Cahn Hilliard Equation

Numerical Solution

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The equation

The Cahn - Hilliard equation is an equation of mathematical physics which describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. If c is the concentration of the fluid, with $\pm c = 1$ indicating domains, then the equation is written as

$$\frac{\partial c}{\partial t} = D\Delta(c^3 - c - \gamma\Delta c)$$

where D is a diffusion coefficient

Applications

Apart from it's traditional application in modelling binary alloy separation, some of the areas where the Cahn Hilliard equation is used for modelling are:

- Explaining the structure of biofilms using two phase biomass-solvent theory. It is also used to model thin film generation.
- Modelling the rings of Saturn
- Image Processing
- In ecology of river beds.

Cahn Hilliard Equation in one dimension

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\frac{\delta G}{\delta u} \qquad x \in (0, L) \subset R, 0 < t \tag{1}$$

$$\frac{\delta G}{\delta u} = pu + ru^3 + q\frac{\partial^2 u}{\partial x^2} \tag{2}$$

where p, q and r are constants with $p < 0$, $q < 0$ and $0 < r$ is a model equation to describe a phase separation phenomenon called the spinodal decomposition. The decomposition phenomenon occurs when binary solutions such as alloys, polymer mixtures are cooled down. Here $u(x, t)$ is a distribution function of the concentration of one component of the binary mixture.

Boundary conditions for the equation are:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \tag{3}$$

$$\left.\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\right|_{x=0} = \left.\frac{\partial}{\partial x}\frac{\delta G}{\delta u}\right|_{x=L} = 0 \tag{4}$$

The functional G means a local free energy called the Ginzburg-Landau free energy.

The proposed difference scheme inherits the properties of :

1. The conservation of mass
2. The decrease of the total energy

We define $U_k^{(n)}(k = -2, -1, 0, ..., N, N+1, N+2; n = 0, 1, 2, ...)$ to be the approximation to u(x, t) at location $x = k\Delta x$ and time $t = n\Delta t$, where x is a space mesh size and Δt is a time mesh size

$$\frac{U_k^{(n+1)} - U_k^{(n)}}{\Delta t} = \delta_k^{(2)} \left(\frac{\delta G_d}{\delta u} \right)_k^{(n+\frac{1}{2})} \tag{5}$$

$$k = 0, 1, ..., N; n = 0, 1, ...,$$

$$\begin{aligned} \left(\frac{\delta G_d}{\delta u} \right)_k^{(n+\frac{1}{2})} & \\ &= p\left\{ \frac{U_k^{(n+1)} + U_k^{(n)}}{2} \right\} \\ &+ r\left\{ \frac{(U_k^{(n+1)})^3 + (U_k^{(n+1)})^2 U_k^{(n)} + (U_k^{(n+1)})(U_k^{(n)})^2 + (U_k^{(n)})^3}{4} \right\} \\ &+ q\delta_k^{(2)}\left\{ \frac{U_k^{(n+1)} + U_k^{(n)}}{2} \right\} \end{aligned} \tag{6}$$

where $N = L/\Delta x$ and $\delta_k^{(2)}$ is a second-order difference operator defined by

$$\delta_k^{(2)} = \frac{1}{(\Delta x)^2}(f_{k-1} - 2f_k + f_{k+1}) \tag{7}$$

The discrete boundary conditions are

$$\delta_k^{(1)} U_k^{(n)}\Bigg|_{k=0} = \delta_k^{(1)} U_k^{(n)}\Bigg|_{k=N} = 0, \tag{8}$$

$$\delta_k^{(1)} \left(\frac{\delta G_d}{\delta u}\right)_k^{(n+\frac{1}{2})}\Bigg|_{k=0} = \Bigg)_k^{(n+\frac{1}{2})}\Bigg|_{k=N} = 0, \tag{9}$$

where

$$\delta_k^{(1)} f_k = \frac{1}{2\Delta x}(f_{k+1} + f_{k-1}) \tag{10}$$

The discrete boundary conditions are:

$$U_{-1}^{(n)} = U_{1}^{(n)}, \;\; U_{N+1}^{(n)} = U_{N-1}^{(n)} \tag{11}$$

$$U_{-2}^{(n)} = U_{2}^{(n)}, \;\; U_{N+2}^{(n)} = U_{N-2}^{(n)} \tag{12}$$

Results

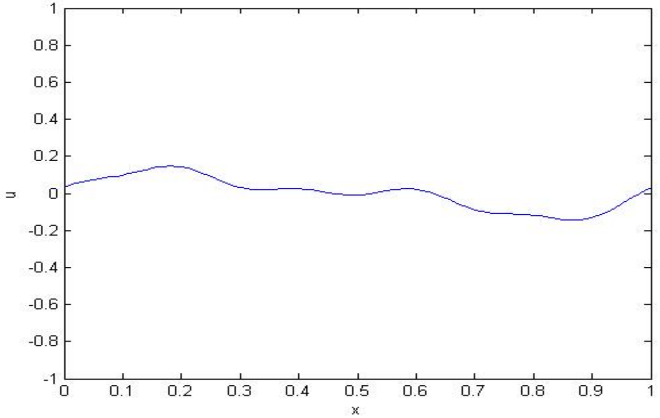

Figure: Initial Condition

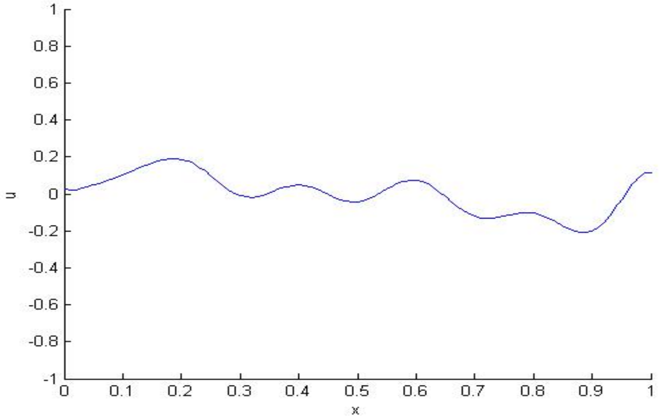

Figure: fifth time step

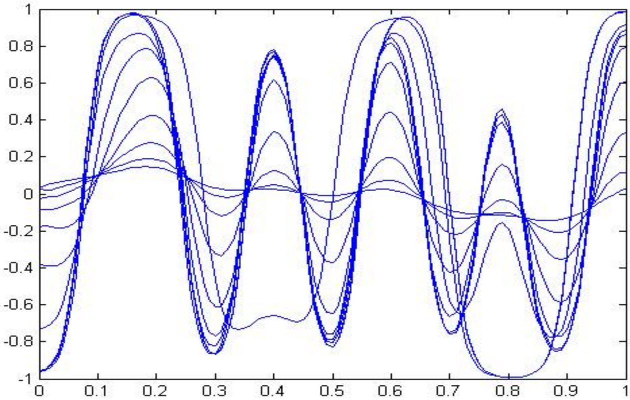

Figure: first 200 time steps

References

D. Furihata. A stable and conservative finite difference scheme for the Cahn-Hilliard equation Numer. Math., 87 (2001), pp. 675699

Thank You