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A new approach to the series expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. The case of arbitrary complete orthonormal systems in Hilbert space. III

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Abstract. The article is Part V of the author's work about a new approach on expansion of iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process. The mentioned approach is based on generalized multiple Fourier series in Hilbert space. In Parts I, II of the work, multiple Fourier–Legendre series and multiple trigonometric Fourier series were used, while in parts III, IV, multiple Fourier series in arbitrary complete orthonormal system of functions in Hilbert space were applied. In addition, sufficient conditions were obtained under which the expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity is valid. The above conditions were verified for integrals of multiplicities 1–6 (Parts I, II) and multiplicities 1–5 (Parts III, IV) under different assumptions on the weight functions of the integrals. In this article, we develop the results of Part IV and obtain an expansion of iterated Stratonovich stochastic integrals of multiplicity 6 with unit weight functions for the case of an arbitrary complete orthonormal system of functions in Hilbert space. Moreover, we propose a simpler (compared to Part IV) proof of the expansion for iterated Stratonovich stochastic integrals of multiplicity 5. The iterated Stratonovich stochastic integrals with respect to components of a multidimensional Wiener process are part of the so-called Taylor–Stratonovich expansion. Therefore, the results of the article will be usefull for constructing efficient high-order strong numerical methods for non-commutative systems of Itô stochastic differential equations.

Key words: iterated Stratonovich stochastic integral, iterated Itô stochastic integral, Itô stochastic differential equation, multidimensional Wiener process, generalized multiple Fourier series, mean-square convergence, expansion.

Contents

1 Introduction

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_t, t\in[0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{w}_t$ be a standard m-dimensional Wiener stochastic process which is F_t-measurable for any $t\in[0,T]$ and has independent components $\mathbf{w}_t^{(i)}$ $(i=1,\ldots,m)$. Consider the system of Itô stochastic differential equations (SDE)

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t \mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\sum_{j=1}^m\int\limits_0^t B_j(\mathbf{x}_\tau,\tau)d\mathbf{w}_\tau^{(j)},\quad \mathbf{x}_0=\mathbf{x}(0,\omega),\quad \omega\in\Omega, \tag{1}$$

where $\mathbf{x}_t\in\mathbb{R}^n$ is a strong solution of (1) [1], the functions $\mathbf{a}, B_j:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ are such that the conditions for the existence of a strong solution of (1) are satisfied [1], the second integral in (1) is the Itô stochastic integral, $\mathbf{x}_0\in\mathbb{R}^n$ is F_0-measurable and $\mathsf{M}\bigl\{|\mathbf{x}_0|^2\bigr\}<\infty$ (M denotes the operator of mathematical expectation and $|\cdot|$ denotes the Euclidean norm of a vector). In addition, we assume that $\mathbf{x}_0$ and $\mathbf{w}_t$ are independent when $t>0$.

The subject of the study in this article is the following iterated Stratonovich stochastic integrals

$$J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{2}$$

where $\psi_1(\tau),\ldots,\psi_k(\tau):[t,T]\to\mathbb{R}$, $i_1,\ldots,i_k=0,\ 1,\ \ldots,\ m$, $\mathbf{w}_\tau^{(0)}=\tau$, $\int^*$ denotes the Stratonovich stochastic integral.

Note that the importance of the problem of numerical integration of Itô SDEs is connected with a wide range of their applications [2]-[4]. One of the effective approaches to solving the above problem is the approach based on the Taylor–Stratonovich expansions [2], [4]–[9]. The stochastic integrals (2) are part of the Taylor–Stratonovich expansions that explains the relevance of the topic of the article.

In addition, note that the so-called classical ($\psi_1(\tau),\ldots,\psi_k(\tau)\equiv 1$, $i_1,\ldots,i_k=0,1,\ldots,m$ in (2)) [2], [4], [5] and unified ($\psi_l(\tau)\equiv(t-\tau)^{q_l}$, $q_l=0,1,\ldots$, $l=1,\ldots,k$, $i_1,\ldots,i_k=1,\ldots,m$ in (2)) [7]-[9] Taylor–Stratonovich expansions are known.

The article is Part V of the work devoted to a new approach on expansion of iterated Stratonovich stochastic integrals (2) ([10]–[13] are Parts I–IV of the above work, respectlively).

Throughout this article we will use the definition of the Stratonovich stochastic integral from [2] or [9] (Sect. 2.1.1).

We will also note other approaches to the expansion of iterated stochastic integrals [2]–[4], [9], [14]–[54].

2 Preliminary Results

2.1 A Review of Some Old Results on the Expansion of Iterated Stratonovich Stochastic Integrals

In this section, we will give a brief review of some old results on the expansion of iterated Stratonovich stochastic integrals obtained by the author (the case of an arbitrary complete orthonormal system (CONS) in $L_2([t,T])$).

Let $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$. Consider the following factorized Volterra–type kernel

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k), & t_1<\ldots<t_k\\ \\ 0, & \text{otherwise}\end{cases}\tag{3}$$

defined on the hupercube $[t,T]^k$ $(k\ge 2)$. The Fourier coefficient for (3) has the form

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k,\tag{4}$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$.

Theorem 1 [9]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$. Moreover, $\psi_1(\tau),\psi_2(\tau)$ are continuous functions on $[t,T]$. Then*

$$J^*[\psi^{(2)}]_{T,t}^{(i_1i_2)}=\underset{p_1,p_2\to\infty}{\text{l.i.m.}}\ \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},\tag{5}$$

where $J^*[\psi^{(2)}]_{T,t}^{(i_1i_2)}$ is defined by (2), $i_1,i_2=0,1,\ldots,m$, l.i.m. *is a limit in the* *mean-square sense,*

$$\zeta_j^{(i)} = \int\limits_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i \neq 0$*), and* $\mathbf{w}_\tau^{(0)} = \tau$*.*

Theorem 2 [9], [13]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t,T])$ *and* $\psi_i(\tau) = (\tau - t)^{l_i}$ $(l_i = 0, 1, 2, \ldots, \ i = 1, \ldots, 4)$*. Then*

$$J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,j_2,j_3=0}^{p} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)},$$

$$J^*[\psi^{(4)}]_{T,t}^{(i_1 \ldots i_4)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_4=0}^{p} C_{j_4 \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_4}^{(i_4)},$$

where $J^*[\psi^{(3)}]_{T,t}^{(i_1 i_2 i_3)}$ *and* $J^*[\psi^{(4)}]_{T,t}^{(i_1 \ldots i_4)}$ *are defined by* (2), $i_1, \ldots, i_4 = 0, 1, \ldots, m$; *another notations as in Theorem* 1.

Theorem 3 [9], [13]. *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t,T])$ *and* $\psi_1(\tau), \ldots, \psi_5(\tau) \equiv 1$*. Then*

$$J^*[\psi^{(5)}]_{T,t}^{(i_1 \ldots i_5)} = \underset{p\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_5=0}^{p} C_{j_5 \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)},$$

where $J^*[\psi^{(5)}]_{T,t}^{(i_1 \ldots i_5)}$ *is defined by* (2), $i_1, \ldots, i_5 = 0, 1, \ldots, m$; *another notations as in Theorem* 1.

2.2 Connection Between Iterated Stratonovich and Itô Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbb{N}$)

Let $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. Introduce the following notations

$$J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)[s_l, \ldots, s_1]} \stackrel{\text{def}}{=} \prod_{q=1}^{l} \mathbf{1}_{\{i_{s_q} = i_{s_q+1} \neq 0\}} \times$$

$$\times \int\limits_t^T \psi_k(t_k) \ldots \int\limits_t^{t_{s_l+3}} \psi_{s_l+2}(t_{s_l+2}) \int\limits_t^{t_{s_l+2}} \psi_{s_l}(t_{s_l+1}) \psi_{s_l+1}(t_{s_l+1}) \times$$

$$
\times \int\limits_{t}^{t_{s_l+1}} \psi_{s_l-1}(t_{s_l-1}) \ldots \int\limits_{t}^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int\limits_{t}^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1}) \psi_{s_1+1}(t_{s_1+1}) \times
$$

$$
\times \int\limits_{t}^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1}) \ldots \int\limits_{t}^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})} \ldots
$$

$$
\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})} dt_{s_l+1} d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{6}
$$

where the iterated integral on the right-hand side of (6) is the iterated Itô stochastic integral, $(s_l, \ldots, s_1) \in \mathrm{A}_{k,l}$,

$$
\mathrm{A}_{k,l} = \Bigl\{(s_l, \ldots, s_1) : \ s_l > s_{l-1} + 1, \ldots, s_2 > s_1 + 1; \ s_l, \ldots, s_1 = 1, \ldots, k-1\Bigr\}, \tag{7}
$$

$l = 1, 2, \ldots, [k/2]$, $i_1, \ldots, i_k = 0, 1, \ldots, m$, $[x]$ is an integer part of a real number x, $\mathbf{1}_A$ is the indicator of the set A.

For $l = 0$, we understand (6) as the following iterated Itô stochastic integral

$$
J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = \int\limits_{t}^{T} \psi_k(t_k) \ldots \int\limits_{t}^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{8}
$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)} = \tau$.

Let us formulate the statement on connection between iterated Stratonovich and Itô stochastic integrals (2) and (8) of arbitrary multiplicity k ($k \in \mathbf{N}$).

Theorem 4 [51] (1997) (also see [9] (Theorem 2.12)). *Suppose that $\psi_1(\tau), \ldots, \psi_k(\tau)$ are continuous functions on $[t, T]$. Then*

$$
J^{*}[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} = J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)} + \sum_{r=1}^{[k/2]} \frac{1}{2^r} \sum_{(s_r, \ldots, s_1) \in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1 \ldots i_k)[s_r, \ldots, s_1]} \quad \text{w. p. 1}, \tag{9}
$$

where $i_1, \ldots, i_k = 0, 1, \ldots, m$, $\sum\limits_{\emptyset}$ is supposed to be equal to zero, w. p. 1 (*here and further*) *means with probability* 1.

Note that the condition of continuity of the functions $\psi_1(\tau), \ldots, \psi_k(\tau)$ is related to the definition of the Stratonovich stochastic integral that we use (see [9], Sect. 2.1.1 for details).

2.3 Expansion of Iterated Stratonovich Stochastic Integrals of Arbitrary Multiplicity k ($k \in \mathbf{N}$) Under the Condition (13). The Case of an Arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_k(\tau) \in L_2([t,T])$

Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{10}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\} = \{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will start this section with an example. Let us assume that $h_1(\tau),\ldots,$ $h_{12}(\tau) \in L_2([t,T])$ and consider the following Lebesgue integral

$$I \stackrel{\mathsf{def}}{=} \int\limits_t^T h_{12}(t_{12}) \int\limits_t^{t_{12}} h_{11}(t_{11}) \ldots \int\limits_t^{t_2} h_1(t_1) dt_1 \ldots dt_{11} dt_{12}.$$

We want to transform the integral I in such a way that

$$I = \int\limits_t^T h_{10}(t_{10}) \int\limits_t^{t_{10}} h_6(t_6) \int\limits_t^{t_6} h_4(t_4) \int\limits_t^{t_4} h_3(t_3) \left(\ldots\right) dt_3 dt_4 dt_6 dt_{10},$$

where $(\ldots)$ is some expression.

Using Fubini's Theorem, we obtain (see [13], Sect. 3.4)

$$I = \int\limits_t^T h_{12}(t_{12}) \int\limits_t^{t_{12}} h_{11}(t_{11}) \int\limits_t^{t_{11}} \underline{h_{10}(t_{10})} \int\limits_t^{t_{10}} h_9(t_9) \int\limits_t^{t_9} h_8(t_8) \int\limits_t^{t_8} h_7(t_7) \int\limits_t^{t_7} \underline{h_6(t_6)} \times$$

$$\times \int\limits_t^{t_6} h_5(t_5) \int\limits_t^{t_5} \underline{h_4(t_4)} \int\limits_t^{t_4} \underline{h_3(t_3)} \int\limits_t^{t_3} h_2(t_2) \int\limits_t^{t_2} h_1(t_1) dt_1 dt_2 dt_3 dt_4 dt_5 dt_6 dt_7 dt_8 \times$$

$$\times dt_9 dt_{10} dt_{11} dt_{12} =$$

$$= \int\limits_t^T \underline{h_{10}(t_{10})} \int\limits_t^{t_{10}} \underline{h_6(t_6)} \int\limits_t^{t_6} \underline{h_4(t_4)} \int\limits_t^{t_4} \underline{h_3(t_3)} \left(\int\limits_t^{t_3} h_2(t_2) \int\limits_t^{t_2} h_1(t_1) dt_1 dt_2 \right) \times$$

$$
\times \left(\int\limits_{t_4}^{t_6} h_5(t_5)dt_5\right) \left(\int\limits_{t_6}^{t_{10}} h_9(t_9) \int\limits_{t_6}^{t_9} h_8(t_8) \int\limits_{t_6}^{t_8} h_7(t_7)dt_7dt_8dt_9\right) \times
$$

$$
\times \left(\int\limits_{t_{10}}^{T} h_{12}(t_{12}) \int\limits_{t_{10}}^{t_{12}} h_{11}(t_{11})dt_{11}dt_{12}\right) dt_3dt_4dt_6dt_{10}. \qquad (11)
$$

Further, let us suppose that $h_l(\tau) = \psi_l(\tau)\phi_{j_l}(\tau)$ $(l = 1, \ldots, 12)$ in (11) (here $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t, T])$. Thus, we get

$$
C_{j_{12}j_{11}\underline{j_{10}}j_9j_8j_7\underline{j_6}j_5\underline{j_4}\underline{j_3}j_2j_1} = \int\limits_{t}^{T} \psi_{10}(t_{10})\phi_{j_{10}}(t_{10}) \int\limits_{t}^{t_{10}} \psi_6(t_6)\phi_{j_6}(t_6) \int\limits_{t}^{t_6} \psi_4(t_4)\phi_{j_4}(t_4)\times
$$

$$
\times \int\limits_{t}^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) C_{j_{12}j_{11}}^{\psi_{12}\psi_{11}}(T, t_{10}) C_{j_9j_8j_7}^{\psi_9\psi_8\psi_7}(t_{10}, t_6) C_{j_5}^{\psi_5}(t_6, t_4) C_{j_2j_1}^{\psi_2\psi_1}(t_3, t)\times
$$

$$
\times dt_3dt_4dt_6dt_{10}, \qquad (12)
$$

where (here and further)

$$
C_{j_k\ldots j_1}^{\psi_k\ldots\psi_1}(s, \tau) = \int\limits_{\tau}^{s} \psi_k(t_k)\phi_{j_k}(t_k) \ldots \int\limits_{\tau}^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1 \ldots dt_k,
$$

where $t \le \tau < s \le T$.

Let $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in (10) and $k > 2r$, $r \ge 1$. Consider $d_1, e_1, \ldots, d_f, e_f$, $f \in \mathbf{N}$ such that

$$
1 \le d_1 - e_1 + 1 < \ldots < d_1 - 1 < d_1 < \ldots < d_f - e_f + 1 < \ldots < d_f - 1 < d_f \le k,
$$

$$
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} =
$$

$$
= \{d_1 - e_1 + 1, \ldots, d_1 - 1, d_1\} \cup \ldots \cup \{d_f - e_f + 1, \ldots, d_f - 1, d_f\},
$$

$$
\{1, \ldots, k\} \setminus \{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\} = \{q_1, \ldots, q_{k-2r}\},
$$

$$
e_1 + e_2 + \ldots + e_f = 2r.
$$

Suppose that

$$
\Biggl|\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Bigl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Bigr)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr|\le K<\infty, \qquad (13)
$$

where constant K does not depend on p and $t_{d_1+1},t_{d_1-e_1},\ldots,t_{d_f+1},t_{d_f-e_f}$ (here $d_1-e_1\ge 1$ and $d_f+1\le k$). In (13): $t_{k+1}\stackrel{\rm def}{=}T$, $t_0\stackrel{\rm def}{=}t$; another notations as above in this section.

Theorem 5 [9], [13]. *Suppose that the condition* (13) *is fulfilled,* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is an arbitrary CONS in* $L_2([t,T])$ *and* $\psi_1(\tau),\ldots,\psi_k(\tau)$ *are continuous functions on* $[t,T]$. *Then*

$$
J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\rm l.i.m.}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \qquad (14)
$$

where $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ *is defined by* (2), $i_1,\ldots,i_k=0,1,\ldots,m$; *another notations as in Sect* 2.1.

Note that the condition (13) can be weakened. Namely, the constant K^2 can be replaced by the function F such that $\psi_{q_1}^2\ldots\psi_{q_{k-2r}}^2F\in L_1([t,T]^{k-2r})$ (integrable majorant). More precisely, the condition (13) can be replaced by the following condition

$$
\Biggl(\sum_{j_{g_1},j_{g_3},\ldots,j_{g_{2r-1}}=0}^{p}\Bigl(C_{j_{d_f}\ldots j_{d_f-e_f+1}}^{\psi_{d_f}\ldots\psi_{d_f-e_f+1}}(t_{d_f+1},t_{d_f-e_f})\ldots
$$

$$
\ldots C_{j_{d_1}\ldots j_{d_1-e_1+1}}^{\psi_{d_1}\ldots\psi_{d_1-e_1+1}}(t_{d_1+1},t_{d_1-e_1})\Bigr)\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}\Biggr)^2\le F(t_{q_1},\ldots,t_{q_{k-2r}}) \qquad (15)
$$

almost everywhere on

$$
X=\{(t_{q_1},\ldots,t_{q_{k-2r}}):t\le t_{q_1}\le\ldots\le t_{q_{k-2r}}\le T\}
$$

with respect to Lebesgue's measure, where the function $F(t_{q_1},\ldots,t_{q_{k-2r}})$ is such that

$$
\psi_{q_1}^2(t_{q_1})\ldots\psi_{q_{k-2r}}^2(t_{q_{k-2r}})F(t_{q_1},\ldots,t_{q_{k-2r}})\in L_1([t,T]^{k-2r}),
$$

where notations are the same as in (13). In this case, instead of $J^*[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ on the left-hand side of (14) there will be

$$J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}+\sum_{r=1}^{[k/2]}\frac{1}{2^r}\sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)[s_r,\ldots,s_1]}$$

(see (9)) and in addition $\psi_1(\tau),\ldots,\psi_k(\tau)\in L_2([t,T])$ (see Theorem 2.60 [9] and the remark at the end of Sect. 2.31 [9]).

3 Main Results

3.1 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 6. The Case of an Arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau),\ldots,\psi_6(\tau)\equiv 1$

This section is devoted to the following theorem.

Theorem 6 [9]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity*

$$J^*[\psi^{(6)}]_{T,t}=\int\limits_t^{*T}\ldots\int\limits_t^{*t_2} d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_6}^{(i_6)}$$

the following expansion

$$J^*[\psi^{(6)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_6=0}^{p} C_{j_6\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_6}^{(i_6)}$$

that converges in the mean-square sense is valid, where $i_1,\ldots,i_6=0,1,\ldots,m$,

$$C_{j_6\ldots j_1}=\int\limits_t^T\phi_{j_6}(t_6)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_6, \tag{16}$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i \neq 0$), $\mathbf{w}_{\tau}^{(0)} = \tau$.

Proof. Our proof will be based on Theorem 5 and verification of the equality (13) under the conditions of Theorem 6 (the case $k = 6 > 2r$, where $r = 1, 2$).

Remark 1. *Recall that the case $k = 2r$ is proved in [13], Sect. 3.2 (also see [55], [56]). Therefore, the case $k = 2r$ is not included in the conditions of Theorem 5. More precisely, the following equality*

$$\lim_{p\to\infty} \sum_{j_{g_1}, j_{g_3}, \ldots, j_{g_{2r-1}}=0}^{p} C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2}, \ldots, j_{g_{2r-1}}=j_{g_{2r}}} =$$

$$= \frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}=\, j_{g_2}, \ldots, j_{g_{2r-1}}=\, j_{g_{2r}}} \tag{17}$$

holds for all possible $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ (see (10)), where $k = 2r$ ($r = 1, 2, \ldots$), $C_{j_k\ldots j_1}$ has the form (4), the value

$$\frac{1}{2^r} \prod_{l=1}^{r} \mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}} C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)\ldots(j_{g_{2r}}j_{g_{2r-1}})\curvearrowright(\cdot), j_{g_1}=\, j_{g_2}, \ldots, j_{g_{2r-1}}=\, j_{g_{2r}}}$$

is defined iteratively using the equality

$$C_{j_k\ldots j_{l+1}j_l j_{l-1}j_{l-2}\ldots j_1}\Biggl|_{(j_l j_{l-1})\curvearrowright(\cdot)} \stackrel{\mathsf{def}}{=}$$

$$\stackrel{\mathsf{def}}{=} \int_t^T \psi_k(t_k)\phi_{j_k}(t_k)\ldots \int_t^{t_{l+2}} \psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1}) \int_t^{t_{l+1}} \psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times \int_t^{t_l} \psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots \int_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_l t_{l+1}\ldots dt_k, \tag{18}$$

where $\{l, l-1\}$ is one of the pairs $\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}$ (see (10)).

Under the conditions of Theorem 6, this means that $k = 6 = 2r$ ($r = 3$) in the equality (17).

Further, let throughout this proof

$$C_{j_k\ldots j_1}(s,\tau)=\int\limits_{\tau}^{s}\phi_{j_k}(t_k)\ldots\int\limits_{\tau}^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_k, \qquad (19)$$

where $k=1,\ldots,4$, $t\le\tau<s\le T$, and $C_{j_6\ldots j_1}$ is defined by (16).

Using Fubini's Theorem and the technique that leads to the formulas (11), (12) (see [13], Sect. 3.4), we obtain (note that we find all possible combinations of pairs using the equality (2.682) (see [9]))

1. $r=1$ (15 combinations):

$$C_{j_1j_5j_4j_3j_2j_1}=$$

$$=\int\limits_{t}^{T}\phi_{j_5}(t_5)\int\limits_{t}^{t_5}\phi_{j_4}(t_4)\int\limits_{t}^{t_4}\phi_{j_3}(t_3)\int\limits_{t}^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(T,t_5)dt_2dt_3dt_4dt_5,$$

$$C_{j_2j_5j_4j_3j_2j_1}=$$

$$=\int\limits_{t}^{T}\phi_{j_5}(t_5)\int\limits_{t}^{t_5}\phi_{j_4}(t_4)\int\limits_{t}^{t_4}\phi_{j_3}(t_3)\int\limits_{t}^{t_3}\phi_{j_1}(t_1)C_{j_2}(t_3,t_1)C_{j_2}(T,t_5)dt_1dt_3dt_4dt_5,$$

$$C_{j_3j_5j_4j_3j_2j_1}=$$

$$=\int\limits_{t}^{T}\phi_{j_5}(t_5)\int\limits_{t}^{t_5}\phi_{j_4}(t_4)\int\limits_{t}^{t_4}\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\phi_{j_1}(t_1)C_{j_3}(t_4,t_2)C_{j_3}(T,t_5)dt_1dt_2dt_4dt_5,$$

$$C_{j_4j_5j_4j_3j_2j_1}=$$

$$=\int\limits_{t}^{T}\phi_{j_5}(t_5)\int\limits_{t}^{t_5}\phi_{j_3}(t_3)\int\limits_{t}^{t_3}\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\phi_{j_1}(t_1)C_{j_4}(t_5,t_3)C_{j_4}(T,t_5)dt_1dt_2dt_3dt_5,$$

$$C_{j_5j_5j_4j_3j_2j_1}=$$

$$=\int\limits_{t}^{T}\phi_{j_4}(t_4)\int\limits_{t}^{t_4}\phi_{j_3}(t_3)\int\limits_{t}^{t_3}\phi_{j_2}(t_2)\int\limits_{t}^{t_2}\phi_{j_1}(t_1)C_{j_5j_5}(T,t_4)dt_1dt_2dt_3dt_4,$$

$$C_{j_6j_5j_4j_3j_1j_1}=$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) C_{j_1 j_1}(t_3, t) dt_3 dt_4 dt_5 dt_6,
$$

$$
C_{j_6 j_5 j_4 j_1 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_4, t_2) dt_2 dt_4 dt_5 dt_6,
$$

$$
C_{j_6 j_5 j_1 j_3 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_5, t_3) dt_2 dt_3 dt_5 dt_6,
$$

$$
C_{j_6 j_1 j_4 j_3 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_6, t_4) dt_2 dt_3 dt_4 dt_6,
$$

$$
C_{j_6 j_5 j_4 j_2 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_1}(t_1) C_{j_2 j_2}(t_4, t_1) dt_1 dt_4 dt_5 dt_6,
$$

$$
C_{j_6 j_5 j_2 j_3 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_2}(t_5, t_3) dt_1 dt_3 dt_5 dt_6,
$$

$$
C_{j_6 j_2 j_4 j_3 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_4}(t_4) \int_t^{t_4} \phi_{j_3}(t_3) \int_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_2}(t_6, t_4) dt_1 dt_3 dt_4 dt_6,
$$

$$
C_{j_6 j_5 j_3 j_3 j_2 j_1} =
$$

$$
= \int_t^T \phi_{j_6}(t_6) \int_t^{t_6} \phi_{j_5}(t_5) \int_t^{t_5} \phi_{j_2}(t_2) \int_t^{t_2} \phi_{j_1}(t_1) C_{j_3 j_3}(t_5, t_2) dt_1 dt_2 dt_5 dt_6,
$$

$$C_{j_6j_3j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_3}(t_4,t_2) C_{j_3}(t_6,t_4) dt_1 dt_2 dt_4 dt_6,$$

$$C_{j_6j_4j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_4j_4}(t_6,t_3) dt_1 dt_2 dt_3 dt_6,$$

2. $r = 2$ (45 combinations):

$$C_{j_6j_5j_3j_3j_1j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) C_{j_3j_3j_1j_1}(t_5,t) dt_5 dt_6,$$

$$C_{j_6j_3j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_4}(t_4) C_{j_3j_1j_1}(t_4,t) C_{j_3}(t_6,t_4) dt_4 dt_6,$$

$$C_{j_6j_4j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_3) C_{j_1j_1}(t_3,t) C_{j_4j_4}(t_6,t_3) dt_3 dt_6,$$

$$C_{j_6j_5j_2j_1j_2j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) C_{j_2j_1j_2j_1}(t_5,t) dt_5 dt_6,$$

$$C_{j_6j_2j_4j_1j_2j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_4}(t_4) C_{j_1j_2j_1}(t_4,t) C_{j_2}(t_6,t_4) dt_4 dt_6,$$

$$C_{j_6j_4j_4j_1j_2j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_2}(t_2) C_{j_1}(t_2,t) C_{j_4j_4j_1}(t_6,t_2) dt_2 dt_6,$$

$$C_{j_6j_5j_1j_2j_2j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_5}(t_5) C_{j_1j_2j_2j_1}(t_5,t) dt_5 dt_6,$$

$$C_{j_6j_2j_1j_3j_2j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_3) C_{j_2j_1}(t_3,t) C_{j_2j_1}(t_6,t_3) dt_3 dt_6,$$

$$C_{j_6 j_3 j_1 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_3 j_1 j_3}(t_6, t_2) dt_2 dt_6,$$

$$C_{j_6 j_1 j_4 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_4}(t_4) C_{j_2 j_2 j_1}(t_4, t) C_{j_1}(t_6, t_4) dt_4 dt_6,$$

$$C_{j_6 j_1 j_2 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_3}(t_3) C_{j_2 j_1}(t_3, t) C_{j_1 j_2}(t_6, t_3) dt_3 dt_6,$$

$$C_{j_6 j_1 j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1 j_3 j_3}(t_6, t_2) dt_2 dt_6,$$

$$C_{j_6 j_4 j_4 j_2 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_1) C_{j_4 j_4 j_2 j_2}(t_6, t_1) dt_1 dt_6,$$

$$C_{j_6 j_3 j_2 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_1) C_{j_3 j_2 j_3 j_2}(t_6, t_1) dt_1 dt_6,$$

$$C_{j_6 j_2 j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_6}(t_6) \int\limits_t^{t_6} \phi_{j_1}(t_1) C_{j_2 j_3 j_3 j_2}(t_6, t_1) dt_1 dt_6,$$

$$C_{j_1 j_5 j_3 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_3 j_3}(t_5, t_2) C_{j_1}(T, t_5) dt_2 dt_5,$$

$$C_{j_1 j_3 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_3}(t_4, t_2) C_{j_1 j_3}(T, t_4) dt_2 dt_4,$$

$$C_{j_1 j_2 j_4 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) C_{j_2 j_1}(t_3, t) C_{j_1 j_2}(T, t_4) dt_3 dt_4,$$

$$C_{j_1 j_5 j_2 j_3 j_2 j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) C_{j_2 j_1}(t_3, t) C_{j_2}(t_5, t_3) C_{j_1}(T, t_5) dt_3 dt_5,$$

$$C_{j_1j_4j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1j_4j_4}(T,t_3)dt_2dt_3,$$

$$C_{j_1j_5j_4j_2j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)C_{j_2j_2j_1}(t_4,t)C_{j_1}(T,t_5)dt_4dt_5,$$

$$C_{j_2j_3j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)C_{j_2j_3}(t_4,t_1)C_{j_2j_3}(T,t_4)dt_1dt_4,$$

$$C_{j_2j_4j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_1}(t_1)C_{j_2}(t_3,t_1)C_{j_2j_4j_4}(T,t_3)dt_1dt_3,$$

$$C_{j_2j_5j_3j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)C_{j_2j_3j_3}(t_5,t_1)C_{j_2}(T,t_5)dt_1dt_5,$$

$$C_{j_2j_1j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_2j_1}(T,t_4)dt_3dt_4,$$

$$C_{j_2j_5j_1j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)C_{j_2j_1}(t_3,t)C_{j_1}(t_5,t_3)C_{j_2}(T,t_5)dt_3dt_5,$$

$$C_{j_2j_5j_4j_1j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)C_{j_1j_2j_1}(t_4,t)C_{j_2}(T,t_5)dt_4dt_5,$$

$$C_{j_3j_2j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_1}(t_1)C_{j_3j_2}(t_4,t_1)C_{j_3j_2}(T,t_4)dt_1dt_4,$$

$$C_{j_3j_4j_4j_3j_2j_1}=\int\limits_t^T\phi_{j_2}(t_2)\int\limits_t^{t_2}\phi_{j_1}(t_1)C_{j_3j_4j_4j_3}(T,t_2)dt_1dt_2,$$

$$C_{j_3j_5j_2j_3j_2j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_1}(t_1)C_{j_2j_3j_2}(t_5,t_1)C_{j_3}(T,t_5)dt_1dt_5,$$

$$C_{j_3j_1j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) C_{j_1}(t_2,t) C_{j_3}(t_4,t_2) C_{j_3j_1}(T,t_4) dt_2 dt_4,$$

$$C_{j_3j_5j_1j_3j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) C_{j_1}(t_2,t) C_{j_1j_3}(t_5,t_2) C_{j_3}(T,t_5) dt_2 dt_5,$$

$$C_{j_3j_5j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) C_{j_3j_1j_1}(t_4,t) C_{j_3}(T,t_5) dt_4 dt_5,$$

$$C_{j_4j_3j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_4j_3j_4j_3}(T,t_2) dt_1 dt_2,$$

$$C_{j_4j_2j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3,t_1) C_{j_4j_2j_4}(T,t_3) dt_1 dt_3,$$

$$C_{j_4j_5j_4j_2j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_1}(t_1) C_{j_4j_2j_2}(t_5,t_1) C_{j_4}(T,t_5) dt_1 dt_5,$$

$$C_{j_4j_1j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2,t) C_{j_4j_1j_4}(T,t_3) dt_2 dt_3,$$

$$C_{j_4j_5j_4j_1j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) C_{j_1}(t_2,t) C_{j_4j_1}(t_5,t_2) C_{j_4}(T,t_5) dt_2 dt_5,$$

$$C_{j_4j_5j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) C_{j_1j_1}(t_3,t) C_{j_4}(t_5,t_3) C_{j_4}(T,t_5) dt_3 dt_5,$$

$$C_{j_5j_5j_3j_3j_2j_1} = \int\limits_t^T \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_5j_5j_3j_3}(T,t_2) dt_1 dt_2,$$

$$C_{j_5j_5j_2j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3,t_1) C_{j_5j_5j_2}(T,t_3) dt_1 dt_3,$$

$$C_{j_5j_5j_4j_2j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) C_{j_2j_2}(t_4, t_1) C_{j_5j_5}(T, t_4) dt_1 dt_4,$$

$$C_{j_5j_5j_1j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_5j_5j_1}(T, t_3) dt_2 dt_3,$$

$$C_{j_5j_5j_4j_1j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1}(t_4, t_2) C_{j_5j_5}(T, t_4) dt_2 dt_4,$$

$$C_{j_5j_5j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) C_{j_1j_1}(t_3, t) C_{j_5j_5}(T, t_4) dt_3 dt_4.$$

It is not difficult to see (based on the above equalities) that the condition (13) will be satisfied under the conditions of Theorem 6 if

$$\left| \sum_{j_1=0}^{p} C_{j_1j_1}(s, \tau) \right| \le K, \tag{20}$$

$$\left| \sum_{j_1=0}^{p} C_{j_1}(s, \tau) C_{j_1}(\theta, u) \right| \le K, \tag{21}$$

$$\left| \sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1j_1}(s, \tau) \right| \le K, \tag{22}$$

$$\left| \sum_{j_1,j_2=0}^{p} C_{j_2j_1j_2j_1}(s, \tau) \right| \le K, \tag{23}$$

$$\left| \sum_{j_1,j_2=0}^{p} C_{j_1j_2j_2j_1}(s, \tau) \right| \le K, \tag{24}$$

$$\left| \sum_{j_1,j_2=0}^{p} C_{j_2j_1j_1}(s, \tau) C_{j_2}(\theta, u) \right| \le K, \tag{25}$$

$$\left| \sum_{j_1,j_2=0}^{p} C_{j_1j_2j_1}(s, \tau) C_{j_2}(\theta, u) \right| \le K, \tag{26}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1}(s,\tau)C_{j_1}(\theta,u)\right| \le K, \tag{27}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1j_1}(s,\tau)C_{j_2j_2}(\theta,u)\right| \le K, \tag{28}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_2j_1}(\theta,u)\right| \le K, \tag{29}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_1j_2}(\theta,u)\right| \le K, \tag{30}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_1}(\rho,v)C_{j_2j_2}(\theta,u)\right| \le K, \tag{31}$$

$$\left|\sum_{j_1,j_2=0}^{p} C_{j_1}(s,\tau)C_{j_2}(\rho,v)C_{j_1j_2}(\theta,u)\right| \le K, \tag{32}$$

where $p \in \mathbf{N}$, $t \le \tau < s \le T$, $t \le u < \theta \le T$, $t \le v < \rho \le T$, constant K does not depend on $p, s, \tau, u, \theta, v, \rho$ (but only on t, T).

The equalities (22)–(24) have been proved earlier (see the proof of (55)–(57) in [13]). Let us prove (20), (21), (25)–(32).

Using Fubini's Theorem and Parseval's equality, we get

$$\left|\sum_{j_1=0}^{p} C_{j_1j_1}(s,\tau)\right| = \frac{1}{2}\sum_{j_1=0}^{p} C_{j_1}^2(s,\tau) \le$$

$$\le \frac{1}{2}\sum_{j_1=0}^{\infty} C_{j_1}^2(s,\tau) = \frac{1}{2}(s-\tau) \le$$

$$\le \frac{1}{2}(T-t) \le K.$$

The equality (20) is proved. Moreover, (28) follows from (20).

Using the inequality of Cauchy–Bunyakovsky and Parseval's equality, we obtain

$$\left(\sum_{j_1=0}^{p} C_{j_1}(s,\tau)C_{j_1}(\theta,u)\right)^2 \le$$

$$\leq \sum_{j_1=0}^{p} C_{j_1}^2(s,\tau) \sum_{j_1=0}^{p} C_{j_1}^2(\theta,u) \leq$$

$$\leq \sum_{j_1=0}^{\infty} C_{j_1}^2(s,\tau) \sum_{j_1=0}^{\infty} C_{j_1}^2(\theta,u) =$$

$$= (s-\tau)(\theta-u) \leq (T-t)^2 \leq K^2,$$

$$\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1}(s,\tau)C_{j_2j_1}(\theta,u)\right)^2 \leq \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2(s,\tau) \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2(\theta,u) \leq$$

$$\leq \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}^2(s,\tau) \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}^2(\theta,u) =$$

$$= \int\limits_{\tau}^{s}\int\limits_{\tau}^{v} dxdv \int\limits_{u}^{\theta}\int\limits_{u}^{v} dxdv \leq \frac{1}{4}(T-t)^4 \leq K^2.$$

Thus, the inequalities (21), (29) are proved. The inequalities (30), (32) are proved similarly to (29). Moreover, (31) follows from (20), (21).

Further, let us prove the equalities (25)–(27). Applying the Cauchy–Bunyakovsky inequality as well as Parseval's equality and (20), we have

$$\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_1j_1}(s,\tau)C_{j_2}(\theta,u)\right)^2 \leq \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_2j_1j_1}(s,\tau)\right)^2 \sum_{j_2=0}^{p} C_{j_2}^2(\theta,u) \leq$$

$$\leq \sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p} C_{j_2j_1j_1}(s,\tau)\right)^2 \sum_{j_2=0}^{\infty} C_{j_2}^2(\theta,u) =$$

$$= \sum_{j_2=0}^{\infty}\left(\int\limits_{\tau}^{s} \phi_{j_2}(v)\sum_{j_1=0}^{p} C_{j_1j_1}(v,\tau)dv\right)^2 \cdot (\theta-u) =$$

$$= (\theta-u)\int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p} C_{j_1j_1}(v,\tau)\right)^2 dv \leq$$

$$\leq K^2(\theta-u)(s-\tau) \leq K^2(T-t)^2 = K_1.$$

The equality (25) is proved.

Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality and (21), we obtain

$$
\left(\sum_{j_1,j_2=0}^{p} C_{j_1j_2j_1}(s,\tau)C_{j_2}(\theta,u)\right)^2 \le \sum_{j_2=0}^{p}\left(\sum_{j_1=0}^{p} C_{j_1j_2j_1}(s,\tau)\right)^2 \sum_{j_2=0}^{p} C_{j_2}^2(\theta,u) \le
$$

$$
\le \sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{s}\phi_{j_1}(z)\int\limits_{\tau}^{z}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dxdydz\right)^2 \sum_{j_2=0}^{\infty} C_{j_2}^2(\theta,u) =
$$

$$
= \sum_{j_2=0}^{\infty}\left(\sum_{j_1=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dx\int\limits_{y}^{s}\phi_{j_1}(z)dzdy\right)^2 \cdot(\theta-u) =
$$

$$
= (\theta-u)\sum_{j_2=0}^{\infty}\left(\int\limits_{\tau}^{s}\phi_{j_2}(y)\sum_{j_1=0}^{p} C_{j_1}(y,\tau)C_{j_1}(s,y)dy\right)^2 =
$$

$$
= (\theta-u)\int\limits_{\tau}^{s}\left(\sum_{j_1=0}^{p} C_{j_1}(y,\tau)C_{j_1}(s,y)\right)^2 dy \le
$$

$$
\le K^2(\theta-u)(s-\tau) \le K^2(T-t)^2 = K_1.
$$

The equality (26) is proved.

Using the Cauchy–Bunyakovsky inequality as well as Fubini's Theorem, Parseval's equality and (20), we get

$$
\left(\sum_{j_1,j_2=0}^{p} C_{j_2j_2j_1}(s,\tau)C_{j_1}(\theta,u)\right)^2 \le \sum_{j_1=0}^{p}\left(\sum_{j_2=0}^{p} C_{j_2j_2j_1}(s,\tau)\right)^2 \sum_{j_1=0}^{p} C_{j_1}^2(\theta,u) \le
$$

$$
\le \sum_{j_1=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_2}(z)\int\limits_{\tau}^{z}\phi_{j_2}(y)\int\limits_{\tau}^{y}\phi_{j_1}(x)dxdydz\right)^2 \sum_{j_1=0}^{\infty} C_{j_1}^2(\theta,u) =
$$

$$
= \sum_{j_1=0}^{\infty}\left(\sum_{j_2=0}^{p}\int\limits_{\tau}^{s}\phi_{j_1}(x)\int\limits_{x}^{s}\phi_{j_2}(y)\int\limits_{y}^{s}\phi_{j_2}(z)dzdydx\right)^2 \cdot(\theta-u) =
$$

$$
= (\theta - u) \sum_{j_1=0}^{\infty} \left(\sum_{j_2=0}^{p} \int\limits_{\tau}^{s} \phi_{j_1}(x) \int\limits_{x}^{s} \phi_{j_2}(z) \int\limits_{x}^{z} \phi_{j_2}(y) dy dz dx \right)^2 =
$$

$$
= (\theta - u) \sum_{j_1=0}^{\infty} \left(\int\limits_{\tau}^{s} \phi_{j_1}(x) \sum_{j_2=0}^{p} C_{j_2 j_2}(s, x) dx \right)^2 =
$$

$$
= (\theta - u) \int\limits_{\tau}^{s} \left(\sum_{j_2=0}^{p} C_{j_2 j_2}(s, x) \right)^2 dx \le
$$

$$
\le K^2 (\theta - u)(s - \tau) \le K^2 (T - t)^2 = K_1.
$$

The equality (27) is proved. The equalities (20)–(32) are proved.

Thus, the condition (13) of Theorem 5 is satisfied under the conditions of Theorem 6. The assertion of Theorem 6 now follows from Theorem 5. Theorem 6 is proved.

3.2 Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicity 5. The Case of an Arbitrary CONS in $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_5(\tau) \equiv 1$. Proof Based on Theorem 5

In this section, we will prove the following theorem.

Theorem 7 [9], [52]–[54]. *Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t, T])$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity*

$$
J^*[\psi^{(5)}]_{T,t}^{(i_1 \ldots i_5)} = \int\limits_{t}^{*T} \ldots \int\limits_{t}^{*t_2} d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_5}^{(i_5)}
$$

the following expansion

$$
J^*[\psi^{(5)}]_{T,t}^{(i_1 \ldots i_5)} = \underset{p \to \infty}{\text{l.i.m.}} \sum_{j_1, \ldots, j_5 = 0}^{p} C_{j_5 \ldots j_1} \zeta_{j_1}^{(i_1)} \ldots \zeta_{j_5}^{(i_5)}
$$

that converges in the mean-square sense is valid, where $i_1, \ldots, i_5 = 0, 1, \ldots, m$,

$$C_{j_5\ldots j_1}=\int\limits_t^T\phi_{j_5}(t_5)\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_5, \tag{33}$$

and

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{w}_\tau^{(i)}$$

are independent standard Gaussian random variables for various i or j (in the case when $i\neq 0$), $\mathbf{w}_\tau^{(0)}=\tau$.

Proof. The following proof will be based on Theorem 5 and verification of the equality (13) under the conditions of Theorem 7 (the case $k=5>2r$, where $r=1$ or $r=2$).

Further, let $C_{j_k\ldots j_1}(s,\tau)$ ($k=1,\ldots,4,\ t\le\tau<s\le T$) is defined by (19) and $C_{j_5\ldots j_1}$ has the form (33).

Applying the technique that leads to (11), (12) (see [13], Sect. 3.4), we obtain (note that we find all possible combinations of pairs using the equality (2.681) (see [9]))

$$C_{j_5j_4j_3j_1j_1}=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)C_{j_1j_1}(t_3,t)dt_3dt_4dt_5,$$

$$C_{j_5j_4j_1j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(t_4,t_2)dt_2dt_4dt_5,$$

$$C_{j_5j_1j_3j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_5}(t_5)\int\limits_t^{t_5}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(t_5,t_3)dt_2dt_3dt_5,$$

$$C_{j_1j_4j_3j_2j_1}=$$

$$=\int\limits_t^T\phi_{j_4}(t_4)\int\limits_t^{t_4}\phi_{j_3}(t_3)\int\limits_t^{t_3}\phi_{j_2}(t_2)C_{j_1}(t_2,t)C_{j_1}(T,t_4)dt_2dt_3dt_4,$$

$$C_{j_5j_4j_2j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_1}(t_1) C_{j_2j_2}(t_4, t_1) dt_1 dt_4 dt_5,$$

$$C_{j_5j_2j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_2}(t_5, t_3) dt_1 dt_3 dt_5,$$

$$C_{j_2j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_1}(t_1) C_{j_2}(t_3, t_1) C_{j_2}(T, t_4) dt_1 dt_3 dt_4,$$

$$C_{j_5j_3j_3j_2j_1} \int\limits_t^T \phi_{j_5}(t_5) \int\limits_t^{t_5} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_3j_3}(t_5, t_2) dt_1 dt_2 dt_5,$$

$$C_{j_3j_4j_3j_2j_1} =$$

$$= \int\limits_t^T \phi_{j_4}(t_4) \int\limits_t^{t_4} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_3}(t_4, t_2) C_{j_3}(T, t_4) dt_1 dt_2 dt_4,$$

$$C_{j_4j_4j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) \int\limits_t^{t_3} \phi_{j_2}(t_2) \int\limits_t^{t_2} \phi_{j_1}(t_1) C_{j_4j_4}(T, t_3) dt_1 dt_2 dt_3,$$

$$C_{j_5j_3j_3j_1j_1} = \int\limits_t^T \phi_{j_5}(t_5) C_{j_3j_3j_1j_1}(t_5, t) dt_5,$$

$$C_{j_5j_2j_1j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5) C_{j_2j_1j_2j_1}(t_5, t) dt_5,$$

$$C_{j_5j_1j_2j_2j_1} = \int\limits_t^T \phi_{j_5}(t_5) C_{j_1j_2j_2j_1}(t_5, t) dt_5,$$

$$C_{j_4j_4j_2j_2j_1} = \int\limits_t^T \phi_{j_1}(t_1) C_{j_4j_4j_2j_2}(T, t_1) dt_1,$$

$$C_{j_3j_2j_3j_2j_1} = \int\limits_t^T \phi_{j_1}(t_1) C_{j_3j_2j_3j_2}(T, t_1) dt_1,$$

$$C_{j_2j_3j_3j_2j_1} = \int\limits_t^T \phi_{j_1}(t_1) C_{j_2j_3j_3j_2}(T, t_1) dt_1,$$

$$C_{j_4j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_3}(t_3) C_{j_1j_1}(t_3, t) C_{j_4j_4}(T, t_3) dt_3,$$

$$C_{j_2j_4j_1j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4) C_{j_1j_2j_1}(t_4, t) C_{j_2}(T, t_4) dt_4,$$

$$C_{j_2j_1j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) C_{j_2j_1}(t_3, t) C_{j_2j_1}(T, t_3) dt_3,$$

$$C_{j_3j_1j_3j_2j_1} = \int\limits_t^T \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_3j_1j_3}(T, t_2) dt_2,$$

$$C_{j_1j_2j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(t_3) C_{j_2j_1}(t_3, t) C_{j_1j_2}(T, t_3) dt_3,$$

$$C_{j_3j_4j_3j_1j_1} = \int\limits_t^T \phi_{j_4}(t_4) C_{j_3j_1j_1}(t_4, t) C_{j_3}(T, t_4) dt_4,$$

$$C_{j_4j_4j_1j_2j_1} = \int\limits_t^T \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_4j_4j_1}(T, t_2) dt_2,$$

$$C_{j_1j_4j_2j_2j_1} = \int\limits_t^T \phi_{j_4}(t_4) C_{j_2j_2j_1}(t_4, t) C_{j_1}(T, t_4) dt_4,$$

$$C_{j_1j_3j_3j_2j_1} = \int\limits_t^T \phi_{j_2}(t_2) C_{j_1}(t_2, t) C_{j_1j_3j_3}(T, t_2) dt_2.$$

It is easy to see (based on the above relations) that (13) will be satisfied (under the conditions of Theorem 7) if (20)–(30) are fulfilled. The equalities

(20)–(30) are proved in Sect. 3.1. The assertion of Theorem 7 now follows from Theorem 5. Theorem 7 is proved.

Recall that for the case $k = 6$, together with (20)–(30), the conditions (31), (32), and the equality (17) ($k = 2r$, $k = 6$, $r = 3$) must be satisfied (see the proof of Theorem 6).

3.3 Partial Proof of the Condition (13)

Let $g_1, g_2, \ldots, g_{2r-1}, g_{2r}$ as in the formula (10). Suppose that $\forall\ \{g_{2l-1}, g_{2l}\}$ ($l = 1, \ldots, r$) $\exists\ h \in \{1, \ldots, f\}$ such that

$$\{g_{2l-1}, g_{2l}\} \subset \{d_h - e_h + 1, \ldots, d_h - 1, d_h\}. \tag{34}$$

Moreover, $\forall\ h \in \{1, \ldots, f\}\ \exists\ \{g_{2l-1}, g_{2l}\}$ ($l = 1, \ldots, r$) such that (34) is fulfilled. Furthermore, assume that $g_{2l} = g_{2l-1} + 1\ \forall\ l = 1, \ldots, r$. Here d_h, e_h ($h = 1, \ldots, f$) are defined in Sect. 2.3.

Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is an arbitrary CONS in $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \equiv 1$.

In this section, we will prove (13) under the above conditions. It is easy to see that (13) will be proved for this case if we prove that

$$\left|\sum_{j_r, j_{r-2}, \ldots, j_2=0}^{p} C_{j_r j_r j_{r-2} j_{r-2} \ldots j_2 j_2}(s,\tau)\right| \le K < \infty, \tag{35}$$

where $p \in \mathbf{N}$, $r = 2, 4, 6, \ldots$, constant K does not depend on p, s, τ (but only on t, T),

$$C_{j_k \ldots j_1}(s,\tau) = \int\limits_{\tau}^{s} \phi_{j_k}(t_k) \ldots \int\limits_{\tau}^{t_2} \phi_{j_1}(t_1) dt_1 \ldots dt_k,$$

where $k \in \mathbf{N}$, $t \le \tau < s \le T$.

Using the equality (139) [12], we obtain

$$C_{j_r j_r j_{r-2} j_{r-2} \ldots j_2 j_2}(s,\tau) + C_{j_2 j_2 \ldots j_{r-2} j_{r-2} j_r j_r}(s,\tau) =$$

$$= C_{j_r}(s,\tau) \cdot C_{j_r j_{r-2} j_{r-2} \ldots j_4 j_4 j_2 j_2}(s,\tau) - C_{j_r j_r}(s,\tau) \cdot C_{j_{r-2} j_{r-2} \ldots j_4 j_4 j_2 j_2}(s,\tau) +$$

$$+ C_{j_{r-2} j_r j_r}(s,\tau) \cdot C_{j_{r-2} j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(s,\tau) - \ldots$$

$$
-C_{j_4 j_4 \ldots j_{r-2} j_{r-2} j_r j_r}(s,\tau) \cdot C_{j_2 j_2}(s,\tau) + C_{j_2 j_4 j_4 \ldots j_{r-2} j_{r-2} j_r j_r}(s,\tau) \cdot C_{j_2}(s,\tau). \qquad (36)
$$

Applying (36), we get

$$
2 \sum_{j_r, j_{r-2}, \ldots, j_4, j_2 = 0}^{p} C_{j_r j_r j_{r-2} j_{r-2} \ldots j_4 j_4 j_2 j_2}(s,\tau) =
$$

$$
= \sum_{j_r=0}^{p} C_{j_r}(s,\tau) \sum_{j_{r-2}, \ldots, j_4, j_2 = 0}^{p} C_{j_r j_{r-2} j_{r-2} \ldots j_4 j_4 j_2 j_2}(s,\tau) -
$$

$$
- \sum_{j_r=0}^{p} C_{j_r j_r}(s,\tau) \sum_{j_{r-2}, \ldots, j_4, j_2 = 0}^{p} C_{j_{r-2} j_{r-2} \ldots j_4 j_4 j_2 j_2}(s,\tau) +
$$

$$
+ \sum_{j_{r-2}=0}^{p} \sum_{j_r=0}^{p} C_{j_{r-2} j_r j_r}(s,\tau) \sum_{j_{r-4}, \ldots, j_4, j_2 = 0}^{p} C_{j_{r-2} j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(s,\tau) - \ldots
$$

$$
- \sum_{j_r, j_{r-2}, \ldots, j_4 = 0}^{p} C_{j_4 j_4 \ldots j_{r-2} j_{r-2} j_r j_r}(s,\tau) \sum_{j_2=0}^{p} C_{j_2 j_2}(s,\tau) +
$$

$$
+ \sum_{j_2=0}^{p} \sum_{j_r, j_{r-2}, \ldots, j_4 = 0}^{p} C_{j_2 j_4 j_4 \ldots j_{r-2} j_{r-2} j_r j_r}(s,\tau) \cdot C_{j_2}(s,\tau). \qquad (37)
$$

Let us prove (35) by induction. The equality (35) is proved for $r = 2, 4$ (see the equalities (53), (55) from [13] (recall that $C_{j_1 j_1}(s,\tau) = \frac{1}{2}\left(C_{j_1}(s,\tau)\right)^2$)). Suppose that

$$
\left| \sum_{j_6, j_4, j_2 = 0}^{p} C_{j_6 j_6 j_4 j_4 j_2 j_2}(s,\tau) \right| \le K < \infty, \qquad (38)
$$

$$
\left| \sum_{j_8, j_6, j_4, j_2 = 0}^{p} C_{j_8 j_8 j_6 j_6 j_4 j_4 j_2 j_2}(s,\tau) \right| \le K < \infty, \qquad (39)
$$

$$
\ldots
$$

$$
\left| \sum_{j_{r-2}, j_{r-4}, \ldots, j_2 = 0}^{p} C_{j_{r-2} j_{r-2} j_{r-4} j_{r-4} \ldots j_2 j_2}(s,\tau) \right| \le K < \infty \qquad (40)
$$

and prove (35).

Using the induction hypothesis (see (38)–(40)), we obtain

$$\left|\sum_{j_r=0}^{p} C_{j_rj_r}(s,\tau)\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right| \le K^2 < \infty, \tag{41}$$

$$\left|\sum_{j_r,j_{r-2}=0}^{p} C_{j_{r-2}j_{r-2}j_rj_r}(s,\tau)\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-4}j_{r-4}\ldots j_4j_4j_2j_2}(s,\tau)\right| \le K^2 < \infty, \tag{42}$$

$$\ldots$$

$$\left|\sum_{j_r,j_{r-2},\ldots,j_4=0}^{p} C_{j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\sum_{j_2=0}^{p} C_{j_2j_2}(s,\tau)\right| \le K^2 < \infty. \tag{43}$$

Applying the inequality of Cauchy–Bunyakovsky, Parseval's equality and the induction hypothesis, we obtain

$$\left(\sum_{j_r=0}^{p} C_{j_r}(s,\tau)\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \le$$

$$\le \sum_{j_r=0}^{p}\left(C_{j_r}(s,\tau)\right)^2\sum_{j_r=0}^{p}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \le$$

$$\le \sum_{j_r=0}^{\infty}\left(C_{j_r}(s,\tau)\right)^2\sum_{j_r=0}^{\infty}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 \le$$

$$\le K_1\sum_{j_r=0}^{\infty}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_rj_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(s,\tau)\right)^2 =$$

$$= K_1\sum_{j_r=0}^{\infty}\left(\int_{\tau}^{s}\phi_{j_r}(u)\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(u,\tau)du\right)^2 =$$

$$= K_1\int_{\tau}^{s}\left(\sum_{j_{r-2},\ldots,j_4,j_2=0}^{p} C_{j_{r-2}j_{r-2}\ldots j_4j_4j_2j_2}(u,\tau)\right)^2 du \le$$

$$\leq K_1 K^2 \int\limits_{\tau}^{s} du \leq (T-t) K_1 K^2 = K_2 < \infty, \tag{44}$$

$$\left(\sum_{j_{r-2}=0}^{p} \sum_{j_r=0}^{p} C_{j_{r-2} j_r j_r}(s,\tau) \sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-2} j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(s,\tau) \right)^2 \leq$$

$$\leq \sum_{j_{r-2}=0}^{p} \left(\sum_{j_r=0}^{p} C_{j_{r-2} j_r j_r}(s,\tau) \right)^2 \sum_{j_{r-2}=0}^{p} \left(\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-2} j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(s,\tau) \right)^2 \leq$$

$$\leq \sum_{j_{r-2}=0}^{\infty} \left(\sum_{j_r=0}^{p} C_{j_{r-2} j_r j_r}(s,\tau) \right)^2 \sum_{j_{r-2}=0}^{\infty} \left(\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-2} j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(s,\tau) \right)^2 =$$

$$= \sum_{j_{r-2}=0}^{\infty} \left(\int\limits_{\tau}^{s} \phi_{j_{r-2}}(u) \sum_{j_r=0}^{p} C_{j_r j_r}(u,\tau) du \right)^2 \times$$

$$\times \sum_{j_{r-2}=0}^{\infty} \left(\int\limits_{\tau}^{s} \phi_{j_{r-2}}(u) \sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(u,\tau) du \right)^2 =$$

$$= \int\limits_{\tau}^{s} \left(\sum_{j_r=0}^{p} C_{j_r j_r}(u,\tau) \right)^2 du \times$$

$$\times \int\limits_{\tau}^{s} \left(\sum_{j_{r-4},\ldots,j_4,j_2=0}^{p} C_{j_{r-4} j_{r-4} \ldots j_4 j_4 j_2 j_2}(u,\tau) \right)^2 du \leq K^4 (T-t)^2 = K_3 < \infty, \tag{45}$$

where constants K_2, K_3 do not depend on p, s, τ (but only on t, T).

Similarly, we get

$$\left(\sum_{j_{r-4}=0}^{p} \sum_{j_r, j_{r-2}=0}^{p} C_{j_{r-4} j_{r-2} j_{r-2} j_r j_r}(s,\tau) \sum_{j_{r-6},\ldots,j_4,j_2=0}^{p} C_{j_{r-4} j_{r-6} j_{r-6} \ldots j_4 j_4 j_2 j_2}(s,\tau) \right)^2 \leq$$

$$\leq K_4 < \infty, \tag{46}$$

$$\ldots$$

$$\left(\sum_{j_4=0}^{p}\sum_{j_r,j_{r-2},\ldots,j_6=0}^{p} C_{j_4j_6j_6\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\sum_{j_2=0}^{p} C_{j_4j_2j_2}(s,\tau)\right)^2 \le K_4 < \infty, \qquad (47)$$

$$\left(\sum_{j_2=0}^{p}\sum_{j_r,j_{r-2},\ldots,j_4=0}^{p} C_{j_2j_4j_4\ldots j_{r-2}j_{r-2}j_rj_r}(s,\tau)\cdot C_{j_2}(s,\tau)\right)^2 \le K_4 < \infty, \qquad (48)$$

where constant K_4 does not depend on p, s, τ (but only on t, T).

Combining (37), (41)–(43), (44), (45), (46)–(48), we obtain (35). The equality (13) is proved under the conditions formulated at the beginning of this section.

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