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Partial differential equations

Control problem for a fourth order pseudo-parabolic equation in a two dimensional domain

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Abstract. In this paper, we consider boundary control problem for a fourth order pseudo-parabolic equation with Dirichlet conditions in the quadratic domain. A control function is given at the boundary of the considered domain. To achieve the average temperature in the domain, it is required to find a control function. Using the Fourier method, the control problem is reduced to the Volterra integral equation of the second type. The existence of the control function is proved using the method of successive approximation.

Keywords: initial-boundary problem; fourth order pseudo-parabolic equation; Volterra integral equation; admissible control.

1. Introduction

It is known that in recent years, due to the increasing interest in physics and mathematics, the boundary problems related to pseudo-parabolic equations were widely studied. For this purpose, various boundary problems for parabolic and pseudo-parabolic equations have been widely studied by many researchers.

It is well known that fourth-order pseudo-parabolic equations describe a variety of important physical processes, such as heat conduction in materials, electric signals in a nonlinear telegraph line with nonlinear damping, viscous flow in materials with memory [1], vibration of a nonlinear elastic rod with viscosity [2], nonlinear bidirectional shallow water waves [3], the velocity evolution of ionacoustic waves in a collisionless plasma when an ion viscosity is invoked [4], and so on.

The boundary control of a linear pseudo-parabolic equation and compare the results to those of parabolic equations was studied in [5]. The stability, uniqueness, and existence of solutions of some classical problems for the considered equation are studied in [6]. In [7], the point control problems for linear pseudo-parabolic and parabolic type equations are considered.

The optimal control problem for the parabolic type equations was studied by Fattorini and Friedman [8, 9]. Control problems for the infinite-dimensional case were studied by Egorov [10], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The boundary control problem for a parabolic equation with a piecewise smooth boundary in an $n-$dimensional domain was studied in [11] and an estimate for the minimum time required to reach a given average temperature was found. In [12], the considered the heat conduction equation with the Robin boundary condition and developed a mathematical model of the process of heating a cylindrical domain. Boundary control problems for fourth-order parabolic equations were studied in [13,14].

The works [15] examine control problems for parabolic equations in bounded two-dimensional domain. In these articles, an estimate was found for the minimum time required to heat a bounded domain to an estimate average temperature. The existence of control function is proved by Laplace transform method. Similar control problems were examined in the one-dimensional domain in [16-18].

Basic information on optimal control problems is given in detail in monographs by Lions and Fursikov [19, 20]. Boundary control problems for linear pseudo-parabolic type equations were studied in works [21-23], and it was proved that there is a control function for heating the domain to the average temperature.

For the fourth order nonlinear pseudo-parabolic equation, there are also some results about initial boundary value problem and Cauchy problem, especially on the global existence, nonexistence and asymptotic behavior of the solutions [24-26].

In 1978, Bakiyevich and Shadrin [24] considered the following problem

$$\begin{cases} u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} = f(x,t), \quad x \in R, \quad t > 0, \\ \qquad\qquad u(x,0) = \varphi(x), \quad x \in R, \end{cases}$$

where $\alpha, \gamma > 0$, $\beta \geq 0$ are constants. They showed that the solutions of this problem are expressed through the sum of convolutions of functions $\varphi(x)$ and $f(x,t)$ with corresponding fundamental solutions of the problem. In 2009, Khudaverdiyev and Farhadova [25] discussed the following fourth order semilinear pseudo-parabolic equation

$$u_t - \alpha u_{xxt} + u_{xxxx} = f(x,t,u,u_x,u_{xx},u_{xxx}), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T < +\infty,$$

where $\alpha > 0$ is a fixed number. They proved the existence in large theorem for generalized solution by means of Schauder stronger fixed point principle. Zhao and Xuan [26] studied the following

$$u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} + f(u)_x = 0, \quad x \in R, \quad t > 0.$$

They obtained the existence and convergence behavior of the global smooth solutions.

In this work, we considered the problem of boundary control for the fourth-order pseudo-parabolic equation with Dirichlet conditions in the quadratic domain. For this, the control problem was first reduced to the Volterra integral equation of the second kind, and the continuity of the integral in the given domain was shown. As a result, the existence of the control function was proved using the method of successive approximation.

2. Statement of problem

In this article, we consider the following fourth order pseudo-parabolic equation in the domain $\Omega = (0,\pi)\times(0,\pi)$

$$u_t - \Delta u - \Delta u_t + \Delta^2 u = 0, \quad (x,y,t)\in\Omega_T := \Omega\times(0,T), \tag{2.1}$$

with boundary value conditions

$$u(0,y,t) = \psi(y)\nu(t), \quad u(\pi,y,t) = 0, \quad u(x,0,t) = u(x,\pi,t) = 0, \tag{2.2}$$

and

$$u_{xx}(0,y,t) = u_{xx}(\pi,y,t) = 0, \quad u_{yy}(x,0,t) = u_{yy}(x,\pi,t) = 0, \tag{2.3}$$

and initial value condition

$$u(x,y,0) = 0, \quad 0 \le x, y \le \pi, \tag{2.4}$$

where $\Delta u = u_{xx}(x,y,t) + u_{yy}(x,y,t)$, $\Delta^2 u = u_{xxxx}(x,y,t) + 2u_{xxyy}(x,y,t) + u_{yyyy}(x,y,t)$, $\psi(y)$ is a given function and $\nu(t)$ is the control function. It is called that the control function $\nu(t)\in C^1[0,T]$ is *admissible*, if it fulfills the following conditions

$$\nu(0) = 0, \quad |\nu(t)| \le 1, \quad t\in[0,T]. \tag{2.5}$$

Assume, the given function $\psi \in C^5[0,\pi]$ satisfies the conditions

$$\psi(0) = \psi(\pi) = \psi^{(2)}(0) = \psi^{(2)}(\pi) = 0. \tag{2.6}$$

Control Problem. Assume that function $\theta(t)$ is a given. Then find the control function $\nu(t)$ from the condition

$$\int_0^\pi\int_0^\pi u(x,y,t)dxdy = \theta(t), \quad 0 \le t \le T, \tag{2.7}$$

where $u(x,y,t)$ is a solution of the problem (2.1)-(2.4) and it depends on the control function $\nu(t)$.

We now offer the primary theorem for demonstrating admissible control's existence.

Theorem 2.1. *There exists a constant $M > 0$ such that for any function $\theta(t)\in C[0,T]$ satisfying the conditions*

$$\theta(0) = 0, \quad |\theta(t)| \le \frac{1}{M}, \quad t\in[0,T],$$

the solution $\nu(t)$ of the equation (2.7) exists, unique and satisfies the conditions (2.5).

We will consider the proof of Theorem 2.1 step by step in the next sections.

3. Main integral equation

In this section, we consider the reduction of the control problem to a Volterra integral equation of the second kind.

Definition 3.1. *By solution to initial-boundary problem (2.1) - (2.4), we understand the function* $u(x, y,t)$ *represented in the form*

$$u(x, y,t) = v(t)\psi(y)\frac{\pi - x}{\pi} - w(x, y,t), \tag{3.1}$$

where the function $w(x, y,t)$ *with the regularity* $w(x, y,t) \in C^{4,4,1}_{x,y,t}(\Omega_T) \cap C(\bar{\Omega}_T)$ *and* $w_{xx}, w_{yy} \in C(\bar{\Omega})$ *is the solution to the problem*

$$w_t - \Delta w - \Delta w_t + \Delta^2 w = \frac{\pi - x}{\pi}\psi(y)v'(t) + \frac{\pi - x}{\pi}\psi^{(4)}(y)v(t)$$
$$-\frac{\pi - x}{\pi}\psi^{(2)}(y)v'(t) - \frac{\pi - x}{\pi}\psi^{(2)}(y)v(t),$$

with boundary value conditions

$$w(0, y,t) = w(\pi, y,t) = 0, \quad w(x,0,t) = w(x,\pi,t) = 0,$$

and

$$w_{xx}(0, y,t) = w_{xx}(\pi, y,t) = 0, \quad w_{yy}(x,0,t) = w_{yy}(x,\pi,t) = 0,$$

and initial value condition

$$w(x, y,0) = 0.$$

To find the solution to the above mixed problem, we use the Fourier method. That is, we look for the solution in the following form:

$$w(x, y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{m,n}(t)\,\vartheta_{m,n}(x, y)\,,$$

where $\vartheta_{m,n}(x, y) = \sin mx \sin ny$ are eigenfunctions of the spectral problem

$$\Delta\vartheta(x, y) + \lambda\vartheta(x, y) = 0, \quad \vartheta\big|_{\partial\Omega} = 0.$$

Consequently, we obtain (see [27])

$$w(x,y,t)=\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1+n^2)\psi_n}{m(1+n^2+m^2)}\left(\int_0^t e^{-\mu_{m,n}(t-s)}\nu'(s)ds\right)\sin mx\sin ny$$
$$+\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(n^2+n^4)\psi_n}{m(1+n^2+m^2)}\left(\int_0^t e^{-\mu_{m,n}(t-s)}\nu(s)ds\right)\sin mx\sin ny, \tag{3.2}$$

where $\mu_{m,n}=m^2+n^2$.

Lemma 3.1. *Let the function $\psi(y)\in C^5[0,\pi]$ satisfy the conditions (2.6). Then the following estimate holds*

$$|\psi_n|\le\frac{C}{n^5},\quad n=1,2,\cdots,$$

where ψ_n is the Fourier coefficient of the function $\psi(y)$ and C is a positive constant.

Proof. It is known that the Fourier coefficient of the function $\psi(y)$ in the interval $(0,\pi)$ is defined as follows

$$\psi_n=\frac{2}{\pi}\int_0^{\pi}\psi(y)\sin ny\,dy,\quad n=1,2,\ldots. \tag{3.3}$$

By (3.3) and condition (2.6), we might write

$$\psi_n=\frac{2}{\pi}\int_0^{\pi}\psi(y)\sin ny\,dy=-\frac{2}{\pi n}\left(\psi(y)\cos ny\Big|_{y=0}^{y=\pi}-\int_0^{\pi}\psi^{(1)}(y)\cos ny\,dy\right)$$
$$=\frac{2}{\pi n^2}\left(\psi^{(1)}(y)\sin ny\Big|_{y=0}^{y=\pi}-\int_0^{\pi}\psi^{(2)}(y)\sin ny\,dy\right)$$
$$=\frac{2}{\pi n^3}\left(\psi^{(2)}(y)\cos ny\Big|_{y=0}^{y=\pi}-\int_0^{\pi}\psi^{(3)}(y)\cos ny\,dy\right)$$
$$=-\frac{2}{\pi n^4}\left(\psi^{(3)}(y)\sin ny\Big|_{y=0}^{y=\pi}-\int_0^{\pi}\psi^{(4)}(y)\sin ny\,dy\right)$$
$$=-\frac{2}{\pi n^5}\left(\psi^{(4)}(y)\cos ny\Big|_{y=0}^{y=\pi}-\int_0^{\pi}\psi^{(5)}(y)\cos ny\,dy\right)$$
$$=\frac{2}{\pi}\frac{\psi^{(4)}(0)-(-1)^n\psi^{(4)}(\pi)}{n^5}+\frac{2}{\pi n^5}\int_0^{\pi}\psi^{(5)}(y)\cos ny\,dy.$$

Then we obtain

$$|\psi_n|\le\frac{C}{n^5},\quad C=const>0,\quad n=1,2,\ldots.$$

Lemma is proved.

Lemma 3.2. *Let* $\nu(t)\in C^1[0,T]$. *Then the solution to problem (2.1)-(2.4) is*

$$u(x,y,t)=\frac{\pi-x}{\pi}\psi(y)\nu(t)$$
$$-\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1+n^2)\psi_n}{m(1+n^2+m^2)}\left(\int_0^t e^{-\mu_{m,n}(t-s)}\nu'(s)ds\right)\sin mx\sin ny$$
$$-\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(n^2+n^4)\psi_n}{m(1+n^2+m^2)}\left(\int_0^t e^{-\mu_{m,n}(t-s)}\nu(s)ds\right)\sin mx\sin ny. \tag{3.4}$$

Proof. Let us prove that the Fourier series of the function $w(x,y,t)$ belongs to the class $C^{4,4,1}_{x,y,t}(\Omega_T)\cap C(\bar{\Omega}_T)$ and $w_{xx},w_{yy}\in C(\bar{\Omega})$. It is sufficient to prove that the corresponding series converge uniformly. Due to conditions imposed on the expression of the function $\nu(t)$ for $w(x,y,t)$ converges uniformly, so $w(x,y,t)\in C(\bar{\Omega}_T)$. It is logically proved that $w_{xx},w_{yy}\in C(\bar{\Omega}_T)$, $w_t,\Delta w,\Delta w_t,\Delta^2 w\in C(\Omega_T)$ and $w(x,y,t)$ satisfy the equation. Q.E.D.

Lemma is proved.

From (3.4) and the condition (2.7), we can write

$$\theta(t)=\int_0^\pi\int_0^\pi u(x,y,t)dxdy=\nu(t)\int_0^\pi\int_0^\pi\psi(y)\frac{\pi-x}{\pi}dxdy$$
$$-\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1+n^2)(1-(-1)^m)(1-(-1)^n)\psi_n}{m^2n(1+n^2+m^2)}\int_0^t e^{-\mu_{m,n}(t-s)}\nu'(s)ds$$
$$-\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(n^2+n^4)(1-(-1)^m)(1-(-1)^n)\psi_n}{m^2n(1+n^2+m^2)}\int_0^t e^{-\mu_{m,n}(t-s)}\nu(s)ds.$$

According to the condition $\nu(0)=0$, we get

$$\theta(t)=\nu(t)\int_0^\pi\int_0^\pi\psi(y)\frac{\pi-x}{\pi}dxdy$$
$$-\nu(t)\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(1+n^2)(1-(-1)^m)(1-(-1)^n)}{m^2n(1+n^2+m^2)}$$
$$+\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(1+n^2)(1-(-1)^m)(1-(-1)^n)}{n(1+n^2+m^2)}\int_0^t e^{-\mu_{m,n}(t-s)}\nu(s)ds. \tag{3.5}$$

According to Parseval equality, we get

$$\int_0^\pi\int_0^\pi \psi(y)\frac{\pi-x}{\pi}dxdy = \frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(1-(-1)^m)(1-(-1)^n)}{m^2 n}. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\theta(t) = \nu(t)\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(1-(-1)^m)(1-(-1)^n)}{n(1+n^2+m^2)}$$

$$+\frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(1+n^2)(1-(-1)^m)(1-(-1)^n)}{n(1+n^2+m^2)}\int_0^t e^{-\mu_{m,n}(t-s)}\nu(s)ds.$$

We set

$$L(t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{m,n}e^{-\mu_{m,n}t}, \quad t>0, \tag{3.7}$$

and

$$\beta = \frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\psi_n(1-(-1)^m)(1-(-1)^n)}{n(1+n^2+m^2)}, \quad m,n=1,2,\cdots, \tag{3.8}$$

where

$$\Psi_{m,n} = \frac{2}{\pi}\frac{\psi_n(1+n^2)(1-(-1)^m)(1-(-1)^n)}{n(1+n^2+m^2)}. \tag{3.9}$$

Then, we get the Volterra integral equation of the second kind

$$\nu(t)+\alpha\int_0^t L(t-s)\nu(s)ds = \alpha\theta(t), \quad t>0, \tag{3.10}$$

where $\alpha=\beta^{-1}$.

4. Proof of Theorem 2.1

In this section, we consider the existence of a solution to the Volterra integral equation of the second kind. Then we prove the admissibility of the control function.

Lemma 4.1. *Let the conditions of Lemma 3.1 hold. Then, the function $L(t)$ is continuous on the half line $t\geq 0$.*

Proof. Using Lemma 3.1 and equality (3.9), we can write

$$|\Psi_{m,n}| \leq \frac{8C}{\pi}\frac{(1+n^2)}{n^6(1+n^2+m^2)}.$$

Then, we have the estimate

$$|L(t)| = \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Psi_{m,n} e^{-\mu_{m,n} t} \right| \le \frac{8C}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1+n^2}{n^6 (1+n^2+m^2)} = A, \tag{4.1}$$

where A is a positive constant.

Therefore, the function $L(t)$ is continuous on the half-line $t \ge 0$.

Lemma is proved.

Lemma 4.2. *Let $\theta(t)$ be continuous on the half-line $t \ge 0$. Then the equation (3.10) has the solution.*

Proof. The Lemma 4.1 states that the function $L(t)$ is bounded in $t \in [0, T]$. Therefore, the integral equation (3.10) can be solved using the method of successive approximation.

Set

$$\nu_0(t) = \alpha\theta(t), \quad \nu_k(t) = \alpha \int_0^t L(t-s)\nu_{k-1}(s)ds, \quad k = 1, 2, ... \tag{4.2}$$

Then, we have the solution

$$\nu(t) = \sum_{k=0}^{\infty} (-1)^k \nu_k(t). \tag{4.3}$$

Indeed, we can see that this function satisfies Eq. (3.10)

$$\alpha \int_0^t L(t-s)\nu(s)ds = \alpha \sum_{k=0}^{\infty} (-1)^k \int_0^t L(t-s)\nu_k(s)ds$$

$$= \sum_{k=0}^{\infty} (-1)^k \nu_{k+1}(t) = -\sum_{k=1}^{\infty} (-1)^k \nu_k(t) = -\sum_{k=0}^{\infty} (-1)^k \nu_k(t) + \nu_0(t) = -\nu(t) + \alpha\theta(t).$$

It is known α is finite. According to Lemma 3.1, β is a finite constant number. Indeed, we may write

$$|\beta| \le \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\psi_n| (1-(-1)^m)(1-(-1)^n)}{n(1+n^2+m^2)} \le \frac{8C}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^6 (1+n^2+m^2)} < +\infty.$$

Lemma is proved.

Set

$$\|\theta\|_T = \max_{0 \le t \le T} |\theta(t)|.$$
3

Lemma 4.3. *Let $\theta \in C[0, T]$. Then, the following estimate is valid:*

$$|\nu_k(t)| \le |\alpha|^{k+1} A^k \frac{t^k}{k!} \|\theta\|_T, \quad t \in [0,T], \quad k = 0,1,\ldots,$$

where $\alpha = \beta^{-1}$ and β, A are defined by (3.8), (4.1), respectively.

Proof. Now we prove the Lemma using the induction method.

I. It is not difficult to see that the inequality holds for $k = 0$.

II. We prove for $k+1$:

$$|\nu_{k+1}(t)| \le |\alpha| \int_0^t |L(t-s)||\nu_k(s)| ds \le |\alpha|^{k+2} A^k \|\theta\|_T \int_0^t |L(t-s)| \frac{s^k}{k!} ds$$

$$\le |\alpha|^{k+2} A^{k+1} \|\theta\|_T \int_0^t \frac{s^k}{k!} ds = |\alpha|^{k+2} A^{k+1} \frac{t^{k+1}}{(k+1)!} \|\theta\|_T .$$

Then, we get the required estimate

$$|\nu_k(t)| \le |\alpha|^{k+1} A^k \frac{t^k}{k!} \|\theta\|_T, \quad k = 0,1,\ldots.$$

Lemma is proved.

Now we present the proof of the main Theorem 2.1.

Proof of Theorem 2.1. Using the Lemma 4.3 and (4.2), (4.3), we get

$$|\nu(t)| \le \sum_{k=0}^{\infty} |\nu_k(t)| \le \|\theta\|_T \sum_{k=0}^{\infty} |\alpha|^{k+1} A^k \frac{t^k}{k!} = |\alpha| e^{|\alpha| At} \|\theta\|_T .$$

Then using the inequality $|\theta(t)| \le \frac{1}{M}$, we have

$$|\nu(t)| \le |\alpha| e^{|\alpha| AT} \|\theta\|_T \le 1, \quad t \in [0,T],$$

where as M we took

$$M = |\alpha| e^{|\alpha| AT}, \quad 0 < T < +\infty.$$

From the integral equation (3.10) and condition $\theta(0) = 0$, we have $\nu(0) = 0$. Thus, we have proved the admissibility of the control function $\nu(t)$.

5. Example

Let $\psi(y) = \sin y$, $y \in [0, \pi]$ in initial-boundary problem (2.1)-(2.4). Then the expressions defined by the equations (3.8) and (3.9) are defined as follows

$$\beta = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1-(-1)^m}{2+m^2},$$

and

$$\Psi_{m,1} = \frac{8}{\pi}\frac{1-(-1)^m}{2+m^2}.$$

Thus, the function $L(t)$ defined by (3.7), we can write

$$L(t) = \sum_{m=0}^{\infty}\Psi_{m,1}e^{-\mu_{m,1}t}, \quad t > 0,$$

where $\mu_{m,1} = m^2 + 1.$

It is clear that

$$\Psi_{2m,1} = 0, \quad \Psi_{2m+1,1} = \frac{16}{\pi}\frac{1}{2+(2m+1)^2}, \quad m = 0,1,2,....$$

We consider the following function:

$$\theta(t) = te^{-t}, \quad t \in [0,T].$$

The physical meaning of the function $\theta(t)$ is the average temperature in the domain.

We can represent the kernel $L(t)$ in the form

$$L(t) = \frac{16}{3\pi}e^{-\mu_{1,1}t} + \frac{16}{\pi}\sum_{m=1}^{\infty}\frac{1}{2+(2m+1)^2}e^{-\mu_{2m+1,1}t}$$

$$= e^{-\mu_{1,1}t}\left(\frac{16}{3\pi} + O(1)e^{-t(\mu_{3,1}-\mu_{1,1})}\right), \quad \mu_{1,1} = 2.$$

Consequently, we can write

$$L(t) \sim \frac{16}{3\pi}e^{-2t}, \quad t > 0,$$

and

$$\alpha = \beta^{-1} = \frac{\pi}{4}\frac{1}{\gamma},$$

where $\gamma=\sum_{m=1}^{\infty}\frac{1-(-1)^m}{2+m^2}$. As we know, since it is $\gamma \ll 1$, we can write $\alpha\approx\frac{\pi}{4}$.

In this case, the main integral equation (3.10) can be replaced by the approximation

$$\nu(t)+\frac{4}{3}\int_0^t e^{-2(t-s)}\nu(s)ds=\frac{\pi}{4}te^{-t}, \quad t>0. \tag{5.1}$$

By solving the integral equation (5.1) using the method of successive approximation, we obtain the following solution:

$$\nu(t)=\frac{3\pi}{196}\left(7te^{-t}+4e^{-t}-4e^{-\frac{10}{3}t}\right), \quad t\geq 0.$$

6. References

[1] Al'shin, A.B., Korpusov, M.O., Siveshnikov, A.G. Blow up in nonlinear Sobolev type equations, 1 edition, Walter Gruyter, Berlin, 2011. https://doi.org/10.1515/9783110255294.

[2] Kolesov, A. Y.,. Mishchenko, E. F., Rozov, N. K. *Asymptotic Methods of Investigation of Periodic Solutions of Nonlinear Hyperbolic Equations*, Trudy Mat. Inst. Steklova, 222, Nauka, MAIK «Nauka», M., 1998, 3–191; Proceedings of the Steklov Institute of Mathematics, vol. 222, pp. 1-189, 1998.

[3] Shivamoggi, B. K. *A symmetric regularized long wave equation for shallow water waves*. Physics of Fluids, vol. 29, pp. 890-891, 1986. https://doi.org/10.1063/1.865895

[4] Rosenau, P. *Evolution and breaking of the ion-acoustic waves*. The Physics of Fluids, vol. 31, pp. 1317-1319, 1988. https://doi.org/10.1063/1.866723

[5] White, L. W. *Controllability properties of pseudo-parabolic boundary control problems*. SIAM Journal on Control and Optimization, vol. 18, pp. 534-539, 1980. https://doi.org/10.1137/031803

[6] Coleman, B. D. *Instability, uniqueness, and nonexistence theorems for the equation* $u_t=u_{xx}-u_{txx}$ *on a strip*. Archive for Rational Mechanics and Analysis, vol. 19, pp. 100-116, 1965. https://doi.org/10.1007/BF00282277

[7] White, L. W. P*oint control of pseudoparabolic problems*. Journal of Differential Equations, vol. 42, pp. 366-374, 1981. https://doi.org/10.1016/0022-0396(81)90110-8

[8] Fattorini, H. O. T*ime-Optimal control of solutions of operational differential equation*. SIAM J. Control, vol. 2, pp. 49-65, 1964.

[9] Friedman, A. *Differential equations of parabolic type*. New Jersey, XVI, Englewood Cliffs, 1964.

[10] Egorov, Yu. V. *Optimal control in Banach spaces*. Dokl. Akad. Nauk SSSR, vol. 150, pp. 241-244, 1963. (In Russian)

[11] Albeverio, S., Alimov, Sh. A. *On one time-optimal control problem associated with the heat exchange process*. Applied Mathematics and Optimization, vol. 57, pp. 58-68, 2008.

[12] Alimov, Sh. A., Komilov, N. M. *Determining the thermal mode setting parameters based on output data*. Differential Equations, vol. 58, pp. 21-35, 2022. https://doi.org/10.1134/S0012266122010049

[13] Dekhkonov, F.N. *On the time-optimal control problem for a fourth order parabolic equation in a two-dimensional domain*. Bulletin of the Karaganda University. Mathematics Series, vol. 114, pp. 57-70, 2024. https://doi.org/10.31489/2024m2/57-70

[14] Dekhkonov, F., W. Li. *On the boundary control problem associated with a fourth order parabolic equation in a two-dimensional domain*. Discrete and Continuous Dynamical Systems - Series S, vol. 17, pp. 2478-2488, 2024. https://doi.org/10.3934/dcdss.2024004

[15] Fayazova, Z.K. *Boundary control of the heat transfer process in the space*. Russian Mathematics, vol. 63, pp. 71-79, 2019. https://doi.org/10.3103/S1066369X19120089

[16] Dekhkonov, F.N. *Boundary control associated with a parabolic equation*. Journal of Mathematics and Computer Science, vol. 33, pp. 146-154, 2024. https://dx.doi.org/10.22436/jmcs.033.02.03

[17] Dekhkonov, F.N. *On the time-optimal control problem for a heat equation*. Bulletin of the Karaganda University. Mathematics Series, vol. 111, pp. 28-38, 2023. https://doi.org/10.31489/2023m3/28-38

[18] Dekhkonov, F.N. *Boundary control problem for a parabolic equation with involution*. Eurasian journal of mathematical and computer applications, vol. 12, pp. 22-34, 2024.

[19] Lions, J. L. Contróle optimal de systèmes gouvernés par des équations aux dérivées partielles. Paris, Dunod Gauthier-Villars, 1968.

[20] Fursikov, A. V. Optimal control of distributed systems, Theory and applications, Translations of Math. Monographs, 187, Amer. Math. Soc., Providence, 2000. http://dx.doi.org/10.1090/mmono/18

[21] Dekhkonov, F.N. *On the control problem associated with a pseudo-parabolic type equation in an one-dimensional domain*. International Journal of Applied Mathematics, 37, pp. 109-118, 2024. http://dx.doi.org/10.12732/ijam.v37i1.9

[22] Dekhkonov, F.N., *On one boundary control problem for a pseudo-parabolic equation in a two-dimensional domain*, Communications in Analysis and Mechanics, vol. 17, pp. 1-14, 2025. http://dx.doi.org/10.3934/cam.2025001

[23] Dekhkonov, F.N. *Boundary control problem associated with a pseudo-parabolic equation*. Stochastic Modelling and Computational Sciences, vol. 3, pp. 117–128, 2023. https://doi.org/10.61485/SMCS.27523829/v3n1P9

[24] Bakiyevich, N. I., Shadrin, G.A. *Cauchy problem for an equation in filtration theory*. Sb. Trudov Mosgospedinstituta, vol. **7**, pp. 47-63, 1978.

[25] Khudaverdiyev, K. I., Farhadova, G.M. *On global existence for generalized solution of one-dimensional non-self-adjoint mixed problem for a class of fourth order semilinear pseudo-parabolic equations*. Proceedings of the Institute of Mathematics and Mechanics (PIMM), National Academy of Sciences of Azerbaijan, vol. 31, pp. 119-134, 2009.

[26] Zhao, H. J., Xuan, B. J. *Existence and convergence of solutions for the generalized BBM-Burgers equations*. Nonlinear Analysis: Theory, Methods and Applications, vol. 28, 1835-1849, 1997. https://doi.org/10.1016/S0362-546X(95)00237-P

[27] Tikhonov, A. N., Samarsky, A. A. Equations of Mathematical Physics, Nauka, Moscow, 1966.