Random dynamical systems

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1 Introduction

The main purpose of this survey is to present and popularize the notion of a *random dynamical system* (RDS) and to give an impression of its scope. The notion of RDS covers the most important families of dynamical systems with randomness which are currently of interest. For instance, products of random maps — in particular products of random matrices — are RDS as well as (the solution flows of) stochastic and random ordinary and partial differential equations.

One of the basic results for RDS is the Multiplicative Ergodic Theorem (MET) of Oseledec [38]. Originally formulated for products of random matrices, it has been reformulated and reproved several times during the past twenty years. Basically, there are two classes of proofs. One makes use of Kingman's Subadditive Ergodic Theorem together with the polar decomposition of square matrices. The other one starts by proving the assertions of the MET for triangular systems, and then enlarges the probability space by the compact group of special orthogonal matrices, so that every matrix cocycle becomes homologous to a triangular one.

Let us emphasize that the MET is a *linear* result. It is possible to introduce Lyapunov exponent-like quantities for nonlinear systems directly à la (9) below, or, much more sophisticated, as by Kifer [27]. However, the wealth of structure provided by the MET is available for linear systems only. Speaking of an "MET for nonlinear systems" always means the MET for the *linearization* of a nonlinear system. What is new for nonlinear systems is the fact that the linearization lives on the tangent bundle of a manifold (instead of the flat bundle $\mathbb{R}^d \times \Omega$ as for products of random matrices). The MET yields nontrivial consequences for deterministic systems already. This case has been dealt with by Ruelle [39]. Ruelle's argument proceeds by trivialization of the nonflat tangent bundle. It is exactly the same argument that works for nonlinear random systems: infer the MET for the linearization of the system from the ordinary MET together with a trivialization argument. We reproduce the argument below.

Stochastic flows have entered the scene a couple of years ago. They are related to RDS, but they are not the same. We describe their relations, and point out their differences.

The final Section briefly reviews all contributions to the present volume.

2 Random Dynamical Systems and Multiplicative Ergodic Theory

2.1 RDS

Consider a set T (time), $T = \mathbb{R}$, $\mathbb{Z}$; $\mathbb{R}^+$, or $\mathbb{N}$, and a family $\{\vartheta_t : \Omega \to \Omega \mid t \in T\}$ of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$ such that $(t, \omega) \mapsto \vartheta_t\omega$ is measurable, $\{\vartheta_t \mid t \in T\}$ is ergodic, and $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $t, s \in T$ with $\vartheta_0 = \mathrm{id}$. Thus $(\vartheta_t)_{t\in T}$ is a flow if $T = \mathbb{R}$ or $\mathbb{Z}$, and a semi-flow if $T = \mathbb{R}^+$ or $\mathbb{N}$. The set-up $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t\in T})$ is a (measurable) dynamical system.

Definition A *random dynamical system* on a measurable space $(X, \mathcal{B})$ over $(\vartheta_t)_{t\in T}$ on $(\Omega, \mathcal{F}, P)$ is a measurable map

$$\varphi : T \times X \times \Omega \to X$$

such that $\varphi(0, \omega) = \mathrm{id}$ (identity on X) and

$$\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega) \tag{1}$$

for all $t, s \in T$ and for all ω outside a P-nullset, where $\varphi(t, \omega) : X \to X$ is the map which arises when $t \in T$ and $\omega \in \Omega$ are fixed, and $\circ$ means composition. A family of maps $\varphi(t, \omega)$ satisfying (1) is called a *cocycle*, and (1) is the cocycle property.

We often omit mentioning $((\Omega, \mathcal{F}, P), (\vartheta_t)_{t\in T})$ in the following, speaking of a random dynamical system (abbreviated RDS) φ.
We do not assume the maps $\varphi(t, \omega)$ to be invertible a priori. By the cocycle property, $\varphi(t, \omega)$ is automatically invertible (for all $t \in T$ and for P-almost all ω) if $T = \mathbb{R}$ or $\mathbb{Z}$, and $\varphi(t, \omega)^{-1} = \varphi(-t, \vartheta_t\omega)$.

The following examples are quite distinct in many respects. However, they all are RDS.

1. The simplest case of a random dynamical system is a non-random — viz., deterministic — dynamical system. An RDS is deterministic if φ does not depend on ω, i. e., $\varphi(t, x, \omega) = \varphi(t, x)$. Then the cocycle property (1) reads $\varphi(t+s) = \varphi(t)\circ\varphi(s)$, hence $(\varphi(t))_{t\in T}$ consists of the iterates of a measurable map on X if $T = \mathbb{Z}^{(+)}$,and $(\varphi(t))_{t\in T}$ is a measurable (semi-) flow if $T = \mathbb{R}^{(+)}$, respectively.

2. Let $\vartheta : \Omega \to \Omega$ be a measure preserving transformation, and let $\psi : X \times \Omega \to X$ be a measurable map. Put $\psi_n = \psi \circ \vartheta^{n-1}$. Then

$$\varphi(n, \omega) = \begin{cases} \psi_n(\omega) \circ \psi_{n-1}(\omega) \ldots \circ \psi_1(\omega) & \text{for} \quad n > 0 \\ \mathrm{id} & \text{for} \quad n = 0 \\ \psi_{n+1}^{-1}(\omega) \circ \psi_{n+2}^{-1}(\omega) \circ \ldots \circ \psi_0^{-1}(\omega) & \text{for} \quad n < 0, \end{cases}$$

defines an RDS (of course, defining $\varphi(n, \omega)$ for $n < 0$ needs ϑ and $\psi(\cdot, \omega)$ invertible P-a. s.). In particular, if $X = \mathbb{R}^d$ and $x \mapsto \psi(x, \omega)$ is linear, then φ is a product of random matrices.

3. Suppose $T = \mathbb{R}$, and M is a C^1 manifold. Denote by TM the total space of the tangent bundle of M, and let $Y : M \times \Omega \to TM$ be a measurable map such that for P-almost all ω the map $Y(\cdot, \omega)$ is a smooth vector field. Then the random differential equation

$$\dot{x}(t) = Y(x(t), \vartheta_t \omega), \qquad x(0) = x_0, \tag{2}$$

induces a map $\varphi(t, \omega) : M \to M$, such that $x(t, \omega) = \varphi(t, \omega)x$ solves (2) with $x(0) = x$ for $t \in (t^-(x, \omega), t^+(x, \omega))$, where $t^-(x, \cdot) < 0 < t^+(x, \cdot)$ (P-a.s.) describe the maximal intervals of definition of solutions. If $t^-(x, \cdot) = -\infty$ and $t^+(x, \cdot) = +\infty$ (for all $x \in M$ P-a.s.) then φ is an RDS. In addition, $x \mapsto \varphi(t, \omega)x$ is a diffeomorphism for all $t \in \mathbb{R}$ (P-a.s.) in this case. The maximal interval of definition is automatically all of $\mathbb{R}$ if M is compact. If $-\infty < t^-(x)$ or $t^+(x) < \infty$ for some x with positive probability we speak of a *local RDS* or *local random flow*.

4. Suppose M is a C^2 manifold, and Y_i, $0 \le i \le n$, are smooth vector fields on M. Then the stochastic differential equation

$$dx(t) = Y_0(x(t))\,dt + \sum_{i=1}^{n} Y_i(x(t)) \circ dW_i(t), \qquad x(0) = x_0, \tag{3}$$

induces a *(local) stochastic flow*. Usually (3) is understood for $t \ge 0$. We will describe below how to give (3) a meaning on the whole time axis. Once this is done, maximal intervals of solutions, containing $t = 0$ as an interior point, exist and have the same properties as for random flows described in the previous example.

We have introduced *local* random and stochastic flows because they play a role in stochastic bifurcation. For details see below.

As for deterministic systems, RDS may be classified according to their spatial properties.

If X is a topological space (with Borel σ-algebra), a random dynamical system is said to be *continuous* if $\varphi(t, \omega) : X \to X$ is continuous for all $t \in T$ and all $\omega \in \Omega$ outside a P-nullset.
If X is a C^r manifold, $r \ge 1$, an RDS φ on X is said to be *differentiable* or *smooth* if $\varphi(t, \omega) : X \to X$ is C^r differentiable for all $t \in T$ and all ω outside a P-nullset.
A random dynamical system on a topological vector space X is said to be linear if $\varphi(t, \omega) : X \to X$ is linear for all $t \in T$ and all ω outside a P-nullset.

If an RDS consists of non-invertible maps then T cannot contain negative times. An RDS φ consisting of invertible maps need not allow negative time, since ϑ_t need not be invertible. So we have to distinguish between two kinds of invertibility. An RDS is said to be *two sided* if $T = \mathbb{R}$ or $T = \mathbb{Z}$. It is said to be *invertible* if, for all $t \in T$ and P-almost all ω, $\varphi(t, \omega)$ is invertible in the corresponding class (measurable, continuous, smooth). Clearly 'two sided' is stronger than 'invertible'.

Any RDS induces a measurable *skew product (semi-) flow*

$$\begin{aligned} \Theta_t : X \times \Omega &\to X \times \Omega \\ (x, \omega) &\mapsto (\varphi(t, \omega)x, \vartheta_t \omega), \end{aligned} \tag{4}$$

$t \in T$, where $\varphi(t,\omega)x = \varphi(t,x,\omega)$. The flow property $\Theta_{t+s} = \Theta_t \circ \Theta_s$ follows from the cocycle property of φ (see (1); we use the term flow for both continuous and discrete time T).

From the point of view of abstract ergodic theory, an RDS is nothing but an ordinary dynamical system $(\Theta_t)_{t\in T}$ with a factor $(\vartheta_t)_{t\in T}$ together with the extra bit of structure provided by the fact that the ergodic invariant measure P for the factor is given a priori. (This observation might serve as an abstract definition of RDS.)

A probability measure μ on $X \times \Omega$ (on the product σ-algebra $\mathcal{B} \otimes \mathcal{F}$) is said to be an *invariant measure for* φ if μ is invariant under Θ_t, $t \in T$, and if it has marginal P on Ω. Invariant measures always exist for continuous RDS on a compact X (which is in complete analogy with deterministic dynamical systems).

Denote by $Pr(X)$ the space of probability measures on X, endowed with the smallest σ-algebra making the maps $Pr(X) \to \mathbb{R}$, $\nu \mapsto \int_X h\, d\nu$, measurable with h varying over the bounded measurable functions on X.
Given a measure $\mu \in Pr(X \times \Omega)$ with marginal P on Ω, a measurable map $\mu_{\bullet} : \Omega \to Pr(X)$, $\omega \mapsto \mu_\omega$ will be called a *disintegration of* μ (*with respect to* P) if

$$\mu(B \times C) = \int\limits_C \mu_\omega(B)\, dP(\omega)$$

for all $B \in \mathcal{B}$ and $C \in \mathcal{F}$.
Disintegrations exist and are unique (P-a. s.), e. g., if X is a Polish space. We will assume existence and uniqueness of a disintegration in the following.
A measure μ is invariant for the RDS φ if and only if

$$E(\varphi(t,\cdot)\mu_{\bullet} \mid \vartheta_t^{-1}\mathcal{F})(\omega) = \mu_{\vartheta_t\omega} \qquad P\text{-a. s. for every } t \in T. \tag{5}$$

If T is two sided then $\vartheta_t^{-1}\mathcal{F} = \mathcal{F}$, hence for T two sided (5) reads

$$\varphi(t,\omega)\mu_\omega = \mu_{\vartheta_t\omega} \qquad P\text{-a. s. for every } t \in T.$$

2.2 Lyapunov exponents and the Multiplicative Ergodic Theorem

For a differentiable manifold M denote by TM the total space of its tangent bundle. The linearization of a differentiable map $\psi : M \to M$ is denoted by $T\psi : TM \to TM$ with $T_x\psi : T_xM \to T_{\psi(x)}M$, $x \in M$, denoting the action of $T\psi$ on individual fibers.

Suppose φ is a smooth RDS on a d-dimensional Riemannian manifold M. The chain rule yields

$$T_x\varphi(t+s,\omega) = T_{\varphi(s,\omega)x}\varphi(t,\omega) \circ T_x\varphi(s,\omega) \tag{6}$$

for all $t, s \in T$ and $x \in M$ with P-measure 1. Consequently, the linearization $T\varphi : T \times TM \times \Omega \to TM$ is a cocycle over the skew product flow $\Theta_t(x,\omega) = (\varphi(t,\omega)x, \vartheta_t\omega)$ on $M \times \Omega$ (cf. (4)).

Suppose μ is an invariant measure for φ such that

$$(x,\omega) \mapsto \sup_{0<t\le t_0} \log^+(\|T_x\varphi(t,\omega)\|) \ \in \ L^1(\mu), \tag{7}$$

where $\log^+ = \max\{\log, 0\}$, and $\|\cdot\|$ denotes the norm induced by the Riemannian metric. Denote by $\lambda_1^\mu(x,\omega) \ge \lambda_2^\mu(x,\omega) \ge \ldots \ge \lambda_d^\mu(x,\omega)$ the *Lyapunov exponents of* φ *associated with* μ, where the $\Theta_\bullet$-invariant maps $(x,\omega) \mapsto \lambda_i^\mu(x,\omega)$ are defined via

$$\sum_{i=1}^k \lambda_i^\mu(x,\omega) = \lim_{t\to\infty} \frac{1}{t} \log \|\wedge^k T_x\varphi(t,\omega)\|, \tag{8}$$

$1 \le k \le d$. Here $\wedge^k$ denotes the k-fold exterior product of $T_x\varphi$. Existence of the limits in (8) follows from Kingman's subadditive ergodic theorem (Kingman [29]). Though (7) guarantees $\lambda_1 < \infty$, the last λ_i's may equal $-\infty$. If μ is ergodic, the Lyapunov exponents do not depend on (x,ω). If M is compact, the Lyapunov exponents do not depend on the choice of the Riemannian metric.

Sometimes it is more convenient to count only the distinct Lyapunov exponents, denoted here by $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_r$, where r is the number of distinct exponents, $1 \le r \le d$ (we assume μ ergodic to ease notation). Denote by $d_i = \max\{p - q + 1 \mid \lambda_p = \lambda_q = \Lambda_i\}$ the multiplicity of Λ_i.

There is another classical way to introduce Lyapunov exponents (see for instance Arnold and Wihstutz [8]). Put

$$\lambda(v,\omega) = \limsup_{t\to\infty} \frac{1}{t} \log \|T_x\varphi(t,\omega)v\|. \tag{9}$$

The map $\lambda(\cdot,\omega) : TM \to \mathbb{R} \cup \{-\infty\}$ satisfies $\lambda(cv) = \lambda(v)$ for all $c \ne 0$, $v \in TM$, and $\lambda(c_1v_1 + c_2v_2) \le \max\{\lambda(v_1), \lambda(v_2)\}$ for all $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2 \in T_xM$, $x \in M$ (sometimes called a *characteristic exponent*); we dropped ω, which is fixed here. These two properties imply that λ takes only finitely many values $\tilde\Lambda_1 > \tilde\Lambda_2 > \ldots > \tilde\Lambda_{\tilde r}$ as v varies over T_xM, $v \ne 0$. The Lyapunov exponents in this approach are the $\tilde\Lambda_i$. By definition of $v \mapsto \lambda(v,\omega)$, the sets

$$V_\delta(x,\omega) = \{v \in T_xM \mid \lambda(v,\omega) \le \delta\}$$

are linear subspaces of T_xM for $\delta \in \mathbb{R}$ arbitrary. Put $V_i = V_{\tilde\Lambda_i}$ and $\tilde d_i = \dim V_i - \dim V_{i+1}$.

The two definitions of Lyapunov exponents presented above are in general not equivalent. However, they are equivalent if (and only if) (x,ω) is a *forward regular point* for $T_x\varphi(t,\omega)$ (see Arnold and Wihstutz [8]) pp. 2–3). In terms of the present paper, (x,ω) is forward regular if $\sum_1^r d_i\Lambda_i = \sum_1^{\tilde r} \tilde d_i \tilde\Lambda_i$. It is clear from (6) that the bundle of linear subspaces

$$V_i(x,\omega) = \{v \in T_xM \mid \lambda(v,\omega) \le \tilde\Lambda_i(x,\omega)\},$$

is invariant under $T\varphi$ in the sense that $T_x\varphi(t,\omega)V_i(x,\omega) \subset V_i(\Theta_t(x,\omega))$. We refer to the family

$$T_xM = V_1(x,\omega) \supset V_2(x,\omega) \supset \ldots \supset V_r(x,\omega) \supset \{0\}$$

as the *Oseledec flag* associated with φ.
In the following we will be concerned with regular systems only, so that we need not distinguish between Λ and $\tilde{\Lambda}$.

We now recall the Multiplicative Ergodic Theorem (MET) of Oseledec for nonlinear RDS and sketch Ruelle's trivialization argument which reduces the nonflat bundle case to the flat bundle one.

Theorem
(i) (Multiplicative Ergodic Theorem without invertibility)
Suppose φ is a smooth RDS on a d-dimensional Riemannian manifold M and let μ be an invariant measure for φ such that (7) is satisfied. Then the linearization $T_x\varphi(t,\omega)$ of φ is forward regular at μ-almost all points $(x,\omega) \in M \times \Omega$.
(ii) (Multiplicative Ergodic Theorem with invertibility)
Suppose φ is a smooth two sided RDS on a d-dimensional Riemannian manifold M and let μ be an invariant measure for φ such that

$$(x,\omega) \mapsto \sup_{0<t\leq t_0} \{\log^+(\|T_x\varphi(t,\omega)\|) + \log^+(\|(T_x\varphi(t,\omega))^{-1}\|)\} \;\in\; L^1(\mu). \tag{10}$$

Then the linearization $T_x\varphi(t,\omega)$ of φ is bi-regular[1] *at μ-almost all points $(x,\omega) \in M \times \Omega$.*

Note that in the invertible case regularity implies that for μ-almost all (x,ω) the spaces

$$E_i(x,\omega) = \{v \in T_xM \mid \lambda^+(v,\omega) = \lambda^-(v,\omega) = \Lambda_i(x,\omega)\}$$

form a splitting of T_xM (with $\lambda^+(v,\omega) = \lambda(v,\omega)$ as in (9), and $\lambda^-(v,\omega)$ defined as in (9) with $t \to -\infty$). $TM = \bigoplus E_i$ is referred to as the *Oseledec splitting*.

Proof of the MET Denote the tangent bundle by (TM,π,M). Choose a countable covering of M by bundle charts (M_i,ψ_i) trivializing TM locally in an isometrical manner. That means, M_i is an open subset of M, and $\psi_i : \pi^{-1}(M_i) \to M_i \times \mathbb{R}^d$, where $\pi : TM \to M$ denotes the canonical bundle projection, such that ψ_i restricted to $\pi^{-1}\{x\}$ is linear for all $x \in M_i$. In addition, ψ_i may be chosen to be an isomorphism with respect to the scalar product on $\pi^{-1}\{x\}$ induced by the Riemannian structure on M and the standard scalar product on $\mathbb{R}^d$ for all $x \in M_i$, see Klingenberg [31] Theorem 1.8.20. (It is not really essential to choose isometric bundle charts, it is simply more convenient.)
Next put $B_0 = M_0$ and $B_n = M_n \setminus \bigcup_{j<n} B_j$ to obtain a countable covering $\{B_n \mid n \in \mathbb{N}\}$ of M by disjoint Borel sets. Putting

$$\begin{array}{rcll} \Sigma : TM & \to & M \times \mathbb{R}^d & \\ u & \mapsto & \psi_i(x) & \quad \text{for } x \in B_i \end{array}$$

yields a bimeasurable bundle map from TM to the flat bundle $M \times \mathbb{R}^d$ such that $\Sigma_x : T_xM \to \{x\} \times \mathbb{R}^d$ is an isomorphism for all $x \in M$. Finally, put

$$\Psi(t;x,\omega) = \Sigma_{\varphi(t,\omega)x} \circ T_x\varphi(t,\omega) \circ \Sigma_x^{-1}.$$

[1] 'regular' in the terminology of Arnold and Wihstutz [8] p. 4

Then Ψ is a linear RDS on $\mathbb{R}^d$ over the (enlarged) probability space $(M \times \Omega, \mu)$. Since Σ_x is an isomorphism for all $x \in M$,

$$\|\wedge^k \Psi(t; x, \omega)\| = \|\wedge^k T_x\varphi(t, \omega)\|$$

for all k, $1 \leq k \leq d$, and

$$\|\Psi(t; x, \omega)y\| = \|T_x\varphi(t, \omega)(\Sigma_x^{-1} y)\|$$

for all $y \in \mathbb{R}^d$, $t \in T$, $x \in M$, and P-almost all $\omega \in \Omega$. Thus regularity of Ψ implies regularity of $T\varphi$. But Ψ satisfies the integrability conditions of the 'ordinary' MET by (7) and (10), respectively, hence Ψ is forward or bi-regular, respectively, for μ-almost all (x, ω). □

Note that the Theorem does not require compactness of the manifold M. For a given smooth RDS with an invariant measure it thus only remains to check the integrability conditions (7) or (10), respectively, to infer the conclusions of the MET. This has been done directly for white noise systems on compact manifolds by Carverhill [19] (without imposing any further assumptions). Later Kifer [28] has shown that white noise systems on compact manifolds satisfy much stronger integrability conditions.

Multiplicative ergodic theory becomes much more difficult when considering *infinite dimensional* RDS. Recall that for a finite dimensional linear deterministic system the Lyapunov exponents are precisely the real parts of the eigenvalues of A (for continuous time, $\dot{x} = Ax$) or the logarithms of the eigenvalues of A (for discrete time, $x_{n+1} = Ax_n$), respectively. Thus, the Lyapunov exponents are determined by the spectrum. The definition (see (8)) suggests that it is essential to have a 'well behaved' top part of the spectrum: isolated eigenvalues of finite multiplicity. Since spectra of infinite dimensional operators in general have a considerably more complicated structure than finite dimensional ones, it is clear that much less is to be expected for infinite dimensional RDS. For a survey on infinite dimensional systems see 4.3 below.

3 Random Dynamical Systems and Markov Processes

3.1 Two cultures

In the theory of RDS two well-established mathematical cultures meet, overlap, and sometimes collide:

- *Dynamical Systems*: the *flow* point of view. Typically $T = \mathbb{R}$ or $\mathbb{Z}$ unless mappings are non-invertible which typically happens only for discrete time. Invariance of a measure is defined as invariance with respect to the mappings of the system.

- *Markov processes, stochastic analysis*: Here time is almost exclusively $\mathbb{Z}^+$ or $\mathbb{R}^+$ (or part of it). Markov processes are defined and studied through their transition semigroups forward in time. The necessity to really *construct* stochastic processes with prescribed transition semigroups (their existence follows from Kolmogorov's theorem) created the theory of stochastic differential equations (SDE's) (which are really ODE's with white noise input). Continuous time is $\mathbb{R}^+$, and a filtration $\mathcal{F}_t$ (i. e., an increasing family of sub σ-algebras of $\mathcal{F}$) collects the information available at time t. Everything has to be adapted, i. e., $\mathcal{F}_t$-measurable. 'Invariant measure' in the Markov context means invariance with respect to the transition semigroup.

The door from Markov processes to dynamical systems was really opened around 1980, when several people (Elworthy [24], Bismut [12], Ikeda and Watanabe [25], Kunita [33]) realized that writing down an SDE for a Markov process means much more than originally thought of by the pioneers K. Itô et al. It means the construction of a *stochastic flow* (or, as we will see, of an RDS with independent increments) whose one-point motions are Markov with the prescribed transition semigroup or its generator, respectively.

Probabilists sometimes criticize moving from an SDE to an RDS as 'forgetting' some of the probabilistic structure of the original, e. g., the fact that coming from an SDE implies in particular that all n-point motions are Markov. We think that the contrary is true, as many contributions to this volume show.

First, the concept of an RDS allows to address completely *new questions on SDE's*, as for instance the problem of finding *all* invariant measures (and not only those solving the Fokker-Planck equation), the problem of random invariant manifolds, random normal forms etc.

Second, the underlying Markov structure gives rise to problems which do not make sense for deterministic dynamical systems, such as the interplay of measurability and adaptedness properties with dynamics, see Crauel [20], [21], [22].

There seems to be some need for describing the connection between RDS and Markov processes in some detail.

3.2 RDS with independent increments, Brownian RDS

An RDS $\varphi(t,\omega)$ over $(\Omega,\mathcal{F},P,(\vartheta_t)_{t\in T})$ is said to have *independent increments* if for all $t_0 \le t_1 \le \ldots \le t_n$ the random variables

$$\varphi(t_1-t_0,\vartheta_{t_0}\omega),\varphi(t_2-t_1,\vartheta_{t_1}\omega),\ldots,\varphi(t_n-t_{n-1},\vartheta_{t_{n-1}}\omega) \tag{11}$$

are independent. If, in addition, for $T=\mathbb{R}^+$ or $\mathbb{R}$ the map $t\mapsto\varphi(t,\omega)x$ is continuous for all $x\in X$ P-a. s., then the RDS or cocycle is said to be a *Brownian* RDS or cocycle.

Remarks (i) An RDS with independent increments automatically has *stationary* (time homogeneous) increments, as, by the ϑ_t invariance of P, $\varphi(h,\vartheta_t\omega)\overset{\mathcal{D}}{=}\varphi(h,\omega)$ for all $t\in T$.
(ii) If $\varphi(t,\omega)$ consists of invertible mappings then, by the cocycle property,

$$\varphi_{s,t}(\omega) := \varphi(t,\omega)\circ\varphi(s,\omega)^{-1} = \varphi(t-s,\vartheta_s\omega)$$

for $s\le t$, so (11) means that for $t_0\le t_1\le\ldots\le t_n$

$$\varphi_{t_0,t_1},\varphi_{t_1,t_2},\ldots,\varphi_{t_{n-1},t_n} \tag{12}$$

are independent.

3.3 RDS and Markov chains, discrete time $T = \mathbb{Z}^+$ or $\mathbb{Z}$

Case $T = \mathbb{Z}^+$: Here $\varphi(n, \omega) = \varphi(1, \vartheta^{n-1}\omega) \circ \ldots \circ \varphi(1, \omega)$, so the cocycle has independent increments if and only if $\varphi(1, \omega), \varphi(1, \vartheta\omega), \ldots$ are iid. We thus have a product of iid random mappings, i. e., a classical 'iterated function system'. The mapping

$$x \mapsto \varphi(n, \omega)x$$

defines a homogeneous Markov chain with transition kernel

$$P(x, B) = P\{\omega \mid \varphi(1, \omega)x \in B\}. \tag{13}$$

Putting $x_n = \varphi(n, \omega)x_0$ we have

$$x_{n+1} = \varphi(1, \vartheta^n\omega)x_n, \tag{14}$$

i. e., a stochastic difference equation generating the Markov chain.

Conversely, given a transition kernel $P(x, B)$ on X, we want to construct an RDS with independent increments over a dynamical system $(\Omega, \mathcal{F}, P, \vartheta)$, i. e., a cocycle $\varphi(n, \omega)$ with $(\varphi(1, \vartheta^n\omega))_{n\in\mathbb{Z}^+}$ iid, such that (13) holds. This question has been dealt with by Kifer [26] Section 1.1. It always has a positive answer as soon as X is a Borel subset of a Polish space and if we are content with a measurable mapping $(x, \omega) \mapsto \varphi(1, \omega)x$. If we want $x \mapsto \varphi(1, \omega)x$ to be continuous or homeomorphisms or smooth etc., a general answer to this representation problem is not known up to now (compare Kifer [26] p. 12).

Case $T = \mathbb{Z}$: Now $\varphi(n, \omega)$ is invertible, and the cocycle has independent increments if and only if $(\varphi(1, \vartheta^n\omega))_{n\in\mathbb{Z}}$ is iid. The mapping $x \mapsto \varphi(n, \omega)x$ defines a homogeneous Markov chain on all of $\mathbb{Z}$ starting at $x_0 = x$, and (14) can be inverted to give

$$x_n = \varphi(1, \vartheta^n\omega)^{-1}x_{n+1} = \varphi(-1, \vartheta^{-n}\omega)x_{n+1}.$$

We can now look at the *forward transition kernel*

$$P^+(x, B) = P\{\omega \mid \varphi(1, \omega)x \in B\}$$

and the *backward transition kernel*

$$P^-(x, B) = P\{\omega \mid \varphi(-1, \omega)x \in B\} = P\{\omega \mid \varphi(1, \omega)^{-1}x \in B\}.$$

Note that in general P^+ and P^- do not have the same invariant measures: a forward invariant measure ν^+ has to satisfy

$$\nu^+ = \int P^+(x, \cdot)\, d\nu^+(x) = E\varphi(1, \omega)\nu^+,$$

whereas a backward invariant measure ν^- is characterized by

$$\nu^- = \int P^-(x, \cdot)\, d\nu^-(x) = E\varphi(-1, \omega)\nu^- = E\varphi(1, \omega)^{-1}\nu^-.$$

How are measures $\nu^{\pm}$ related to invariant measures of the RDS, i. e., measures μ on $X \times \Omega$ whose disintegration satisfies $\varphi(1,\omega)\mu_\omega = \mu_{\vartheta\omega}$? For one sided time Ohno [37] has proved that if $\nu = \nu^+$ is an invariant measure for the forward transition kernel P^+, then $\mu = \nu \times P$ is invariant for the RDS. Conversely, if a product measure $\nu \times P$ is invariant for the RDS, then ν is invariant for P^+.

For two sided time, a product measure is invariant for the RDS if and only if ν is fixed under φ, i. e., $\varphi(1,\omega)\nu = \nu$ P-almost surely. If a measure $\nu^{\pm}$ is $P^{\pm}$ invariant then the measures μ^+ and μ^-, given by

$$\mu^{\pm}_\omega = \lim_{n \to \mp\infty} \varphi(n,\omega)^{-1}\nu^{\pm},$$

are invariant for the RDS (so-called *Markov* measures). They are the ones which 'remember' the Markov kernels $P^{\pm}$. This equally applies for continuous time $T = \mathbb{R}^+$ or $\mathbb{R}$. These questions have been studied systematically by Crauel [21].

However, typically an RDS has more invariant measures not coming from the Markov chain — and those measures are needed for a systematic study of the RDS.

3.4 RDS and Markov processes, continuous time $T = \mathbb{R}^+$ or $\mathbb{R}$

Here the situation is much nicer and its description more complete than in the discrete time case. Basic results are due to Baxendale [10] and Kunita [33], [34], [35]. We describe the situation conceptually, i. e., without stating all technical assumptions, and quote freely from the above sources. We mainly emphasize the RDS point of view.

Case $T = \mathbb{R}^+$: Assume $X = \mathbb{R}^d$ (similar things hold on manifolds). Let $\varphi(t,\omega)$ be a Brownian RDS of homeomorphisms (or diffeomorphisms of some smoothness class). Then $(\varphi(t,\omega))_{t\in\mathbb{R}^+}$ is a Brownian motion with values in the group $\mathrm{Hom}(\mathbb{R}^d)$ (or $\mathrm{Diff}^*(\mathbb{R}^d)$ with a suitable $*$) in the sense of Baxendale [10], or

$$\varphi_{s,t}(\omega) = \varphi(t,\omega) \circ \varphi(s,\omega)^{-1},$$

$s, t \in \mathbb{R}^+$, is a temporally homogeneous Brownian flow in the sense of Kunita [35] p. 116.

By studying the infinitesimal mean

$$\lim_{h \searrow 0} \frac{1}{h} E(\varphi_{t,t+h}(\omega)x - x) = b(x)$$

and the infinitesimal covariance

$$\lim_{h \searrow 0} \frac{1}{h} E(\varphi_{t,t+h}(\omega)x - x)(\varphi_{t,t+h}(\omega)y - y)' = a(x,y),$$

Kunita constructs a vector field valued Brownian motion $(F(x,t,\omega))_{x\in\mathbb{R}^d, t\in\mathbb{R}^+}$, i. e., a continuous (in t) Gaussian process $(F(\cdot,t,\omega))_{t\in\mathbb{R}^+}$ with values in the space of vector fields on $\mathbb{R}^d$ (so $x \mapsto F(x,t,\omega)$ is a vector field), which has additively stationary independent increments and satisfies $F(\cdot,0,\omega) = 0$ (P-a. s.). The Brownian motion F is related to the

Brownian flow φ by $EF(x,t,\omega) = tb(x)$ and $\operatorname{cov}(F(x,t,\omega), F(y,s,\omega)) = \min\{t,s\}\, a(x,y)$. This implies that for all $s \in \mathbb{R}^+$

$$\varphi_{s,t}(\omega)x = x + \int_s^t F(\varphi_{s,u}(\omega)x, du, \omega), \quad t \in [s,\infty), \tag{15}$$

which has to be understood in the sense that (15) has a solution which coincides in distribution with the original Brownian flow φ. In short: The (forward) flow satisfies an Itô SDE driven by vector field valued Gaussian white noise. F is called the random infinitesimal generator of φ.

All n-point motions $(\varphi(t,\omega)x_1, \dots, \varphi(t,\omega)x_n)$ are homogeneous Feller-Markov processes. In particular, $(\varphi(t,\omega)x)_{t\in\mathbb{R}^+}$ is a Markov process whose transition semigroup has generator

$$L = \sum_{i=1}^d b^i(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x,x)\frac{\partial^2}{\partial x^i \partial x^j}. \tag{16}$$

The backward flow $\varphi_{s,t}(\omega)^{-1} = \varphi(s,\omega)\circ\varphi(t,\omega)^{-1}$, $0 \le s \le t$, satisfies for each $t \in \mathbb{R}^+$ a backward Itô equation in $s \in [0,t]$,

$$\varphi_{s,t}(\omega)^{-1}x = x - \int_s^t \hat{F}(\varphi_{u,t}(\omega)^{-1}x, \hat{du}, \omega),$$

where

$$\hat{F}(x,t,\omega) = F(x,t,\omega) - t\,c(x), \qquad c_i(x) = \sum_{j=1}^d \frac{\partial a^{ij}}{\partial x^j}(x,y)\Big|_{y=x}, \tag{17}$$

and the backward integral $\int_s^t \hat{F}(\varphi_{u,t}(\omega)^{-1}x, \hat{du}, \omega)$ is formally defined by the same definition as the forward integral — the difference being the inverted measurability counting from t backward to s.

As usual, things get more symmetric if we use Stratonovich forward and backward integrals. Put

$$F^0(x,t,\omega) = F(x,t,\omega) - \frac{t}{2}c(x),$$

then F^0 is the forward as well as the backward Stratonovich infinitesimal generator of φ.

Conversely, given a temporally homogeneous $\mathcal{V}(\mathbb{R}^d)$ (= vector fields on $\mathbb{R}^d$) -valued Brownian motion F, we can write down the SDE (15) to generate a Brownian flow with generator F. We can easily construct a Brownian RDS describing the same object. Indeed, put

$$\begin{aligned}
\Omega &= \{\omega \mid \omega(0) = 0, \omega(\cdot) \in \mathcal{C}(\mathbb{R}^+, \mathcal{V}(\mathbb{R}^d))\} \\
\mathcal{F} &= \text{Borel field} \\
P &= \text{distribution of } F = \text{`Wiener measure'} \\
\vartheta_t\omega(\cdot) &= \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}^+,
\end{aligned}$$

so ϑ_t leaves P invariant and is ergodic. Moreover, representing $F(\cdot,t,\omega) \equiv \omega(t)$, the uniqueness of the solution flow of (15) yields

$$\varphi_{s,t}(\omega) = \varphi(t-s, \vartheta_s\omega), \qquad 0 \le s \le t$$

(for a rigorous proof see, e. g., Wentzell [40] p. 192), so that the flow property $\varphi_{0,t+s} = \varphi_{s,t+s} \circ \varphi_{0,s}$ reads

$$\varphi(t+s,\omega) = \varphi(t,\vartheta_s\omega) \circ \varphi(s,\omega),$$

which is nothing but the cocycle property. As (12) is satisfied, and (12) is equivalent to (11), we have constructed a Brownian RDS from the given $\mathcal{V}(\mathbb{R}^d)$-valued temporally homogeneous Brownian motion F.

The *result* is (modulo technical assumptions):
There is a one to one relation between Brownian RDS and temporally homogeneous vector field valued Brownian motions (defined via the SDE (15)).

Since the law of $F(\cdot,t)$ (Gaussian!) is uniquely determined by the infinitesimal characteristics $b(x)$ and $a(x,y)$, it follows that (a,b) uniquely determine the law of the corresponding Brownian RDS. Consequently, as a and b appear in the generator of the two-point motion, the law of the two-point motion uniquely determines the law of the Brownian RDS.
(The last statement is in general not true in case of discrete time.)

Note that, however, the law of the Brownian RDS is in general not determined by the law of the one-point motion. The point is that the generator (16) only contains $a(x,x)$ instead of $a(x,y)$. In general there are thus many distinct Brownian RDS whose one-point motions have the same law. For instance, if $a(x,x) \in \mathcal{C}_b^2(\mathbb{R}^d;\mathbb{R}^{d\times d})$ is given, it can be factorized as $a(x,x) = \sigma(x)\sigma(x)'$ with $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ Lipschitz. Now put $a(x,y) = \sigma(x)\sigma(y)'$, and generate a Brownian RDS by solving

$$d\varphi_{s,t}(\omega)x = b(\varphi_{s,t}(\omega)x)\,dt + \sigma(\varphi_{s,t}(\omega)x)\,dB(t,\omega),$$

where B is a standard Brownian motion in $\mathbb{R}^d$. In this case

$$F(x,t,\omega) = t\,b(x) + \sigma(x)B(t,\omega).$$

In general countably many B_j's are necessary to represent F.

Remarks: (i) By replacing the driving Brownian motion F in (15) by a continuous vector field valued semimartingale, Kunita [35] constructs a more general class of flows. However, these flows do not correspond to RDS, unless the semimartingale has stationary increments. A stochastic flow φ generated by (15) is a Brownian RDS if and only if F is a Brownian motion.
(ii) There are more stochastic flows than covered by (i). See Example 3 in Section 2.1.

Case $T = \mathbb{R}$: As working on $T = \mathbb{R}$ is quite unusual in stochastic analysis, we develop the connection between Brownian RDS and SDE on $\mathbb{R}$ in detail.

Suppose we are given a Brownian RDS on $\mathbb{R}$. Then $\varphi_{s,t}(\omega) := \varphi(t,\omega) \circ \varphi(s,\omega)^{-1}$, $s,t \in \mathbb{R}$ is a temporally homogeneous Brownian flow in the sense of Kunita [35] on *all* of $\mathbb{R}$. Our aim is to write down an SDE for φ on $\mathbb{R}$. We split the cocycle $(\varphi(t,\omega))_{t\in\mathbb{R}}$ into two independent halves which are both temporally homogeneous Brownian cocycles on $\mathbb{R}^+$:

(i) $(\varphi^+(t,\omega))_{t\in\mathbb{R}^+}$, defined by $\varphi^+(t,\omega) = \varphi(t,\omega)$, is a cocycle on $\mathbb{R}^+$ over the dynamical system $(\Omega,\mathcal{F},P,(\vartheta_t)_{t\in\mathbb{R}^+})$,

(ii) $(\varphi^-(t,\omega))_{t\in\mathbb{R}^+}$, defined by $\varphi^-(t,\omega) = \varphi(-t,\omega)$, is a cocycle on $\mathbb{R}^+$ over the dynamical system $(\Omega, \mathcal{F}, P, (\vartheta_{-t})_{t\in\mathbb{R}^+})$.

The two cocycles φ^+ and φ^- are tied together by

$$\varphi^+(t,\omega)^{-1} = \varphi^-(t, \vartheta_t\omega), \qquad t \in \mathbb{R}^+.$$

Both $\varphi^\pm$ generate homogeneous Brownian flows

$$\varphi^\pm_{s,t}(\omega) = \varphi^\pm(t,\omega) \circ \varphi^\pm(s,\omega)^{-1} \qquad s,t \in \mathbb{R}^+.$$

There are thus two independent homogeneous vector field valued Brownian motions $(F^\pm(\cdot,t,\omega))_{t\in\mathbb{R}^+}$ such that for each fixed $s \in \mathbb{R}^+$

$$\varphi^\pm_{s,t}(\omega)x = x + \int_s^t F^\pm(\varphi^\pm_{s,u}(\omega)x, du, \omega), \qquad t \in [s,\infty), \tag{18}$$

and for each fixed $t \in \mathbb{R}^+$

$$\varphi^\pm_{s,t}(\omega)^{-1}x = x - \int_s^t \hat{F}^\pm(\varphi^\pm_{u,t}(\omega)^{-1}x, \hat{du}, \omega), \qquad s \in [0,t]. \tag{19}$$

We want to determine the covariances $\mathrm{cov}(F^\pm(x,t,\omega), F^\pm(y,t,\omega)) = \min\{t,s\}\, a^\pm(x,y)$ and the means $EF^\pm(x,t,\omega) = t\, b^\pm(x)$. We have for $h > 0$

$$E\varphi^-_{t,t+h}(\omega)x = E\varphi^+_{t,t+h}(\omega)^{-1}x$$

and

$$E(\varphi^-_{t,t+h}(\omega)x)(\varphi^-_{t,t+h}(\omega)y)' = E(\varphi^+_{t,t+h}(\omega)^{-1}x)(\varphi^+_{t,t+h}(\omega)^{-1}y)'.$$

Using the representation (18) for $\varphi^-(h,\omega)x$ and (19) for $\varphi^+(h,\omega)^{-1}x$, we obtain

$$a^-(x,y) = \hat{a}^+(x,y) = a^+(x,y) =: a(x,y)$$

and

$$b^-(x) = -\hat{b}^+(x) = -b^+(x) + c(x)$$

with $c(x)$ defined in (17). For the Stratonovich generators $F^{0\pm}$ this means $b^{0-}(x) = -b^{0+}(x)$, $a^{0-} = a^{0+} = a$, where $b^{0+} = b^+ - \frac{1}{2}c$.

We now piece the two independent vector field valued Brownian motions $F^\pm$ on $\mathbb{R}^+$ together to produce one on all of $\mathbb{R}$:

$$F^0(x,t,\omega) = \begin{cases} F^{0+}(x,t,\omega), & t \geq 0, \\ F^{0-}(x,-t,\omega), & t \leq 0. \end{cases} \tag{20}$$

This is a homogeneous Brownian motion on $T = \mathbb{R}$ with $F^0(x,0,\omega) = 0$, $EF^0(x,t,\omega) = t\, b^{0+}(x)$, and covariance given by $a(x,y)$.

As an intermediate step we introduce the notation

$$\int_0^t F^0(\varphi(u,\omega)x, \circ du, \omega) := \begin{cases} \int_0^t F^{0+}(\varphi^+(u,\omega)x, \circ du, \omega), & t \geq 0 \\ \int_0^{-t} F^{0-}(\varphi^-(u,\omega)x, \circ du, \omega), & t \leq 0. \end{cases}$$

As a result, the original cocycle $(\varphi(t,\omega))_{t\in\mathbb{R}}$ satisfies

$$\varphi(t,\omega)x = x + \int_0^t F^0(\varphi(u,\omega)x, \circ du, \omega)$$

on all of $\mathbb{R}$.

Similarly we put

$$\int_0^t F^0(\varphi_{u,t}(\omega)^{-1}x, \circ \hat{du}, \omega) := \begin{cases} \int_0^t F^{0+}(\varphi^+_{u,t}(\omega)^{-1}x, \circ \hat{du}, \omega), & t \geq 0 \\ \int_0^{-t} F^{0-}(\varphi^-_{u,-t}(\omega)x, \circ \hat{du}, \omega), & t \leq 0. \end{cases}$$

Again as a result, the inverse $(\varphi(t,\omega)^{-1})_{t\in\mathbb{R}}$ of the original cocycle satisfies

$$\varphi(t,\omega)^{-1}x = x - \int_0^t F^0(\varphi_{u,t}(\omega)^{-1}x, \circ \hat{du}, \omega)$$

on all of $\mathbb{R}$.

Our last step is the convention

$$\int_t^0 F^0(\varphi_{t,u}(\omega)x, \circ du) = -\int_0^t F^0(\varphi_{u,t}(\omega)^{-1}x, \circ \hat{du}), \qquad t \in \mathbb{R}.$$

Now we can write down a unified SDE-representation of the Brownian flow $\varphi_{s,t}(\omega) = \varphi(t,\omega) \circ \varphi(s,\omega)^{-1}$, $s,t \in \mathbb{R}$, as follows

$$\varphi_{s,t}(\omega)x = x + \int_s^t F^0(\varphi_{s,u}(\omega)x, \circ du, \omega), \qquad s,t \in \mathbb{R}, \tag{21}$$

or, symbolically,

$$d\varphi = F^0(\varphi, \circ dt).$$

If properly interpreted, (21) is satisfied for all $s,t \in \mathbb{R}$. For instance, if $s < 0 < t$, then

$$x + \int_s^t F^0(\varphi_{s,u}(\omega)x, \circ du) = x + \int_s^0 F^0(\varphi_{s,u}(\omega)x, \circ du) + \int_0^t F^0(\varphi_{s,u}(\omega)x, \circ du)$$

$$= x - \int_0^s F^0(\varphi_{u,s}(\omega)^{-1}x, \circ \hat{du}) + \int_0^t F^0(\varphi(u,\omega)\varphi(s,\omega)^{-1}x, \circ du)$$

$$= \varphi(s,\omega)^{-1}x + \int_0^t F^0(\varphi(u,\omega)\varphi(s,\omega)^{-1}x, \circ du) = (\varphi(t,\omega)\circ\varphi(s,\omega)^{-1})x = \varphi_{s,t}(\omega)x.$$

Conversely, given a vector field valued homogeneous Brownian motion $(F(x,t,\omega))_{x\in\mathbb{R}^d, t\in\mathbb{R}}$, characterized by $EF(x,t,\omega) = tb(x)$ and

$$\mathrm{cov}(F(x,t,\omega), F(y,s,\omega)) = \begin{cases} \min\{|t|,|s|\}\, a(x,y), & ts \ge 0, \\ 0, & ts < 0, \end{cases}$$

we write down equation (21), which consistently defines a Brownian flow on $\mathbb{R}$. The corresponding RDS is one over

$$\begin{aligned} \Omega &= \{\omega \mid \omega \in \mathcal{C}(\mathbb{R}, \mathcal{V}(\mathbb{R}^d)), \omega(0) = 0\} \\ P &= \text{‘Wiener measure’ on } \Omega \text{ generated by } F \\ \vartheta_t\omega(\cdot) &= \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R}. \end{aligned}$$

The cocycle property for $\varphi(t,\omega) := \varphi_{0,t}(\omega)$, $t \in \mathbb{R}$, follows from the fact that

$$\varphi_{s,t}(\omega) = \varphi(t-s, \vartheta_s\omega), \qquad s,t \in \mathbb{R}.$$

As a *result*, we again have a one to one correspondence between Brownian RDS φ for $T = \mathbb{R}$ and homogeneous Brownian motions F on $T = \mathbb{R}$. Again, (a,b) uniquely determines the law of the Brownian RDS.

All n-point motions of a Brownian RDS are homogeneous Markov processes on $\mathbb{R}$. In particular, the one-point motions $(\varphi(t,\omega)x)_{t\in\mathbb{R}}$ have the forward semigroup

$$P_t^+(x,B) = P\{\omega \mid \varphi(t,\omega)x \in B\} = P\{\omega \mid \varphi^+(t,\omega)x \in B\}, \qquad t \ge 0,$$

with generator

$$L^+ = \sum_{i=1}^d b^{+i}(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x,x)\frac{\partial^2}{\partial x^i \partial x^j},$$

while for $-t \le 0$ they have the backward semigroup

$$\begin{aligned} P_t^-(x,B) &= P\{\omega \mid \varphi(-t,\omega)x \in B\} = P\{\omega \mid \varphi^-(t,\omega)x \in B\} \\ &= P\{\omega \mid \varphi(t,\omega)^{-1}x \in B\} = P\{\omega \mid \varphi^+(t,\omega)^{-1}x \in B\}, \qquad t \ge 0, \end{aligned}$$

with generator

$$L^- = \sum_{i=1}^d b^{-i}(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x,x)\frac{\partial^2}{\partial x^i \partial x^j},$$

where $b^- = -b^+ + c$. Note that both L^+ and L^- describe the *same* Markov process, but L^+ forward and L^- backward in time. In general $L^+ \neq L^-$, so the Fokker-Planck equations $L^{+*}\nu = 0$ and $L^{-*}\nu = 0$ usually have different solutions. For the relation between $\nu^\pm$ and invariant measures for the RDS φ cf. the discrete time case $T = \mathbb{Z}$.

Example Suppose a homogeneous vector field valued Brownian motion is given by

$$F^0(x,t,\omega) = tX_0(x) + \sum_{i=1}^m B_i(t,\omega)X_i(x), \qquad t \in \mathbb{R},$$

with $X_0, \ldots, X_m$ vector fields and $B-1, \ldots, B_m$ independent standard Brownian motions on $\mathbb{R}$ (this is the lucky case: the general one needs $m = \infty$). Then

$$d\varphi = X_0(\varphi)\,dt + \sum_{i=1}^{d} X_i(\varphi) \circ dB_i(t), \qquad t \in \mathbb{R},$$

gives a Brownian flow (and RDS) on $T = \mathbb{R}$, and the one-point motions have generators

$$L^{\pm} = \pm X_0 + \frac{1}{2} \sum_{i=1}^{m} X_i^2.$$

Remark: We have not discussed a generalization of the above case which runs under 'real noise case' (Arnold, Kliemann, and Oeljeklaus [6], Arnold and Kliemann [5]) or 'Markovian multiplicative system' (Bougerol [14], [15]; Carmona and Lacroix [18] Chapter IV) or 'Markovian RDS' (Crauel [21]). Here we have a stationary Markov process $(\xi_t)_{t\in T}$, and, for discrete time, a difference equation $x_{n+1} = f(x_n, \xi_n)$, so that (x_n, ξ_n) is Markov for suitable x_0, e. g., for x_0 deterministic. In the continuous time case, e. g., $\dot{x} = f(x, \xi_t)$; again $(x(t), \xi_t)$ is Markov for suitable x_0. This reduces to the independent increments case if (ξ_n) is iid or if ξ_t is white, respectively.

4 Recent Developments

We will now briefly review the contributions to this volume and put them into the context of RDS laid out above.

4.1 Linear RDS

The MET is a statement about a *linear* cocycle (which may be a linearization) over a dynamical system. So the linear theory is the basis for any nonlinear extension.

An introduction into recent developments around the Multiplicative Ergodic Theorem is given by GOLDSHEID[2]. In particular, for products of iid matrices he discusses the Central Limit Theorem, and introduces into algebraic conditions for simplicity of the Lyapunov spectrum found recently. These yield one of the most beautiful formulations of a hyperbolicity result: If the support of the distribution of the matrices is Zariski dense in $\mathrm{Gl}(d, \mathbb{R})$, then their products have d distinct Lyapunov exponents $\lambda_1 \ldots > \lambda_d$.

A linear invertible RDS on $\mathbb{R}^d$ induces nonlinear RDS on the Graßmann manifolds $\mathrm{Gr}(k, d)$ of k-dimensional subspaces of $\mathbb{R}^d$. CRAUEL shows how to calculate the Lyapunov exponents of the induced Graßmann systems in terms of the exponents of the original linear one.

LEIZAROWITZ generalizes the theory of p-moment Lyapunov exponents (see Arnold, Kliemann, and Oeljeklaus [6], Arnold, Oeljeklaus, and Pardoux [7], and Baxendale [11]) to a

[2]Names in SMALL CAPITALS refer to contributions in the present volume

certain class of Markov processes in $\mathbb{R}^d$, which includes linear random differential equations driven by a finite step Markov chain. He shows that under his more general conditions the p-moment Lyapunov exponents are approximate eigenvalues for an associated differential operator.

Products of iid matrices with finitely many values $\{A_i \mid 1 \leq i \leq N\}$ with $p_i = P\{A = A_i\}$ are being addressed by PERES. He proves several results on the dependence of the Lyapunov exponents on the probabilities p_i. For instance, the top Lyapunov exponent $p \mapsto \lambda_1(p)$ depends on $p = (p_1, \ldots, p_N)$ locally real-analytically, provided $\lambda_1(p)$ is simple (i. e., $\lambda_1(p) > \lambda_2(p)$ in the notation of (8)).

LE JAN proves a theorem on the asymptotic behaviour of linear second order quantities and suggests possible applications to curvature of stable foliations etc.

KNILL considers $\mathrm{Sl}(2, \mathbb{R})$-valued cocycles (over an aperiodic automorphism). Previously he has proved that the set of cocycles with positive (top) Lyapunov exponent is dense in the set of essentially bounded ones (Knill [32]), which provides another of the few genericity results on existence of non-vanishing Lyapunov exponents. Here he shows that there always exist essentially bounded $\mathrm{Sl}(2, \mathbb{R})$ cocycles where the Lyapunov exponent depends discontinuously on variation of the cocycle.

LATUSHKIN AND STEPIN are concerned with the spectral theory of Linear Skew Product Flows. Starting from a cocycle of operators of a Hilbert space over a flow on a compact metric space with a quasi-invariant measure, they investigate the relations between spectra of weighted composition operators, the Lyapunov spectrum of the cocycle, and the Sacker-Sell spectrum of the cocycle.

BOUGEROL proves exponential stability of the Kalman filter for a linear system with stationary (additive and multiplicative) coefficients using the MET together with a result of M. Wojtkowski (see WOJTKOWSKI).

4.2 Nonlinear RDS

The papers in this section touch upon several areas of the theory of differentiable nonlinear stochastic systems, in which important progress has been made recently. But let us first give a brief comment on the measurable case.

As mentioned before, a measurable RDS is not much more than a dynamical system with a factor. A considerable part of Kifer's book [26] is devoted to this setting. Kifer considers exclusively products of iid maps, restricting attention to one sided time invariant Markov measures μ. Hence $\mu = \rho \times P$, where P is a one sided infinite product measure and ρ is a probability measure on X satisfying

$$E(\varphi(1)\rho) = \rho.$$

For this setting Kifer [26] generalizes the notion of fiber entropy introduced by Abramov and Rohlin [1], and investigated further by Ledrappier and Walters [36]. Bogenschütz [13] extends this notion to general measurable RDS, and observes that certain properties of the factor entropy — in particular, a generalization of the Shannon-McMillan-Breiman theorem for relative entropy — need the two sided time point of view.

Although not made explicit by the author, we consider BAXENDALE's contribution as important for stochastic bifurcation theory. One of the basic pictures of stochastic bifurcation is the following: Suppose we have a family $\varphi_\alpha(\omega)$ of cocycles with a family of invariant reference measures μ_α and corresponding top Lyapunov exponents $\lambda(\mu_\alpha)$. Suppose further $\lambda(\mu_\alpha) < 0$ for $\alpha < \alpha_0$, $\lambda(\mu_{\alpha_0}) = 0$, and $\lambda(\mu_\alpha) > 0$ for $\alpha > \alpha_0$, so that the reference measure μ_α looses its stability at $\alpha = \alpha_0$. We then expect bifurcation of a new invariant measure $\nu_\alpha \neq \mu_\alpha$ for $\alpha > \alpha_0$ with $\nu_\alpha \longrightarrow \mu_{\alpha_0}$ for $\alpha \searrow \alpha_0$ and, hopefully, $\lambda(\nu_\alpha) < 0$.

There are instructive examples in dimension $d = 1$ supporting this picture (see Arnold and Boxler [2]). It also turns out that necessarily $\lambda(\mu_{\alpha_0}) = 0$ if there is a family ν_α bifurcating from μ_α at $\alpha = \alpha_0$ (see Boxler [17]).

BAXENDALE now gives more evidence to the above paradigma by proving that an SDE in $\mathbb{R}^d$ with invariant (reference) measure $\delta_0 \times P$ (i. e., with a fixed point at $x = 0$), and corresponding top exponent λ has the following property: $\lambda < 0$, or $\lambda = 0$, or $\lambda > 0$ according to the diffusion process on $\mathbb{R}^d \backslash \{0\}$ being transient, or null-recurrent, or recurrent. In particular, for $\lambda > 0$ there exists another invariant measure besides the unstable $\delta_0 \times P$.

Considerable progress has recently been made on the problem of invariant manifolds for RDS. Boxler [16] has developed a stochastic center manifold theory. The most general results on invariant manifolds (e. g., those tangent to individual spaces of the Oseledec splitting) were obtained by Dahlke [23].

The method of Boxler [16] works on the level of the cocycle. However, if an RDS is given by a random or stochastic differential equation (as in many applications), one would like to obtain an approximation of the stochastic center manifold by manipulating the corresponding vector fields. This is not (yet) possible for the white noise case due to measurability problems (which call for the use of a stochastic calculus which can handle non-adapted processes). It is, however, possible for the real noise case, see BOXLER.

We mention in passing that the problem of simplifying a random diffeomorphism by smooth coordinate transformations, i. e., a *stochastic normal form theory*, has been dealt with by Xu Kedai [41] and Arnold and Xu Kedai [9].

ARNOLD AND BOXLER give a stochastic analogue of the well-known fact that small perturbations turn a hyperbolic fixed point into a bounded solution. The proof uses the MET, random norms and a recent result of Arnold and Crauel [3] for the affine case.

XUERONG MAO proves exponential stability of $x = 0$ of a general stochastic flow in the sense of Kunita [35] (see Section 3.4), and provides applications to Brownian flows.

KIFER discusses large deviations for products of random expanding maps in Markovian dependence. Since here 'state dependent transition probabilities' are allowed, the processes under consideration in this contribution are more general than RDS.

4.3 Infinite dimensional RDS

Multiplicative Ergodic Theory in infinite dimensions, if possible at all, should simultaneously generalize the finite dimensional random case (i. e., the MET) and the infinite dimensional deterministic spectral theory of operators. So far, not much of such a theory has come to existence. An up-to-date account of what has come to existence is provided by the survey paper of SCHAUMLÖFFEL. Multiplicative Ergodic Theory for a class of stochastic parabolic PDE's is worked out by FLANDOLI.

The top Lyapunov exponent of a product of iid infinite matrices is investigated by DARLING. These matrices arise in percolation theory.

4.4 Deterministic dynamical systems

One dimensional systems, in particular interval maps, have been one of the most active areas of interest during the past decade. KELLER gives a brief introduction into recent developments on piecewise monotone maps (usually understood as maps which are monotone on each of finitely many open intervals whose union is the whole unit interval except for finitley many one point sets, the endpoints of the intervals). In particular, Keller discusses the 'Lyapunov exponent maximizes entropy' formula $h_\mu \leq \max\{\lambda_\mu, 0\}$. It turns out that in case the Lyapunov exponent vanishes more subtle notions are needed, and Keller introduces and discusses algorithmic complexity for this kind of maps.

Also HOFBAUER is concerned with piecewise monotone interval maps. He proves $h_\mu \leq \max\{\lambda_\mu, 0\}$ for these maps, where the assumptions on the maps are weakened insofar countably many instead of finitely many intervals of monotonicity are allowed.

THIEULLEN introduces α-entropy for a smooth dynamical system, which is a modification of Katok's local entropy. He then extends Pesin's formula for α-entropy.

WOJTKOWSKI discusses Hamiltonian systems. He shows existence of non-zero Lyapunov exponents for two systems: a gas of hard balls interacting by elastic collisions, and a system of falling balls on a vertical line interacting by elastic collisions, with the bottom ball bouncing back elastically from a hard floor.

HOLZFUSS AND PARLITZ deal with the 'inverse problem' of extracting Lyapunov exponents of a linearized (deterministic) flow from a 'time series' of observations of the nonlinear system. They propose a method to extract the whole spectrum.

4.5 Engineering applications and control theory

The present volume gathers four contributions of engineers who investigate stability and other qualitative features of systems with random loads, random impurities etc., by means of Multiplicative Ergodic Theory.

ARIARATNAM AND XIE's contribution can serve as a survey. They present very well worked-out case studies. SRI NAMACHCHIVAYA, PAI, AND DOYLE look at stochastic stability of an electric power system with harmonically and stochastically varying network conditions. The (nonlinear) random oscillator is dealt with by WEDIG. He proposes several techniques for calculating invariant measures and Lyapunov exponents. BUCHER presents a method for the approximate calculation of the top exponent of a linear system in $\mathbb{R}^d$ and shows that it yields good results for certain models.

The survey paper of Colonius and Kliemann describes the use of the concept of Lyapunov exponents in nonlinear control theory. Originally, nonlinear control theory and Multiplicative Ergodic Theory came into touch via support theorems: the supports of solutions of the Fokker-Planck equation can be characterized as invariant control sets. This way it is possible to obtain uniqueness results for invariant (Markov) measures, see Kliemann [30] and Arnold and Kliemann [5]. Here Colonius and Kliemann consider control systems on a smooth manifold M, given by differential equations whose right hand sides depend on a control parameter from a compact subset U of some $\mathbb{R}^k$. This system turns out to be a (topological) skew product flow over the set of Lebesgue measurable $\{t \mapsto u(t) \in U\}$ with time shift. Colonius and Kliemann present a spectral theory for the linearization (in $x \in M$) of the system, which they use for discussing stabilizability, stability radii, and robustness. This view opens the door to an area which is very rich in structure and sheds new light on the MET.

References

[1] L. M. Abramov and V. A. Rohlin, The entropy of a skew product of measure-preserving transformations, *Amer. Math. Soc. Transl. Ser. 2*, 48 (1966) 255–265

[2] L. Arnold and P. Boxler, Stochastic bifurcation: Instructive examples in dimension one, in *Stochastic Flows*, M. Pinsky and V. Wihstutz (eds.). Birkhäuser, Bosten 1991

[3] L. Arnold and H. Crauel, Iterated function systems and multiplicative ergodic theory, in *Stochastic Flows*, M. Pinsky and V. Wihstutz (eds.). Birkhäuser, Bosten 1991

[4] L. Arnold and W. Kliemann, Large deviations of linear stochastic differential equations, pp. 117–151 in *Stochastic Differential Systems*, H. J. Engelbert and W. Schmidt (eds.), Lecture Notes in Control and Information Sciences 96. Springer, Berlin 1987

[5] L. Arnold and W. Kliemann, On unique ergodicity for degenerate diffusions, *Stochastics* 21 (1987) 41–61

[6] L. Arnold, W. Kliemann, and E. Oeljeklaus, pp. 85–125 in *Lyapunov Exponents. Proceedings. Bremen 1984*, L. Arnold and V. Wihstutz (eds.), Lecture Notes in Mathematics 1186. Springer, Berlin 1986

[7] L. Arnold, E. Oeljeklaus, and E. Pardoux, Almost sure and moment stability for linear Itô equations, pp. 129–159 in *Lyapunov Exponents. Proceedings. Bremen 1984*, L. Arnold and V. Wihstutz (eds.), Lecture Notes in Mathematics 1186, Springer, Berlin 1986

[8] L. Arnold and V. Wihstutz, pp. 1–26 in *Lyapunov Exponents. Proceedings. Bremen 1984*, L. Arnold and V. Wihstutz (eds.), Lecture Notes in Mathematics 1186, Springer, Berlin 1986

[9] L. Arnold and Xu Kedai, Normal forms for random dynamical systems, Preprint, Bremen 1991

[10] P. Baxendale, Brownian motions in the diffeomorphism group I, *Compositio Mathematica* 53 (1984) 19–50

[11] P. Baxendale, Moment stability and large deviations for linear stochastic differential equations, pp. 31–54 in *Proceedings of the Taneguchi Symposium on Probabilistic Methods in Mathematics*, N. Ikeda (ed.), Katata & Kyoto 1985

[12] M. Bismut, A generalized formula of Itô and some other properties of stochastic flows, *Z. Wahrscheinlichkeitstheorie Verw. Geb.* 55 (1981) 331–350

[13] T. Bogenschütz, Entropy, pressure and a variational principle for random dynamical systems, Preprint, Bremen 1991

[14] P. Bougerol, Comparaison des exposants de Lyapounov des processus markoviens multiplicatifs, *Ann. Inst. Henri Poincaré* 24 (1988) 439–489

[15] P. Bougerol, Théorèmes limites pour les systèmes linéaires à coefficients markoviens, *Probab. Th. Rel. Fields* 78 (1988) 193–221

[16] P. Boxler, A stochastic version of center manifold theory, *Probab. Th. Rel. Fields 83* (1989) 509–545

[17] P. Boxler, A necessary condition for a stochastic bifurcation, Preprint, Bremen 1990

[18] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston 1990

[19] A. Carverhill, Flows of stochastic dynamical systems: ergodic theory, *Stochastics* 14 (1985) 273–317

[20] H. Crauel, Extremal exponents of random dynamical systems do not vanish, *J. Dynamics Differential Equations* 2 (1990) 245–291

[21] H. Crauel, Markov measures for random dynamical systems, Preprint, Bremen 1989

[22] H. Crauel, Non-Markovian invariant measures are hyperbolic, Preprint, Bremen 1990

[23] S. Dahlke, *Invariante Mannigfaltigkeiten für Produkte zufälliger Diffeomorphismen*, Dissertation, Universität Bremen 1989

[24] K. D. Elworthy, Stochastic dynamical systems and their flows, pp. 79–95 in *Stochastic Analysis*, A. Friedman and M. Pinsky (eds.). Academic Press 1978

[25] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*. North Holland – Kodansha, Tokyo 1981

[26] Yu. Kifer, *Ergodic Theory of Random Transformations*, Birkhäuser, Boston 1986

[27] Yu. Kifer, Characteristic exponents for random homeomorphisms of metric spaces, pp. 74–84 in *Lyapunov Exponents. Proceedings. Bremen 1984*, L. Arnold and V. Wihstutz (eds.), Lecture Notes in Mathematics 1186, Springer, Berlin 1986

[28] Yu. Kifer, A note on integrability of C^r-norms of stochastic flows and applications, pp. 125–131 in *Stochastic Mechanics and Stochastic Processes, Proceedings Swansea 1986*, Lecture Notes in Mathematics 1325, Springer, Berlin 1988

[29] J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Roy. Statist. Soc. Ser. B* 30 (1968) 499–510

[30] W. Kliemann, Recurrence and invariant measures for degenerate diffusions, *Ann. Probab.* 15 (1987) 690–707

[31] W. Klingenberg, *Riemannian Geometry*, de Gruyter, Berlin 1982

[32] O. Knill, Positive Lyapunov exponents for a dense set of bounded measurable $\mathrm{Sl}(2, \mathbb{R})$ cocycles, Preprint, Zürich 1990

[33] H. Kunita, *Stochastic Differential Equations and Stochastic Flows of Diffeomorphisms*, pp. 143–303 in Ecole d'Eté de Probabilités de Saint-Flour 1982, Lecture Notes in Mathematics 1097. Springer, Berlin 1984

[34] H. Kunita, *Lectures on Stochastic Flows and Applications*, Tata Institute of Fundamental Research, Bombay. Springer, Berlin 1986

[35] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press 1990

[36] F. Ledrappier and P. Walters, A relativised variational principle for continuous transformations, *J. London Math. Soc. (2)*, 16 (1977) 568–576

[37] T. Ohno, Asymptotic behaviors of dynamical systems with random parameters, *Publ. RIMS, Kyoto Univ.* 19 (1983) 83–98

[38] V. I. Oseledec, A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* 19 (1968) 197–231

[39] D. Ruelle, Ergodic theory of differentiable dynamical systems, *Publ. Math. I.H.E.S.* 50 (1979) 27–58

[40] A. D. Wentzell, *Theorie zufälliger Prozesse*. Akademie-Verlag, Berlin 1979

[41] Xu Kedai, *Normalformen für zufällige dynamische Systeme*, Dissertation, Universität Bremen 1990