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Delay differential equations

Observer-based controller design for delayed T-S fuzzy nonlinear systems

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Abstract. The issue of stabilization by an estimated state feedback for a family of Takagi-Sugeno fuzzy systems with nonlinear time delays is the focus of this research. It is intended that the delay remain constant. The closed-loop systems are found to be globally exponentially stable based on the Lyapunov-Krasovskii functionals. Even with modeling uncertainty and state delay, it is possible to design the controller and observer gains independently. Lastly, a numerical example is provided to demonstrate how the primary finding can be used.

Keywords: Time-delay fuzzy systems, controller, stabilization, separation principle.

1 Introduction

A nonlinear system's control design and stability analysis are challenging tasks. In actuality, the control system's controlled object is never linear. One of the primary areas of study for the control community is the stability and stabilization of systems, with or without delays. The theoretical definitions of asymptotic stability and stability in the sense of Lyapunov used in the majority of contributions are too restrictive when taking into account real-world problems: a system may be stable or asymptotically stable in theory, but in practice it is unstable because the stable domain or the domain of the desired attractor is too small; occasionally, a system's state may be mathematically unstable, but the system may oscillate close enough to this state for its performance to be acceptable. The state estimation problem is a crucial one for control theory. An observer's goal is to recreate a system's state using their understanding of the both the system's input and output. The Kalman filter or the Luenberger observer are commonly used to estimate the system state for linear systems. Extending the linear Kalman filter or the linear Luenberger observer to nonlinear systems is a popular method for estimating the state of nonlinear systems. Nevertheless, there aren't many state estimate findings for nonlinear systems with delayed states in the literature. These systems frequently show up when modeling systems that require data propagation and transit, like biological systems, chemical reactors, and communication systems. Complex nonlinear systems can occasionally be challenging to characterize using linearization or recognition approaches. Consequently, nonlinear objects are approximated using the so-called fuzzy model, which expresses the local linear input-output relationship of nonlinear systems via fuzzy implication. Since it enables the systematic application of linear matrix inequality (LMI) techniques and the parallel distributed compensation idea to complicated nonlinear systems, Takagi-Sugeno (T-S) fuzzy model-based control has drawn a lot of attention recently. The output feedback control problem can also be resolved with T-S fuzzy model-based control. The overall fuzzy model of the system is achieved by smoothly blending the local linear models together through the membership functions. Then, based on this fuzzy model, the control design is worked out by taking full advantage of the strength of modern linear control theory. Such models can approximate exactly a wide class of nonlinear systems. Hence it is important to study their stability ([1]-[5]) or the synthesis of stabilizing controllers in the case of systems with delay ([7]-[17], [23], [27], [28]). Since time delays frequently appear in many practical systems, such as chemical processes, and telecommunication systems, the control problem for time-delay systems has received considerable attention [14]. While there is an extensive literature on this topic most of the reported studies focus on linear time-delay systems [3], due to the difficulty of the stability analysis and controller design. The relative existing literature on the technique can be roughly divided into three types: static output feedback control [18], dynamic control [6], and observer-based control ([21],[26]). Therefore, fuzzy observer-based control is the most popular scheme because the state variables can be reconstructed by a T-S fuzzy observer. In our recent paper ([15]), under the fact that both the estimator dynamics and the state feedback dynamics are stable we propose a separation principle for Takagi-Sugeno fuzzy control systems with Lipschitz nonlinearities. The considered nonlinearities are Lipschitz or meets an integrability condition which have no influence on the LMI to prove the stability of the associated closed-loop system.

In this paper, we investigate the problem of global stabilization of a class of nonlinear time-delay Takagi-Sugeno fuzzy systems with constant delay. We use the parallel distributed compensation concept to propose state and output feedback controllers. Consequently, we establish global exponential stability of the closed-loop systems using suitable Lyapunov-Krasovskii functionals. After that, linear matrix inequality (LMI) issues are solved using the exponential stability requirements that have been derived. Even with modeling uncertainty and state delay, the controller and observer gains can be independently built using the newly discovered LMI methods.

This is how the remainder of the paper is structured. A summary of some first findings and a description of the system are provided in Section 2. Section 3 presents the main findings. To guarantee the nonlinear time-delay system's global exponential stability, linear state and output feedback controllers are first created. Section 4 then provides a simulation result that illustrates the efficacy of the suggested strategy.

2 Takagi-Sugeno fuzzy delayed system

It is challenging to construct precise mathematical models of the majority of physical systems due to their complexity and uncertainties. For small range motion, these systems' dynamics could exhibit either linear or nonlinear characteristics. Lyapunov's linearization technique is frequently used to create local linearized approximations and address the local dynamics of nonlinear systems. In other words, a collection of local mathematical models can be used to partition the complex system. Takagi and Sugeno have suggested a useful method for combining these models by building with the fuzzy inferences.

2.1 Preliminaries

Consider the following time-delay system:

$$\begin{cases} \dot{x} = G(x, x(t-\tau)) \\ x(\theta) = \phi(\theta), \theta \in [-\tau, 0] \end{cases} \tag{1}$$

where $\tau > 0$ denotes the time delay and $\phi \in \mathcal{C}$ is the initial function, where $\mathcal{C}$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0] \longrightarrow \mathbb{R}^n$ equipped with the supremum-norm:

$$\|\phi\|_\infty = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|$$

$\|.\|$ being the Euclidean-norm. The map $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth and satisfies $g(0,0) = 0$. The function segment x_t is defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$.

Definition 1 *[25] The zero solution of system* (1) *is said to be globally exponentially stable with a decay $\sigma > 0$, if there exists a positive real β such that, for any initial condition $\phi \in \mathcal{C}$, the following inequality holds:*

$$\|x(t)\| \leq \beta \|\phi\|_\infty e^{-\sigma t}; \;\; \textit{for all} \;\; t \geq 0.$$

Sufficient conditions for stability of time-delay systems are provided by the theory of Lyapunov-Krasovskii functionals [14], a generalization of the classical Lyapunov theory of ordinary differential equations [19]. The following theorem gives sufficient conditions to ensure that the origin of system (1) is globally exponentially stable [25].

Theorem 1 *If there exist positive numbers λ_1, λ_2 ϱ and a continuous differentiable functional $V : \mathcal{C} \longrightarrow \mathbb{R}_+$ such that:*

$$\lambda_1 \|x(t)\|^2 \leq V(x_t) \leq \lambda_2 \|x_t\|_\infty, \tag{2}$$

$$\dot{V}(x_t) + \varrho V(x_t) \leq 0, \tag{3}$$

then, the zero solution of (1) *is globally exponentially stable with the decay rate ϱ.*

2.2 Fuzzy control system

The T-S fuzzy model is given by:

Rule i : If $z_1(t)$ is F_{i1} and $z_2(t)$ is F_{i2} ... and $z_p(t)$ is F_{ip}, then

$$\dot{x} = A_i x + B_i u + \tilde{F}_i(x, x(t-\tau), u), \quad i = 1, ..., r \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $A_i(n,n)$ constant matrix, $B_i(n,m)$ matrix control input and the functions $\tilde{F}_i$ represent the delayed perturbations of each fuzzy subsystem for $i = 1, ..., r$. F_{ij} is the fuzzy set $(j = 1, 2, ..., p)$, $z(t) =^T (z_1(t), ..., z_p(t))$ is the premise variable vector associated with the system states and inputs and r is the number of fuzzy rules. Center of gravity defuzzification yields the output of fuzzy system:

$$\dot{x} = \frac{\sum_{i=1}^{r} w_i(z)(A_i x(t) + B_i u(t) + \tilde{F}_i(x, x(t-\tau), u))}{\sum_{i=1}^{r} w_i(z)}$$

where

$$w_i(z) = \prod_{j=1}^{p} F_{ij}(z_j)$$

and $F_{ij}(z_j)$ denotes the grade of the number ship function F_{ij}, corresponding to $z_j(t)$. Let $\mu_i(z)$ be defined as:

$$\zeta_i(z) = \frac{w_i(z)}{\sum_{i=1}^{r} w_i(z)}.$$

Then, the fuzzy system has the state-space form:

$$\dot{x} = \sum_{i=1}^{r} \zeta_i(z)(A_i x(t) + B_i u(t) + \tilde{F}_i(x, x(t-\tau), u))$$

Clearly,

$$\sum_{i=1}^{r} \zeta_i(z) = 1$$

and

$$\zeta_i(z) \geq 0 \ \ for \ \ i = 1, ..., r.$$

It is assumed that the T-S fuzzy system is as follows:

The pairs (A_i, B_i), $i = 1, ..., r$ are controllable. That is, the nominal fuzzy system is locally controllable.

Based on this assumption, a state feedback control gain K_i can be obtained by pole placement design or Ackerman's formula, such that each local dynamics is stably controlled. The representation of the global control input matrix, denoted by B, is in the form: $B = \sum_{i=1}^{r} \zeta_i B_i$. This means that the global control input matrix dominates the control performance. The design of the fuzzy controller can be taken as a linear state feedback control can defined as:

Rule i : If $z_1(t)$ is F_{i1} and $z_2(t)$ is F_{i2} ... and $z_p(t)$ is F_{ip}, then

$$u(t) = -K_i x(t), \quad i = 1, 2, ..., r,$$

where K_i is the local state feedback gain. Consequently, the defuzzified result is:

$$u(t) = -\sum_{i=1}^{r} \zeta_i(z) K_i x(t).$$

Taking $y = \sum_{i=1}^{r} \zeta_i(z) C_i x$, $y \in \mathbb{R}^q$ and C_i has an appropriate dimension and $\hat{y}$ defined by

$$\hat{y} = \sum_{i=1}^{r} \zeta_i(z) C_i \hat{x}.$$

In this case, an observer can be designed which has the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \zeta_i(z) \big(A_i \hat{x} + B_i u) - L_i(y - \hat{y}).$$

We wish to find a gain matrix L such that the error $e(t) = \hat{x}(t) - x(t)$ converge exponentially to zero as t goes to infinity. We assume that the following rules are given concerning the observer of each subsystem.

Rule l : If $z_i(t)$ is F_{i1} and $z_2(t)$ is F_{i2} ... and $z_p(t)$ is F_{ip}, then

$$\dot{\hat{x}} = \big(A_i \hat{x} + B_i u\big) - L(y - \hat{y}), \quad i = 1, 2, ..., r.$$

It suffices to take $L = \sum_{i=1}^{r} \zeta_i(z) L_i$. So,

$$\dot{\hat{x}} = \sum_{i=1}^{r} \zeta_i(z) \big(A_i \hat{x} + B_i u\big) - \sum_{i=1}^{r} \zeta_i(z) L_i (y - \hat{y}).$$

When we want to control systems using an estimated controller that is available via an observer design, the idea of observers for fuzzy systems is crucial because the building of the closed-loop fuzzy system does not guarantee its stability. The following assumptions on the matrices $(A_i; B_i; C_i)$, can now be used to summarize this fact in order to demonstrate a stabilization result using an observer. First, $(A_i; B_i)$ are controllable and second $(C_i; A_i)$ are observable for $i = 1, 2, ..., r$. The state-feedback gain K_i and the state-observer gain L_i can be obtained by formulating a synthesis tool based on some LMIs.

The aim of this paper is to design an estimated state feedback to stabilize the origin of the following time-delay perturbed fuzzy system depending on a parameter:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \zeta_i(z)\big(A_i x + B_i u + \tilde{F}_i(x, x(t-\tau), u(t))\big) \\ y(t) = \sum_{i=1}^{r} \zeta_i(z) C_i x(t) \end{cases} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^q$ is the output and τ is a positive known scalar that denotes the time delay affecting the state variables. $r \geq 2$ is the number of If-then rules, and F_{ij} are the fuzzy sets $(j = 1, ..., p)$. $z_1, ..., z_p$ are the premise variables which are supposed to be measurable. We set $z = [z_1, ..., z_p]$.

It is assumed that $\zeta_i(z) \geq 0$, for all $i = 1, ..., r$ and $\sum_{i=1}^{r} \zeta_i(z) = 1$, for all $t \geq 0$.

The matrices A_i, B_i, C_i are of appropriate dimension and the perturbed term

$$\tilde{F}_i(x, x(t-\tau), u(t)) = [(\tilde{F}_{i_1}(x(t), x(t-\tau), u(t)), ..., \tilde{F}_{i_n}(x(t), x(t-\tau), u(t))]^T,$$

where for all $i \in \{1, ..., r\}$ and $j \in \{1, ..., n\}$ the function $\tilde{F}_{i_j} : \mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$ are smooth such and satisfy the following assumption:
$\mathcal{A}_1$: There exist constants $\gamma_{i_1} > 0$ and $\gamma_{i_2} > 0$ such that

$$\|\tilde{F}_i(x(t), x(t-\tau), u(t))\| \leq \gamma_{i_1}\|x(t)\| + \gamma_{i_2}\|x(t-\tau)\|.$$

The parallel distributed compensation (PDC) principle serves as the foundation for a number of published findings pertaining to fuzzy system control ([20], [22], [24]). It is expected that the fuzzy system can be controlled locally. The fuzzy controller's design uses a linear state feedback control in the subsequent section and has the same antecedent as the fuzzy system. The controller is defined as:

$$u = -\sum_{i=1}^{r} \zeta_i(z) K_i x \tag{6}$$

where $K_i \in \mathbb{R}^{n\times m}$ is the gain matrix.

To guarantee global exponential stability of the nonlinear time-delay system (5) under Assumption $\mathcal{A}_1$, we shall provide time-delay independent conditions in this work. We shall suggest a state controller and an output feedback controller using parallel distributed compensation. Thus, we will create conditions for linear matrix inequalities (LMIs). It is important to note that when the nonlinearities satisfy a linear growth condition or are Lipschitz, several results on the stabilization of the nonlinear system (5) proposing linear matrix inequality conditions were developed ([15], [16]).
Throughout the paper, the time argument is omitted and the delayed state vector $x(t-\tau)$ is noted by x^τ. A^T means the transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and minimal eigenvalue of a matrix A, respectively and I is the matrix identity.

3 Main results

We first establish a stabilization result and then take into consideration a fuzzy observer based on a fuzzy controller in order to stabilize the fuzzy system via a state estimated feedback law. It is commonly known that a stabilizing estimated feedback controller for linear systems is produced when an observer and a stabilizing state feedback are combined. The separation principle is the name given to this. However, it is unknown if the separation principle holds true because the fuzzy system under consideration is nonlinear. Additionally, keep in mind that a TS model with uncertainties does not make this separation concept any more accessible.

3.1 Global exponential stabilization

The state feedback controller is given by (6). Thus, the closed-loop system is given by:

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{r}\sum_{j=1}^{r}\zeta_i(z)\zeta_j(z)\big((A_i - B_iK_j)x + \tilde{F}_i(x,x^\tau,u)\big)\\ &= \sum_{i=1}^{r}\zeta_i^2 G_{ii}x + 2\sum_{i<j}\zeta_i\zeta_j G_{ij}x + \sum_{i=1}^{r}\zeta_i\tilde{F}_i(x,x^\tau,u), \qquad (7)\end{aligned}$$

where

$$G_{ii} = A_i - B_i K_i$$

and

$$G_{ij} = \frac{1}{2}(A_i - B_i K_j + A_j - B_j K_i).$$

Theorem 2 *Suppose that* $(\mathcal{A}_1)$ *hold and there exist symmetric and positive definite matrices* P *and positive constant* ϱ *such that the following inequalities hold,*

$$G_{ii}^T P + P G_{ii} \leq -I, \quad i = 1, ..., r,$$

$$G_{ij}^T P + P G_{ij} \leq -I, \quad 1 \leq i < j \leq r,$$

and

$$P < a(\varrho) I,$$

where

$$a(\varrho) = \frac{1}{2} \min\Big(\frac{1}{2\varrho + 2\gamma_1 + \gamma_2}, \frac{e^{-2\varrho\tau}}{2\gamma_2}\Big),$$

$\gamma_1 = \sum_{i=1}^{r} \gamma_{i_1}$ *and* $\gamma_2 = \sum_{i=1}^{r} \gamma_{i_2}$, *then the fuzzy closed-loop system* $(7) - (6)$ *is globally exponentially stable.*

Proof 1 *Let us choose a Lyapunov-Krasovskii functional candidate as follows:*

$$V(x_t) = x^T P x + \frac{1}{2} \int_{-\tau}^{0} e^{2\varrho s} \|x(s+t)\|^2 ds. \tag{8}$$

First, it is easy to see that

$$\begin{aligned} V(x_t) &\leq \lambda_{\max}(P)\|x\|^2 + \frac{1}{2} \int_{-\tau}^{0} e^{2\varrho s} \|x_t(s)\|^2 ds \\ &\leq \lambda_{\max}(P)\|x_t\|_\infty^2 + \frac{1}{2} \int_{-\tau}^{0} e^{2\varrho s} \|x_t\|_\infty^2 ds \\ &\leq \Big(\lambda_{\max}(P) + \frac{1}{4\varrho}\Big)\|x_t\|_\infty^2 \end{aligned}$$

and

$$V(x_t) \geq \lambda_{\min}(P)\|x(t)\|^2.$$

Thus condition (2) *of Theorem* (1) *is satisfied with*

$$\lambda_1 = \lambda_{\min}(P) \quad \textit{and} \quad \lambda_2 = \lambda_{\max}(P) + \frac{1}{4\varrho}.$$

One can see that by using the change of variable $u = t + s$ *one gets*

$$\int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 = \int_{t-\tau}^{t} \|x(u)\|^2 du.$$

Therefore, the time derivative of $V(x_t)$ *along the trajectories of system* (7) *is*

$$\dot{V}(x_t) = \sum_{i=1}^{r} \zeta_i^2 x^T(G_{ii}^T P + PG_{ii})x + 2\sum_{i<j}^{r} \zeta_i\zeta_j x^T(G_{ij}^T P + PG_{ij})x + 2x^T P \sum_{i=1}^{r} \zeta_i \tilde{F}_i(x, x^\tau, u)$$
$$+ \frac{1}{2}\|x\|^2 - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 - \varrho \int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds.$$

On the one hand, we have

$$x^T(G_{ii}^T P + PG_{ii})x \le -\|x\|^2,\ i = 1, 2, ..., r,$$

and

$$x^T(G_{ij}^T P + PG_{ij})x \le -\|x\|^2,\ 1 \le i < j \le r.$$

It follows that,

$$\dot{V}(x_t) \le -\|x\|^2 \sum_{i=1}^{r}\sum_{i=1}^{r} \zeta_i\zeta_j + 2x^T P \sum_{i=1}^{r} \zeta_i \tilde{F}_i(x, x^\tau, u)$$
$$+ \frac{1}{2}\|x\|^2 - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 - \varrho \int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds.$$

Since,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \zeta_i\zeta_j = 1,$$

then,

$$\begin{aligned}\dot{V}(x_t) \le\ & -\|x\|^2 + 2x^T P \sum_{i=1}^{r} \zeta_i \tilde{F}_i(x, x^\tau, u) + \frac{1}{2}\|x\|^2 - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 \\ & - \varrho \int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds \\ \le\ & -\frac{1}{2}\|x\|^2 + 2\|x\|\|P\| \sum_{i=1}^{r} \zeta_i \|\tilde{F}_i(x, x^\tau, u)\| - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 \\ & - \varrho \int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds.\end{aligned}$$

On the other hand by assumption $\mathcal{A}_1$*, we obtain*

$$\begin{aligned}\dot{V}(x_t) &\leq -\frac{1}{2}\|x\|^2 - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 + 2\|x\|\|P\|\sum_{i=1}^{r}\zeta_i\Big(\gamma_{i_1}\|x\| + \gamma_{i_2}\|x^\tau\|\Big) \\ &- \varrho\int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds \\ &\leq -\frac{1}{2}\|x\|^2 - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 + 2\|x\|\|P\|\sum_{i=1}^{r}\Big(\gamma_{i_1}\|x\| + \gamma_{i_2}\|x^\tau\|\Big) \\ &- \varrho\int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds \\ &\leq -\frac{1}{2}\|x\|^2 - \frac{e^{-2\varrho\tau}}{2}\|x^\tau\|^2 + 2\|P\|\gamma_1\|x\|^2 + 2\|P\|\gamma_2\|x\|\|x^\tau\| \\ &- \varrho\int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds.\end{aligned}$$

Using the fact that

$$2\|x\|\|x^\tau\| \leq \|x\|^2 + \|x^\tau\|^2,$$

we deduce that

$$\begin{aligned}\dot{V}(x_t) &\leq -\Big[\frac{1}{2} - 2\|P\|\gamma_1 - \|P\|\gamma_2\Big]\|x\|^2 \\ &- \Big[\frac{e^{-2\varrho\tau}}{2} - \|P\|\gamma_2\Big]\|x^\tau\|^2 \\ &- \varrho\int_{-\tau}^{0} e^{2\varrho s}\|x(s+t)\|^2 ds.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\dot{V}(x_t) + 2\varrho V(x_t) &\leq 2\varrho x^T P x - \Big[\frac{1}{2} - 2\|P\|\gamma_1 - \|P\|\gamma_1\Big]\|x\|^2 \qquad (9) \\ &- \Big[\frac{e^{-2\varrho\tau}}{2} - \|P\|\gamma_2\Big]\|x^\tau\|^2 \\ &\leq -c\|x\|^2 - d\|x^\tau\|^2, \qquad (10)\end{aligned}$$

where

$$c = \frac{1}{2} - 2\varrho\|P\| - 2\|P\|\gamma_1 - \|P\|\gamma_2$$

and

$$d = \frac{e^{-2\varrho\tau}}{2} - \|P\|\gamma_2.$$

Since,

$$P < a(\varrho)I,$$

then $c > 0$ *and* $d > 0$. *Consequently,*

$$\dot{V}(x_t) + 2\varrho V(x_t) \leq 0.$$

Thus, condition (3) *of theorem (2.1) is satisfied. Therefore, the system* $(7)-(6)$ *is globally exponentially stable.*

Corollary 1 *Suppose that* $(\mathcal{A}_1)$ *hold and there exists symmetric and positive definite matrices* P *such that the following inequalities hold,*

$$G_{ii}^T P + PG_{ii} \leq -I, \;\; i,j = 1,...,r,$$

$$G_{ij}^T P + PG_{ij} \leq -I, \;\; 1 \leq i < j \leq r,$$

and

$$P < bI, \tag{11}$$

where

$$b = \frac{1}{2\Big(2\gamma_1 + \gamma_2\Big)},$$

$\gamma_1 = \sum_{i=1}^r \gamma_{i_1}$ *and* $\gamma_2 = \sum_{i=1}^r \gamma_{i_2}$, *then the fuzzy closed-loop system* $(7) - (6)$ *is globally exponentially stable.*

Proof 2 *From theorem* 2, *we have the closed loop system* (7) *is globally exponentially stable if* $\frac{1}{2} - 2\varrho\|P\| - 2\|P\|\gamma_1 - \|P\|\gamma_2 > 0$ *and* $\frac{e^{-2\varrho\tau}}{2} - \|P\|\gamma_2 > 0$. *therefore, the constant* ϱ *should satisfies*

$$\varrho < \frac{\alpha}{2\|P\|} \;\; and \;\; -2\varrho\tau > \ln\Big(2\|P\|\gamma_2\Big), \tag{12}$$

where $\alpha = \frac{1}{2} - 2\|P\|\gamma_1 - \|P\|\gamma_2$. *Since we have*

$$P < bI,$$

then, $\alpha > 0$. *Thus,* $1 - 2\|P\|\gamma_2 > 4\|P\|\gamma_1 > 0$. *It follows that*

$$\ln\Big(2\|P\|\gamma_2\Big) < 0.$$

Consequently,

$$\varrho < \frac{1}{2\tau}\ln\Big(2\|P\|\gamma_2\Big).$$

Just take

$$\varrho = \frac{1}{2}\min\Big[\frac{\alpha}{2\|P\|}, \frac{1}{2\tau}\ln\Big(2\|P\|\gamma_2\Big)\Big] > 0.$$

Since sensing devices cannot access the state variables and transducers are either unavailable or prohibitively expensive, the physical state variables of systems are either completely or partially unavailable for measurement in many real-world control situations. Control schemes based on observers should be developed in these situations in order to estimate the state. Newt, for the perturbed fuzzy system, an observer-based controller is suggested to stabilize the closed-loop dynamic. by fusing the state fuzzy feedback control rule with a Luenberger-like state fuzzy observer.

3.2 Observer based controller

It is demonstrated that T-S models with a PDC control law exhibit exponential stabilization. This control law's primary characteristic is that it is based on the same assumptions as the T-S model. Conditions of stabilization with a fuzzy observer exist if the state cannot be measured, but they assume that the premises can be measured. The system that we suggest is as follows:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \zeta_i(z)\big(A_i\hat{x} + B_i u - L_i(y - \hat{y})\big), \tag{13}$$

Where $\hat{y}$ given by $\hat{y} = \sum_{i=1}^{r} \zeta_i(z) C_i \hat{x}$ and the estimated feedback controller is given by

$$u = -\sum_{i=1}^{r} \zeta_i K_i \hat{x}. \tag{14}$$

We wish to find L_i, $i = 1, ..., r$ such that $e = x - \hat{x}$ converges to zero exponentially as t tends to infinity.
Subtracting (7) from (13), we have the system error

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r} \zeta_i(z)\zeta_j(z)(A_i - L_j C_i)e + \sum_{i=1}^{r} \zeta_i \tilde{F}_i(x, x^{\tau}, u). \tag{15}$$

Thus,

$$\dot{e} = \sum_{i=1}^{r} \zeta_i^2 \Upsilon_{ij} e + 2\sum_{i<j} \zeta_i\zeta_j(z)\Upsilon_{ij} e + \sum_{i=1}^{r} \zeta_i \tilde{F}_i(x, x^{\tau}, u),$$

where

$$\Upsilon_{ii} = A_i - L_i C_i,$$

and

$$\Upsilon_{ij} = \frac{1}{2}(A_i - L_jC_i + A_j - L_jC_i).$$

Then let consider the following theorem.

Theorem 3 *Suppose that* $(\mathcal{A}_1)$ *hold and there exist symmetric and positive definite matrices* $\tilde{P}$ *and* $\tilde{Q}$ *and positive constant* ϱ *such that the following inequalities hold,*

$$\Upsilon_{ii}^T\tilde{P} + \tilde{P}\Upsilon_{ii} \le -I,\ i = 1, ..., r,$$
$$\Upsilon_{ij}^T\tilde{P} + \tilde{P}\Upsilon_{ij} \le -I,\ 1 \le i < j \le r,$$

and

$$\tilde{P} < \tilde{a}(\varrho)I,$$

where

$$\tilde{a}(\varrho) = \frac{1}{2\big(2\varrho + \gamma_1 + \gamma_2\big)},$$

$\gamma_1 = \sum_{i=1}^r \gamma_{i_1}$ *and* $\gamma_2 = \sum_{i=1}^r \gamma_{i_2}$*, then the closed loop system* $(7)-(14)$*is globally exponentially stable.*

Proof 3 *Let us choose a Lyapunov-Krasovskii functional candidate as follows*

$$V(e_t) = e^T\tilde{P}e + \frac{1}{2}\int_{-\tau}^{0} e^{2\varrho s}\|e(s+t)\|^2 ds.$$

As in the proof of theorem 1, we have

$$\lambda_{\min}(\tilde{P})\|e(t)\| \le W(e_t) \le \Big(\lambda_{\min}(\tilde{P}) + \frac{1}{4\varrho}\Big)\|e_t\|_\infty.$$

The time derivative of $W(e_t)$ *along the trajectories of system* (13) *is given by*

$$\dot{W}(e) = \sum_{i=1}^r \zeta_i^2 e^T(\Upsilon_{ii}^T\tilde{P} + \tilde{P}\Upsilon_{ii})e + 2\sum_{i<j}\zeta_i\zeta_j e^T(\Upsilon_{ij}^T\tilde{P} + \tilde{P}\Upsilon_{ij})e$$
$$+ 2e^T\tilde{P}\sum_{i=1}^r \zeta_i\tilde{F}_i(x, x^\tau, u) + \frac{1}{2}\|e\|^2 - \frac{e^{-2\varrho\tau}}{2}\|e^\tau\|^2 - \varrho\int_{-\tau}^{0} e^{2\varrho s}\|e(s+t)\|^2 ds.$$

On the one hand, we have

$$e^T(\Upsilon_{ii}^T\tilde{P} + \tilde{P}\Upsilon_{ii})e \le -\|e\|^2,\ i = 1, ..., r,$$

and

$$e^T(\Upsilon_{ij}^T\tilde{P} + \tilde{P}\Upsilon_{ij})e \le -\|e\|^2,\ 1 < i < j < r.$$

Then, one gets

$$\dot{W}(e_t) \leq -\|e\|^2 \sum_{i=1}^{r} \sum_{i=1}^{r} \zeta_i \zeta_j + 2\|e\| \|\tilde{P}\| \sum_{i=1}^{r} \zeta_i \|\tilde{F}_i(x, x^\tau, u)\| + \frac{1}{2}\|e\|^2 - \frac{e^{-2\varrho\tau}}{2}\|e^\tau\|^2 - \varrho \int_{-\tau}^{0} e^{2\varrho s} \|e(s+t)\|^2 ds.$$

It follows that

$$\dot{W}(e_t) \leq -\frac{1}{2}\|e\|^2 + 2\gamma_1 \|\tilde{P}\| \|e\| \|x\| + 2\gamma_2 \|\tilde{P}\| \|e\| \|x^\tau\| - \frac{e^{-2\varrho\tau}}{2}\|e^\tau\|^2 - \varrho \int_{-\tau}^{0} e^{2\varrho s} \|e(s+t)\|^2 ds.$$

Using the fact that

$$2\|e\| \|x\| \leq \|e\|^2 + \|x\|^2 \ \ \textit{and} \ \ 2\|e\| \|x^\tau\| \leq \|e\|^2 + \|x^\tau\|^2,$$

we deduce that

$$\dot{W}(e_t) \leq -\Big[\frac{1}{2} - \gamma_1 \|\tilde{P}\| - \gamma_2 \|\tilde{P}\|\Big] \|e\|^2 - \frac{e^{-2\varrho\tau}}{2}\|e^\tau\|^2 + \gamma_1 \|\tilde{P}\| \|x\|^2 + \gamma_2 \|\tilde{P}\| \|x^\tau\|^2 - \varrho \int_{-\tau}^{0} e^{2\varrho s} \|e(s+t)\|^2 ds. \tag{16}$$

Let

$$U(x_t, e_t) = \eta W(e_t) + V(x_t),$$

where V is given by (8). *Using* (9) *and* (16), *we get*

$$\begin{aligned} \dot{U}(x_t, e_t) + 2\varrho U(x_t, e_t) \leq & -\eta \Big[\frac{1}{2} - \gamma_1 \|\tilde{P}\| - \gamma_2 \|\tilde{P}\|\Big] \|e\|^2 \\ & - \Big[c - \eta \gamma_1 \|\tilde{P}\|\Big] \|x\|^2 \\ & - \Big[d - \eta \gamma_2 \|\tilde{P}\|\Big] \|x^\tau\|^2. \end{aligned}$$

Finally, we select η such that

$$\eta < \min\Big(\frac{c}{\gamma_1 \|\tilde{P}\|}, \frac{d}{\gamma_2 \|\tilde{P}\|}\Big).$$

to get

$$\dot{U}(x_t, e_t) + 2\varrho U(x_t, e_t) \leq 0.$$

Therefore, the closed-loop system (7) – (14) *is globally exponentially stable.*

Corollary 2 *Suppose that* $(\mathcal{A}_1)$ *hold and there exist symmetric and positive definite matrices* $\tilde{P}$ *and* $\tilde{Q}$ *and positive constant* ϱ *such that the following inequalities hold,*

$$\Upsilon_{ii}^T \tilde{P} + \tilde{P}\Upsilon_{ii} \leq -I,\ i = 1, ..., r,$$
$$\Upsilon_{ij}^T \tilde{P} + \tilde{P}\Upsilon_{ij} \leq -I,\ 1 \leq i < j \leq r,$$

and

$$\tilde{P} < \tilde{b}I, \tag{17}$$

where

$$\tilde{b} = \frac{1}{2\big(\gamma_1 + \gamma_2\big)},$$

$\gamma_1 = \sum_{i=1}^{r} \gamma_{i_1}$ *and* $\gamma_2 = \sum_{i=1}^{r} \gamma_{i_2}$, *then the system error* $(7) - (14)$ *is guaranteed to globally exponentially stable.*

Proof 4 *The fact that the LMI inequality* $\tilde{P} < \tilde{a}(\varrho)I$, *implies that*

$$\varrho < \frac{\tilde{c}}{2}.$$

where $\tilde{c} = \frac{1}{2} - \gamma_1\|\tilde{P}\| - \gamma_2\|\tilde{P}\|$. *Since,* $\tilde{P} < \tilde{b}I$, *then* $c > 0$. *Therefore, in view of* (12), *we choose*

$$\varrho = \frac{1}{2}\min\Big[\frac{\tilde{c}}{2}, \frac{\alpha}{2\|P\|}, \frac{1}{2\tau}\ln\Big(2\|P\|\gamma_2\Big)\Big] > 0.$$

Remark 1 *The conditions* (11) *and* (17) *do not depend on the delay* τ.

We will then look at an example that presents an observer-based controller for a delay fuzzy system in order to demonstrate the main result's application. Based on two issues pertaining to the fuzzy controller's idea and the fuzzy observer's design, closed-loop stability is ensured.

4 Numerical example

Consider the following nonlinear fuzzy planar system,

$$\begin{cases} \dot{x}_1 = x_2 + 0.2x_3\sin(x_2)\sin(x_3) + 0.2x_3(t-\tau)\cos(u) \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \\ y = x_1 \end{cases} \tag{18}$$

Now one can represent exactly the system by the following two-rule fuzzy model:

Rule 1 : If z is F_{11} then

$$\begin{cases} \dot{x}(t) = A_1 x + B_1 u + \tilde{F}_1(x, x^\tau, u) \\ y(t) = C_1 x, \end{cases}$$

Rule 2 : If z is F_{21} then

$$\begin{cases} \dot{x}(t) = A_2 x + B_2 u + \tilde{F}_2(x, x^\tau, u) \\ y(t) = C_2 x, \end{cases}$$

where

$$z = \sin(x_2),$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\tilde{F}_1(x, x^\tau, u) = 0.2 x_3 \sin(x_2) \sin(x_3) \cos(u) + \frac{1}{0.4} x_3(t-\tau) \sin(u),$$

$$\tilde{F}_2(x, x^\tau, u) = -0.2 x_3 \sin(x_2) \sin(x_3) \cos(u) + \frac{1}{0.4} x_3(t-\tau) \sin(u),$$

$$F_{11} = \frac{\sin(x_2) + 1}{2} \text{ and } F_{21} = \frac{1 - \sin(x_2)}{2}.$$

We define the membership functions for rule 1 and 2 as:

$$\zeta_1(t) = \frac{1 - \sin(x_2(t))}{2} \text{ and } \zeta_2(t) = 1 - \zeta_1(t).$$

It is easy to check that system (18) satisfies Assumption $\mathcal{A}_1$ with $\gamma_{i_1} = \gamma_{i_2} = 0.2$, $i = 1, 2$. Using an LMI optimization algorithm, we obtain:

$$P = \begin{bmatrix} 1.5145 & 0.7341 & 0.0265 \\ 0.7341 & 1.4339 & 0.0548 \\ 0.0265 & 0.0548 & 0.0446 \end{bmatrix},$$

and the following feedback gains

$$K_1 = \begin{bmatrix} 17.4927 & 22.4490 & 8.5714 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} 78.1250 & 62.500 & 12.5000 \end{bmatrix}.$$

Now concerning the observer, let's suppose the following fuzzy observer rules:
Rule 1: If z is F_{11} then $\begin{cases} \dot{\hat{x}} = A_1\hat{x} + B_1u - L_1(y - \hat{y}) \\ \hat{y} = C_1\hat{x} \end{cases}$
Rule 2: If z is F_{21} then $\begin{cases} \dot{\hat{x}} = A_2\hat{x} + B_2u - L_2(\hat{y} - y) \\ \hat{y} = C_2\hat{x} \end{cases}$
Then, we obtain the following observer gain:

$$L_1 = \begin{bmatrix} -34.9854 & -34.9854 & -34.9854 \end{bmatrix}^T \text{ and } L_2 = \begin{bmatrix} -34.9854 & -34.9854 & -34.9854 \end{bmatrix}^T$$

and the positive symmetric definite matrices:

$$\tilde{P} = \begin{bmatrix} 0.8079 & -0.3374 & -0.4560 \\ -0.3374 & 0.7030 & -0.3425 \\ -0.4560 & -0.3425 & 0.7890 \end{bmatrix},$$

For the numerical simulation results given in Figure 1, the constant delay τ is equal to 1.

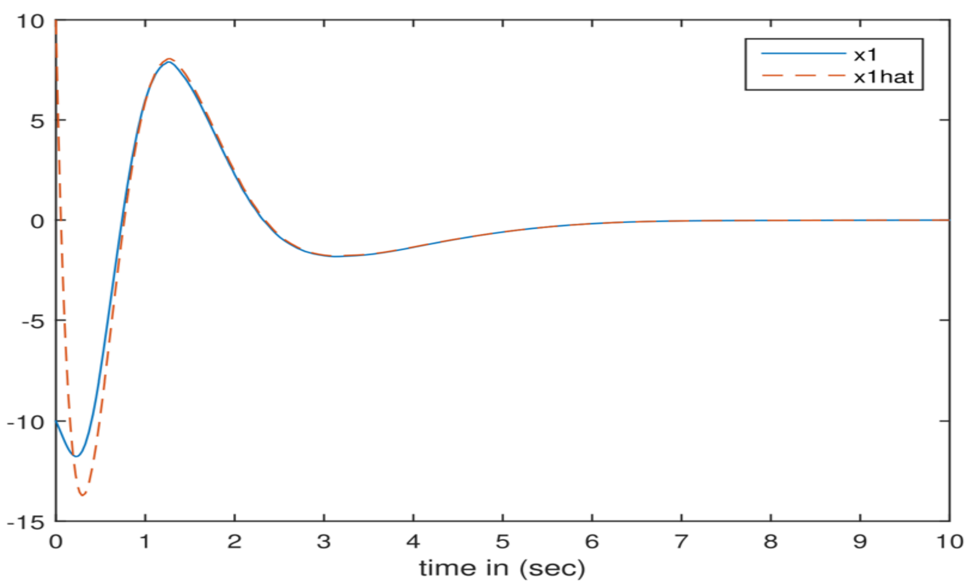

Figure 1: The time evolution of the state $x_1(t)$ and its estimate $\hat{x}_1(t)$

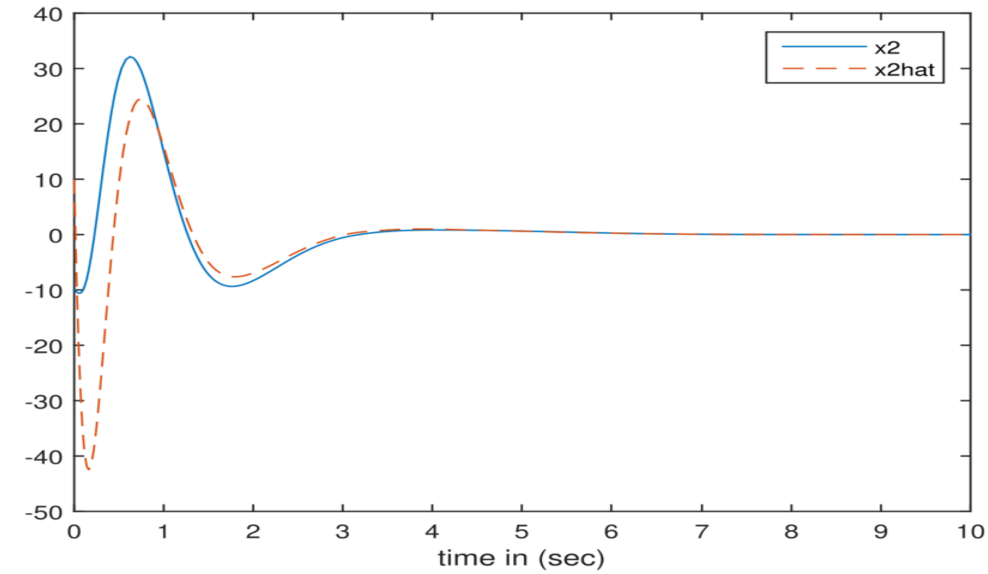

Figure 2: The time evolution of the state $x_2(t)$ and its estimate $\hat{x}_2(t)$

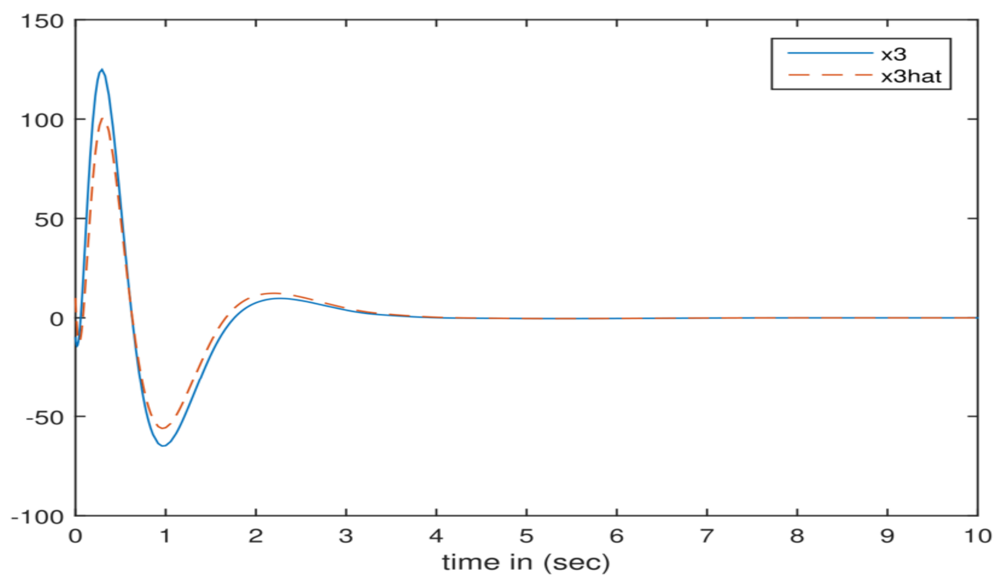

Figure 3: The time evolution of the state $x_3(t)$ and its estimate $\hat{x}_3(t)$

5 Conclusion

We have introduced state and estimated feedback controllers for a certain type of nonlinear time-delay fuzzy systems in this paper. To guarantee the global exponential stability of the resulting closed-loop systems, we have deduced delay-independent requirements. We have demonstrated that both the observer and the controller can be created independently using the conventional LMIs conditions, as there is a separation concept accessible. A computer simulation of a theoretical case demonstrates the efficacy of the suggested hypothesis.

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Conflict of Interest: The authors declare that they have no conflict of interest.

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