Using the Schwarz-Christoffel transform to smooth over complex regions

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Outline

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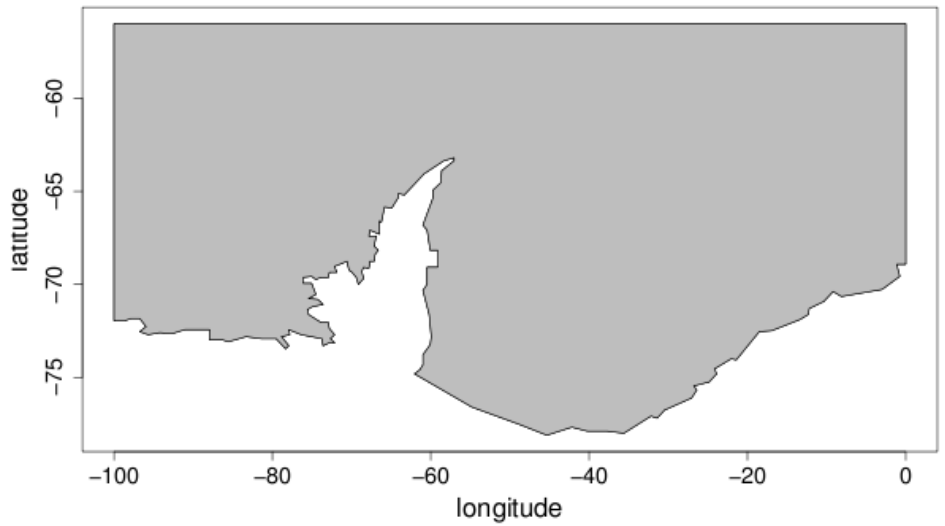
Smoothing in 2 dimensions

- Have some geographical region and wish to find out something about the biological population in it.
- Response is eg. animal distribution, wish to predict based on (x, y) and other covariates eg. habitat, size, sex, etc.
- This problem is relatively easy if the domain is simple.



Smoothing with GAMs

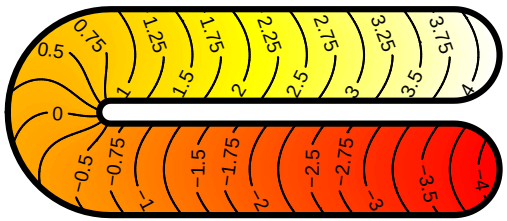
- Take a GLM, throw in some non-parametric components.
- Objective function takes the form:

$$\sum_{i=1}^{n}(z_i - f(x_i, y_i; \theta))^2 + \lambda \int_{\Omega} Pf(x, y; \theta) d\Omega$$

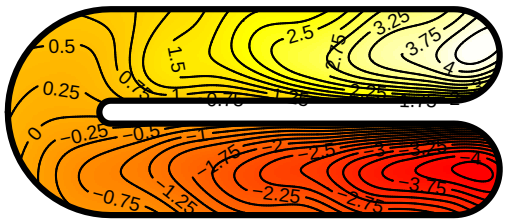
- f is the function you want to estimate, made up of basis functions, so $z_i = f(x_i, y_i; \theta) + \epsilon_i$.
- P is some penalty function, usually second squared derivative.

Smoothing over complex domains

- Smoothing of complex domains makes this a lot more difficult.
- Problem of leakage.
- Euclidean distance doesn't always make sense.
- Models need to incorporate information about the intrinsic structure of the domain.



Ramsay test function



Thin plate spline fit

Possible solutions to leakage problems

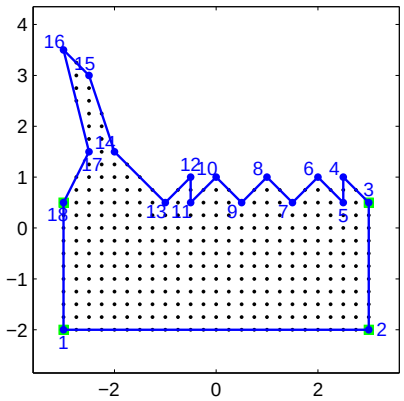
- FELSPLINE (Ramsay, (2002).)
- Within-area distance (Wang and Ranalli, (2007).)
- Soap film smoothers (Wood *et al.* (2008).)
- Domain morphing (me!)

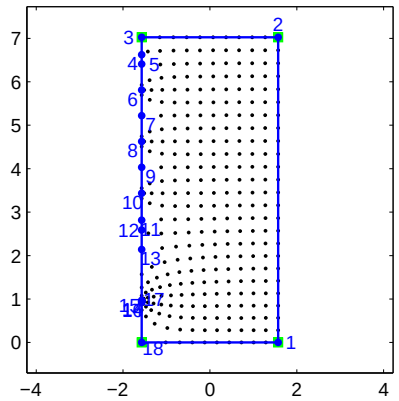
Why morph the domain?

- Takes into account within-area distance.
- Gives a known domain that is easier to smooth over.
- Potentially less computationally intensive.

However:

- Don't maintain isotropy - distribution of points odd.
- Not clear what this does to the smoothness penalty.

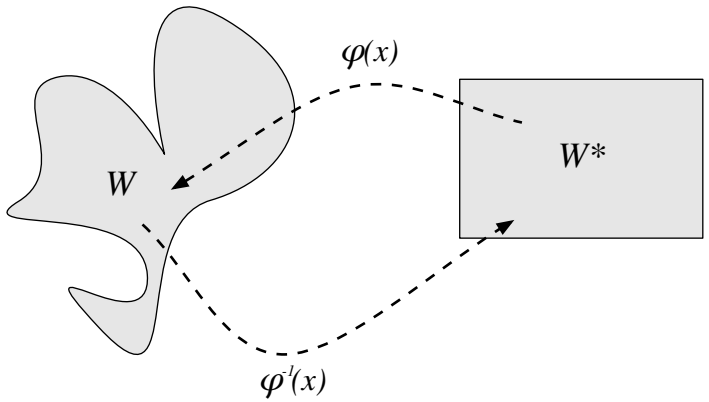




Outline

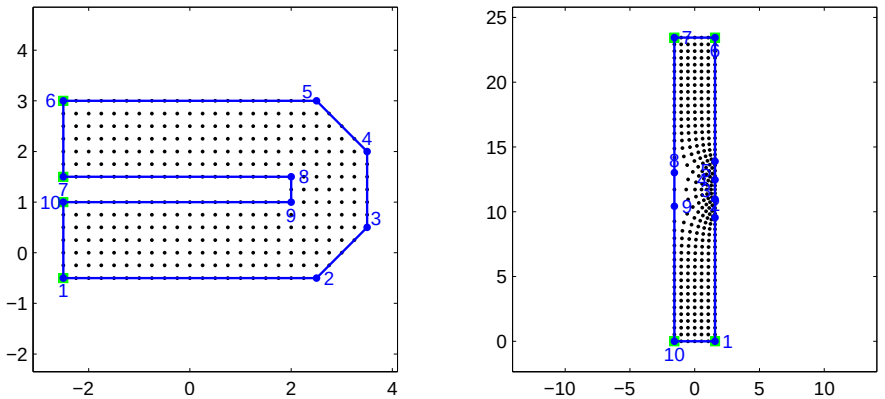
The Schwarz-Christoffel transform

- Take a polygon in some domain W and morph it to a new domain W^*.
- Do this by starting at the new domain and working back to the polygon.
- Can draw a polygonal bounding box around some arbitrary shape.



Schwarz-Christoffel algorithm

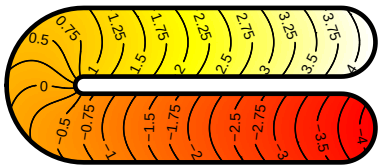
- Start with a rectangle.
- Add vertices.
- Iteratively deform by changing angles.
- Continue until the new shape is identical to the polygon.
- We then have a function for the mapping, $\varphi(x, y)$.
- $\varphi(x, y)$ is a conformal mapping.



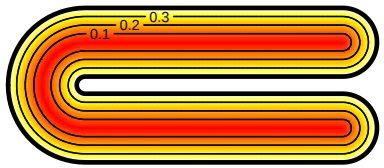
Outline

Ramsay's horseshoes

- Ramsay proposed a horseshoe test function that precisely demonstrates the problem.
- An alternate function was also proposed.
- Simulation consisted of taking a sample from the functions, adding noise and then trying to fit a transformed model along with a soap film for comparison.



Ramsay test function

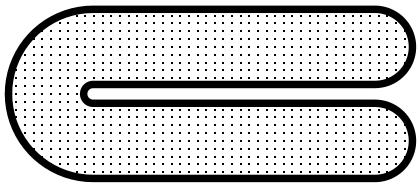


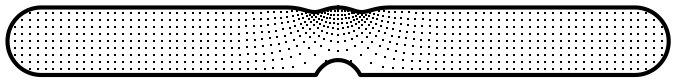
Alternate Ramsay test function

The mapping

- Use a bounding box around the horseshoe.

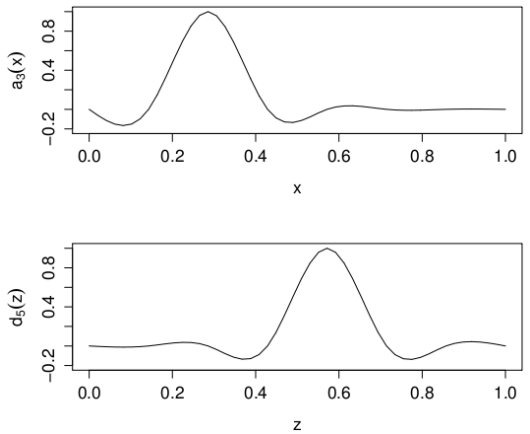
- Morphing the horseshoe shape still gives a slightly odd domain however, we are still doing better than before.

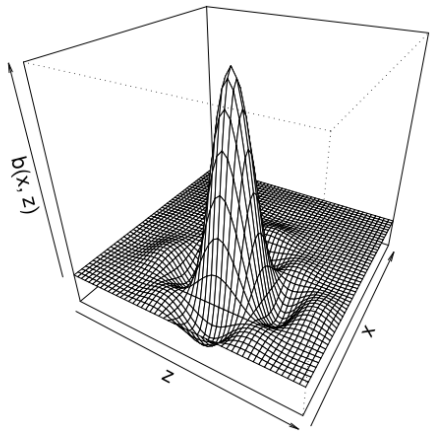




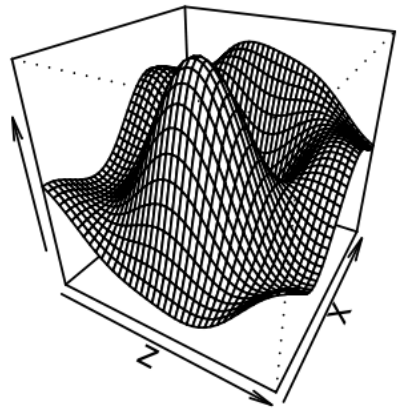
Results

- Fit using both thin plate regression splines and P-splines with a comparison to the soap film (current best.)
- MSE used to compare over a grid of predicition points.
- Transformation method does well on the first horseshoe.
- Not so well for the alternate horseshoe.





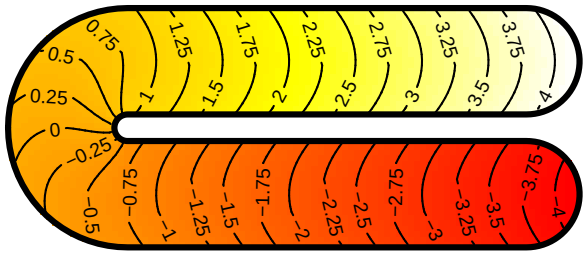
Tensor product



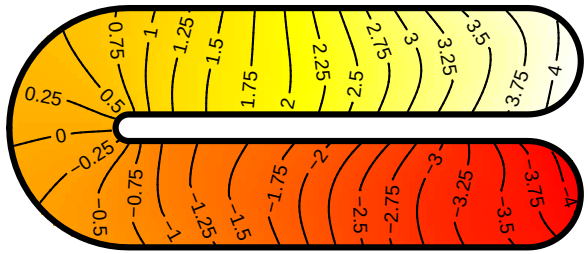
Thin plate

Horseshoe plots

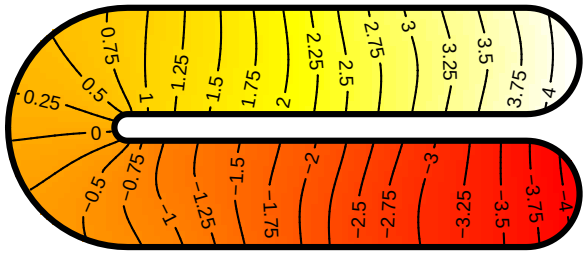
Truth



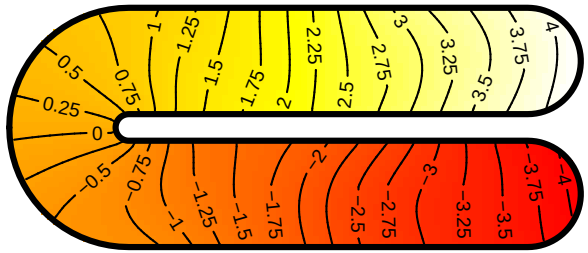
SC+PS



SC+TPRS

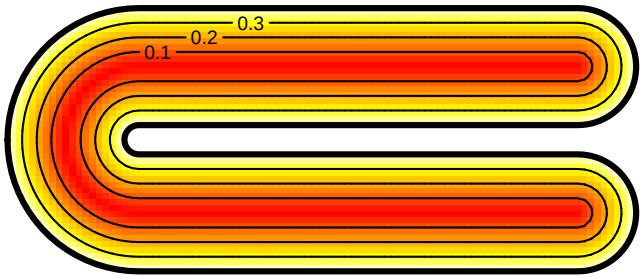


Soap film

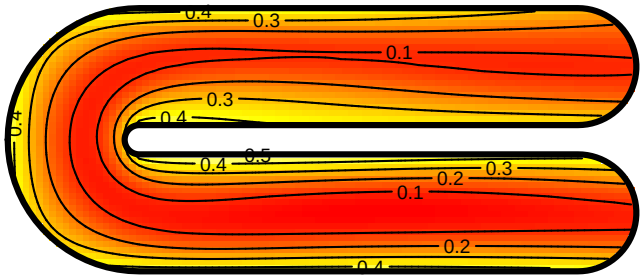


Alternate horseshoe plots

Truth



SC+PS



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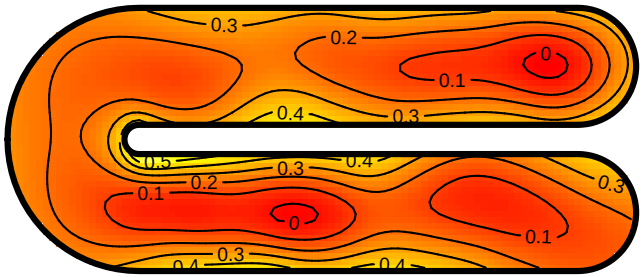
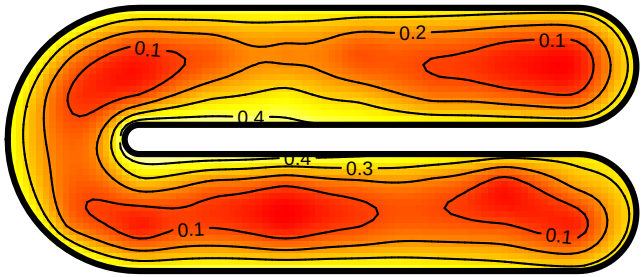
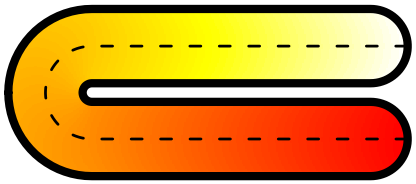


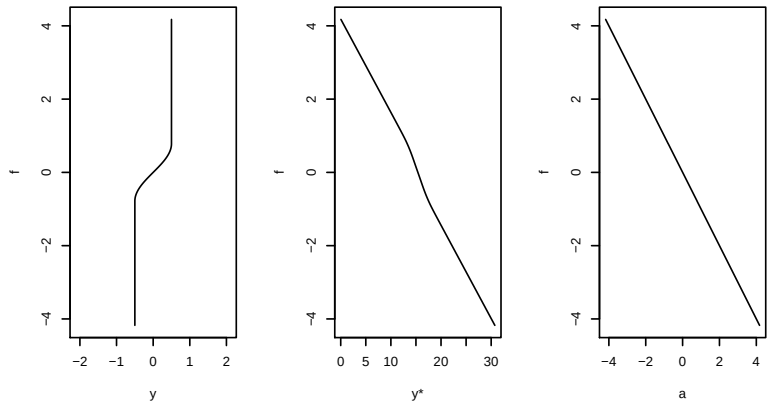
Soap film



Why does it (sometimes) do better?

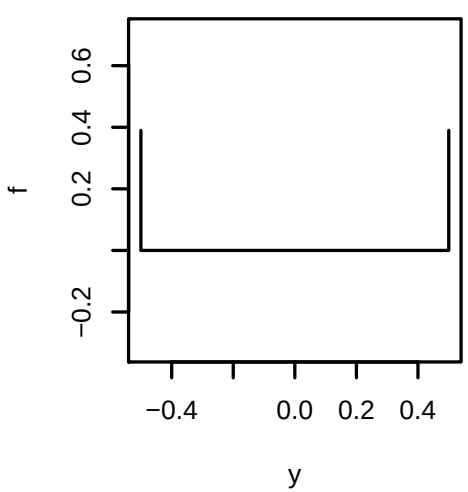
- Looking at line plots, we can see the difference in gradient.
- SC method seems to approximate the gradient better.

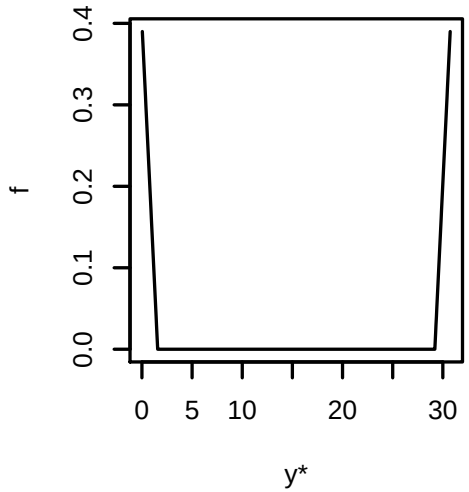


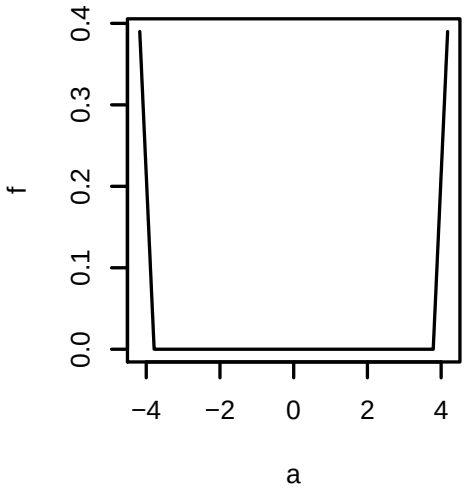


Why does it (sometimes) do worse?

- SC method still seems to approximate the gradient very well here.
- Ignoring the two ends (problem with tensor products.)
- Get around this with additional splines?







Where next?

- Real data!
- Looking at alternative penalties for P-splines.
- Implementation is Matlab+R, would be nice to have something more coherent (and not Matlab.)

References

- S.N. Wood, M.V. Bravington, and S.L. Hedley. *Soap film smoothing. JRSSB, 2008*
- H. Wang and M.G. Ranalli. *Low-rank smoothing splines on complicated domains. Biometrics, 2007*
- T.A. Driscoll and L.N. Trefethen. *Schwarz-Christoffel Mapping. Cambridge, 2002*
- T. Ramsay. *Spline smoothing over difficult regions. JRSSB, 2001*