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Nonlinear problem involving the fractional $p(x)-$Laplacian operator by topological degree

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Abstract. This paper is concerned with the study of a nonlinear problem involving the fractional $p(x)-$Laplacian operator. By means of the Berkovits degree theory, we prove the existence of nontrivial weak solutions for this problem. The appropriate functional framework for this problem is the fractional Sobolev spaces with variable exponent.
Keywords: Nonlinear elliptic problem, fractional $p(x)-$Laplacian operator, fractional Sobolev spaces with variable exponent, Degree theory.

1 Introduction

Great attention was paid to the study of elliptic problems including the fractional operator during the last years. The study of this type of problems is motivated by their abilities to model several physical phenomena such as those of phase transition, continuum mechanics and dynamics. Fractional operator is also present in game theory and probability which he provides a simple model to describe stochastic stabilization of Lévy jump processes (see [**?**, **?**, **?**, **?**] and the references therein).

Let Ω be a smooth bounded open set in $\mathbb{R}^N$, $s \in (0, 1)$ and let $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent with $sp(x, y) < N$. We assume that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le p(x,y) \le p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty, \tag{1}$$

and p is symmetric i.e.

$$p(x,y) = p(y,x), \ \ \forall (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{2}$$

Let us denote

$$q(x) = p(x,x), \ \ \forall x \in \overline{\Omega}.$$

Let us consider the fractional $p(x)-$Laplacian operator given by

$$(-\Delta_{p(x)})^s u(x) = p.v. \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy, \ \ \forall x \in \Omega,$$

where $p.v.$ is a commonly used abbreviation in the principal value sense.

In this paper, we are concerned with the study of the following nonlinear elliptic problem,

$$\begin{cases} (-\Delta_{p(x)})^s u(x) + |u(x)|^{q(x)-2}u(x) = \lambda |u(x)|^{r(x)-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

where $r(\cdot) \in C(\bar{\Omega})$ and λ is a real parameter. We also assume that

$$1 < r^- \le r(x) \le r^+ < p^-. \tag{3}$$

Note that $(-\Delta_{p(x)})^s$ is the fractional version of well-known $p(x)-$Laplacian operator $-\Delta_{p(x)}(u) = -div(|\nabla u|^{p(x)-2}\nabla u)$ for which Alsaedi in [**?**] establishes sufficient conditions for the existence of nontrivial weak solutions for a problem similar to (**??**) that is the following problem:

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2}u + |u|^{q(x)-2}u & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases}$$

The proofs combine the Ekeland variational principle, the mountain pass theorem and energy arguments.

A first introduction to the operator $(-\Delta_{p(x)})^s$ can be found in [**?**], where the authors extended Sobolev spaces with variable exponents to the fractional case with a compact embedding theorem, and proved the existence and uniqueness of weak solutions for the following fractional $p(x)-$Laplacian problem

$$\begin{cases} (-\Delta_{p(x)})^s u(x) + |u|^{q(x)-2}u(x) = f(x) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \end{cases}$$

where $f \in L^{a(x)}(\Omega)$ for some $a(x) > 1$. Subsequently, Bahrouni and Rădulescu in [?] are interested in some qualitative properties on both the variable exponent Sobolev fractional space and the operators $(-\Delta_{p(x)})^s$. The results presented abstractly in their paper are used to prove that the problem (??) admits at least one nontrivial weak solution by a variational analysis. More recently, Azroul et al. [?] study the problem (??) as an eigenvalue problem using adequate variational techniques, mainly based on Ekeland's variational principle. They establish the existence of a continuous family of eigenvalues lying in a neighborhood at the right of the origin.

Note that the fractional Laplacian considered in this work and in [?, ?, ?] is in fact "regional" since the integration is over Ω but not over $\mathbb{R}^N$. In the case of the "global" fractional Laplacian, where the integral is defined on $\mathbb{R}^N \backslash B(x, \varepsilon)$ by tending ε to 0, similar problems have been studied (see, for instance, [?, ?, ?]).

Using another technique based on topological degree theory, notably the recent Berkovits degree, we prove in this paper the existence of at least one nontrivial weak solution the problem (??).

The paper is divided into four sections. In the second section, we introduce some preliminary results about Lebesgue and fractional Sobolev spaces with variable exponent, some classes of operators and an outline of the recent Berkovits degree. The third section is reserved for some technical lemmas. Finally, in the fourth section we give our main results concerning the weak solutions of the problem (??).

2 Some preliminary results

2.1 Lebesgue and fractional Sobolev spaces with variable exponent

In this subsection, we first recall some useful properties of the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$. For more details we refer the reader to [?, ?, ?] for more details.

Denote

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) | \inf_{x \in \bar{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ := max\{h(x), x \in \bar{\Omega}\}, h^- := min\{h(x), x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}.$$

Endowed with *Luxemburg norm*

$$\|u\|_{p(x)} = \inf\{\lambda > 0/\rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\}.$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega),$$

$(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space, separable and reflexive. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$.
We have also the following result

Proposition 1 *For any $u \in L^{p(x)}(\Omega)$ we have*

(i) $\|u\|_{p(x)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$,

(ii) $\|u\|_{p(x)} \geq 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^+}$,

(iii) $\|u\|_{p(x)} \leq 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^-}$.

From this proposition, we can deduce the inequalities

$$\|u\|_{p(x)} \leq \rho_{p(\cdot)}(u) + 1, \tag{4}$$

$$\rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}. \tag{5}$$

If $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \bar{\Omega}$, then there exists the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

Next, we present the definition and some results on fractional Sobolev spaces with variable exponent that was introduced in [**?**, **?**, **?**]. Let s be a fixed real number such that $0 < s < 1$ and lets the assumptions (**??**) and (**??**) with $sp(x, y) < N$ be satisfied, we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$\begin{aligned} W &= W^{s,p(x,y)}(\Omega) \\ &= \{u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dxdy < +\infty, \text{ for some } \lambda > 0\}, \end{aligned}$$

where $q(x) = p(x,x)$. We equip the space W with the norm

$$\|u\|_W = \|u\|_{q(x)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = \inf\{\lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dxdy \le 1\}.$$

The space $(W, \|\cdot\|_W)$ is a Banach space (see [**?**]), separable and reflexive (see [**?**, Lemma 3.1]).

We also define W_0 as the subspace of W which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_W$. From [**?**, Theorem 2.1 and Remark 2.1],

$$\|\cdot\|_{W_0} := [\cdot]_{s,p(x,y)}$$

is a norm on W_0 which is equivalent to the norm $\|\cdot\|_W$, and we have the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$. So the space $(W_0, \|\cdot\|_{W_0})$ is a Banach space separable and reflexive.
We define the modular $\rho_{p(\cdot,\cdot)} : W_0 \to \mathbb{R}$ by

$$\rho_{p(\cdot,\cdot)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy.$$

The modular $\rho_{p(\cdot,\cdot)}$ checks the following results, which is similar to Proposition **??**(see [**?**, Lemma 2.1])

Proposition 2 *For any $u \in W_0$ we have*

(i) $\|u\|_{W_0} \ge 1 \quad \Rightarrow \quad \|u\|_{W_0}^{p^-} \le \rho_{p(\cdot,\cdot)}(u) \le \|u\|_{W_0}^{p^+}$,

(ii) $\|u\|_{W_0} \le 1 \quad \Rightarrow \quad \|u\|_{W_0}^{p^+} \le \rho_{p(\cdot,\cdot)}(u) \le \|u\|_{W_0}^{p^-}$.

2.2 Some classes of operators and an outline of Berkovits degree

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle . \,,\, . \rangle$ and let Ω be a nonempty subset of X.

Let Y be a real Banach space. We recall that a mapping $F : \Omega \subset X \to Y$ is *bounded,* if it takes any bounded set into a bounded set. F is said to be *demicontinuous,* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$. F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact.

A mapping $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+), if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.
F is said to be *quasimonotone* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$
\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \mid G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\}.
\end{aligned}
$$

Here, $T \in \mathcal{F}_1(\bar{G})$ is called an *essential inner map* to F.

Lemma 1 *[?, Lemma 2.3] Suppose that $T \in \mathcal{F}_1(\bar{G})$ is continuous and $S : D_S \subset X^* \to X$ is demicontinuous such that $T(\bar{G}) \subset D_S$, where G is a bounded open set in a real reflexive Banach space X. Then the following statements are true:*

(i) *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\bar{G})$, where I denotes the identity operator.*

(ii) *If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\bar{G})$*

Definition 1 *Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\bar{G})$ be continuous and let $F, S \in \mathcal{F}_T(\bar{G})$. The affine homotopy $H : [0, 1] \times \bar{G} \to X$ defined by*

$$H(t, u) := (1 - t)Fu + tSu \text{ for } (t, u) \in [0, 1] \times \bar{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 1 *[?, Lemma 2.5] The above affine homotopy satisfies condition $(S_+)_T$.*

We introduce the topological degree for the class $\mathcal{F}_B(X)$ constructed by Berkovits in [**?**].

Theorem 1 *[**?**, Theorem 3.1] There exists a unique degree function*

$$d : \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_{T,B}(\bar{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties

1. *(Existence) If $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
2. *(Additivity) Let $F \in \mathcal{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*
$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$
3. *(Homotopy invariance) If $H : [0, 1] \times \bar{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0, 1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0, 1]$,then the value of $d(H(t, .), G, h(t))$ is constant for all $t \in [0, 1]$.*
4. *(Normalization) For any $h \in G$, we have $d(I, G, h) = 1$.*

3 Technical lemmas

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded open set, $s \in (0, 1)$ and we assume that (**??**), (**??**) and (**??**) hold. In this section, we present two technical lemmas that we will need to study our problem (**??**).

Let us denote by $L : W_0 \to W_0^*$ the operator associated to the $(-\Delta_{p(x)})^s$ defined by

$$\begin{aligned} \langle Lu, \varphi \rangle &= \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} \, dxdy \\ &+ \int_{\Omega} |u(x)|^{q(x)-2}u(x)\varphi(x) \, dx, \end{aligned}$$

for all $u, \varphi \in W_0$, where W_0^* is the dual space of W_0.

Lemma 2 *[**?**, Remark 4.3]*

(i) *L is bounded and strictly monotone operator,*

(ii) *L is a mapping of type (S_+),*

(iii) *L is a homeomorphism.*

Lemma 3 *The operator $S : W_0 \to W_0^*$ setting by*

$$\langle Su, \varphi \rangle = -\lambda \int_\Omega |u(x)|^{r(x)-2} u(x)\varphi(x)dx, \quad \forall u, \varphi \in W_0$$

is compact.

Proof
Let $\phi : W_0 \to L^{q'(x)}(\Omega)$ be the operator defined by

$$\phi u(x) := -\lambda |u(x)|^{r(x)-2} u(x) \text{ for } u \in W_0 \text{ and } x \in \Omega.$$

It's obvious that ϕ is continuous. We prove that ϕ is bounded.
For each $u \in W_0$, we have by the inequalities (**??**) and (**??**) that

$$\begin{aligned} \|\phi u\|_{q'(x)} &\leq \rho_{q'(\cdot)}(\phi u) + 1 \\ &= \int_\Omega |\lambda |u|^{r(x)-1}|^{q'(x)}\, dx + 1 \\ &\leq const\ \rho_{\alpha(\cdot)}(u) + 1 \\ &\leq const(\|u\|_{\alpha(x)}^{\alpha^-} + \|u\|_{\alpha(x)}^{\alpha^+}) + 1 \end{aligned}$$

where $\alpha(x) = (r(x) - 1)q'(x) \in C_+(\overline{\Omega})$ with $\alpha(x) \leq q(x)$. By the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ and the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, we obtain

$$\|\phi u\|_{q'(x)} \leq const((\|u\|_{W_0}^{\alpha^-} + \|u\|_{W_0}^{\alpha^+}) + 1$$

This implies that ϕ is bounded on W_0.
Since the embedding $I : W_0 \to L^{q(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{q'(x)}(\Omega) \to W_0^*$ is also compact. Therefore, the composition $S = I^* \circ \phi$ is compact.

4 Main Result

In this section, we study the nonlinear problem (**??**) based on the Berkovits degree theory introduced in subsection 2.2, where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded open, $s \in (0, 1)$ and under assumptions (**??**), (**??**) with $sp(x, y) < N$ and (**??**) .

Let L and $S : W_0 \to W_0^*(\Omega)$ be as in Section 3.

Definition 2 *We say that $u \in W_0$ is a weak solution of (**??**) if*

$$\langle Lu, \varphi\rangle + \langle Su, \varphi\rangle = 0, \ \ \forall \varphi \in W_0.$$

Theorem 2 *Under assumptions* (**??**), (**??**) *and* (**??**), *the problem (**??**) has a weak solution u in W_0.*

Proof $u \in W_0$ is a weak solution of (**??**) if and only if

$$Lu = -Su. \tag{6}$$

Thanks to the properties of the operator L seen in Lemma **??** and in view of Minty-Browder Theorem [**?**, Theorem 26A], the inverse operator
$T := L^{-1} : W_0^* \to W_0$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma **??** that the operator S is bounded, continuous and quasimonotone.
Consequently, equation (**??**) is equivalent to

$$u = Tv \text{ and } v + S \circ Tv = 0. \tag{7}$$

To solve (**??**), we will apply the degree theory introduced in section 2. To do this, we first claim that the set

$$B := \{v \in W_0^* | v + tS \circ Tv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded. Indeed, let $v \in B$. Set $u := Tv$, then $\|Tv\|_{W_0} = \|u\|_{W_0}$.
If $\|u\|_{W_0} \leq 1$, then $\|Tv\|_{W_0}$ is bounded.
If $\|u\|_{W_0} > 1$, then we get by the implication (i) in Proposition **??** and the inequality (**??**) the estimate

$$\begin{array}{rcl} \|Tv\|_{W_0}^{p^-} & = & \|u\|_{W_0}^{p-} \\ & \leq & \rho_{p(\cdot,\cdot)}(u) \\ & \leq & \rho_{p(\cdot,\cdot)}(u) + \displaystyle\int_\Omega |u(x)|^{q(x)}\,dx \\ & = & \langle Lu, u\rangle \\ & = & \langle v, Tv\rangle \\ & = & -t\langle S \circ Tv, Tv\rangle \\ & \leq & t \displaystyle\int_\Omega |\lambda|u(x)|^{r(x)}|\,dx \\ & \leq & const(\|u\|_{r(x)}^{r^-} + \|u\|_{r(x)}^{r^+}). \end{array}$$

From the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ and the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, we can deduce the estimate

$$\|Tv\|_{W_0}^{p^-} \leq const\ \|Tv\|_{W_0}^{r^+}.$$

It follows that $\{Tv|v \in B\}$ is bounded.
By writing $B = \{v \in W_0^*|v = -tS \circ Tv \text{ for some } t \in [0,1]\}$ and taking into account the boundedness of Tv $(v \in B)$, S and t $(t \in [0,1])$, we can see that B is bounded in W_0^*. Consequently, there exists $R > 0$ such that

$$\|v\|_{W_0^*} < R \text{ for all } v \in B.$$

Therefore, if $v \notin B_R(0)$, then $v \notin B$, that is $v + tS \circ Tv \neq 0$ for all $t \in [0,1]$. In particular

$$v + tS \circ Tv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0,1],$$

where $B_R(0) = \{v \in W_0^*; \|v\|_{W_0^*} < R\}$ the open ball of W_0^* of centre 0 and radius R.
From Lemma **??** it follows that

$$I + S \circ T \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = L \circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + S \circ T$ is also bounded. We conclude that

$$I + S \circ T \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to W_0^*$ given by

$$H(t,v) := v + tS \circ Tv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem **??**, we get

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + S \circ Tv = 0.$$

We conclude that $u = Tv$ is a weak solution of (**??**).

References

[1] Alsaedi R. *Perturbed subcritical Dirichlet problems with variable exponents.* Electron. J. Differential Equations, 2016, **295**, 1–12.

[2] Applebaum D. *Lévy processes and stochastic calculus, Second edition.* Cambridge Studies in Advanced Mathematics, Vol. 116, Cambridge University Press, Cambridge, 2009.

[3] Azroul E., Benkirane A., Shimi M. *Eigenvalue problems involving the fractional $p(x)-$Laplacian operator.* Adv. Oper. Theory, 2019, **4 (2)**, 539–555.

[4] Bahrouni A., Ho K. Y. *Remarks on eigenvalue problems for fractional $p(\cdot)-$Laplacian.* Asymptot. Anal., 2021, **123 (1-2)**, 139–156.

[5] Bahrouni A., Rădulescu V. *On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent.* Discrete Contin. Dyn. Syst., 2018, **11**, 379–389.

[6] Berkovits J. *Extension of the Leray-Schauder degree for abstract Hammerstein type mappings.* J. Differential Equations, 2007, **234**, 289–310.

[7] Bisci G.M., Rădulescu V., Servadi R. *Variational methods for nonlocal fractional problems, With a foreword by Jean Mawhin.* Encyclopedia of Mathematics and its Applications, Vol. 162, Cambridge University Press, Cambridge, 2016.

[8] Bucur C., Valdinoci E. *Nonlocal diffusion and applications.* Lecture Notes of the Unione Matematica Italiana, Vol. 20, Springer, [Cham], Unione Matematica Italiana, Bologna, 2016.

[9] Caffarelli L. *Nonlocal diffusions, drifts and games.* Nonlinear partial differential equations, Abel Symp., Vol. 7: 37–52, Springer, Heidelberg, 2012.

[10] Chen Y., Levine S., Rao M. *Variable exponent, linear growth functionals in image processing.* SIAM J. Appl. Math., 2006, **66**, 1383–1406.

[11] Fan X.L., Zhao D. *On the Spaces* $\mathrm{L}^{p(x)}(\Omega)$ *and* $\mathrm{W}^{m,p(x)}(\Omega)$. J. Math. Anal. Appl., 2001, **263**, 424–446.

[12] Ho K., Kim Y. H. *A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p(\cdot)-$Laplacian.* Nonlinear Anal., 2019, **188**, 179–201.

[13] Ho K., Kim Y. H. *The concentration-compactness principles for $W^{s,p(\cdot,\cdot)}(\mathbb{R}^N)$ and application.* Adv. Nonlinear Anal., 2021, **10**, 816–848.

[14] Kaufmann U., Rossi J.D., Vidal R. *Fractional Sobolev spaces with variable exponents and fractional $p(x)-$Laplacians.* Electron. J. Qual. Theory Differ. Equ., 2017, **76**, 1–10.

[15] Kim I. S., Hong, S. J. *A topological degree for operators of generalized (S_+) type.* Fixed Point Theory and Appl., 2015, 2015:194.

[16] Kováčik O., Rákosník J. *On spaces* $\mathrm{L}^{p(x)}$ *and* $\mathrm{W}^{1,p(x)}$. Czechoslovak Math. J., 1991, **41**, 592–618.

[17] Zeidler E. *Nonlinear Functional Analysis and its Applications.* II/B: Nonlinear monotone Operators, Springer, New York, 1990.

[18] Zhang C., Zhang X. *Renormalized solutions for the fractional $p(x)-$Laplacian equation with L^1 data.* Nonlinear Anal., 2020, **190**, 111610.

[19] Zhao D., Qiang W.J., Fan X.L. *On generalized Orlicz spaces* $\mathrm{L}^{p(x)}(\Omega)$. J. Gansu Sci., 1996, **9 (2)**, 1–7.