Hints for using the MASS Library

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Abstract

The MASS library was created to provide a functional toolbox for handling mass properties of rigid bodies. This includes functions for computing the total mass of an object $\mathcal{M}$ the center of mass position $\mathbf{r}$ and the inertia tensor $\mathcal{I}$. These values depend on the chosen frame of reference. To achieve a correct physical simulation it is important to apply the correct mass values. In this short paper we will address the issues in computing the values of mass properties of rigid bodies and compounds hereof. Our contribution includes a description of how to deal with initialization and the visualization synchronizations of rigid bodies and the construction of compound bodies.

Creation of a Rigid Body

When creating content in rigid body simulators several different coordinate frames are used. We will work with three different coordinate frames

The world frame is the world coordinate frame wherein everything can be absolutely placed.

The body frame refers to the coordinate frame with its origin at the center of mass of a given rigid body and the orientation of the frame chosen such that the inertia tensor of the given body is a constant and diagonal tensor.

The model frame is some convenient frame used to describe the geometry in. The frame is fixed in the sense that it always will follow the same motion as the geometry.

The model frame needs not be physically founded, but can in principle be defined anywhere where it make sense from an artistic or modelling viewpoint. From a modelling viewpoint one would define the geometry in some model reference frame and then place the model frames in the world frame. However, when connecting to a rigid body simulator one must account find the body frame as this is the one the simualtor works with. This leads to the following steps of actions

1. Compute model frame values $\mathcal{M}$, $\mathbf{r}_M$, and $\mathcal{I}_M$
2. Transform from model frame values into the body frame values $\mathcal{I}_B$.
3. Given model frame placement in the world $\mathbf{r}$ and $\mathbf{R}$ compute the body frame placement in the world $\mathbf{r}_B$ and $\mathbf{R}_B$.

The MASS library provides routines for the first step. For the second step one would have to first transform the reference point to the center of mass position using the parallel-axis theorem (**p-a-t**)[1],

$$\mathcal{I}' = \textbf{p-a-t}(\mathcal{I}_M, -\mathbf{r}_M). \tag{1}$$

Next we can apply eigen-value-decomposition to find the body frame values[2],

$$\mathcal{I}_M = \mathbf{R}_M \mathcal{I}_B \mathbf{R}_M^T. \tag{2}$$

Now given the model frame placement in the world frame we compute,

$$\mathbf{r}_B = \mathbf{r} - \mathbf{R}\mathbf{r}_M, \tag{3a}$$
$$\mathbf{R}_B = \mathbf{R}\mathbf{R}_M. \tag{3b}$$

[1]See $mass :: translate_inertia$ function

[2]See $mass :: compute_orientation$ function

There is one more subtlety in the creation process. The rigid body simulator must know where its geometry is relative to its body frame. Thus, one should apply the transformation from model frame to body frame onto the geometry. That is first translate by $-\mathbf{r}_M$ and then rotation by $\mathbf{R}_M^{-1}$[3].

Visualization Update of a Rigid Body

During simulation a rigid body simulator will compute new values of $\mathbf{r}_B$ and $\mathbf{R}_B$. However, for the visualization one needs to find the new placement of the model frame in the world. Thus one must solve

$$\mathbf{r} = \mathbf{r}_B + \mathbf{R}\mathbf{r}_M, \tag{4a}$$

$$\mathbf{R} = \mathbf{R}_B\mathbf{R}_M^{-1}. \tag{4b}$$

Thus, one must always store the body frame to model frame transformation given by $\mathbf{r}_M$ and $\mathbf{R}_M$.

The Curious Parallel Axis Theorem

One has to be carefull when applying the parallel axis theorem. By its definition it transforms a body frame inertia tensors $\mathcal{I}$ into a modified tensor $\mathcal{I}'$ having the same orientation but a different reference point given by the translation $\mathbf{d}$

$$\mathcal{I}'_{\alpha\alpha} = \mathcal{I}_{\alpha\alpha} + \mathcal{M}(\mathbf{d}_\beta^2 + \mathbf{d}_\gamma^2) \tag{5a}$$

$$\mathcal{I}'_{\alpha\beta} = \mathcal{I}_{\alpha\beta} - \mathcal{M}(\mathbf{d}_\alpha\mathbf{d}_\beta) \tag{5b}$$

Here the primed quantities would be the modified inertia tensor that no longer lives in the body space frame. The translation given here is the vector from the body space frame origin to the new origin of the new model frame.

However, here comes the tricky part say one which to do another translation to get $\mathcal{I}''$ then one can not just apply the above formula to $\mathcal{I}'$. Instead one must first transform $\mathcal{I}'$ back to the body space frame using

$$\mathcal{I}_{\alpha\alpha} = \mathcal{I}'_{\alpha\alpha} - \mathcal{M}(\mathbf{d}_\beta^2 + \mathbf{d}_\gamma^2) \tag{6a}$$

$$\mathcal{I}_{\alpha\beta} = \mathcal{I}'_{\alpha\beta} + \mathcal{M}(\mathbf{d}_\alpha\mathbf{d}_\beta) \tag{6b}$$

and first then may one apply the transform taking one from the body space inertia tensor into the $\mathcal{I}''$ tensor.

[3] Some simultors allow one to specify these transformations directly. Other simulators assume geometry is living in body space in which case one actually has to transform the geometry.

Creating Compounds

As mass properties are defined by volume integrals it is easy to see that all we need to do is to make sure all properties are given with respect to the same reference frame. When this is the case we can simply sum up all the mass properties.

Here we will give an example of two rigid bodies their body frame inertia tensors are given by the constant diagonal tensors $\mathcal{I}_A$ and $\mathcal{I}_B$. The center of mass positions are given by $\mathbf{r}_A$ and $\mathbf{r}_B$ and the orientatio of the body frames wrt. the world frame is given by $\mathbf{R}_A$ and $\mathbf{R}_B$.

In our first step we will transform inertia tensors into the world frame

$$\mathcal{I}'_A \leftarrow \mathbf{R}_A\mathcal{I}_A\mathbf{R}_A^T, \tag{7a}$$

$$\mathcal{I}'_B \leftarrow \mathbf{R}_B\mathcal{I}_B\mathbf{R}_B^T, \tag{7b}$$

$$\mathcal{I}'^{\prime}_A \leftarrow \textbf{p-a-t}(\mathcal{I}'_A, \mathbf{r}_A), \tag{7c}$$

$$\mathcal{I}'^{\prime}_B \leftarrow \textbf{p-a-t}(\mathcal{I}'_B, \mathbf{r}_B). \tag{7d}$$

Next we may find the compund inertia tensor with reference to the world frame

$$\mathcal{I}'_C = \mathcal{I}'^{\prime}_A + \mathcal{I}'^{\prime}_B \tag{8}$$

The total mass is simply

$$\mathcal{M}_C = \mathcal{M}_A + \mathcal{M}_B \tag{9}$$

and the center of mass position is

$$\mathbf{r}_C = \frac{\mathcal{M}_A\mathbf{r}_A + \mathcal{M}_B\mathbf{r}_B}{\mathcal{M}_C} \tag{10}$$

What remains is to find the body frame inertia tensor

$$\mathcal{I}''_C \leftarrow \textbf{p-a-t}(\mathcal{I}'_C, -\mathbf{r}_C). \tag{11a}$$
$$\mathbf{R}_C \mathcal{I}_C \mathbf{R}_C^T \leftarrow \mathcal{I}''_C \tag{11b}$$

The recipe can be incremental extended straigthforwardly. Observe that there is one more snag as with the creation of rigid bodies one must ensure that geometries in the rigid body simulator is given writh respect to the new compound bodies body frame.

Handling a Deformed Box

Imagine we are given a deformed box shape. We assume that the deformation can be specified by some linear coordinate transformation and write thie mathematically as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{\Phi} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where the linear transformation is given by the matrix $\mathbf{A}$ that maps coordinates the (X, Y, Z) from a regular unit box with its center placed at the origin into the deformed coordinates (x, y, z). Here we assume that the mapping is bijective and thus $\mathbf{\Phi}$ must be invertible implying we can find the inverse mapping given by $\mathbf{\Phi}^{-1}$

We now wish to find closed form solutions for the mass properties of the deformed box. That is we wish to solve the volume integrals

$$\begin{aligned}
\mathcal{M} &= \int_v \rho dv \\
\mathbf{r}_x &= \frac{1}{\mathcal{M}} \int_v \rho \mathbf{x} dv \\
\mathbf{r}_y &= \frac{1}{\mathcal{M}} \int_v \rho \mathbf{y} dv \\
\mathbf{r}_z &= \frac{1}{\mathcal{M}} \int_v \rho \mathbf{z} dv \\
\mathcal{I}_{xx} &= \int_v \rho \left(y^2 + z^2\right) dv \\
\mathcal{I}_{yy} &= \int_v \rho \left(x^2 + y^2\right) dv \\
\mathcal{I}_{zz} &= \int_v \rho \left(x^2 + y^2\right) dv \\
\mathcal{I}_{xy} &= -\int_v \rho \left(xy\right) dv \\
\mathcal{I}_{xz} &= -\int_v \rho \left(xz\right) dv \\
\mathcal{I}_{yz} &= -\int_v \rho \left(yz\right) dv
\end{aligned}$$

Our approach to finding the closed form solutions we seek is to rewrite the volume integrals such that we are integrating over the undeformed volume. The machinery is pretty much the same for all equations so we will here just treat one term in detail and leave the remaining terms for the reader.

First we will make a change of variables using the formula $dv = jdV$ where $j = \det(\mathbf{\Phi})$

$$\begin{aligned}
\mathcal{I}_{xx} &= \int_v \rho \left(y^2 + z^2\right) dv \\
\mathcal{I}_{xx} &= j\rho \int_V \left(y^2 + z^2\right) dV
\end{aligned}$$

Secondly we observe that the deformed coordinates are a linear mapping of undeformed coordinates that means

$$\begin{aligned}
x &= \Phi_{11} X + \Phi_{12} Y + \Phi_{13} Z \\
y &= \Phi_{21} X + \Phi_{22} Y + \Phi_{23} Z \\
z &= \Phi_{31} X + \Phi_{32} Y + \Phi_{33} Z
\end{aligned}$$

using this we have that

$$y^2 + z^2 = \left(\Phi_{21}X + \Phi_{22}Y + \Phi_{23}Z\right)^2 + \left(\Phi_{31}X + \Phi_{32}Y + \Phi_{33}Z\right)^2 = P_2(X, Y, Z)$$

where P_2 denotes a general second order polynomial in X, Y and Z. So now the volume integral reads

$$\begin{aligned}\mathcal{I}_{xx} &= j\rho \int_V P_2(X, Y, Z) dV \\ &= j\rho \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} P_2(X, Y, Z) dX dY dZ\end{aligned}$$

Which is straightforward to solve for a closed form solution. All that are missing in order to do this is formulas for the coefficients of P_2 in the given integral these are

$$\begin{aligned}P_2(X, Y, Z) = &\underbrace{\left(\Phi_{21}^2 + \Phi_{31}^2\right)}_{a_{XX}} X^2 \\ &+ \underbrace{\left(\Phi_{22}^2 + \Phi_{32}^2\right)}_{a_{YY}} Y^2 \\ &+ \underbrace{\left(\Phi_{23}^2 + \Phi_{33}^2\right)}_{a_{ZZ}} Z^2 \\ &+ \underbrace{2\left(\Phi_{21}\Phi_{22} + \Phi_{31}\Phi_{32}\right)}_{a_{XY}} XY \\ &+ \underbrace{2\left(\Phi_{21}\Phi_{23} + \Phi_{31}\Phi_{33}\right)}_{a_{XZ}} XZ \\ &+ \underbrace{2\left(\Phi_{22}\Phi_{23} + \Phi_{32}\Phi_{33}\right)}_{a_{YZ}} YZ\end{aligned}$$

From this we find

$$\mathcal{I}_{xx} = j\rho \left[\begin{array}{c} \frac{a_{XX}}{3} X^3 Y Z + \frac{a_{yy}}{3} X Y^3 Z + \\ \frac{a_{zz}}{3} X Y Z^3 + \frac{a_{XY}}{4} X^2 Y^2 Z + \\ \frac{a_{XZ}}{4} X^2 Y Z^2 + \frac{a_{YZ}}{4} X Y^2 Z^2 \end{array} \right]_{(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})}^{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}$$

From this we can generalize the recipe for deriving a closed form solution for all the volume intergrals. The other principal moments of inertia will be permutations of the above derivation. The products of moments are all similar and will also result in integrals of second order polynomials. The center of mass integrals all results in integrals of linear polynomials and the mass volume integral is straightforward.