Two new approaches to smoothing over complex regions

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Outline

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Smoothing in 2 dimensions

- Have some geographical region and wish to find out something about the biological population in it.
- Response is eg. animal distribution, wish to predict based on (x, y) and other covariates eg. habitat, size, sex, etc.
- This problem is relatively easy if the domain is simple.

Smoothing with penalties

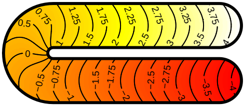
- Objective function takes the form:

$$\sum_{i=1}^{n}(z_i - f(x_i, y_i; \theta))^2 + \lambda \int_{\Omega} Pf(x, y; \theta) d\Omega$$

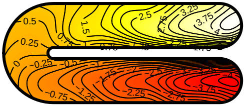
- f is the function you want to estimate, made up of some combination of basis functions.
- P is some squared derivative penalty operator, usually $P = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^2$.
- This can be generalized to an additive model or GAM.

Smoothing over complex domains

- Smoothing of complex domains makes this a lot more difficult.
- Problem of leakage.
- Euclidean distance doesn't always make sense.
- Models need to incorporate information about the intrinsic structure of the domain.



(modified) Ramsay test function



Thin plate spline fit

Possible solutions to leakage problems

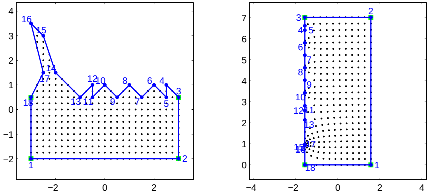
- FELSPLINE (Ramsay, (2002).)
- Within-area distance (Wang and Ranalli, (2007).)
- Soap film smoothers (Wood *et al.* (2008).)
- Domain morphing (Eilers (2006).)

Why morph the domain?

- Takes into account within-area distance.
- Gives a known domain that is easier to smooth over.
- Potentially less computationally intensive.

However:

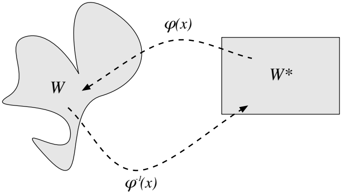
- Don't maintain isotropy - distribution of points odd.
- Not clear what this does to the smoothness penalty.



Outline

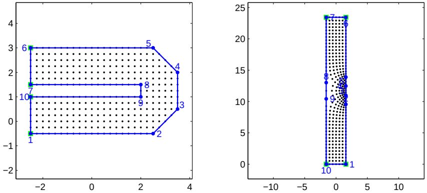
The Schwarz-Christoffel transform

- Take a polygon in some domain W and morph it to a new domain W^*.
- Do this by starting at the new domain and working back to the polygon.
- Can draw a polygonal bounding box around some arbitrary shape.



Schwarz-Christoffel algorithm

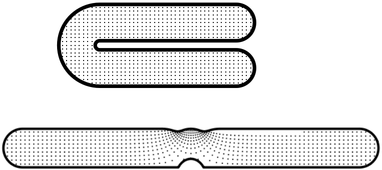
- Start with a rectangle.
- Add vertices.
- Iteratively deform by changing angles.
- Continue until the new shape is identical to the polygon.
- We then have a function for the mapping, $\varphi(x, y)$.
- $\varphi(x, y)$ is a conformal mapping.



The mapping

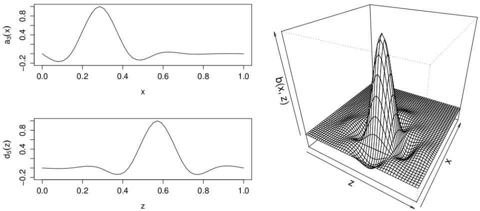
- Use a bounding box around the horseshoe.

- Morphing the horseshoe shape still gives a slightly odd domain however, we are still doing better than before.

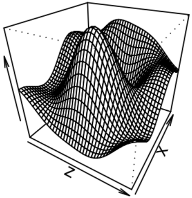


Results

- Fit using both thin plate regression splines and P-splines with a comparison to the soap film (current best.)
- MSE used to compare over a grid of predicition points.
- Transformation method does well on the horseshoe.



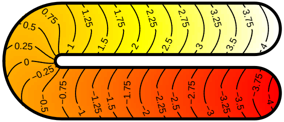
Tensor product



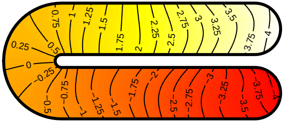
Thin plate

Horseshoe plots

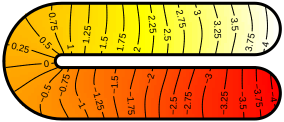
Truth



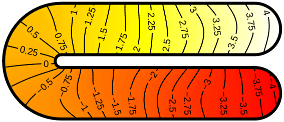
SC+PS



SC+TPRS

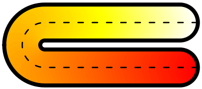


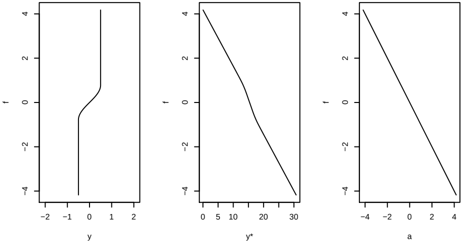
Soap film



Why does it do well?

- Looking at line plots, we can see the difference in gradient.
- SC method seems to approximate the gradient better.

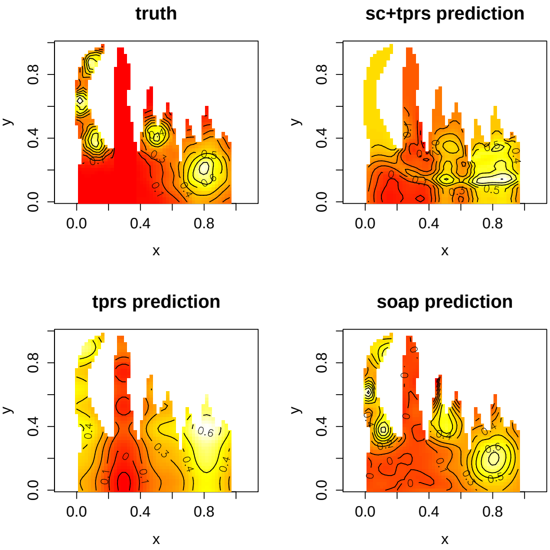




Problems

- Implementation is Matlab+R.
- Problems with more realistic situations:
  - Weird artifacts.
  - Arbitrary selection of vertices.
  - Morphing of domain appears to cause features to be smoothed over.

A more realistic domain



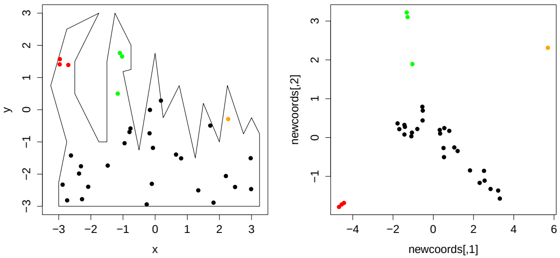
Outline

Multidimensional scaling and within-area distances

- Idea: use MDS to to arrange points in the domain according to their “within-domain distance.”
- First need to find the within area distances.
- Can then smooth over the new points.

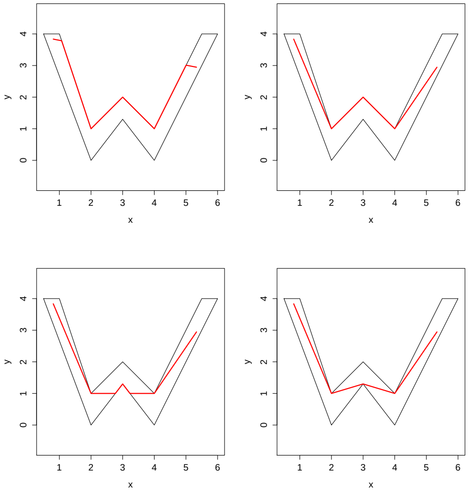
Multidimensional scaling refresher

- Double centre matrix of between point distances, D, (subtract row and column means) then find DD^T.
- Finds a configuration of points such that
- Euclidean distance between points in new arrangement is approximately the same as distance in the domain.
- Prediction outside the original data is fine.
- Already implemented in R by `cmdscale`.



Finding within-area distances

- Use a new algorithm of Simon's to find the within area distances.



Ramsey simulations

- again test on Ramsey's horseshoe. [lots of pics]

Other domains

- test on other domains [lots of pics]

Outline

Conclusions

- Seems that the S-C transform does not have much utility.
- MDS shows more promise.
- MDS easier to transfer to higher dimensions.

Further work

- Real data!

References

- S.N. Wood, M.V. Bravington, and S.L. Hedley. *Soap film smoothing. JRSSB, 2008*
- H. Wang and M.G. Ranalli. *Low-rank smoothing splines on complicated domains. Biometrics, 2007*
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- C. Chatfield and A.J. Colins *Introduction to multivariate analysis. CRC, 1980*