

Using the Schwarz-Christoffel transform to smooth over complex regions

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Outline

Smoothing over complex regions

- Intro

- Solutions

Domain morphing with the Schwarz-Christoffel transform

- Details

Simulation experiment

- Ramsay's horseshoes

- Results

- Analysis

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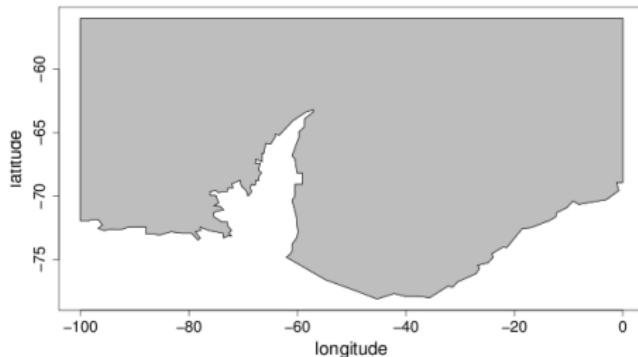
Ramsay's horseshoes

Results

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Smoothing in 2 dimensions

- ▶ Have some geographical region and wish to find out something about the biological population in it.
- ▶ Response is eg. animal distribution, wish to predict based on (x, y) and other covariates eg. habitat, size, sex, etc.
- ▶ This problem is relatively easy if the domain is simple.



Smoothing with GAMs

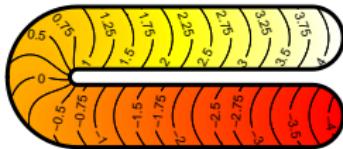
- ▶ Take a GLM, throw in some non-parametric components.
- ▶ Objective function takes the form:

$$\sum_{i=1}^n (z_i - f(x_i, y_i; \theta))^2 + \lambda \int_{\Omega} P f(x, y; \theta) d\Omega$$

- ▶ f is the function you want to estimate, made up of basis functions, so $z_i = f(x_i, y_i; \theta) + \epsilon_i$.
- ▶ P is some penalty function, usually second squared derivative.

Smoothing over complex domains

- ▶ Smoothing of complex domains makes this a lot more difficult.
- ▶ Problem of leakage.
- ▶ Euclidean distance doesn't always make sense.
- ▶ Models need to incorporate information about the intrinsic structure of the domain.



Ramsay test function



Thin plate spline fit

Possible solutions to leakage problems

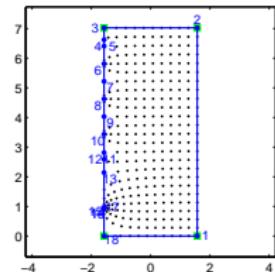
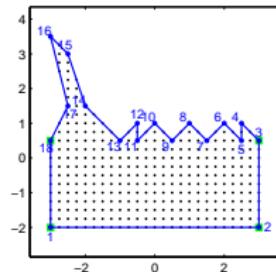
- ▶ FELSPLINE (Ramsay, (2002).)
- ▶ Within-area distance (Wang and Ranalli, (2007).)
- ▶ Soap film smoothers (Wood *et al.* (2008).)
- ▶ Domain morphing (me!)

Why morph the domain?

- ▶ Takes into account within-area distance.
- ▶ Gives a known domain that is easier to smooth over.
- ▶ Potentially less computationally intensive.

However:

- ▶ Don't maintain isotropy - distribution of points odd.
- ▶ Not clear what this does to the smoothness penalty.



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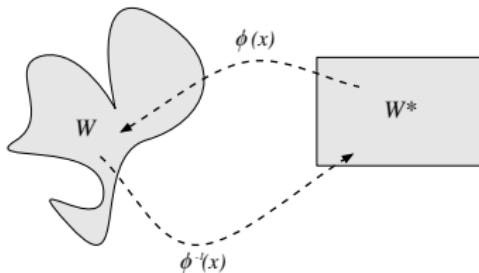
Ramsay's horseshoes

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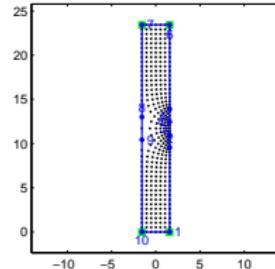
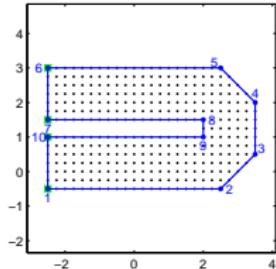
The Schwarz-Christoffel transform

- ▶ Take a polygon in some domain W and morph it to a new domain W^* .
- ▶ Do this by starting at the new domain and working back to the polygon.
- ▶ Can draw a polygonal bounding box around some arbitrary shape.



Schwarz-Christoffel algorithm

- ▶ Start with a rectangle.
- ▶ Add edges.
- ▶ Iteratively deform by changing angles.
- ▶ Continue until the new shape is identical to the polygon.
- ▶ We then have a function for the mapping, $\varphi(x, y)$.
- ▶ $\varphi(x, y)$ is a conformal mapping.



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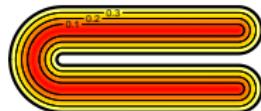
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Ramsay's horseshoes

- ▶ Ramsay proposed a horseshoe test function that precisely demonstrates the problem.
- ▶ An alternate function was also proposed.
- ▶ Simulation consisted of taking a sample from the functions, adding noise and then trying to fit a transformed model along with a soap film for comparison.



Ramsay test function



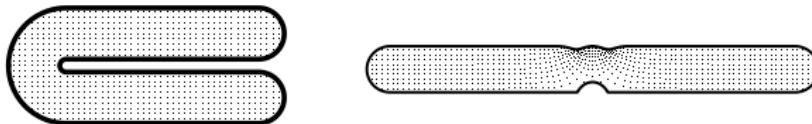
Alternate Ramsay
test function

The mapping

- ▶ Use a bounding box around the horseshoe.

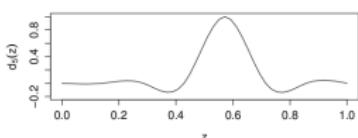
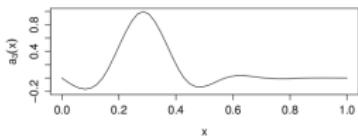


- ▶ Morphing the horseshoe shape still gives a slightly odd domain however, we are still doing better than before.

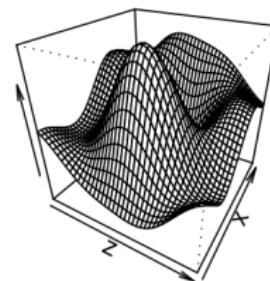
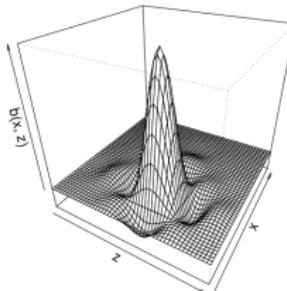


Results

- ▶ Fit using both thin plate regression splines and P-splines with a comparison to the soap film (current best.)
- ▶ MSE used to compare over a grid of prediction points.
- ▶ Transformation method does well on the first horseshoe.
- ▶ Not so well for the alternate horseshoe.



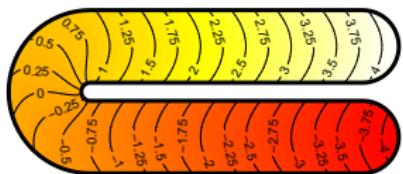
Tensor product



Thin plate

Horseshoe plots

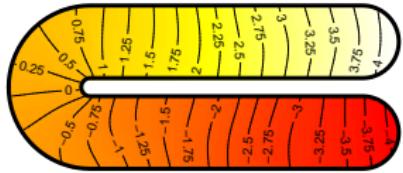
Truth



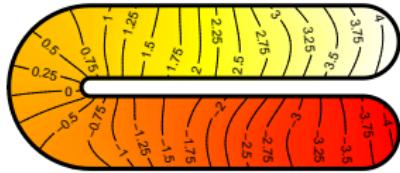
SC+PS



SC+TPRS

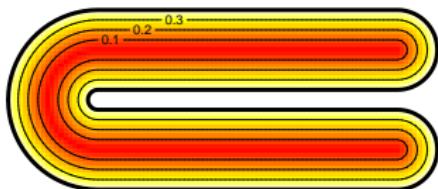


Soap film

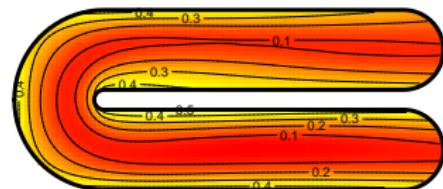


Alternate horseshoe plots

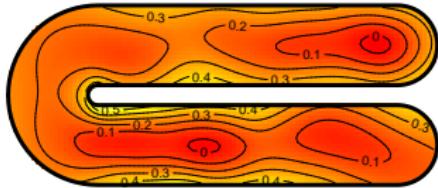
Truth



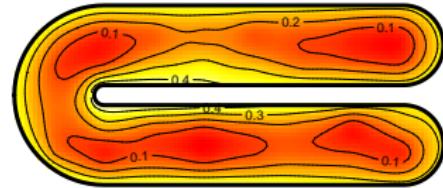
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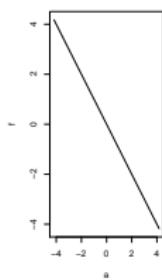
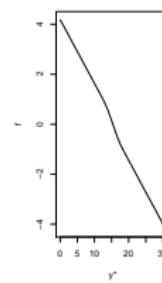
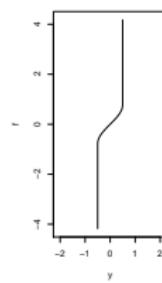
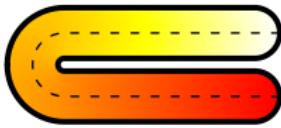


Soap film



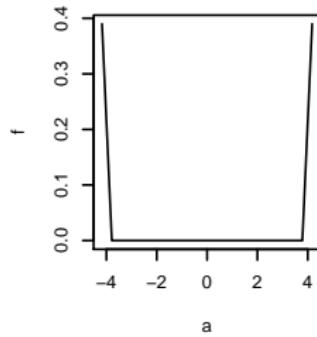
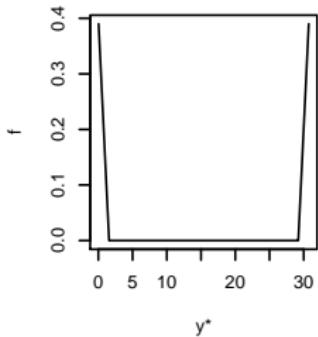
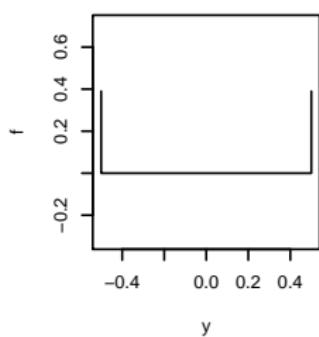
Why does it (sometimes) do better?

- ▶ Looking at line plots, we can see the difference in gradient.
- ▶ SC method seems to approximate the gradient better.



Why does it (sometimes) do worse?

- ▶ SC method still seems to approximate the gradient very well here.
- ▶ Ignoring the two ends (problem with tensor products.)
- ▶ Get around this with additional splines?



Where next?

- ▶ Real data!
- ▶ Looking at alternative penalties for P-splines.
- ▶ Implementation is Matlab+R, would be nice to have something more coherent (and not Matlab.)

References

- ▶ S.N. Wood, M.V. Bravington, and S.L. Hedley. *Soap film smoothing*. *JRSSB*, 2008
- ▶ H. Wang and M.G. Ranalli. *Low-rank smoothing splines on complicated domains*. *Biometrics*, 2007
- ▶ T.A. Driscoll and L.N. Trefethen. *Schwarz-Christoffel Mapping*. Cambridge, 2002
- ▶ T. Ramsay. *Spline smoothing over difficult regions*. *JRSSB*, 2001