Demand function with constant income elasticity

SH

Consider the following maximization problem:

$$\max_{\{h,c\}} u(h,c) \qquad \text{s.t.} \quad rh + pc = y$$

where h and c are goods (housing and non-housing consumption), $u(\cdot,\cdot)$ is a standard utility function, r and p are prices of h and p in dollars, and y is income in dollars.

The FOCs of this problem yield the following optimality condition

$$\frac{u_h(h,c)}{u_c(h,c)} = \frac{p}{r} \tag{1}$$

where u_x represents the partial derivative of the utility function with respect to x. We represent this condition as

$$c = f(h,r)$$

where $f(\cdot,r)$ gives the value of c for each h such that equation (1) holds. We assume that this value is unique.

Replacing in the budget constraint of the problem we have

$$rh + pf(h,r) = y. \tag{2}$$

A constant income elasticity function would be defined as $\tilde{h} = g(p,r)y^{\xi}$ where $\xi \equiv \frac{d\ln\tilde{h}}{d\ln y}$ is the elasticity. Can we obtain this function form in this problem?

Fully differentiating (2) with respect to $\ln h$ and $\ln y$ yields

$$rhd\ln(h) + pf_h hd\ln(h) = yd\ln(y)$$

$$hd\ln(h)(r + pf_h) = yd\ln(y) \implies \frac{d\ln(h)}{d\ln(y)} = \frac{y}{h}\frac{1}{(r+pf_h)}$$

and using the budget constraint

$$\frac{d\ln(h)}{d\ln(y)} = \frac{r + p\,(c/h)}{r + pf_h} \equiv \xi$$

By definition, $f_h = \frac{dc}{dh}$.Thus, for the income elasticity to be constant we require a condition on the marginal rate of substitution given in (1), namely that

$$\frac{c}{h} = \frac{r}{p}\,(\xi - 1) - \xi\frac{dc}{dh}$$

for any y and given p and r.