mathbf{P}_1 \le \mathbf{P}_2 & \stackrel{\text{def}}{\iff} & \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_1 \\ & \mathbf{P}_1 \vee \mathbf{P}_2 & = & \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \\ & \mathbf{P}_1 \wedge \mathbf{P}_2 & = & \mathbf{P}_1\mathbf{P}_2 \end{array}$$



Example B.7 (Lattice of a single topology). [19] Let X be a set, τ a topology on X, $\subseteq$ the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $(\tau,\ \cup,\ \cap\ ;\ \subseteq)$ is a lattice.

Example B.8 (Lattice of topologies). [20]Let X be a set and $\{\tau_1,\ \tau_2,\ \tau_3, \ldots\}$ all the possible topologies on X. Let $\subseteq$ be the set inclusion relation, $\cup$ the set union operator, and $\cap$ the set intersection operator. Then the tuple $\left(\left\{(X, \tau_n)\right\},\ \cup,\ \cap\ ;\ \subseteq\right)$ is a lattice.



Proposition B.2. [21] *Let X_n be a finite set with order $n = |X_n|$. Let L_n be the number of labeled lattices on X_n, l_n the number of unlabeled lattices, and p_n the number of unlabeled posets.*

PRP

n	0	1	2	3	4	5	6	7	8	9	10
L_n	1	1	2	6	36	380	6390	157962	5396888	243, 179, 064	13, 938, 711, 210
l_n	1	1	1	1	2	5	15	53	222	1078	5994
p_n	1	1	2	5	16	63	318	2045	16, 999	183, 231	2, 567, 284

[17] Isham (1999) pages 21–22
[18] Isham (1999) pages 21–22, Dunford and Schwartz (1957), pages 481–482
[19] Burris and Sankappanavar (1981) page 9, Birkhoff (1936) page 161
[20] Isham (1999) page 44, Isham (1989), page 1515
[21] Sloane (2014) ⟨http://oeis.org/A055512⟩, Sloane (2014) ⟨http://oeis.org/A006966⟩, Sloane (2014) ⟨http://oeis.org/A000112⟩, Heitzig and Reinhold (2002)

B.4 Bounded lattices

Let $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice. By the definition of a *lattice* (Definition B.3 page 23), the *upper bound* $(x \vee y)$ and *lower bound* $(x \wedge y)$ of any two elements in X is also in X. But what about the upper and lower bounds of the entire set X ($\bigvee X$ and $\bigwedge X$)[22]? If both of these are in X, then the lattice $\mathbf{L}$ is said to be *bounded* (next definition). All *finite* lattices are bounded (next proposition). However, not all lattices are bounded—for example, the lattice $(\mathbb{Z}, \le)$ (the lattice of integers with the standard integer ordering relation) is *unbounded*. Proposition B.4 (page 30) gives two properties of bounded lattices. Boundedness is one of the "*classic 10*" properties of *Boolean algebras*. Conversely, a bounded and complemented lattice that satisfies the conditions $1' = 0$ and *Elkan's law* is a *Boolean algebra*.

Definition B.4. *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice. Let $\bigvee X$ be the least upper bound of $(X, \le)$ and let $\bigwedge X$ be the greatest lower bound of $(X, \le)$.*

DEF

L** is **upper bounded	*if $\left(\bigvee X\right) \in X$.*
L** is **lower bounded	*if $\left(\bigwedge X\right) \in X$.*
L** is **bounded	*if **L** is both upper and lower bounded.*

A BOUNDED *lattice is optionally denoted $(X, \vee, \wedge, 0, 1; \le)$, where $0 \triangleq \bigwedge X$ and $1 \triangleq \bigvee X$.*

Proposition B.3. *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice.*

PRP

***L** is* FINITE $\Longrightarrow$ ***L** is* BOUNDED

Proposition B.4. *Let $\mathbf{L} \triangleq (X, \vee, \wedge; \le)$ be a lattice with $\bigvee X \triangleq 1$ and $\bigwedge X \triangleq 0$.*

PRP

$$\left\{ \begin{array}{l} \mathbf{L} \text{ is BOUNDED} \\ \text{(Definition B.4 page 30)} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lclll} x \vee 1 &=& 1 & \forall x \in X & \text{(UPPER BOUNDED)} \quad \text{and} \\ x \wedge 0 &=& 0 & \forall x \in X & \text{(LOWER BOUNDED)} \quad \text{and} \\ x \vee 0 &=& x & \forall x \in X & \text{(JOIN-IDENTITY)} \quad \text{and} \\ x \wedge 1 &=& x & \forall x \in X & \text{(MEET-IDENTITY)} \end{array} \right\}$$

✎PROOF:

$x \vee 1 = x \vee \left(\bigvee X\right)$	by definition of 1 (Definition B.4 page 30)
$= \bigvee X$	because $x \in X$
$= 1$	by definition of 1 (Definition B.4 page 30)
$x \wedge 0 = x \wedge \left(\bigwedge X\right)$	by definition of 0 (Definition B.4 page 30)
$= \bigwedge X$	because $x \in X$
$= 0$	by definition of 0 (Definition B.4 page 30)
$\boxed{x} = \bigvee \{x\}$	
$\le \bigvee \{x, 0\}$	because $\{x\} \subseteq \{0, x\}$ and *isotone* property (Proposition A.3 page 20)
$= \boxed{x \vee 0}$	by definition of $\vee$ (Definition A.9 page 20)
$= x \vee \left(\bigwedge X\right)$	by definition of 0 (Definition B.4 page 30)
$\le x \vee \left(\bigwedge \{x\}\right)$	because $\{x\} \subseteq X$ and *isotone* property (Proposition A.3 page 20)
$\le x \vee \left(\bigwedge \{x, x\}\right)$	by definition of $\{\cdot\}$
$= x \vee (x \wedge x)$	by definition of $\wedge$ (Definition A.10 page 20)

[22] $\bigvee X$: Definition A.9 page 20, $\bigwedge X$:Definition A.10 (page 20)

$$
\begin{aligned}
&= \boxed{x} && \text{by \textit{absorptive} property of lattices (Theorem B.3 page 24)}\\
&= x \wedge (x \vee x) && \text{by \textit{absorptive} property of lattices (Theorem B.3 page 24)}\\
&\triangleq x \wedge \left(\bigvee \{x, x\}\right) && \text{by definition of } \vee \text{ (Definition A.9 page 20)}\\
&\triangleq x \wedge \left(\bigvee \{x\}\right) && \text{by definition of set } \{\cdot\}\\
&\le x \wedge \left(\bigvee X\right) && \text{because } \{x\} \subseteq \{x, 1\} \text{ and by \textit{isotone} property of } \bigwedge \text{ (Proposition A.3 page 20)}\\
&= \boxed{x \wedge 1} && \text{by definition of } 1 \text{ (Definition B.4 page 30)}\\
&= \bigwedge \{x, 1\} && \text{by definition of } \wedge \text{ (Definition A.10 page 20)}\\
&\le \bigwedge \{x\} && \text{because } \{x\} \subseteq \{x, 1\} \text{ and by \textit{isotone} property of } \bigwedge \text{ (Proposition A.3 page 20)}\\
&= \boxed{x}
\end{aligned}
$$

✏

Definition B.5. *Let* $\mathbf{L} \triangleq (X, \vee, \wedge, 0, 1; \le)$ *be a* BOUNDED LATTICE *(Definition B.4 page 30).*

DEF

A set $\{x_1, x_2, \ldots\}$ *is a* ***partition*** *of an element* $y \in X$ *if*

1.	$x_n \neq 0$	$\forall n$	NON-EMPTY	*and*
2.	$x_n \wedge x_m = 0$	$\forall n \neq m$	MUTUALLY EXCLUSIVE	*and*
3.	$\bigvee_n x_n = 1$			

APPENDIX C

SET STRUCTURES

C.1 General set structures

Similar to the definition of a *relation* on a set X as being any subset of the *Cartesian product* $X \times X$, a *set structure* on a set X is simply any subset of the *power set* 2^X (Definition C.3 page 33) of the set X (next definition and Figure C.1 page 34).

Definition C.1. [1] *Let 2^X be the* POWER SET *(Definition C.3 page 33) of a set X.*

DEF	*A set $S(X)$ is a* ***set structure*** *on X if* $S(X) \subseteq 2^X$. *A set structure $Q(X)$ is a* ***paving*** *on X if* $\varnothing \in Q(X)$.

Definition C.2. [2] *Let $Q(X)$ be a* PAVING *(Definition C.1 page 33) on a set X. Let Y be a set containing the element 0.*

DEF	*A function* $\mathsf{m} \in Y^{Q(X)}$ *is a* ***set function*** *if* $\mathsf{m}(\varnothing) = 0$.

Definition C.3.

DEF	*The* ***power set*** 2^X *on a set X is defined as* $2^X \triangleq \{A \mid A \subseteq X\}$ *(the set of all subsets of X)*

Definition C.4. [3] *Let 2^X be a set. Let $|X|$ be a function in the function space $[0 : +\infty]^X$.*

DEF	$|X|$ *is the* ***cardinality*** *or* ***order*** *of X if* $$|X| \triangleq \begin{cases} \text{number of elements in } X & \text{if } X \text{ is FINITE} \\ +\infty & \text{otherwise} \end{cases}$$

Definition C.5 (next) introduces seven standard set operations: two *nullary* operations, one *unary* operation, and four *binary operations*.

Definition C.5. [4] *Let 2^X be the* POWER SET *(Definition C.3 page 33) on a set X. Let $\neg$ represent the* LOGICAL NOT *operation, $\vee$ represent the* LOGICAL OR *operation, $\wedge$ represent the* LOGICAL AND *operation, and*

[1] Molchanov (2005) page 389, Pap (1995) page 7, Hahn and Rosenthal (1948) page 254

[2] Pap (1995) page 8 ⟨Definition 2.3: extended real-valued set function⟩, Halmos (1950) page 30 ⟨§7. MEASURE ON RINGS⟩, Hahn and Rosenthal (1948), Choquet (1954)

[3] Tao (2011) page 12 ⟨Example 3.6⟩, Tao (2010) page 7 ⟨Example 1.1.14⟩

[4] Aliprantis and Burkinshaw (1998), pages 2–4

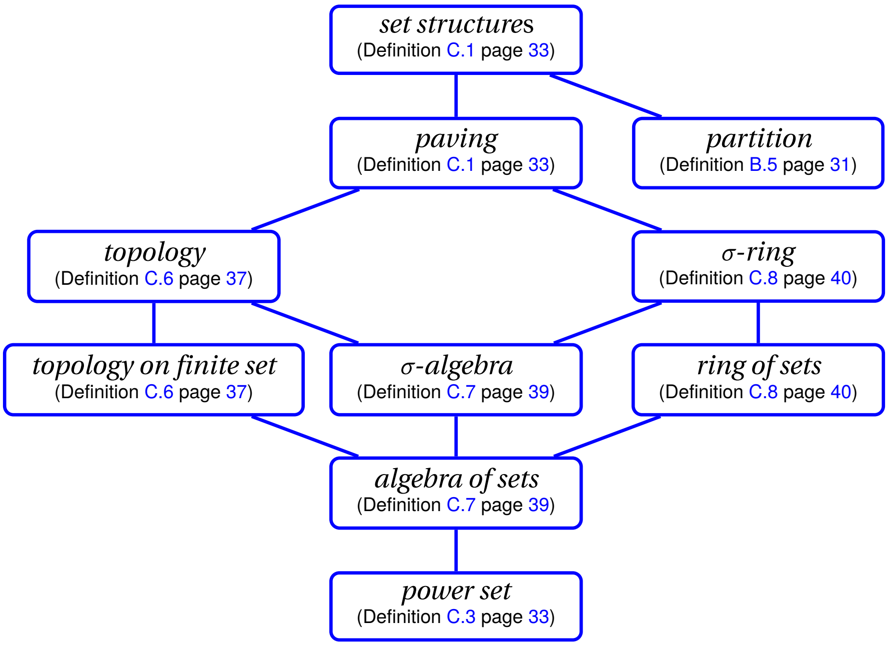


Figure C.1: some standard set structures

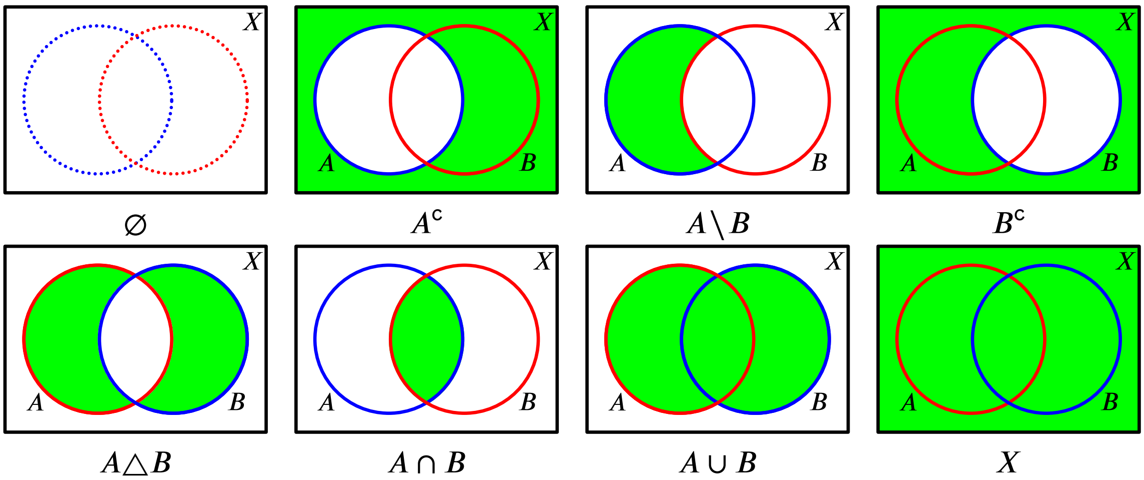
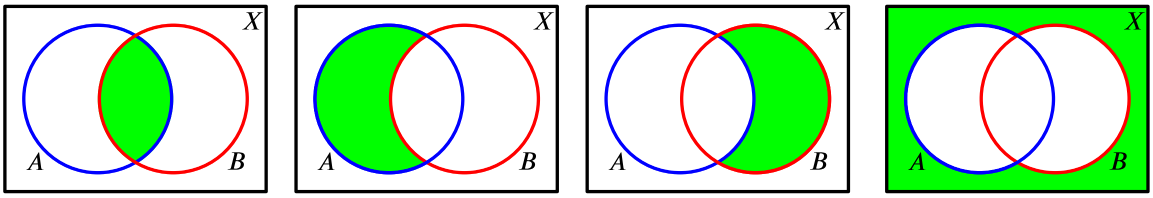


Figure C.2: Venn diagrams for standard set operations (Definition C.5 page 33)



$A \cap B$ (in both A and B) | $A \cap B^c$ (in A but not in B) | $A^c \cap B$ (not in A but in B) | $A^c \cap B^c$ (neither in A nor B)

Figure C.3: The partition of a set X into 4 regions by subsets A and B

$\oplus$ *represent the* LOGICAL EXCLUSIVE-OR *operation.*

DEF

name/symbol		*arity*	*definition*	*domain*
emptyset	$\varnothing$	0	$\varnothing \triangleq \{x \in X \mid x \neq x\}$	
universal set	X	0	$X \triangleq \{x \in X \mid x = x\}$	
complement	c	1	$A^c \triangleq \{x \in X \mid \neg(x \in A)\}$	$\forall A \in 2^X$
union	$\cup$	2	$A \cup B \triangleq \{x \in X \mid (x \in A) \vee (x \in B)\}$	$\forall A, B \in 2^X$
intersection	$\cap$	2	$A \cap B \triangleq \{x \in X \mid (x \in A) \wedge (x \in B)\}$	$\forall A, B \in 2^X$
difference	$\setminus$	2	$A \setminus B \triangleq \{x \in X \mid (x \in A) \wedge \neg(x \in B)\}$	$\forall A, B \in 2^X$
symmetric difference	$\triangle$	2	$A \triangle B \triangleq \{x \in X \mid (x \in A) \oplus (x \in B)\}$	$\forall A, B \in 2^X$

With regards to the standard seven set operations only, Theorem C.1 (next) expresses each of the set operations in terms of pairs of other operations.

Theorem C.1.

THM

$$\begin{array}{rclclclcl}
X &=& \varnothing^c \\
\varnothing &=& X^c = (A \cup A^c)^c &=& A \cap A^c &=& A \setminus A &=& A \triangle A \\
X &=& A \cup A^c &=& (A \cap A^c)^c \\
A^c &=& X \setminus A &=& X \triangle A \\
A \cup B &=& (A^c \cap B^c)^c &=& (A \triangle B) \triangle (A \cap B) &=& (A \setminus B) \triangle B \\
A \cap B &=& (A^c \cup B^c)^c &=& (A \cup B) \triangle A \triangle B &=& A \setminus (A \setminus B) \\
A \setminus B &=& (A^c \cup B)^c &=& A \cap B^c &=& (A \cup B) \triangle B &=& (A \triangle B) \cap A \\
A \triangle B &=& \left[(A^c \cup B)^c\right] \cup \left[(A \cup B^c)^c\right] & & &=& \left[(A^c \cap B^c)^c\right] \cap (A \cap B)^c \\
&=& (A \setminus B) \cup (B \setminus A)
\end{array}$$

Two subsets A and B of a set X that are intersecting but yet one is not contained in the other, partition the set X into four regions, as illustrated in Figure C.3 (page 35). Because there are four regions, the number of ways we can select one or more of them is $2^4 = 16$. Therefore, a binary operator on sets A and B can likewise result in one of $2^4 = 16$ possibilities. The 16 set operations under the inclusion relation $\subseteq$ form a lattice; this lattice is illustrated by a *Hasse diagram* in Figure C.4 (page 36).

C.2 Standard set structures

Set structures are typically designed to satisfy some special properties— such as being closed with respect to certain set operations. Examples of commonly occurring set structures include

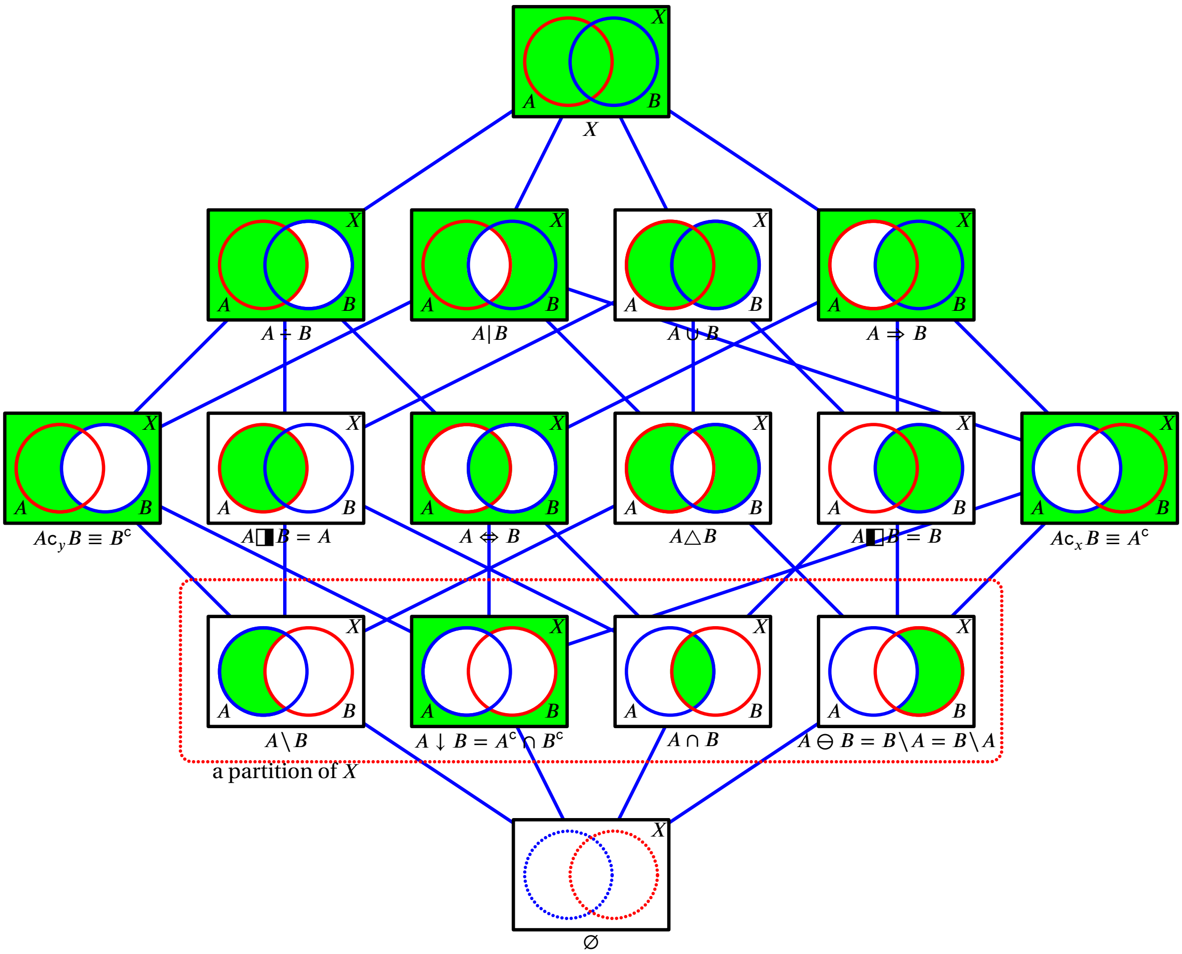


Figure C.4: lattice of set operations

- *power set* (Definition C.3 page 33)
- *topologies* (Definition C.6 page 37)
- *algebra of sets* (Definition C.7 page 39)
- *ring of sets* (Definition C.8 page 40)
- *partitions* (Definition C.9 page 42)

C.2.1 Topologies

Definition C.6. [5] *Let Γ be a set with an arbitrary (possibly uncountable) number of elements. Let 2^X be the* POWER SET *of a set X.*

DEF

*A family of sets $\boldsymbol{T} \subseteq 2^X$ is a **topology** on a set X if*

1. $\varnothing \in \boldsymbol{T}$ *($\varnothing$ is in $\boldsymbol{T}$)* and
2. $X \in \boldsymbol{T}$ *(X is in $\boldsymbol{T}$)* and
3. $U, V \in \boldsymbol{T} \implies U \cap V \in \boldsymbol{T}$ *(the intersection of a finite number of open sets is open)* and
4. $\{U_\gamma | \gamma \in \Gamma\} \subseteq \boldsymbol{T} \implies \bigcup_{\gamma\in\Gamma} U_\gamma \in \boldsymbol{T}$ *(the union of an arbitrary number of open sets is open).*

*A **topological space** is the pair $(X, \boldsymbol{T})$. An **open set** is any member of $\boldsymbol{T}$.*
*A **closed set** is any set D such that D^c is open.*
The set of topologies on a set X is denoted $\mathcal{T}(X)$. That is,

$$\mathcal{T}(X) \triangleq \left\{\boldsymbol{T} \subseteq 2^X | \boldsymbol{T} \text{ is a topology}\right\}.$$

If X is FINITE, *then $\boldsymbol{T}$ is a **topology on a finite set**, and (4.) can be replaced by*

$$U, V \in \boldsymbol{T} \implies U \cup V \in \boldsymbol{T}.$$

Example C.1. [6] Let $\mathcal{T}(X)$ be the set of topologies on a set X and 2^X the *power set* (Definition C.3 page 33) on X.

EX

$\{\varnothing, X\}$	is a *topology* in	$\mathcal{T}(X)$	(*indiscrete topology* or *trivial topology*)
2^X	is a *topology* in	$\mathcal{T}(X)$	(*discrete topology*)

Example C.2. [7] There are four topologies on the set $X \triangleq \{x, y\}$:

EX

topologies on $\{x, y\}$	corresponding closed sets
$\boldsymbol{T}_0 = \{\varnothing, X\}$	$\{\varnothing, X\}$
$\boldsymbol{T}_1 = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y\}, X\}$
$\boldsymbol{T}_2 = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x\}, X\}$
$\boldsymbol{T}_3 = \{\varnothing, \{x\}, \{y\}, X\}$	$\{\varnothing, \{x\}, \{y\}, X\}$

The topologies $(X, \boldsymbol{T}_1)$ and $(X, \boldsymbol{T}_2)$, as well as their corresponding closed set topological spaces, are all *Serpiński spaces*.

Example C.3. There are a total of 29 *topologies* (Definition C.6 page 37) on the set $X \triangleq \{x, y, z\}$:

topologies on $\{x, y, z\}$	**corresponding closed sets**
$\boldsymbol{T}_{00} = \{\varnothing, X\}$	$\{\varnothing, X\}$
$\boldsymbol{T}_{01} = \{\varnothing, \{x\}, X\}$	$\{\varnothing, \{y, z\}, X\}$
$\boldsymbol{T}_{02} = \{\varnothing, \{y\}, X\}$	$\{\varnothing, \{x, z\} X\}$
$\boldsymbol{T}_{04} = \{\varnothing, \{z\}, X\}$	$\{\varnothing, \{x, y\}, X\}$
$\boldsymbol{T}_{10} = \{\varnothing, \{x, y\}, X\}$	$\{\varnothing, \{z\}, X\}$
$\boldsymbol{T}_{20} = \{\varnothing, \{x, z\}, X\}$	$\{\varnothing, \{y\}, X\}$
$\boldsymbol{T}_{40} = \{\varnothing, \{y, z\}, X\}$	$\{\varnothing, \{x\}, X\}$

[5] Munkres (2000) page 76, Riesz (1909), Hausdorff (1914), Tietze (1923) ⟨cited by Thron page 18⟩, Hausdorff (1937) page 258

[6] Munkres (2000), page 77, Kubrusly (2011) page 107 ⟨Example 3.J⟩, Steen and Seebach (1978) pages 42–43 ⟨II.4⟩, DiBenedetto (2002) page 18

[7] Isham (1999), page 44, Isham (1989), page 1515

$\boldsymbol{T}_{11} = \{\varnothing, \{x\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{y,z\}, X\}$
$\boldsymbol{T}_{21} = \{\varnothing, \{x\}, \{x,z\}, X\}$	$\{\varnothing, \{y\} \{y,z\}, X\}$
$\boldsymbol{T}_{41} = \{\varnothing, \{x\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y,z\}, X\}$
$\boldsymbol{T}_{12} = \{\varnothing, \{y\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,z\} X\}$
$\boldsymbol{T}_{22} = \{\varnothing, \{y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,z\}, X\}$
$\boldsymbol{T}_{42} = \{\varnothing, \{y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,z\}, X\}$
$\boldsymbol{T}_{14} = \{\varnothing, \{z\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,y\}, X\}$
$\boldsymbol{T}_{24} = \{\varnothing, \{z\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,y\}, X\}$
$\boldsymbol{T}_{44} = \{\varnothing, \{z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,y\}, X\}$
$\boldsymbol{T}_{31} = \{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$
$\boldsymbol{T}_{52} = \{\varnothing, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$
$\boldsymbol{T}_{64} = \{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$
$\boldsymbol{T}_{13} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, X\}$	$\{\varnothing, \{z\}, \{x,z\}, \{y,z\}, X\}$
$\boldsymbol{T}_{25} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{x,y\}, \{y,z\}, X\}$
$\boldsymbol{T}_{46} = \{\varnothing, \{y\}, \{z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{x,y\}, \{x,z\}, X\}$
$\boldsymbol{T}_{33} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$\boldsymbol{T}_{53} = \{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$
$\boldsymbol{T}_{35} = \{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	$\{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$
$\boldsymbol{T}_{65} = \{\varnothing, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$
$\boldsymbol{T}_{56} = \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$
$\boldsymbol{T}_{66} = \{\varnothing, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$
$\boldsymbol{T}_{77} = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	$\{\varnothing, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$

Theorem C.2. *Let* $\boldsymbol{L} \triangleq (X, \vee, \wedge; \leq)$ *be a* LATTICE.

THM $\boldsymbol{T}$ *is a* TOPOLOGY $\implies$ $(\boldsymbol{T}, \cup, \cap; \subseteq)$ *is a* DISTRIBUTIVE LATTICE

✎PROOF:

1. $(\boldsymbol{S}, \subseteq)$ is an *ordered set.*
2. $\cup$ is *least upper bound* operation on $(\boldsymbol{S}, \subseteq)$. and $\cap$ is *greatest lower bound* operation on $(\boldsymbol{S}, \subseteq)$.
3. Therefore, by Definition B.3 (page 23), $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice.
4. By Theorem B.3 (page 24), $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is *idempotent, commutative, associative,* and *absorptive.*
5. Proof that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is *distributive*:
   (a) Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:

$$\begin{aligned}
&A \cap (B \cup C) \\
&= \{x \in X | x \in A \wedge x \in (B \cup C)\} && \text{by definition of } \cap \\
&= \{x \in X | x \in A \wedge x \in \{x \in X | x \in B \vee x \in C\}\} && \text{by definition of } \cup \\
&= \{x \in X | x \in A \wedge (x \in B \vee x \in C)\} \\
&= \{x \in X | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\
&= \{x \in X | x \in A \wedge x \in B\} \cup \{x \in X | x \in A \wedge x \in C\} && \text{by definition of } \cup \\
&= (A \cap B) \cup (A \cap C) && \text{by definition of } \cap
\end{aligned}$$

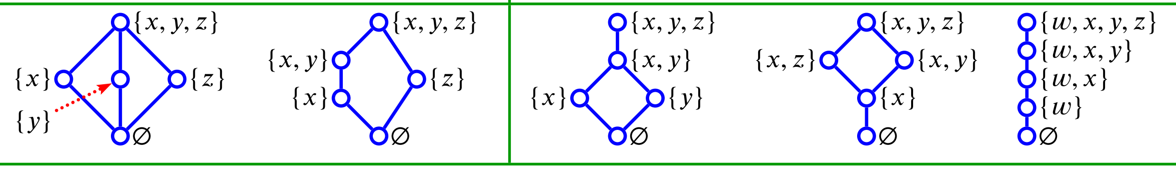
   (b) Proof that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$:
   This follows from the fact that $(\boldsymbol{S}, \cup, \cap; \subseteq)$ is a lattice (item (3) page 38), that $\cap$ distributes over $\cup$.

✎

Example C.4. There are five unlabeled lattices on a five element set (Proposition B.2 page 29). Of these five, three are *distributive*. The following illustrates that the distributive lattices are isomorphic to topologies, while the non-distributive lattices are not.

EX

non-distributive/ *not* topologies	*distributive*/ *are* topologies



✎PROOF:

1. The first two lattices are non-distributive by *Birkhoff distributivity criterion*.
   (a) This lattice is not a topology because, for example,
   $\{x\} \vee \{y\} = \{x,y,z\} \neq \{x,y\} = \{x\} \cup \{y\}$.
   That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.
   (b) This lattice is not a topology because, for example,
   $\{x\} \vee \{y\} = \{y\} \neq \{x,y\} = \{x\} \cup \{y\}$
2. The last three lattices are distributive by *Birkhoff distributivity criterion*.
   (a) This lattice is the topology $\boldsymbol{T}_{13}$ of Example C.3 (page 37). On the set $\{x,y,z\}$, there are a total of three topologies that have this order structure (see Example C.3):
   $$\begin{array}{lcl}\boldsymbol{T}_{13} &=& \{\ \varnothing,\ \{x\},\ \{y\},\ \{x,y\},\ \{x,y,z\}\ \}\\ \boldsymbol{T}_{25} &=& \{\ \varnothing,\ \{x\},\ \{z\},\ \{x,z\},\ \{x,y,z\}\ \}\\ \boldsymbol{T}_{46} &=& \{\ \varnothing,\ \{y\},\ \{z\},\ \{y,z\},\ \{x,y,z\}\ \}\end{array}$$
   (b) This lattice is the topology $\boldsymbol{T}_{31}$ of Example C.3 (page 37). On the set $\{x,y,z\}$, there are a total of three topologies that have this order structure (see Example C.3):
   $$\begin{array}{lcl}\boldsymbol{T}_{31} &=& \{\ \varnothing,\ \{x\},\ \{x,y\},\ \{x,z\},\ \{x,y,z\}\ \}\\ \boldsymbol{T}_{52} &=& \{\ \varnothing,\ \{y\},\ \{x,y\},\ \{y,z\},\ \{x,y,z\}\ \}\\ \boldsymbol{T}_{64} &=& \{\ \varnothing,\ \{z\},\ \{x,z\},\ \{y,z\},\ \{x,y,z\}\ \}\end{array}$$
   (c) This lattice is a topology by Definition C.6 (page 37).

✎

C.2.2 Algebras of sets

Definition C.7. [8] *Let X be a set with* POWER SET 2^X (Definition C.3 page 33).

DEF

$\boldsymbol{A} \subseteq 2^X$ *is an* ***algebra of sets*** *on X if*

1. $A \in \boldsymbol{A} \implies A^c \in \boldsymbol{A}$ *(closed under complement operation)* and
2. $A, B \in \boldsymbol{A} \implies A \cap B \in \boldsymbol{A}$ *(closed under $\cap$)*

The set of all algebra of sets on a set X is denoted $\mathcal{A}(X)$ such that
$$\mathcal{A}(X) \triangleq \left\{\boldsymbol{A} \subseteq 2^X \,|\, \boldsymbol{A} \text{ is an algebra of sets}\right\}.$$
An ALGEBRA OF SETS $\boldsymbol{A}$ *on X is a* ***σ-algebra*** *on X if*

3. $\left\{A_n \middle| n\in\mathbb{Z}\right\} \subseteq \boldsymbol{A} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \boldsymbol{A}$ *(closed under countable union operations).*

[8] Aliprantis and Burkinshaw (1998) page 95, Aliprantis and Burkinshaw (1998) page 151, Halmos (1950) page 21, Hausdorff (1937) page 91

On every set X with at least 2 elements, there are always two particular algebras of sets: the *smallest algebra* and the *largest algebra*, as demonstrated by Example C.5 (next).

Example C.5. [9] Let $\mathcal{A}(X)$ be the set of *algebras of sets* (Definition C.7 page 39) on a set X and 2^X the *power set* (Definition C.3 page 33) on X.

EX

$$\begin{array}{lcll} \{\varnothing,\ X\} & \in & \mathcal{A}(X) & \text{(smallest algebra)} \\ 2^X & \in & \mathcal{A}(X) & \text{(largest algebra)} \end{array}$$

Isomorphically, all *algebras of sets* are *boolean algebras* and all boolean algebras are algebras of sets (next theorem).

Theorem C.3 (Stone Representation Theorem). [10] *Let* $\boldsymbol{L} \triangleq (X,\ \vee,\ \wedge\ ;\ \le)$ *be a* LATTICE.

THM

$$\boldsymbol{L} \text{ is BOOLEAN} \iff \left\{ \begin{array}{l} \boldsymbol{L} \text{ is isomorphic to } (\boldsymbol{A},\ \cup,\ \cap,\ \varnothing,\ X\ ;\ \subseteq) \\ \text{for some ALGEBRA OF SETS (Definition C.7 page 39) } \boldsymbol{A} \end{array} \right\}$$

C.2.3 Rings of sets

A *ring of sets* (next definition) is a family of subsets that is closed under an "addition-like" set union operator $\cup$ and "subtraction-like" set difference operator $\setminus$. Using these two operations, it is not difficult to show that a ring of sets is also closed under a "multiplication-like" set intersection operator $\cap$. Because of this, a ring of sets behaves like an *algebraic ring*. Note however that a ring of sets is not necessarily a *topology* (Definition C.6 page 37) because it does not necessarily include X itself.

Definition C.8. [11] *Let X be a set with* POWER SET 2^X *(Definition C.3 page 33).*

DEF

$\boldsymbol{R} \subseteq 2^X$ *is a* ***ring of sets*** *on X if*

1. $A, B \in \boldsymbol{R} \implies A \cup B$ (closed under $\cup$) and
2. $A, B \in \boldsymbol{R} \implies A \setminus B \in \boldsymbol{R}$ (closed under $\setminus$)

The set of all rings of sets on a set X is denoted $\mathcal{R}(X)$ such that

$$\mathcal{R}(X) \triangleq \left\{ \boldsymbol{R} \subseteq 2^X \,|\, \boldsymbol{R} \text{ is a ring of sets} \right\}.$$

A RING OF SETS $\boldsymbol{R}$ *on X is a* ***σ-ring*** *on X if*

3. $\left\{ A_n \middle| n \in \mathbb{Z} \right\} \subseteq \boldsymbol{R} \implies \bigcup_{n\in\mathbb{Z}} A_n \in \boldsymbol{R}$ (closed under countable union operations).

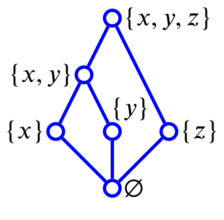
Example C.6. Table C.2 (page 41) lists some *rings of sets* on a finite set X.

Example C.7. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets

$$\boldsymbol{R} \triangleq \{\varnothing,\ X,\ \{x\},\ \{y\},\ \{z\},\ \{x, y\}\}.$$

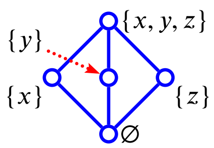
Note that $(\boldsymbol{R},\ \subseteq,\ \cup,\ \cap)$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

$$\{x, y, z\} \setminus \{x\} = \{y, z\} \notin \boldsymbol{R}.$$



Example C.8. Let $X \triangleq \{x, y, z\}$ be a set and $\boldsymbol{R}$ be the family of sets $\boldsymbol{R} \triangleq \{\varnothing,\ X,\ \{x\},\ \{y\},\ \{z\}\}$. Note that $(\boldsymbol{T}, \subseteq) \cup \cap$ is a lattice as illustrated in the figure to the right. However, $\boldsymbol{R}$ is *not* a ring of sets on X because, for example,

$$\{x, y, z\} \setminus \{x\} = \{y, z\} \notin \boldsymbol{R}.$$



[9] Stroock (1999) page 33, Aliprantis and Burkinshaw (1998) pages 95–96

[10] Levy (2002) page 257, Grätzer (2003) page 85, Joshi (1989) page 224, Saliĭ (1988) page 32 ⟨"Stone's Theorem"⟩, Stone (1936)

[11] Berezansky et al. (1996) page 4, Halmos (1950) page 19, Hausdorff (1937) page 90

rings $\mathcal{R}(X)$ on a set X

$$\mathcal{R}(\varnothing) = \left\{ \boldsymbol{R}_1 = \{\varnothing\} \right\}$$

$$\mathcal{R}(\{x\}) = \left\{ \begin{array}{lllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\} & \} \end{array} \right\}$$

$$\mathcal{R}(\{x,y\}) = \left\{ \begin{array}{llllllll} \boldsymbol{R}_1 & = & \{ & \varnothing, & & & & \} \\ \boldsymbol{R}_2 & = & \{ & \varnothing, & \{x\}, & & & \} \\ \boldsymbol{R}_3 & = & \{ & \varnothing, & & \{y\}, & & \} \\ \boldsymbol{R}_4 & = & \{ & \varnothing, & & & \{x,y\} & \} \\ \boldsymbol{R}_5 & = & \{ & \varnothing, & \{x\}, & \{y\}, & \{x,y\} & \} \end{array} \right\}$$

$$\mathcal{R}(\{x,y,z\}) = \left\{ \begin{array}{lllllllllll} \boldsymbol{R}_1 & = \{ & \varnothing, & & & & & & & & \} \\ \boldsymbol{R}_2 & = \{ & \varnothing, & \{x\}, & & & & & & & \} \\ \boldsymbol{R}_3 & = \{ & \varnothing, & & \{y\}, & & & & & & \} \\ \boldsymbol{R}_4 & = \{ & \varnothing, & & & \{z\}, & & & & & \} \\ \boldsymbol{R}_5 & = \{ & \varnothing, & & & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_6 & = \{ & \varnothing, & & & & & \{x,z\}, & & & \} \\ \boldsymbol{R}_7 & = \{ & \varnothing, & & & & & & \{y,z\}, & & \} \\ \boldsymbol{R}_8 & = \{ & \varnothing, & \{x\}, & \{y\}, & & \{x,y\}, & & & & \} \\ \boldsymbol{R}_9 & = \{ & \varnothing, & \{x\}, & & \{z\}, & & \{x,z\}, & & & \} \\ \boldsymbol{R}_{10} & = \{ & \varnothing, & & \{y\}, & \{z\}, & & & \{y,z\}, & & \} \\ \boldsymbol{R}_{11} & = \{ & \varnothing, & & & & & & & X & \} \\ \boldsymbol{R}_{12} & = \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X & \} \\ \boldsymbol{R}_{13} & = \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X & \} \\ \boldsymbol{R}_{14} & = \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X & \} \\ \boldsymbol{R}_{15} & = \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X & \} \end{array} \right\}$$

Table C.2: some *rings of sets* on a finite set X (Example C.6 page 40)

Proposition C.1. [12] *Let $\mathcal{R}(X)$ be the set of* RINGS OF SETS *(Definition C.8 page 40) on a set X.*

PRP

$$\left\{\begin{array}{l}\boldsymbol{R}_1 \text{ and } \boldsymbol{R}_2\\ \text{are rings of sets}\end{array}\right\} \implies \left\{\begin{array}{l}(\boldsymbol{R}_1 \cap \boldsymbol{R}_2)\\ \text{is a ring of sets}\end{array}\right\}$$

C.2.4 Partitions

The following definition is a special case of *partition* defined on lattices (Definition B.5 page 31).

Definition C.9. [13]

DEF

A SET STRUCTURE $\left\{P_n \in 2^X \,\middle|\, n=1,2,\ldots,N\right\}$ *is a **partition** of the set X if*

1. $P_n \neq \varnothing \quad \forall n \in \{1,2,\ldots,N\}$ NON-EMPTY *and*
2. $P_n \cap P_m = \varnothing \quad \forall n \neq m$ MUTUALLY EXCLUSIVE *and*
3. $\displaystyle\bigcup_{n\in\mathbb{Z}} P_n = X$

Example C.9. Let $A, B \subseteq X$, as illustrated in Figure C.3 (page 35). There are a total of 15 partitions of X induced by A and B. Here are 5 of these partitions:

EX

1. $\{X\}$ (1 region)
2. $\{A,\ A^c\}$ (2 regions)
3. $\{A \cup B,\ A^c \cap B^c\}$ (2 regions)
4. $\{A \cap B,\ A \triangle B,\ A^c \cap B^c\}$ (3 regions)
5. $\{A \cap B,\ A \cap B^c,\ A^c \cap B,\ A^c \cap B^c\}$ (4 regions) [see also Figure C.3 page 35 and Figure C.4 page 36]

Proposition C.2. [14] *Let $\mathcal{P}(X)$ be the set of partitions on a set X.*

PRP

The relation $\trianglelefteq \in 2^{\mathbb{P}\mathbb{P}}$ defined as

$$\boldsymbol{P} \trianglelefteq \boldsymbol{Q} \quad \stackrel{\text{def}}{\iff} \quad \forall B \in \boldsymbol{Q}, \exists A \in \boldsymbol{P} \quad \text{such that} \quad B \subseteq A$$

is an ordering relation on $\mathcal{P}(X)$.

Example C.10. Table C.3 (page 43) lists some partitions $\boldsymbol{P}(X)$ on a finite set X.

C.3 Lattices of set structures

C.3.1 Ordering relations

The *set inclusion* relation $\subseteq$ (Definition C.10 page 42) is an *order relation* (Definition A.2 page 14) on set structures, as demonstrated by Proposition C.3 (next proposition).

Definition C.10. *Let S be a* SET STRUCTURE *(Definition C.1 page 33) on a set X.*

DEF

The relation $\subseteq \in 2^{SS}$ is defined as

$$A \subseteq B \quad \text{if} \quad x \in A \implies x \in B \quad \forall x \in X$$

12 Kolmogorov and Fomin (1975) page 32, Bartle (2001) page 318

13 Munkres (2000), page 23, Rota (1964), page 498, Halmos (1950) page 31

14 Roman (2008) page 111, Comtet (1974) page 220, Grätzer (2007), page 697

partitions $\mathcal{P}(X)$ on a set X

$$\mathcal{P}(\varnothing) = \left\{ \boldsymbol{P}_1 = \varnothing \right\}$$

$$\mathcal{P}(\{x\}) = \left\{ \boldsymbol{P}_1 = \{ \{x\} \} \right\}$$

$$\mathcal{P}(\{x,y\}) = \left\{ \begin{array}{lcl} \boldsymbol{P}_1 &=& \{ \{x\}, \{y\}, \} \\ \boldsymbol{P}_2 &=& \{ \{x,y\} \} \end{array} \right\}$$

$$\mathcal{P}(\{x,y,z\}) = \left\{ \begin{array}{lcl} \boldsymbol{P}_1 &=& \{ \{x,y,z\} \} \\ \boldsymbol{P}_2 &=& \{ \{x\}, \{y,z\}, \} \\ \boldsymbol{P}_3 &=& \{ \{y\}, \{x,z\}, \} \\ \boldsymbol{P}_4 &=& \{ \{z\}, \{x,y\} \} \\ \boldsymbol{P}_5 &=& \{ \{x\}, \{y\}, \{z\} \} \end{array} \right\}$$

$$\mathcal{P}(\{w,x,y,z\}) = \left\{ \begin{array}{lcl} \boldsymbol{P}_1 &=& \{ X \} \\ \boldsymbol{P}_2 &=& \{ \{w\}, \{x,y,z\} \} \\ \boldsymbol{P}_3 &=& \{ \{x\}, \{w,y,z\} \} \\ \boldsymbol{P}_4 &=& \{ \{y\}, \{w,x,z\} \} \\ \boldsymbol{P}_5 &=& \{ \{z\}, \{w,x,y\} \} \\ \boldsymbol{P}_6 &=& \{ \{w,x\}, \{y,z\} \} \\ \boldsymbol{P}_7 &=& \{ \{w,y\}, \{x,z\} \} \\ \boldsymbol{P}_8 &=& \{ \{w,z\}, \{x,y\}, \} \\ \boldsymbol{P}_9 &=& \{ \{w\}, \{x\} \; \{y,z\} \} \\ \boldsymbol{P}_{10} &=& \{ \{w\}, \{y\}, \{x,z\} \} \\ \boldsymbol{P}_{11} &=& \{ \{w\}, \{z\}, \{x,y\} \} \\ \boldsymbol{P}_{12} &=& \{ \{x\}, \{y\}, \{w,z\} \} \\ \boldsymbol{P}_{13} &=& \{ \{x\}, \{z\}, \{w,y\} \} \\ \boldsymbol{P}_{14} &=& \{ \{y\}, \{z\}, \{w,x\} \} \\ \boldsymbol{P}_{15} &=& \{ \{w\}, \{x\}, \{y\}, \{z\}, \} \end{array} \right\}$$

Table C.3: some partitions $\boldsymbol{P}(X)$ on a finite set X (Example C.10 page 42)

Proposition C.3 (order properties). *Let S be a* SET STRUCTURE *(Definition C.1 page 33) on a set X.*

PRP *The pair $(S, \subseteq)$ is an* ORDERED SET. *In particular,*

$$\begin{array}{llllllllllll}
A & \subseteq & A & & & & & & & \forall A \in S & \text{(REFLEXIVE)} & \text{and}\\
A & \subseteq & B & \text{and} & B & \subseteq & C & \Longrightarrow & A \subseteq C & \forall A,B,C \in S & \text{(TRANSITIVE)} & \text{and}\\
A & \subseteq & B & \text{and} & B & \subseteq & A & \Longrightarrow & A = B & \forall A,B \in S & \text{(ANTI-SYMMETRIC).} &
\end{array}$$

✎PROOF: By Definition A.2 (page 14), a relation is an *order relation* if it is *reflexive*, *transitive*, and *anti-symmetric*.

1. Proof that $\subseteq$ is *reflexive* on 2^X:

$$\begin{aligned}
x \in A &\implies x \in A\\
&\implies A \subseteq A
\end{aligned}$$

2. Proof that $\subseteq$ is *transitive* on 2^X:

$$\begin{aligned}
x \in A &\implies x \in B && \text{by first left hypothesis}\\
&\implies x \in C && \text{by second left hypothesis}\\
&\implies A \subseteq C
\end{aligned}$$

3. Proof that $\subseteq$ is *anti-symmetric* on 2^X:

$$\begin{aligned}
A \subseteq B &\implies (x \in A \implies x \in B)\\
B \subseteq A &\implies (x \in B \implies x \in A)\\
A \subseteq B \text{ and } B \subseteq A &\implies (x \in A \iff x \in B)\\
&\implies A = B
\end{aligned}$$

✏

In a set structure that is *closed* under the *union* operation $\cup$ and *intersection* operation $\cap$, the *greatest lower bound* of any two elements A and B is simply $A \cap B$ and *least upper bound* is simply $A \cup B$ (Proposition C.4 page 44). However, this may not be true for a set structure that is *not* closed under these operations (Example C.11 page 45).

Proposition C.4. *Let S be a* SET STRUCTURE *(Definition C.1 page 33) on a set X.*

PRP ***If** S is closed under $\cup$ and $\cap$ **then***

$$\begin{array}{llll}
A \cup B & \textit{is the } \text{LEAST UPPER BOUND} & \textit{of } A \textit{ and } B \textit{ in } (S, \subseteq) & (\cup = \vee) \quad \text{and}\\
A \cap B & \textit{is the } \text{GREATEST LOWER BOUND} & \textit{of } A \textit{ and } B \textit{ in } (S, \subseteq) & (\cap = \wedge).
\end{array}$$

✎PROOF:

1. Proof that $A \cup B$ is the least upper bound:

$$\begin{aligned}
A &= \{x \in X | x \in A\} \\
&\subseteq \{x \in X | x \in A \text{ or } x \in B\} \\
&= A \cup B && \text{by Definition C.5 page 33} \\
B &= \{x \in X | x \in B\} \\
&\subseteq \{x \in X | x \in A \text{ or } x \in B\} \\
&= A \cup B && \text{by Definition C.5 page 33} \\
A \subseteq C \text{ and } B \subseteq C &\implies \left\{x \in A \text{ and } y \in B \implies x, y \in C\right\} \\
&\implies \{x \in A \text{ or } x \in B \implies x \in C\} \\
&\implies \{x \in A \cup B \implies x \in C\} \\
&\implies A \cup B \subseteq C
\end{aligned}$$

2. Proof that $A \cap B$ is the greatest lower bound:

$$\begin{aligned}
A \cap B &= \left\{x \in X | x \in A \text{ and } x \in B\right\} && \text{by Definition C.5 page 33} \\
&\subseteq \{x \in X | x \in A\} \\
&= A \\
A \cap B &= \left\{x \in X | x \in A \text{ and } x \in B\right\} && \text{by Definition C.5 page 33} \\
&\subseteq \{x \in X | x \in B\} \\
&= B \\
C \subseteq A \text{ and } C \subseteq B &\implies \left\{x \in C \implies x \in A \quad \text{and} \quad x \in C \implies x \in B\right\} \\
&\implies \{x \in C \implies x \in A \text{ or } x \in B\} \\
&\implies \{x \in C \implies x \in A \cap B\} \\
&\implies C \subseteq A \cap B
\end{aligned}$$

✏

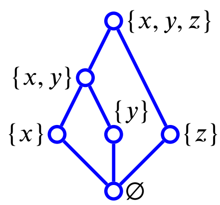
Example C.11. The set structure

$$\boldsymbol{S} \triangleq \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, y, z\}\}$$

ordered by the set inclusion relation $\subseteq$ is illustrated by the Hasse diagram to the right. Note that

$$\{x\} \vee \{z\} = \{x, y, z\} \neq \{x, z\} = \{x\} \cup \{z\}.$$

That is, the set union operation $\cup$ is *not* equivalent to the order join operation $\vee$.

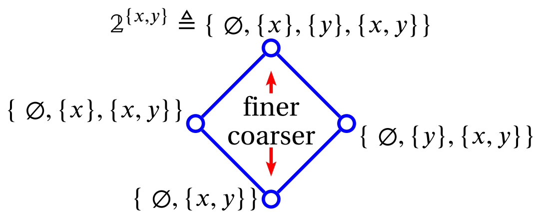


C.3.2 Lattices of topologies

Example C.12. [15] Example C.2 (page 37) lists the four topologies on the set $X \triangleq \{x, y\}$. The lattice of these topologies

$$\left(\left\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{T}_4\right\}, \cup, \cap; \subseteq\right)$$

is illustrated by the *Hasse diagram* to the right.



[15] Isham (1999), page 44, Isham (1989), page 1515

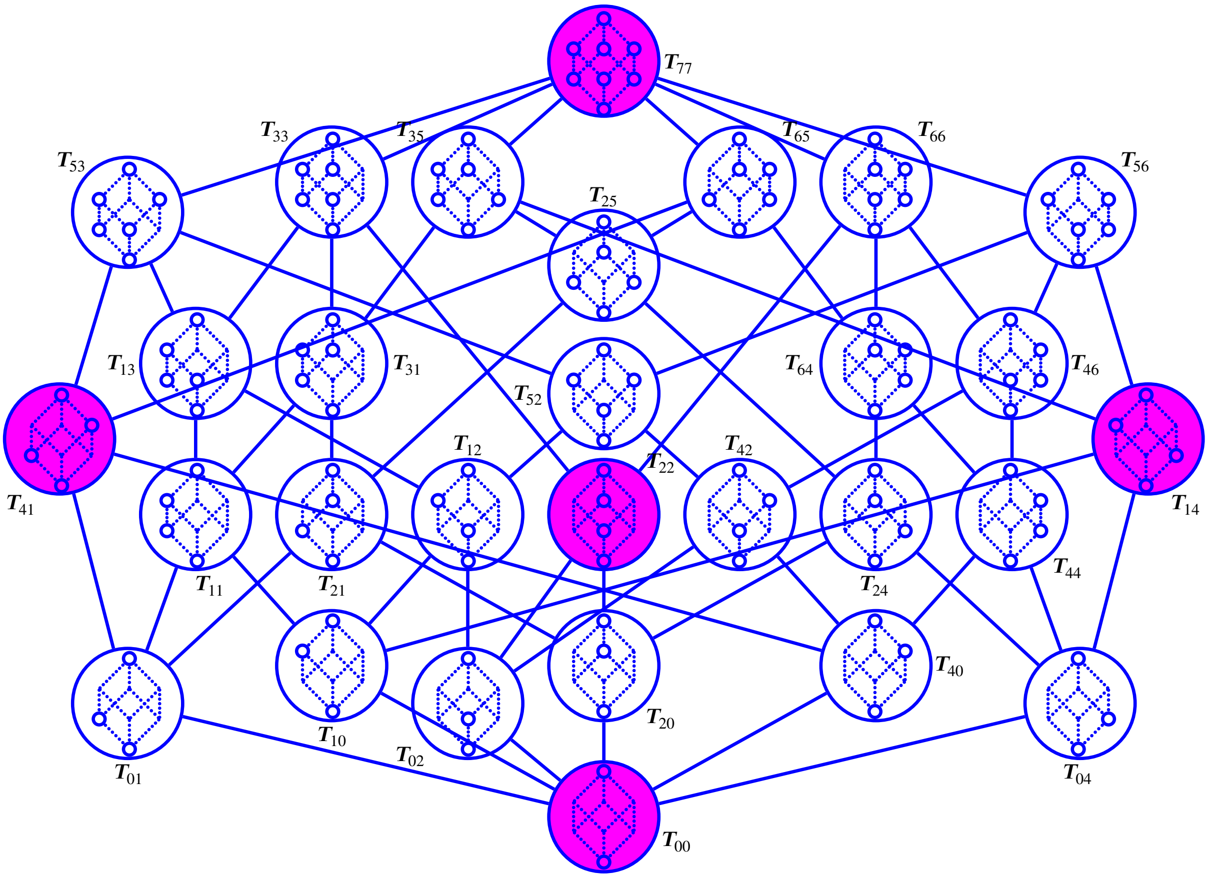
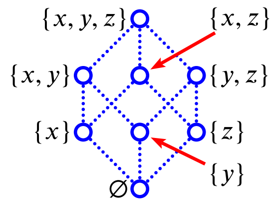


Figure C.5: Lattice of *topologies* on $X \triangleq \{x, y, z\}$ (see Example C.13 page 46)



Example C.13. [16]Let a given topology in $\mathcal{T}(\{x, y, z\})$ be represented by a Hasse diagram as illustrated to the right, where a circle present means the indicated set is in the topology, and a circle absent means the indicated set is not in the topology. Example C.3 (page 37) lists the 29 topologies $\mathcal{T}(\{x, y, z\})$. The lattice of these 29 topologies $(\mathcal{T}(\{x, y, z\}), \cup, \cap; \subseteq)$ is illustrated in Figure C.5 (page 46). The five topologies $\boldsymbol{T}_1$, $\boldsymbol{T}_{41}$, $\boldsymbol{T}_{22}$, $\boldsymbol{T}_{14}$, and $\boldsymbol{T}_{77}$ are also *algebras of sets*; these five sets are shaded in Figure C.5.

Theorem C.4. [17] *Let $\mathcal{T}(X)$ be the* ***lattice of topologies*** *on a set X with $|X|$ elements.*

THM	$|X| \leq 2 \implies \mathcal{T}(X)$ *is* DISTRIBUTIVE $|X| \geq 3 \implies \mathcal{T}(X)$ *is* NOT MODULAR *(and not distributive)*

Theorem C.5. [18] *Let $\mathcal{T}(X)$ be the* ***lattice of topologies*** *on a set X.*

THM	$\mathcal{T}(X)$ *is* SELF-DUAL $\iff$ $|X| \leq 3$

Theorem C.6. [19]

THM	*Every lattice of topologies is complemented.*

[16] Greenhoe (2016), page 226 ⟨Example 14.14⟩, Greenhoe (2017), pages 10–11 ⟨Example 1.13⟩, Isham (1999), page 44, Isham (1989), page 1516, Steiner (1966), page 386
[17] Steiner (1966), page 384
[18] Steiner (1966), page 385
[19] van Rooij (1968), Steiner (1966), page 397, Gaifman (1961), Hartmanis (1958)

topologies on $\{x, y, z\}$	1st complement	2nd compl.
$\mathsf{T}_{00} = \{\emptyset, X\}$	T_{77}	
$\mathsf{T}_{01} = \{\emptyset, \{x\}, X\}$	T_{56}	T_{66}
$\mathsf{T}_{02} = \{\emptyset, \{y\}, X\}$	T_{65}	T_{35}
$\mathsf{T}_{04} = \{\emptyset, \{z\}, X\}$	T_{53}	T_{33}
$\mathsf{T}_{10} = \{\emptyset, \{x,y\}, X\}$	T_{65}	T_{66}
$\mathsf{T}_{20} = \{\emptyset, \{x,z\}, X\}$	T_{53}	T_{56}
$\mathsf{T}_{40} = \{\emptyset, \{y,z\}, X\}$	T_{33}	T_{35}
$\mathsf{T}_{11} = \{\emptyset, \{x\}, \{x,y\}, X\}$	T_{64}	T_{46}
$\mathsf{T}_{21} = \{\emptyset, \{x\}, \{x,z\}, X\}$	T_{52}	T_{46}
$\mathsf{T}_{41} = \{\emptyset, \{x\}, \{y,z\}, X\}$	T_{22}	T_{14}
$\mathsf{T}_{12} = \{\emptyset, \{y\}, \{x,y\}, X\}$	T_{64}	T_{25}
$\mathsf{T}_{22} = \{\emptyset, \{y\}, \{x,z\}, X\}$	T_{41}	T_{14}
$\mathsf{T}_{42} = \{\emptyset, \{y\}, \{y,z\}, X\}$	T_{31}	T_{25}
$\mathsf{T}_{14} = \{\emptyset, \{z\}, \{x,y\}, X\}$	T_{41}	T_{22}
$\mathsf{T}_{24} = \{\emptyset, \{z\}, \{x,z\}, X\}$	T_{52}	T_{13}
$\mathsf{T}_{44} = \{\emptyset, \{z\}, \{y,z\}, X\}$	T_{31}	T_{13}
$\mathsf{T}_{31} = \{\emptyset, \{x\}, \{x,y\}, \{x,z\}, X\}$	T_{42}	T_{44}
$\mathsf{T}_{52} = \{\emptyset, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{21}	T_{24}
$\mathsf{T}_{64} = \{\emptyset, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{11}	T_{12}
$\mathsf{T}_{13} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, X\}$	T_{24}	T_{44}
$\mathsf{T}_{25} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, X\}$	T_{12}	T_{42}
$\mathsf{T}_{46} = \{\emptyset, \{y\}, \{z\}, \{y,z\}, X\}$	T_{11}	T_{21}
$\mathsf{T}_{33} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, X\}$	T_{04}	T_{40}
$\mathsf{T}_{53} = \{\emptyset, \{x\}, \{y\}, \{x,y\}, \{y,z\}, X\}$	T_{04}	T_{20}
$\mathsf{T}_{35} = \{\emptyset, \{x\}, \{z\}, \{x,y\}, \{x,z\}, X\}$	T_{02}	T_{40}
$\mathsf{T}_{65} = \{\emptyset, \{x\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{02}	T_{10}
$\mathsf{T}_{56} = \{\emptyset, \{y\}, \{z\}, \{x,y\}, \{y,z\}, X\}$	T_{01}	T_{20}
$\mathsf{T}_{66} = \{\emptyset, \{y\}, \{z\}, \{x,z\}, \{y,z\}, X\}$	T_{01}	T_{10}
$\mathsf{T}_{77} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}, X\}$	T_{00}	

Table C.4: the 29 topologies on a set $\{x, y, z\}$ along with their respective complements (Example C.14 page 47)

Theorem C.7. [20]

THM *Every* TOPOLOGY *(Definition C.6 page 37) except the* DISCRETE TOPOLOGY *and* INDISCRETE TOPOLOGY *(Example C.1 page 37) in the* ***lattice of topologies*** *on a set X has at least $|X| - 1$* COMPLEMENTS.

Example C.14. Example C.3 (page 37) lists the 29 topologies on a set $X \triangleq \{x, y, z\}$. By Theorem C.7 (page 47), with the exception of T_{00} (the indiscrete topology) and T_{77} (the discrete topology), each of those topologies has exactly $|X| - 1 = 3 - 1 = 2$ complements. Table C.4 (page 47) lists the 29 topologies on $\{x, y, z\}$ along with their respective complements.

Theorem C.8. [21]

THM $\mathcal{T}(X)$ *is a topology of sets* $\implies \begin{cases} \mathcal{T}(X) \text{ is atomic.} \\ \mathcal{T}(X) \text{ is anti-atomic.} \end{cases}$

Theorem C.9. [22] *Let $\mathcal{T}(X)$ be the lattice of topologies on a set X and let $n \triangleq |X|$.*

---

[20] Hartmanis (1958), Schnare (1968), page 56, Watson (1994), Brown and Watson (1996), page 32

[21] Larson and Andima (1975), page 179, Frölich (1964), Vaidyanathaswamy (1960), Vaidyanathaswamy (1947)

[22] Larson and Andima (1975), page 179, Frölich (1964)

THM
$\mathcal{T}(X)$ *contains* $2^n - 2$ *atoms* *for finite* X.
$\mathcal{T}(X)$ *contains* $2^{|X|}$ *atoms* *for infinite* X.
$\mathcal{T}(X)$ *contains* $n(n-1)$ *anti-atoms* *for finite* X.
$\mathcal{T}(X)$ *contains* $2^{2^{|X|}}$ *anti-atoms* *for infinite* X.

C.3.3 Lattices of algebra of sets

Example C.15. The following table lists some algebras of sets on a finite set X. Lattices of algebras of sets are illustrated in Figure C.8 (page 50) and Figure C.6 (page 49).

algebra of sets $\mathcal{A}(X)$ on a set X

$$\mathcal{A}(\varnothing) = \left\{ \boldsymbol{A}_1 = \{\varnothing\} \right\}$$

$$\mathcal{A}(\{x\}) = \left\{ \boldsymbol{A}_1 = \{\varnothing,\ \{x\}\} \right\}$$

$$\mathcal{A}(\{x,y\}) = \left\{ \begin{array}{lcl} \boldsymbol{A}_1 &=& \{\varnothing,\ X\} \\ \boldsymbol{A}_2 &=& \{\varnothing,\ \{x\},\ \{y\},\ X\} \end{array} \right\}$$

$$\mathcal{A}(\{x,y,z\}) = \left\{ \begin{array}{lclllllllll} \boldsymbol{A}_1 &=& \{ & \varnothing, & & & & & & & X \ \} \\ \boldsymbol{A}_2 &=& \{ & \varnothing, & \{x\}, & & & & & \{y,z\}, & X \ \} \\ \boldsymbol{A}_3 &=& \{ & \varnothing, & & \{y\}, & & & \{x,z\}, & & X \ \} \\ \boldsymbol{A}_4 &=& \{ & \varnothing, & & & \{z\}, & \{x,y\}, & & & X \ \} \\ \boldsymbol{A}_5 &=& \{ & \varnothing, & \{x\}, & \{y\}, & \{z\}, & \{x,y\}, & \{x,z\}, & \{y,z\}, & X \ \} \end{array} \right\}$$

$$\mathcal{A}(\{w,x,y,z\}) =$$
$$\left\{ \begin{array}{lcllllll} \boldsymbol{A}_1 &=& \{ & \varnothing, & & & & X \ \} \\ \boldsymbol{A}_2 &=& \{ & \varnothing, & \{w\}, & & \{x,y,z\}, & X \ \} \\ \boldsymbol{A}_3 &=& \{ & \varnothing, & \{x\}, & & \{w,y,z\}, & X \ \} \\ \boldsymbol{A}_4 &=& \{ & \varnothing, & \{y\}, & & \{w,x,z\}, & X \ \} \\ \boldsymbol{A}_5 &=& \{ & \varnothing, & \{z\}, & & \{w,x,y\}, & X \ \} \\ \boldsymbol{A}_6 &=& \{ & \varnothing, & & \{w,x\},\{y,z\}, & & X \ \} \\ \boldsymbol{A}_7 &=& \{ & \varnothing, & & \{w,y\},\{x,z\}, & & X \ \} \\ \boldsymbol{A}_8 &=& \{ & \varnothing, & & \{w,z\},\{x,y\}, & & X \ \} \\ \boldsymbol{A}_9 &=& \{ & \varnothing, & \{w\},\{x\}, & \{w,x\},\{y,z\}, & \{w,y,z\},\{x,y,z\}, & X \ \} \\ \boldsymbol{A}_{10} &=& \{ & \varnothing, & \{w\},\{y\}, & \{w,y\},\{x,z\}, & \{w,x,z\},\{x,y,z\}, & X \ \} \\ \boldsymbol{A}_{11} &=& \{ & \varnothing, & \{w\},\{z\}, & \{w,z\},\{x,y\}, & \{w,x,y\},\{x,y,z\}, & X \ \} \\ \boldsymbol{A}_{12} &=& \{ & \varnothing, & \{x\},\{y\}, & \{w,z\},\{x,y\}, & \{w,x,z\},\{w,y,z\}, & X \ \} \\ \boldsymbol{A}_{13} &=& \{ & \varnothing, & \{x\},\{z\}, & \{w,y\},\{x,z\}, & \{w,x,y\},\{w,y,z\}, & X \ \} \\ \boldsymbol{A}_{14} &=& \{ & \varnothing, & \{y\},\{z\}, & \{w,x\},\{y,z\}, & \{w,x,y\},\{w,x,z\}, & X \ \} \\ \boldsymbol{A}_{15} &=& 2^X \end{array} \right\}$$

C.3.4 Lattices of rings of sets

Example C.16. There are a total of **15** rings of sets on the set $X \triangleq \{x,y,z\}$. These rings of sets are listed in Example C.6 (page 40) and illustrated in Figure C.7 (page 49). The five rings containing X ($\boldsymbol{R}_{11}$–$\boldsymbol{R}_{15}$) are also *algebras of sets*, and thus also *Boolean algebras* (Theorem C.3 page 40). The five algebras

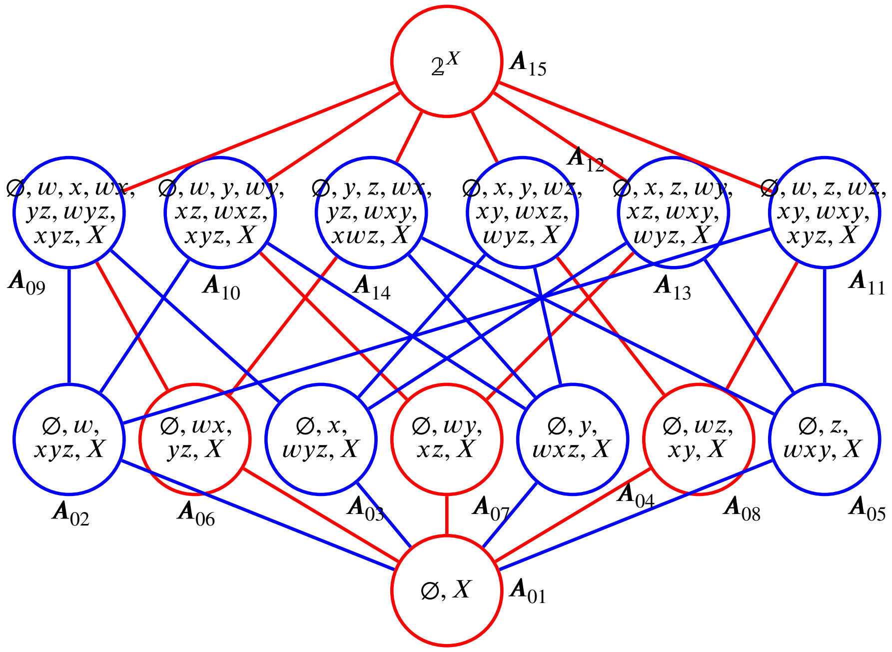


Figure C.6: lattice of *algebras of sets* on $\{w, x, y, z\}$ (Example C.15 page 48)

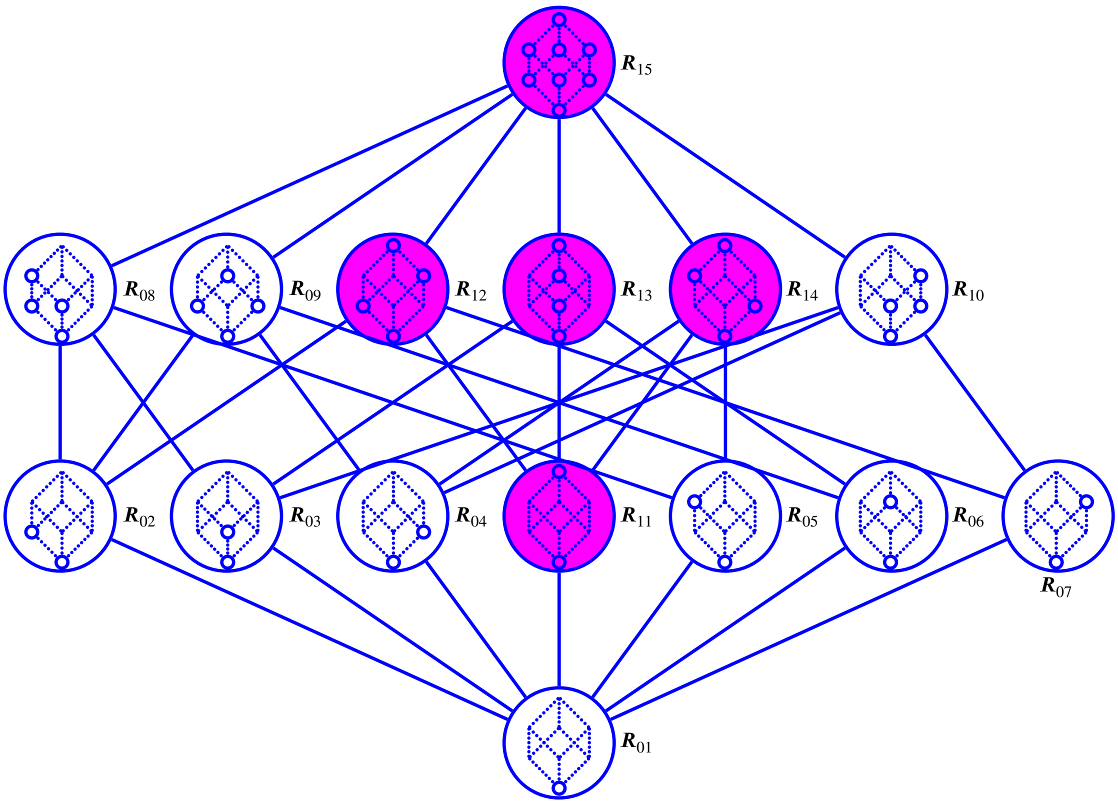


Figure C.7: Lattice of rings of sets on $X \triangleq \{x, y, z\}$ (Example C.16 page 48)

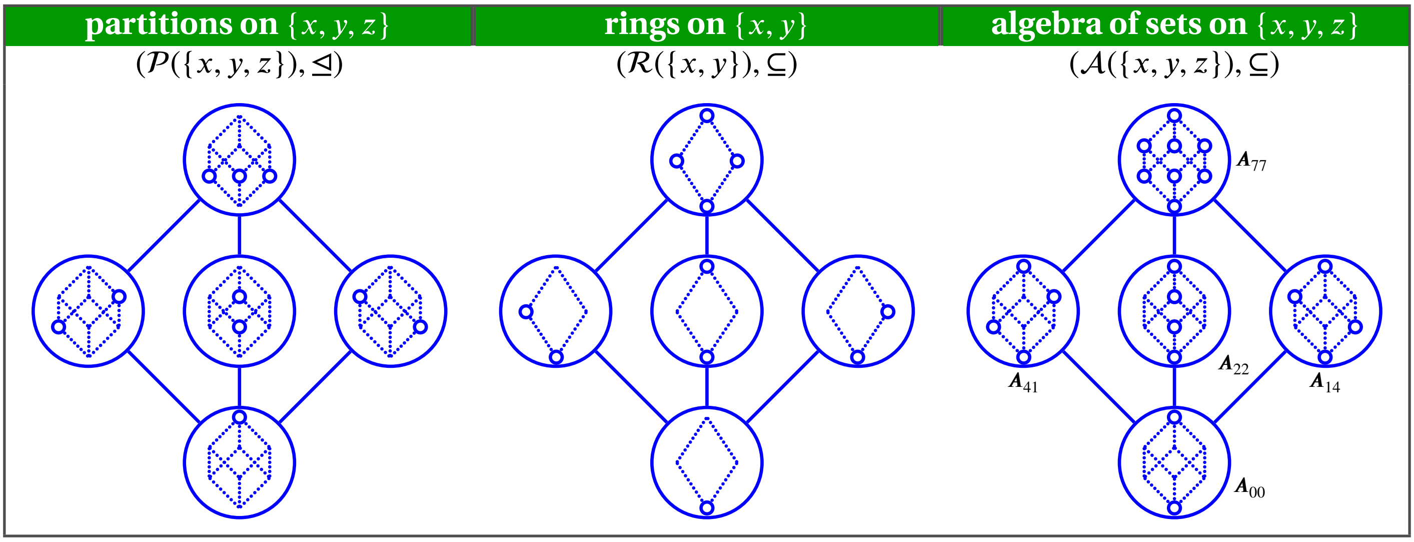


Figure C.8: Lattices of set structures (see Example C.17 (page 50), Example C.6 (page 40), and Example C.15 (page 48))

of sets are shaded Figure C.7.

C.3.5 Lattices of partitions of sets

Example C.17. There are a total of **5** partitions of sets on the set $X \triangleq \{x, y, z\}$. These sets are listed in Example C.10 (page 42) and illustrated in Figure C.8 (page 50).

Example C.18. There are a total of **15** partitions of sets on the set $X \triangleq \{w, x, y, z\}$. These sets are listed in Example C.10 (page 42) and illustrated in Figure C.9 (page 51).

In 1946, Philip Whitman proposed an amazing conjecture—that all finite lattices are isomorphic to a lattice of partitions. A proof for this was published some 30 years later by Pavel Pudlák and Jiří Tůma (next theorem).

Theorem C.10. [23] *Let* $\mathbf{L}$ *be a lattice.*

THM			
THM	$\mathbf{L}$ *is* FINITE	$\Longrightarrow$	$\mathbf{L}$ *is isomorphic to a* LATTICE OF PARTITIONS

Example C.19. There are five unlabeled lattices on a five element set as stated in Proposition B.2 (page 29) and illustrated in Example 1.3 (page 4). All of these lattices are isomorphic to a lattice of partitions (Theorem C.10 page 50), as illustrated in Figure C.10 (page 51).

[23] Pudlák and Tůma (1980) ⟨improved proof⟩, Pudlák and Tůma (1977) ⟨proof⟩, Whitman (1946) ⟨conjecture⟩, Saliĭ (1988) page vii ⟨list of lattice theory breakthroughs⟩

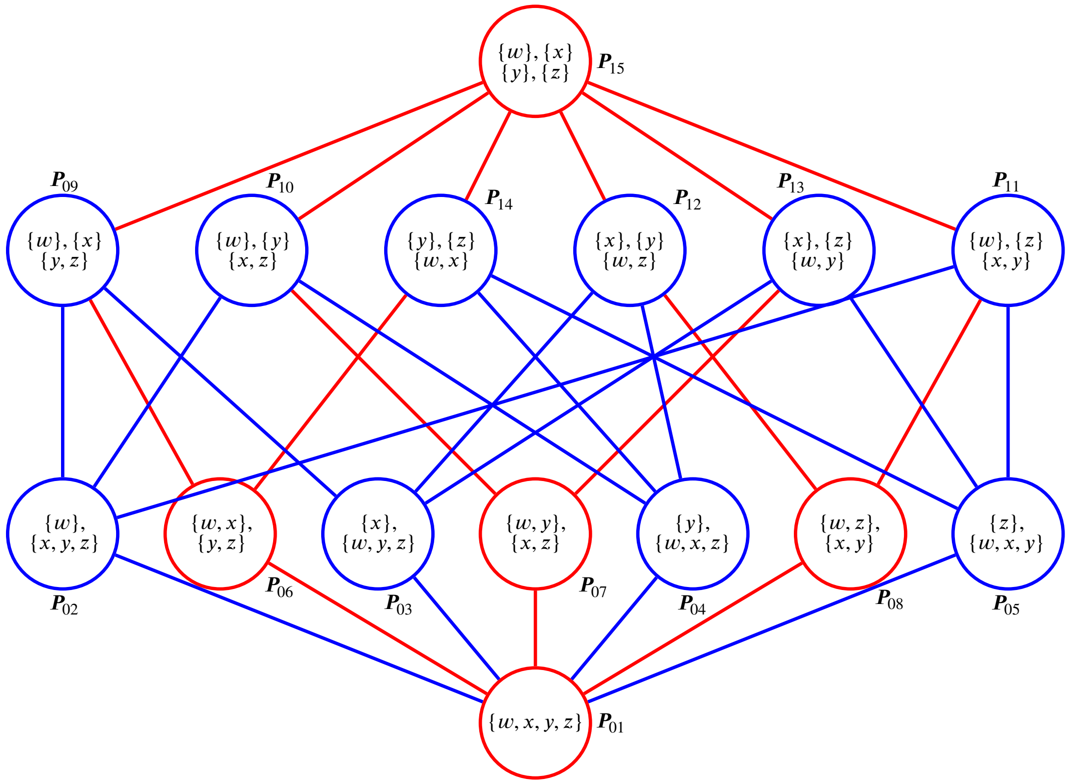


Figure C.9: Lattice of partitions of sets on $X \triangleq \{w, x, y, z\}$ (Example C.18 page 50)

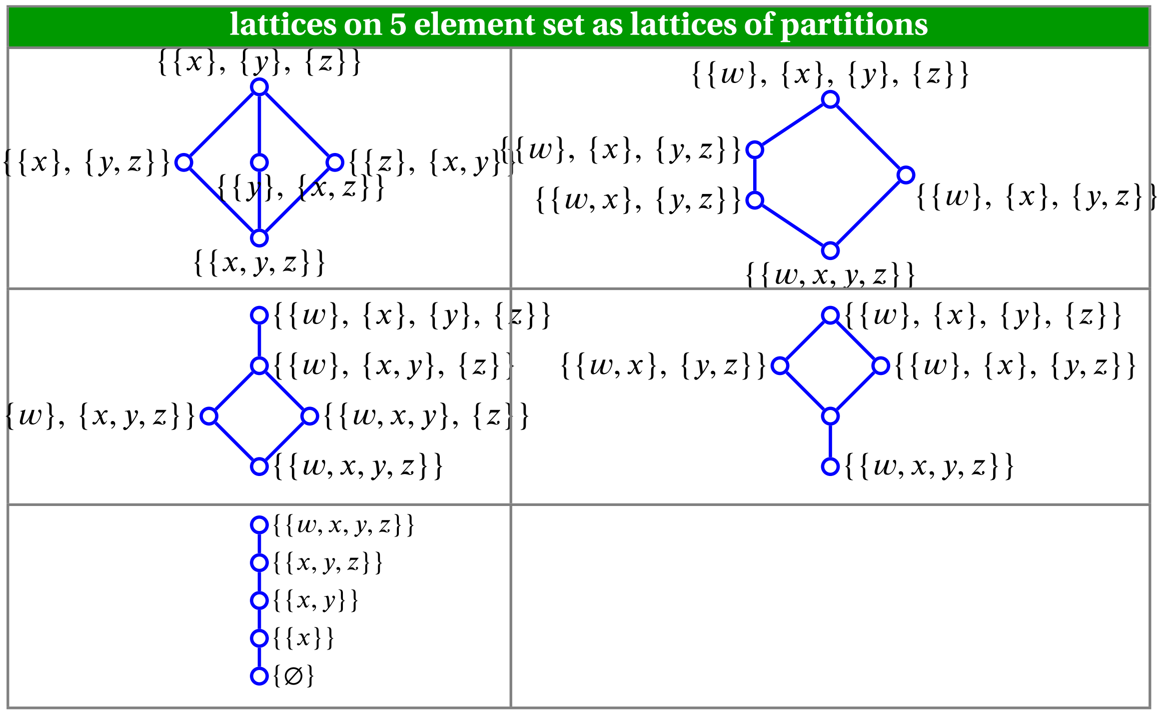


Figure C.10: Lattice of partitions

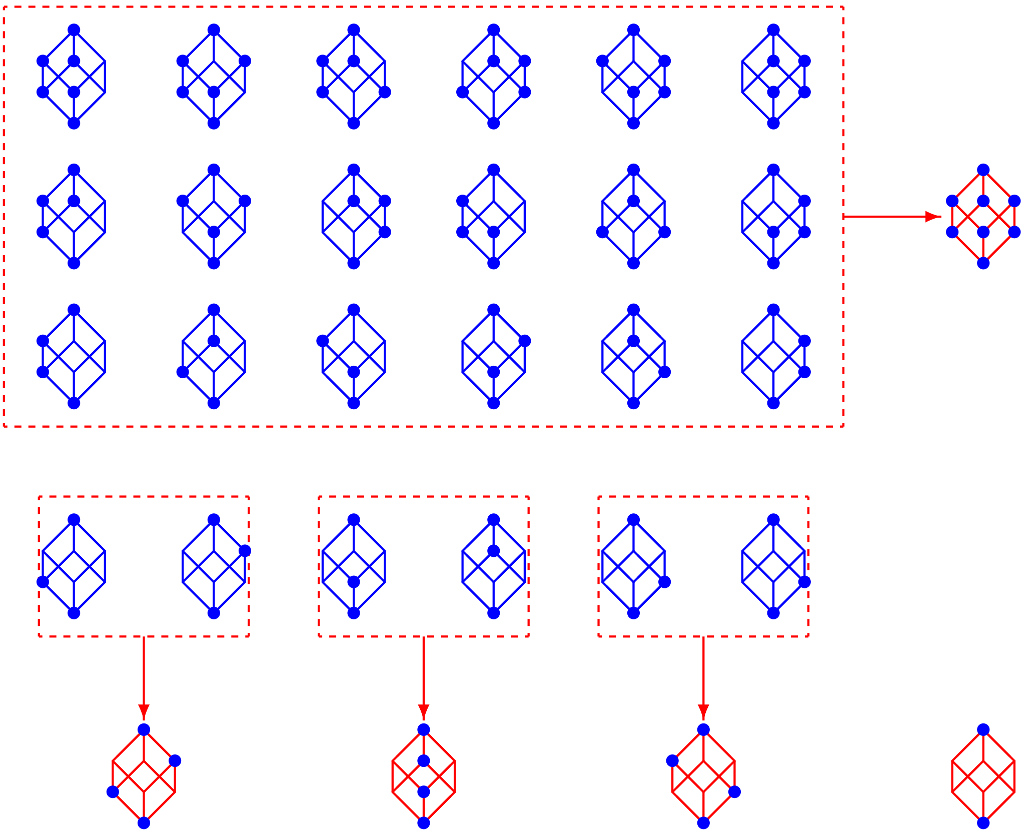


Figure C.11: Algebras of sets generated by topologies on the set $X \triangleq \{x, y, z\}$ (see Example C.20 page 52)

C.4 Relationships between set structures

Proposition C.5. [24]

PRP	$\left\{ \begin{array}{l} \boldsymbol{R} \textit{ is a } \textbf{\textit{ring of sets}} \\ \textit{on a set } X \end{array} \right\}$	$\Longrightarrow$	$\left\{ \begin{array}{l} \boldsymbol{R} \cup X \textit{ is an } \textbf{\textit{algebra of sets}} \\ \textit{on } X \end{array} \right\}$

Example C.20. There are a total of 29 *topologies* on the set $X \triangleq \{x, y, z\}$; and of these, 5 are also *algebras of sets*, 24 are not. Figure C.11 (page 52) illustrates the 24 topologies on the set $\{x, y, z\}$ that are *not* algebras of sets and the 5 algebras of sets that they generate.

[24] Berezansky et al. (1996) page 4, Halmos (1950) page 21

APPENDIX D

SOURCE CODE

D.1 Unlabeled lattices

D.1.1 Lattices on 1, 2, and 3 element sets

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice (2^{x,y,z}, subseteq)
%============================================================================
\begin{pspicture}(-\latbot,-\latbot)(\latbot,\latbot)%
  %---------------------------------
  % settings
  %---------------------------------
  %\psset{%
  %  %
  %  }%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  %\ncline{t}{b}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](b){$\bid$}%
  %\uput[180](b){$\bzero$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice (2^{x,y,z}, subseteq)
% nominal unit = 5mm
%============================================================================
\begin{pspicture}(-0.3,-\latbot)(0.3,1.3)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,1){t}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
```

```
  \ncline{t}{b}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[0](t)  {$\bid$}%
  %\uput[0](b)  {$\bzero$}%
\end{pspicture}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 10mm
%==========================================================================
{%\psset{yunit=0.75\psunit}%
\begin{pspicture}(-\latbot,-\latbot)(\latbot,2.3)
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(0,2){t}%
  \Cnode(0,1){x}%
  \Cnode(0,0){b}%
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{x}%
  \ncline{b}{x}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[0](t){$\setn{a,b}$}%
  %\uput[0](x){$\setn{a}$}%
  %\uput[0](b)  {$\emptyset$}%
\end{pspicture}
}%
```

D.1.2 Lattices on 4 element sets

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on M2
% nominal unit = 10mm
%==========================================================================
{%
\begin{pspicture}(-0.5,-\latbot)(0.5,3.5)%
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(0,3){t}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}%
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{d}{t}%
  \ncline{c}{d}%
  \ncline{b}{c}%
\end{pspicture}
}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2
% nominal unit = 10mm
%==========================================================================
{%
\begin{pspicture}(-1.5,-\latbot)(1.5,2.5)
  %-----------------------------------
  % nodes
```

```
  %--------------------------------
  \Cnode(0,2){t}
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %--------------------------------
  % node connections
  %--------------------------------
  \ncline{t}{x}\ncline{t}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %--------------------------------
  % node labels
  %--------------------------------
  %\uput[0](t) {$\bid$}%
  %\uput[0](y) {$y$}%
  %\uput[0](x) {$x$}%
  %\uput[0](b) {$\bzero$}%
\end{pspicture}
}%
```

D.1.3 Lattices on 5 element sets

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
% nominal unit = 10mm
%==========================================================================
{%
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %--------------------------------
  % nodes
  %--------------------------------
  \Cnode(0,3){t}%
  \Cnode(0,2){c}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %--------------------------------
  % node connections
  %--------------------------------
  \ncline{t}{c}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{x}\ncline{b}{y}%
\end{pspicture}
}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
% nominal unit = 10mm
%==========================================================================
{%
\begin{pspicture}(-0.5,-\latbot)(0.5,4.5)
  %--------------------------------
  % nodes
  %--------------------------------
  \Cnode(0,4){t}%
  \Cnode(0,3){e}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %--------------------------------
  % node connections
  %--------------------------------
  \ncline{e}{t}%
  \ncline{d}{e}%
  \ncline{c}{d}%
  \ncline{b}{c}%
  %--------------------------------
  % node labels
  %--------------------------------
  %\uput[0](t) {$\setn{w,x,y,z}$}%
```

```
  %\uput[0](e) {$\setn{w,x,y}$}%
  %\uput[0](d) {$\setn{w,x}$}%
  %\uput[0](c) {$\setn{w}$}%
  %\uput[0](b) {$\emptyset$}%
\end{pspicture}
}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
% nominal unit = 10mm
%==========================================================================
{%
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){x}\Cnode(1,2){y}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{x}\ncline{t}{y}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x) {$\setn{x}$}%
  %\uput[0](z) {$\setn{z}$}%
  %\uput[-90](b) {$\szero$}%
\end{pspicture}
}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M3
% nominal unit = 10mm
%==========================================================================
{%
\begin{pspicture}(-1.3,-\latbot)(1.3,2.3)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,2){t}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
\end{pspicture}
}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice N5
% nominal unit = 10mm
%==========================================================================
{%
```

```
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}
  \Cnode(-1,2){d}%
  \Cnode( 1,1.5){m}%
  \Cnode(-1,1){c}
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{d}\ncline{t}{m}%
  \ncline{c}{d}%
  \ncline{b}{c}\ncline{b}{m}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,    0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}
}%
```

D.1.4 Lattices on 6 element sets

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
% nominal unit = 10mm
%============================================================================
{%
\begin{pspicture}(-1.5,-\latbot)(1.5,4.5)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}%
  \Cnode(0,3){d}%
  \Cnode(-1,2){x}\Cnode(1,2){y}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{d}%
  \ncline{d}{x}\ncline{d}{y}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
%============================================================================
{%\psset{unit=0.667\psunit}%
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
```

```
  %----------------------------------
  % nodes
  %----------------------------------
  \Cnode(0,3){t}%
  \Cnode(0,2){c}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}%
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{c}%
  \ncline{c}{x}\ncline{c}{y}\ncline{c}{z}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %----------------------------------
  % node labels
  %----------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice N5
%============================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,4.3)%
  %----------------------------------
  % nodes
  %----------------------------------
                  \Cnode(0,4){t}%
                  \Cnode(0,3){e}%
  \Cnode(-1,2){d}%
                                 \Cnode( 1,1.5){m}%
  \Cnode(-1,1){c}%
                  \Cnode(0,0){b}%
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{e}%
  \ncline{e}{d}\ncline{e}{m}%
  \ncline{c}{d}%
  \ncline{b}{c}\ncline{b}{m}%
  %----------------------------------
  % node labels
  %----------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
% nominal unit = 10mm
%============================================================================
{%
\begin{pspicture}(-1.4,-\latbot)(1.4,4.5)
  %----------------------------------
  % nodes
  %----------------------------------
  \Cnode(0,4){t}%
  \Cnode(0,3){d}%
  \Cnode(0,2){c}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{d}%
  \ncline{c}{d}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %----------------------------------
  % node labels
  %----------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
  %\uput[0](c) {$\setn{x,y}$}%
```

```
  %\uput[0](y)  {$\setn{y}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](b)  {$\emptyset$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice L4 in M2
% nominal unit = 10mm
%============================================================================
{%
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}
  \Cnode(0,2){d}
  \Cnode(-1,1.5){x}\Cnode(1,1.5){z}%
  \Cnode(0,1){c}
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{x}\ncline{t}{d}\ncline{t}{z}%
  \ncline{c}{d}%
  \ncline{b}{x}\ncline{b}{c}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
% nominal unit = 10mm
%============================================================================
{%
\begin{pspicture}(-0.3,-\latbot)(0.3,5.5)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,5){t}%
  \Cnode(0,4){f}%
  \Cnode(0,3){e}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{f}{t}%
  \ncline{e}{f}%
  \ncline{d}{e}%
  \ncline{c}{d}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t)  {$\setn{w,x,y,z}$}%
  %\uput[0](e)  {$\setn{w,x,y}$}%
  %\uput[0](d)  {$\setn{w,x}$}%
  %\uput[0](c)  {$\setn{w}$}%
```

```
  %\uput[0](b)  {$\emptyset$}%
\end{pspicture}
}%
```

```
%======================================================================
% Daniel J. Greenhoe
% LaTeX file
%======================================================================
{%
\begin{pspicture}(-1.3,-\latbot)(2.3,3.5)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(1,3){t}
  \Cnode(0,2){y}\Cnode(2,2){z}%
  \Cnode(-1,1){w}\Cnode(1,1){x}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{y}\ncline{t}{z}%
  \ncline{x}{y}\ncline{x}{z}%
  \ncline{w}{y}%
  \ncline{b}{w}\ncline{b}{x}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%======================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 10mm
%======================================================================
{%
\begin{pspicture}(-1.4,-\latbot)(1.4,4.5)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}%
  \Cnode(-1,3){x}\Cnode(1,3){y}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{x}\ncline{t}{y}%
  \ncline{d}{x}\ncline{d}{y}%
  \ncline{c}{d}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}
}%
```

```
%======================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on L2
%======================================================================
{%\psset{unit=0.667\psunit}%
```

```
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){x}\Cnode(0,2){y}\Cnode(1,2){z}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{c}{x}\ncline{c}{y}\ncline{c}{z}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t)  {$\setn{x,y,z}$}%
  %\uput[0](c)  {$\setn{x,y}$}%
  %\uput[0](y)  {$\setn{y}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](b)  {$\emptyset$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M4
% nominal unit = 10mm
%============================================================================
{%\psset{xunit=0.75\psxunit,yunit=0.75\psyunit}%
\begin{pspicture}(-1.9,-\latbot)(1.9,2.2)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,2){t}%
  \Cnode(-1.5,1){w}\Cnode(-0.5,1){x}\Cnode(0.5,1){y}\Cnode(1.5,1){z}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{w}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice N5
%============================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}
  \Cnode(-1,3){d}%
  \Cnode( 1,2.5){m}%
  \Cnode(-1,2){c}
  \Cnode(0,1){bn}
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{d}\ncline{t}{m}%
  \ncline{c}{d}%
  \ncline{bn}{c}\ncline{bn}{m}%
  \ncline{b}{bn}%
  %---------------------------------
  % node labels
```

```
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy) {$\setn{x,y}$}%
  %\uput[0](yz) {$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice N5
%============================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.3,-\latbot)(1.3,4.5)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}%
  \Cnode(-1,3){e}%
  \Cnode( 1,2){m}%
  \Cnode(-1,2){d}%
  \Cnode(-1,1){c}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{e}\ncline{t}{m}%
  \ncline{c}{d}\ncline{d}{e}%
  \ncline{b}{c}\ncline{b}{m}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy) {$\setn{x,y}$}%
  %\uput[0](yz) {$\setn{y,z}$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice O6
% nominal unit = 10mm
%============================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{c}{x}\ncline{d}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy) {$\setn{x,y}$}%
  %\uput[0](yz) {$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
%=====================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.5)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{c}{x}\ncline{d}{y}%
  \ncline{c}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
%=====================================================================
{%
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{c}{x}\ncline{d}{y}%
  \ncline{x}{d}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice primorial 3
% nominal unit = 10mm
%=====================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-0.5,2){m}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{m}\ncline{t}{z}%
  \ncline{m}{x}\ncline{m}{y}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
```

```
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice primorial 3 dual
% nominal unit = 10mm
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){x}\Cnode(0,2){y}\Cnode(1,2){z}%
  \Cnode(-0.5,1){m}%
  \Cnode(0,0){b}%
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{m}{x}\ncline{m}{y}%
  \ncline{b}{m}\ncline{b}{z}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

D.1.5 Lattices on 7 element sets

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice L4 in M2
% nominal unit = 10mm
%=========================================================================
{%
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(0,4){T}
  \Cnode(0,3){t}
  \Cnode(0,2){d}
  \Cnode(-1,1.5){x}\Cnode(1,1.5){z}%
  \Cnode(0,1){c}
  \Cnode(0,0){b}
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{T}%
  \ncline{t}{x}\ncline{t}{d}\ncline{t}{z}%
  \ncline{c}{d}%
  \ncline{b}{x}\ncline{b}{c}\ncline{b}{z}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on M2
%=========================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.2,-\latbot)(2.2,4.2)
  %-----------------------------------
  % nodes
```

```
  %-----------------------------------
  \Cnode(1,4){t}
  \Cnode(1,3){s}%
  \Cnode(0,2){u}\Cnode(2,2){v}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{s}%
  \ncline{s}{u}\ncline{s}{v}%
  \ncline{y}{u}\ncline{y}{v}%
  \ncline{x}{u}%
  \ncline{b}{x}\ncline{b}{y}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on M2
%=========================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.2,-\latbot)(2.2,4.2)
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(1,4){t}
  \Cnode(0,3){u}\Cnode(2,3){v}%
  \Cnode(-1,2){x}\Cnode(1,2){y}%
  \Cnode(0,1){c}
  \Cnode(0,0){b}
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{u}\ncline{t}{v}%
  \ncline{y}{v}%
  \ncline{u}{x}\ncline{u}{y}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{c}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,5.4)%
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(0,5){t}%
  \Cnode(0,4){e}%
  \Cnode(-1,3){x}\Cnode(1,3){y}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}
  \Cnode(0,0){b}
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{e}%
  \ncline{e}{x}\ncline{e}{y}%
  \ncline{d}{x}\ncline{d}{y}%
  \ncline{c}{d}%
  \ncline{b}{c}%
  %-----------------------------------
  % node labels
  %-----------------------------------
```

```
  %\uput[0](t)  {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=======================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice N5
%=======================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
                \Cnode(0,4){t}%
                \Cnode(0,3){tt}%
  \Cnode(-1,2){x}\Cnode(0,2){y}\Cnode(1,2){z}%
                \Cnode(0,1){bb}%
                \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
               \ncline{t}{tt}%
  \ncline{tt}{x}\ncline{tt}{y}\ncline{tt}{z}%
  \ncline{bb}{x}\ncline{bb}{y}\ncline{bb}{z}%
               \ncline{b}{bb}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=======================================================================
% Daniel J. Greenhoe
% LaTeX file
%=======================================================================
\begin{pspicture}(-1.9,-\latbot)(1.9,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
                            \Cnode(0,3){t}%
                            \Cnode(0,2){s}%
  \Cnode(-1.5,1){w}\Cnode(-0.5,1){x}\Cnode(0.5,1){y}\Cnode(1.5,1){z}%
                            \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{s}%                                        L2 sublattice
  \ncline{s}{w}\ncline{s}{x}\ncline{s}{y}\ncline{s}{z}% top    half of m4
  \ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}% bottom half of m4
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=======================================================================
% Daniel J. Greenhoe
% LaTeX file
%=======================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,5.4)%
  %---------------------------------
  % nodes
  %---------------------------------
                \Cnode(0,5){t}%
                \Cnode(0,4){tt}%
  \Cnode(-1,3){d}%
                               \Cnode( 1,2.5){m}%
  \Cnode(-1,2){c}%
                \Cnode(0,1){bb}%
                \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
         \ncline{t}{tt}%
  \ncline{tt}{d}\ncline{tt}{m}%
  \ncline{d}{c}%
```

```
  \ncline{bb}{c}\ncline{bb}{m}%
          \ncline{b}{bb}%
  %--------------------------------
  % node labels
  %--------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
%=====================================================================
\begin{pspicture}(-1.5,-\latbot)(1.5,5.4)%
  %--------------------------------
  % nodes
  %--------------------------------
  \Cnode(0,5){T}%
                  \Cnode(0,4){t}%
  \Cnode(-1,3){e}%
                                    \Cnode( 1,2){m}%
  \Cnode(-1,2){d}%
  \Cnode(-1,1){c}%
                  \Cnode(0,0){b}%
  %--------------------------------
  % node connections
  %--------------------------------
  \ncline{T}{t}%
  \ncline{t}{e}\ncline{t}{m}%
  \ncline{c}{d}\ncline{d}{e}%
  \ncline{b}{c}\ncline{b}{m}%
  %--------------------------------
  % node labels
  %--------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
%=====================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)%
  %--------------------------------
  % nodes
  %--------------------------------
  \Cnode(0,4){t}
  \Cnode(0,3){s}
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %--------------------------------
  % node connections
  %--------------------------------
  \ncline{t}{s}%
  \ncline{s}{c}\ncline{s}{d}%
  \ncline{c}{x}\ncline{d}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %--------------------------------
  % node labels
  %--------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
%=====================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %--------------------------------
  % nodes
  %--------------------------------
  \Cnode(0,4){t}
  \Cnode(0,3){s}
  \Cnode(-1,2){c}\Cnode(1,2){d}%
```

```
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{s}%
  \ncline{s}{c}\ncline{s}{d}%
  \ncline{c}{x}\ncline{d}{y}\ncline{x}{d}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice primorial 3
% nominal unit = 10mm
%============================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}%
  \Cnode(0,3){c}%
  \Cnode(-0.5,2){m}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}%
  \ncline{c}{m}\ncline{c}{z}%
  \ncline{m}{x}\ncline{m}{y}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[90](t) {$1$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
                    \Cnode(0,4){t}%
                    \Cnode(0,3){s}%
  \Cnode(-1,2){x}\Cnode(0,2){y}\Cnode(1,2){z}%
           \Cnode(-0.5,1){m}%
                    \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{s}%
  \ncline{s}{x}\ncline{s}{y}\ncline{s}{z}%
  \ncline{m}{x}\ncline{m}{y}%
  \ncline{b}{m}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[90](t) {$1$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 10mm
%============================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
```

```
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(0,1.5){m}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{m}\ncline{t}{d}%
  \ncline{c}{x}\ncline{d}{y}%
  \ncline{b}{x}\ncline{b}{m}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice L3 in O6
% nominal unit = 5mm
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(0,1.5){m}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}\ncline{t}{m}%
  \ncline{x}{c}\ncline{y}{d}%
  \ncline{x}{m}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x) {$\setn{x}$}%
  %\uput[0](z) {$\setn{z}$}%
  %\uput[-90](b) {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100, 0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 10mm
%=========================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,3.4)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(0,1){m}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{m}\ncline{t}{d}%
  \ncline{c}{x}\ncline{d}{y}%
```

```
  \ncline{c}{y}%
  \ncline{b}{x}\ncline{b}{m}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
                \Cnode(0,4){t}%
                \Cnode(0,3){e}%
  \Cnode(-1,2){x}\Cnode(0,2){d}\Cnode(1,2){y}%
                \Cnode(0,1){c}%
                \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{x}\ncline{x}{b}%                             linear 3 element component
  \ncline{t}{e}\ncline{e}{d}\ncline{d}{c}\ncline{c}{b}%   linear 5 element component
  \ncline{e}{y}\ncline{y}{c}%                             linear 4 element component
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.5,-\latbot)(1.5,5.4)%
  %---------------------------------
  % nodes
  %---------------------------------
          \Cnode(0,5){t}%
          \Cnode(0,4){e}%
          \Cnode(0,3){d}%
  \Cnode(-1,2){x}\Cnode(1,2){y}%
          \Cnode(0,1){c}
          \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{e}%
  \ncline{e}{d}%
  \ncline{d}{x}\ncline{d}{y}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %---------------------------------
  % nodes
  %---------------------------------
                \Cnode(0,4){t}%
                \Cnode(0,3){d}%
                \Cnode(0,2){c}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
                \Cnode(0,0){b}%
  %---------------------------------
```

```
  % node connections
  %-----------------------------------
  \ncline{t}{d}%
  \ncline{d}{c}%
  \ncline{c}{x}\ncline{c}{y}\ncline{c}{z}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[0](t){$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice N5
%=========================================================================
\begin{pspicture}(-1.5,-\latbot)(1.5,5.4)%
  %-----------------------------------
  % nodes
  %-----------------------------------
                   \Cnode(0,5){t}%
                   \Cnode(0,4){f}%
                   \Cnode(0,3){e}%
  \Cnode(-1,2){d}%
                                     \Cnode( 1,1.5){m}%
  \Cnode(-1,1){c}%
                   \Cnode(0,0){b}%
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{f}%
  \ncline{f}{e}%
  \ncline{e}{d}\ncline{e}{m}%
  \ncline{c}{d}%
  \ncline{b}{c}\ncline{b}{m}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[ 90](t){$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,4.4)%
  %-----------------------------------
  % nodes
  %-----------------------------------
                  \Cnode(0,4){t}%
                  \Cnode(0,3){e}\Cnode(1,3){y}%
                  \Cnode(0,2){d}%
  \Cnode(-1,1){x}\Cnode(0,1){c}%
                  \Cnode(0,0){b}%
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{y}\ncline{y}{c}%                         linear 4 element component
  \ncline{t}{e}\ncline{e}{d}\ncline{d}{c}\ncline{c}{b}% linear 5 element component
  \ncline{b}{x}\ncline{x}{e}%                         linear 4 element component
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[0](t){$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 5mm
%=========================================================================
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)
  %-----------------------------------
  % nodes
```

```
  %---------------------------------
   \Cnode(0,3){t}%
   \Cnode(-1.5,2){d}%
   \Cnode(-1.5,1){c}%
   \Cnode(-0.5,1.5){x}\Cnode(0.5,1.5){y}\Cnode(1.5,1.5){z}%
   \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{t}{d}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
   \ncline{c}{d}%
   \ncline{b}{c}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.5,-\latbot)(1.5,5.4)%
  %---------------------------------
  % nodes
  %---------------------------------
   \Cnode(0,5){t}%
   \Cnode(0,4){e}%
   \Cnode(0,3){d}%
   \Cnode(0,2){c}%
   \Cnode(-1,1){x}\Cnode(1,1){y}%
   \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{t}{e}%
   \ncline{e}{d}%
   \ncline{c}{d}%
   \ncline{c}{x}\ncline{c}{y}%
   \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,4.4)
  %---------------------------------
  % nodes
  %---------------------------------
                    \Cnode(0,4){t}%
                    \Cnode(0,3){e}%
   \Cnode(-1,2){x}\Cnode(0,2){d}\Cnode(1,2){y}%
                    \Cnode(0,1){c}%
                    \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{t}{x}\ncline{t}{y}%                               top    half of m2
   \ncline{b}{x}\ncline{b}{y}%                               bottom half of m2
   \ncline{t}{e}\ncline{e}{d}\ncline{d}{c}\ncline{c}{b}%  middle linear 5 element component
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
```

```
{%
\begin{pspicture}(-0.3,-\latbot)(1.3,5.4)%
  %---------------------------------
  % nodes
  %---------------------------------
   \Cnode(0,5){t}%
   \Cnode(0,4){f}%
   \Cnode(0,3){e}%
                  \Cnode(1,2.5){m}%
   \Cnode(0,2){d}%
   \Cnode(0,1){c}%
   \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{f}{t}%
   \ncline{e}{f}%
   \ncline{d}{e}%
   \ncline{c}{d}%
   \ncline{b}{c}%
   \ncline{b}{m}\ncline{t}{m}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$\setn{w,x,y,z}$}%
  %\uput[0](e) {$\setn{w,x,y}$}%
\end{pspicture}
}%
```

```
%======================================================================
% Daniel J. Greenhoe
% LaTeX file
%======================================================================
{%\psset{unit=0.667\psunit}%
\begin{pspicture}(-0.3,-\latbot)(0.3,6.5)
  %---------------------------------
  % nodes
  %---------------------------------
   \Cnode(0,6){t}%
   \Cnode(0,5){g}%
   \Cnode(0,4){f}%
   \Cnode(0,3){e}%
   \Cnode(0,2){d}%
   \Cnode(0,1){c}%
   \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{g}{t}%
   \ncline{f}{g}%
   \ncline{e}{f}%
   \ncline{d}{e}%
   \ncline{d}{e}%
   \ncline{c}{d}%
   \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%======================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 5mm
%======================================================================
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)
  %---------------------------------
  % nodes
  %---------------------------------
   \Cnode(0,3){t}%
   \Cnode(0,2){xy}%
   \Cnode(-1.5,1){w}\Cnode(-0.5,1){x}\Cnode(0.5,1){y}\Cnode(1.5,1){z}%
   \Cnode(0,0){b}%
  %---------------------------------
```

```
   % node connections
   %-------------------------------------
   \ncline{t}{w}\ncline{t}{xy}\ncline{t}{z}%
   \ncline{xy}{x}\ncline{xy}{y}%
   \ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
   %-------------------------------------
   % node labels
   %-------------------------------------
   %\uput[ 90](t)  {$\setn{x,y,z}$}%
   %\uput[-90](b)  {$\szero$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)
   %-------------------------------------
   % nodes
   %-------------------------------------
   %\Cnode(0,3){t}
   %\Cnode(0,2){xy}%
   %\Cnode(-1.5,1){w}\Cnode(-0.5,1){x}\Cnode(0.5,1){y}\Cnode(1.5,1){z}%
   %\Cnode(0,0){b}
   \Cnode(0,3){t}
   \Cnode(-0.75,2){xy}%
   \Cnode(0.5,1){w}\Cnode(-1.5,1){x}\Cnode(-0.5,1){y}\Cnode(1.5,1){z}%
   \Cnode(0,0){b}
   %-------------------------------------
   % node connections
   %-------------------------------------
   \ncline{t}{w}\ncline{t}{xy}\ncline{t}{z}
   \ncline{xy}{x}\ncline{xy}{y}%
   \ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
   %-------------------------------------
   % node labels
   %-------------------------------------
   %\uput[ 90](t)  {$\setn{x,y,z}$}%
   %\uput[180](xy){$\setn{x,y}$}%
   %\uput[0](yz){$\setn{y,z}$}%
   %\uput[180](x)  {$\setn{x}$}%
   %\uput[0](z)  {$\setn{z}$}%
   %\uput[-90](b)  {$\szero$}%
   %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
   %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
   %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
   %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 5mm
%============================================================================
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)
   %-------------------------------------
   % nodes
   %-------------------------------------
   \Cnode(0,3){t}
   \Cnode(-1.5,2){w}\Cnode(-0.5,2){x}\Cnode(0.5,2){y}\Cnode(1.5,2){z}%
   \Cnode(0,1){xy}%
   \Cnode(0,0){b}
   %-------------------------------------
   % node connections
   %-------------------------------------
   \ncline{t}{w}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
   \ncline{xy}{x}\ncline{xy}{y}%
   \ncline{b}{w}\ncline{b}{xy}\ncline{b}{z}
   %-------------------------------------
   % node labels
   %-------------------------------------
   %\uput[ 90](t)  {$\setn{x,y,z}$}%
   %\uput[180](xy){$\setn{x,y}$}%
   %\uput[0](yz){$\setn{y,z}$}%
```

```
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
\begin{pspicture}(-1.9,-\latbot)(1.9,4.4)%
  %----------------------------------
  % nodes
  %----------------------------------
                                \Cnode(0,4){t}%
  \Cnode(-1.5,3){d}%
                  \Cnode(-0.5,2){x}\Cnode(0.5,2){y}\Cnode(1.5,2){z}%
  \Cnode(-1.5,1){c}%
                                \Cnode(0,0){b}%
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{d}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{c}{d}%
  \ncline{b}{c}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %----------------------------------
  % node labels
  %----------------------------------
  %\uput[0](t)  {$1$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 5mm
%============================================================================
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)%
  %----------------------------------
  % nodes
  %----------------------------------
  \Cnode(0,3){t}%
  \Cnode(-0.5,2){xy}%
  \Cnode(-1.5,1){w}\Cnode(-0.5,1){x}\Cnode(0.5,1){y}\Cnode(1.5,1){z}%
  \Cnode(0,0){b}%
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{xy}\ncline{t}{z}%
  \ncline{xy}{w}\ncline{xy}{x}\ncline{xy}{y}%
  \ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %----------------------------------
  % node labels
  %----------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 5mm
%============================================================================
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)
  %----------------------------------
```

```
  % settings
  %---------------------------------
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1.5,2){w}\Cnode(-0.5,2){x}\Cnode(0.5,2){y}\Cnode(1.5,2){z}%
  \Cnode(-0.5,1){xy}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{w}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{xy}{w}\ncline{xy}{x}\ncline{xy}{y}%
  \ncline{b}{xy}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t)  {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x)  {$\setn{x}$}%
  %\uput[0](z)  {$\setn{z}$}%
  %\uput[-90](b)  {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100,   0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
%==========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
                  \Cnode(0,3){t}%
  \Cnode(-0.5,2){p}                \Cnode(0.5,2){q}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
                  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{p}              \ncline{t}{q}%
  \ncline{p}{x}\ncline{p}{y}\ncline{q}{y}\ncline{q}{z}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t)  {$\setn{w,x,y,z}$}%
  %\uput[0](e)  {$\setn{w,x,y}$}%
\end{pspicture}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on M2
%==========================================================================
\begin{pspicture}(-1.25,-\latbot)(2.25,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(1,3){t}
  \Cnode(0,2){u}\Cnode(2,2){v}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{u}\ncline{t}{v}%
  \ncline{v}{z}%
  \ncline{u}{x}\ncline{u}{y}\ncline{u}{z}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
```

```
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice O6
% nominal unit = 10mm
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
   \Cnode(0,3){t}%
   \Cnode(-1,2){c}\Cnode(1,2){d}%
   \Cnode(0,1.5){m}%
   \Cnode(-1,1){x}\Cnode(1,1){y}%
   \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{t}{c}\ncline{t}{m}\ncline{t}{d}%
   \ncline{c}{x}\ncline{d}{y}%
   \ncline{x}{m}%
   \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[-90](b) {$\szero$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice O6
%=========================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.1,-\latbot)(1.1,4.2)
  %---------------------------------
  % nodes
  %---------------------------------
   \Cnode(0,4){t}%
   \Cnode(-1,3){c}\Cnode(1,3){d}%
   \Cnode(-1,2){p}\Cnode(1,2){q}%
   \Cnode(-1,1){x}\Cnode(1,1){y}%
   \Cnode(0,0){b}%
   \Cnode(0,1.5){m}%
  %---------------------------------
  % node connections
  %---------------------------------
   \ncline{t}{c}\ncline{t}{d}%
   \ncline{p}{c}\ncline{q}{d}%
   \ncline{x}{p}\ncline{x}{q}%
   \ncline{b}{x}\ncline{b}{y}%
   \ncline{t}{m}\ncline{m}{x}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$1$}%
  %\uput[0](d) {$x^\ocop$}%
  %\uput[180](c) {$y^\ocop$}%
  %\uput[0](q) {$p^\ocop$}%
  %\uput[180](p) {$p$}%
  %\uput[0](y) {$y$}%
  %\uput[180](x) {$x$}%
  %\uput[0](b) {$0$}%
\end{pspicture}
}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
```

```
%==========================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,5.4)%
  %---------------------------------------
  % nodes
  %---------------------------------------
  \Cnode(0,5){t}%
  \Cnode(-1,4){x}\Cnode(1,4){y}%
  \Cnode(0,3){e}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}%
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{x}\ncline{t}{y}%
  \ncline{e}{x}\ncline{e}{y}%
  \ncline{d}{e}%
  \ncline{c}{d}%
  \ncline{b}{c}%
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on M2
% nominal unit = 10mm
%==========================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %---------------------------------------
  % nodes
  %---------------------------------------
           \Cnode(0,4){t}
  \Cnode(-1,3){c}\Cnode(1,3){d}%
           \Cnode(0,2){m}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
           \Cnode(0,0){b}
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{m}{c}\ncline{m}{d}%
  \ncline{m}{x}\ncline{m}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%==========================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M2 on M2
%==========================================================================
\begin{pspicture}(-1.3,-\latbot)(2.3,3.3)%
  %---------------------------------------
  % nodes
  %---------------------------------------
  \Cnode(1,3){t}%
  \Cnode(0,2){u}\Cnode(1,2){v}\Cnode(2,2){w}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}%
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{u}\ncline{t}{v}\ncline{t}{w}%
  \ncline{y}{u}\ncline{y}{v}\ncline{y}{w}%
  \ncline{x}{u}\ncline{y}{w}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------------
  % node labels
  %---------------------------------------
```

```
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
\begin{pspicture}(-1.9,-\latbot)(1.9,3.3)%
  %---------------------------------------
  % nodes
  %---------------------------------------
                               \Cnode(0,3){t}%
                      \Cnode(-0.5,2){tt}%
                                                              \Cnode(1.5,1.5){m}%
  \Cnode(-1.5,1){x}\Cnode(-0.5,1){y}\Cnode(0.5,1){z}%
                               \Cnode(0,0){b}%
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{tt}\ncline{t}{m}%
  \ncline{tt}{x}\ncline{tt}{y}\ncline{tt}{z}%
                                                     \ncline{b}{m}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[0](t) {$\setn{w,x,y,z}$}%
  %\uput[0](e) {$\setn{w,x,y}$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %---------------------------------------
  % nodes
  %---------------------------------------
  \Cnode(0,4){t}%
  \Cnode(-1,3){x}\Cnode(0,3){y}\Cnode(1,3){z}%
  \Cnode(0,2){d}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{d}{x}\ncline{d}{y}\ncline{d}{z}%
  \ncline{c}{d}%
  \ncline{b}{c}%
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
\begin{pspicture}(-1.8,-\latbot)(1.8,3.3)%
  %---------------------------------------
  % nodes
  %---------------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1.5,2){w}\Cnode(-0.5,2){x}\Cnode(0.5,2){y}\Cnode(1.5,2){z}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}%
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{w}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{c}{w}\ncline{c}{x}\ncline{c}{y}\ncline{c}{z}%
  \ncline{b}{c}%
  %---------------------------------------
```

```
  % node labels
  %-----------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M5
% nominal unit = 5mm
%============================================================================
\begin{pspicture}(-2.4,-\latbot)(2.4,2.2)%
  %-----------------------------------
  % nodes
  %-----------------------------------
                                  \Cnode(0,2){t}%
  \Cnode(-2,1){v}\Cnode(-1,1){w}\Cnode(0,1){x}\Cnode(1,1){y}\Cnode(2,1){z}%
                                  \Cnode(0,0){b}%
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{v}\ncline{t}{w}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{b}{v}\ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
{%\psset{unit=0.667\psunit}%
\begin{pspicture}(-1.3,-\latbot)(1.3,4.3)
  %-----------------------------------
  % nodes
  %-----------------------------------
  \Cnode(0,4){t}%
  \Cnode(-0.5,3){d}%
  \Cnode(-1.5,2){x}\Cnode(0.5,2){y}\Cnode(1.5,2){z}%
  \Cnode(-0.5,1){c}%
  \Cnode(0,0){b}
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{d}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{c}{d}%
  \ncline{b}{c}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %-----------------------------------
  % node labels
  %-----------------------------------
  %\uput[0](t) {$\setn{x,y,z}$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
%============================================================================
\begin{pspicture}(-1.4,-\latbot)(2.4,4.4)%
  %-----------------------------------
  % nodes
  %-----------------------------------
                  \Cnode(1,4){t}%         top node of lattice
          \Cnode(0,3){d}%
                          \Cnode(2,2.5){m}%  middle right of n5 sublattice
          \Cnode(0,2){c}%                 top of m2 sublattice
  \Cnode(-1,1){x}\Cnode(1,1){y}%          left and right nodes of m2 sublattice
          \Cnode(0,0){b}%                 bottom node of lattice
  %-----------------------------------
  % node connections
  %-----------------------------------
  \ncline{t}{d}\ncline{t}{m}%
  \ncline{c}{d}\ncline{y}{m}%
```

```
  \ncline{c}{x}\ncline{c}{y}% m2 top half
  \ncline{b}{x}\ncline{b}{y}% m2 bottom half
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)%
  %---------------------------------------
  % nodes
  %---------------------------------------
                \Cnode(0,4){t}%
                \Cnode(0,3){e}%
  \Cnode(-1,2){x}\Cnode(0,2){d}%
                \Cnode(0,1){c}\Cnode(1,1){y}%
                \Cnode(0,0){b}%
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{x}\ncline{x}{b}%                         linear 3 element component
  \ncline{t}{e}\ncline{e}{d}\ncline{d}{c}\ncline{c}{b}% linear 5 element component
  \ncline{d}{y}\ncline{y}{b}%                         linear 4 element component
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[0](t) {$\setn{w,x,y,z}$}%
  %\uput[0](b) {$\emptyset$}%
\end{pspicture}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
{%
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)
  %---------------------------------------
  % nodes
  %---------------------------------------
                \Cnode(0,4){t}%
                \Cnode(0,3){e}%
  \Cnode(-1,2){x}\Cnode(0,2){d}\Cnode(1,2){y}%
                \Cnode(0,1){c}%
                \Cnode(0,0){b}
  %---------------------------------------
  % node connections
  %---------------------------------------
  \ncline{t}{x}\ncline{x}{b}%                         linear 3 element component
  \ncline{t}{e}\ncline{e}{d}\ncline{d}{c}\ncline{c}{b}% linear 5 element component
  \ncline{b}{y}\ncline{y}{e}%                         linear 4 element component
  %---------------------------------------
  % node labels
  %---------------------------------------
  %\uput[0](t) {$\setn{w,x,y,z}$}%
  %\uput[0](e) {$\setn{w,x,y}$}%
  %\uput[0](d) {$\setn{w,x}$}%
  %\uput[0](c) {$\setn{w}$}%
  %\uput[0](b) {$\emptyset$}%
\end{pspicture}
}%
```

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX file
%=========================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,5.4)%
  %---------------------------------------
  % nodes
  %---------------------------------------
               \Cnode(0,5){t}%
  \Cnode(-1,4){f}%
```

```
  \Cnode(-1,3){e}%
                              \Cnode(1,2.5){m}%
  \Cnode(-1,2){d}%
  \Cnode(-1,1){c}%
              \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{f}\ncline{t}{m}%
  \ncline{f}{e}%
  \ncline{e}{d}%
  \ncline{d}{c}%
  \ncline{b}{c}\ncline{b}{m}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[0](t) {$\setn{w,x,y,z}$}%
  %\uput[0](e) {$\setn{w,x,y}$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
%=====================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}%
  \Cnode(-1,3){u}\Cnode(1,3){v}%
  \Cnode(-1,2){x}\Cnode(1,2){y}%
  \Cnode(0,1){c}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{u}\ncline{t}{v}%
  \ncline{u}{x}\ncline{v}{y}\ncline{x}{v}%
  \ncline{c}{x}\ncline{c}{y}%
  \ncline{b}{c}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice O7 with slash
% nominal unit = 10mm
%=====================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}
  \Cnode(-1,3){c}\Cnode(1,3){d}%
  \Cnode(-1,2){m}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{c}{m}\ncline{m}{x}\ncline{d}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$1$}%
\end{pspicture}%
```

```
%=====================================================================
% Daniel J. Greenhoe
```

```
% LaTeX file
% lattice O7 with slash
% nominal unit = 10mm
%============================================================================
\begin{pspicture}(-1.4,-\latbot)(1.4,4.4)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,4){t}%
  \Cnode(-1,3){c}\Cnode(1,3){d}%
  \Cnode(-1,2){m}%
  \Cnode(-1,1){x}\Cnode(1,1){y}%
  \Cnode(0,0){b}%
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{c}{m}\ncline{m}{x}\ncline{d}{y}%
  \ncline{c}{y}%
  \ncline{b}{x}\ncline{b}{y}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[-90](b) {$\szero$}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice O6
%============================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.1,-\latbot)(1.1,3.2)
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}
  \Cnode(-1,2){c}\Cnode(1,2){d}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}
  %---------------------------------
  % node connections
  %---------------------------------
  \ncline{t}{c}\ncline{t}{d}%
  \ncline{c}{x}\ncline{c}{y}\ncline{d}{y}\ncline{d}{z}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %---------------------------------
  % node labels
  %---------------------------------
  %\uput[ 90](t) {$\setn{x,y,z}$}%
  %\uput[180](xy){$\setn{x,y}$}%
  %\uput[0](yz){$\setn{y,z}$}%
  %\uput[180](x) {$\setn{x}$}%
  %\uput[0](z) {$\setn{z}$}%
  %\uput[-90](b) {$\szero$}%
  %\uput[0](100,300){\rnode{xzlabel}{$\setn{x,z}$}}%
  %\uput[0](100, 0){\rnode{ylabel}{$\setn{y}$}}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{xzlabel}{xz}%
  %\ncline[linestyle=dotted,nodesep=1pt]{->}{ylabel}{y}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% recommended unit = 5mm
%============================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %---------------------------------
  % nodes
  %---------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){xy} \Cnode(0,2){xz} \Cnode(1,2){yz}%
  \Cnode(-1,1){x}  \Cnode(0,1){y}  \Cnode(1,1){z}%
  \Cnode(0,0){b}%
```

```
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}%
  \ncline{x}{xy}\ncline{x}{xz}%
  \ncline{y}{xy}\ncline{y}{yz}%
  \ncline{z}{xz}\ncline{z}{yz}%
  \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}%
\end{pspicture}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% nominal unit = 5mm
%============================================================================
\begin{pspicture}(-1.3,-\latbot)(1.3,3.3)%
  %----------------------------------
  % nodes
  %----------------------------------
  \Cnode(0,3){t}%
  \Cnode(-1,2){xy}\Cnode(0,2){xz}\Cnode(1,2){yz}%
  \Cnode(-1,1){x}\Cnode(0,1){y}\Cnode(1,1){z}%
  \Cnode(0,0){b}%
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}%
  \ncline{x}{xy}\ncline{x}{xz}%
  \ncline{y}{xy}\ncline{y}{yz}%
  \ncline{z}{xz}\ncline{z}{yz}%
  \ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
\end{pspicture}%
```

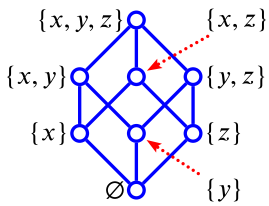
```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice M4
%============================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-2.7,-\latbot)(2.7,2.2)
  %----------------------------------
  % nodes
  %----------------------------------
  \Cnode(0,2){t}%
  \Cnode(-2.5,1){u}\Cnode(-1.5,1){v}\Cnode(-0.5,1){w}\Cnode(0.5,1){x}\Cnode(1.5,1){y}\Cnode(2.5,1){z}%
  \Cnode(0,0){b}%
  %----------------------------------
  % node connections
  %----------------------------------
  \ncline{t}{u}\ncline{t}{v}\ncline{t}{w}\ncline{t}{x}\ncline{t}{y}\ncline{t}{z}%
  \ncline{b}{u}\ncline{b}{v}\ncline{b}{w}\ncline{b}{x}\ncline{b}{y}\ncline{b}{z}%
  %----------------------------------
  % node labels
  %----------------------------------
  %\uput[0](t)  {$\bid$}%
  %\uput[0](u)  {$u$}%
  %\uput[0](v)  {$v$}%
  %\uput[0](w)  {$w$}%
  %\uput[0](x)  {$x$}%
  %\uput[0](y)  {$y$}%
  %\uput[0](z)  {$z$}%
  %\uput[0](b)  {$\bzero$}%
\end{pspicture}
}%
```

```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% lattice O6
%============================================================================
{%\psset{unit=0.5\psunit}%
\begin{pspicture}(-1.1,-\latbot)(1.1,4.2)
  %----------------------------------
  % nodes
  %----------------------------------
```

```
 \Cnode(0,4){t}
 \Cnode(-1,3){c}\Cnode(1,3){d}%
 \Cnode(-1,2){p}\Cnode(1,2){q}%
 \Cnode(-1,1){x}\Cnode(1,1){y}%
 \Cnode(0,0){b}
 %---------------------------------
 % node connections
 %---------------------------------
 \ncline{t}{c}\ncline{t}{d}%
 \ncline{p}{c}\ncline{q}{d}%
 \ncline{x}{p}\ncline{x}{q}%
 \ncline{b}{x}\ncline{b}{y}%
 %---------------------------------
 % node labels
 %---------------------------------
 %\uput[0](t) {$1$}%
 %\uput[0](d) {$x^\ocop$}%
 %\uput[180](c) {$y^\ocop$}%
 %\uput[0](q) {$p^\ocop$}%
 %\uput[180](p) {$p$}%
 %\uput[0](y) {$y$}%
 %\uput[180](x) {$x$}%
 %\uput[0](b) {$0$}%
\end{pspicture}
}%
```

D.2 Labeled lattices

Alternatively, one can append labels to the lattice as illustrated to the right and as coded below:




```
%============================================================================
% Daniel J. Greenhoe
% LaTeX file
% recommended unit = 7.5mm
%============================================================================
\begin{pspicture}(-2.4,-.3)(2.4,3.3)
 %---------------------------------
 % nodes
 %---------------------------------
 \Cnode(0,3){T}
 \Cnode(-1,2){ab} \Cnode(0,2){ac} \Cnode(1,2){bc}
 \Cnode(-1,1){a}  \Cnode(0,1){b}  \Cnode(1,1){c}
 \Cnode(0,0){B}
 %---------------------------------
 % node connections
 %---------------------------------
 \ncline{T}{ab}\ncline{T}{ac}\ncline{T}{bc}%
 \ncline{a}{ac}\ncline{a}{ab}%
 \ncline{b}{ab}\ncline{b}{bc}%
 \ncline{c}{ac}\ncline{c}{bc}%
 \ncline{B}{a} \ncline{B}{b} \ncline{B}{c}%
 %---------------------------------
 % node labels
 %---------------------------------
 \pnode(1,2.5){Lac}%
 \pnode(1,0.5){Lb}%
 \uput[180](T){$\setn{x,y,z}$}%
 \uput[-180](ab){$\setn{x,y}$}%
 \uput[0](bc){$\setn{y,z}$}%
 \uput[180](a){$\setn{x}$}%
 \uput[0](c) {$\setn{z}$}%
 \uput[180](B) {$\emptyset$}%
 \uput[0](1,3){\rnode{aclabel}{$\setn{x,z}$}}%
```

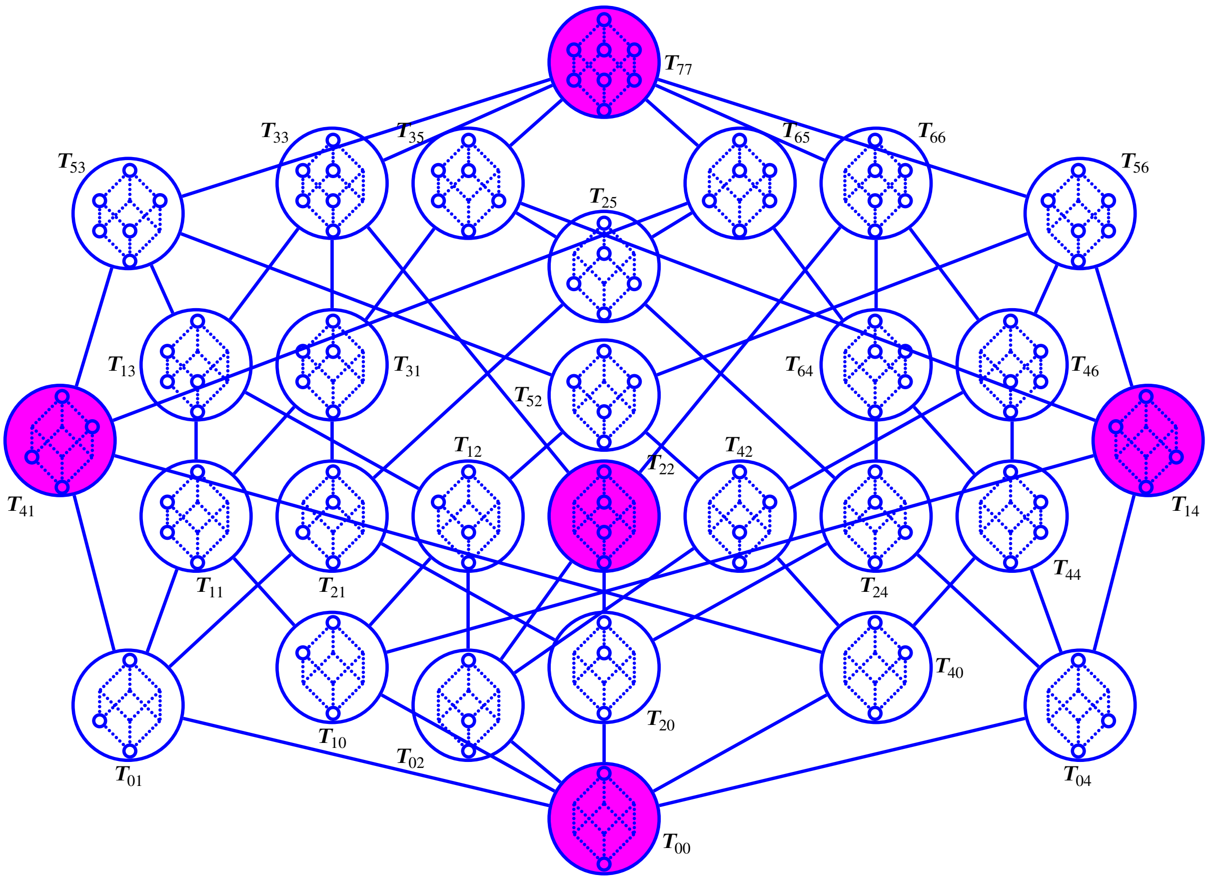


Figure D.1: Lattice of *topologies* on $X \triangleq \{x, y, z\}$ (see Example C.13 page 46)

```
   \uput[0](1,   0){\rnode{blabel}{$\setn{y}$}}%
   \ncline[linestyle=dotted,linecolor=red,nodesep=1pt]{->}{aclabel}{ac}%
   \ncline[linestyle=dotted,linecolor=red,nodesep=1pt]{->}{blabel}{b}%
 \end{pspicture}%
```

D.3 Lattice of lattices

It is even possible to draw lattices within lattices (draw Hasse diagrams within Hasse diagrams). A Hasse diagram for the *lattice of topologies* on a 3 element set was described in Example C.13 (page 46) and illustrated in Figure C.5 (page 46). Here is some LaTeX source for rendering such a diagram:

```
%=========================================================================
% Daniel J. Greenhoe
% LaTeX File
% lattice of topologies over the set {x,y,z}
%=========================================================================
%{\psset{xunit=0.18mm,yunit=0.20mm}%
\begin{pspicture}(-450,-40)(450,540)%
  %---------------------------------
  % settings
  %---------------------------------
  \fns%
  %\psset{labelsep=1.5mm,radius=75\psunit}
  \psset{
    labelsep=8mm,
    radius=7.5mm,
    linearc=100\psxunit,
    }
```

```
%---------------------------------
% developement tools
%---------------------------------
%\psgrid[xunit=100\psxunit,yunit=100\psyunit](-5,-1)(5,5)%
%---------------------------------
% nodes
%---------------------------------
\Cnode[fillstyle=solid,fillcolor=latlatshade](   0,500){T77}%
%
\Cnode( 200,420){T66}%
\Cnode( 350,400){T56}%
\Cnode( 100,420){T65}%
\Cnode(-100,420){T35}%
\Cnode(-350,400){T53}%
\Cnode(-200,420){T33}%
%
\Cnode( 300,300){T46}%
\Cnode(   0,365){T25}%\Cnode( 200,300){T25}%
\Cnode(-300,300){T13}%\Cnode( 100,300){T13}%
\Cnode( 200,300){T64}%\Cnode(-100,300){T64}%
\Cnode(   0,280){T52}%\Cnode(-200,300){T52}%
\Cnode(-200,300){T31}%
%
\Cnode( 300,200){T44}%
\Cnode( 200,200){T24}%
\Cnode[fillstyle=solid,fillcolor=latlatshade]( 400,250){T14}%
\Cnode( 100,200){T42}%
\Cnode[fillstyle=solid,fillcolor=latlatshade](   0,200){T22}%
\Cnode(-100,200){T12}%
\Cnode[fillstyle=solid,fillcolor=latlatshade](-400,250){T41}%
\Cnode(-200,200){T21}%
\Cnode(-300,200){T11}%
%
\Cnode( 200,100){T40}%%\Cnode( 300,100){T40}%
\Cnode(   0,100){T20}%%\Cnode( 200,100){T20}%
\Cnode(-200,100){T10}%%\Cnode( 100,100){T10}%
\Cnode( 350, 75){T04}%  %\Cnode(-100,100){T04}%
\Cnode(-100, 75){T02}%  %\Cnode(-200,100){T02}%
\Cnode(-350, 75){T01}%  %\Cnode(-300,100){T01}%
\Cnode[fillstyle=solid,fillcolor=latlatshade](   0,0)  {T00}%
%---------------------------------
% node connections
%---------------------------------
\ncline{T77}{T33}%
\ncline{T77}{T53}%
\ncline{T77}{T35}%
\ncline{T77}{T65}%
\ncline{T77}{T56}%
\ncline{T77}{T66}%
%
\ncline{T33}{T31}%
\ncline{T33}{T22}%
\ncline{T33}{T13}%
\ncline{T53}{T41}%
\ncline{T53}{T52}%
\ncline{T53}{T13}%
\ncline{T35}{T31}%
\ncline{T35}{T14}%
\ncline{T35}{T25}%
\ncline{T65}{T64}%
\ncline{T65}{T41}%
\ncline{T65}{T25}%
\ncline{T56}{T52}%
\ncline{T56}{T14}%
\ncline{T56}{T46}%
\ncline{T66}{T64}%
\ncline{T66}{T22}%
\ncline{T66}{T46}%
%
\ncline{T31}{T11}%
\ncline{T31}{T21}%
\ncline{T52}{T12}%
\ncline{T52}{T42}%
\ncline{T64}{T24}%
\ncline{T64}{T44}%
\ncline{T13}{T11}%
\ncline{T13}{T12}%
```

```
 \ncline{T25}{T21}%
 \ncline{T25}{T24}%
 \ncline{T46}{T42}%
 \ncline{T46}{T44}%
%
 \ncline{T01}{T11}%
 \ncline{T01}{T21}%
 \ncline{T01}{T41}%
 \ncline{T02}{T12}%
 \ncline{T02}{T22}%
 \ncline{T02}{T42}%
 \ncline{T04}{T14}%
 \ncline{T04}{T24}%
 \ncline{T04}{T44}%
 \ncline{T10}{T11}%
 \ncline{T10}{T12}%
 \ncline{T10}{T14}%
 \ncline{T20}{T21}%
 \ncline{T20}{T22}%
 \ncline{T20}{T24}%
 \ncline{T40}{T41}%
 \ncline{T40}{T42}%
 \ncline{T40}{T44}%
%
 \ncline{T00}{T01}%
 \ncline{T00}{T02}%
 \ncline{T00}{T04}%
 \ncline{T00}{T10}%
 \ncline{T00}{T20}%
 \ncline{T00}{T40}%
%---------------------------------------
% node labels
%---------------------------------------
 \uput[  0](T77){$\topT_{77}$}%
 \uput[ 45](T66){$\topT_{66}$}%
 \uput[ 45](T56){$\topT_{56}$}%
 \uput[ 45](T65){$\topT_{65}$}%
 \uput[135](T35){$\topT_{35}$}%
 \uput[135](T53){$\topT_{53}$}%
 \uput[135](T33){$\topT_{33}$}%
 \uput[  0](T46){$\topT_{46}$}%
 \uput[ 90](T25){$\topT_{25}$}%
 \uput[180](T13){$\topT_{13}$}%
 \uput[180](T64){$\topT_{64}$}%
 \uput[180](T52){$\topT_{52}$}%
 \uput[  0](T31){$\topT_{31}$}%
 \uput[-45](T44){$\topT_{44}$}%
 \uput[-90](T24){$\topT_{24}$}%
 \uput[-60](T14){$\topT_{14}$}%
 \uput[ 90](T42){$\topT_{42}$}%
 \uput[ 45](T22){$\topT_{22}$}%
 \uput[ 90](T12){$\topT_{12}$}%
 \uput[240](T41){$\topT_{41}$}%
 \uput[-90](T21){$\topT_{21}$}%
 \uput[-80](T11){$\topT_{11}$}%
 \uput[  0](T40){$\topT_{40}$}%
 \uput[-45](T20){$\topT_{20}$}%
 \uput[-90](T10){$\topT_{10}$}%
 \uput[-90](T04){$\topT_{04}$}%
 \uput[225](T02){$\topT_{02}$}%
 \uput[-90](T01){$\topT_{01}$}%
 \uput[-20](T00){$\topT_{00}$}%
%---------------------------------------
% discriptions
%---------------------------------------
%\rput[bl](-450,0){%left N5 lattice
%  \psframe[linestyle=dashed,linecolor=red](0,0)(200,450)%
%  \uput[-45](200,0){$N5$ lattice}
%  }%
%\rput[br](450,0){%right N5 lattice
%  \psframe[linestyle=dashed,linecolor=red](0,0)(-200,450)%
%  \uput[-45](-200,0){$N5$ lattice}
%  }%
% \ncbox[nodesep=20\psyunit,boxsize=100\psxunit,linestyle=dashed,linecolor=red]{T01}{T53}%
% \ncbox[nodesep=20\psyunit,boxsize=100\psxunit,linestyle=dashed,linecolor=red]{T04}{T56}%
%  \rput[t](-350,10){N5 lattice}
%  \rput[t]( 350,10){N5 lattice}
```

```
%---------------------------------
% node inner lattices
%---------------------------------
\psset{
   unit=0.04mm,
   radius=1mm,
   dotsep=0.5pt,
   linecolor=blue,
   }%
\rput(T77){\begin{pspicture}(-100,0)(100,300)
                        \Cnode(0,300){t}
     \Cnode(-100,200){xy} \Cnode(0,200){xz}  \Cnode(100,200){yz}
     \Cnode(-100,100){x}  \Cnode(0,100){y}   \Cnode(100,100){z}
                        \Cnode(0,   0){b}
     \psset{linestyle=dotted}%
     \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
     \ncline{x}{xy}\ncline{x}{xz}
     \ncline{y}{xy}\ncline{y}{yz}
     \ncline{z}{xz}\ncline{z}{yz}
     \ncline{b}{x} \ncline{b}{y}  \ncline{b}{z}
   \end{pspicture}}%
\rput(T66){\begin{pspicture}(-100,0)(100,300)
                        \Cnode(0,300){t}%
     \pnode(-100,200){xy} \Cnode(0,200){xz}  \Cnode(100,200){yz}%
     \pnode(-100,100){x}  \Cnode(0,100){y}   \Cnode(100,100){z}%
                        \Cnode(0,   0){b}%
     \psset{linestyle=dotted}%
     \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
     \ncline{x}{xy}\ncline{x}{xz}
     \ncline{y}{xy}\ncline{y}{yz}
     \ncline{z}{xz}\ncline{z}{yz}
     \ncline{b}{x} \ncline{b}{y}  \ncline{b}{z}
   \end{pspicture}}%
\rput(T56){\begin{pspicture}(-100,0)(100,300)
                        \Cnode(0,300){t}%
     \Cnode(-100,200){xy} \pnode(0,200){xz}  \Cnode(100,200){yz}%
     \pnode(-100,100){x}  \Cnode(0,100){y}   \Cnode(100,100){z}%
                        \Cnode(0,   0){b}%
     \psset{linestyle=dotted}%
     \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
     \ncline{x}{xy}\ncline{x}{xz}
     \ncline{y}{xy}\ncline{y}{yz}
     \ncline{z}{xz}\ncline{z}{yz}
     \ncline{b}{x} \ncline{b}{y}  \ncline{b}{z}
   \end{pspicture}}%
\rput(T65){\begin{pspicture}(-100,0)(100,300)
                        \Cnode(0,300){t}%
     \pnode(-100,200){xy} \Cnode(0,200){xz}  \Cnode(100,200){yz}%
     \Cnode(-100,100){x}  \pnode(0,100){y}   \Cnode(100,100){z}%
                        \Cnode(0,   0){b}%
     \psset{linestyle=dotted}%
     \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
     \ncline{x}{xy}\ncline{x}{xz}
     \ncline{y}{xy}\ncline{y}{yz}
     \ncline{z}{xz}\ncline{z}{yz}
     \ncline{b}{x} \ncline{b}{y}  \ncline{b}{z}
   \end{pspicture}}%
\rput(T35){\begin{pspicture}(-100,0)(100,300)
                        \Cnode(0,300){t}%
     \Cnode(-100,200){xy} \Cnode(0,200){xz}  \pnode(100,200){yz}%
     \Cnode(-100,100){x}  \pnode(0,100){y}   \Cnode(100,100){z}%
                        \Cnode(0,   0){b}%
     \psset{linestyle=dotted}%
     \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
     \ncline{x}{xy}\ncline{x}{xz}
     \ncline{y}{xy}\ncline{y}{yz}
     \ncline{z}{xz}\ncline{z}{yz}
     \ncline{b}{x} \ncline{b}{y}  \ncline{b}{z}
   \end{pspicture}}%
\rput(T53){\begin{pspicture}(-100,0)(100,300)
                        \Cnode(0,300){t}%
     \Cnode(-100,200){xy} \pnode(0,200){xz}  \Cnode(100,200){yz}%
     \Cnode(-100,100){x}  \Cnode(0,100){y}   \pnode(100,100){z}%
                        \Cnode(0,   0){b}%
     \psset{linestyle=dotted}%
     \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
     \ncline{x}{xy}\ncline{x}{xz}
```

```
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T33){\begin{pspicture}(-100,0)(100,300)
                                         \Cnode(0,300){t}%
        \Cnode(-100,200){xy}  \Cnode(0,200){xz}  \pnode(100,200){yz}%
        \Cnode(-100,100){x}   \Cnode(0,100){y}   \pnode(100,100){z}%
                                         \Cnode(0,   0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T46){\begin{pspicture}(-100,0)(100,300)
                                         \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \pnode(0,200){xz}  \Cnode(100,200){yz}%
        \pnode(-100,100){x}   \Cnode(0,100){y}   \Cnode(100,100){z}%
                                         \Cnode(0,   0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T25){\begin{pspicture}(-100,0)(100,300)
                                         \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \Cnode(0,200){xz}  \pnode(100,200){yz}%
        \Cnode(-100,100){x}   \pnode(0,100){y}   \Cnode(100,100){z}%
                                         \Cnode(0,   0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T13){\begin{pspicture}(-100,0)(100,300)
                                         \Cnode(0,300){t}%
        \Cnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
        \Cnode(-100,100){x}   \Cnode(0,100){y}   \pnode(100,100){z}%
                                         \Cnode(0,   0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T64){\begin{pspicture}(-100,0)(100,300)
                                         \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \Cnode(0,200){xz}  \Cnode(100,200){yz}%
        \pnode(-100,100){x}   \pnode(0,100){y}   \Cnode(100,100){z}%
                                         \Cnode(0,   0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T52){\begin{pspicture}(-100,0)(100,300)
                                         \Cnode(0,300){t}%
        \Cnode(-100,200){xy}  \pnode(0,200){xz}  \Cnode(100,200){yz}%
        \pnode(-100,100){x}   \Cnode(0,100){y}   \pnode(100,100){z}%
                                         \Cnode(0,   0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T31){\begin{pspicture}(-100,0)(100,300)
```

```
                          \Cnode(0,300){t}%
    \Cnode(-100,200){xy}  \Cnode(0,200){xz}  \pnode(100,200){yz}%
    \Cnode(-100,100){x}   \pnode(0,100){y}   \pnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
    \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
    \ncline{x}{xy}\ncline{x}{xz}
    \ncline{y}{xy}\ncline{y}{yz}
    \ncline{z}{xz}\ncline{z}{yz}
    \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
  \end{pspicture}}%
\rput(T44){\begin{pspicture}(-100,0)(100,300)
                          \Cnode(0,300){t}%
    \pnode(-100,200){xy}  \pnode(0,200){xz}  \Cnode(100,200){yz}%
    \pnode(-100,100){x}   \pnode(0,100){y}   \Cnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
    \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
    \ncline{x}{xy}\ncline{x}{xz}
    \ncline{y}{xy}\ncline{y}{yz}
    \ncline{z}{xz}\ncline{z}{yz}
    \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
  \end{pspicture}}%
\rput(T24){\begin{pspicture}(-100,0)(100,300)
                          \Cnode(0,300){t}%
    \pnode(-100,200){xy}  \Cnode(0,200){xz}  \pnode(100,200){yz}%
    \pnode(-100,100){x}   \pnode(0,100){y}   \Cnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
    \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
    \ncline{x}{xy}\ncline{x}{xz}
    \ncline{y}{xy}\ncline{y}{yz}
    \ncline{z}{xz}\ncline{z}{yz}
    \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
  \end{pspicture}}%
\rput(T14){\begin{pspicture}(-100,0)(100,300)
                          \Cnode(0,300){t}%
    \Cnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
    \pnode(-100,100){x}   \pnode(0,100){y}   \Cnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
    \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
    \ncline{x}{xy}\ncline{x}{xz}
    \ncline{y}{xy}\ncline{y}{yz}
    \ncline{z}{xz}\ncline{z}{yz}
    \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
  \end{pspicture}}%
\rput(T42){\begin{pspicture}(-100,0)(100,300)
                          \Cnode(0,300){t}%
    \pnode(-100,200){xy}  \pnode(0,200){xz}  \Cnode(100,200){yz}%
    \pnode(-100,100){x}   \Cnode(0,100){y}   \pnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
    \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
    \ncline{x}{xy}\ncline{x}{xz}
    \ncline{y}{xy}\ncline{y}{yz}
    \ncline{z}{xz}\ncline{z}{yz}
    \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
  \end{pspicture}}%
\rput(T22){\begin{pspicture}(-100,0)(100,300)
                          \Cnode(0,300){t}%
    \pnode(-100,200){xy}  \Cnode(0,200){xz}  \pnode(100,200){yz}%
    \pnode(-100,100){x}   \Cnode(0,100){y}   \pnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
    \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
    \ncline{x}{xy}\ncline{x}{xz}
    \ncline{y}{xy}\ncline{y}{yz}
    \ncline{z}{xz}\ncline{z}{yz}
    \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
  \end{pspicture}}%
\rput(T12){\begin{pspicture}(-100,0)(100,300)
                          \Cnode(0,300){t}%
    \Cnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
    \pnode(-100,100){x}   \Cnode(0,100){y}   \pnode(100,100){z}%
                          \Cnode(0,  0){b}%
    \psset{linestyle=dotted}%
```

```
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
    \end{pspicture}}%
  \rput(T41){\begin{pspicture}(-100,0)(100,300)
                         \Cnode(0,300){t}%
      \pnode(-100,200){xy} \pnode(0,200){xz} \Cnode(100,200){yz}%
      \Cnode(-100,100){x}  \pnode(0,100){y}  \pnode(100,100){z}%
                         \Cnode(0,  0){b}%
      \psset{linestyle=dotted}%
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
    \end{pspicture}}%
  \rput(T21){\begin{pspicture}(-100,0)(100,300)
                         \Cnode(0,300){t}%
      \pnode(-100,200){xy} \Cnode(0,200){xz} \pnode(100,200){yz}%
      \Cnode(-100,100){x}  \pnode(0,100){y}  \pnode(100,100){z}%
                         \Cnode(0,  0){b}%
      \psset{linestyle=dotted}%
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
    \end{pspicture}}%
  \rput(T11){\begin{pspicture}(-100,0)(100,300)
                         \Cnode(0,300){t}%
      \Cnode(-100,200){xy} \pnode(0,200){xz} \pnode(100,200){yz}%
      \Cnode(-100,100){x}  \pnode(0,100){y}  \pnode(100,100){z}%
                         \Cnode(0,  0){b}%
      \psset{linestyle=dotted}%
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
    \end{pspicture}}%
  \rput(T40){\begin{pspicture}(-100,0)(100,300)
                         \Cnode(0,300){t}%
      \pnode(-100,200){xy} \pnode(0,200){xz} \Cnode(100,200){yz}%
      \pnode(-100,100){x}  \pnode(0,100){y}  \pnode(100,100){z}%
                         \Cnode(0,  0){b}%
      \psset{linestyle=dotted}%
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
    \end{pspicture}}%
  \rput(T20){\begin{pspicture}(-100,0)(100,300)
                         \Cnode(0,300){t}%
      \pnode(-100,200){xy} \Cnode(0,200){xz} \pnode(100,200){yz}%
      \pnode(-100,100){x}  \pnode(0,100){y}  \pnode(100,100){z}%
                         \Cnode(0,  0){b}%
      \psset{linestyle=dotted}%
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
    \end{pspicture}}%
  \rput(T10){\begin{pspicture}(-100,0)(100,300)
                         \Cnode(0,300){t}%
      \Cnode(-100,200){xy} \pnode(0,200){xz} \pnode(100,200){yz}%
      \pnode(-100,100){x}  \pnode(0,100){y}  \pnode(100,100){z}%
                         \Cnode(0,  0){b}%
      \psset{linestyle=dotted}%
      \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
      \ncline{x}{xy}\ncline{x}{xz}
      \ncline{y}{xy}\ncline{y}{yz}
      \ncline{z}{xz}\ncline{z}{yz}
      \ncline{b}{x} \ncline{b}{y} \ncline{b}{z}
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      \end{pspicture}}%
    \rput(T04){\begin{pspicture}(-100,0)(100,300)
                                 \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
        \pnode(-100,100){x}    \pnode(0,100){y}    \Cnode(100,100){z}%
                                 \Cnode(0,    0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T02){\begin{pspicture}(-100,0)(100,300)
                                 \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
        \pnode(-100,100){x}    \Cnode(0,100){y}    \pnode(100,100){z}%
                                 \Cnode(0,    0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T01){\begin{pspicture}(-100,0)(100,300)
                                 \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
        \Cnode(-100,100){x}    \pnode(0,100){y}    \pnode(100,100){z}%
                                 \Cnode(0,    0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
    \rput(T00){\begin{pspicture}(-100,0)(100,300)
                                 \Cnode(0,300){t}%
        \pnode(-100,200){xy}  \pnode(0,200){xz}  \pnode(100,200){yz}%
        \pnode(-100,100){x}    \pnode(0,100){y}    \pnode(100,100){z}%
                                 \Cnode(0,    0){b}%
        \psset{linestyle=dotted}%
        \ncline{t}{xy}\ncline{t}{xz}\ncline{t}{yz}
        \ncline{x}{xy}\ncline{x}{xz}
        \ncline{y}{xy}\ncline{y}{yz}
        \ncline{z}{xz}\ncline{z}{yz}
        \ncline{b}{x}  \ncline{b}{y}  \ncline{b}{z}
      \end{pspicture}}%
  \end{pspicture}%
%}%
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