Demand function with constant income elasticity

SH

Problem set-up

Consider the following maximization problem:

$$\max_{\{h,c\}} u(h,c) \qquad \text{s.t.} \quad rh + pc = y$$

where h and c are goods (housing and non-housing consumption), $u(\cdot,\cdot)$ is a standard utility function, r and p are prices of h and p in dollars, and y is income in dollars.

The FOCs of this problem yield the following optimality condition

$$\frac{u_h(h,c)}{u_c(h,c)} = \frac{p}{r} \tag{1}$$

where u_x represents the partial derivative of the utility function with respect to x. We represent this condition as

$$c = f(h,r)$$

where $f(\cdot,r)$ gives the value of c for each h such that equation (1) holds. We assume that this value is unique.

Replacing in the budget constraint of the problem we have

$$rh + pf(h,r) = y. \tag{2}$$

A constant income elasticity function would be defined as $\tilde{h} = g(p,r)y^{\xi}$ where $\xi \equiv \frac{d\ln\tilde{h}}{d\ln y}$ is the elasticity. Can we obtain this function form in this problem?

Fully differentiating (2) with respect to $\ln h$ and $\ln y$ yields

$$rhd\ln(h) + pf_h hd\ln(h) = yd\ln(y)$$

$$hd\ln(h)(r+pf_h) = yd\ln(y) \implies \frac{d\ln(h)}{d\ln(y)} = \frac{y}{h}\frac{1}{(r+pf_h)}$$

and using the budget constraint

$$\frac{d\ln(h)}{d\ln(y)} = \frac{r+p(c/h)}{r+pf_h} \equiv \xi$$

By definition, $f_h = \frac{dc}{dh}$.Thus, for the income elasticity to be constant we require a condition on the relationship between the ratio of c and h and the marginal rate of substitution, namely that

$$\frac{c}{h} = \frac{r}{p}(\xi - 1) + \xi\frac{dc}{dh} \tag{3}$$

for any y and given p and r.

Solving equation (3)

Re-write equation (3) as

$$c(h) - \xi hc'(h) = \frac{r}{p}(\xi - 1)h.$$

Implement the solution from this website, then we have

$$c(h) = \frac{\int \mu(h)\frac{r}{p}(\xi-1)hdh + k}{\mu(h)}$$

where k is a constant and

$$\mu(h) = e^{-\int \xi hdh}.$$

Solving we have that $\mu(h) = -\frac{1}{\xi}e^{-\xi h}$, so

$$\begin{aligned} c(h) &= \frac{\frac{r}{p}\frac{(\xi-1)}{\xi}\int \mu(h)hdh + k}{\mu(h)} \\ &= \frac{\frac{r}{p}\frac{(\xi-1)}{\xi^2}\frac{e^{-\xi h}}{\xi^2}(1+\xi h) + k}{-\frac{1}{\xi}e^{-\xi h}} \end{aligned}$$

and so

$$c(h) = \frac{r}{p}\frac{(1-\xi)}{\xi^3}(1+\xi h) - k\xi e^{-\xi h} \tag{4}$$

which is the general solution to the problem.

For the elasticity of income to be constant we need equation (1) to be of the form in (4).