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Filtration and identification

Local parameter identifiability of an infinite-dimensional parameter in discrete linear dynamical systems

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Abstract. In this paper, we study the problem of local parameter identifiability for discrete linear dynamical systems with discrete infinite-dimensional parameter. We establish conditions under which the system is locally parameter identifiable for any parameter and initial data, i.e., parameters of the system can be uniquely determined in a neighborhood of a fixed parameter by observing functions of trajectories. Sufficient conditions under which the system is not locally parameter identifiable are also obtained. Two cases of a parameter are considered: the case where values of parameters are restricted by linear subspaces and the case where parameters are arbitrary. For the last case, the genericity of the property of local identifiability is proved.

Keywords: local parameter identifiability, discrete dynamical system, residual set, genericity.

1. Introduction. The problem of parameter identification consists of determining parameters of a dynamical system based on observations of trajectories or functions of trajectories. Parameter identification is an important part of the applied theory of dynamical systems. Within the framework of this problem, one of the key concepts is the property of local parameter identifiability.

This property means that parameter of the system can be uniquely determined by observations of trajectories if the parameter belong to some neighborhood of a fixed parameter.

The property of local parameter identifiability for systems of differential equations was studied in detail in the case of a finite-dimensional parameter in the monograph [1]. In the studies on the topic, the case of a finite-dimensional parameter was mainly studied. In the paper [2], sufficient conditions for local parameter identifiability of a parameter-function in a system of differential equations by observing a solution at the endpoint of a fixed interval were obtained. The paper [3] develops the idea of the previous article: a class of systems of differential equations has been specified such that the parameter-function can be uniquely determined by observing a solution at a finite number of points of a fixed interval. The paper [4] discusses several formulations of the local parameter identifiability problem for discrete-time dynamical systems. Sufficient conditions for local parameter identifiability of a parameter-sequence in the general formulation are obtained for the case of observing a trajectory at all points of the trajectory and at an arbitrary countable set of points; the case of a linearly perturbed diffeomorphism in a neighborhood of a hyperbolic set was considered.

We note the paper [5] in which dynamical systems of the type "input–output" are considered. The authors proved that the property of local parameter identifiability for such systems with finite-dimensional parameter is generic, i.e., almost all systems of this type are locally parameter identifiable. In the paper [3], the genericity of the local parameter identifiability property is established for the case of linear dependence of a system on a parameter-function.

In this paper, we study the problem of local parameter identifiability for the discrete-time linear dynamical system in the space $\mathbb{R}^n$,

$$x_{k+1} = A_k x_k + B_k p_k,$$

depending on a parameter-sequence $(p_k \in \mathbb{R}^l)_{k\geq 0}$. We assume that observations are linear functions of x_k.

In Section 2, we consider parameters p_k belonging to linear subspaces of the space $\mathbb{R}^l$ and find sufficient conditions for local parameter identifiability and find sufficient conditions for the absence of this property.

In Section 3, we consider the same system but do not restrict parameter values to specific subspaces and assume that $p_k \in \mathbb{R}^l$. We provide sufficient conditions for local parameter identifiability and obtain one more sufficient condition fro which the system is not locally parameter identifiable.

In Section 4, we prove that for the case where $p_k \in \mathbb{R}^l$ locally parameter identifiable systems are generic, i.e., in certain sense almost all systems possess this property.

2. The case of parameter belonging to a linear subspace. Consider a discrete-time linear dynamical system depending on parameter:

$$x_{k+1} = A_k x_k + B_k p_k, \quad k \geq 0, \tag{1}$$

where $x_k \in \mathbb{R}^n$, $A_k \in M_{n\times n}$, $B_k \in M_{n\times l}$, $M_{q\times r}$ is the normed space of matrices of size $q \times r$, and parameter p_k belongs to V_k that is a nonzero linear subspace of the space $\mathbb{R}^l$. We denote by $P = (p_k)_{k\geq 0}$ the parameter-sequence in the system. It is assumed that parameter P belongs to the space of bounded sequences with the norm $\|P\| = \sup\limits_{k\geq 0} |p_k|$.

We fix an initial vector $x_0 \in \mathbb{R}^n$ and observe vectors $y_k \in \mathbb{R}^m$ defined as follows:

$$y_k = C_k x_k, \quad k \geq 1, \tag{2}$$

where $C_k \in M_{m\times n}$. The corresponding values $y_k(P)$ and $x_k(P)$ depend only on the first k components of parameter P.

Definition 1 *System* (1)–(2) *is called locally parameter identifiable for parameter $P^0 = (p_k^0)$ and initial data $x_0 \in \mathbb{R}^n$ if there is an $\varepsilon > 0$ such that for any parameter P with $0 < \|P - P^0\| < \varepsilon$ there exists an index k for which*

$$y_k(P) \neq y_k(P^0).$$

For any $k \geq 0$ there is a matrix $D_k \in M_{l\times l}$ such that an image $\operatorname{im} D_k$ of the corresponding linear map $D_k : \mathbb{R}^l \to \mathbb{R}^l$ is equal to V_k. We will identify a matrix and the corresponding linear operator. Then

$$\operatorname{rank} D_k = \dim V_k.$$

Let us introduce the notation

$$\Delta P = P - P^0 = (\Delta p_k) = (p_k - p_k^0),$$

$$\Delta x_k(P) = x_k(P) - x_k(P^0),$$

$$\Delta y_k(P) = C_k \Delta x_k(P).$$

It is shown in [4] that for any $k \geq 0$

$$\Delta x_{k+1}(P) = A_k \cdots A_1 B_0 \Delta p_0 + \cdots + A_k B_{k-1} \Delta p_{k-1} + B_k \Delta p_k.$$

Hence,

$$\Delta y_{k+1} = C_{k+1}(A_k \cdots A_1 B_0 \Delta p_0 + \cdots + A_k B_{k-1} \Delta p_{k-1} + B_k \Delta p_k). \tag{3}$$

First we establish a sufficient condition for local parameter identifiability.

Theorem 1 *If*

$$\ker C_{k+1} B_k \cap \operatorname{im} D_k = \{0\} \tag{4}$$

for any $k \geq 0$, then system (1)–(2) *is locally parameter identifiable for any parameter P^0 and any initial data x_0.*

Proof. Suppose that system (1)–(2) is not locally parameter identifiable for some parameter P^0 and some initial data x_0. Then there is a sequence of parameters $P^s = (p_k^s) \to P^0$, $s \to \infty$, and $P^s \neq P^0$ such that

$$y_k(P^s) = y_k(P^0)$$

for any $s, k \geq 1$. If $k = 1$, then $\Delta y_1(P^s) = C_1 B_0 \Delta p_0^s = 0$ by formula (3); this is possible only if $\Delta p_0^s = 0$ for any s due to relation (4).

Let $\Delta p_k^s = 0$ for each $k = 0, 1, \ldots, v-1$, $v \geq 1$, and for all s. Then formula (3) implies that

$$\Delta y_{v+1}(P) = C_{v+1} B_v \Delta p_v^s = 0.$$

Applying the above reasoning, we can conclude that $\Delta p_v^s = 0$ for all s. Therefore, $\Delta p_k^s = 0$ for all k, s. This contradicts the fact that $P^s \neq P^0$. $\square$

Now we establish two sufficient conditions under which the system is not locally parameter identifiable.

Theorem 2 *Let one of the following conditions be satisfied:*

1. *There is a number k such that*

$$\ker B_k \cap \operatorname{im} D_k \neq \{0\};$$

2. $\operatorname{rank} D_k > \operatorname{rank} B_k$ *for some k.*

Then system (1)–(2) *is not locally parameter identifiable for any parameter P^0 and any initial data x_0.*

Proof. To get a contradiction, assume that system (1)–(2) is locally parameter identifiable for some parameter P^0 and some initial data x_0. Hence, there is a number $\varepsilon > 0$ such that for any P with $0 < \|\Delta P\| < \varepsilon$ one can find an index j for which $\Delta y_{j+1}(P) \neq 0$.

Suppose that condition 1 of the theorem is satisfied. If $\Delta p_i = 0$ for $i \neq k$, $0 < |\Delta p_k| < \varepsilon$, and $\Delta p_k \in \ker B_k \cap \operatorname{im} D_k$, then formula (3) implies that

$$\Delta y_{i+1}(P) = 0, \quad i < k,$$
$$\Delta y_{k+1}(P) = C_{k+1} B_k \Delta p_k = 0,$$
$$\Delta y_{i+1}(P) = C_{i+1} A_i \cdots A_{k+1} B_k \Delta p_k = 0, \quad i > k.$$

Hence, there is a P with $0 < \|\Delta P\| < \varepsilon$ for which $\Delta y_{j+1}(P) = 0$ for any j, which is a contradiction.

Let condition 2 be satisfied. Since $\operatorname{rank} D_k > \operatorname{rank} B_k$, we conclude that $l - \operatorname{rank} B_k + \operatorname{rank} D_k > l$. It follows from the rank–nullity theorem that

$$\dim \ker B_k + \dim \operatorname{im} D_k > l.$$

Hence, the sum $\ker B_k + \operatorname{im} D_k$ of the linear subspaces coincides with $\mathbb{R}^l$. Then $\dim(\ker B_k \cap \operatorname{im} D_k)$ is estimated as follows:

$$\dim \ker B_k + \dim \operatorname{im} D_k - \dim(\ker B_k + \operatorname{im} D_k) > l - l = 0.$$

Hence, $\ker B_k \cap \operatorname{im} D_k \neq \{0\}$, and we can refer to the proof of our theorem in the case where condition 1 is fulfilled. □

Comment. Let us comment "informal" meaning of conditions of Theorem 2. Condition 1 means that the space V_k (the image of D_k) contains "inessential" parameters such that some variations of these parameters do not influence the values x_i and hence y_i. Condition 2 means that we have "too many" parameters to be identified.

3. A particular case in which $V_k = \mathbb{R}^l$. In this section, we consider an important particular case of the dynamical system (1)–(2) studied in the previous section; now we assume that $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$, $A_k \in M_{n\times n}$, $B_k \in M_{n\times l}$, $C_k \in M_{m\times n}$, and parameters p_k belong to $\mathbb{R}^l$, i.e. $\dim V_k = \operatorname{rank} D_k = l$ for all k (for definiteness, we agree that D_k is the identity matrix for all k).

Let us reformulate Theorems 1 and 2 for the considered case.

Theorem 3 *If* $\operatorname{rank} C_{k+1} B_k = l$ *for all* k*, then system* (1)–(2) *is locally parameter identifiable for any parameter* P^0 *and any initial data* x_0.

Remark. Note that $\operatorname{rank} C_{k+1}B_k$ can be equal to l only if $l \leq \min(m, n)$.

Theorem 4 *If there is an index k such that* $\operatorname{rank} B_k < l$, *then system* (1)–(2) *is not locally parameter identifiable for any parameter P^0 and any initial data x_0.*

Now we obtain one more condition under which system (1)–(2) is not locally parameter identifiable for the case where $l = n$ (we study the case where the assumption of Theorem 4 is not satisfied but impose additional conditions not only on the matrices B_k and C_k but also on the matrices A_k).

Theorem 5 *Let $l = n$, assume that* $\operatorname{rank} C_{k+1} < n$ *for some k,* $\operatorname{rank} B_i = l$ *for all i, and there exist constants $M > 0$ and $\mu \in (0, 1)$ such that*

$$\|B_i\|, \|B_i^{-1}\| \leq M, \quad i \geq k,$$
$$\|A_i\| \leq \mu, \quad i \geq k,$$
$$\mu(1 + M^2) < 1.$$

Then system (1)–(2) *is not locally parameter identifiable for any parameter P^0 and any initial data x_0.*

Proof. Set $\Delta p_i = 0$ for $i < k$. Then

$$\Delta y_{i+1}(P) = 0, \quad i < k. \tag{5}$$

Since $\operatorname{rank} C_{k+1} < n = l$ and $\operatorname{rank} B_k = l = n$, for any $\varepsilon > 0$ there exists a $z_k \in \ker C_{k+1}B_k$ such that $0 < |z_k| < \varepsilon$. Set $\Delta p_k = z_k$. Then $0 < |\Delta p_k| < \varepsilon$ and

$$\Delta y_{k+1}(P) = C_{k+1}B_k\Delta p_k = 0$$

due to formula (3) and equalities (5).

Set

$$\Delta p_{k+i} = -B_{k+i}^{-1} \sum_{s=0}^{i-1} A_{k+i} \cdots A_{k+s+1} B_{k+s} \Delta p_{k+s} \tag{6}$$

for $i > 0$.

Formula (3) implies that

$$\Delta y_{k+i+1} = C_{k+i+1} \left(\sum_{s=0}^{i-1} A_{k+i} \cdots A_{k+s+1} B_{k+s} \Delta p_{k+s} + B_{k+i} \Delta p_{k+i} \right) = 0$$

for $i > 0$.

Let us show that $|\Delta p_{k+i}| < \varepsilon$ for $i > 0$. First we apply induction to prove the inequality

$$|\Delta p_{k+i}| \leq \mu M^2(\mu^{i-1}(1+M^2)^{i-1})\varepsilon, \quad i > 0. \tag{7}$$

If $i = 1$, then

$$|\Delta p_{k+1}| = |B_{k+1}^{-1}A_{k+1}B_k\Delta p_k| \leq \mu M^2\varepsilon.$$

Let inequalities (7) hold for $i = 1, 2, \ldots, \nu$; then it follows from (6) and (7) that

$$|\Delta p_{k+\nu+1}| \leq \mu M^2 \sum_{s=0}^{\nu} \mu^{\nu-s}|\Delta p_{k+s}| \leq \mu M^2\varepsilon\mu^\nu \left(1 + \sum_{s=1}^{\nu} M^2(1+M^2)^{s-1}\right). \tag{8}$$

Note that $(1+M^2)^\nu = M^2(1+M^2)^{\nu-1} + (1+M^2)^{\nu-1}$. Iteratively applying this equality, we show that

$$(1+M^2)^\nu = M^2(1+M^2)^{\nu-1} + \ldots + M^2(1+M^2) + M^2 + 1. \tag{9}$$

Therefore, comparing the right-hand side of inequality (8) and equality (9), we show that

$$|\Delta p_{k+\nu+1}| \leq \mu M^2\varepsilon\mu^\nu(1+M^2)^\nu,$$

which is inequality (7) with $i = \nu + 1$.

Since $\mu(1+M^2) < 1$, $\mu M^2 < 1 - \mu < 1$. Hence, it follows from estimate (7) that $|\Delta p_{k+i}| < \varepsilon$, $i > 0$, as required. $\square$

Comment. Conditions of Theorem 5 mean that trajectories of the "homogeneous" system

$$x_{k+1} = A_k x_k \tag{10}$$

converge to 0 exponentially fast as k grows, while the constant M estimating the norms of B_k and B_k^{-1} is not very large compared to the exponential rate of convergence of trajectories of system (10).

We show that in this case, it is possible to find arbitrarily small perturbations P of parameter-sequence P^0 for which the observed values $y_j(P)$ coincide with $y_j(P^0)$ for all j.

4. Genericity of the local parameter identifiability property. In this section, we show that, in a certain sense, almost all systems of type (1)–(2) are locally parameter identifiable for all parameters and initial data.

Denote by $\mathcal{G}$ the linear space of sequences of pairs of matrices

$$\mathcal{G} = \{((C_1, B_0), (C_2, B_1), \ldots) \, : \, (C_{k+1}, B_k) \in M_{m\times n} \times M_{n\times l}\}$$

with the norm $\|G\| = \sup\limits_{k\geq 0} \|G_k\|$, where $G_k = (C_{k+1}, B_k)$, $G = (G_0, G_1, \ldots)$ and $\|G_k\| = \|(C_{k+1}, B_k)\|$ is an arbitrary norm. Since $M_{m\times n} \times M_{n\times l}$ is a Banach space then the space $\mathcal{G}$ as a countable Cartesian product of copies of $M_{m\times n} \times M_{n\times l}$ with sup norm is also a Banach space (see [6]).

A subset of a topological set is called residual (or comeagre) if it is a countable intersection of dense open subsets. In literature, residual subsets are sometimes called sets of the second Baire category. A property of elements of a topological space X for which there exists a residual subset Y of X such that any element of Y has this property is called generic.

Since $\mathcal{G}$ is a complete metric space, it is a Baire space (see [7]), i.e., every residual subset of $\mathcal{G}$ is dense in $\mathcal{G}$.

Proposition 1 *Let $l \leq \min(n, m)$, then the set*

$$\mathcal{G}' = \{((C_1, B_0), (C_2, B_1), \ldots) \, : \, \operatorname{rank} C_{k+1} B_k = l, \ k \geq 0\}$$

is residual in $\mathcal{G}$.

Proof. Rank estimation of a matrix product gives us the following inequality:

$$\operatorname{rank} C_{k+1} B_k \leq \min(\operatorname{rank} C_{k+1}, \operatorname{rank} B_k) \leq \min(\min(m, n), \min(n, l)) = l.$$

Hence, the set $\mathcal{G}'$ is not empty since the product of matrices $m \times n$ and $n \times l$ in which all the entries on the main diagonals are equal to 1 and the remaining entries are equal to 0 has rank equal to l. Consider the set

$$\mathcal{G}_i = \{((C_1, B_0), (C_2, B_1), \ldots) \, : \, \operatorname{rank} C_{i+1} B_i = l\}$$

where $i \geq 0$ is fixed. We show that $\mathcal{G}_i$ is open and dense in $\mathcal{G}$. Consider the mapping

$$g : \mathcal{G} \to \mathbb{R}_{\geq 0}, \quad G = ((C_1, B_0), (C_2, B_1), \ldots) \mapsto |\kappa(C_{i+1} B_i)|^2,$$

where $\kappa(C_{i+1} B_i)$ is a vector composed of all $l \times l$ minors of the matrix $C_{i+1} B_i \in M_{m\times l}$ ordered in a fixed way and $|\cdot|$ is the Euclidean norm. The function g is continuous since it is a composition of the projection $G \mapsto G_i$ and the map $G_i \mapsto |\kappa(C_{i+1} B_i)|^2$ which is a polynomial depending on entries of the matrices C_{i+1}, B_i.

If $G \in \mathcal{G}_i$, then $g(G) > 0$. There is an open neighborhood U of the point $g(G)$ such that $0 \notin U$. Then $G \in g^{-1}(U) \subset \mathcal{G}_i$, i.e., the set $\mathcal{G}_i$ is open.

Let us prove the density of the set $\mathcal{G}_i$. Fix an arbitrary pair of matrices C^0_{i+1}, B^0_i and fix $G^0 \in \mathcal{G}_i$ with $G^0_i = (C^0_{i+1}, B^0_i)$. If $\operatorname{rank} C^0_{i+1} B^0_i = l$, then any neighborhood of G^0 has a nonempty intersection with $\mathcal{G}_i$.

Otherwise, if $\operatorname{rank} C^0_{i+1} B^0_i < l$, consider an arbitrary $\varepsilon > 0$ and matrices C_{i+1}, B_i such that $\operatorname{rank} C_{i+1} B_i = l$.

Define a family G^t, $t \in [0, 1]$, such that $G^t_k = G^0_k$ for all $t \in [0, 1]$, $k \neq i$, and

$$G^t_i = (tC_{i+1} + (1-t)C^0_{i+1}, tB_i + (1-t)B^0_i), \quad t \in [0, 1].$$

Then $g(G^t)$ is a polynomial in variable t which has a finite number of zeros since $g(G^1) = |\kappa(C_{i+1}B_i)|^2 > 0$.

Since $g(G^0) = 0$, there is a positive t_0 such that

$$t_0 < \frac{\varepsilon}{\|(C_{i+1} - C^0_{i+1}, B_i - B^0_i)\|}$$

and $g(G^t) > 0$ for any $0 < t < t_0$. Then

$$\|G^t - G^0\| = t\|(C_{i+1} - C^0_{i+1}, B_i - B^0_i)\| < \varepsilon.$$

Hence, in any ε-neighborhood of the element $G^0 \notin \mathcal{G}_i$ there is an element G^t belonging to $\mathcal{G}_i$ for some t.

Thus, $\mathcal{G}' = \bigcap_{i \geq 0} \mathcal{G}_i$ is a countable intersection of dense open subsets, hence $\mathcal{G}'$ is residual. □

Therefore, we can formulate the following theorem which directly follows from Proposition 1 and Theorem 3.

Theorem 6 *If $l \leq \min(m, n)$, then there exists a residual subset $\mathcal{G}'$ of the space $\mathcal{G}$ of sequences of matrix pairs (C_{k+1}, B_k), $k \geq 0$, with the following property: for any matrices C_{k+1}, B_k, $k \geq 0$, such that the sequence $((C_{k+1}, B_k))_{k \geq 0}$ belongs to $\mathcal{G}'$ and for any matrices A_k, $k \geq 0$, system (1)–(2) is locally parameter identifiable for any parameter P^0 and any initial data x_0.*

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