Ergodic theory, geometry and dynamics

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1 Introduction

These notes address topics in geometry and dynamics, and make contact with some related results in number theory and Lie groups as well.

The basic setting for dynamics is a bijective map $T : X \to X$. One wishes to study the behavior of orbits $x, T(x), T^2(x), \ldots$ from a topological or measurable perspective. In the former cases T is required to be at least a homeomorphism, and in the latter case one usually requires that T preserves a probability measure m on X. This means:

$$m(T^{-1}(E)) = m(E) \tag{1.1}$$

for any measurable set E, or equivalently

$$\int f = \int f \circ T$$

for any $f \in L^1(X, m)$.

More generally one can consider a flow $T : \mathbb{R} \times X \to X$ or the action of a topological group $T : G \times X \to X$. In topological dynamics, one requires that T is a continuous map in the product topology; in measurable dynamics, that T is a measurable map.

One can also consider the dynamics of an *endomorphism* T. Equation (1.1) defines the notion of measure preservation in this case as well.

Ergodicity. A basic issue is whether or not a measurable dynamical system is 'irreducible'. We say T is *ergodic* if whenever X is split into a disjoint union of measurable, T^{-1}-invariant sets,

$$X = A \sqcup B,$$

either $m(A) = 0$ or $m(B) = 0$.

Note: this definition makes sense for group actions too, and it makes sense even when $m(X)$ is infinite, and when m is not preserved by T.

Rotations. A simple and yet surprisingly rich family of examples of measure–preserving maps is given by the rotations of the circle $X = S^1 = \mathbb{R}/\mathbb{Z}$, define by

$$T(x) = x + a \bmod 1.$$

Theorem 1.1 *The rotation T is ergodic if and only if a is irrational.*

Proof. If $a = p/q$ is rational then T has order q, $Y = S^1/\langle T\rangle$ is a circle of length $1/q$, and any measurable subset of Y pulls back to give a T–invariant measurable subset of S^1.

Otherwise, T has infinite order (and Y is the 'tiny circle', inaccessible by traditional measure theory and topology, but amenable to the methods of non–commutative geometry). In this case, it is easy to see that every orbit $[x] = (T^i x \ : \ i \in \mathbb{Z})$ is dense in S^1. Otherwise, T would have to permute the intervals forming the complement of the orbit closure, preserving their length, which quickly implies that T is periodic.

Now suppose $A \subset S^1$ is T–invariant and $m(A) > 0$. Then, by the Lebesgue density theorem, for any $\epsilon > 0$ we can find an interval I such that $m(I \cap A)/m(I) > 1 - \epsilon$. Since the orbit of I covers the circle, we conclude that $m(A) > 1 - \epsilon$. Hence $m(A) = 1$ and we have ergodicity. ■

We will examine the irrational rotation from other perspectives in §2.

Breadth of the topic. To indicate the range of topics related to ergodic theory, we now turn to some examples and applications.

Examples of measure-preserving dynamical systems.

1. *Endomorphism of* S^1. For $S^1 = \mathbb{R}/\mathbb{Z}$, the group endomorphisms $T(x) = dx$, $d \neq 0$, are also measure preserving.

2. *Blaschke products on the circle.* A typical proper holomorphic map $B : \Delta \to \Delta$ with $B(0) = 0$ has the form $B(z) = z \prod (z - a_i)/(1 - \overline{a_i} z)$. This map preserves the linear measure $m = |dz|/2\pi$ on the unit circle S^1. For the proof, observe that the measure of a set $E \subset S^1$ is given by $u_E(0)$, where $u_E(z)$ is the harmonic extension of the indicator function of E to the disk. Since $B(0) = 0$, we have
$$m(B^{-1})(E) = u_{B^{-1}E}(0) = u_E \circ B(0) = u_E(0) = m(E).$$

3. *Interval exchange transformations.* Let $I = [0,1] = \bigcup_1^n I_i$ be a tiling of the unit interval, and let $f : I \to I$ be the map obtained by reordering these intervals. On each interval we have $f(x) = x + t_i$ for some i, and so linear measure is preserved. The case $n = 2$ is essentially the same as a rotation of S^1.

4. *Dynamics on the torus.* The torus $X = \mathbb{R}^n/\mathbb{Z}^n$ also admits many interesting measure–preserving maps. In addition to the translations $T(x) = x + a$, each $T \in \mathrm{GL}_n(\mathbb{Z})$ determines a measure–preserving automorphism of X. In fact any matrix $T \in M_n(\mathbb{Z})$ with nonzero determinant gives a measure–preserving endomorphism of the torus.

5. *Continued fractions.* The continued fraction map $x \mapsto \{1/x\}$ on $[0,1]$ preserves the Gauss measure $m = dx/(1+x)$ (with total mass $\log 2$). This (together with ergodicity) allows us to describe the behavior of the continued fraction of a typical real number.

6. *Stationary stochastic processes.* A typical example is a sequence of independent, equally distributed random variables $X_1, X_2, \ldots$. The model space is then $X = \mathbb{R}^{\mathbb{Z}_+}$ with the product measure $m = \prod m_i$, with all the factors m_i the same probability measure on $\mathbb{R}$. The relevant dynamics then comes from the *shift map*,

   $$\sigma(X_1, X_2, \ldots) = (X_2, X_3, \ldots),$$

   which is measure–preserving.

   In this setting the *Kolmogorov* 0/1 *law* states than any tail event E has probability zero or one. A tail event is one which is independent of X_i for each individual i; for example, the event "$X_i > 0$ for infinitely many i" is a tail event. The zero–one law is a manifestation of ergodicity of the system (X, σ).

7. *The Hénon map.* The Jacobian determinant $J(x,y) = \det DT(x,y)$ of any polynomial automorphism $T : \mathbb{C}^2 \to \mathbb{C}^2$, is constant. Indeed, $J(x,y)$ is itself a polynomial, which will vanish at some point unless it is constant, and vanishing would contradict invertibility.

   (The Jacobian conjecture asserts the converse: a polynomial map $T : \mathbb{C}^n \to \mathbb{C}^n$ with constant Jacobian has a polynomial inverse. It is know for polynomials of degree 2 in n variables, and for polynomials of degree ≤ 100 in 2 variables.)

   If T is defined over $\mathbb{R}$, and $\det DT = 1$, then T gives an area–preserving map of the plane.

   In the Hénon family $H(x,y) = (x^2 + c - ay, x)$, measure is preserved when $a = 1$.

8. *The full shift* $\Sigma_d = (\mathbb{Z}/d)^{\mathbb{Z}}$.

9. *The baker's transformation.* This is the discontinuous map given in base two by

   $$f(0.x_1x_2\ldots, 0.y_1y_2\ldots) = (0.x_2x_3\ldots, 0.x_1y_1y_2\ldots).$$

   This map acts on the unit square $X = [0,1] \times [0,1]$ by first cutting X into two vertical rectangles, A and B, then stacking them (with B on top) and finally flattening them to obtain a square again. It is measurably conjugate to the full 2-shift.

10. *Hamiltonian flows.* Let (M^{2n}, ω) be a symplectic manifold. Then any smooth function $H : M^{2n} \to \mathbb{R}$ gives rise to a natural vector field X, characterized by $dH = i_X(\omega)$ or equivalently

    $$\omega(X,Y) = YH$$

    for every vector field Y. In the case of classical mechanics, M^{2n} is the cotangent bundle to a manifold N^n and $\omega = \sum dp_i \wedge dq_i$ is the canonical symplectic form in position – momentum coordinates. Then H gives the energy of the system at each point in phase space, and $X = X_H$ gives its time evolution. The flow generated by X preserves both H (energy is conserved) and ω. To see this, note that

    $$\mathcal{L}_X(\omega) = di_X\omega + i_X d\omega = d(dH) = 0 \quad \text{and} \quad XH = \omega(X,X) = 0.$$

    In particular, the volume form $V = \omega^n$ on M^{2n} is preserved by the flow.

    When M^{2n} is a Kähler manifold, we can write $X_H = J(\nabla H)$, where $J^2 = -I$. In particular, in the plane with $\omega = dx \wedge dy$, any function H determines an area–preserving flow along the level sets of H, by rotating ∇H by 90 degrees so it becomes parallel to the level sets. The flow accelerates when the level lines get closer together.

11. *Area–preserving maps on surfaces.* The dynamics of a Hamilton flow on a surface are very tame, since each level curve of H is invariant under the flow. (In particular, the flow is never ergodic with respect to area measure.) One says that such a system is *completely integrable.*

    General area–preserving maps on surfaces exhibit remarkable universality, and many phenomena can be consistently observed (elliptic islands and a stochastic sea), yet almost none of these phenomena has been mathematically justified. (An exception is the existence of elliptic islands, which follows from KAM theory and generalizes to symplectic maps in higher dimensions.)

12. *Algebraic dynamics.* Maps that preserve a measure arise naturally in algebraic geometry. For example, any automorphism or endomorphism T of an elliptic curve $E = \mathbb{C}/\Lambda$ preserves the area form inherited from $\mathbb{C}$. If T and E are defined over $\mathbb{R}$, then T preserves a suitable linear measure on $E(\mathbb{R})$.

    Going to dimension 2, any automorphism T of a K3 surface X preserves a holomorphic $(2,0)$ η up to a scalar of modulus one, and hence preserves the volume form $\eta \wedge \overline{\eta}$. Over the reals, T gives rise to an area–preserving map on $X(\mathbb{R})$.

    More generally, when X is a Calabi–Yau manifold of dimension 3 or more, any automorphism of X preserves a similar volume for $\eta \wedge \overline{\eta}$.

    As a simple, special case, any endomorphism of a complex torus is volume–preserving.

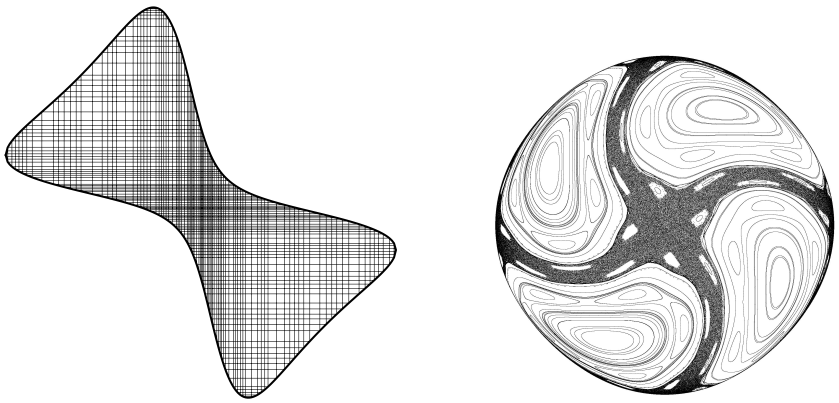


Figure 1. Algebraic dynamics over $\mathbb{R}$ on an elliptic curve and on a K3 surface.

13. *Algebraic examples.* Figure 1 shows two simple examples of algebraic dynamical systems over $\mathbb{R}$.

The boundary of the bowtie in the first example is a curve $X \subset \mathbb{R}^2$ defined by

$$(1+x^2)(1+y^2)+Axy=3,$$

with $A=7$. This is an elliptic curve, best regarded as a curve of degree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. That is, the equation above has degree two in each variable. Consequently we have a pair of involutions ι_x and ι_y on X, preserving the x and y coordinates respectively. Their composition $f=\iota_x \circ \iota_y$ is a translation in the group law on the elliptic curve, and consequently $f|X$ is smoothly conjugate to a rotation.

The second example similarly describes the orbits of the composition f of three involutions on a K3 surface $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$(1+x^2)(1+y^2)(1+z^2)+Axyz=2$$

with $A=5/2$ [Mc3]. In this case one sees the elliptic islands typical of KAM theory, as well as a stochastic sea which, possibly, corresponds to an invariant set $A \subset X$ of positive measure with $f|A$ ergodic.

14. *The geodesic flow.* Let $X = \mathrm{T}_1 M$ be the unit tangent bundle of a compact Riemannian manifold M. Then the *geodesic flow* $g_t : X \to X$ preserves *Liouville measure.*

    One can view the geodesic flow as an example of a Hamiltonian flow on $\mathrm{T}^* M$ with its natural symplectic structure. The Riemannian metric, $H=\sum g_{ij} q_i q_j$, gives the kinetic energy of a particle and is preserved under its frictionless motion; and ω^n gives the Liouville measure on the cotangent bundle of M. The measure can then be conditioned to the unit tangent bundle.

15. *Hyperbolic surfaces.* The unit tangent bundle of a compact hyperbolic surface can be identified with the quotient space $X = \mathrm{SL}_2(\mathbb{R})/\Gamma$ for some cocompact lattice Γ, and then the geodesic flow becomes the action of the diagonal subgroup A.

16. *Billiards.* Cf. [KMS]. The motion of a billiard ball in a smoothly bounded plane region also preserves the natural measure on the unit tangent bundle to the region. One can imagine the double of the region as a closed surface, with curvature concentrated along the edge. In the case of a polygon, the curvature is concentrated at single points (the

vertices). Many easily stated questions about billiards in polygons are open. For example, does every triangle have a periodic trajectory? Is the motion of billiards in almost every triangle ergodic?

Billiards in polygons can be related to the dynamics of a holomorphic 1-form on a Riemann surface.

17. *Holomorphic 1–forms on Riemann surfaces.* If (X, ω) is a holomorphic 1–form on a compact Riemann surface X, then $|\omega|$ determines a flat metric on X with singularities. The geodesic flow for this metric is closely related to both billiards and interval exchange transformations.

A panorama of applications. We conclude this section by mentioning some geometric and arithmetic applications of ergodic theory.

1. Mostow rigidity. Thurston and Perelman proved that most compact topological 3-manifolds M are hyperbolic; that is, M is homeomorphic to a quotient $\Gamma\backslash\mathbb{H}^3$ of hyperbolic space by a discrete group of isometries.

This 'uniformization theorem' is all the more remarkable because the hyperbolic structure, when it exists, is unique. More precisely, we have:

Theorem 1.2 (Mostow) *Let $f : M^3 \to N^3$ be a homotopy equivalence between a pair of compact hyperbolic manifolds. Then f can be deformed to an isometry.*

The proof of this theorem is based on studying the ergodic theory Γ acting on the sphere at infinity S^2 forming the boundary of hyperbolic space. The key point is that the action on $S^2 \times S^2$ is ergodic, or equivalently that the geodesic flow on $T_1 M^3$ is ergodic.

As emphasized already by Klein in his Erlangen program, the study of hyperbolic 3-manifolds is closely related to the study of the Lie group $G = \mathrm{SO}(3,1)$. Indeed, a hyperbolic 3-manifold is nothing more than a quotient

$$M = \Gamma\backslash G/K,$$

where $\Gamma \subset G$ is a discrete, torsion–free group and $K = \mathrm{SO}(3)$ is a maximal compact subgroup. The study of the geodesic flow on the frame bundle of M is equivalent to the study of the action of A on $\Gamma\backslash G$, where A is the diagonal subgroup of $\mathrm{SO}(3,1)$. And the study of the action of Γ on S^2 is equivalent to the study of its action on G/AN, where N is a maximal unipotent subgroup of G.

From this perspective, 'geometry' is the study of $\Gamma/\backslash G/K$ where K is compact, and 'dynamics' is the study of Γ acting on G/H or of H acting on $\Gamma\backslash G$, where H is noncompact. In both cases we assume that Γ is discrete, and often that $\Gamma\backslash G$ has finite volume or is even compact.

2. The Oppenheim conjecture. A *quadratic form* in n variables over K is a homogeneous polynomial $Q(x)$ of degree two in $K[x_1, \ldots, x_n]$. Usually we will assume that Q is *nondegenerate*, i.e. that the matrix $Q(e_i, e_j)$ has nonzero determinant.

We say Q *represents* y if there exists an $x \in K^n$, $x \neq 0$, such that $Q(x) = y$.

Quadratic forms over $\mathbb{R}$ are determined by their signature (p, q), $0 \leq p + q <= n$. If Q represents zero over $\mathbb{R}$, it is said to be *indefinite*. This is equivalent to requiring that its signature is not $(n, 0)$ or $(0, n)$.

Meyer proved that an indefinite quadratic form over $\mathbb{Z}$ in 5 or more variables always represents zero. In other words, $Q \in \mathbb{Z}[x_1, \ldots, x_5]$ represents zero iff Q represents zero over $\mathbb{R}$.

In 1929, Oppenheim conjectured that an indefinite quadratic form over $\mathbb{R}$ in 5 or more variables 'nearly' represents zero over $\mathbb{Z}$: for any $\epsilon > 0$, there is an $x \in \mathbb{Z}^n$, $x \neq 0$, such that $|Q(x)| < \epsilon$. A strong version of this result was finally proved by Margulis in 1987.

Theorem 1.3 *If $Q(x_1, x_2, x_3)$ is a real quadratic form of signature $(1, 2)$, then either:*

- *Q is proportional to an integral form, and $Q(\mathbb{Z}^3)$ is discrete, or*
- *$Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.*

The proof is an instance of a much stronger rigidity phenomenon for actions of unipotent groups, whose most general formulation is given by *Ratner's theorem.* In the case at hand, one is reduced to analysis of the orbits of $H = \mathrm{SO}(2, 1)$ on $\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})$. It is shown that such an orbit is either closed, or dense, giving the two possibilities above.

Remark on an open problem: Littlewood's conjecture. The original conjecture is that for any 2 real numbers a, b, we have

$$\inf n\|na\| \cdot \|nb\| = 0,$$

where $\|x\|$ is the fractional part of x. It is now known that the conjecture holds for all (a, b) outside a set of Hausdorff dimension zero (Einsiedler, Katok, Lindenstrauss).

The full conjecture would follow if one had a version of Ratner's theorem for the semisimple group of diagonal matrices $A \subset \mathrm{SL}_3(\mathbb{R})$, acting again on $\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})$. More precisely, it would be sufficient to show that any *bounded* A orbit is closed (and hence homeomorphic to a torus $\mathbb{R}^2/\mathbb{Z}^2$).

An equivalent conjecture is that whenever a lattice $\Lambda \subset \mathbb{R}^3$ has the property that $N(x) = |x_1x_2x_3|$ is bounded below (on its nonzero elements), the lattices comes from the ring of integers in a totally real cubic field.

If (α, β) *satisfy* Littlewood's conjecture, then the lattice generated by $(1, \alpha, \beta)$, $(0, 1, 0)$ and $(0, 0, 1)$ has vectors of the form $(n, \|n\alpha\|, \|n\beta\|)$ with arbitrarily small norm, consistent with the conjecture just stated.

3. Expanding graphs. We mention in passing that the study of unitary representations of discrete groups like $\mathrm{SL}_n(\mathbb{Z})$, which is a facet of ergodic theory since it includes the action of Γ on $L^2(G/K)$, has led to the construction of explicit *finite graphs* with good expansion properties.

Indeed, $\mathrm{SL}_3(\mathbb{Z})$ has Kazhdan's property T, and thus once we fix a generating set S, the Cayley graphs for the quotients $\mathrm{SL}_3(\mathbb{Z}/p)$ are a sequence of graphs of bounded degree with uniform expansion. This fact was noted by Margulis in 1975, when he was employed by the Institute of Problems of Information Transmission in the former Soviet Union. (Expanding graphs are useful, at least in principle, in networking problems.)

4. Optimal billiards. The problem here is to construct a polygon billiard tables $P \subset \mathbb{R}^2$ such that every billiard trajectory is either periodic, or uniformly distributed. (The simplest example of a table with optimal dynamics is a rectangle).

This problem is closely related to the study of the action of $H = \mathrm{SL}_2(\mathbb{R})$, not on $\Gamma\backslash G$, but on

$$\Omega\mathcal{M}_g = \mathrm{Mod}_g \backslash \Omega\mathcal{T}_g,$$

the moduli space of holomorphic 1-forms on Riemann surfaces of genus g. One expects a theorem like Ratner's in this setting, and indeed such a result was proved very recently by Eskin and Mirzakhani.

2 Ergodic theory

In this section we begin the systematic study of measurable dynamics. We will introduce the idea of analyzing $T : X \to X$ via its action on $L^2(X)$, and prove two versions of the ergodic theorem. We will also study spectral invariants, mixing and ergodicity in examples such as rotations of the circle and linear maps on tori.

Useful reference for this section include [CFS] and [KT].

Measure spaces. A *measure* space is a triply $(X, \mathcal{B}, m)$ consisting of a σ-algebra $\mathcal{B}$ of subsets of X together with a countably–additive measure $m : \mathcal{B} \to [0, \infty]$. If $m(X) = 1$ then X is a *probability space.*

Two measure spaces are *isomorphic* if there is a bijection $\phi : X \to X'$ sending $\mathcal{B}$ to $\mathcal{B}'$ and m to m'.

An *atom* is a point $x \in X$ with $m(x) > 0$. Most measure spaces without atoms of concern to us are isomorphic to Lebesgue measure on an interval $[0, a)$ with $a = m(X)$. For example, $\mathbb{R}^n$ is isomorphic to $[0, \infty)$; $[0, 1]^n$ is isomorphic to $[0, 1]$; and any manifold with a Riemannian metric of volume 1 is isomorphic to $[0, 1]$.

If we allow a countable set of atoms as well, we obtain the notion of a *standard Lebesgue space.* The upshot is that, in practical terms, measure spaces have no geometry; they all look the same (up to atoms). Thus in the study of measurable dynamics $T : X \to X$, all the action takes place with T.

Nonstandard example. Let m be a probability measure on $\mathbb{R}$, say a Gaussian measure. A 'random function' on $[0, 1]$, or 'naive white noise', is defined by the product measure on $X = \mathbb{R}^{[0,1]}$. This is *not* a standard Lebesgue space; e.g. $L^2(X)$ is inseparable. Moreover a random function $f : [0, 1] \to \mathbb{R}$ in X is almost surely not measurable.

For related foundational points, see [Mac], [Me, Prop 12.6], or [Roy].

Measurable dynamics. The basic setting of ergodic theory is a *measure-preserving* transformation T on X. This means T is measurable and

$$m(T^{-1}A) = m(A) \; \forall \; A \in \mathcal{B}.$$

Usually we assume T is invertible and we thus obtain a measure–preserving action of $\mathbb{Z}$ on X.

More generally, the action of a topological group G on a measure space $(X, \mathcal{B}, m)$ is *measurable* if $T : G \times X \to X$ is measurable. This means $T^{-1}(E)$ is measurable for all measurable E. The Borel structure on G is used to defined measurability in $G \times X$.

One of the fundamental questions in ergodic theory is: when are two measurable dynamical systems isomorphic? In this section we will study basic properties of these dynamical systems:

> *Ergodicity; mixing;* and *spectral invariants.*

We will apply these ideas in concrete examples coming from the shift and dynamics on the circle and a torus.

Note that when X has atoms, T simply permutes the atoms of a given mass $m_0 > 0$ — and there are only finitely many such atoms, since $m(X) = 1$. Thus we may almost always reduce to the case where X has no atoms, and indeed to the case where $(X, m) \cong ([0, 1], dx)$.

Poincaré recurrence. Let $T : X \to X$ be an automorphism of a probability space. Here is one of the most general results concern measure–preserving dynamics. It was discovered by Poincaré in his analysis of the three–body problem in celestial dynamics.

Theorem 2.1 *If $m(A) > 0$ then almost every $x \in A$ is recurrent (we have $T^i(x) \in A$ for infinitely many $i > 0$).*

Proof. Let N be the set of points in A whose forward orbits *never* return to A. Then $T^i(N) \cap N \subset T^i(N) \cap A = \emptyset$ for any $i > 0$. It follows that $\langle T^i(N)\rangle$ are disjoint for all $i \in \mathbb{Z}$; since $\sum m(T^i(N)) \leq 1$, we conclude $m(N) = 0$. Thus almost every $x \in A$ returns at least once to A. Replacing T with T^n for $n \gg 0$, we see almost x returns to A infinitely many times. ∎

The same reasoning shows:

Theorem 2.2 *If $m(A) > 0$ and T is ergodic, then almost every orbit of T visits A infinitely often.*

Later we will see that the amount of time spent in A is proportional to $m(A)$; see Theorem 2.21.

The Rohlin–Halmos lemma. Let us state another general result about measure–preserving maps.

Theorem 2.3 *Let T be a measure–preserving transformation whose periodic points have measure zero. Then for any $n, \epsilon > 0$ there exists a measurable set E such that $E, T(E), \ldots, T^n(E)$ are disjoint and $E \cup T(E) \cup \cdots \cup T^n(E)$ has measure at least $1 - \epsilon$*

See [CFS, §10.5]. It is interesting to work this out for particular cases, e.g. an irrational rotation of the circle.

Ergodic theory and Hilbert spaces. We now turn to a central theme in ergodic theory.

Let H be a Hilbert space. Recall that H is determined, up to isomorphism, by its dimension (the cardinality of an orthonormal basis). The space of all bounded linear operators on H will be denoted by $\mathcal{B}(H)$. An operator $U : H \to H$ is *unitary* if it is invertible and preserves the norm (and hence the inner product) on H. Since

$$\langle x, y\rangle = \langle Ux, Uy\rangle = \langle U^*Ux, y\rangle$$

for all y, U is unitary if and only if

$$U^* = U^{-1}.$$

Any measure space canonically determines a Hilbert space via the association

$$(X, m) \to H = L^2(X, m).$$

We can regard this transform as a *functor*. In particular, on the level of automorphisms, it sends a measure–preserving bijection $T : X \to X$ to a unitary operator $U : H \to H$.

The Hilbert space $L^2(X, m)$ comes *equipped* with a natural linear functional, namely

$$f \mapsto \int f = \langle f, 1\rangle.$$

It is frequently useful to eliminate the obvious invariant subspace of $L^2(X, m)$, namely the constant functions, by passing to the kernel:

$$L^2_0(X, m) = \{f \in L^2(X, m) \ : \ \int f\, dm = 0\}.$$

Unitary representations. For any group G, the Hilbert space functor also makes sense for measurable G–actions, with the target the category of *unitary representations* of G.

Now the irreducible representations of a locally compact abelian group (such as $\mathbb{Z}$ or $\mathbb{R}$) are all *one-dimensional.* So an ergodic action is *never* irreducible from the point of view of representation theory, and finding its spectral decomposition gives additional structure and invariants.

Review: Spectral theorem for normal operators. For simplicity, assume H is a *separable* Hilbert space. Since a Hilbert space is determined up to isomorphism by the cardinality of an orthonormal basis, the space H is isomorphic to $\mathbb{C}^N$ or $\ell^2(\mathbb{Z})$.

We let $\mathcal{B}(H)$ denote the Banach space of bounded linear operators $S : H \to H$, with the operator norm. The adjoint of an operator is defined by $\langle S^* x, y\rangle = \langle x, Sy\rangle$ for all $x, y \in H$. The *spectrum* of an operator is the compact subset of $\mathbb{C}$ defined by

$$\sigma(S) = \{\lambda \in \mathbb{C} \,:\, (\lambda - S) \text{ is not invertible}\}.$$

Clearly $\sigma(S)$ contains the eigenvalues of S, but it is often much richer as we will soon see.

For a unitary operator, we have $\sigma(U) \subset S^1$. A self–adjoint operator has real spectrum.

There is a good description of S whenever S is a self–adjoint ($S^* = S$), unitary ($S^* = S^{-1}$) or more generally, normal operator (the commutator $[S, S^*] = 0$). Namely, S is uniquely determined, up to isomorphism (unitary conjugacy), by:

- A *spectral measure* μ, which is a σ–finite Borel measure on $\sigma(S)$; and
- A *multiplicity function* $M : \sigma(S) \to \{0, 1, 2, \ldots, \infty\}$, Borel measurable, which specifies the dimension of each eigenspace.

The measure μ is allowed to be infinite for convenience; it can always be taken to be finite.

From this data we can construct a Hilbert space bundle $\mathcal{H} \to S^1$ with fibers satisfying

$$\dim \mathcal{H}_\lambda = M(\lambda),$$

where $H_\lambda \cong \ell^2(\mathbb{Z})$ when $M(\lambda) = \infty$. We can then form the Hilbert space of sections $L^2(\mu, M)$, where $f : \sigma(S) \to \mathcal{H}$ is measurable and satisfies

$$\|f\|^2 = \int_{\sigma(S)} \|f(\lambda)\|^2 \, \mu < \infty.$$

On $L^2(\mu, M)$ we have a natural multiplication operator, namely $L(f) = \lambda f(\lambda)$. Note that L is normal. We can now state the spectral theorem.

Theorem 2.4 *For any normal operator $S \in \mathcal{B}(H)$, there exists a measure and multiplicity function (μ, M) on the spectrum $\sigma(S)$, and an isometric isomorphism*

$$\psi : H \cong L^2(M, \mu),$$

sending S to the operator $L(f) = \lambda f(\lambda)$.

Thus (μ, M) determines S up to isomorphism. (See Appendix A for more details.)

Variant. Another way to describe $L^2(\mu, M)$ is that it consists of measurable maps $f : \sigma(S) \to \ell^2(\mathbb{N})$, with $f(\lambda) = (f_0(\lambda), f_1(\lambda), \ldots)$, such that $f_i(\lambda) = 0$ for $i \geq M(\lambda)$, and with

$$\|f\|_2^2 = \int_{\sigma(S)} \|f(\lambda)\|_2^2 \, \mu(\lambda) < \infty.$$

Measure classes. Two measures are in the same *measure class* if they have the same sets of measure zero. It is not hard to show that (μ, M) and (μ', M) determine isomorphic operators iff μ and μ' are in the same measure class, and $M = M'$ a.e.

Although one speaks of 'the spectral measure of S', actually it is only the measure class that is determined by S.

Direct sums. To understand multiplicity, it is useful to note that if $S_i \in \mathcal{B}(H)$ have spectral data (μ_i, M_i), $i = 1, 2$, then $S_1 \oplus S_2 \in \mathcal{B}(H \oplus H)$ has spectral data $(\mu_1 + \mu_2, M_1 + M_2)$.

In fact every normal operator can be written as a finite or countable direct sum of operators with multiplicity one. To see this, write $M = \sum M_i$ where each M_i assumes only the values 0 and 1, and observe that

$$L^2(\mu, M) \cong \oplus_i L^2(\mu, M_i).$$

Example 1: The finite–dimensional case. If H is finite–dimensional, then S has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicities $m_1, \ldots, m_n$. In this case $\mu = \sum \delta_{\lambda_i}$ and $M(\lambda_i) = m_i$. The values of M on the rest of the circle are irrelevant. Thus the spectral theorem diagonalizes S.

Example 2: Multiplication operators. Any $g \in L^\infty[0, 1]$ determines an operator on $H = L^2[0, 1]$ by setting

$$S_g(f) = g(x)f(x).$$

Note that $S_g^* = S_{\overline{g}}$, and thus S is normal.

In this case $\mu = g_*(dx)$ is the pushforward of Lebesgue measure, and $\sigma(S_g)$ is the support of μ. When g is well–behaved, $M(\lambda) = |g^{-1}(\lambda)|$. For example, if g is injective, then $M(\lambda) = 1$. If g only assumes finite or countably many values (a_i), each on a set of positive measure, then $\mu = \sum \delta_{a_i}$ and $M(\lambda) = \infty$.

The spectral theorem essentially says that all normal operators essential come from multiplication by functions. Indeed, we can reduce to the case where $g(\lambda) = \lambda$ acts on $L^2(\sigma(S), \mu)$; the only nuance is that we also need to take multiplicity into account.

Example 3: Lebesgue spectrum. Consider the unitary operator U on $\ell^2(\mathbb{Z})$ defined by the left shift:

$$U(a_i) = (a_{i-1}).$$

Clearly this operator has no eigenvectors. What is its spectral measure?

Theorem 2.5 *The shift operator on $\ell^2(\mathbb{Z})$ has Lebesgue spectrum of multiplicity one.*

(This means its spectral measure (class) μ is given by Lebesgue measure $|d\lambda|$ on the unit circle S^1.)

Proof. Observe that the map

$$(a_i) \mapsto \sum a_i z^i$$

gives an isomorphism between $\ell^2(\mathbb{Z})$ and $L^2(S^1)$. Since

$$zf(z) = z \sum a_i z_i = \sum a_{i-1} z^i,$$

this isomorphism sends U to the operator $L(f) = zf(z)$ acting on $L^2(S^1)$. Consequently the spectral measure of U is given simply by Lebesgue measure on S^1, with multiplicity one. ■

Lebesgue spectrum arises frequently in ergodic theory as we will soon see.

Irreducible representations of $\mathbb{Z}$. Next we explain how the spectral theorem gives the decomposition of unitary representations of $\mathbb{Z}$ into irreducibles.

All irreducible unitary representations of $\mathbb{Z}$ are 1-dimensional; they are naturally parameterized by $\lambda \in S^1$. A general a unitary representation of $\mathbb{Z}$ is simply given by $\rho(n) = U^n$ for some unitary operator U. We have $\sigma(U) \subset S^1$, since U is unitary. Thus the spectral theorem for a unitary operator can be thought of as a way of decomposing the action of U on H into 1–dimensional irreducible representations, with multiplicities:

$$H \cong \int_{S^1} \mathbb{C}_\lambda^{M(\lambda)} \, \mu,$$

with $U|\mathbb{C}_\lambda$ given by multiplication by λ.

The spectrum in ergodic theory. Returning to the theory of measure–preserving maps, we observe that:

Theorem 2.6 *The spectral measure and the multiplicity function (μ, M) on S^1 of the unitary operator $T|L^2(X, m)$ are invariants of T, up to equivalence.*

Here are two examples at opposite extremes.

Examples 1: The rotation. Let $T(z) = \lambda z$ on S^1, where $|\lambda| = 1$. Recall that the functions $e_k(z) = z^k$ form an orthonormal basis for $L^2(S^1, m)$, where m is normalized Lebesgue measure. The unitary operator $U(f) = f \circ T$ satisfies, evidentially,

$$U(e_k) = (\lambda^k)z^k = \lambda_k e_k.$$

Thus U is diagonal with respect to this basis. Assuming λ is irrational, its spectral measure is given by

$$\mu = \sum_{n \in \mathbb{Z}} \delta_{\lambda^n},$$

with $M(\lambda) = 1$. In brief, we have:

Theorem 2.7 *An irrational rotation T of the circle has pure point spectrum of multiplicity one, supported on $\lambda^{\mathbb{Z}}$ for some $\lambda \in S^1$ of infinite order.*

Corollary 2.8 *Irrational rotations by λ_1 and λ_2 on S^1 are isomorphic as measurable systems if and only if $\lambda_1 = \lambda_2^{\pm 1}$.*

Note: when T is a periodic rotation, it is determined up to isomorphism by its period p. Its spectral measure consists of δ masses at the pth roots of unity on S^1, and $M(\lambda) = \infty$.

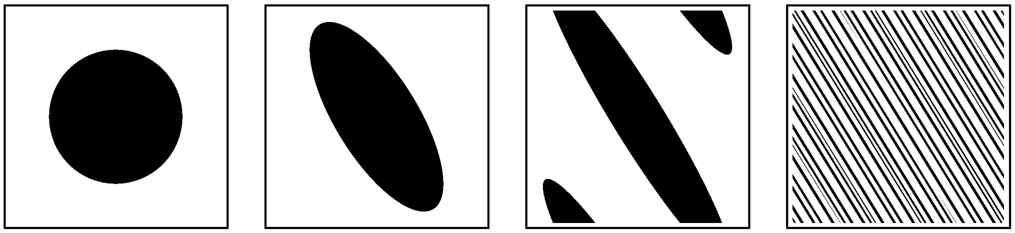


Figure 2. A mixing map on the torus.

Example 2: Automorphism of a 2-torus. Now let $X = \mathbb{R}^2/\mathbb{Z}^2$ and let T be an automorphism of X specified by a matrix $T \in \mathrm{GL}_2(\mathbb{Z})$. Recall that T is *hyperbolic*, *parabolic* or *elliptic.* It is elliptic if it has finite order; hyperbolic if it has two distinct real eigenvalues; and parabolic if it has one eigenvalue of multiplicity two (and has infinite order).

In this case we have $L^2(X) \cong \ell^2(\mathbb{Z}^2)$, with the Fourier orthonormal basis given, for $(a, b) \in \mathbb{Z}^2$, by

$$e_{ab}(x, y) = \exp(2\pi i(ax + by)).$$

The action of T is not diagonal in this basis; rather, it satisfies $T(e_v) = e_{T^*v}$, where T^* is the adjoint of $T|\mathbb{R}^2$. We are thus led to study the action of T^* on $\mathbb{Z}^2$.

Suppose T is hyperbolic. In this case we find:

Every orbit of T^* *on* $\mathbb{Z}^2 - \{(0, 0)\}$ *is infinite.*

Indeed, if T^* were to have a periodic orbit, then $T|\mathbb{R}^2$ would have an eigenvalue which is a root of unity, which it does not.

Now each infinite orbit of T^* gives a subspace of $L^2_0(X)$ isomorphic to $\ell^(\mathbb{Z})$, on which T acts by the shift. The direct sum of these subspace is the whole Hilbert space. This shows:

Theorem 2.9 *The action of a hyperbolic element* $T \in \mathrm{SL}_2(\mathbb{Z})$ *on the torus has Lebesgue spectrum of infinite multiplicity.*

Corollary 2.10 *All hyperbolic toral automorphisms are unitarily conjugate.*

It is an interest challenge to show they are not all *spatial* conjugate. Here a useful invariant is the *entropy*, introduced by Kolmogorov for exactly this purpose.

We will later see that hyperbolic toral automorphisms are *mixing.* These are also called (linear) *Anosov maps.* The image of a round disk under of T^i for $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $i = 0, 1, 2, 8$ is shown in Figure 2.

Open problem. No example is known of a measure–preserving map with Lebesgue spectrum of *finite* multiplicity.

Parabolic and elliptic maps. If $T \in \mathrm{SL}_2(\mathbb{Z})$ is parabolic then it preserves a foliation of $X = \mathbb{R}^2/\mathbb{Z}^2$ by closed circles, and any function constant on these leaves is invariant. If T is elliptic then X/T is a nice space and hence there are abundant invariant functions. In either case T is not ergodic.

Properties of the spectrum. Not every unitary operator can arise from a measure–preserving transformation. For example, using the Rohlin–Halmos Theorem 2.3, one can show:

Theorem 2.11 *If U comes from a measure–preserving map T without period points, then $\sigma(U) = S^1$.*

Ergodicity and mixing. We now turn to the properties of *ergodicity* and *mixing* for the map T.

Recall that T is ergodic if any T–invariant measure set A satisfies $m(A) = 0$ or $m(X - A) = 0$. We say T is *mixing* if whenever $A, B \subset X$ have positive measure, we have

$$m(A \cap T^{-i}(B)) \to m(A)m(B)$$

as $i \to \infty$. Clearly mixing implies ergodicity.

These properties are in fact detected by the unitary operator $U = T|L^2_0(X, m)$; they are *spectral invariants* of T. Indeed, the next two results are easy to show:

Theorem 2.12 *Let $U = T|L^2_0(X, m)$ be the unitary operator associated to a measure–preserving automorphism of (X, m). The following are equivalent:*

1. *T is ergodic.*
2. *If $m(A) > 0$ then $\bigcup_{-\infty}^{\infty} T^i(A) = X$ is a set of full measure.*
3. *Any T-invariant measurable function is constant a.e.*
4. *The only U–invariant vector is $f = 0$.*
5. *The spectral measure of U assigns no mass to $\lambda = 1$.*

Theorem 2.13 *The following are equivalent:*

1. *For all pairs of measurable sets A, B in X, we have $m(A \cap T^{-i}(B)) \to m(A)m(B)$ as $i \to \infty$.*
2. *For all $f, g \in L^2_0(X)$, we have*
$$\lim_{i\to\infty} \langle U^i f, g\rangle = 0. \tag{2.1}$$

In classical terminology, one says the matrix coefficients of U^i decay as $i \to \infty$, or $U^i \to 0$ in the weak topology on $\mathcal{B}(H)$.

Put differently, mixing is equivalent to the assertion that
$$\int (f \circ T^i) g \to \int f \int g$$
for all $f, g \in L^2(X)$.

Example: the shift operator. Let ν be a probability measure on $\mathbb{R}$, and let m be the product measure on $X = \mathbb{R}^{\mathbb{Z}}$. This space models an infinite sequence of independent random variables $(f(i))$ with the same distribution.

Theorem 2.14 *The shift operator on $\mathbb{R}^{\mathbb{Z}}$ is mixing.*

Proof. Functions $f : \mathbb{Z} \to \mathbb{R}$ that depend on only finitely many coordinates are dense in $L^2(X)$, and by Fubini's theorem they satisfy
$$\int (f \circ T^i) g = \int f \int g$$
once i is large. ■

Mixing and spectrum. Let us say a unitary operator is mixing if it satisfies equation (2.1). We then have:

Theorem 2.15 *Any unitary operator with Lebesgue spectrum is mixing.*

Proof. First consider the case of multiplicity one. Then U is modeled on the shift on $\ell^2(\mathbb{Z})$. If f and g have finite support, then $\langle U^i f, g\rangle = 0$ whenever $|i| \gg 0$; since such functions are dense, we have mixing.

The case of infinite multiplicity is similar. Finally any unitary operator with Lebesgue spectrum is the restriction of the operator with infinite multiplicity, and mixing passes to restrictions. ■

Remark: decay of L^1 Fourier coefficients. Mixing with Lebesgue spectrum also follows from the Riemann–Lebesgue lemma, which says that for any $f \in L^1(S^1)$ we have

$$\int z^i h \to 0$$

as $|i| \to \infty$.

Corollary 2.16 *If $U = T|L^2_0(X)$ has an eigenvector (point spectrum), then T is not mixing. On the other hand, if U has Lebesgue spectrum (of any multiplicity), then T is mixing.*

Examples. We turn to our two basic examples.

1. *The irrational rotation.* We have seen that $T|L^2(S^1)$ has pure point spectrum of multiplicity one supported on the infinite subgroup $\lambda^{\mathbb{Z}}$. Thus T is ergodic but not mixing.

2. *Automorphisms of a torus.* On the other hand, $T|L^2_0(X)$ has Lebesgue spectrum (of infinite multiplicity) for a hyperbolic automorphism of $\mathbb{R}^2/\mathbb{Z}^2$. Thus T is mixing.

Measure endomorphisms and Hilbert space isometries A bounded linear map $U : H \to H$ is an *isometry* if it preserves the inner product, but is not necessary invertible. A measure–preserving map need not be a bijection, and in general it induces an isometry, rather than a unitary transformation, on $L^2(X)$. In the case of an isometry, we still have $U^*U = I$ but we need not have $UU^* = I$; in particular, U and U^* may generate a non–commutative subalgebra of $\mathcal{B}(H)$.

Group endomorphisms: The doubling map. An nice example of mixing, even simpler than a hyperbolic toral automorphism, is provided by the measure–preserving *endomorphism* of $G = S^1 = \mathbb{R}/\mathbb{Z}$ given by $T(x) = 2x \bmod 1$. Then $\widehat{T}$ acts on $\widehat{G} = \mathbb{Z}$ by $n \mapsto 2n$. From this it is immediate that T is mixing. Intuitively, if f is a function of mean zero, then $f(T^n x) = f(2^n x)$ is a sound wave with very high frequency, that is hard to hear with a fixed listening device.

Harmonic analysis, compact groups and Pontryagin duality. Let G be a compact abelian topological group (such as $\mathbb{R}^n/\mathbb{Z}^n$. Its *Pontryagin dual* is the discrete group of continuous characters

$$\widehat{G} = \{\chi : G \to S^1\}.$$

For example, if $G = \mathbb{R}^n/\mathbb{Z}^n$ then $\widehat{G} = \mathbb{Z}^n$.

We are interested in these groups because they are naturally probability spaces with respect to normalized Haar measure.

The *Plancheral theorem* states that the pairing between characters and functions establishes an isomorphism

$$L^2(G) \cong L^2(\widehat{G}),$$

generalizing Fourier series for functions on the circle or a torus.

The general setting includes compact groups such as $G = (\mathbb{Z}/2)^{\mathbb{Z}}$. Its dual $\widehat{G}$ is generated by the characters $\chi_j(a) = (-1)^{a_j}$. Other examples include the d–adic integers, $G = \mathbb{Z}_d = \varprojlim \mathbb{Z}/d^n$, and the d–adic solenoid

$$G = \varprojlim \mathbb{R}/d^i\mathbb{Z},$$

consisting of those sequences $(x_i) \in \prod_0^\infty S^1$ such that $dx_i = x_{i-1}$. The solenoid is a $\mathbb{Z}_d$ bundle over S^1 with monodromy $x \mapsto x+1$. The dual of the solenoid is generated by the characters $\chi_{ik}(z) = z_i^k$, $i \geq 0$, $k \in \mathbb{Z}$; in fact it is isomorphic to $\mathbb{Z}[1/d]$ (the rationals whose denominators are powers of d), via $\chi_{ik} \mapsto i/d^k$.

The Pontryagin dual can be defined for any locally compact abelian group, and the duality theorem states that $\widehat{\widehat{G}} \cong G$. In particular, the classification of compact abelian groups is the same as the classification of discrete abelian groups — which, outside the finitely generated case, is a difficult open problem.

Dynamics on compact abelian groups. The group G carries a unique G–invariant *Haar measure.* By uniqueness, any continuous automorphism T of G is also measure-preserving. Generalizing our two basic examples, we have:

Theorem 2.17 *The translation map $T(x) = x + g$ acts diagonally on $L^2(\widehat{G})$ with respect to its basis of characters. Thus it has pure point spectrum, supported on the set*

$$\{\chi(g) \,:\, \chi \in \widehat{G}\}.$$

Corollary 2.18 *The map T is ergodic if and only if g generates a dense subgroup of G. It is mixing only when G is the trivial group.*

Proof. Let H be the closure of the group generated by G. Then $\chi(g) = 1$ if and only if χ is a character of G/H. Thus $H = G$ iff the number of characters satisfying $\chi(g) = 1$ is exactly one (the trivial character). This is the same as saying $\lambda = 1$ has multiplicity one in the spectrum of $T|L^2(G)$. ■

Theorem 2.19 *An automorphism T of a compact Abelian group G is ergodic iff $\widehat{T}$ has no periodic points in $\widehat{G}$, other than the trivial character. In this case T is also mixing and has Lebesgue spectrum of infinite multiplicity, provided G is nontrivial.*

Proof. The orbits of $T|\widehat{G}$ give a decomposition of $L^2(\widehat{G})$. The finite orbits give eigenfunctions, while the infinite orbits provide subspaces on which T is conjugate to the shift of $\ell^2(\mathbb{Z})$. For ergodicity we can only have the latter type of orbit, outside of the trivial character, in which case we have Lebesgue spectrum and mixing. The fact that $T|\widehat{G}$ has infinitely many orbits is an exercise about abelian groups. ■

Corollary 2.20 *An automorphism of $X = \mathbb{R}^d/\mathbb{Z}^d$ given by $T \in \mathrm{GL}_d(\mathbb{Z})$ is ergodic if and only if no eigenvalue of T is a root of unity.*

When T is ergodic it is also mixing, with Lebesgue spectrum of infinite multiplicity.

The ergodic theorem: behavior of orbits. A central issue in ergodic theory is to study how an orbit of $T : X \to X$ is distributed in X. For example, when T is ergodic and $A \subset X$ has positive measure, we know by Poincaré recurrence that $T^i(x)$ returns infinitely often to A. But how often? A basic fact, to be proved below, is:

Theorem 2.21 *If T is ergodic, then for almost every $x \in X$, the percentage of time that the orbit of x spends in A is given by $m(A)$.*

For a more general analysis, we consider $f \in L^1(X)$ and then form the sequence of sums:

$$S_n(f, x) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x). \tag{2.2}$$

Suppressing x, we obtain a function $S_n(f) \in L^1(X)$. Note that

$$\int S_n(f,x) = \int f \quad \text{and} \quad \|S_n(f)\|_1 \le \|f\|_1.$$

The first inequality follows from the fact that T is measure–preserving. The second inequality holds for any T–invariant norm; for example, it holds for L^p if $f \in L^p(X)$. In fact we have

$$\|S_n(f,Tx) - S_n(f,x)\| \le (2/n)\|f\|.$$

In other words the functions $S_n(f)$ are becoming more and more nearly T–invariant.

The *ergodic theorem* asserts that if T is ergodic, then

$$S_n(f) \to \int f.$$

The right hand side denotes the constant function whose value is the average of f.

More generally, if T is not ergodic, $S_n(f)$ converges to a T–invariant function $\overline{f}(x)$.

In this section we will prove two versions of this theorem: one where convergence takes place in $L^2(X)$, and another where convergence takes place pointwise. Taking $f = \chi_A$, the pointwise result gives Theorem 2.21.

The ergodic theorem I: Hilbert space. We now turn to the study of the averages $S_n(f)$ defined by (2.2) from the perspective of Hilbert space. We will formulate a theorem which describes the behavior of the associated operator $U = T|L^2(X)$; indeed, this is simply a result about unitary operators.

To state it, let U be a unitary operator on H and let $H^U = \operatorname{Ker}(I - U)$ denote the set of *invariant vectors*, satisfying $U(f) = f$. We will be interested, for general $f \in H$, in the sums

$$S_n(f) = \frac{1}{n}\sum_0^{n-1} U^i(f).$$

Of course $S_n(f) = f$ for all n if $f \in H^U$. Note that H^U is a closed subspace of H. We let $P : H \to H^U$ be orthogonal projection. Then von Neumann's ergodic theorem states:

Theorem 2.22 *For all $f \in H$ we have $S_n(f) \to P(f)$ in norm.*

Coboundaries. We say f is a *coboundary* if it has the form

$$f = g - U(g)$$

for some $g \in H$. Equivalently, $f \in \operatorname{Im}(I - U)$. If f is a coboundary, then the sum $S_n(f)$ telescopes and we have

$$S_n(f) \to 0.$$

Also $P(Ug) = P(g)$, so $P(f) = 0$. Thus the ergodic theorem holds for coboundaries. The same holds true for any limit of coboundaries. Thus von Neumann's theorem follows from:

Lemma 2.23 *The Hilbert space H decomposes as an orthogonal direct sum,*

$$H = H^U \oplus \overline{\operatorname{Ker}(I - U)}.$$

Proof. Suppose f is perpendicular to all coboundaries. Then we find f is U^*-invariant, since

$$\langle f, g - Ug \rangle = 0 = \langle f - U^* f, g \rangle$$

for all $g \in H$. Since $U^* = U^{-1}$, we are done. ∎

Endomorphisms. In the case where T is an endomorphism, U is an *isometry* but we need not have $U^* = U^{-1}$; in fact U^{-1} may not exist. We only know that $\langle f, g \rangle = \langle Uf, Ug \rangle$ for all $f, g \in H$. The ergodic theorem still holds in this case.

Proof for an isometry. Since $\langle f, g \rangle = \langle Uf, Ug \rangle$ for all $f, g \in H$, we have $U^*U = I$. Now if f is U^*–invariant, then

$$\|f - Uf\|^2 = \langle f, f \rangle + \langle Uf, Uf \rangle - 2\langle f, Uf \rangle.$$

But $\langle Uf, Uf \rangle = \langle f, f \rangle$ since U is an isometry, and $\langle f, Uf \rangle = \langle U^* f, f \rangle$ since f is U^* invariant. Thus f is U-invariant as desired. ∎

A second proof. The spectral theorem for a unitary operator provides an equally transparent proof of the ergodic theorem above.

For convenience we assume H is separable. Then by the spectral theorem, there is an isomorphism from H to $\oplus L^2(S^1, \mu, m)$ where μ is a probability measure and m is the multiplicity function. With this model for H, the nth sum in the ergodic theorem is given by

$$S_n(f) = \frac{1}{n}\left(1 + z + \cdots + z^{n-1}\right) f(z) = \frac{1}{n}\frac{1-z^n}{1-z} f(z).$$

Clearly

$$\|(S_n f)(z)\| \leq \|f(z)\|,$$

and $(S_n f)(z) \to 0$ pointwise, for $z \neq 1$. By the dominated convergence theorem, this implies that $S_n(f)$ converges to the unique function with $F(1) = f(1)$ and $F(z) = 0$ for $z \neq 1$. This is exactly the projection of f onto the space of U-invariant functions.

The ergodic theorem II: pointwise limits. While von Neumann's theorem suffices for many applications, it does *not* address the behavior of $S_n(f, x)$ for any particular x. In fact we can have $F_n \to 0$ in $L^2(X)$ while $\limsup F_n(x) = \infty$ for all x. From the perspective of physics, the universe itself corresponds to a particular, arguably random point x, and we only have access to the values of $S_n(f, x)$, not to any averages of this function as x varies.

From the point of view of probability theorem, we want to know what happens for a typical, infinite sequence of trials. These questions are settled by the pointwise ergodic theorem of Birkhoff and Khinchin.

Theorem 2.24 *Let $T : X \to X$ be a measure–preserving endomorphism of a probability space. Then for any $f \in L^1(X)$, there exists a T-invariant function $\overline{f} \in L^1(X)$ such that*

$$(S_n f)(x) \to \overline{f}(x)$$

for almost every $x \in X$. Moreover, $\|S_n f - \overline{f}\|_1 \to 0$.

Note: when f is in $L^2(X)$, von Neumann's theorem tells us that $\overline{f}$ is simply the projection of f to the space of invariant functions. In general, $\overline{f}$ can be thought of as the average of f along the leaves of the foliation of X by its ergodic components. In any case, we have:

Corollary 2.25 *If T is ergodic, then $S_n(f,x) \to \int f$ for almost every x.*

The maximal theorem. We will give two proofs of the pointwise ergodic theorem. The first hinges on the following result.

Theorem 2.26 (The maximal theorem) *Given $f \in L^1(X,\mu)$, let $A = \{x : \sup_n (S_n f)(x) > 0\}$. Then $\int_A f \, d\mu \geq 0$.*

Proof. We will use the fact that the integral of a coboundary is zero. Let

$$F_n(x) = f(x) + f(Tx) + \cdots f(T^{n-1}x)$$

be the running sum of f, let

$$M_n(x) = \max\{F_1(x), \ldots, F_n(x)\},$$

and let

$$M_n^*(x) = \max\{F_0(x), \ldots, F_n(x)\}.$$

Note that $M_n^*(x) \geq 0$, that $M_1 \leq M_2 \leq \cdots$, and that A is the increasing union of the sets $A_n = \{x : M_n(x) > 0\}$. Note also that we have

$$M_{n+1}(x) = M_n^*(Tx) + f(x),$$

which nearly expresses f as a coboundary. We now observe that

$$\begin{aligned}
\int_{A_n} f &= \int_{A_n} M_{n+1} - M_n^* \circ T \\
&\geq \int_{A_n} M_n - M_n^* \circ T \quad \text{because } M_n \text{ increases with } n \\
&= \int_{A_n} M_n^* - M_n^* \circ T \quad \text{since } M_n = M_n^* \text{ on } A_n \\
&= \int_X M_n^* - \int_{A_n} M_n^* \circ T \quad \text{since } M_n^*(x) = 0 \text{ outside } A_n \\
&\geq \int_X M_n^* - M_n^* \circ T \quad \text{since } M_n^* \geq 0 \\
&= 0 \quad \text{since } T \text{ is measure-preserving.}
\end{aligned}$$

Since $A = \bigcup A_n$ the Theorem follows. ■

Proof of Theorem 2.24. Suppose to the contrary that $(S_n f)(x)$ does not converge a.e. Then for some $a < b$, the set

$$E = \{x \,:\, \liminf(S_n f)(x) < a < b < \limsup(S_n f)(x)\} \tag{2.3}$$

has positive measure. Note that E is also T-invariant. Since we are aiming at a contradiction, we may now assume $X = E$. We may also rescale and translate f so that $a = -1 < 1 = b$.

Clearly $\sup(S_n f)(x) > 0$ for every $x \in E$; in fact $\sup(S_n f)(x) > 1$. By the maximal theorem,

$$\int_E f \geq 0.$$

By the same reasoning, $\inf(S_n f)(x) < 0$ and hence

$$\int_E f \leq 0.$$

Thus $\int_E f = 0$. But we may apply the same reasoning to $f - c$ for any small value of c, and hence $\int_E f = c$. This is a contradiction, so $(S_n f)(x)$ converges pointwise a.e.

Let $F(x) = \lim(S_n f)(x)$. We wish to show that $F \in L^1(X)$ and $\|S_n f - F\|_1 \to 0$. Since any L^1 function is the difference of two positive functions, it suffices to treat the case where $f \geq 0$. Then by Fatou's Lemma, we have

$$\|F\|_1 = \int F = \int \lim S_n f \leq \liminf \int S_n f = \int f = \|f\|_1,$$

so F is in L^1. Moreover, if f is bounded then $S_n f \to F$ in $L^1(X)$ by the dominated convergence theorem. For the general case, write f as the sum $f_0 + r$ of two positive functions, with f_0 bounded and $\|r\|_1 < \epsilon$. we have $F \in L^1(X)$. Moreover, if f is bounded then $S_n f \to F$ in L^1 by the dominated convergence theorem. But we can write $f = f_0 + r$ where f_0 is bounded and $\|r\|_1 < \epsilon$. Then $S_n f_0 \to F_0$ in $L^1(X)$, $R(x) = \lim(S_n r)(x)$ satisfies $F_0 + R = F$, and $\|R\|_1 \leq \|r\|_1 < \epsilon$. Thus:

$$\limsup \|S_n f - F\|_1 \leq \limsup \|S_n f_0 - F_0\|_1 + \limsup |S_n r - R\|_1 \leq 2\epsilon.$$

It follows that $S_n f \to F$ in $L^1(X)$ and the proof is complete. ∎

Corollary 2.27 *If T is ergodic, then for every $f \in L^1(X)$, we have $(S_n f)(x) \to \int_X f\, d\mu$ for almost every $x \in X$.*

The strong law of large numbers. To illustrate the power of this result, we will use it to prove an important theorem in probability theory. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables, with $E(|X_1|) < \infty$.

Theorem 2.28 *The averages $(X_1 + \cdots + X_n)/n$ converge to $E(X_1)$ almost surely.*

Proof. Choose a function $F : [0,1] \to \mathbb{R}$ with the same distribution as X_1, and apply the ergodic theorem to the shift space $[0,1]^{\mathbb{Z}}$, $f(x) = F(x_1)$. ■

Example. For almost every $x \in [0,1]$, exactly 10% of its decimal digits are equal to 7.

Flows. We remark that the ergodic theorem also holds for flows: given a measure–preserving action H^t of $\mathbb{R}$, and $f \in L^1(X,\mu)$, there exists a flow invariant function $F \in L^1(X,\mu)$ such that

$$(S_t f)(x) = \frac{1}{2t} \int_{-t}^{s} f(H^s x)\, ds \to F(x)$$

a.e. and in L^1. This version will be very useful when we study actions of Lie groups and, in particular, their 1-parameter subgroups, as well as geodesic flows.

Forward and backward sums. By von Neumann's result we now can succinctly say what the ergodic averages $S_n(f)$ converge *to*: for $f \in L^2$ it is just the projection of f onto the T-invariant functions.

In particular, we have:

Corollary 2.29 *For any measure-preserving map T, the averages of f along the forward and backward orbits of T agree a.e.*

Another approach to the Birkhoff–Khinchin ergodic theorem. (cf. Keane, [BKS, p.42], who attributes the idea to Kamae (1982)).

Let $A \subset X$ be a measurable set, and let

$$S_n(x) = S_n(\chi_A, x) = \frac{1}{n} \sum_{0}^{n-1} \chi_A(T^i x)$$

be the amount of time the orbit of x spends in A up to time n. We already know that $S_n(x)$ converges in L^2 to the projection of χ_A to the invariant functions, and to $m(A)$ when T is ergodic. What we want to know is that it also converges pointwise; if it does, the limit must be the same as the L^2 limit. So the main point is to show:

Theorem 2.30 *$S_n(x)$ converges almost everywhere.*

Proof. Let $\overline{S}(x) = \limsup S_n(x)$; our goal is to show that

$$\int \overline{S}(x)\, d\mu(x) \leq \mu(A).$$

The same argument will show the average of the liminf is at least $\mu(A)$, so the limsup and liminf agree a.e. and we'll be done.

Fixing $\epsilon > 0$ let $\tau(x)$ be the least $i > 0$ such that $S_i(x) > \overline{S}(x) - \epsilon$. In other words it is how long we have to wait to get the average close to its limsup. Obviously $\tau(x)$ is finite.

Now suppose the wait is bounded, i.e. $\tau(x) < N$ for all x. Then we can compute $S_n(x)$ for $n \gg N$ by waiting (at most N steps) until the average exceeds $\overline{S}(x) - \epsilon$, waiting that long again, and so on. Thus for all n large enough, we have $S_n(x) \geq \overline{S}(x) - 2\epsilon$ (more precisely, $S_n(x) \geq (1 - N/n)(\overline{S}(x) - \epsilon)$). Since $\int S_n(x) = \mu(A)$ for any n we are done.

Now suppose the wait is not bounded. Still, we can choose N large enough that the points where $\tau(x) > N$ have measure less than ϵ. Adjoin these points to A to obtain A', and let $S'_n(x)$ denote the average number of visits to A'. This time we can again compute $S'_n(x)$ by waiting at most N steps for the average to exceed $\overline{S}(x) - \epsilon$, since if $\tau(x) > N$ then x is already in A' (and one step suffices). Thus once again, $S'_n(x) \geq \overline{S}(x) - 2\epsilon$ for all n sufficiently large. Since $\int S'_n(x) = \int A' \leq \mu(A) + \epsilon$, we have shown $\int \overline{S}(x) \geq \mu(A)$ in this case as well. ■

Historical remarks. The proof of von Neumann's theorem is easier than the proof of the Birkhoff–Khinchin ergodic theorem. In mid–1931, Koopman made the connection between measurably dynamics and unitary operators. By October of the same year, von Neumann had proved the ergodic theorem below and communicated it to Birkhoff. By December, Birkhoff had completed the proof of his pointwise result, and arranged for it to be published before von Neumann's result. (Both appeared in the Proc. Nat. Acad. Sci.)

Appendix: Entropy and information. We have seen that many measurable dynamical systems such as shifts and toral automorphisms are spectrally equivalent (since they are Lebesgue). Are they actually spatially conjugate? Kolmogorov and Shannon invented an invariant to distinguish them.

Suppose you learn that $x \in X$ belongs to $A \subset X$; how much information $I(A)$ have you gained about x? If we postulate that:

- $I(A) = f(m(A))$; the amount of information depends only on the measure of A; and
- $f(xy) = f(x)+f(y)$; so the information provided by independent events (in the sense of probability theory) is additive;

then we find that $f(x) = C \log x$; and to maintain positivity we set $I(A) = \log(1/m(A))$.

The *entropy* of a partition $\mathcal{A}$ of a measure space X is the expected amount of information gained when you pick a point at random, and then learn which element of $\mathcal{A}$ contains it. It is thus given by

$$h(\mathcal{A}) = \int_X \sum -\chi_A \log m(A) = \sum_{\mathcal{A}} m(A) \log(1/m(A)).$$

Entropy of a dynamical system. Given a partition $\mathcal{A}$ and dynamics $T : X \to X$, consider the information gained by measuring the position not only of x but of Tx, T^2x, $\ldots T^{n-1}x$ relative to $\mathcal{A}$. This is the same as measuring x relative to the partition

$$\mathcal{A}(T,n) = \mathcal{A} \vee T^{-1}(\mathcal{A}) \vee \ldots \vee T^{-n+1}(\mathcal{A}).$$

We can measure the rate of growth of information along an orbit by:

$$h(T,\mathcal{A}) = \limsup \frac{h(\mathcal{A}(T,n))}{n}$$

and finally define the *entropy* of T by

$$h(T) = \sup_{\mathcal{A}} h(T,\mathcal{A})$$

where the supremum is over all finite partitions.

Basic property of entropy. The most basic property of entropy is

$$h(\mathcal{A} \vee \mathcal{B}) \leq h(\mathcal{A}) + h(\mathcal{B}).$$

In other words, the information from two measurements is maximized when the measurements are independent.

Proof. Let $\phi(x) = x \log x$; then ϕ is convex; the inequality will come from the fact that $\phi(\sum a_i b_i) \leq \sum a_i \phi(b_i)$ when $\sum a_i = 1$, $a_i \geq 0$. Indeed, we have

$$\begin{aligned}
h(\mathcal{A} \vee \mathcal{B}) &= -\sum_{i,j} m(A_i \cap B_j) \log m(A_i \cap B_j) \\
&= -\sum_i m(A_i) \sum_j \frac{m(A_i \cap B_j)}{m(A_i)} \left(\log \frac{m(A_i \cap B_j)}{m(A_i)} + \log m(A_i) \right) \\
&= h(\mathcal{A}) - \sum_{i,j} m(A_i)\, \phi\left(\frac{m(A_i \cap B_j)}{m(A_i)} \right) \\
&\leq h(\mathcal{A}) - \sum_j \phi(m(B_j)) \\
&= h(\mathcal{A}) + h(\mathcal{B}).
\end{aligned}$$

■

From this inequality it follows that the lim sup in the definition of $h(T, \mathcal{A})$ is actually a limit.

Entropy calculations. Suppose the smallest T-invariant σ-algebra generated by $\mathcal{A}$ is equal to all the measurable sets, up to sets of measure zero. Then we say $\mathcal{A}$ is a *generating* partition.

Theorem 2.31 (Kolmogorov-Sinai) *If $\mathcal{A}$ is a generating partition, then $h(T, \mathcal{A}) = h(T)$.*

Idea of the proof. Given an arbitrary partition $\mathcal{B}$, suppose we can find an i such that $\mathcal{A}(T, i) > \mathcal{B}$ (the first partition is finer). Then $\mathcal{A}(T, n+i) > \mathcal{B}(T, n)$ and thus

$$h(T, \mathcal{A}) = \lim \frac{h(\mathcal{A}(T, i+n)}{n} \geq \lim \frac{h(\mathcal{B}(T, n)}{n} = h(\mathcal{B}).$$

Since $h(T) = \sup h(T, \mathcal{B})$ we would be done. The idea is that since $\mathcal{A}$ is a generating partition, $\mathcal{A}(T, i)$ does almost refine any given finite partition $\mathcal{B}$.

■

Note: we always have $h(T) \leq h(\mathcal{A}) \leq |\mathcal{A}|$. Thus entropy constrains the size of a generating partition.

Examples.

Let $(T, X) = (S^1, z \mapsto \lambda z)$. Then $h(T) = 0$.

Proof. Take a pair of intervals as the partition $\mathcal{A}$. Then $\mathcal{A}(T, n)$ consists of at most $2n$ intervals, since the endpoints of the original intervals give at most $2n$ possible endpoints. The entropy of such a partition is maximized when the intervals have equal length; thus

$$\frac{h(\mathcal{A}(T, n))}{n} \leq \frac{2n \cdot (2n)^{-1} \log 2n}{n} \to 0.$$

Theorem 2.32 *Let $(T, X) = (\sigma, \Sigma_d)$. Then $h(T) = \log d$.*

Proof. Let $\mathcal{A}$ be the partition into d blocks according to the first symbol. Then $\mathcal{A}$ is a generating partition, $\mathcal{A}(T, n)$ consists of d^n blocks each of measure d^{-n}, and thus

$$\frac{h(\mathcal{A}(T, n))}{n} = \frac{d^n \cdot d^{-n} \log d^n}{n} = \log d.$$

■

Bernoulli shifts. The measure space $B(p_1, \ldots, p_n)$ is Σ_n with the product measure assigning probability p_i to the ith symbol ($\sum p_i = 1$). The *Bernoulli shift* is the dynamical system in which the shift σ acts on $B(p_1, \ldots, p_n)$. Its entropy is given by:

$$h(\sigma) = \sum -p_i \log p_i.$$

Theorem 2.33 (Ornstein) *Entropy is a complete invariant for Bernoulli shifts; two are measurably conjugate iff their entropies agree.*

Toral automorphisms. Let $T \in \mathrm{SL}_n \mathbb{Z}$ represent a toral automorphism with eigenvalues λ_i. Then it can be shown that

$$h(T) = \sum_{|\lambda_i|>1} \log |\lambda_i|.$$

Conjecture: there is a constant h_0 such that $h(T) > h_0 > 0$ for any positive entropy toral automorphism of any dimension.

This is directly related to Lehmer's conjecture, that for an algebraic integer λ, the product of the conjugates of λ of norm greater than one is bounded away from one.

Theorem 2.34 *If $h(T) > 0$ then the spectral decomposition of U_T includes Lebesgue spectrum of infinite multiplicity.*

See [Par, p.71].

3 Measures and topological dynamics

In this section we add a topological element to our discussion. Let X be a compact metric space. We consider a continuous map $T : X \to X$ — usually a homeomorphism — and study its space of invariant probability measures, $P(X)^T$.

This allows us to formulate the important notion of *unique ergodicity* — where every orbit is uniformly distributed. We show, for example, that an irrational rotation of S^1 is uniquely ergodic.

We also study a hyperbolic automorphism T of a torus $X = \mathbb{R}^2/\mathbb{Z}^2$ using its invariant foliations, and give Hopf's proof of ergodicity and mixing. This argument is of central importance; it will occur again in the analysis of the geodesic and horocycle flows on a hyperbolic surface in §4.

Functional analysis. For later use, let us briefly recall some general results in functional analysis:

The space of measures. Let X be a compact metric space, and let $C(X)$ denote the Banach space of continuous functions on a compact metric space X, with the sup-norm. Its dual $M(X) = C(X)^*$ is naturally identified with the space of finite, signed Borel measures on X. (This is called the Riesz representation theorem; in Bourbaki, it becomes the *definition* of a measure.)

The *weak* topology* on $M(X)$ is defined by $\mu_n \to \mu$ if and only if we have

$$\int f\mu_n \to \int f\mu$$

for all $f \in C(X)$. (Our assumptions on X implies that the weak* topology is metrizable; in particular, one can work with sequences.)

By *Alaoglu's theorem*, the unit ball in $M(X)$ is compact in the weak* topology. (This is easy to see directly here: given a sequence of measures μ_n of total mass one, for each $f \in C(X)$ the sequence $\mu_n(f)$ is bounded; hence it has a convergent subsequence. Taking a countable dense set of f and diagonalizing, we obtain a convergent subsequence of μ_n.)

In the weak* topology, the space of probability measure

$$P(X) = \{m \in M(X) \; : \; m \geq 0 \quad \text{and} \quad m(X) = 1\}$$

is a compact, convex set. By the Krein–Milman theorem, $P(X)$ is the closed convex hull of its extreme points. These points are simply the δ–masses on X.

Remark. The norm topology on $C(X)^*$ is rarely used. Note that X embeds continuously into $C(X)^*$ in the weak* topology, by the map $x \mapsto \delta_x$. On the other hand, in the norm topology, the image of X in $C(X)^*$ is a closed, discrete set.

Invariant measures. Now let $T : X \to X$ be a continuous map. Then T acts on $M(X)$ by $\mu \mapsto T_*(\mu)$; here

$$\int f \, T_*(\mu) = \int (f \circ T) \, \mu;$$

equivalently, $(T_*\mu)(A) = \mu(T^{-1}A)$. This action is continuous in the weak* topology. Its set of fixed points in $P(X)$ will be denoted by $P(X)^T$; these are the T–invariant probability measures.

Theorem 3.1 *Any continuous map $T : X \to X$ on a (nonempty) compact metric space admits an invariant probability measure.*

Proof. Given $x \in X$, let μ_n be the average of δ-masses at the points x, $T(x)$, $\ldots T^{n-1}(X)$. Then for any continuous function $f \in C(X)$, the values of $\mu_n(f)$ and $\mu_n(f \circ T)$ nearly agree. It follows that any weak* limit of μ_n is T-invariant. Such a limit exists by compactness. ■

There is also an infinite–dimensional version of the Brouwer fixed–point theorem, due to Schauder, that insures, by general principles, that $T|P(X)$ has a fixed point.

We note that ergodicity can be recognized as follows:

Proposition 3.2 *The map T is ergodic on (X, μ) if and only if μ is an extreme point of $P(X)^T$.*

Ergodic components of a general invariant measure. In fact every invariant measure can be decomposed unique as a direct integral of ergodic measures, by Choquet theory. (For uniqueness one uses the fact that $P(X)$ is a *Choquet simplex.*)

Absence of invariant measures. When T is invertible, the result above gives an invariant measure for the action of $G = \mathbb{Z}$ on X. There are other group actions that admit *no* invariant measure; for example, the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The existence of an invariant measure follows from *amenability* of G.

Examples of the space of invariant measures.

1. Let $X = (\mathbb{Z}/n)^{\mathbb{Z}}$, $n > 1$, and let $T(x)_i = x_{i+1}$ be the shift map. The probability measures on $\mathbb{Z}/n$ form a simplex, and each of these gives a point in $P(X)^T$. But T also has infinitely many periodic points, and these also give (atomic) invariant measures. Many other invariant measures are possible — they correspond to stationary stochastic processes with values in $\mathbb{Z}/n$.

2. Let $X = S^1 = \mathbb{R}/\mathbb{Z}$ considered as a group, and let $X[n] \cong \mathbb{Z}/n$ denote torsion points of order n, i.e. solutions to $nx = 0$. Let $T(x) = dx$, $d > 1$. Then T sends $X[n]$ into itself; in fact T acts by a permutation when $\gcd(n, d) = 1$, and hence T has infinitely many periodic points. Thus $P(X)^T$ has lots of ergodic invariant measures besides Lebesgue measure.

   In fact the space of expanding covering maps on S^1 of degree d also embeds into the space of extreme points of $P(X)^T$. See §22

3. Let $X = \mathbb{R}^2/\mathbb{Z}^2$ and let $T \in \mathrm{SL}_2(\mathbb{Z})$ define a hyperbolic automorphism of X. Then $T|X[n]$ is a permutation for each n, again yielding lots of ergodic invariant measures aside from Lebesgue measure.

Remark: Hyperbolicity. In all three examples above, one can actually show that the (ergodic) measures coming from periodic cycles are dense in $P(X)^T$. In particular, the space of ergodic measures is not closed. This density is a general feature of *hyperbolic dynamics*, where any orbit can be approximated by a periodic orbit.

Open problem: Times 2 times 3. As we have just seen, the doubling map $T(x) = 2x \bmod 1$ has a huge space of ergodic invariant measures. The same is true for the tripling map, $S(x) = 3x \bmod 1$.

But what happens if we put them together? The following famous open problem is due to Furstenberg:

Conjecture 3.3 *Any ergodic probability measures on $S^1 = \mathbb{R}/\mathbb{Z}$ that is invariant under both $x \mapsto 2x$ and $x \mapsto 3x$ is either Lebesgue measure or a measure with finite support.*

In other words, the pair of commuting transformations $\langle S, T\rangle$ is *almost* uniquely ergodic; it has only one invariant measure with no atoms.

Unique ergodicity. The situation for an irrational rotation is radically different. We say $T : X \to X$ is *uniquely ergodic* if T admits a *unique* invariant probability measure. Note that this is a *topological* notion.

Theorem 3.4 (Weyl, Sierpiński, Bohl) *An irrational rotation T of S^1 is uniquely ergodic.*

Proof. Let $T(z) = \lambda z$ on $S^1 = \{z : |z| = 1\}$, and let $f_k(z) = z^k \in C(S^1)$. Let $\mu \in C(S^1)^*$ be a T–invariant probability measure. Then $\mu(f_0) = 1$, while for $k \neq 0$ we have

$$\mu(f_k) = \mu(f_k \circ T) = \mu(\lambda^k f_k) = \lambda^k \mu(f_k),$$

so $\mu(f_k) = 0$. Since $\langle f_k \rangle$ span a dense subspace of $C(S^1)$, we conclude that $\mu(f) = \int_{S^1} f(z)|dz|/2\pi$ for all f, and hence μ is normalized Lebesgue measure on S^1. ■

Here is a more general argument.

Theorem 3.5 *Let G be a compact topological group. Then $T(x) = gx$ is uniquely ergodic if and only if g generates a dense subgroup of G.*

Proof. Let H be the closure of $\langle g \rangle$. Any g–invariant measure μ is H invariant; so if $H = G$ then μ must be Haar measure and hence T is uniquely ergodic.

On the other hand, if $H \neq G$ then every coset of H supports a T–invariant measure, and hence T is not uniquely ergodic. First suppose $\langle g \rangle$ is dense in G. ■

Rotations of spheres. If G is a nonabelian group like $\mathrm{SO}(n)$, $n > 2$, then T cannot be ergodic since it generates an Abelian subgroup. Similarly a rotation of S^n, $n \geq 2$, is never ergodic. (How large can an orbit closure be on S^3?)

Cocycles and coboundaries. Let us return to the general setting of a compact metric space X equipped with a homeomorphism $T : X \to X$. A function $f \in C(X)$ is a *coboundary* if $f = g - g \circ T$ for some $g \in C(X)$.

Clearly $S_n(g, x) \to 0$ for a coboundary, in fact $|S_n(g, x)| = O(1/n)$. (Note that a constant function cannot be coboundary.)

Theorem 3.6 *The following are equivalent.*

1. *T is uniquely ergodic.*
2. *For all $f \in C(X)$, $(S_n f)(x)$ converges* pointwise *to a constant function.*
3. *For all $f \in C(X)$, $(S_n f)(x)$ converges* uniformly *to a constant function.*
4. *We have $C(X) = \mathbb{R} \oplus \overline{\mathrm{Im}(I - T)}$. That is, every continuous function is the sum of a constant function and a uniform limit of coboundaries.*

Proof. By definition, the space of invariant measures is given by

$$M(X)^T = \mathrm{Im}(T - I)^\perp \cong (C(X)/\overline{\mathrm{Im}(T - I)})^*.$$

This shows that (1) and (4) are equivalent. Approximating f by the sum of a constant and a coboundary, we see that (4) implies (3), which implies (2). Now suppose (2) holds. Then for all $f \in C(X)$, the constant limit $S_n(f) \to \mu(f)$ defines an T–invariant element of $C(X)^*$ and hence a measure on X. For any other invariant measure ξ, the fact that $S_n(f) \to \mu(f)$ pointwise implies, by the dominated convergence theorem, that $S_n(f) \to \mu(f)$ in $L^1(X, \xi)$; and hence

$$\int f \, \xi = \int S_n(f) \, \xi \to \mu(f),$$

which shows $\xi = \mu$. Thus μ is unique and (2) implies (1). ■

Uniform distribution. Let μ be a probability measure on X. In general, one says an orbit $\langle T^n x\rangle$ is *uniformly distributed* (with respect to μ) if the measures probability μ_n defined using δ-masses at the first n points in the orbit converge to μ as $n \to \infty$. This is equivalent to the condition that $S_n(f,x) \to \int f\, mu$.

Using this terminology, we see that T is uniquely ergodic on (X,μ) iff *every* T–orbit is uniformly distributed with respect to μ. Since $C(X)$ is dense in $L^1(X,\mu)$, the ergodic theorem can be formulated as the following weaker statement:

Theorem 3.7 *A homeomorphism T is ergodic with respect to μ if and only if* almost every *point in X is uniformly distributed.*

One can also speak of a sequence of finite sets being uniformly distributed or *equidistributed*; for example, when X is a torus, its torsion points $X[n]$ are uniformly distributed as $n \to \infty$.

Visits to open sets. It is tempting to think that a uniformly distributed set spends that right amount of time, $m(U)/m(X)$, in each open set $U \subset X$. For example, we do have:

Proposition 3.8 *If T is in irrational rotation, and $I \subset S^1$ is an interval, then $S_n(\chi_I, x) \to m(I)$ uniformly for all x. In other words, the amount of time an orbit of T spends in I is proportional to the length of I.*

However, this statement is *false* if we replace I with a suitable open set U! For example, one can take U to be a union of open intervals of length $\epsilon/2^n$ around $T^n(x)$. Then $T^n(x)$ spends all its time in U, even though $m(U) \asymp \epsilon$.

One can prove Proposition 3.8 by approximating χ_I from above and below by continuous functions f and g with $\int |f-g| < \epsilon$. Such an approximation is impossible for χ_U. The difference is that χ_I is *Riemann integrable*, while χ_U is not. In general one can show:

Proposition 3.9 *Let $f : S^1 \to \mathbb{R}$ be Riemann integrable, and let T be an irrational rotation. Then $S_n(f,x) \to \int f$ uniformly in x.*

Distribution of decimal digits. (Arnold.) Write the powers of 2 in base 10:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536\ldots$$

(The last two numbers are well–known to computer scientists of a certain generation, since they are one more than the largest signed and unsigned integers that an be represent in 16 bits.)

How often is the first digit equal to 1? Using uniform distribution of multiplication by 2 on the circle $S^1 = \mathbb{R}/10^{\mathbb{Z}}$, it is easy to see that the answer is about 30%. In this case normalized Haar measure is $m = (\log 10)^{-1} dx/x$. For a complete proof, one must show that $\log 2/\log 10$ is irrational! (For simpler proof, show that for each $n \geq 1$ there is a unique power of 2 that is n digits long in base 10 and begins with 1.)

Other questions of uniform distribution. Weyl's theorem shows that $(n\theta)$ is uniformly distributed whenever θ is irrational. A great many other sequences are known to be uniformly distributed; for example, the sequence (n^α) for $0 < \alpha < 1$. It is suspected that the sequence $(3/2)^n \bmod 1$ and similar sequences are uniformly distributed, but even their density is unknown.

Pisot and Salem. An algebraic integer $\lambda > 1$ is a *Pisot number* if all its Galois conjugates satisfy $|\lambda'| < 1$. (The simplest example is the golden ratio, satisfying $\lambda^2 = \lambda + 1$.) It is a *Salem number* if λ is conjugate to $1/\lambda$ and any other conjugate lies on the unit circle.

It is known that $\lambda^n \bmod 1$ is dense but not equidistributed when λ is a Salem number of degree 4 or more. Even worse, $\lambda^n \to 0 \bmod 1$ when λ is a Pisot number. On the other hand, one can show that λ^n is uniformly distributed mod 1 for almost every λ.

Benford's Law. The same predominance of 1s occurs in many different types of real–world data, and is used in forensic finance to help detect fraud. It has also been observed that the beginning of a table of logarithms is usually much more worn than the end (Simon Newcomb).

Temperatures. We remark that average U.S. temperatures, in Fahrenheit, tend to violate Benford's law.

Aside: Finite sums of coboundaries. Given a function f on S^1 of mean zero, can we always write f as a coboundary rather than as a limit of coboundaries? For simplicity we focus on the case where $f(z) = \sum a_n z^n$ is in $L^2(S^1)$. Then if $f = g - g \circ T$, where $g = \sum b_n z^n$, we find

$$b_n = \frac{a_n}{1 - \lambda^n} \cdot$$

Since λ^n is dense on the circle, the denominators accumulate at zero and in general we cannot have $\sum |b_n|^2 < \infty$. So the answer is no.

However every function $f \in L^2(S^1)$ with mean zero is the sum of 3 boundaries for the full rotation group.

Theorem 3.10 *Any function $f \in L^2(S^1)$ with $\int f = 0$ is the sum of 3 coboundaries for the rotation group: that is, $f = \sum_1^3 g_i - g_i \circ R_i$, with $g_i \in L^2(S^1)$ and $R_i(z) = \lambda_i z$ a rotation.*

Corollary 3.11 *Any rotation-invariant linear functional on $L^2(S^1)$ is a multiple of Lebesgue measure.*

Proof of Theorem. Write $f(z) = \sum a_n z^n$ with $\sum |a_n|^2 < \infty$. We wish to find $\lambda_i \in S^1$ and $g_i = \sum b_n^i z^n \in L^2(S^1)$, $i = 1, 2, 3$, such that

$$a_n = \sum_1^3 b_n^i (1 - \lambda_i^n).$$

Thinking of b_n and λ^n as vectors in $\mathbb{C}^3$, we can choose b_n to be a multiple of λ^n, in which case $|b_n|^2 = |a_n|^2/|1 - \lambda^n|^2$.

Now consider, on the torus $(S^1)^3 \subset \mathbb{C}^3$, the function $F(\lambda) = |1-\lambda|^2$. Near its singularity $\lambda = (1, 1, 1)$ this function behaves like $1/r^2$, which is integrable on $\mathbb{R}^3$. (This is where the 3 in the statement of the Theorem comes from.) Also, since $\lambda \mapsto \lambda^n$ is measure-preserving, we find that

$$\int_{(S^1)^3} F(\lambda)\, d\lambda = \int_{(S^1)^3} F(\lambda^n)\, d\lambda$$

and thus

$$\int_{(S^1)^3} \sum_n |b_n|^2 = \int_{(S^1)^3} \sum_n \frac{|a_n|^2}{|1 - \lambda^n|^2}\, d\lambda = \sum |a_n|^2 \|F\|_1 < \infty.$$

Thus almost every triple of rotations $(\lambda_1, \lambda_2, \lambda_3)$ works. ∎

Minimality and unique ergodicity. A topological dynamical system is *minimal* if every orbit is dense. (Compare *transitive*, meaning there exists a dense orbit.) The following result is clear for group translations, but holds more generally.

Theorem 3.12 *If T is uniquely ergodic, and its invariant measure has full support, then T is minimal.*

Proof. Let f be supported in an arbitrary open set $U \neq \emptyset$, with $\int f\, d\mu = 1$. Then $S_n(x, f) \to 1$ for any $x \in X$. Therefore the orbit of X enters U, so it is dense. ∎

Minimal but not uniquely ergodic. Furstenberg constructed an analytic diffeomorphisms of the torus which is minimal but not uniquely ergodic; see [Me, §II.7]. There are also well–studied examples of interval exchange maps with the same property.

The Hopf argument: Foliations on a torus. We now return to the study of a hyperbolic toral endomorphism $T : X \to X$, $X = \mathbb{R}^2/\mathbb{Z}^2$. For convenience we assume T has eigenvalues $\lambda > 1 > \lambda^{-1} > 0$. The expanding and contracting directions of T determine foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ of the torus by parallel lines. Two points x, y lie in the same leaf of the *stable foliation* $\mathcal{F}^s$ if and only if $d(T^n x, T^n y) \to 0$ as $n \to +\infty$. Thus the leaves of $\mathcal{F}^s$ are those contracted by T. The unstable foliation has a similar characterization.

Since their leaves are contracted and expanded by T, these foliations have no closed leaves. In other words their slopes on $\mathbb{R}^2/\mathbb{Z}^2$ are irrational.

Theorem 3.13 *Let $T \in \mathrm{SL}_2(\mathbb{Z})$ be a hyperbolic automorphism of a torus X. Then T is ergodic.*

Proof. Given $f \in C(X)$, the ergodic theorem implies that the forward and backwards limits

$$F_\pm(x) \lim_{n\to\pm\infty} S_n(f,x)$$

exist for almost every x, and $F_+(x) = F_-(x)$ a.e. If x_1 and x_2 are on the same leaf of the contracting foliation $\mathcal{F}^s$, then $d(T^n x_1, T^n x_2) \to 0$ as $n \to \infty$, so by uniform continuity of f we have $F_+(x_1) = F_+(x_2)$. Thus means F_+ is constant along the leaves of $\mathcal{F}^s$, in the sense that F_+ is constant on almost every leaf.

Similarly, F_- is constant along the leaves of $\mathcal{F}^u$. But the forward and backward averages agree, so $F_+ = F_-$ is constant.

Since continuous functions are dense in L^2, and their ergodic averages are constants, we find that all T-invariant functions are constant, and hence T is ergodic. ■

Aside: Fubini foiled for non–measurable sets. Hopf's proof relies on Fubini's theorem to see that a set of positive measure saturated with respect to $\mathcal{F}^u$ and $\mathcal{F}^s$ is the whole space. However, using the continuum hypothesis it is easy to construct a set $A \subset [0,1]\times[0,1]$ such that A meets horizontal line in sets of measure zero and vertical lines in sets of measure one. Namely, let

$A = \{(x, y) : x < y\}$ with respect to a well-ordering coming from a bijection $I \cong \Omega$ where Ω is the first uncountable ordinal.

The irrational flow. It is easy to see that an irrational flow on a torus is ergodic, indeed uniquely ergodic. To formulate this precisely, let $e \in \mathbb{R}^2$ be a nonzero vector, and define an action of $\mathbb{R}$ on $X = \mathbb{R}^2/\mathbb{Z}^2$ by the flow

$$H^t(x) = x + te.$$

The behavior of this flow is very similar to the behavior of an irrational rotation, and of a translation in a compact Abelian group; namely, it satisfies:

Theorem 3.14 *The translation flow $H^t(x)$ on a torus is either periodic or uniquely ergodic.*

Proof. Use the fact that any H^t–invariant measure is invariant under the closure of $\mathbb{R} \cdot e \subset X$. ■

For a flow, unique ergodicity implies that for all $f \in C(X)$, the average for time $|s| \leq t$, given by

$$S_t(f, x) = \frac{1}{2T} \int_{-t}^{t} f(H^s x)\, ds,$$

satisfies

$$S_t(f, x) \to \int f$$

uniformly in x as $t \to \infty$.

One can also observe that H^t is the suspension of an irrational rotation on S^1.

Foliations. The orbits of H^t give a *foliation* $\mathcal{F}$ of the torus by parallel geodesics for its flat metric. When the slope of e is rational, all leaves of $\mathcal{F}$ are closed; otherwise, they are all infinite and dense. The foliation its is *ergodic* in the sense that any measurable set $A \subset X$, saturated by the leaves of $\mathcal{F}$, has either full measure or measure zero.

Mixing of a toral automorphism. We also use these irrational foliations to give another proof that T is mixing.

The idea of the proof is that for any small box B, the image $T^n(B)$ is just a long rectangle along one of the leaves of the expanding (unstable) foliation.

Since that foliation is ergodic, B becomes equidistributed, so $m(A \cap T^n B) \to mAmB$.

To make this precise let $T \in \mathrm{SL}_2(\mathbb{Z})$ be a hyperbolic automorphism of the torus $X = \mathbb{R}^2/\mathbb{Z}^2$. For simplicity assume that the leading eigenvalue of T is $\lambda > 1$, with unit eigenvector e. Consider the translation flow

$$H^t(x) = x + te.$$

This flow is uniquely ergodic.

Solvability. The key point is that T and H^t together generate a *solvable group*; as automorphisms of X, they satisfy:

$$TH^t = H^{\lambda t}T.$$

Indeed, we have

$$TH^t(x) = T(x + te) = T(x) + \lambda te = H^{\lambda t}T(x).$$

On the level of functions, with $Tf = f \circ T$, the relation just shown implies that the averages of f along flow lines of H^t satisfy

$$S_t T = T S_{\lambda t} \tag{3.1}$$

as operators on $L^2(X)$. Indeed, we have

$$\begin{aligned}(S_t T f)(x) &= S_t(f(Tx)) = \frac{1}{2T}\int_{-t}^{t} f(TH^s x)\,ds \\ &= \frac{1}{2T}\int_{-t}^{t} f(H^{\lambda s}Tx)\,ds = (S_{\lambda t}f)(Tx) = (TS_{\lambda t}f)(x).\end{aligned}$$

Note also that $T^* = T^{-1}$ and $(H^t)^* = H^{-t}$ but $S_t^* = S_t$, since the interval $[-t, t]$ is symmetric.

Theorem 3.15 *T is mixing.*

Proof. Consider $f, g \in C(X)$ with $\int f = \int g = 0$. Since f is continuous, we can choose a small $t > 0$ such that the norm of $f - S_t f$ is also small. Then for all n we have:

$$\langle f, T^n g\rangle \approx \langle S_t f, T^n g\rangle = \langle f, S_t T^n g\rangle = \langle f, T^n S_{\lambda^n t}, g\rangle = \langle T^{-n} f, S_{\lambda^n t} g\rangle.$$

By ergodicity of the flow H^t, as $n \to \infty$ we have $S_{\lambda^n t} g \to 0$ in $L^2(X)$, while $\|T^{-n} f\| = \|f\|$. Thus $\langle f, T^n g\rangle \to 0$, which is mixing. ∎

Anosov diffeomorphisms and flows. The flow H^t is the analogue of the *horocycle flow* on a hyperbolic surface, to be studied in the next section.

One virtue of the Hopf argument is that, if we start with a linear toral automorphism T, and smoothly perturb it among area–preserving maps, its foliations persists and the proof of ergodicity and mixing goes through as before. These smooth *Anosov* maps have been axiomatized as *hyperbolic dynamical systems* and go well beyond the homogeneous examples we will study; for example, they include geodesic flows in variable negative curvature.

Staying with surfaces, the Hopf argument also shows that *pseudo–Anosov maps* ψ on surfaces of genus $g > 1$ are ergodic. An interesting and trickier generalization from the torus to higher genus shows that the expanding (or contracting) foliation $\mathcal{F}$ of ψ is also *uniquely ergodic*; all its leaves are uniformly distributed. In particular, this foliation is *minimal.*

Density of periodic measures. It is a general fact that, in suitably hyperbolic dynamical systems, the ergodic measures with finite support are dense in $P(X)^T$; see [Sig].

4 Geometry of the hyperbolic plane

In this section and the next we discuss the geometry of surfaces of constant negative curvature, in preparation for the study of their dynamics.

Geometric surfaces. The uniformization theorem states that every simply–connected Riemann surface X is isomorphic to $\mathbb{C}$, $\widehat{\mathbb{C}}$ or $\mathbb{H}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{H} = \{z : \operatorname{Im}(z) > 0\}$. The same theorem holds for (most) complex orbifolds of dimension one. (We must exclude those with no universal cover, which are genus 0 and signature (n) or (n, m) with $1 < n < m$.)

Each of these spaces carries a conformal metric of constant curvature, which we can normalize to be 1, 0 and -1. These metrics are given by:

$$\frac{2|dz|}{1+|z|^2} \text{ on } \widehat{\mathbb{C}}, \;\; |dz| \text{ on } \mathbb{C}, \text{ and } \;\frac{2\,|dz|}{1-|z|^2} \text{ on } \Delta \cong \mathbb{H}.$$

The last metric is canonical, but the other two are not, i.e. the conformal automorphism groups of $\widehat{\mathbb{C}}$ and $\mathbb{C}$ are larger than their isometry groups. Nevertheless:

> *Any Riemann surface X inherits a conformal metric of constant curvature from its universal cover.*

We will now consider these covering spaces and their quotient surfaces in turn.

Spherical geometry. The Riemann sphere $\widehat{\mathbb{C}}$ and the unit sphere $S^2 \subset \mathbb{R}^3$ can be naturally conformally identified by stereographic projection, in such way that 0 and ∞ are the south and north poles of S^2, and $S^1 = \{z : |z| = 1\}$ is the equator. The group

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

acts on $\widehat{\mathbb{C}} \cong \mathbb{CP}^1$ by Möbius transformations. The action is not quite faithful, but it is often convenient to take the action as it comes rather than passing to $\mathrm{PSL}_2(\mathbb{C})$, which acts faithfully.

If we think of $\widehat{\mathbb{C}}$ as $\mathbb{PC}^2$, then the basic Hermitian metric

$$\langle z, w \rangle = z_1 \overline{w}_1 + z_2 \overline{w}_2$$

gives rise to the spherical metric. Thus its isometry group corresponds to the subgroup preserving this form, viz.

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}.$$

Using a and b as coordinates, we see $\mathrm{SU}(2) \cong S^3 \subset \mathbb{C}^2$; in particular, SU(2) is simply–connected.

Alternatively, since S^2 is simply the unit sphere in $\mathbb{R}^3$ with the Euclidean metric, its group of orientation preserving isometries is given by

$$\mathrm{SO}(3) = \{A \in \mathrm{SL}_3(\mathbb{R}) : A^t I A = I\},$$

where I is the identity matrix.

Geodesics. The geodesics on the sphere are *great circles.*

For example, any line of longitude C is a geodesic. To see this, observe that C is that sliding along latitudes provides a distance–decreasing retraction to C, so the shortest path between nearby points p and q lies on C. In other words, C is the fastest path to the north pole.

For another the proof, observe that any two nearby points p, q lie on a great circle C, and any great circle is the fixed–point set of an isometric involution on S^2, Since the unique geodesic from p to q must also be fixed, it lies on C.

A typical geodesic, parameterized by arclength, is the equator:

$$\gamma(s) = (\sin s, \cos s, 0),$$

of total length 2π. If we replace S^2 by $\mathbb{RP}^2$, then these circles give an honest instance of planar geometry: there is a unique line through any 2 points, two lines meet in a single point, etc. An oriented geodesic circle C_p is uniquely determined by its 'center' p, via

$$C_p = p^\perp \cap S^2.$$

Other circles. The other circles on S^2 have a natural geometric meaning: they are *parallels* to geodesics, but not geodesics themselves. This is a manifestation of curvature.

Triangles. In this geometry the interior angles of a triangle T satisfy

$$\alpha + \beta + \gamma > \pi;$$

in fact, the angle defect satisfies

$$\alpha + \beta + \gamma - \pi = \mathrm{area}(T).$$

(For example, the area of a quadrant — an equilateral right triangle – is $4\pi/8 = \pi/2 = 3(\pi/2) - \pi$.)

Meaning of the inner product. For any 2 points $p, q \in S^2$, it is clear that

$$\langle p, q\rangle = \cos d(p, q).$$

where distance is measured in the spherical metric. Similarly, the dihedral angle between two great circles satisfies

$$\langle p, q\rangle = \cos\theta(C_p, C_q)$$

For example, if C_p and C_q are the same circle with opposite orientations, then $q = -p$, $\theta(C_p, C_q) = \pi$ and $\langle p, q\rangle = -1$.

Constructing a triangle with given angles. It is easy to convince oneself, by a continuity argument, that for any angles $0 < \alpha, \beta, \gamma < \pi$ whose sum exceeds π, there exists a spherical triangle T with these interior angles.

Here is a rigorous proof. We see the construct oriented great circles C_a, C_b, C_c that define the sides of T with a cyclic orientation. With respect to the basis (a, b, c) for $\mathbb{R}^3$, the inner product would have to become

$$Q(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & -\cos\alpha & -\cos\beta \\ -\cos\alpha & 1 & -\cos\gamma \\ -\cos\beta & -\cos\gamma & 1 \end{pmatrix} \cdot \tag{4.1}$$

We note that

$$\det Q(\alpha, \beta, \gamma) = 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma - 2\cos(\alpha)\cos(\beta)\cos(\gamma). \tag{4.2}$$

As an exercise, one can then check that $\det Q(\alpha, \beta, \gamma) > 0$ iff $\alpha + \beta + \gamma > \pi$.

Thus in the case of excess angle, Q has signature $(1, 2)$ or $(3, 0)$. But the first two vectors clearly span a subspace of signature $(2, 0)$, so the signature is $(3, 0)$. Thus there is a change of basis which sends Q to the standard Euclidean metric, and sends a, b, c to three points on S^2 defining the required circles.

Exercise. Give an explicit $(2, 3, 5)$ triangle on S^2, i.e. a triangle with inner angles $(\alpha, \beta, \gamma) = (\pi/2, \pi/3, \pi/5)$.

Solution. Let us take $a = (1, 0, 0)$ and $b = (0, 1, 0)$ to be the centers of the first two circles, so that $\theta(C_a, C_b) = \pi/2$. Then the final circle C_c must satisfy $\langle c, a\rangle = -\cos\beta$ and $\langle c, b\rangle = -\cos\gamma$; thus

$$c = (-\cos\pi/3, -\cos\pi/5, \sqrt{1 - \cos^2\pi/3 - \cos^2\pi/5}).$$

To find the vertices, we observe that C_a and C_b meet along the intersection of S^2 with the line through the cross-product $a \times b$. Thus the vertices come from the 3 cross products $a \times b$, $b \times c$ and $c \times a$. Their projections to the (x, y) plane are approximately $(0, 0)$, $(0.5257, 0)$ and $(0, 0.3568)$.

Orbifolds. The nontrivial discrete subgroups of SO(3) all have torsion, and hence the corresponding quotients $X = \Gamma\backslash S^2$ are all orbifolds. They are easily enumerated; aside from the cyclic and dihedral groups, which yield the (n, n) and $(2, 2, n)$ orbifolds, the 5 Platonic solids (3 up to duality) give the orbifolds $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. These are simply the compact orientable 2-dimensional orbifolds with

$$\chi(X) = 2 - 2g - n + \sum_{i=1}^{n} \frac{1}{p_i} > 0,$$

with the bad orbifolds (p) and (p, q), $1 < p < q$, excluded.

Hyperbolic geometry. We now turn to constant curvature -1, which will be our main concern. There are at least 5 useful models for the hyperbolic plane $\mathbb{H}$.

I. The first is the upper halfplane $\mathbb{H} \subset \mathbb{C}$, with the metric

$$\rho = \frac{|dz|}{y},$$

where $z = x+iy$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations and gives all of its conformal symmetries. These symmetries are *all* isometries; thus any Riemann surface covered by $\mathbb{H}$ has a *canonical* hyperbolic metric. The geodesics are circles perpendicular to the real axis. A typical geodesic parameterized by arclength is the imaginary axis:

$$\gamma(s) = i \exp(s).$$

Another is the unit circle:

$$\gamma(s) = \tanh s + i \operatorname{sech} s.$$

(Recall that $\tanh^2(s) + \operatorname{sech}^2(s) = 1$.) Note also that

$$\begin{aligned} \|\gamma'(s)\|^2_\rho &= \frac{|\gamma'(s)|^2}{(\operatorname{Im}\gamma(s))^2} = \cosh^2(s)(\operatorname{sech}^4(s) + \operatorname{sech}^2(s)\tanh^2(s)) \\ &= \operatorname{sech}^2(s) + \tanh^2(s) = 1. \end{aligned}$$

In particular, the 'parallel' at distance s from the imaginary axis is the line through the origin and $\gamma(s)$, hence its angle α with the real axis satisfies

$$\cos\alpha = \tanh(s)\cdot$$

As another consequence, using the fact that $\rho = |dz|/y$, we see that passing to the parallel at distance s from a geodesic expands distances by a factor of $\cosh s$.

We remark that $\mathbb{H}$ can be interpreted as the space of lattices $L \subset \mathbb{C}$ with a chosen, oriented basis, modulo the action of $\mathbb{C}^*$, via

$$\tau \in \mathbb{H} \iff \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}.$$

II. The second model is the *unit disk*:

$$\mathbb{H} \cong \Delta = \{z \,:\, |z| < 1\} \subset \mathbb{C},$$

with the metric

$$\frac{2|dz|}{1-|z|^2}.$$

This is also called the *Poincaré model* for hyperbolic space.

Any Möbius transformation $g : \mathbb{H} \to \Delta$, such as $g(z) = (z-i)/(z+i)$, gives an isometry between these two models. Thus the geodesics remain circles perpendicular to the boundary S^1.

The symmetry group becomes

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \,:\, |a|^2 - |b|^2 = 1 \right\}.$$

Indeed, as a subset of $\widehat{\mathbb{C}} \cong \mathbb{PC}^2$, the unit disk is the space of *positive lines* for the Hermitian metric

$$\langle z, w \rangle = z_1 \overline{w}_1 - z_2 \overline{w}_2$$

on $\mathbb{C}^2$, where the slope $z = z_2/z_1$ of a complex line gives the complex coordinate on $\mathbb{PC}^2$.

From the perspective of Hodge theory, this explains why $\mathbb{H}$ is the space of marked complex tori. If Σ_1 is a smooth oriented surface of genus 1, then a complex structure is the same as a splitting

$$H^1(\Sigma_1, \mathbb{C}) = H^{1,0} \oplus H^{0,1}.$$

The intersection form gives the first space a Hermitian metric of signature $(1,1)$, and the line $H^{1,0}$ must be positive since its classes are represented by holomorphic 1-forms.

We remark that the unit disk is also a model for $\mathbb{CH}^1$. It can be argued that the 'correct' invariant metric on $\mathbb{CH}^n \cong B(0,1) \subset \mathbb{C}^n$ is given by

$$\rho = \frac{|dz|}{1-|z|^2}.$$

This metric has constant curvature -1 on the totally real locus $\mathbb{RH}^n \subset \mathbb{CH}^n$. It then has curvature -4 on complex tangent planes; in particular, $\mathbb{CH}^1$ has curvature -4.

We also note that any simply–connected region $U \subset \mathbb{C}$, other than $\mathbb{C}$ itself, carries a natural hyperbolic metric ρ_U provided by the Riemann mapping to U. Using the Schwarz lemma and the Koebe 1/4-theorem, one can show that ρ_U is comparable to the $1/d$–metric; that is,

$$\frac{1}{2d(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{d(z, \partial U)},$$

where $d(\cdot, \cdot)$ is Euclidean distance. Knowing the hyperbolic metric up to a bounded factor is enough for many purposes, e.g. it allows one to define quasigeodesics, which are always a bounded distance from actual geodesics.

III. The third model is the *Minkowski model.* We give $\mathbb{R}^3$ the metric of signature $(2, 1)$ with quadratic form

$$(x, y, t)^2 = x^2 + y^2 - t^2.$$

A model of $\mathbb{H}$ is provided by the upper sheet of the 'sphere of radius i', i.e. the locus

$$\mathcal{H} = \{p \,:\, p \cdot p = -1 \quad \text{and} \quad t > 0\} = \{(x, y, t) \,:\, x^2 + y^2 = t^2 - 1, t > 0\}.$$

Now the tangent space to $\mathcal{H}$ at p is naturally identified with $p^\perp$, which is a positive–definite space; thus $\mathcal{H}$ carries a natural metric, which is invariant under $\mathrm{SO}(2, 1)$.

A typical geodesic in this model looks like a geodesic on the sphere, with hyperbolic functions replacing the spherical ones:

$$\gamma(s) = (\sinh(s), 0, \cosh(s))$$

is parameterized by arclength, since

$$\|\gamma'(s)\|^2 = \cosh^2(s) - \sinh^2(s) = 1.$$

Every oriented geodesic corresponds to a unique point on the 1-sheeted hyperboloid

$$\mathcal{G} = \{p \,:\, p \cdot p = 1\},$$

by

$$\gamma_p = p^\perp \cap \mathcal{H}.$$

III'. *Polar coordinates.* If we fix a geodesic ray $R = [p, \infty)$ in $\mathbb{H}$, then any other point q can be describe by the coordinates $s = d(p, q)$ and $\theta =$ the angle between $[p, q]$ and R.

In these coordinates, the hyperbolic metric is given by

$$\rho^2 = ds^2 + \sinh^2(s)\, d\theta^2. \tag{4.3}$$

Note that to first order we have $\rho^2 = ds^2 + s^2 d\theta^2$ as in Euclidean space.

We can thus easily compute the length of a sphere and the area of a ball of radius R:

$$\begin{aligned} L(R) &= 2\pi \sinh(R), \quad \text{and} \\ A(R) &= 2\pi(\cosh(R) - 1). \end{aligned}$$

(Equivalently, $A(R) = 4\pi \sinh^2(R/2)$).

Note that, as is always the case for an embedded Riemannian ball, we have $dA/dR = L$. We also observe that as $R \to \infty$, we have

$$A(R) \sim L(R) \sim 2\pi \exp(R).$$

To check equation (4.3), it suffices to show $L(R) = \sinh(R)$. But this is easy: in the Minkowski model, the restriction of the metric $dx^2 + dy^2 - dt^2$ to any horizontal slice $t = t_0$ agrees with the Euclidean metric in (x, y) coordinates. We have seen that for $p = (0, 0, 1)$ as our basepoint in $\mathcal{H}$, a point at distance s from p is given by $(x, y, t) = (\sinh(s), 0, \cosh(s))$. The corresponding circle lies in the plane $t = \cosh(s)$ and agrees with a Euclidean circle of radius $\sinh(s)$, so we are done.

We can also use the Gauss–Bonnet formula to calculate the geodesic curvature $k(R)$ of a circle of radius R: we have

$$2\pi\chi(\Omega) = \int_\Omega K\, dA + \int_{\partial\Omega} k\, ds,$$

which gives, for $\Omega = B(p, R)$ and $K = -1$, $2\pi = -A(R) + L(R)k(R)$, and hence

$$1/k(R) = \tanh(R).$$

The conversion between Minkowski coordinates and polar coordinates is given simply by the equation

$$(x, y, t) = (\sinh(s) \cos\theta, \sinh(s) \sin\theta, \cosh(s)).$$

The observations above highlight the following important isoperimetric inequality:

Proposition 4.1 *For any bounded region Ω in $\mathbb{H}$, we have* $\operatorname{area}(\Omega) \leq \operatorname{length}(\partial\Omega)$.

Proof. The proof is conveniently given in the upper halfspace model: if we set $\omega = dx/y$, then $|\omega|/ = \rho$, while $|d\omega| = |dx\,dy/y^2| = \rho^2$. Thus by Stokes' theorem,

$$\operatorname{area}(\Omega) = \int_\Omega d\omega = \int_{\partial\Omega} \omega \leq \int_{\partial\Omega} |\omega| = \operatorname{length}(\partial\Omega).$$

■

IV. The fourth model is the *Klein model* $B \subset \mathbb{RP}^2$. It is obtained by projectivizing the Minkowski model and observing that $\mathcal{H}$ becomes the unit ball in the coordinates $[x, y, 1]$. In this model the group $\mathrm{SO}(2,1) \subset \mathrm{GL}_3(\mathbb{R})$ acts by isometries. The geodesics are *straight lines* in B and the space of *unoriented* geodesics can be identified with the Möbius band

$$\mathcal{G}/(\pm 1) \cong \mathbb{RP}^2 - \overline{B}.$$

Geometrically, through any point p outside $\overline{B}$ we have 2 lines tangent to ∂B, say L_1 and L_2, meeting it at q_1 and q_2; then the line $q_1 q_2$ is the geodesic γ_p.

V. The fifth model is the *homogeneous space* model, $\mathbb{H} = G/K$. Here $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R})$. The identification with $\mathbb{H}$ is transparent in the upper space picture, where $\mathrm{SO}_2(\mathbb{R})$ is the stabilizer of $z = i$.

One can think of $\mathbb{H}$ as the space ellipses of area 1 centered at the origin in $\mathbb{R}^2$. Equivalently, $\mathbb{H}$ is the space of inner products on $\mathbb{R}^2$ up to scale, or the space of complex structures on $\mathbb{R}^2$ agreeing with the standard orientation.

V.' Let us elaborate the identification between G/K and the space of unimodular, positive–define quadratic forms on $\mathbb{R}^2$.

A quadratic form on $\mathbb{R}^n$ is specified by a symmetric matrix Q, with $Q(v) = v^t Q v$. The Euclidean inner product corresponds to the matrix $Q = I$. We say Q is *unimodular* when $\det(Q)^2 = 1$.

An automorphism g of $\mathbb{R}^n$ gives a new quadratic form Q', defined by $Q'(v) = Q(gv)$, whose matrix is given by

$$Q' = g^t Q g.$$

The group $G = \mathrm{SL}_n(\mathbb{R})$ acts transitively on the positive–definite, unimodular quadratic forms, and the stabilizer of $Q = I$ is the compact group

$K = \mathrm{SO}(n)$. Thus we can identify the space of all such forms with G/K, provided G acts on the left on the space of forms. To obtain a left action, we must set

$$(g \cdot Q)(v) = Q(g^{-1}v),$$

and thus on the level of matrices we have

$$g \cdot Q = (g^{-1})^t Q g^{-1}. \tag{4.4}$$

Going between the models. A direct relation between the Minkowski model and G can be obtained by considering the adjoint action of G on its Lie algebra $sl_2\mathbb{R}$, in which the trace form $\langle a, b\rangle = -\operatorname{tr}(a \cdot b)$ of signature $(2, 1)$ is preserved.

To convert from the Klein model to the Poincaré disk model Δ, think of Δ as the southern hemisphere of S^2 (via stereographic projection). Then S^1_∞ is the equator. Under orthogonal projection to the equatorial plane, the geodesics in Δ become the straight lines of the Klein model. In other words, the Klein model is just the southern hemisphere as seen from (an infinite distance) above.

With this perspective, it is clear that hyperbolic balls near S^1_∞ in the Klein model are ellipses with major axes nearly parallel to S^1_∞, becoming more eccentric as we near the circle.

Remark: Hyperbolic space $\mathbb{H}^3$. We remark that a similar discussion holds for the hyperbolic space $\mathbb{H}^n$ of dimension n, with $\mathbb{H}$ and Δ replaced by the upper halfspace and unit ball in $\mathbb{R}^n$, and with the Minkowski geometry associated to with $\mathbb{R}^{n,1}$ and $\mathrm{SO}(n, 1)$.

In particular, $\partial\mathbb{H}^3$ can be identified with $\widehat{\mathbb{C}} \cong S^2_\infty$, with

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{SL}_2(\mathbb{C})/(\pm I).$$

The study of hyperbolic 3–manifolds and orbifolds is thus identified with the study of *Kleinian groups*, i.e. discrete subgroups of Möbius transformations acting on $\widehat{\mathbb{C}}$.

Hyperbolic trigonometry. We now turn to the geometric meaning of the inner product in the Minkowski model. This model is often the most efficient one for actual computations.

If $p, q \in \mathcal{H}$, so $p^2 = q^2 = -1$, then we have

$$\langle p, q\rangle = -\cosh(d(p, q)).$$

To check this, just note that

$$\langle \gamma(0), \gamma(s) \rangle = -\cosh(s).$$

On the other hand, if $p, q \in \mathcal{G}$, then the angle θ between the oriented lines they determine satisfies

$$\cos(\theta(\gamma_p, \gamma_q)) = \langle p, q \rangle.$$

To see this, consider the case where $p, q \in \mathcal{G}$ lie on the circle $t = 0$, so $x^2 + y^2 = 1$; then $\langle p, q \rangle$ is just the usual inner product in the (x, y) plane.

What if the lines don't meet? Then we have

$$\langle p, q \rangle = \pm \cosh d(\gamma_p, \gamma_q),$$

where the sign is determined by the orientations. Finally for $p \in \mathcal{H}$ and $q \in \mathcal{G}$, we have

$$\langle p, q \rangle = \pm \sinh d(p, \gamma_q).$$

This can be checked by letting $p = (\sinh s, 0, \cosh s)$ and $q = (1, 0, 0)$.

Geodesics. There is an elegant connection between geodesics in the Minkowski model and the Poinca˚e model.

Proposition 4.2 *Let $p = (x, y, t)$ satisfy $p^2 = x^2 + y^2 - t^2 = 1$. Then the corresponding geodesic $\gamma = p^\perp$ in the Poincaré model is an arc of the circle C of radius $1/t$ centered at $z = (x + iy)/t$.*

The proof is indicated in Figure 5; note that $[x : y : t] = [x/t : y/t : 1]$ in $\mathbb{RP}^2$. Forming a right triangle with vertices 0, $(x + iy)/t$ and one endpoint of γ, we see the radius r of C satisfies

$$1 + r^2 = \frac{x^2 + y^2}{t^2} = \frac{1 + t^2}{t^2} = 1 + \frac{1}{t^2}.$$

Classification of isometries of $\mathbb{H}$. There are three types of isometries of $\mathbb{H}$, which are conveniently distinguished in terms of the *translation length*:

$$T(g) = \inf_{x \in \mathbb{H}} d(x, gx).$$

The right hand side is a convex function of x. Provided $g \neq \mathrm{id}$. There are 3 cases:

- *Elliptic:* $T(g) = 0$, realized. In this case g is rotation about its unique fixed point $p \in \mathbb{H}$.

- *Parabolic:* $T(g) = 0$, not realized. In this case g has a unique fixed point $p \in S^1_\infty$, which arises as the limit of nay sequence of points x_n with $d(x_n, gx_n) \to 0$. It has no fixed points in $\mathbb{H}$.

- *Hyperbolic:* $T(g) > 0$. In this case g has two distinct fixed points $p, q \in S^1_\infty$, and the translation distance is realized along the geodesic γ joining them.

Every elliptic transformation is conjugate to $g(z) = \exp(2\pi i\theta)z$ in the disk model Δ, with $z = 0$ fixed. The parabolic transformations form two conjugacy classes, represented by $g(z) = z \pm 1$ acting on $\mathbb{H}$. Finally any hyperbolic transformation is conjugate to $g(z) = \lambda z$ acting on $\mathbb{H}$, with $\lambda > 1$.

The trace of a matrix is a conjugacy invariant, and in fact g is *elliptic* when $|\operatorname{Tr}(g)| < 2$, *parabolic* when $|\operatorname{Tr}(g)| = 2$ and *hyperbolic* when $|\operatorname{Tr}(g)| > 2$. The trace uniquely determines each non–parabolic conjugacy class, and carries the same information as the invariants θ and λ.

Classification of homogeneous spaces for *G*. Next we briefly examine the other homogeneous spaces for $G = \mathrm{PSL}_2(\mathbb{R})$, aside from $\mathbb{H}$. These will all play a role in the sequel.

As above, we let

$$G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/(\pm I) = \mathrm{Isom}^+(\mathbb{H}).$$

Within G we have the following important subgroups:

$$\begin{aligned}
K &= \left\{ k_t = \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix} \,:\, t \in \mathbb{R} \right\}, \quad \text{and} \\
A &= \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \,:\, t \in \mathbb{R} \right\}, \quad \text{and} \\
N &= \left\{ n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,:\, t \in \mathbb{R} \right\}.
\end{aligned}$$

It is understood that the matrices on the right are representatives for equivalence classes in G. The following result is well–known and readily checked:

Theorem 4.3 *Every connected subgroup of G is conjugate to $\{1\}$, K, A, N, or AN.*

As we will see below, each of these subgroups H is associated to a geometric object, so the associated homogeneous spaces G/H all have geometric interpretations, which we will elaborate below. In brief, we will find:

Corollary 4.4 *Up to coverings, G has 5 homogeneous spaces, namely $T_1\mathbb{H}$, $\mathbb{H}$, S^1_∞, $\mathcal{G} = (S^1_\infty \times S^1_\infty - \mathrm{diag})$ and $\mathcal{L} = (\mathbb{R}^2 - \{0\})/(\pm I)$.*

1. *K*: The hyperbolic plane. We have already seen that K is the stabilizer of $z = i$ in $\mathbb{H}$, giving a natural identification

$$\mathbb{H} = G/K.$$

2. *AN*: The circle at infinity S^1_∞. The space $S^1_\infty = G/AN$ is the space of *asymptotic* equivalence classes of geodesics on $\mathbb{H}$.

In the Minkowski model, S^1_∞ is the space of lines in the light cone.

3. *A*: The space of geodesics $\mathcal{G}$. The space $\mathcal{G}$ of all oriented geodesics in $\mathbb{H}$ can be described as

$$\mathcal{G} = (S^1_\infty \times S^1_\infty - \mathrm{diag}) = \widetilde{\mathbb{RP}^2 - \overline{\Delta}} = G/A.$$

The Möbius band $\mathbb{RP}^2 - \Delta$ is a natural model for the space of *unoriented* geodesics in the Klein model. Namely two points on S^1_∞ determine a pair of tangent lines meeting in $\mathbb{RP}^2$.

Identifying S^1_∞ with $\mathbb{R} \cup \infty$, the invariant measure on $\mathcal{G}$ is given by

$$\frac{dx\,dy}{|x-y|^2}.$$

In the Minkowski model, $\mathcal{G}$ can be described as the one-sheeted hyperboloid $x^2 + y^2 = t^2 + 1$ in the Minkowski model, consisting of the vectors with $\ell(v) = 1$. Under projectivization it gives the Möbius band. The geodesic corresponding to v is the intersection of the plane $v^\perp$ with $\mathbb{H}$.

Upon projectivization to $\mathbb{RP}^2$, the space of geodesics $\mathcal{G}$ becomes the double cover of the Möbius band $\mathcal{G}' = \mathbb{RP}^2 - \overline{\mathbb{H}}$. The points $p \in \mathcal{G}'$ correspond to unoriented geodesics $\gamma = [a, b] \subset \overline{\mathbb{H}}$, such that the lines ap and bp are tangent to S^1_∞.

The Minkowski form induces a Lorentz metric on $\mathcal{G}$, and the null geodesics through p are the extensions of the lines ap and bp. A pair of points $p, q \in \mathcal{G}$ are causally related (i.e. connected by a timelike path) iff the corresponding geodesics are disjoint.

4. $\boldsymbol{N}$: The light cone $\boldsymbol{\mathcal{L}}$. If we regard N as a subgroup of $\mathrm{SL}_2(\mathbb{R})$, then it is simply the stabilizer of the vector $(1,0) \in \mathbb{R}^2$. Thus

$$\mathrm{SL}_2(\mathbb{R})/N \cong \mathbb{R}^2 - \{(0,0\}.$$

In other words, N corresponds to the usual linear action of $\mathrm{SL}_2(\mathbb{R})$.

To obtain homogeneous space for $G = \mathrm{PSL}_2(\mathbb{R})$, we must take the quotient:

$$\mathcal{L} = (\mathbb{R}^2 - \{(0,0)\}/(\pm I) \cong G/N.$$

This space can be naturally identified with the *future light cone* in Minkowski space, i.e. the vectors $p = (x, y, t)$ with $p^2 = 0$ and $t > 0$.

Horocycles. Here is a more geometric interpretation. A *horocycle* $H \subset \mathbb{H}$ is, in the Poincaré model, a circle tangent to $S^1_\infty = \partial\mathbb{H}$.

Geometrically, a horocycle through p can be defined in any space by taking the limit of a sequence of spheres $S(c_n, r_n)$, where $c_n \to \infty$ and $r_n = d(p, c_n)$. In spherical geometry, there is no way for the centers to escape; and in Euclidean geometry, the horocycles are just straight lines. But in hyperbolic geometry, the horocycles have geodesic curvature $+1$. They interpolate between *spheres* and *parallels of geodesics.*

The (unoriented) horocycles correspond bijectively to points in the light cone: to each $q \in \mathcal{L}$ we associate the locus

$$H_q = \{p \in \mathcal{H} \, : \, \langle p, q \rangle = -1\}.$$

Recall that for a center $c \in \mathcal{H}$ we have $\langle c, c \rangle = -1$ and the sphere of radius r corresponds to the locus

$$S(c, r) = \{p \in \mathcal{H} \, : \, \langle p, c \rangle = -\cosh(r)\}.$$

Now if $c_n \to \infty$ its limiting direction lies in the light cone, and if $r_n = \langle p_0, c_n \rangle$, $p_0 \in \mathcal{H}$, then

$$\frac{c_n}{\cosh r_n} \to q \in \mathcal{L}, \quad \text{and} \quad S(c_n, r_n) \to H_q.$$

One can also think of $\mathcal{L}$ as the bundle of nonzero vectors over S^1_∞, up to sign. The horocycle H_v corresponding to a vector v is the set of points in $\mathbb{H}$

from which v has visual measure 1. The direction of v gives the orientation of H_v.

The restriction of the Minkowski form gives a degenerate metric on the space of horocycles $\mathcal{L} = \mathbb{R}^2 - \{0\}$. Given $v \in \mathbb{R}^2$, the length of a vector w at v is $|v \wedge w|$. Since the area form on $\mathbb{R}^2$ is preserved, so is this metric.

Geometrically, if we have a family of horocycles H_t with centers $c_t \in S^1_\infty$, then the rate of motion dH_t/dt is equal to the visual length of the vector dc_t/dt as seen from any point on H_t. (Note that the visual length is the same from all points.)

5. *G* itself: The unit tangent bundle. Finally we note that G acts simply transitively on the unit tangle bundle $\mathrm{T}_1\mathbb{H}$ by $g \cdot v = Dg(v)$. Choosing a base vector, such as the vertical vector at $z = i$, yields an isomorphism

$$G = \mathrm{T}_1\mathbb{H}.$$

Note that it is the *left* action of G that is compatible with the fibration $G \to G/K$, which is itself a model for the fibration $\mathrm{T}_1(\mathbb{H}) \to \mathbb{H}$.

Invariant measures on $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{G}$. Every homogeneous space for G carries a G–invariant measure that is unique up to scale. Let us compute these measures explicitly.

Theorem 4.5 *The natural* $\mathrm{SO}(2,1)$*-invariant measure on the light cone is given by* $dx\,dy/t$.

Proof. Let $q(x, y, t) = x^2 + y^2 - t^2$. Then q is $\mathrm{SO}(2,1)$ invariant, the light-cone $\mathcal{L}$ is a level set of q, and $dq \neq 0$ along $\mathcal{L}$. Therefore an invariant volume form ω on $\mathcal{L}$ is uniquely determined by the requirement that

$$\omega \wedge dq = dV = dx\,dy\,dt$$

(since the latter is also $\mathrm{SO}(2,1)$-invariant). But clearly

$$\frac{dx\,dy}{t} \wedge dq = \frac{dx\,dy}{t} \wedge (2x\,dx + 2y\,dy - 2t\,dt) = -2\,dx\,dy\,dt,$$

so ω is proportional to $dx\,dy/t$. ∎

This vector field is clearly $\mathrm{SO}(2,1)$-invariant, and by duality it determines the invariant measure on $\mathcal{H}$. Since the vector field blows up like $1/t$ near the origin, so does the invariant volume form on $\mathcal{H}$.

Yet another point of view: for any level set $L = \{v : q(v) = r\}$, consider the band $L_\epsilon = \{v : |q(v) - r| < \epsilon\}$. Then $dV|L_\epsilon$ is $\mathrm{SO}(2,1)$ invariant, and as $\epsilon \to 0$, it converges (after rescaling) to dV/dq. This yields:

Theorem 4.6 *The* $\mathrm{SO}(2,1)$*-invariant measures on* $\mathcal{H}$*,* $\mathcal{L}$ *and* $\mathcal{G}$ *are given by*

$$\omega = \frac{dV}{dq} = \frac{dx\,dy}{\sqrt{x^2+y^2-r}},$$

where $r = -1, 0, 1$ *respectively.*

Corollary 4.7 *The invariant measure of the Euclidean ball* $B(0,R)$ *intersected with* $\mathcal{H}$*,* $\mathcal{L}$ *or* $\mathcal{G}$ *is asymptotic to* CR *for* $R \gg 0$*.*

Proof. The measure of a ball $B(0,R)$ in the plane with respect to the measure $dx\,dy/\sqrt{x^2+y^2}$ grows like R. ∎

Remark. By the same reasoning, the invariant measure on $\mathcal{H}$ for $\mathrm{SO}(n,1)$ is $dx_1\,dx_2\ldots dx_n/t$, and the measure of $B(0,R)\cap\mathcal{H}$, $B(0,R)\cap\mathbb{H}^n$ and $B(0,r)\cap\mathcal{G}$ all grow like R^{n-1}. (For $n=1$ they grow like $\log R$.)

These estimates are useful for Diophantine counting problems related to quadratic forms.

Appendix: Euclidean space. In this addendum we explain how Euclidean space interpolates between spherical and hyperbolic geometry.

Imagine we are standing at the north pole N of a very large sphere. Then spherical geometry is nearly the same as flat geometry. To keep the north pole in sight in $V = \mathbb{R}^3$, we can rescale so that $N = (0,0,1)$. Then our large sphere becomes the large, flat pancake determined by the equation

$$q_R(x,y,z) = (x^2+y^2)/R^2 + z^2 = 1,$$

with $R \gg 0$ the radius of the pancake. We would like to describe the limit G of the group $\mathrm{SO}(q_r)$ as $r \to \infty$. In the limit for the form $q(x,y,z) = z^2$ will be preserved. The 'sphere' $z^2 = 1$ has two components; let us stick to group elements which preserve each component, and hence preserve the z–coordinate.

Now in the dual space V^* we have a dual quadratic form

$$q_R^*(x,y,z) = R^2(x^2+y^2)+z^2.$$

Rescaling, we say see for any $A \in G = \lim_{R\to\infty} SO(q_R)$, A preserves $q(x,y,z) = z^2$ and A^t preserves $q^*(x,y,z) = x^2+y^2$. The set of all such matrices, with our convention that z itself is fixed in V, is given by

$$G = \left\{ \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix} \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}(2) \right\} \subset \mathrm{GL}(V).$$

On the plane $z=1$, any g as above acts by $(x,y,1) \mapsto (x',y',1)$, where

$$(x',y') = g(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}.$$

Such g are exactly the orientation–preserving isometries of the Euclidean plane.

Alternatively, and more directly, the limiting g preserve (i) the limiting form $q = z^2$, and (ii) its rescaled limit $q' = x^2+y^2$ *on* the plane annihilated by q.

Appendix: Special relativity. The space $\mathbb{R}^{1,1}$ with the metric x^2-t^2 is a model for 2–dimensional space time that is already sophisticated enough to demonstrate the Lorentz contraction.

In this model, the speed of light is given by $c=1$. The coordinates (x,t) describe an object at rest, say the segment $L=[0,1]$ at time $t=0$. In spacetime, this stationary rod sweeps out the rectangle $R=[0,1]\times\mathbb{R}$.

To see the Lorentz contraction, suppose instead we have a rod moving at velocity v. Its path in spacetime is simply the image of R under a suitable isometry f. This isometry should send the stationary path $x=0$ to the path $x=vt$ with velocity v. It follows that f sends the spacelike slice $t=0$ to the line $vx=t$. Thus $f(1,0)=(p,q)$ where $vp=q$ and $p^2-q^2=1$. A solution to these equations is conveniently given by

$$(p,q) = (\cosh s, \sinh s), \text{ where } v = \tanh s.$$

The apparent size of the moving rod is the length of the segment $[0,r] = R' \cap (t=0)$ along the x–axis. To compute r, we observe that the line

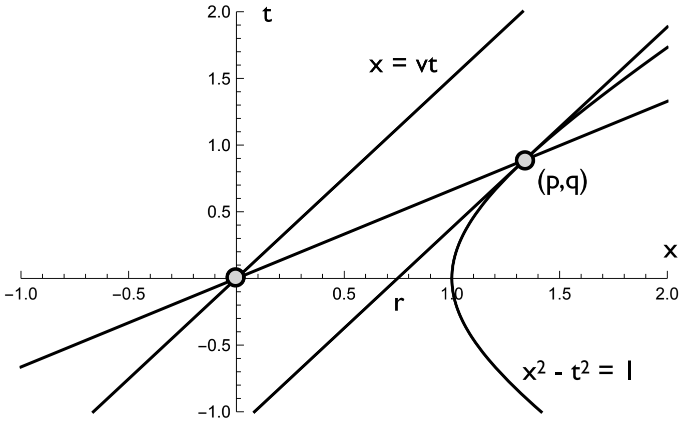


Figure 3. A rod of unit length, in motion.

$x = r + vt$ is one edge of R', passing through (p, q); thus:

$$r = p - vq = \operatorname{sech} s = \sqrt{1 - v^2}.$$

Here we have used the fact that

$$\cosh s - \tanh s \sinh s = \frac{\cosh^2 s - \sinh^2 s}{\sinh s} = \operatorname{sech} s = \sqrt{1 - \tanh^2 s}.$$

This r represents the famous Lorentz contraction: the length of a rod moving at velocity v is shorter than its stationary length by a factor of r.

We note that from the moving rod's perspective, the measurement of its length is mistaken: the two ends of the rod are not observed at the same time. It is the relative character of simultaneity at play here.

A related thought experiment is to picture a long, rapidly moving rocket passing through the door of a small barn. When the foreshortened rocket is entirely inside the barn, both doors are shut. What happens from the perspective of the rocket?

5 Hyperbolic surfaces

In this section we discuss hyperbolic surfaces and, more generally, hyperbolic orbifolds.

Discrete groups. We begin with generalities.

Let $\Gamma \subset G = \text{Isom}^+ \mathbb{H}$ be a discrete group. Then Γ acts properly discontinuously on $\mathbb{H}$, with finite point stabilizers, so

$$X = \Gamma \backslash \mathbb{H}$$

is a hyperbolic orbifold. If Γ is torsion–free, then X is a complete hyperbolic surface. We say Γ is a *lattice* if X has finite volume, and Γ is *cocompact* if X is compact. In the lattice case, the area and orbifold Euler characteristic of X are related by:

$$\text{area}(X) = 2\pi |\chi(X)|.$$

Fundamental group. Since $\mathbb{H}$ is simply–connected, we have

$$\pi_1(X) \cong \Gamma.$$

We can thus classify elements of the fundamental group according to their type in Γ: hyperbolic, parabolic and elliptic.

Theorem 5.1 *Let $g \in \Gamma \cong \pi_1(X)$. Suppose $g \neq \text{id}$. Then either:*

1. *g is hyperbolic, and there is a unique closed geodesic on X representing the conjugacy class $[g] \subset \pi_1(X)$; or*

2. *g is parabolic, and $[g]$ corresponds to a closed horocycle around a cusp of X; or*

3. *g is elliptic, and $[g]$ corresponds to a unique orbifold point on X.*

Note that in cases (2) and (3), the corresponding element of $\pi_1(X)$ has no geodesic representative; its translation distance is zero. When X is a compact surface, only case (1) can occur.

Proof. Use the fact that g stabilizes a unique geodesic; a unique point on S^1_∞; and a unit point in $\mathbb{H}$; respectively, in the three cases above. ■

Cusps. Let us explain in more detail the parabolic case. Every parabolic in G is conjugate to n or n^{-1}, where $n(z) = z + 1$. We have a covering map

$$e : \mathbb{H} \cong \Delta^* = \{q \in \mathbb{C} \;:\; 0 < |q| < 1\},$$

given by $q = e(z) = \exp(2\pi i z)$, whose deck group is

$$\langle n \rangle \cong \mathbb{Z} \cong \pi_1(\Delta^*).$$

There are no closed geodesics in the quotient space; instead, the generator of π_1 is represented by a family of *closed horocycles* $\eta_y \subset \Delta^*$ covered by the lines $\mathrm{Im}(z) = y$ in $\mathbb{H}$. The radius of η_y is $\exp(-2\pi y)$. These horocycles shrink so rapidly that a small neighborhood of $q = 0$ in Δ^* has *finite hyperbolic volume.*

Whenever $X = \Gamma\backslash\mathbb{H}$ and Γ has a parabolic element, there is a corresponding covering space $\pi : \Delta^* \to X$ that is injective and proper near $q = 0$. We refer to the corresponding end of X as a *cusp* of X.

The group Γ is a lattice if and only if there exists a compact subsurface $K \subset X$ such that $X - K$ is a finite union of horoball neighborhoods of cusps.

The thick–thin decomposition. The geometry of a surface near a short, closed geodesic is very similar to the geometry of cusp. To formulate a precise statement, we describe the basic structure theorem regarding the *thin part* of a hyperbolic surface, $X_\epsilon \subset X$. This is the set where the *injectivity radius* of X is less than ϵ. If $X = \mathbb{H}/\Gamma$, then

$$\widetilde{X}_\epsilon = \{x \in \mathbb{H} \; : \; \inf_{\gamma \neq \mathrm{id}} d(x, \gamma x) \leq 2\epsilon\}.$$

We then have:

Proposition 5.2 *There exists an $\epsilon > 0$ such that for all hyperbolic surfaces, every component U of X_ϵ is either a collar neighborhood of a short geodesic, or a horoball neighborhood of a cusp.*

Caveats. In particular, we have $\pi_1(U) \cong \mathbb{Z}$, and $\mathrm{area}(U) > C(\epsilon) > 0$. Thus if X has finite volume, its thin part has only finitely many components.

Sketch of the proof. Let $X = \Gamma\backslash\mathbb{H}$. Suppose $d(x, gx) < \epsilon$ and $d(x, hx) < \epsilon$, with neither g nor h equal to the identity. Then both g and h are close to the identity in G; any significant twist would make them elliptic.

Suppose that the discrete group $\langle g, h \rangle$ is not cyclic. Then the centralizers $Z(g)$ and $Z(h)$ meet only at the identity. Consequently, the commutator

$$g_1 = [h, g] = hgh^{-1}g^{-1}$$

is even closer to the identity; on the order of ϵ^2. It can belong to the centralizer of g or h, but not both; taking the commutator repeatedly, we obtain a sequence of elements $g_n \in \Gamma$, $g_n \to \mathrm{id}$, $g_n \neq \mathrm{id}$, contradicting discreteness.

Thus for each point $x \in \widetilde{X}_\epsilon$ there a *unique* maximal cyclic subgroup $\langle g \rangle \subset \Gamma$, generated by a group element that translates x a small distance. The components associated to different g are disjoint, and hence the quotient of each such component by its cyclic group injects into X. ■

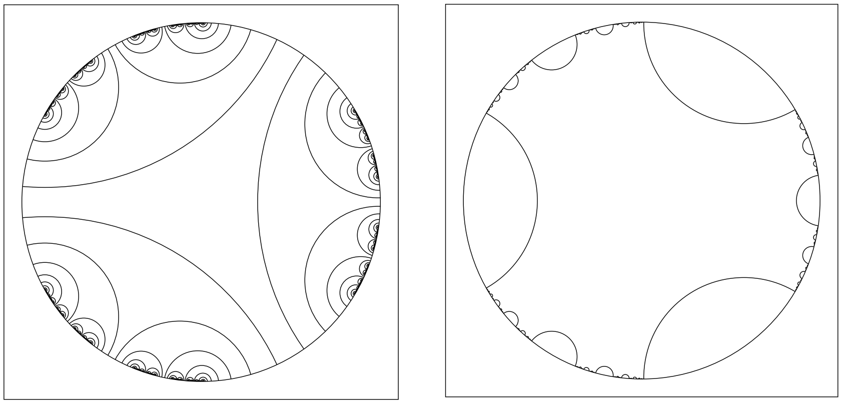


Figure 4. A reflection group and the convex hull of its limit set.

The limit set $\Lambda \subset S^1_\infty$. How does a discrete group Γ act on the circle at infinity for $\mathbb{H}$?

From a topological perspective, the answer is neatly expressed in terms of the *limit set* $\Lambda \subset S^1_\infty$, defined by

$$\Lambda = \overline{\Gamma p} \cap S^1_\infty$$

for any $p \in \mathbb{H}$. It is easy to see that Λ is independent of the choice of p, closed, and Γ–invariant.

A discrete group Γ is *elementary* if it contains an abelian subgroup of finite index. It is not hard to show that Γ is elementary iff $|\Lambda| \leq 2$.

Theorem 5.3 *Suppose Γ is nonelementary. Then Λ is the smallest closed, nonempty, Γ-invariant subset of S^1_∞. In particular, every Γ orbit in Λ is dense.*

Proof. Let $K \subset S^1_\infty$ be a closed, nonempty, Γ–invariant set. Our aim is to show that $\Lambda \subset K$.

It is easy to see that $|K| \leq 2$; otherwise Γ is elementary (it is conjugate to a subgroup of AN).

Let $\mathrm{hull}(K) \subset \mathbb{H}$ be the convex hull of K, i.e. the smallest closed convex set containing all geodesics with both endpoints in K. Choose $p \in \mathrm{hull}(K)$. Since $\mathrm{hull}(K)$ is Γ-invariant, and its closure meets the circle exactly in K, we have

$$\overline{\Gamma p} \cap S^1_\infty = \Lambda \subset K,$$

as desired. ■

Proposition 5.4 *If $X = \Gamma\backslash\mathbb{H}$ has finite volume, then the limit set of Γ is the whole circle S^1_∞. In particular, every Γ orbit on the circle is dense.*

Proof. First suppose X is compact. Choose $p \in \mathbb{H}$. By compactness, there exists an $R > 0$ such that every ball $B(q, R)$ in $\mathbb{H}$ meets Γp. But for every point $x \in S^1_\infty$, we can find a sequence of hyperbolic balls $B(q_n, R) \to x$. Hence Γp accumulates on x, and therefore $\Lambda = S^1_\infty$.

A similar argument applies when X has finite volume, using the fact that the thick part of X is compact and carries the full fundamental group of X. ■

With some additional argument one can show:

Theorem 5.5 *If $\Lambda = S^1_\infty$, then closed geodesics are dense in $\mathrm{T}_1 X$ and a generic geodesic is dense.*

Here generic is in the sense of Baire category. (It may be true, at the same time, that dense geodesics have measure zero.)

To complete the picture, we add the following remark:

Theorem 5.6 *The action of Γ on the open set $\Omega = S^1_\infty - \Lambda$ is properly discontinuous. In fact*

$$\overline{X} = (\mathbb{H} \cup \Omega)/\Gamma$$

is a smooth surface or orbifold with boundary.

Proof. Projection from Ω to the nearest point on $\partial\,\mathrm{hull}(\Lambda)$ gives a proper, equivariant map from Ω to $\mathbb{H}$. Since the action of Γ on $\mathbb{H}$ is properly discontinuous, so is the action on Ω. ■

We also note that when Γ is nonelementary, Γ is a perfect set, which yields the following trichotomy: either $\Lambda = S^1_\infty$, Λ is a Cantor set, or $|\Lambda| \leq 2$. See Figure 4 for an example where $\Gamma \subset \operatorname{Isom} \mathbb{H}$ is generated by reflections in 3 disjoint geodesics. The orientable double cover of $\overline{X}$ is a compact pair of pants.

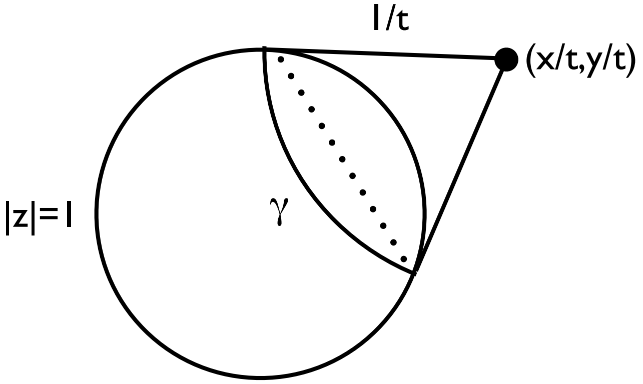


Figure 5. The Klein and Poincaré geodesics specified by a Minkowski unit vector (x, y, t).

Finite volume hyperbolic surfaces. We now turn to the construction of lattices in $G = \mathrm{SL}_2(\mathbb{R})$. Many constructions are possible; we mention those related to tessellations, pairs of pants, and arithmetic groups.

Constructing a triangle with given angles. Consider again the matrix $Q(\alpha, \beta, \gamma)$ defined in equation (4.1), but now with $0 \leq \alpha + \beta + \gamma < \pi$. Then $\det Q < 0$, and again there is an evident plane of signature $(2, 0)$, so we can conclude that Q has signature $(2, 1)$. Changing coordinates to obtain the standard quadratic form of this signature, the original basis elements become points of $\mathcal{G}$ defining the 3 sides of a triangle with inner angles α, β and γ.

The $(2, 3, 7)$ triangle. One of the most significant triangles in hyperbolic geometry is the one with inner angles $\pi/2$, $\pi/3$, and $\pi/7$. Reflections in the sides of this triangle give the discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H})$ of minimal covolume. The associated tiling is shown in Figure 6.

Let us construct this triangle $T(2, 3, 7)$ concretely in the Poincaré model, Δ. We can assume two sides are given by the lines $x = 0$ and $y = 0$. We just

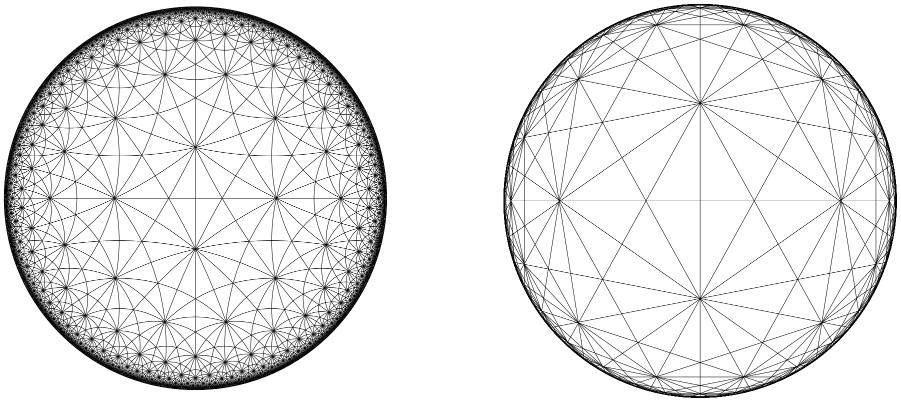


Figure 6. Tessellation of the disk by $(2,3,7)$ triangles, in the Poincaré and Klein models.

need to compute the center and radius of the circle C defining the third side.

Now the first 2 sides correspond, in the Minkowski model, to the planes normal to $c_1 = (1,0,0)$ and $c_2 = (0,1,0)$. The third circle should be normal to $c_3 = (x,y,t)$, where c_3 satisfies

$$\langle c_3, c_i\rangle_{i=1}^{3} = (\cos\pi/3, \cos\pi/7, 1).$$

This gives immediately

$$c_3 = (\cos\pi/3, \cos\pi/7, t)$$

where

$$1 + t^2 = \cos^2\pi/3 + \cos^2\pi/7.$$

Projectivizing, we find C should be centered at $z = x + iy$ where

$$(x,y) = \left(\frac{\cos\pi/3}{t}, \frac{\cos\pi/7}{t}\right),$$

and its radius is given by $r = 1/t$. In the case at hand, we find

$$c = (2.01219, 3.62585), r = 4.02438.$$

The group Γ generated by reflections in the sides of $T(2,3,7)$, and then taking its orientation–preserving subgroup of index two, is cocompact lattice in G. The quotient space $X = \Gamma\backslash X$ is called the (2,3,7) orbifold; geometrically, it is simply the double of $T(2,3,7)$. In particular it has genus zero and three singular points. The orders of these singular points determine it uniquely.

Tessellation. By similar calculations, it is easy to find circles whose reflections generate tessellations of $\mathbb{H}$. The examples coming from a triangle with 0° angles, a square with 45° angles, and a pentagon with 90° angles are shown in Figure 7.

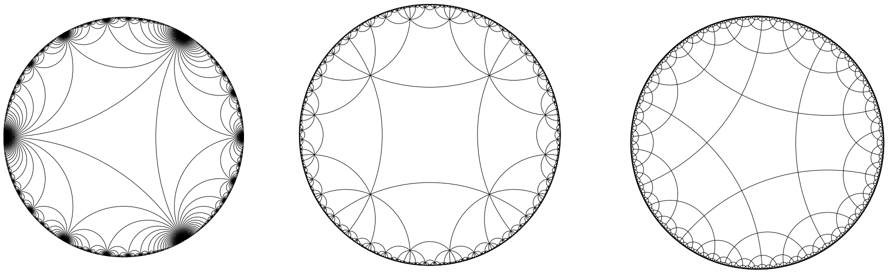


Figure 7. Groups and Circles.

Compact surfaces. Now it is easy to divide a surface of genus two into 8 cells, each of which is a pentagon, meeting 4 to a vertex. By making these into regular right pentagons, we obtain a hyperbolic structure on a surface of genus 2. Every surface of higher genus covers one of genus 2, we have show:

Theorem 5.7 *Every compact topological surface of genus $g \geq 2$ admits a metric of constant negative curvature.*

Covers and automorphism groups. One can also construct compact surfaces by starting with compact orbifolds and then passing to a finite cover.

The smallest volume hyperbolic orbifold is the $(2,3,7)$ orbifold, with Euler characteristic $1-(1/2)-(1/3)-(1/7) = 1/42$. As a complex space, it is obtained by compactifying $X(7)$. It is covered by the Klein quartic (a canonical curve in $\mathbb{P}^2$), which has genus 2, Euler characteristic 4 and hence deck group of order 168 as mentioned above. Since $X/\operatorname{Aut}(X)$ is a hyperbolic orbifold for any compact X, the symmetry group of a hyperbolic Riemann surface satisfies

$$|\operatorname{Aut}(X)| \leq 42(2g-2) = 84(g-1).$$

Equality is only obtained when X covers the $(2,3,7)$ orbifold. For example, equality is not obtained for genus 2; in this case, the largest possible automorphism group is of order 48 (and it is related to $X(4)$).

Constructing a pair of pants. By the same token, we can construct a pair of pants with cuffs (and waist) of arbitrary length. That is, we can find 3 geodesics bounding a region with specified positive distances L_1, L_2, L_3 between consecutive geodesics. This amounts to showing that Q has signature $(2,1)$, where

$$Q(L_1,L_2,L_3) = \begin{pmatrix} 1 & -\cosh L_1 & -\cosh L_2 \\ -\cosh L_1 & 1 & -\cosh L_3 \\ -\cosh L_2 & -\cosh L_3 & 1 \end{pmatrix}.$$

This indeed the case, since the first 2×2 block has signature $(1,1)$, and $\det Q < 0$. In fact, since $\cosh(L_i) \geq 1$ for all i, we have $\det(Q) \leq -4$.

Pairs of pants the basic building blocks of compact hyperbolic surfaces, and the origin of Fenchel–Nielsen coordinates on Teichmüller space, which allow one to show:

Theorem 5.8 *The space $\mathcal{T}_g$ of marked hyperbolic surfaces of genus $g \geq 2$ is naturally parameterized by length and twist coordinates: for any fixed topological pair of pants decomposition, we have*

$$\mathcal{T}_g \cong \mathbb{R}+^{3g-3} \times \mathbb{R}^{3g-3} \cong \mathbb{R}^{6g-6}.$$

More examples of hyperbolic surfaces. It is also easy to give concrete, finite-volume examples of hyperbolic surfaces using arithmetic. Start with the group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$; its quotient $X(1) = \Gamma(1)\backslash\mathbb{H}$ is the $(2,3,\infty)$ orbifold. By the 'highest point' algorithm, a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ is given by the region

$$F = \{z \ : \ |\operatorname{Re}(z)| \leq 1/2 \quad \text{and} \quad |z| \geq 1\}.$$

Clearly

$$F \subset F' = \{z \ : \ |\operatorname{Re}(z)| \leq 1/2 \quad \text{and} \quad |\operatorname{Im}(z)| > \sqrt{3}/2\},$$

and it is easy to see that the integral of $|dz|^2/y^2$ over F' is finite, so $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ has finite volume. The set F' is a substitute for a strict fundamental domain, called a *Siegel set*. It has the property that every orbit meets F' at least once and at most N times, for some N. For such a set, finiteness of the area of F' is equivalent to finiteness of the volume of the quotient space.

It is a general fact that an arithmetic quotient of a semisimple group, $G(\mathbb{Z})\ G(\mathbb{R})$, has finite volume. This was proved by Borel and Harish–Chandra, using Siegel sets. Borel once said is he spent a good part of his career figuring out how to never compute a fundamental domain.

For more examples one can consider the covers $X(n) = \Gamma(n)\backslash\mathbb{H}$ of $X(1)$, defined using the congruence subgroup

$$\Gamma(n) = \{A \in \Gamma(1) \ : \ A \equiv \mathrm{id} \bmod n\}.$$

In general the symmetry group of $X(n)/X(1)$ is given by

$$G(n) = \mathrm{PSL}_2(\mathbb{Z}/n),$$

which has order 6 when $n = 2$ (since $I = -I$ in this case). When $n = p > 2$ is prime, we have

$$|G(p)| = \frac{(p^2-1)(p^2-p)}{2(p-1)} = \frac{p(p^2-1)}{2},$$

and the number of cusps is $(p^2 - 1)/2$. For example:

1. $X(2)$ is the triply-punctured sphere, with symmetry group S_3;

2. $X(3)$ is a tetrahedron with its 4 face centers removed, and symmetry group A_4;

3. $X(4)$ is a cube with its 6 face centers removed, and symmetry group S_4; and

4. $X(5)$ is a dodecahedron with its 12 face centers removed, and symmetry group A_5. But we can continue past the Platonic solids; e.g.

5. $X(7)$ is the Klein surface of genus 3, built out of 24 heptagons the face centers removed, and a symmetry group of order 168.

It is harder to give arithmetic examples of cocompact subgroups in G. These will be discussed in the next section.

Example. What is the fractional part of $\boldsymbol{\pi}$? An extreme example, for the case of a cusp, is provided by the triply–punctured sphere $X =$

$\mathbb{H}/\Gamma(2)$. The balls $B(p/q)$ resting on $z = p/q$ with height $1/q^2$, together with $B(1/0) = \{z : \mathrm{Im}(z) \geq 1\}$, form a symmetric packing by maximal horoballs. They descend to horoballs H_1, H_2, H_3 on X, with 2 complementary triangular regions, T_1 and T_2.

The stabilizer of $B(1/0)$ is generated by $z \mapsto z + 2$, so the length of $\partial H_i = 2$. Thus the area of H_i is 2 for each i. By Gauss–Bonnet, the area of X is $2\pi = 6.28318\ldots$. Hence we have

$$\mathrm{area}(T_i) = (2\pi - 6)/2 = \{\pi\} = 0.14159265\ldots$$

This example is extreme, in the following sense:

> *For any hyperbolic surface X, there exist disjoint horoball neighborhoods B_i of the cusps of X such that the length of ∂B_i is 2 for all i.*

6 Arithmetic hyperbolic manifolds

In this section we discuss connections between hyperbolic manifolds, lattices in $\mathbb{R}^n$, and the arithmetic of quadratic forms. Arithmetic provides concrete examples of finite volume hyperbolic surfaces and orbifolds such as $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$; it also provides examples of compact surfaces, as we will see below.

A central role is played by Mahler's compactness criterion, which describes when a collection of lattices in $\mathbb{R}^n$ has a convergent subsequence.

Lattices in Euclidean space. We can think of

$$\mathcal{L}_n = \mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$$

as the space of unimodular lattices $\Lambda \subset \mathbb{R}^n$, i.e. discrete subgroups isomorphic to $\mathbb{Z}^n$ of covolume one. The correspondence is given by

$$\Lambda = \mathbb{Z}^n g,$$

where we think of vectors $x \in \mathbb{Z}^n$ as *row vectors.*

One can also picture $\mathcal{L}_n$ as the moduli space of baseframed, n-dimensional flat tori T, normalized to have volume 1. The correspondence is by $\Lambda \mapsto \mathbb{R}^n/\Lambda$.

Geometric limits. It is convenient to formulate a percursor to Mahler's theorem, due to Hausdorff. Let (X, d) be a compact metric space, and let

$K(X)$ be the set of all compact subsets $K \subset X$. Then $K(X)$ becomes a compact metric space in its own right, if we set

$$D(K, K') = \inf\{r > 0 \ : \ K_1 \subset B(K_2, r) \quad \text{and} \quad K_2 \subset B(K_1, r)\}.$$

The definitions extend to locally compact spaces such as $\mathbb{R}^n$ in an obvious way: we let $K(\mathbb{R}^n)$ denote the space of all closed subsets $F \subset \mathbb{R}^n$, and embed this space into $K(S^n)$ by taking the one–point compactification and sending F to $F \cup \{\infty\}$. Then $K(\mathbb{R}^n)$ is also compact.

Thought of as a subset of $K(\mathbb{R}^n)$, the space of closed subgroups is readily seen to be compact. But the subset of lattices is not. For example, $\Lambda_n = \mathbb{Z}(1/n, 0) \oplus \mathbb{Z}(0, n)$ converges, in $\mathbb{R}^2$, to the subgroup $\mathbb{R} \times \{0\}$.

Inradius and diameter bounds. To prevent this kind of collapsing, it is useful to control

$$L(\Lambda) = \inf\{|v| \ : \ v \in \Lambda, v \neq 0\}.$$

Given $r > 0$, let

$$\mathcal{L}_n(r) = \{\Lambda \in \mathcal{L}_n \ : \ L(\Lambda) \geq r\}.$$

Since Λ is unimodular, there exists an $R = R(r)$ such that

$$B(0, R) + \Lambda = \mathbb{R}^n \tag{6.1}$$

for all $\Lambda \in \mathcal{L}_n(r)$. Indeed, the standard convex fundamental domain F for Λ contains $B = B(0, r/2)$, so it also contains the convex hull $B(x)$ of B and x for any $x \in F$. The condition $\operatorname{vol}(B(x)) \leq \operatorname{vol} F = 1$ gives an upper bound R on x.

Put differently, once the injectivity radius of the unit volume torus $T = \mathbb{R}^n/\Lambda$ is bounded below by $r/2$, its diameter is bounded above. This is easily proved geometrically.

Mahler's thoerem. We are now in a position to prove:

Theorem 6.1 (Mahler) *The space of unimodular lattices $\mathcal{L}_n(r)$ with a lower bound of r on the shortest vector is compact.*

Proof. The proof is conveniently given using the Hausdorff topology on closed subsets of $\mathbb{R}^n$. Let Λ_n be a sequence of lattices with $L(\Lambda) \geq r$. Pass to a subsequence such that $\Lambda_n \to \Delta$ as a closed subset of $\mathbb{R}^n$. It is easy to see that Δ is an additive group. Since $B(0, r) \cap \Lambda_n = \{0\}$ for all n, the same is true for Δ, and thus Δ is discrete.

Choosing $R = R(r)$ as in equation (6.1), the condition $B(0, R) + \Lambda_n = \mathbb{R}^n$ also passes in the limit to Δ. Thus $\mathbb{R}^n/\Delta$ has finite volume, and hence Δ is a lattice. ■

See §23 and its sequels for more on lattices in $\mathbb{R}^n$.

Real quadratic forms. Now let $Q : \mathbb{R}^n \to \mathbb{R}$ be a real quadratic form of signature (p, q). Thinking of $\mathbb{R}^n$ as row vectors, there is a unique real symmetric matrix Q such that

$$Q(x) = xQx^t.$$

The *discriminant* of Q is given by $\operatorname{disc}(Q) = \det Q_{ij}$.

Note that $\mathrm{SL}_n(\mathbb{R})$ acts on the space of quadratic forms by

$$(gQ)(x) = Q(xg),$$

or by $(gQ) = gQg^t$ on the level of matrices. This action preserves the signature and discriminant, and it is otherwise transitive. Letting $\mathrm{SO}(p, q)$ denote the isometry group of

$$Q^{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

we find the space of quadratic forms with fixed invariants can be identified with the homogeneous space

$$\mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}(p, q).$$

The matrix entries of Q provide explicit coordinates on this space.

A compactness criterion. We now bring Mahler's criterion into play. Let

$$m(Q) = m(Q, \mathbb{Z}^n) = \inf\{|Q(x)| \ : \ 0 \neq x \in \mathbb{Z}^n\}.$$

We then have:

Theorem 6.2 *If $m(Q) > 0$, then the orbit of $\mathbb{Z}^n$ under $\mathrm{SO}(Q, \mathbb{R})$ is bounded in the space of lattices.*

(This means the orbit has compact closure.)

Proof. By continuity, there exists an $r > 0$ such for all $x \in \mathbb{R}^n$, $|x| < r$ implies $|Q(x)| < m(Q)$. Since $m(Q, \Lambda) = m(Q, \mathbb{Z}^n)$ for all Λ in the orbit of $\mathrm{SO}(Q, \mathbb{R})$, we have $L(\Lambda) \geq r$. Hence by Mahler's criterion, Λ ranges in a compact set of lattices. ■

Integral quadratic forms. We say Q is *integral* if its matrix has integral entries, i.e. if the associated bilinear form takes integral values on $\mathbb{Z}^n$.

Since $\mathbb{Z}$ is discrete, we have:

Proposition 6.3 *The space of integral forms (with fixed discriminant and signature) is discrete in* $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(p,q)$.

It is also clear that the integral forms are preserved by $\mathrm{SL}_n(\mathbb{Z})$.

Now it is a general fact that, for two closed subgroups H_1 and H_2 of a Lie group G, and $x \in G$, we have

> $H_1 x$ *is closed in* $G/H_2 \iff xH_2$ *is closed in* $H_1\backslash G \iff$
> $H_1 x H_2$ *is closed* G .

Thus we have:

Corollary 6.4 *Let Q be an integral quadratic form on $\mathbb{R}^n$ of signature (p,q). Then $[\mathbb{Z}^n]\cdot\mathrm{SO}(Q,\mathbb{R})$ is closed in the space of lattices.*

Proof 1. This is equivalent to saying the orbit of $\mathrm{SL}_n(\mathbb{Z})\cdot[Q]$ is closed in the space of quaratic forms; but this is clear, because it is contained in the discrete set of integral forms. ■

Proof 2. Suppose $\Lambda_n \to \Lambda$ in the orbit of $\mathbb{Z}^n$ under $\mathrm{SO}(Q,\mathbb{R})$. Choose a basis (e_i) for Λ over $\mathbb{Z}$. Then there is a nearby basis (e_i') in Λ_n for any $n \gg 0$. The Q–inner products on e_i' take integral values, so these values stabilize as $n \to \infty$. It follows that (Λ, Q) is isometric to (Λ_n, Q) for all $n \gg 0$, which means these lattices are in the same orbit of $\mathrm{SO}(Q,\mathbb{R})$. ■

Compact orbits. This bring us to our main result. Let Q be an integral quadratic form. We say Q *represents zero* if $Q(x) = 0$ for some $0 \neq x \in \mathbb{Z}^n$. Clearly Q represents zero iff $m(Q) = 0$. In fact, if Q does not represent zero, then $m(Q) \geq 1$.

Theorem 6.5 *Let Q be a integral form of signature (p,q) that does not represent zero. Then $\mathrm{SO}(Q,\mathbb{Z})$ is a cocompact lattice in $\mathrm{SO}(Q,\mathbb{R})$.*

Proof. By taking the orbit of the lattice $\mathbb{Z}^n$ under $\mathrm{SO}(Q,\mathbb{R})$, we obtain an injective, continuous map

$$f : X = \mathrm{SO}(Q,\mathbb{Z})\backslash\mathrm{SO}(Q,\mathbb{R}) \to \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}).$$

By Corollaries 6.2 and 6.4, the orbit $[\mathbb{Z}^n]\cdot\mathrm{SO}(Q,\mathbb{R})$ is compact. Therefore f is a homeomorphism to its image, and hence its domain is also compact. ■

Finite volume. With some extra analysis, one can also treat the case where Q *does* represent zero. We state without proof:

Theorem 6.6 *Let Q be an integral quadratic form of signature (p, q) on $\mathbb{R}^n$, $n \geq 3$. If Q represents zero, then $\mathrm{SO}(Q, \mathbb{Z})$ is a lattice in $\mathrm{SO}(Q, \mathbb{R})$, but the quotient is not compact.*

This result depends on the fact that $\mathrm{SO}(p, q)$ is a semisimple Lie group when $n = p + q \geq 3$. For $n = 2$ it is false. If Q is an integral quadratic form of signature $(1, 1)$ and $m(Q) = 0$, then $\mathrm{SO}(Q, \mathbb{Z})$ is not a lattice; it is a finite group.

Dimension one: Dirichlet's unit theorem. Let us apply this theorem to produce some arithmetic hyperbolic manifolds of *dimension one.*

Recall that the *units* of a number field K are the algebraic integers of norm ± 1; equivalently, they are the invertible elements $\mathcal{O}_K^\vee$ of the ring of integers.

From the perspective of number theory, we will show:

Proposition 6.7 *Let $K = \mathbb{Q}(\sqrt{D})$, $D > 0$, be a real quadratic number field. Then its group of units is infinite.*

Here we assume D is a square–free integer.

In fact we will see that the group of units of K, modulo ± 1, is isomorphic to $\mathbb{Z}$. In general, Dirichlet's unit theorem says that the group of units in a number field with r real places and s complex places has rank $r + s - 1$. (Here $\deg(K/\mathbb{Q}) = r + 2s$.)

Proof. Since D is square–free, the integral quadratic form $Q(x, y) = x^2 - Dy^2$ does not represent zero. Thus $\mathrm{SO}(Q, \mathbb{Z})$ is a cocompact subgroup of $\mathrm{SO}(Q, \mathbb{R}) \cong \mathrm{SO}(1, 1) \cong \mathbb{R} \times \mathbb{Z}/2$. It follows that

$$\mathrm{SO}(Q, \mathbb{Z})/(\pm I) \cong \mathbb{Z}.$$

Note that $Q(1, 0) = 1$. Thus for any $(x, y) \in (1, 0) \cdot \mathrm{SO}(Q, \mathbb{Z})$, we have

$$N_{\mathbb{Q}}^K(x + y\sqrt{D}) = Q(x, y) = 1$$

as well. Since the orbit of $(1, 0)$ under $\mathrm{SO}(Q, \mathbb{Z})$ is infinite, so is the set of units in K. ∎

From a geometric perspective, the quadratic forms $Q(x,y)$ considered above give rise to a family of 1–dimensional hyperbolic manifolds, $\mathrm{SO}(Q,\mathbb{Z})\backslash\mathbb{H}^1$.

Examples. The units of $K \subset \mathbb{R}$, modulo ± 1, form an infinite cyclic group under multiplication. Its unique generator $\epsilon > 1$ is called a *fundamental unit* for K.

For example, if we let $q(x,y) = x^2 - 2y^2$, then $3 - 2 \cdot 2^2 = 1$ gives a unit $\omega = 3 + 2\sqrt{2}$, and $\omega^2, \omega^3, \ldots$ give solutions

$$\begin{aligned} 17^2 - 2 \cdot 12^2 &= 289 - 2 \cdot 144 = 1, \\ 99^2 - 2 \cdot 70^2 &= 9801 - 2 \cdot 4900 = 1, \end{aligned}$$

etc.

The equation

$$x^2 - Dy^2 = 1$$

with $D > 0$ an integer is called *Pell's equation.* Its smallest nontrivial solution can be surprisingly large; for $D = 61$, it is given by $(x,y) = (1766319049, 226153980)$. Integral solutions give good rational approximations to $\sqrt{D}$, and can be found explicitly using the continued fraction expansion of $\sqrt{D}$.

Compact hyperbolic surfaces. We now pass to the case $n = 3$. Consider a quadratic form

$$q(x,y,t) = x^2 + y^2 - Dt^2$$

on $\mathbb{R}^3$ with $D > 0$ an integer. Suppose q does not represent zero over $\mathbb{Z}$; that is, the light cone for q is disjoint from $\mathbb{Z}^3$. This is the case if $D \equiv 3 \bmod 4$. Then we have:

Theorem 6.8 *The group* $\mathrm{SO}(q,\mathbb{Z})$ *gives a cocompact lattice in* $\mathrm{SO}(q,\mathbb{R}) \cong \mathrm{SO}(2,1)$.

We obtain in this way infinitely many examples of cocompact hyperbolic orbifolds. Suitable finite covers of these give compact hyperbolic surfaces.

Theorem 6.9 *The group* $\mathrm{SO}(q,\mathbb{Z})$ *contains a torsion-free subgroup of finite index.*

Proof. It suffices to prove the same for $\mathrm{SL}_n(\mathbb{Z})$. But if $A \in \mathrm{SL}_n(\mathbb{Z})$ has finite order, then its eigenvalues are roots of unity of degree at most n over $\mathbb{Q}$. There are only finitely many such roots of unity (since they satisfy a monic equation of degree n with integral coefficients that are symmetric functions of roots of unity, hence bounded). Thus we can find a finite set of irreducible polynomials $p_i(X)$ such that for any A of finite order (greater than one), $\det p_i(A) = 0$ for some i. We can also assume $\det p_i(I) \neq 0$.

Let p be a prime which is large enough that $\det p_i(I) \neq 0 \bmod p$ for all i. Let $\Gamma_n(p) \subset \mathrm{SL}_n(\mathbb{Z})$ be the finite index subgroup of matrices congruent to the identity $\bmod p$. Then $\Gamma_n(p)$ is torsion free. ■

Parabolics. For general D (with $q(x)$ possibly representing zero) the construction above gives a lattice (but possibly noncompact).

Example. The case $D = 1$ gives $\mathrm{PSL}_2(\mathbb{Z})$. Indeed, the adjoint action of $\mathrm{SL}_2(\mathbb{R})$ on its Lie algebra $sl_2(\mathbb{R})$ preserves the form $q(A, B) = \mathrm{tr}(AB)$. (This trace form comes from the identification of $sl(V)$ with $\mathrm{Hom}(V, V)$ associated to the tautological representation of $\mathrm{SL}(V)$ acting on V.) With respect to the basis $k = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $a = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $b = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the form becomes

$$q(k, a, b) = 2(a^2 + b^2 - k^2).$$

Note that the compact direction k is distinguished as being time-like, while the geodesic directions a and b are space-like. We can think of the $\mathrm{sl}_2(\mathbb{R})$ with the form q as a picture of an infinitesimal neighborhood of the identity in $\mathrm{SL}_2(\mathbb{R})$, with the light-cone of parabolic elements bounding two cones of elliptic elements (distinguished by direction of rotation) and a connected region of hyperbolic elements.

Finiteness of orbits. Here is a concrete case of a general finiteness theorem for arithmetic groups.

Theorem 6.10 *For general D, the integral solutions to $q(x, y, t) = -u$ fall into a finite orbits under the action of* $\mathrm{SO}(q, \mathbb{Z})$.

Proof. The integral points just described lie on a copy of $\mathbb{H}$ for q. When the quotient is compact, this Theorem is clear, since the integral points are a discrete subset of $\mathbb{H}$.

When the quotient is noncompact but finite volume, the key point is that each cusp corresponds to an integral point v on the light-cone $q(v, v) = 0$.

Since $q(v,w)$ assumes integral values at integral points, the horoball neighborhood $|q(v,w)| < 1$ of the cusp is devoid of integral points. So again the integral points are discrete and confined to a compact region in the quotient, and hence finite in number. ■

Beyond $\mathbb{Q}$. The surfaces just constructed are *arithmetic*; they are also related to quaternion algebras. The construction can be generalized to yield compact arithmetic hyperbolic n-manifolds for every $n \geq 2$. For $n = 3$ we can still use an integral indefinite form q that does not represent zero. For $n \geq 4$ one can continue to use very simple, diagonal quadratic forms, provided one is willing to pass to number fields Cf. [BP, §E.3] and below.

Restriction of scalars. Finally, we describe how compact hyperbolic surfaces can be constructed using quadratic forms over more general number fields than $\mathbb{Q}$.

For example, the subgroup of Γ of $\mathrm{SL}_3(\mathbb{Z}[\sqrt{2}])$ that preserves the quadratic form

$$q(x,y,t) = x^2 + y^2 - \sqrt{2}t^2$$

embeds in $\mathrm{SO}(2,1) \times \mathrm{SO}(3)$, using the two embeddings of $\mathbb{Q}(\sqrt{2})$ into $\mathbb{R}$. The projection to the first factor gives a discrete group $\Gamma \subset \mathrm{SO}(2,1)$, which has no unipotent elements because the second factor is compact.

Note that q form does not represent zero over $\mathbb{Z}[\sqrt{2}]$; if it did, so would its Galois conjugate q', but q' is positive definite.

We claim $\mathbb{H}/\Gamma$ is compact. The proof will combine Mahler's compactness criterion with the method of *restriction of scalars*.

We have a quadratic space $(L,q) = (\mathbb{Z}[\sqrt{2}]^3, q)$ defined over $K = \mathbb{Q}(\sqrt{2})$. The two real places of K give a map $v \mapsto (v, v')$ sending

$$L \to \mathbb{R}^3 \oplus \mathbb{R}^3$$

whose image is a discrete lattice. Thus $L \in X_6$. On this lattice we have a 'quadratic form'

$$Q : L \to \mathbb{Z}[\sqrt{2}] \cong \mathbb{Z}^2$$

given by $Q(v,w) = (q(v), q'(w))$, which extends to a form $Q : \mathbb{R}^6 \to \mathbb{R}^2$. Letting $\mathrm{SO}(Q)$ denote the symmetries of this form, we have

$$\mathrm{SO}(Q) \cong \mathrm{SO}(q) \times \mathrm{SO}(q') \cong \mathrm{SO}(2,1) \oplus \mathrm{SO}(3).$$

(Note that the two summands of $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$ can be distinguished as the radicals of q and q' respectively, so they must be preserved by any element of $\mathrm{SO}(Q)$.)

Finally we have $\Gamma = \mathrm{SO}(Q) \cap \mathrm{SL}_6(\mathbb{Z})$, and thus

$$\mathrm{SO}(2,1)/\Gamma \cong \mathrm{SO}(Q) \cdot L \subset X_6.$$

Since Q does not represent zero, the orbit $\mathrm{SO}(Q) \cdot L$ is *bounded* in X_6.

To complete the proof we need to show this orbit is *closed* So suppose $L_n \to L'$. Then for L_n large enough there is an obvious isomorphism $\iota_n : L_n \to L'$ close to the identity. Since the quadratic form takes integral values, this map is an isometry for n large enough. Moreover L_n is T–invariant so the same is true for L' and the isometry commutes with the action of T. Thus L' is in the $\mathrm{SO}(Q)$ orbit of L.

General arithmetic groups. More generally and more functorially, one can consider an order A in a totally real field $K \subset \mathbb{R}$ with n real places. Then using these places to give maps $A \to \mathbb{R}$, we obtain a lattice

$$A \subset A \otimes_A \mathbb{R}^n \cong \mathbb{R}^n.$$

Now given a quadratic form $q : L \to L$, where L is a rank d A–module, we similarly have a lattice

$$L \subset L \otimes_A \mathbb{R}^n \cong \mathbb{R}^{nd},$$

and a quadratic form

$$Q : \mathbb{R}^{nd} \to \mathbb{R}^n$$

obtained by tensoring q with $\mathbb{R}^n$ over A. We then have

$$\mathrm{SO}(Q, \mathbb{R}) \cong \mathrm{SO}(L \otimes_A \mathbb{R}^n) \cong \prod_1^d \mathrm{SO}(p_i, q_i),$$

where (p_i, q_i) is the signature of q at the ith place of K. Using L to give $\mathrm{SO}(Q)$ an integral structure, we then have

$$\Gamma = \mathrm{SO}(q, A) \cong \mathrm{SO}(Q, \mathbb{Z}) \subset \mathrm{SO}(Q, \mathbb{R}).$$

It is then a general fact that Γ is a lattice in $\mathrm{SO}(Q, \mathbb{R})$; so if $q_i = 0$ for all $i > 1$, by projection we get a lattice in $\mathrm{SO}(p_1, q_1)$.

Parabolics and representations of zero. In the construction above, the fact that q did not represent zero played an important role in the cocompactness of Γ. Conversely, if q *does* represent zero, then we can find parabolic elements in Γ; cf. Artin's *Geometric Algebra*.

7 Dynamics on hyperbolic surfaces

We now turn to the study the geodesic and horocycle flows on hyperbolic surfaces.

In this section we will show that on a finite volume surface X, these flows are mixing on T_1X. This type of generic ergodic theory on surfaces generalizes to higher dimensions, and leads into Mostow rigidity (§20), the theory of unitary representations (§10), and property T (§19).

In later sections we will show that on a compact surface, the horocycle flow is minimal (§9) and uniquely ergodic (§12). The topological statement leads into the classification of geodesic planes in hyperbolic 3–manifolds (§14), and the proof of the Oppenheim conjecture (§13). The unique ergodicity statement also holds much more generally, and both are subsumed under Ratner's results on unipotent flows and actions.

The unit tangent bundle to $\mathbb{H}$. We begin by describing the geodesic, horocyclic and elliptic flows on the unit tangent bundle of a hyperbolic surface.

Let $G = \mathrm{Isom}^+ \mathbb{H} \cong \mathrm{PSL}_2(\mathbb{R})$. Recall that $\mathbb{H} = G/K$, where K is the stabilizer of $z = i$.

Let us choose a base unit tangent vector $v_0 \in T_i\mathbb{H}$, pointing vertically along the imaginary axis. Since G acts transitively on $\mathbb{H}$, and K acts transitively on the unit vectors based at $z = i$, we obtain a natural isomorphism

$$G \cong T_1\mathbb{H}$$

by $g \mapsto (Dg)(v_0)$. Note that the bundle map $T_1\mathbb{H} \to \mathbb{H}$ corresponds to the map $G \mapsto G/K$.

The *left* action of G on $T_1\mathbb{H}$ is given by $g \cdot v = (Dg)(v)$; it covers the isometric action of G on $\mathbb{H} = G/K$. What about the right action?

Dynamics. The *geodesic flow* $g^t : T_1\mathbb{H} \to T_1\mathbb{H}$ is defined by moving distance t along the oriented geodesic tangent to a given vector v.

The *horocycle flow* $h^s : T_1\mathbb{H} \to T_1\mathbb{H}$ is defined by moving distance s along the horocycle perpendicular to v with v pointing inward. (This means the horocycle rests on the positive endpoint of the geodesic through v.)

Finally the *elliptic flow* $e^r : T_1\mathbb{H} \to T_1\mathbb{H}$ is defined by rotating v through angle r in its tangent space.

All three flows preserve the Liouville measure on $T_1\mathbb{H}$, which is the product of area measure in the base and angular measure in the fiber.

Theorem 7.1 *The geodesic, horocyclic and elliptic flows on $G = T_1\mathbb{H}$ correspond to right actions of the 1-parameter subgroups A, N and K respectively.*

Proof. Since the flows g_t, h_s and e_r on $T_1\mathbb{H}$ are defined geometrically, they commute with the action of isometries of $\mathbb{H}$. Similarly, the right and left actions of G on $G = T_1\mathbb{H}$ also commute. Thus it suffices to check that the action of each 1-parameter subgroups above is correct at the base vector v_0, corresponding to the identify element in the identification $G = T_1\mathbb{H}$.

But the right and left actions *agree* at the identity element, so it suffices to verify the Theorem for the *isometric* (or left) action of each subgroup on v_0. This is clear:

(a) We have $g_t(z) = e^t z$ for $g_t \in A$, giving translation along the vertical geodesic tangent to v_0.

(b) We have $h_s(z) = z + s$ for $h_s \in N$, giving translation along the horocycle $\operatorname{Im} z = 1$ orthogonal to v_0.

(c) Finally $e_r \in K$ stabilizes $z = i$ and rotates v_0 through angle r. ■

We can thus represent these geometric flows by the corresponding right actions on $\mathrm{T}_1\mathbb{H} = G$:

$$g_t(v) = va_t,\ h_s(v) = vn_s, \quad \text{and} \quad e_r(v) = vk_r.$$

The geometric behavior of these subgroups allows one to visualize some of the standard decompositions of G, as follows:

1. *The polar decomposition $G = KAK$.* Given two vectors $x, y \in T_1\mathbb{H}$, rotate the first so it points along the geodesic to the second; then apply the geodesic flow, then rotate once more. This decomposition is not quite unique (since $K = KK$).

2. *The Iwasawa decomposition $G = KAN$.* Draw the horocycle H normal to x, and then find the geodesic normal to H passing through y; finally rotate, upon arrival, so the image vector lines up with y. This decomposition is unique.

3. *The Bruhat decomposition $G = BWB$.* It is also not hard to see that for $B = AN$ (a Borel subgroup), and $W = \langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \rangle$ (the Weyl group), we have $G = BWB$. It is nearly true, but not quite right, that $G = N^t AN = B^t B$.

Hyperbolic surfaces and unitary representations of G. We now pass from $\mathbb{H}$ to a general hyperbolic surface or orbifold $X = \Gamma\backslash\mathbb{H}$. Its unit tangent bundle is given by can be naturally described by

$$\mathrm{T}_1 X = \Gamma\backslash G.$$

The actions of the geodesic, horocyclic and elliptic flows are again given by the right actions of A, N and K.

The right action of G provides a natural representation

$$\rho : G \to L^2(\Gamma\backslash G).$$

This representation is given explicitly by

$$\rho(g) \cdot f(x) = f(xg). \tag{7.1}$$

Note that, because this is a right action, we have

$$\rho(g)\rho(h)f(x) = \rho(g)f(xh) = f(xgh) = (\rho(gh)f)(x)$$

as desired. (For a left action, we would set $\rho(g)f(x) = f(g^{-1}x)$).

The decomposition of this representation into irreducibles carries profound information about harmonic functions and automorphic forms on the Riemann surface $X = \Gamma\backslash\mathbb{H}$.

Basic relations between the geodesic, horocyclic and elliptic flows. Our analysis of the geodesic and horocycle flows will depend in a crucial way on the structure of the group G itself.

First, note that $h_s(x)$ and $g_t(x)$ converge to the same point on S^1_∞ as $s, t \to \infty$. However the vector x is rotated by almost π when it is moved along the horocycle. Imagine a vector x in $\mathbb{H}$, resting on the horizontal horocycle $\{z \ : \ \operatorname{Im} z = 1\}$. Drawing the geodesic between x and $h^s(x)$, we find

$$h^s(x) = e^{\pi - r} g^t e^{-r}(x), \tag{7.2}$$

where $t(s) \to \infty$ and $r(s) \to 0$ as $s \to \infty$.

Second, note that the inward parallel to a horocycle at distance t is contracted in length by the factor $\exp(t)$. This implies

$$g_t h^{\exp(t)s}(x) = h^s g_t(x). \tag{7.3}$$

The second equation has an exact analogue for a toral automorphism, while the first does not. The second equation reflects the Anosov nature of the geodesic flow.

Theorem 7.2 *The geodesic and horocycle flows on a hyperbolic surface of finite area are ergodic and mixing.*

The proof will be in steps: a) g^t is ergodic; b) h^s is ergodic; c) g^t is mixing; d) h^s is mixing.

Proof of (a): Ergodicity of the geodesic flow. Following Hopf's argument for the ergodicity of a toral endomorphism, we first consider a compactly supported continuous function f on $T_1 Y$. By density in L^2, it suffices to show the ergodic average

$$F(x) = \lim_{T\to\infty} \int_0^T f(xa_t)dt$$

is constant a.e. for all such f.

Now we can study this average as a function of $x \in T_1(\mathbb{H})$ by lifting to the universal cover. Clearly $F(x)$ only depends on the geodesic γ through x; this geodesic can be labeled $\gamma_{a,b}$ for $a, b \in S^1_\infty$, so F becomes a function $F(a, b)$.

Any two geodesics with the same forward endpoint are asymptotic, so $F(a, b)$ is independent of a (when it exists). On the other hand, the ergodic averages as $t \to -\infty$ agree with the positive time ones a.e., so $F(a, b)$ is also independent of b. Therefore F is constant a.e., and thus the geodesic flow is ergodic.

Put differently, the space of oriented geodesics $\mathcal{G} = (S^1_\infty \times S^1_\infty) - \text{diag}$ has an obvious pair of foliations coming from the product structure. The Hopf argument shows the ergodic averages are constant a.e. along almost every leaf, so F is constant. ■

Proof of (b): Ergodicity of the horocycle flow. Now let $f \in L^2_0(\mathrm{T}_1 X)$ be invariant under the horocycle flow. We will show $f = 0$.

Let A_t, N_t and K_t denote the operators on $L^2(X)$ corresponding to the various flows via (7.1). By equation (7.2), we have

$$N_s = K_{\pi-r} A_t K_{-r}$$

where $r \to 0$ as $s, t \to \infty$. Since $N_s f = f$, we have for any $T > 0$,

$$f = \frac{1}{T}\int_0^T K_{\pi-r} A^t K^{-r} f\, dt.$$

It follows that

$$\langle f, f\rangle = \lim_{T\to\infty} \langle (1/T) \int_0^T G^t K^{-r} f \, dt, K_{\pi-r} f\rangle.$$

Since $K_{\pi-r}g \to K_\pi g$ and $K_{-r}f \to f$ as $t \to \infty$, the first term in the inner product converges in $L^2(X)$ to zero, by ergodicity of the geodesic flow and von Neumann's ergodic theorem. The second term has norm $\|f\|$, since the action of K is unitary. Therefore $\langle f, f\rangle = 0$ and hence $f = 0$. ■

Proof of (c): Mixing of the geodesic flow. We have seen:

$$h^s g^t = g^t h^{\exp(t)s}.$$

This implies, on the level of operators, that

$$N_{\exp(t)s} A_t = A_t N_s.$$

(One can check this equation using the matrices a_t and n_s.)

The proof of mixing of the geodesic flow now follows exactly as in the case of a toral automorphism, using this equation. First we associate to the horocycle flow the self–adjoint averaging operator

$$S_T = \frac{1}{2T} \int_{-T}^{T} N_s(f) \, ds.$$

Then for any $f \in L^2_0(\mathrm{T}_1 X)$, we have $S_T f \approx f$ when T is small, but $S_T f \approx 0$ when T is large (by ergodicity of the horocycle flow). We then have

$$S_{\exp(t)s} A_t = A_t S_s.$$

Compare equation (3.1).

Thus for $f, g \in L^2_0(\mathrm{T}_1 X)$, and s small, we have

$$\langle f, A_{-t} g\rangle \;\approx\; \langle S_s f, A^{-t} g\rangle = \langle A_t S_s f, g\rangle = \langle S_{\exp(t)s} A_t f, g\rangle = \langle A_t f, S_{\exp(t)s} g\rangle.$$

By ergodicity of the horocycle flow, as $t \to \infty$ we have $S_{\exp(t)s} g \to 0$, while $\|A_t f\| = \|f\|$ remains constant. Hence

$$\langle f, A_{-t} g\rangle = \langle A^t f, g\rangle \to 0,$$

which is mixing. ■

Here is the intuitive picture of mixing of g^t. Consider a small box B in the unit tangent bundle; B is a packet of vectors all pointing in approximately the same direction, say with spread θ. Then $g^t(B)$ is concentrated along an arc of length $t\theta$ along a circle of radius t. This circular arc approximates a horocycle; since the horocycle flow is ergodic, we obtain mixing.

Proof of (d): Mixing of G. We conclude by observing that the action of the whole group $G = \mathrm{SL}_2(\mathbb{R})$ on T_1X is mixing.

Theorem 7.3 *If $g_n \to \infty$, then $\langle g_n\alpha, \beta\rangle \to \langle \alpha, 1\rangle\langle\beta, 1\rangle$.*

Proof. Use the KAK decomposition of G. If $g_n = k_n a_n k_n' \to \infty$, we can pass to a subsequence such that $k_n \to k$ and $k_n' \to k'$. Then

$$\langle k_n a_n k_n' \alpha, \beta\rangle = \langle a_n k_n' \alpha, k_n^{-1}\beta\rangle \approx \langle a_n k' \alpha, k^{-1}\beta\rangle \to \langle\alpha, 1\rangle\langle\beta, 1\rangle,$$

by mixing of the geodesic flow. ∎

Corollary 7.4 *The horocycle flow is mixing.*

Corollary 7.5 *The action of any closed subgroup of G on T_1X is mixing.*

Example. Let $T : \mathrm{T}_1X \to \mathrm{T}_1X$ be the geodesic or horocycle flow for time $s > 0$. Then T is mixing.

8 Orbit counting and equidistribution

The classical Gauss circle problem concerns estimating

$$N(R) = |\mathbb{Z}^2 \cap B(0, R)|.$$

By associating to the $\mathbb{Z}^2$ lattice the tiling of $\mathbb{R}^2$ by unit squares, each centered at an integral point, it is easy to see that

$$N(R) = \pi R^2 + O(R).$$

The error term comes from the squares meeting $\partial B(0, R)$, which has total length $2\pi R$. A long standing problem is to establish:

Conjecture 8.1 *We have $N(R) = \pi R^2 + O(R^{1/2+\epsilon})$ for all $\epsilon > 0$.*

It is known that one cannot take $\epsilon = 0$.

In this section we will develop equidistribution theorems that follow from mixing and that lead to similar estimates of the number of points in an orbit Γp that lie in a hyperbolic ball $B(0, R)$. An important new feature in negative curvature is that the area of $B(0, R)$ and the length of its boundary are comparable: they are both on the order of e^R when R is large. Thus a more subtle argument is required even to get the leading term for $N(R)$.

Reference for this section: [EsM].

Equidistribution of spheres.

Theorem 8.2 *Let $p \in X$ be a point on a hyperbolic surface of finite area. Then the spheres about p are equidistributed on X. That is, for any compactly supported continuous function f on X,*

$$\frac{1}{\text{length}(S(p,R))} \int_{S(p,R)} f(s)\, ds \to \frac{1}{\text{area}(X)} \int_X f(x)\, dx$$

as $R \to \infty$.

Proof. Consider over p a small symmetric ball A in $T_1 Y$. Then the geodesic flow transports A to a set $g_R(A)$, symmetric about p and concentrated near $S(p, R)$. Pulling f back to $T_1 X$ we have $\langle g_R \chi_A, f\rangle \to \langle \chi_A, 1\rangle\langle f, 1\rangle$ by mixing of the geodesic flow. But $(1/mA)\langle g_R\chi_A, f\rangle$ is almost the same as the average of f over $S(p, R)$, by continuity. ■

Orbit counting.

Theorem 8.3 *For any lattice Γ in the isometry group of the hyperbolic plane, and any $p, q \in \mathbb{H}^2$,*

$$|B(p,R) \cap \Gamma q| \sim \frac{\text{area}\, B(p,R)}{\text{area}(\mathbb{H}/\Gamma)}.$$

Proof. Replace q by a bump function f of total mass 1 concentrated near q, and let $F = \sum_\Gamma f \circ \gamma$. Then the orbit count is approximately $\int_{B(p,R)} F(x)\, dx$. But area measure on $B(p, R)$ is a continuous linear combination of the probability measures on the circles $S(p, r)$, $0 < r < R$. For

r large, the average of $F(x)$ over $S(p, r)$ is close to the average of $F(x)$ over $X = \mathbb{H}/\Gamma$, by equidistribution. The latter is $1/\operatorname{area}(X)$. Since the total measure of $B(p, R)$ is $\operatorname{area}(B(p, R))$, we find the orbit count is asymptotic to $\operatorname{area}(B(p, R))/\operatorname{area}(X)$. ■

The error term in the hyperbolic orbit counting problem is studied in [PR].

For a Euclidean lattice $L \subset \mathbb{R}^n$, it is easy to see that

$$|L \cap B(0, R)| = \frac{\operatorname{vol} B(0, R)}{\operatorname{vol}(\mathbb{R}^n/L)} + O(R^{n-1}).$$

For $L = \mathbb{Z}^2$ the classical *circle problem* is to estimate the error term, which is usually written in the form

$$P(x) = \sum_{n \leq x} r(n) - \pi x$$

where $r(n)$ is the number of integer solutions to $a^2 + b^2 = n$. The estimate above gives $P(x) = O(x^{1/2})$. A typical modern bound is $P(x) = O(x^{7/22+\epsilon})$ (Iwaniec and Mozzochi, 1988). Numerical evidence supports $P(x) = O(x^{1/4} \log x)$.

Counting lifts of closed geodesics. Let $\gamma \subset Y$ be a closed geodesic on a surface of finite area. Let $\widetilde{\gamma} \subset \mathbb{H}$ be its lift to the universal cover; it is a locally finite configuration of geodesics. Fixing a point $p \in \mathbb{H}$, let $N(R)$ denote the number of distinct geodesics in $\widetilde{\gamma}$ meeting $B(p, R)$.

In terms of the Minkowski model, γ gives a point x in the 1-sheeted hyperboloid $\mathcal{G}$ with a discrete orbit. The counting problem translates into knowing the number of points of Γx meeting a compact region corresponding to $B(p, R)$. But $\Gamma \backslash \mathcal{G}$ is not even Hausdorff! Although the orbit Γx is discrete, typical orbits in $\mathcal{G}$ are dense (by ergodicity of the geodesic flow). So how to solve the counting problem?

Theorem 8.4 *We have*

$$N(R) \sim \frac{\operatorname{area} B(\gamma, R)}{\operatorname{area} Y}.$$

Here $\operatorname{area} B(\gamma, R)$ is the immersed area of an R-neighborhood of γ on Y. One can show that

$$\operatorname{area}(B(\gamma, R)) \sim \operatorname{length}(\gamma) \operatorname{area}(B(x, R)/\pi.$$

Proof. Using mixing of the geodesic flow, one can show the parallels L_t of γ at distance t are equidistributed on Y. (The proof is similar to that for equidistribution of spheres.) Now consider the covering space $Z \to Y$ corresponding to $\pi_1(\gamma) \subset \pi_1(Y)$. Let $E \subset Z$ be the projection of Γp to Z. Then it is not hard to see that $N(R)$ is the same as $|E \cap B(\gamma', R)|$, where γ' is the canonical lift of γ from Y to Z. Projecting Dirichlet regions based at Γp to Z gives the heuristic for the count; it is justified by equidistribution of parallels to γ. ■

Counting integral points on level sets of quadratic forms. Fix $A \neq 0$ and $D > 0$. Consider the Diophantine problem of counting the number of *integral solutions* $N(R)$ to the equation

$$x^2 + y^2 = A + Dt^2$$

with $x^2 + y^2 \leq R^2$. For $A < 0$ this corresponds to counting orbits; for $A > 0$, to counting geodesics.

To fix the ideas, suppose $D \equiv 3 \bmod 4$, and $A < 0$. Then the quotient $\mathrm{SO}(q, \mathbb{R}) / \mathrm{SO}(q, \mathbb{Z})$ is compact, so only finitely many points on $Y = \mathbb{H} / \mathrm{SO}(q, \mathbb{Z})$ correspond to integral points on the hyperboloid $x^2 + y^2 - Dt^2 = A$. For a given orbit $\mathcal{O}$, the preceding discussion shows

$$N(R, \mathcal{O}) \sim C \operatorname{vol}(B(0, R) \cap \mathbb{H}) \sim CR$$

(by our earlier estimate of the volume form on $\mathbb{H}$ and $\mathcal{H}$); and since there are only finitely many orbits we have

$$N(R) \sim C(A, D) R.$$

For the case $A > 0$ the geodesic counting argument leads to the same conclusion.

In fact the same estimate holds for any $A \neq 0$, since even in the finite volume case, there are only finitely many orbits of integral points.

Counting integral points in $\mathbb{R}^{2+1}$. Consider the simplest case, namely the problem of counting solutions to

$$x^2 + y^2 + 1 = t^2,$$

associated to the level set $q = -1$ for the form $q(x, y, t) = x^2 + y^2 - t^2$.

Theorem 8.5 *Let $N(R)$ be the number of (x, y) with $x^2+y^2 \leq R^2$ that occur in integral solutions to $x^2 + y^2 + 1 = t^2$. Then $N(R) \sim R$ as $R \to \infty$.*

Proof. Let $\Gamma = \mathrm{SO}(q, \mathbb{Z})$. We first show that all the integral solutions belong to a single Γ-orbit, namely the orbit of $(0, 0, 1)$.

Indeed, Γ is essentially the group $\mathrm{SL}_2(\mathbb{Z})$. Passing to $\mathbb{RP}^2$, consider the ideal quadrilateral $F \subset \mathbb{H} \cong \Delta$ which has vertices $(\pm 1, 0), (0, \pm 1)$. Then F is a fundamental domain for a subgroup of Γ (in fact for $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$.) The geodesic from $(1, 0)$ to $(0, 1)$ bounding F is just the line $x + y = 1$, or $x + y = t$ in homogeneous coordinates. By moving solutions into F, we see the orbit of every integral point has a representative with $|x \pm y| \leq |t|$. But squaring this equation and using the assumption that $x^2 + y^2 + 1 = t^2$, we get $|2xy| \leq 1$. The only integral point with this property is $(x, y) = (0, 0)$.

By tiling $B(R) \cap \mathbb{H}^2$ with copies of F, we conclude that $N(R)$ grows like $\mathrm{vol}\, B(R)/\,\mathrm{vol}(F)$.

Now projecting to the (x, y)-plane, F becomes the region $|2xy| \leq 1$, $B(R)$ becomes the region $x^2 + y^2 \leq R^2$, and the hyperbolic area element becomes $dA = dx\, dy/\sqrt{x^2 + y^2 + 1}$. We then calculate that $\mathrm{area}(F) = 2\pi$. Since $dA \sim dr d\theta$, we get $\mathrm{area}\, B(R) \sim 2\pi R$, and thus $N(R) \sim R$. ■

Example: for radius $R = 100$, there are 101 solutions, generated from (x, y) in the set

$$\{(0,0), (2,2), (8,4), (12,12), (18,6), (30,18), (32,8), (38,34),$$
$$(46,22), (50,10), (68,44), (70,70), (72,12), (76,28), (98,14)\}$$

by changing signs and swapping the coordinates. A few other values: $N(50) = 41$; $N(200) = 197$; $N(1000) = 993$; see Figure 8.

Quaternion algebras. A *quaternion algebra* D over a field K is a central simple associative algebra of rank 4. Alternatively, passing to the algebraic closure we have $D \otimes \overline{k} \cong M_2(\overline{k})$, a matrix algebra.

Every element $x \in D - k$ generates a quadratic field extension $K = k(x)$ of k, since this is true in the matrix algebra. Thus we obtain a trace, norm and conjugation involution on D.

Assume $\mathrm{char}\, k \neq 2$. Then we can normalize $x \in D$ by subtracting $\mathrm{tr}(x)/2$ to arrange that $\mathrm{tr}(x) = 0$.

Let $i \in D - K$ be an element of trace zero. Then $i^2 = a \in k^*$. It is known that there exists an element $j \in k$ such that $ij = -ji$, and we can

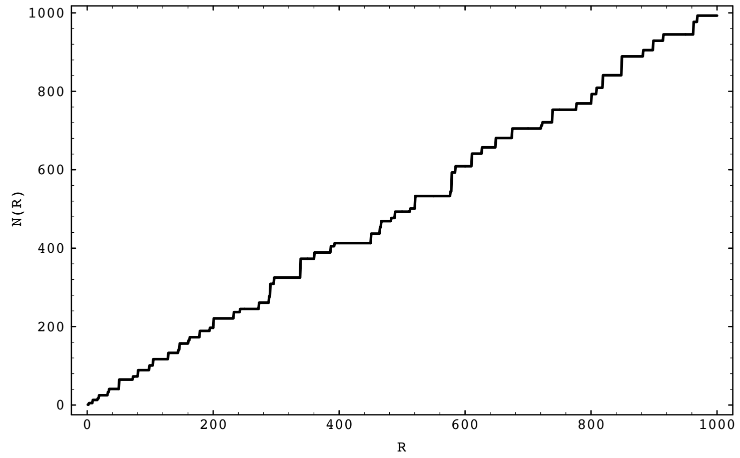


Figure 8. Orbit counts.

also arrange that $\mathrm{tr}(j) = 0$. Write $j^2 = b \in k^*$. Then if we let $k = ij$ (not to be confused with the base field), we have $k^2 = -ab$.

In short, *a quaternion algebra is specified by a pair of elements* $(a, b) \in k^*$*.*

Examples: The Hamilton quaternions correspond to $(a, b) = (-1, -1)$. The algebra $M_2(k)$ comes from $(a, b) = (1, 1)$. For the matrix algebra we can take

$$i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A quaternion algebra is *split* if $D \cong M_2(k)$; otherwise it is *ramified*.

Let $D_0 \subset D$ be the trace zero subset. Then $D_0 = ki \oplus kj \oplus k(ij)$, with the norm given by

$$N(xi + yj + tk) = q(x, y, t) = ax^2 + by^2 - abt^2.$$

Thus a quaternion algebra D determines a ternary quadratic form q, whose zero locus gives a conic C in $\mathbb{P}^2$. The algebra D is split iff q represents zero iff $C(k)$ has a point.

For example, a quaternion algebra over $\mathbb{R}$ is unramified iff the form $q(x, y, t)$ is *indefinite*.

To discuss 'integral points', we need to take a maximal order $\mathcal{O} \subset D$, that is, a maximal subring that is finitely-generated as a $\mathbb{Z}$-module. For concreteness let us suppose $k = \mathbb{Q}$. Then the group of units $U \subset \mathcal{O}$ consists of elements with norm $N(u) = \pm 1$. We can also discuss the units U_1 with norm 1, and the quotient $U' = U_1/(\pm 1)$.

For example, if $\mathcal{O} = M_2(\mathbb{Z})$, then $U = GL_2(\mathbb{Z})$, $U_1 = \mathrm{SL}_2(\mathbb{R})$ and $U' = \mathrm{PSL}_2(\mathbb{R})$.

If we suppose further that $D \otimes \mathbb{R}$ is split, i.e. that the form q is indefinite, then we obtain an embedding $U_1 \subset \mathrm{SL}_2(\mathbb{R})$ realizing U_1 is an *arithmetic group.* This groups is commensurable to $\mathrm{SO}(q, \mathbb{Z})$.

At the same time, U_1 acts on the set X_n of elements of norm n and trace 0 in $\mathcal{O}$, i.e. on the solutions to the Diophantine equation $q(x, y, t) = n$. The space of orbits X_n/U_1 is finite.

Example: the integral solutions X to $x^2 + y^2 + 1 = t^2$ form the level set X_{-1} for the algebra $D \cong M_2(\mathbb{Z})$ with structure constants $(a, b) = (1, 1)$. Indeed, we can think of $X \subset M_2(\mathbb{Z})$ as the set of matrices with trace zero and determinant one, with $U_1 \cong \mathrm{SL}_2(\mathbb{Z})$ acting by conjugation. Any matrix $A \in X$ is conjugate to $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, explaining again why X consists of a single orbit.

By localizing, we can talk about the set of places of k such that D_v is ramified. Class field theory shows over a number field k, a quaternion algebra is ramified at a finite set of places R, that $|R|$ is even, that any even set arises, and that R determines D up to isomorphism.

More abstractly, we can regard the data (a, b) as a pair of classes in $k^*/(k^*)^2 \cong H^1(k, \mathbb{Z}/2)$ in Galois cohomology, and the isomorphism class of D is the cup product $a \wedge b$ in the Brauer group $H^2(k, \mathbb{Z}/2)$. For the rationals, we have

$$H^2(\mathbb{Q}, \mathbb{Z}/2) \quad = \quad \{x \in \oplus_v (\mathbb{Z}/2)_v \ : \ \sum x_v = 0\}.$$

9 Double cosets and geometric configurations

In this section we introduce the method of *double cosets* for studying the geodesic and horocycle flow on hyperbolic surfaces, and more generally for studying the dynamics and representations of Lie groups.

We will use this method for 3 applications:

1. To prove minimality of the horocycle flow on a compact hyperbolic surface;

2. To study invariant vectors for unitary representations of $\mathrm{SL}_2(\mathbb{R})$ and, more generally, for $\mathrm{SL}_n(\mathbb{R})$; and

3. To study geodesic planes in hyperbolic 3–manifolds.

The first application is included at the end of this section; we will prove:

Theorem 9.1 (Hedlund) *Every horocycle on a compact hyperbolic surface is dense.*

A similar statement holds in the finite volume case, and will be proved as well.

Geometry and cosets. We begin with generalities. Let G be a Lie group and H a closed subgroup of G. Following Klein, one can regard G as the full symmetry group of a 'geometry', and H as the stabilizer of a particular geometric object. Assuming G acts transitively on objects of that type, the space of all such objects can be identified with G/H.

As a prime example, let

$$G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/(\pm I) = \mathrm{Isom}^+(\mathbb{H}).$$

Within G we have the group K, A and N. As we have seen, we have a naturally identification

$$G/K \cong \mathbb{H},$$

given by $[g] \mapsto g(i)$.

The semisimple group A is the stabilizer of the oriented *geodesic* $\gamma = i\mathbb{R}_+ \subset \mathbb{H}$ running from 0 to ∞. The space of all oriented geodesics can be naturally identified with

$$G/A = \{(x,y) \in \widehat{\mathbb{R}} \,:\, x \neq y\},$$

via $[g] \mapsto (g(0), g(\infty))$. In other words, an oriented geodesic is specified by a pair of distinct points

$$x, y \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}.$$

Finally, the unipotent group N is the stabilizer of *horocycle* $\eta = \{z : \mathrm{Im}(z) = 1\}$. The space of all horocycles can be naturally identified with

$$G/N = (\mathbb{R}^2 - \{0\})/((x,y) \sim (-x,-y)).$$

For this model, we think of g as a matrix; the map is given by

$$g \mapsto g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

The equivalence on $\mathbb{R}^2$ relation takes into account the fact that G is the quotient of $\mathrm{SL}_2(\mathbb{R})$ by $\pm I$.

Pairs of objects. The *configuration space* of pairs of geometric objects, corresponding to subgroups $H_1, H_2 \subset G$, is just the space

$$(G/H_1) \times (G/H_2).$$

The *moduli space* records to the geometric invariants of such a pair: it is given by taking the quotient of the space above by the group of isometries G. This moduli space can be naturally identified, via $(g_1, g_2) \mapsto g_2^{-1} g_1$, with the *double coset space*

$$H_1 \backslash G / H_2.$$

As we will see in examples, this moduli space is often *not Hausdorff.* Because of this, it is often more informative to try to visualize the *orbits* of H_1 acting on G/H_2, and regard the moduli space as the *space of orbits.*

Far from being a defect, the failure of the moduli space to be Hausdorff gives rise to important geometric and dynamical phenomena and explains their origin.

The case of $\mathbf{SL_2}(\mathbb{R})$. In the remainder of this section we will analyze the case of $\mathrm{SL}_2(\mathbb{R})$ in detail. To indicate the spirit of the results we will obtain, we formulate:

Theorem 9.2 *For $G = \mathrm{SL}_2(\mathbb{R})$:*

1. *Any continuous function $f : N\backslash G/N \to \mathbb{R}$ is constant on A;*
2. *Any continuous function $f : A\backslash G/A \to \mathbb{R}$ is constant on N; and*
3. *Any continuous function $f : AN\backslash G/AN \to \mathbb{R}$ is constant on G.*

Note that A represents infinitely many double cosets in case (1), and similar statements hold in the other cases. The point here is that these different double cosets cannot be separated by a real–valued function.

We now turn to a case–by–case analysis.

Pairs of points: $K\backslash G/K$. The moduli space of pairs of points in $\mathbb{H}$ turns out to be Hausdorff. Indeed, the distance function

$$d : K\backslash G/K \to [0,\infty),$$

is a homeomorphism. This case is particularly tame because K is compact.

From a dynamical perspective, $G/K = \mathbb{H}$, and the left action of K is given simply by rotation about the point i. Its orbits are circles centered at i in the hyperbolic metric.

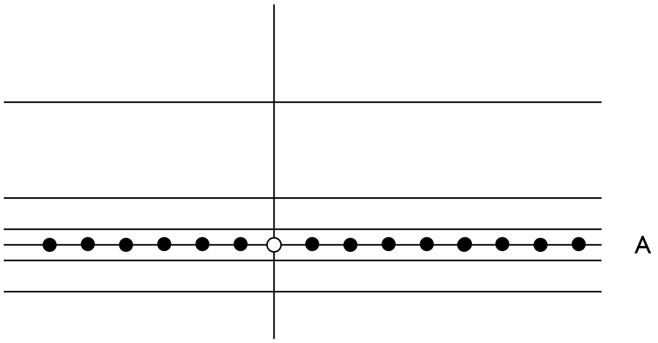


Figure 9. Orbits of $n_t(x,y) = (x+ty,y)$ on $\mathbb{R}^2 - \{(0,0)\}$.

Pairs of horocycles: $N\backslash G/N$. A richer example of a configuration space comes about when we consider pairs of horocycles. The corresponding coset space Ignoring the issue of $\pm I$ for the moment, this quotient describes the orbits of $(x,y) \mapsto (x+ty,y)$ on $\mathbb{R}^2 - \{0,0\}$. These orbits are of two types:

- *Lines* of constant height $y \neq 0$; and
- *Points* $(x,0)$ with $y = 0$.

See Figure 9.

Now recall that a horocycle η can be regarded as a circle with a center x at infinity. The first type of orbit corresponds to pairs of horocycles (η_1,η_2) with different centers x_1 and x_2. We can always normalize so that $x_1 = 0$, $x_2 = \infty$ and η_1 is a circle of radius 1. Then η_2 is given by $\operatorname{Im}(z) = y$, and the number $\log(y) \in \mathbb{R}$ can be regarded as the *renormalized distance* $D(\eta_1,\eta_2)$. This distance provides an approximate isomorphism

$$D : N\backslash G/N \dashrightarrow \mathbb{R}.$$

However this is not the full story. The *points* $(x, 0)$ in $\mathbb{R}^2$ correspond to horocycles with the *same* center. In this case, η_1 and η_2 are *parallel* in $\mathbb{H}$, and we can naturally associate to them a new invariant, the *signed distance* $\delta(\eta_1, \eta_2)$ between the horocycles themselves.

These parallel configurations arise in the limit as $D(\eta_1, \eta_2) \to -\infty$. They correspond to line orbits in $\mathbb{R}^2$ converging towards the point orbits on the x–axis. Moreover, it is clear that *every point on the x–axis* arises as a limit.

This important phenomena – that a sequence of configurations can have infinitely many limits – arises because $N\backslash G/N$ is not Hausdorff. It is the fundamental mechanism underlying many rigidity results.

Let us state this result geometrically. Note that when $D(\eta, \eta') < 0$, the horocycles η and η' cross one another with a certain angle θ. We choose this angle so that $\theta \to 0$ as $D(\eta, \eta') \to -\infty$.

Theorem 9.3 *Let $\eta_n \neq \eta'_n$ be a sequence of horocycles in $\mathbb{H}$ meeting in angles $\theta_n \to 0$. Pick $\delta \in \mathbb{R}$. Then we can find $g_n \in G$ such that $g_n \cdot (\eta_k, \eta'_k)$ converges to a pair of parallel horocycles, distance δ apart.*

Proof. This theorem has a very elementary proof. We can normalize so that η'_n is, for all n, simply our basic horocycle η, given by $\mathrm{Im}(z) = 1$. Its stabilizer N acts by on $\mathbb{H}$ $z \mapsto z + t$. The horocycles η_n have finite centers $x_n \in \mathbb{R}$ and Euclidean diameters $d_n \to \infty$, and all of them cross η.

If, for example, we want to we want to achieve $\delta = 0$, we simply translate η_n by an element of N such that one of its crossings is at $z = i$. Then the centers $x_n \to \infty$, and $\eta_n \to \eta$.

Similarly, if we want to achieve distance δ, we simply translate so that η_n crosses the imaginary axis at iy, where $y = \exp(\delta)$. Then η_n converges to the horocycle parallel to η given by $\mathrm{Im}(z) = y$, as desired. ∎

Now let us restate the same result in group–theoretic language. This type of statement is what one often finds in papers on homogeneous dynamics.

Theorem 9.4 *Let $g_n \to I$ in $G - AN$. Then for every $a \in A$, there exist sequences $u_n, u'_n \in N$ such that*

$$u_n g_n u'_n \to a.$$

Note that parallel horocycles correspond to the cosets NAN.

The Euclidean case. We remark that the preceding result has an analogue in Euclidean space, which is very easy to prove.

Theorem 9.5 *Let $L_n \neq L'_n$ be a sequence of lines in $\mathbb{R}^2$ meeting in angles $\theta_n \to 0$. Then for all $\delta > 0$, there exists a sequence of isometries $g_n \in \operatorname{Isom}(\mathbb{R}^2)$ such that $g_n \cdot (L_n, L'_n)$ converges to a pair of parallel lines distance δ apart.*

Proof. Normalize so the L'_n is the x–axis M for all n. Draw a circular arc of length δ with endpoints on M and L'_n, centered at the point where M and L'_n cross. Translate horizontally so that one endpoint of this arc is at the origin. Then in the limit, δ becomes a segment along the y–axis, perpendicular to both M and $M' = \lim L'_n$. ■

Pairs of oriented geodesics. When a pair of oriented geodesics cross, they do so at a well–defined angle $\theta(\gamma_1, \gamma_2) \in \mathbb{R}/2\pi\mathbb{Z}$; this is the angle you need to rotate γ_1 so it becomes γ_2. When they don't cross, there is a well–defined *signed* distance between them, $\delta(\gamma_1, \gamma_2)$, that takes into account their relative orientations. These quantities can in turn be related to the Minkowski inner product $\langle v_1, v_2 \rangle$ of vectors v_i on the 1–sheeted hyperboloid satisfying $v_i^\perp = \gamma_i$.

The main nuance in the configuration space $A\backslash G/A$ is that geodesics at distance zero need not coincide; they need only be *parallel.* This means that they share an endpoints on $\widehat{\mathbb{R}} = \partial\mathbb{H}$. Because of this, we have:

Theorem 9.6 *Suppose $\gamma_n = [x_n, y_n]$ is a sequence of geodesic with endpoints in $\mathbb{R}^*$, converging to the geodesic $[0, \infty]$. Then for any $x \in \mathbb{R}$, there exists a sequence $a_n \in A$ such that $a_n\gamma_n \to [x, \infty]$; and a sequence $b_n \in A$ such that $b_n\gamma_n \to [0, x]$.*

Indeed, we can just take $a_n(z) = z(x/x_n)$ and $b_n(z) = (x/y_n)$.

Proof of Theorem 9.2. (1) First we recall that G/N is identified with $\mathbb{R}^2/(\pm I)$ with G acting linearly as usual. The basepoint is taken to be $(1, 0)$, so the stabilizer of this basepoint is N. Thus A is sent to the orbit of $(1, 0)$ under the diagonal group, which is the locus $y = 0$ (the x–axis).

A continuous function $f : N\backslash G/N \to \mathbb{R}$ gives a continuous function on $\mathbb{R}^2 - \{(0, 0)\}$ that is constant along the horizontal lines with $y \neq 0$. It is therefore constant along the line $y = 0$, and hence constant on A.

(2) In the space of pairs of geodesics $f : A\backslash G/A \to \mathbb{R}$, the subgroup N collapses to two points: forward parallel geodesics, and identical geodesics. The second point corresponds to the identity element. Thus f must be constant on $N - \{\mathrm{id}\}$, and hence on N itself.

(3) A continuous function $f : AN\backslash G/AN \to \mathbb{R}$ gives a continuous map on the circle $\widehat{\mathbb{R}} = G/AN$ that is constant on the single AN–orbit $\mathbb{R}$. It is therefore constant on the whole space, G. ■

Other configurations. There are 3 other types of configurations one may wish to consider:

1. $K\backslash G/N$. This space is Hausdorff. There is only one invariant of a horocycle η and a point p: the signed distance $d(\eta, p) \in \mathbb{R}$. One can also visualize this space in terms of K acting on G/N: the orbits are circles in $\mathbb{R}^2 - \{(0,0)\}$.

2. $K\backslash G/A$. Again, there is only one invariant of an oriented geodesic and a point: the signed distance $d(\gamma, p) \in \mathbb{R}$. One can visualize this space as the orbits of A acting on $\mathbb{H} = G/K$.

3. $A\backslash G/N$. The last case is more interesting. It can be visualized as the orbits of A acting on $G/N = \mathbb{R}^2 - \{(0,0\}/(\pm I)$. These orbits are either hyperbolas or rays. The rays correspond to geodesics that begin or end at the 'center' of a horocycle.

Pairs of points at infinity. One can also consider configuration involving points on S^1, such as $K\backslash G/AN$, but these are always finite sets since S^1 is already 1–dimensional.

For example, $AN\backslash G/AN$ is the moduli space of pairs on points (p, q) on S^1_∞. There are only two different orbits, depend on whether $p = q$ or not. Put differently, if $g \notin AN$ then we have

$$(AN)g(AN) = G - AN.$$

Orbits of *AN*. We now turn to the study of topological dynamics on $\mathrm{T}_1 X$. We first recall:

Theorem 9.7 *If X has finite volume, then every AN orbit in $\mathrm{T}_1 X$ is dense.*

Proof. This is equivalent to the density of Γ–orbits on $G/AN = S^1_\infty$, which follows from Theorem 5.3 and Proposition 5.4. ■

Minimality of the horocycle flow. We conclude with the proof of Theorem 9.1, which we reformulate as follows.

Theorem 9.8 (Hedlund) *Let* $\mathrm{T}_1 X$ *be the unit tangent bundle to a compact hyperbolic surface* $X = \Gamma\backslash\mathbb{H}$*. Then every orbit of the horocycle flow on* $\mathrm{T}_1 X$ *is dense. Equivalently, every* N*–orbit in* $\Gamma\backslash G$ *is dense.*

Proof. By the axiom of choice, every orbit closure $\overline{xN}$ in $T_1 X$ contains a minimal set Z, i.e. a nonempty compact set where every N orbit is dense. It suffices to show that $Z = T_1 X$.

We have just seen that every AN orbit is dense. Thus if we can show that Z is A–invariant, we will be done.

Let

$$G_Z = \{g \in G \,:\, Zg \cap Z \neq \emptyset\},$$

and let

$$A_Z = G_Z \cap A.$$

Both sets are closed by compactness of Z.

Since A normalizes N, we find that Za is N–minimal for every $a \in A_Z$. But any two minimal sets that meet must coincide. Thus A_Z is a closed subgroup of A, and $ZA_Z = Z$. So to complete the proof, it suffices to show that A_Z contains elements of A arbitrarily close to the identity.

Since Z is N–invariant, we have $NG_Z N = G_Z$. In other words, we can analyze G_Z by the method of double cosets.

Since X is compact, it has no cusps, so there is no closed N–orbit in $\mathrm{T}_1 X$. This implies there exist $g_n \to \mathrm{id}$ in $G_Z - N$. By our analysis of $N\backslash G/N$, it follows that either (i) $g_n \in AN$ for infinitely many n; or (ii) given $a \in A$, we can choose $u_n, u_n' \in N$ such that

$$u_n g_n u_n' \to a \in A.$$

In case (ii) we are done, and in case (i) as well, since $g_n = a_n u_n$ with $a_n \to \mathrm{id}$ in A. ■

Parallel horocycles. Intuitively, by compactness of X any horocycle $\eta \subset \mathrm{T}_1 X$ is recurrent; thus its closure contains two nearly parallel horocycles. Passing to a limit we find the closure contains two exactly parallel horocycles; by minimality, this gives A–invariance which completes the proof.

Finite volume. We now turn to the case where $X = \Gamma\backslash\mathbb{H}$ has finite volume. In this case the cusps of X, if any, give rise to *closed orbits* for the horocycle flow. We will show:

Theorem 9.9 *If* $\mathrm{T}_1 X = \Gamma\backslash G$ *has finite volume, then any orbit of the horocycle flow is closed or dense.*

Proof. Let $Z = \overline{xN} \subset \mathrm{T}_1 X$ be the closure of an orbit of the horocycle flow. We now distinguish three cases.

(i) Z contains no closed orbits. It is easy to show there is a compact set $K \subset \mathrm{T}_1 X$ that meets every orbit of N that it not closed. From this it follows that Z contains a (nonempty) minimal set, and the proof proceeds as before.

(ii) Z itself is a closed orbit. Then we are done.

(iii) xN is not closed, but Z contains a closed orbit, say $W = yN \cong S^1$. We now consider

$$G_{WZ} = \{g \in G \ : \ Wg \cap Z \neq \emptyset\}.$$

Since W and Z are both N–invariant, and W is compact, we find that G_{WZ} is a closed collection of $N\backslash G/N$ cosets. Since W is not isolated in Z, there exist $g_n \to \mathrm{id}$ in $G_{WZ} - AN$. It follows that for any $a \in A$ we can find u_n, u_n' such that

$$u_n' g_n u_n \to a \in G_{WZ}.$$

In other words, $A \subset G_{WZ}$. But since A normalizes N, this implies that $W(AN) \subset Z$. Since every AN orbit is dense, this shows $W = \mathrm{T}_1 X$. ■

Spiraling and parallels. Put geometrically, this argument shows that once Z contains a closed horocycle W, we can find noncompact horocycles very close to it; passing to a limit, we can find parallels to W in Z at any prescribed distance. Putting all of these together, we obtain orbits of the geodesic flow inside Z, and use these to show Z is dense.

10 Unitary representations of simple Lie groups

In this section we revisit mixing of the geodesic horocycle flows for general representations of $\mathrm{SL}_2(\mathbb{R})$ and more general Lie groups, using the method of double cosets. For more on the Howe–Moore theorem, see [Zim].

Unitary representations.

In this section we study mixing for unitary representations of more general Lie groups, modeling the discussion on $G = \mathrm{SL}_2(\mathbb{R})$. The main result we will prove is:

Theorem 10.1 (Howe–Moore) *Let $\rho : G \to U(V)$ be a unitary representation of a simple Lie group G. If G is ergodic, then G is mixing.*

Here $U(V)$ is the group of unitary operators on a Hilbert space V. We say G is *ergodic* if it has no nonzero fixed–vectors; equivalently, if ρ does not contain the trivial representation; and we say G is *mixing* if for all $f \in V$, we have

$$\langle g_n f, f\rangle \to 0$$

as $g_n \to \infty$ in G. (This is equivalent to the requirement that $\langle g_n f_1, f_2\rangle \to 0$ for all $f_1, f_2 \in V$.)

Note that mixing implies ergodicity, except when G is compact.

Measurable dynamics. The terminology is of course borrowed from the case where G acts by measure–preserving transformations on a probability space (X, μ). In that case we take $V = L^2_0(X)$, and recover the usual notions of ergodicity and mixing.

One case where ergodicity is obvious is when G acts transitively. Thus the Theorem implies:

Corollary 10.2 *Let Γ be a lattice in G, and let $H \subset G$ be a non–compact subgroup. Then the action of H on $\Gamma\backslash G$ is ergodic sand mixing.*

As a special case, this shows the geodesic and horocycle flows on $\mathrm{T}_1 X = \Gamma\backslash G$ are mixing when $X = \Gamma\backslash\mathbb{H}$ is a finite volume hyperbolic surface.

Semisimple groups. Let us now explain the terminology. A Lie group G is *simple* if G/G^0 is finite, G^0 has finite center, and G^0 has no proper normal subgroup of positive dimension. (Here G^0 is the connected component of the identity).

Examples: $\mathrm{SL}_n\mathbb{R}$, $n \geq 2$; $\mathrm{SO}(n)$, $n \geq 3$; and $\mathrm{SO}(n,1)$, $n \geq 2$, are all simple. The groups $\mathrm{SO}(2)$ and $\mathrm{SO}(1,1)$ are not simple, because their centers are infinite.

A Lie group G is *semisimple* if G/G^0 is finite, and G^0 is finitely covered by a product of simple groups, $\prod G_i$. The Howe–Moore theorem can be extended to cover this case as well:

Theorem 10.3 *Let $\rho : G = \prod_1^n G_i \to U(V)$ be a unitary representation of a product of simple Lie groups G_i. Suppose each G_i acts ergodically. Then G is mixing.*

Mixing for $\mathbf{SL_2}(\mathbb{R})$. To begin the proof, we will treat the case $G = \mathrm{SL}_2(\mathbb{R})$. We begin by showing:

Theorem 10.4 *Let $G = \mathrm{SL}_2(\mathbb{R})$ act unitarily on V, and let $f \in V$. Then the following are equivalent:*

1. *The vector f is A–invariant.*
2. *The vector f is N–invariant.*
3. *The vector f is G–invariant.*

Proof. We may assume $\|f\| = 1$. Consider the continuous function $\phi(g) = \mathrm{Re}\langle gf, f\rangle$. Note that $gf = f$ if and only if $\phi(g) = 1$. Note also that if f is H–invariant for some subgroup $H \subset G$, then ϕ factors through the double coset space $H\backslash G/H$, since

$$\langle h_1 g h_2 f, f\rangle = \langle g h_2 f, h_1^{-1} f\rangle = \langle gf, f\rangle.$$

Applying Theorem 9.2, if f is A–invariant then $\phi|N$ is constant, which implies that f is N–invariant. Similarly N invariance implies A invariance, and AN invariance implies G invariance. ■

Proof of Theorem 10.1 for $\mathbf{SL_2}(\mathbb{R})$. The argument closely follows the discussion in §7. Let $\mathrm{T}_1 X = \Gamma\backslash G$ have finite volume, and let $V = L^2_0(\mathrm{T}_1 X)$. Clearly the action of G is ergodic. By the preceding theorem, the action of N is ergodic. Thus, by von Neumann's ergodic theorem, the averages $S_s f$ of $f \in V$ over an interval $[n_{-s}, n_s] \subset N$ satisfy $S_s f \approx f$ when s is small, and $S_s f \approx 0$ when s is large. We now use the fact that conjugation by a_t changes S_s into $S_{e^t s}$ to conclude that A is mixing. Then G itself is mixing, by the polar decomposition $G = KAK$. ■

Corollary 10.5 *Let* $\mathrm{SL}_2(\mathbb{R})$ *act by automorphisms on a probability space* (X, μ)*. If one of the groups* A*,* N *or* G *acts ergodically, then they all do.*

In this next section we will use this observation to prove the Teichmüller geodesic flow on $Q\mathcal{M}_g$ is mixing.

Fixed subspace. Given a subgroup $H \subset G$, and a unitary representation of G on V, we let

$$V^H = \{f \in V \,:\, hf = f \;\forall h \in H\}.$$

Mautner's Lemma. The passage from A to N invariance is an instance of *Mautner's Lemma.* The general principle is that if $a_n, h \in G$, $a_n f = f$, and

$$g_n = a_n h a_n^{-1} \to \mathrm{id},$$

then $hf = f$. For the proof, just observe that $\langle g_n f, f\rangle = \langle hf, f\rangle$ for all n. In particular this shows:

Theorem 10.6 *Let* $\rho : G = AN \to U(V)$ *be a unitary representation of* $AN \cong \mathbb{R}_+ \ltimes \mathbb{R}$*. Then* $V^A = V^G$*.*

Note that N is normal in AN. For normal subgroups we have a slightly weaker conclusion.

Proposition 10.7 *Let* N *be a normal subgroup of* G *which acts unitarily on* V*. Then* V^N *is* G*–invariant.*

Proof. Given $g \in G$, let $n' = gng^{-1}$ denote the induced automorphism of the normal subgroup N. Then for any $n \in N$ and $f \in V^N$, we have $n(gf) = (gn'f) = gf$, and hence $gf \in V^N$. ■

Structure of semisimple Lie groups: *KAK* and *KAN*. We now return to the general Howe–Moore theorem.

Let G be a connected semisimple Lie group. Then $G = KAK$, where K is a maximal compact subgroup and A is a maximal $\mathbb{R}$-torus. This is the *polar decomposition.*

Examples: for $G = \mathrm{SO}(n, 1)$, the isometries of $\mathbb{H}^n$, this says any two points in $\mathbb{H}^n = G/\mathrm{SO}(n)$ are joined by a geodesic. For $G = \mathrm{SL}_n \mathbb{R}$, this says any two unit volume ellipsoids (points in $\mathrm{SL}_n \mathbb{R}/\mathrm{SO}_n \mathbb{R}$) are related by a rotation and an affine stretch. (Note on the other hand that a typical solvable group cannot be expressed as KAK.)

We can similarly write $G = KAN$ where N is a maximal *unipotent* subgroup associated to A. This is the *Iwasawa decomposition.* For $\mathrm{SL}_n \mathbb{R}$, N is the group of upper-triangular matrices with 1's on the diagonal. For $\mathrm{SO}(n, 1)$, N is the group of horocycle flows for the horocycle at the positive end of the geodesic corresponding to A.

The case $G = \mathrm{SL}_n(\mathbb{R})$. We now prove the Howe–Moore theorem for $\mathrm{SL}_n(\mathbb{R})$, $n \geq 2$.

It is useful to consider the parabolic subgroup

$$P = \left\{ \begin{pmatrix} I_{n-1} & x \\ 0 & I_1 \end{pmatrix}, \ x \in \mathbb{R}^{n-1} \right\} \subset \mathrm{SL}_n \mathbb{R}.$$

Then

$$AP \cong (\mathbb{R}^*)^{n-1} \ltimes \mathbb{R}^{n-1}$$

is a solvable group, with A acting multiplicatively on P.

For any $1 < i \leq n$, we can embed $\mathrm{SL}_2 \mathbb{R}$ as $G_i \subset \mathrm{SL}_n \mathbb{R}$ so it acts on the subspace $\mathbb{R}e_i \oplus \mathbb{R}e_n$. Writing $G_i = K_i A_i N_i$, we note that $A_i \subset A$ and $N_i \subset P$. Generalizing Theorem 10.4, we have:

Theorem 10.8 *Given a unitary representation of $G = \mathrm{SL}_n(\mathbb{R})$, any P or A invariant vector f is also G invariant.*

Proof. Suppose f is P invariant. It is then N_i invariant for each i, since $N_i \subset P$. Applying the result for $\mathrm{SL}_2(\mathbb{R})$, we conclude that f is G_i–invariant. But the groups G_i generate G, so f is G–invariant. The argument for A is similar. ∎

Lemma 10.9 *Let $\rho : G \to U(V)$ be a unitary representation. Then for any $1 \leq i < n$, the subspace V^{A_i} of all vectors fixed by A_i is G–invariant.*

Proof. For $1 \leq j \neq k \leq n$, let e_{jk} denote the nilpotent matrix with its (j,k)th entry equal to one and the rest equal to zero, and let $U_{jk} \subset \mathrm{SL}_n(\mathbb{R})$ denote the unipotent subgroup of matrices of the form $I + te_{jk}$, $t \in \mathbb{R}$. Any diagonal matrix normalizes U_{jk}. Thus $A_i U_{jk}$ is a solvable group.

We claim V^{A_i} is invariant under U_{jk}. This is immediate (e.g. from Proposition 10.7) $A_i U_{jk}$ is abelian; otherwise, it follows from Mautner's Lemma, which shows $V^{A_i} = V^{U_{jk}}$. But the subgroups U_{jk} generate G, so G stabilizes V^{A_i}. ■

Theorem 10.10 *Any ergodic unitary representation of $G = \mathrm{SL}_n \mathbb{R}$ is mixing.*

Proof. Let $\rho : G \to U(V)$ be an ergodic unitary representation. Let $f \in V$ be a unit vector. Since $G = KAK$, to prove mixing it suffices to show that

$$\langle a_n f, f \rangle \to 0$$

as $a_n \to \infty$ in A.

By restricting ρ to the abelian subgroup $P \cong \mathbb{R}^{n-1}$, we obtain a decomposition of V into a Hilbert space bundle over $\widehat{P} \cong \mathbb{R}^{n-1}$ (see Appendix A). The unit vector f gives rise to a measure μ_f on $\widehat{P}$. Since A normalizes P, it acts on $\widehat{P}$, with the property

$$\mu_{af} = a_*(\mu_f).$$

First suppose μ is supported on a compact subset $K \subset \mathbb{R}^{n-1}$ disjoint from the coordinate hyperplanes. The set of a such that aK meets K itself forms a compact subset of A. Once A is outside this subset, μ_f and μ_{af} have disjoint support, so $\langle af, f \rangle = 0$, and hence A is mixing on f.

A similar argument applies provided μ_f assigns no mass to the coordinate hyperplanes; in this case, we can find a compact set K as above that carries most of the mass of μ_f, and again complete the argument.

What happens when μ_f assigns positive mass to the ith hyperplane in $\widehat{P}$, $1 \le i \le n$? This hyperplane is the kernel of the natural projection $\widehat{P} \to \widehat{N}_i$. This means f has a nonzero projection to the subspace V^{N_i}. In particular, V^{N_i} is nonempty.

By Theorem 10.4, we have $V^{N_i} = V^{A_i} = V^{G_i}$. By Lemma 10.9, this subspace is stabilized by the full group G, and thus we have a restriction homomorphism

$$\rho_i : G \to U(V^{G_i}),$$

including G_i in its kernel. But G is a connected simple Lie group, so $\mathrm{Ker}(\rho_i) = G$. This implies G has fixed vectors, contradicting ergodicity of ρ. ■

11 Dynamics on moduli space

We briefly recount the related theory of the $\mathrm{SL}_2(\mathbb{R})$ action of the bundle $Q\mathcal{M}_g$ of quadratic differentials (X, q) with $X \in \mathcal{M}_g$ and $q \in Q(X)$. Here $\dim Q(X) = 3g - 3$ by Riemann–Roch. The fact that $3g-3$ is also the maximum number of disjoint simple geodesics on a Riemann surface of genus g already hints at the close connection we will see between quadratic differentials and simple closed curves.

Any such differential can be presented in the form $(X, q) = (P, dz^2)/\sim$ with P a polygon in $\mathbb{C}$, and the action is given by

$$A \cdot (X, q) = (A(P), dz^2).$$

A beautiful feature of this action is that we have a natural map

$$Q\mathcal{M}_g \to \mathcal{ML}_g \times \mathcal{ML}_g$$

sending (X, q) to the pair of foliations $(\alpha, \beta) = (\mathcal{F}(q), \mathcal{F}(-q))$. We have $i(\alpha, \beta) = \int_X |q|$. In these coordinates the geodesic flow acts by sending (α, β) to $(t\alpha, (1/t)\beta)$, $t \in \mathbb{R}_+$. Thus geodesics in unit sphere bundle of $Q\mathcal{M}_g$ can be identified with points in

$$\mathbb{P}\mathcal{ML}_g \times \mathbb{P}\mathcal{ML}_g.$$

Using the Hopf argument (with some added finesse, because not *all* geodesic rays with the same endpoint are asymptotic), we obtain:

Theorem 11.1 (Masur,Veech) *The Teichmüller geodesic flow on $Q\mathcal{M}_g$ is ergodic.*

Here we have not explained the invariant measure; it comes from periods and is invariant under $\mathrm{SL}_2(\mathbb{R})$ as well. Due to the basic theory of unitary representation of $\mathrm{SL}_2(\mathbb{R})$ we have already discussed, an immediate corollary is:

Theorem 11.2 *The geodesic flow on $Q\mathcal{M}_g$ is mixing. In fact the full $\mathrm{SL}_2(\mathbb{R})$ action is mixing.*

Curve systems. The connection to elements of $\mathcal{ML}_g$ can be made simple and explicit in the case of differentials such that both q and $-q$ are *Strebel*, that is they decompose the surface into cylinders.

Start with $\alpha = \sum a_i A_i$ and $\beta = \sum b_j B_j$, systems of disjoint simple curves on Σ_g with positive real weights. Assume these curves *bind* the surface, in the sense that the complementary regions are disks. At each point of intersection between A_i and B_j, introduce a rectangle with dimensions $a_i \times b_j$. These glue together to give a Riemann surface X and a quadratic differential q. Each complementary $4+2n$–gon gives rise to a zero of multiplicity n for q.

Affine dynamics. Using this construction one can find quadratic differentials whose stabilizers $\mathrm{SL}(X,q)$ are large. To this end, first suppose q decomposes into cylinders C_i with heights and circumferences h_i and c_i. Their moduli are given by $m_i = h_i/c_i$. If all these moduli are equal to a single constant $m > 0$, then:

$$\begin{pmatrix} 1 & 1/m \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(X,q).$$

More generally, if the moduli are given by $(m_i) = m(n_i)$, with n_i relatively prime integers, then the same is true. The resulting affine transformation is a product of Dehn twists, $\prod \tau_i^{n_i}$. Higher cylinders are twisted more.

Now let us return to a curve system (A_i, B_j) and try to arrange that the moduli of the vertical and horizontal cylinders are all the same. For this it is useful to put the curves together into a single system C_i, let the weights (heights) be h_i, and let $J_{ij} = i(C_i, C_j)$. Then for any choice of heights we have

$$c_i = \sum_j J_{ij} h_i.$$

So if we would like $1/m_i = c_i/h_i = \mu > 0$ to be a positive constant, we need to solve the eigenvalue equation

$$\mu h_i = c_i = \sum_j J_{ij} h_i.$$

By the Perron–Frobenius theorem, there is a unique solution up to scale. With these weights we obtain a quadratic differential such that

$$\Gamma = \langle \left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ -\mu & 1 \end{smallmatrix}\right) \rangle \subset \mathrm{SL}(X,q).$$

These elements are represented by right Dehn multitwists τ_A and τ_B. Note that

$$\operatorname{tr}(\tau_A \tau_B^{-1}) = 2 + \mu^2 > 2,$$

so this product gives a *pseudo–Anosov* automorphism of X. Similarly

$$\operatorname{tr}(\tau_A \tau_B) = 2 - \mu^2$$

will give a pseudo–Anosov map if $\mu^2 > 2$.

General pseudo–Anosov maps. Remarkably, any irreducible mapping–class f of infinite order can be represented by an essentially unique pseudo–Anosov map. That is, these elements of $\pi_1(\mathcal{M}_g)$ are the projects of closed A–orbits. By the Hopf argument applied to the stable and unstable manifolds of f, we obtain:

Theorem 11.3 *Any pseudo–Anosov map is ergodic.*

In fact these maps are mixing and their foliations are uniquely ergodic, just as for the torus case. More generally, if the Teichmüller ray determined by $\mathcal{F}(q)$ is recurrent in $\mathcal{M}_g$, then the foliation $\mathcal{F}(q)$ is uniquely ergodic (Masur).

The golden mean and the regular pentagon. The simplest curve system is the A_4 diagram in genus two. In general a curve system gives rise to a rectangular intersection matrix K, indexed by the A_i's and the B_j's. We then have $\mu^2 = \rho(KK^t)$. This makes computations easy. For A_4 we have $K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and thus

$$\mu^2 = \rho(KK^t) = \gamma^2 = \frac{3+\sqrt{5}}{2}.$$

Thus μ is the golden mean. Since $\mu < 2$ the group Γ is a lattice, in fact it is the $(2, 5, \infty)$ triangle group. To see we have an element of order 5, note that

$$|\operatorname{tr}(\tau_A \tau_B)| = \mu^2 - 2 = \gamma^{-1},$$

and $\gamma^{-1} = 2\cos(2\pi/5) = \zeta_5 + \zeta_5^{-1}$.

The golden table. The differential q can be realized using the symmetric L–shaped polygon with long sides of length γ and short sides of length 1.

This corresponds to the curve $y^2 = p(x)$ where the roots of $p(x)$ are at the 5 points

$$\zeta_5^k - \zeta_5^{-k} = \sin(2\pi k/5), \quad k = 1, 2, \ldots, 5.$$

The regular pentagon. We can deduce by symmetry that the order 5 point in the $\mathrm{SL}_2(\mathbb{R})$ orbit of (X, q) corresponds to the algebraic curve $y^2 = x^5 - 1$, with q the differential dx^2/y^2. Its unique zero comes from the point at infinity.

Using the fact that $\mathrm{SL}(X, q)$ is a lattice, and the unique ergodicity criterion of Masur, one can then deduce:

Theorem 11.4 *Billiards in a regular pentagon have optimal dynamics. That is, every trajectory is either periodic or uniformly distributed.*

Weighted curve systems. We can also attach positive integer weights to each of the curve A_i and B_j, and then τ_A and τ_B become products of multitwists. If the weights are given by diagonal matrices W_A and W_B, and the incidence matrix is K, then

$$\mu^2 = \rho(W_A K W_B K^t).$$

For example, with the weights $(3, 1, 1, 1)$ on the A_4 diagram we obtain

$$\mu^2 = \rho\left(\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right) = \rho\left(\begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}\right) = \frac{5 + \sqrt{13}}{2} = 4.30277\ldots$$

Thus $4 - \mu^2$ is just less than -2, and in fact $\tau_A \tau_B$ gives the shortest geodesic on $\mathcal{M}_2$. (For $g \geq 10$ the length of the shortest geodesic on $\mathcal{M}_g$ is unknown.) We remark that

$$\mu^2 - 4 = \frac{1 + \sqrt{13}}{2} = \lambda + \lambda^{-1},$$

where λ is the smallest Salem number of degree 4 (satisfying $x^4 - x^3 - x^2 - x + 1 = 0$).

12 Unique ergodicity of the horocycle flow

In this section we establish unique ergodicity of the horocycle flow on a compact surface, as well as some related rigidity results. In the next section we will see these results are special cases of Ratner's general theorems.

Rigidity of the horocycle foliation. Unlike the geodesic foliation, the *topology* of the horocycle foliation *changes* whenever we vary the metric on X.

Theorem 12.1 (Marcus, [Mrc]) *Let X and Y be closed hyperbolic surfaces. If the horocycle foliations of T_1X and T_1Y are topologically conjugate, then X and Y are isometric.*

Sketch of the proof. Let $f : T_1X \to T_1Y$ be a homeomorphism respecting the horocycle foliations, and let $F : T_1\mathbb{H} \to T_1\mathbb{H}$ be its lift to the unit tangent bundle of the universal covers of X and Y. (This lift exists because f preserves the *center* of $\pi_1(T_1X)$ which corresponds to the fibers.) Then F is a quasi-isometry, it conjugates Γ_X to Γ_Y, and it sends horocycles to horocycles.

Now two horocycles rest on the same point on S^1_∞ if and only if they are a bounded distance apart. Since F is a quasi-isometry, it respects this relationship between horocycles.

Moreover F descends to a map between the *spaces* of horocycles, which is naturally $\mathbb{R}^2$ with the groups Γ_X and Γ_Y acting linearly. Thus we have a homeomorphism

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

such that (i) f sends rays to rays and (ii) $f(\gamma v) = \gamma' f(v)$ for all $\gamma \in \Gamma_X$.

Because of (i), we have a function $T(t, v)$ on $\mathbb{R}_+ \times \mathbb{R}^2$ characterized by

$$f(tv) = T(t, v)f(v).$$

Since f is continuous, so is $T(t, v)$. Moreover, since the actions of Γ_X and Γ_Y on $\mathbb{R}^2$ are *linear*, for each fixed t we have $T(t, \gamma v) = T(t, v)$. Since Γ has a dense orbit on $\mathbb{R}^2$ (e.g. by ergodicity), this implies $T(t, v)$ depends only on t. Then from the definition we have

$$T(st) = T(s)T(t),$$

and thus $T(t) = t^\alpha$ for some $\alpha \in \mathbb{R}_+$.

Now consider the expanding eigenvector v for $\gamma \in \Gamma_X$, satisfying $\gamma v = \lambda v$ with $\lambda > 1$. It is related to the hyperbolic length of the corresponding geodesic on X by $2 \log \lambda = L(\gamma)$. On the other hand, $v' = f(v)$ must be the expanding eigenvector for γ', with eigenvalue

$$\lambda' = T(\lambda) = \lambda^\alpha.$$

This shows lengths on X and Y are related by

$$L(\gamma') = \alpha\, L(\gamma);$$

i.e. corresponding geodesics on X and Y have the proportional lengths.

But it is known that the number of geodesics with length $\ell(\gamma) \le L$ on a compact hyperbolic surface grows like e^L/L. Therefore $\lambda = \pm 1$ and we have shown that corresponding geodesics have the same length. The length function on $\pi_1(X)$ determines the metric on X uniquely, so we are done. ■

Topological invariants of the horocycle foliation. Here is a useful exercise: just given the topological action of Γ on $\mathbb{R}^2$, and $g, h \in \Gamma$, determine if $L(g) > L(h)$. For the solution, let α and β be the eigenvalues of g and h, and observe that the eigenvectors of the *conjugates* of g are dense in $\mathbb{R}^2$. Thus the expanding eigenvector b for g_2 is a limit of expanding eigenvectors a_n for g_1. We can then compare the limit of the segments $[a_n, \alpha a_n]$ to $[b, \beta b]$ and see which one contains the other.

Unique ergodicity of the horocycle flow. We now come to a strengthening of Hedlund's topological result.

Theorem 12.2 (Furstenberg [Fur2]) *The horocycle flow on a closed hyperbolic surface X is uniquely ergodic.*

We already know the horocycle flow is ergodic; now we want to show *every* orbit is distributed the same way that *almost* every orbit is.

Irrational rotations. To convey the spirit of the argument, we first reprove unique ergodicity of an irrational rotation $T : S^1 \to S^1$. Consider an interval $I = [a, b] \subset S^1$. Given $y \in S^1$, we need to show

$$A_n(y, I) = \frac{|\{i \;:\; T^i(y) \in I, 1 \le i \le n\}|}{n} \to m(I).$$

Fix any $\epsilon > 0$. Consider intervals $I' \supset I \supset I''$, slightly larger and smaller by ϵ. By ergodicity, $A_n(x, I) \to m(I)$ for almost every x, and similarly for I' and I''. Thus we can find a set of measure at least $1 - \epsilon$ and an N such that $|A_n(x) - m(I)| < \epsilon$ for all $x \in E$ and $n \ge N$, and similarly for I' and I''.

Since $m(E) > 1 - \epsilon$, there is a point $x \in E$ such that $|x - y| < \epsilon$. Then the orbits of these points are close: $|T^i x - T^i y| < \epsilon$ for all i. Therefore when $n > N$ we have

$$A_n(y, I) < A_n(x, I') < m(I') + \epsilon < m(I) + 2\epsilon,$$

and a reverse bound also holds by considering I''.

Therefore $A_n(y) \to m(I)$ and T is uniquely ergodic.

Proof of unique ergodicity of the horocycle flow. (Adapted from Ratner and [Ghys, §3.3].) Through any x in T_1X, consider the flow box $Q(x, s_-, s_+, t)$ obtained by first applying the negative, then the positive horocycle flows for time $[-s_-, s_-]$ and $[-s_+, s_+]$ respectively, and then applying the geodesic flow for time $[-t, t]$. Since X is compact, when the parameters s_-, s_+, t are small enough we have $Q(x, s_-, s_+, t)$ embedded in T_1X for any x.

Fix one such flow box Q_0. Let $H(x,t) = h_+^{[0,t]}(x)$ denote horocycle flow line of length t starting at x. Then by ergodicity of the horocycle flow with respect to Lebesgue measure $m(\cdot)$, we have

$$\lim \frac{\text{length}(H(x,t) \cap Q_0)}{t} = m(Q_0)$$

for almost every x. Moreover the limit is almost uniform: for any ϵ there exists a T and an $X_0 \subset T_1X$, $m(X_0) > 1 - \epsilon$, such that

$$\left| m(Q_0) - \frac{\text{length}(H(x,T) \cap Q_0)}{T} \right| < \epsilon$$

for all $x \in X_0$. We can also ensure that the same statement holds for slightly smaller and larger boxes, $Q_0' \supset Q_0 \supset Q_0''$.

Now consider any point $y \in T_1X$. We will show

$$\frac{1}{T} m\{t \in [0,T] \,:\, h_+^t(y) \in Q_0\} \approx m(Q_0)$$

as well, the approximation becoming better as $T \to \infty$.

To this end, transform the whole picture by applying the geodesic flow $g^{\log T}$, which compresses the positive horocycle flow by the factor T. Under the geodesic flow we have

$$g^u Q(x, s_-, s_+, t) = Q(g^u x, e^u s_-, e^{-u} s_+, t),$$

and

$$g^u H(x,t) = H(g^u x, e^{-u} t).$$

Thus

$$Q_1 = g^{\log T} Q_0 = Q(x', T s_-, s_+/T, t),$$

and $H(y,T)$ is transformed to a unit segment $H(y',1)$. Finally X_0 is transformed to a set X_1, still of measure $1-\epsilon$, such that the length of the part of $H(x,1)$ inside Q_1 is almost exactly $m(Q_1)=m(Q_0)$.

Since X_1 almost has full measure, there is an $x \in X_1$ very close to y'. Now $H(x,1)$ consists of very many short segments (of length about s_+/T) inside the highly flattened box Q_1. Moving x slightly to y', $H(x,1)$ moves slightly to $H(y',1)$. The edge effects can be ignored by the fact that $H(x,1)$ also works for slightly larger and smaller boxes. More precisely, the length of $H(y,1)\cap Q_1$ is bounded above by the length of $H(x,1+\epsilon)\cap Q_1' \supset Q_1$, and there is a similar lower bound. Thus the lengths of $H(x,1)\cap Q_1$ and $H(y',1)\cap Q_1$ is almost the same, so the horocycle orbit through y is equidistributed. ■

Note: In this last step we have used the fact that the injectivity radius of X is bounded below to study the local picture of Q_1 and $H(x,1)$.

Slow divergence. Here is an explanation of Ratner's proof without applying $g^{\log T}$.

Consider an arbitrary point $y \in T_1Y$ and a set of good points $E \subset T_1Y$, of measure $1-\epsilon$. We want to show the h_+-orbit, $H(y,T)$, cuts Q_0 in about the same length as $H(x,T)$.

Now any set F with $m(F)>\epsilon$ meets E. The key idea is *not* to take F to be a ball around y. Instead, F is taken to be a long, narrow region around the segment $H(y,S)$, where $S=\delta T$ and δ is small. The point is that we are willing to sacrifice $H(x,T)$ following $H(y,T)$ for time $2S$, as long as we can get the two segments to be close for the rest of the time.

More precisely, F is taken to have dimension δ in the g^t direction, and δ/T in the h_--direction. The latter two conditions insure that $H(x,T)$ follows $H(y,T)$ to within $O(\delta)$. Now the total volume of F is about $S\times\delta\times\delta/T=\delta^3$, which does not depend on T. Thus when ϵ is small enough, F meets E and we are done.

To make this rigorous, we have to show F maps *injectively* into the unit tangent bundle. For this it is useful to flow by $g^{\log T}$; then the dimensions of F are independent of T.

The case of finite area. We just state the results:

Theorem 12.3 (Dani [Dani]) *Any ergodic measure on T_1Y, invariant under the horocycle flow, is either Liouville measure or concentrated on a single closed horocycle.*

Theorem 12.4 (Dani and Smillie [DS]) *Any horocycle is either closed, or equidistributed in $T_1 Y$.*

See [Dani] and [DS] for details.

Example. Consider $\Gamma = \mathrm{SL}_2 \mathbb{Z}$ acting on $\mathbb{R}^2 - \{0\} = G/N$. Then the orbit of a vector $(x, y) \in \mathbb{Z}^2$ is discrete; more generally, the orbit is discrete if $x/y \in \mathbb{Q}$. Otherwise, the orbit is dense.

Proof. Let μ be an invariant measure and let y be a point whose orbit is distributed according to μ. Then the preceding proof works to show $\mu = m$, so long as $y' = g^{\log T}(y)$ is recurrent as $T \to \infty$ (this recurrence allows us to work in a fixed compact subset of X, where the injectivity radius is bounded below). But if the horocycle through y is not recurrent, it is closed. Thus the only other ergodic invariant measures are uniform length measure supported on closed orbits. ■

<table>
<tr><th colspan="2">Dynamics on compact hyperbolic surfaces</th></tr>
<tr><th>Geodesic flow</th><th>Horocycle flow</th></tr>
<tr><td>Ergodic</td><td>Ergodic</td></tr>
<tr><td>Mixing</td><td>Mixing</td></tr>
<tr><td>Countably many closed orbits</td><td>No closed orbits</td></tr>
<tr><td>Not minimal</td><td>Minimal (all orbits dense)</td></tr>
<tr><td>Positive entropy</td><td>Zero entropy</td></tr>
<tr><td>Topology of orbits depends only on genus of Y</td><td>Topology of orbits determines geometry of Y</td></tr>
<tr><td>Many ergodic measures</td><td>Uniquely ergodic</td></tr>
</table>

Table 10.

In conclusion, various properties of the geodesic and horocycle flows that we have established are collected in Table 10.

13 Ratner's theorem and the Oppenheim conjecture

In this section we state Ratner's general theorem about unipotent actions, and apply it to prove the Oppenheim conjecture on quadratic forms. For more details, see e.g. [Ghys], [Rn], [Mg] and [BM, Ch. VI].

Some interesting applications of Ratner's theorem are given in [Mc4, §2], [ElM], and [Kap].

Statement of Ratner's theorem. We have seen that the horocycle flow on a finite volume surface obeys considerable rigidity, compared to the geodesic flow. One of the striking properties of the horocycle flow is that we can describe *every orbit closure* $\overline{xN} \subset \mathrm{T}_1 X$: either xN is already a closed orbit, or xN is dense.

In contrast, for the geodesic flow, the Hausdorff dimension of an orbit closure, $\mathrm{H.\dim}\, \overline{xA}$ can assume any value in $[1, 2]$. While it is possible to describe the behavior of *almost every* geodesic in some detail, *particular* geodesics exhibit complicated, fractal behavior.

Ratner's theorem, initially conjectured by Raghunathan, shows this rigidity of unipotent dynamics persists in a very general homogeneous setting.

Theorem 13.1 (Ratner) *Let $\Gamma \subset G$ be a lattice in a connected Lie group G, and let $U \subset G$ be a subgroup generated by unipotent elements. Then*

> *(a) The closure of any U–orbit in $\Gamma\backslash G$ is a finite-volume J orbit, for some Lie group J with where $U \subset J \subset G$.*
>
> *(b) Any U-invariant ergodic probability measure on Z is given by Haar measure on a finite–volume J orbit, where J is as above.*

Moreover, if U is a 1-parameter group, then any orbit xU is uniformly distributed with respect to the natural Haar measure on $\overline{xU}$ given by (a) and (b).

Here $g \in G$ is *unipotent* if its adjoint action has 1 as its only eigenvalue.

The finite volume condition in case (a) means

$$\Gamma_J = J \cap x^{-1}\Gamma x$$

is a lattice in J, and we have $xJ \cong \Gamma_J\backslash J$. In particular, J must be a unimodular group (since it contains a lattice).

First cases. The horocycle flow on a finite volume hyperbolic surface $X = \Gamma\backslash\mathbb{H}$ is the first interesting case of Ratner's theorem. In this case $G = \mathrm{SL}_2(\mathbb{R})/(\pm I)$, $\Gamma\backslash G = \mathrm{T}_1 X$, and $U = N$.

The topological part (a) says that any orbit of the horocycle flow is closed or dense; the only possibilities are $J = N$ and $J = G$. (The case $J = AN$ is ruled out by the fact that J is unimodular.)

Tthe measure–theoretic part (b) says the ergodic measures comes from closed horocycles and Liouville measure; and the final statement says that every horocycle is closed or uniformly distributed.

Diophantine problems and homogeneous forms. Next we set the stage for the Oppenheim conjecture. We will use the terminology for quadratic forms Q on $\mathbb{R}^n$ introduced in §6; in particular,

$$m(Q) = \inf\{|Q(x)| \,:\, 0 \neq x \in \mathbb{Z}^n\}.$$

Clearly $m(Q) \geq 1$ if Q is an integral quadratic form that does not represent zero. Here is a converse (now proved):

Conjecture 13.2 (Oppenheim) *Let $Q(x)$ be a quadratic form of indefinite signature (p, q) on $\mathbb{R}^n$, $n \geq 3$. Then $m(Q) > 0$ if and only Q is a multiple of an integral form.*

By Theorem 6.5, we have the following equivalent formulation:

> *A real quadratic form satisfies $m(Q) > 0$ if and only if* $\mathrm{SO}(Q, \mathbb{Z})$ *is a cocompact lattice in* $\mathrm{SO}(Q, \mathbb{R})$.

By straightforward slicing arguments, it suffices to prove this theorem for $n = 3$ and $(p, q) = (2, 1)$. The proof will pivot on Ratner's theorem, with $U = \mathrm{SO}(Q, \mathbb{R})$.

The case of $\mathbb{R}^2$. The Oppenheim conjecture is false for $n = 2$. Let us discuss this case first.

For an indefinite form on $\mathbb{R}^2$, we have $\mathrm{SO}(Q, \mathbb{R}) \cong \mathrm{SO}(1, 1) \cong A \subset \mathrm{SL}_2(\mathbb{R})$. This group is not generated by unipotents! Thus Ratner's theorem does not apply.

Instead, we have a correspondence between quadratic forms and geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. This correspondence is easy to make explicit: the geodesic γ_{ab} with endpoints (a, b) corresponds to the quadratic form

$$Q_{ab}(x, y) = (x - ay)(x - by).$$

As we have seen in §6 that the condition $m(Q_{ab}) > 0$ is equivalent to the condition that the project of the geodesic γ_{ab} to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is bounded, i.e. it does not accumulate on the cusp.

In general $m(Q)$ may be difficult to determine. For example, $m(Q_a)$ is *unknown* for the form

$$Q_a(x, y) = x^2 - a^2y^2,$$

when $a = 2^{1/3}$, although it is expected that $m(Q_a) = 0$.

The following are equivalent:

1. $m(Q_a) > 0$.
2. The continued fraction expansion $a = a_0 + 1/a_1 + 1/a_2 + \cdots$ is bounded ($\max a_i < \infty$), and a is irrational.
3. There exists a constant $C > 0$ such that $|a - p/q| > C/q^2$ for all rationals p/q.
4. The geodesic ray $\gamma \subset \mathbb{H}$ with endpoints $[-a, a]$ has a bounded image on the modular surface $\Gamma\backslash\mathbb{H}$.

Numbers with this property are said to be of *bounded type*; they form a subset $B \subset \mathbb{R}$ of measure zero but Hausdorff dimension two. In particular, B is uncountable.

The asymptotic behavior of a geodesic ray only depends on its endpoint in $\partial\mathbb{H}$. Consequently:

Theorem 13.3 *We have $m(Q_{ab}) > 0$ if and only if a and b are both irrationals of bounded type.*

Corollary 13.4 *In dimension $n = 2$, there are many indefinite quadratic forms with $m(Q) > 0$ that are not multiples of integral forms.*

The case $n = 3$. We now study the case of forms Q of signature $(2, 1)$.

Let $G = \mathrm{SL}_3(\mathbb{R})$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, and $H = \mathrm{SO}(2, 1)$. By mixing of the action of H on $\Gamma\backslash G$, we immediately obtain:

Theorem 13.5 *For almost every indefinite quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, $m(Q) = 0$. In fact $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}^n$.*

Thus forms with $m(Q) > 0$ are rare — but can we classify them?

In this case, Ratner's theorem implies:

Theorem 13.6 *Any $H = \mathrm{SO}(2,1)$ orbit on $\Gamma\backslash G = \mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})$ is closed or dense. In the closed case, xH has finite volume, and x is defined over $\mathbb{Q}$.*

This result was first proved by Dani and Margulis. It yields the Oppenheim conjecture, in the following stronger form:

Corollary 13.7 *Let Q be a real quadratic form of signature $(2,1)$. Then either Q is proportional to an integral form, or $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.*

Recognizing integral forms. To prove the Oppenheim conjecture, we need to understand the case where xH is closed.

Lemma 13.8 *Suppose Q is a real quadratic form of signature $(2,1)$, and $\mathrm{SO}(Q,\mathbb{Z})$ is a lattice in $\mathrm{SO}(Q,\mathbb{R})$. Then Q is proportional to an integral form.*

Proof. Let $V \subset \mathrm{Sym}(\mathbb{R}^3)$ be the subspace of quadratic forms F invariant under $\mathrm{SO}(Q,\mathbb{Z})$. Since the matrices in $\mathrm{SO}(Q,\mathbb{Z})$ have integral entries, the linear equations $g^tFg = F$ defining V are rational, i.e. V is defined over $\mathbb{Q}$.

We claim that V is one-dimensional; in other words, that $\mathrm{SO}(Q,\mathbb{Z})$ determines Q up to scale. To see this, one can note that the lattice $\mathrm{SO}(Q,\mathbb{Z})$ gives a Kleinian group acting on $\mathbb{RP}^2$, whose limit set is the ellipse C defined by $Q = 0$. Any other homogeneous quadratic equation F defining C must be proportional to V.

Alternatively, the Borel density theorem states that a lattice in a semisimple Lie group with no compact factor is Zariski dense; thus any group $\mathrm{SO}(F,\mathbb{R})$ containing $\mathrm{SO}(Q,\mathbb{Z})$ must equal $\mathrm{SO}(Q,\mathbb{R})$, and hence $F \in \mathbb{R}\cdot Q$.

Since V is one dimensional and rational, some real multiple of $Q \in V$ must have integral coefficients. ■

Proof of Theorem 13.6. There is no closed, connected Lie subgroup between $H = \mathrm{SO}(2,1,\mathbb{R})$ and $G = \mathrm{SL}_3(\mathbb{R})$. Thus for any $x \in \Gamma\backslash G$, xH is either closed or dense. In the closed case, the corresponding quadratic form Q has the property that

$$\mathrm{SO}(Q,\mathbb{Z}) \cong H \cap x^{-1}\Gamma x \subset \mathrm{SO}(Q,\mathbb{R}) \cong H$$

is a lattice, and hence Q is proportional to a integral form by the preceding Lemma. ■

An analogue of the Oppenheim conjecture for products on linear forms on $\mathbb{R}^3$ is still open, although much progress has been made. See Conjecture 25.1 below.

14 Geodesic planes in hyperbolic 3–manifolds

In this section we continue the analysis of double cosets to prove [Sh]:

Theorem 14.1 (Shah) *A totally geodesic plane in a hyperbolic 3–manifold is either closed or dense.*

This result is a special case of Ratner's theorem, and the proof serves to illustrate some of the ideas behind the general theory.

As usual, one obtains density not just in the 3-manifold but in the frame bundle. The proof we describe is modeled on [MMO, §8, §9].

Configurations in $\mathbb{H}^3$. We begin by studying planes and horocycles in hyperbolic 3–space.

The boundary of hyperbolic space $\mathbb{H}^3$ can be naturally identified with the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The metric geometric of $\mathbb{H}^3$, when rescaled, becomes conformal geometry on $\partial\mathbb{H}^3$. Indeed, we can identify the isometry group of $\mathbb{H}^3$ with the conformal automorphism group of $\widehat{\mathbb{C}}$: we have

$$G = \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C}).$$

Within G the following subgroups will play an important role:

$$\begin{aligned}
H &= \mathrm{PGL}_2(\mathbb{R}), \\
A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \ : \ a \in \mathbb{R}_+ \right\}, \\
K &= \mathrm{SU}(2)/(\pm I) \cong \mathrm{SO}(3), \\
N &= \left\{ n_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \ : \ s \in \mathbb{C} \right\}, \\
U &= \{n_s \ : \ s \in \mathbb{R}\}, \quad \text{and} \\
V &= \{n_s \ : \ s \in i\mathbb{R}\}.
\end{aligned}$$

Note that the groups $N = UV$ and AN is the group of affine automorphism of $\mathbb{C}$, i.e. the map $g(z) = az + b$. The group K is the isometric group of $\widehat{\mathbb{C}}$

in its round metric $2|dz|/(1+|z|^2)$ of constant curvature 1; equivalently, it is the stabilizer of the 'center' of the hyperbolic ball $\mathbb{H}^3$, where the geodesic from 0 to ∞ crosses the equatorial plane spanning $S^1 \subset \mathbb{C}$.

Homogeneous spaces. Each of these subgroups has an associated homogeneous space. For example $\mathbb{H}^3 = G/K$, with G acting isometrically. Since AN is the stabilizer of ∞, we can write

$$S^2_\infty = \widehat{\mathbb{C}} = G/AN.$$

The quotient $\mathcal{C} = G/H$ can be regarded as the space of oriented circles on $\widehat{\mathbb{C}}$, since H is the stabilizer of $\widehat{\mathbb{R}}$ as an oriented circle.

A discrete group $\Gamma \subset G$ yields a quotient hyperbolic manifold or orbifold,

$$M = \Gamma\backslash\mathbb{H}^3 = \Gamma\backslash G/K,$$

and we can identify $\gamma\backslash G$ itself with the oriented frame bundle $\mathrm{F}M \to M$. Note that the fibers of FM are copies $K \cong \mathrm{SO}(3)$.

The geodesic flow on $\mathrm{F}M$ is given by the (right) action of A. Similarly, the orbits of H describe immersed, totally geodesic planes in M, lifted to the frame bundle. The orbits of N are horospheres; and the orbits of U are horocycles.

Configuration spaces. Let us now describe the non–Hausdorff geometry of the three configuration spaces that relate to horocycles and planes.

I. Pairs of planes. The simplest is the space $H\backslash G/H$, which describes pairs of circles C_1 and C_2 on $\widehat{\mathbb{C}}$, or equivalently pairs of planes P_1 and P_2 in $\mathbb{H}^3$.

This space is very similar to the space of pairs of geodesics in $\mathbb{H}^2$. Namely, we have a map

$$D : H\backslash G/H \to \mathbb{R}$$

which sends a pair of planes that cross into $(-1,1)$ and a pair of planes that do not cross into $\pm[1,\infty)$. In the first case D records $\pm\cos\theta$, the cosine of the dihedral angle between P_1 and P_2; in the second case, it records $\pm\cosh d(P_1,P_2)$. In both cases the sign of $D(P_1,P_2)$ reflects their relative orientations.

In the Minkowski model, we can write $P_i = v_i^\perp$ where $\langle v_i, v_i\rangle = 1$, and $D(P_1,P_2) = \langle v_1, v_2\rangle$.

One might expect that $\dim H\backslash G/H = 0$, because $2\dim H = 6 = \dim G$. The reason the dimension is one is that for any pair of planes, their common stabilizer in G is at least one dimensional. For example the common stabilizer is a copy of A when the planes meet in a line.

The configuration space is not Hausdorff because of *parallel planes*, or equivalently because of pairs of *tangent circles*.

Provided the orientations agree, parallel planes satisfy $D(P_1, P_2) = 1$; however there are actually two points in $H\backslash G/H$ represented here, one where $P_1 = P_2$ and the other where $P_1 \neq P_2$.

Let $C = \widehat{\mathbb{R}}$. The planes parallel to $P = \operatorname{hull}(C)$ and passing through ∞ correspond to the circle $\widehat{\mathbb{R}} + iy$, $y \in \mathbb{R}$. It is now elementary to see:

Theorem 14.2 *Let $C_n \to \widehat{\mathbb{R}}$ be a convergent sequence of oriented circles distinct from $\widehat{\mathbb{R}}$. Then for any $y > 0$, there exist $h_n \in H$ such that*

$$h_n(C_n) \to \widehat{\mathbb{R}} \pm iy.$$

Proof. Passing to a subsequence, we can arrange that either,

(i) C_n is contained in $\mathbb{H}$ for all n;

(ii) C_n is contained in $-\mathbb{H}$ for all n; or

(iii) C_n meets $\widehat{\mathbb{R}}$ for all n.

In case (i) we can regard C_n as a circle in the hyperbolic metric on $\mathbb{H}$. Using the action of $H = \operatorname{Isom}^+(\mathbb{H})$, we can arrange that the diameter of C_n is an interval lying along the imaginary axis, of the form $i[y, y_n]$ with $y_n \to \infty$. Then $C_n \to \widehat{\mathbb{R}} + iy$ as $n \to \infty$. Case (ii) is similar.

In case (iii), we can apply the action of H to arrange that C_n and $\widehat{\mathbb{R}}$ meet at ∞. Then C_n is a sequence of straight lines with slopes tending to zero. We can further apply the action of $AU \subset H$ to arrange that C_n passes through iy; then again we have $C_n \to \widehat{\mathbb{R}} + iy$. ■

In terms of Lie groups an equivalent formulation is this:

Theorem 14.3 *Suppose $g_n \to I$ in $G - H$. Give $v \in V$, there exist $h_n, h'_n \in H$ such that*

$$h_n g h'_n \to v^{\pm 1}.$$

II. Planes and horocycles. We now turn to the space $U\backslash G/H$. This space describes oriented circles (or lines) in $\mathbb{C}$ up to horizontal translation. For example, a circle of radius r can be translated so that highest point $iy \in \mathbb{C}$ lies on the imaginary axis; this yields a map

$$D : U\backslash G/H \to (-\infty, infty] \times (0, \infty],$$

defined by $D(C) = (y, r)$, with the convention that $y = r = \infty$ when C is a line.

Theorem 14.4 *Suppose $C_n \to \widehat{\mathbb{R}}$ and no C_n is a horizontal line. Then for any $y \geq 0$ there exist $x_n \in \mathbb{R}$ such that*

$$C_n + x_n \to \widehat{\mathbb{R}} \pm iy.$$

Proof. Very similar to the preceding proof. As before the radius of C_n tends to infinity with n. If C_n crosses $\widehat{\mathbb{R}}$ then it does so with slope tending to zero; thus we can translate C_n by x_n so it passes through iy to achieve the desired limit. If $C_n \subset \mathbb{H}$ for all n, then its lowest point tends to zero and its highest points tends to infinity, so again we can translate so it passes through iy with slope tending to zero. (If $y = 0$ we normalize so its lowest point is on the imaginary axis.) Finally if $C_n \subset -\mathbb{H}$ for all n, the same argument works for $-iy$. ∎

In terms of Lie groups this gives:

Theorem 14.5 *Suppose $g_n \to I$, $g_n \in G - VH$. Then for any $v \in V$, there exist $u_n \in U$ and $h_n \in H$ such that*

$$u_n g_n h_n \to v^{\pm 1}.$$

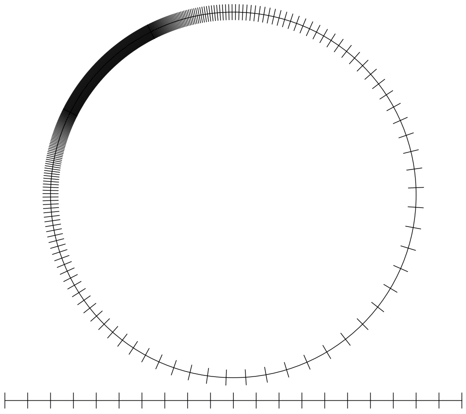


Figure 11. A circle density (C, α) close to $(\widehat{\mathbb{R}}, dx)$.

III. Pairs of horocycles. Finally we analyze the configuration space for pairs of horocycles, $U\backslash G/U$. For this we need a geometric picture to attach to the space of horocycles $\mathcal{H} = G/U$.

To start with, observe that if we equip $C = \widehat{\mathbb{R}}$ with the standard 1–form $\alpha = dx$, then the stabilizer of (C, α) in G is simply U. Thus, we can regard G/U as the space of *circle densities* (C, α) that arise as the image of $(\widehat{\mathbb{R}}, dx)$ under G.

By definition, the group G acts transitively on such pairs. One can visualize a circle density by first integrating α, to obtain a Möbius transformation $g : C \to \widehat{\mathbb{R}}$, and then marking the points $g^{-1}(\mathbb{Z})$ on C. For an example see Figure 11. Adjacent tickmarks are distance one apart in the metric $|\alpha|$ on C.

Let us say a circle density is *horizontal* if it invariant under U. This happens exactly if it has the form $(C, \alpha) = (\widehat{\mathbb{R}} + iy, a\, dx)$.

Theorem 14.6 *Let (C_n, α_n) be a sequence of non–horizontal circle densities converging to $(\widehat{\mathbb{R}}, dx)$. Then we can find $u_n \in U$ such that for some a and y,*

$$\lim u_n \cdot (C_n, \alpha_n) = (\widehat{\mathbb{R}} + iy, a\, dx) \neq (\widehat{\mathbb{R}}, dx).$$

Moreover we can arrange that $|a - 1|$ and $|y|$ are as small as we like.

Proof. For large n the circle C_n is very close to $\widehat{\mathbb{R}}$, and its density α_n is nearly dx, on a large ball about the origin. As we move to the left or right, either the circle is eventually at height 1 above or below $\widehat{\mathbb{R}}$, or the density is well–approximated by $(1 + a)\, dx$, $|a| = 1/2$, or both. (For example, if C_n is horizontal, then its density has a pole in the finite part of C_n, and it increases as we move towards the pole.)

If we stop at the first moment one or the other of these conditions is achieved, we obtain a translate $u_n \cdot (C_n, \alpha_n)$ that is a good approximation to a horizontal line $(\widehat{\mathbb{R}} + iy, (1 + a)\, dx)$ with either $|y| = 1$ or $|a| = 1/2$. Passing to a subsequence, we obtain convergence to such a horizontal line.

The proves the first part of the theorem; for the second part, repeat the argument for $|y| = \epsilon$ and $|a| = \epsilon$. ■

Here is the group theory formulation.

Theorem 14.7 *Suppose $g_n \to I$ in $G - AN$ and G_0 is a neighborhood of I in G. Then there exist $u_n, u_n' \in U$ such that*

$$u_n g_n u_n' \to g \in AV - \{id\},$$

and $g \in G_0$.

Hyperbolic planes. We are now ready to carry out the proof of Theorem 14.1. Here is a group–theory formulation.

Theorem 14.8 *Let $\Gamma \subset G = \mathrm{SL}_2(\mathbb{C})$ be a cocompact lattice. Then every H–orbit in $\Gamma\backslash G$ is closed or dense.*

Geometrically, we can regard $\Gamma\backslash G$ as the frame bundle $\mathrm{F}M$ of the compact hyperbolic 3–manifold $M = \Gamma\backslash\mathbb{H}^3$. An H–orbit corresponds to a totally geodesic immersion $f : \mathbb{H} \to M$. The H orbit is closed exactly when there is an immersed, compact hyperbolic surface $\iota : N \to M$ such that f factors through the universal covering map $\pi : \mathbb{H} \to N$. When H is dense, not only is $f(\mathbb{H})$ dense in M, but normal vectors to its image are density in the unit tangent bundle of M.

For the proof of Theorem 14.8, we need two preliminary facts:

1. If $\overline{xH}$ contains a closed H–orbit, then xH is closed or dense.

2. If $\overline{xH}$ contains a path x_t that is not contained in a single H–orbit, then $\overline{xH} = \mathrm{F}M$.

For brevity we omit the proofs, which are not difficult, because we wish to emphasize the similarities to Hedlund's proof.

Proof of Theorem 14.8. Let $Z = \overline{xH} \subset \mathrm{F}M$. Using fact (1) above, we can assume that Z contains no closed H–orbit. We also can reduce to the case where Z is minimal. Our goal is to show that $Z = \mathrm{F}M$.

The main problem is that the normalizer of H in G is too small (finite). This makes it hard to find g such that $Zg = Z$.

So instead, we choose a minimal U–invariant $Y \subset Z$. We claim that there exists a 1–parameter subgroup $L \subset AV$ such that $LY \subset Y$. To see this, let

$$S = \{g \in G \ : \ Yg \cap Y \neq \emptyset\}.$$

The set S is closed and satisfies $USU = S$. Then, using Theorem 14.7, we find that $S \cap AV$ contains I as an accumulation point. But AV normalizes U, so

$$S \cap AV = \{g \in AV \ : \ Yg = Y\}.$$

This is a closed subgroup of AV, so once it contains elements accumulating at the identity, it contains a 1–parameter subgroup L.

Summing up, we have $L \subset AV$ such that $YL = Y$. We also have $YU = Y$. Thus Y is invariant under the group $LU \subset AN$. Provided LU is not contained in H, we are now done: A path g_t in LU, not contained in H, provides a path $x_t = xg_t \subset \overline{xH}$ that is not contained in an H–orbit, and hence $\overline{xH} = \mathrm{F}M$.

It remains to treat the case where $LU \subset H$. It is easy to see that in this case, $L = A$. To handle it, we consider a new set

$$S = \{g \in G \ : \ Yg \cap Z \neq \emptyset\}.$$

Since $ZH = Z$ and $YU = Y$, we have $USH = S$. In other words, S describes a closed subset of $U\backslash G/H$.

Choose $y \in Y$. Since yH is dense but not closed in Z, we can find $g_n \to I$ in $G - H$ with $g_n \in S$. Then, Theorem 14.5 implies there exists a $v \neq I$ in $S \cap V$. This means $y'v \subset Z$ for some $y' \in Y$. Since Z is U–invariant and U commutes with v, we have

$$Yv = \overline{y'U}v = \overline{y'v}U \subset Z.$$

Since Y is A–invariant, Yv is $v^{-1}Av$ invariant. The latter subgroup does not lie in H; it contains a path g_t not contained in H, and as before this suffices to complete the proof. ■

15 Amenability

In this section we discuss the notion of *amenability* of a discrete group G. In a certain sense, amenable groups are small.

We wil see that amenability can be formulated in many equivalent ways: (i) in terms of finitely–additive invariant measures; (ii) in terms of an isoperimetric inequality for the Cayley graph of G; and (iii) in terms of the regular representation of G on $L^2(G)$

From (ii) we obtain connections to eigenvalues of the Laplacian, as we will see in subsequent sections; while (iii) motivates the definition of Kazhdan's property T.

Reference for this section: [Gre].

Amenable groups. Let G be a discrete group. A *mean* is a linear functional $m : L^\infty(G) \to \mathbb{R}$ on the space of bounded real-valued functions such that $m(1) = 1$ and $f \geq 0 \implies m(f) \geq 0$.

A mean is the same as a *finitely additive probability measure* on G, this measure being given by $\mu(A) = m(\chi_A)$.

A group is *amenable* iff it admits a (right or left) G-invariant mean. Here (right) *invariance* means $\mu(Ag) = \mu(A)$ for all $A \subset G$ and $g \in G$. Equivalently, $m(g \cdot f) = m(f)$ for all $f \in L^\infty(G)$, where $(g \cdot f)(x) = f(xg)$. Left invariance is defined similarly.

Note that if $|G| = \infty$, then an invariant measure satisfies $\mu(A) = 0$ for any finite set $A \subset G$.

Basic properties. Working directly from this definition, the following facts are readily verified.

- Finite groups are amenable.

  The integers $\mathbb{Z}$ are amenable. To see this, take any weak* limit of the means $m_n(f) = \frac{1}{n}\sum_1^n f(i)$.

- An amenable group is 'small': If G is amenable then any quotient $H = G/N$ is amenable. (Pull back functions from H to G and average there.)

- If $H \subset G$ and G is amenable, then so is H. (Foliate G by cosets of H, choose a transversal and use it to spread functions on H out to G.)

- More generally, if we have an exact sequence of groups,

  $$1 \to A \to B \to C \to 1,$$

  and A and C are amenable, then so is B. (Given a function on B, average over cosets of A to obtain a function on C, then average over C.)

- G is amenable iff every finitely generated subgroup of G is amenable. (For a finitely generated subgroup H, we get an H-invariant mean on G with the aid of a transversal. In the limit as H exhausts G we get an invariant mean on G.)

- Abelian groups are amenable (since finitely generated ones are). Solvable groups are amenable by virtue of the statement on exact sequences above.

What groups are large?

Theorem 15.1 *The free group $G = \langle a, b\rangle$ is* not *amenable.*

Proof. Every $g \in G$ can be represented by a reduced word in the generators a, b and their inverses. Let $A \subset G$ be the subset represented by words beginning with A. We then have

$$A =\supset abA \sqcup a\overline{b}A.$$

Thus any (left) invariant finitely–additive measure μ on G must satisfy

$$\mu(A) \geq \mu(abA) + \mu(a\overline{b}A) = 2\mu(A) \geq 0,$$

and hence $\mu(A) = 0$. By a similar argument, the set of words beginning with $\overline{a}, b$ or $\overline{b}$ have measure zero. Since the identity element also has measure zero, this shows $\mu(G) = 0$, contrary to the assumption $\mu(G) = 1$. ■

Since any group that contains a free group must also be nonamenable, we have:

Corollary 15.2 *The fundamental group of a surface of genus $g \geq 2$ is nonamenable.*

The same holds the fundamental group of any finite volume hyperbolic manifold of dimension $n \geq 2$.

In fact, the Tits alternative asserts that any finitely generated group $G \subset \mathrm{GL}_n(\mathbb{R})$ is either virtually solvable, or it contains a free group on two generators. In the first case G is amenable, and in the second case it is nonamenable.

Geometry of graphs. We now turn to a more geometric characterization of amenability, in terms of the Cayley graph of the group.

Let $\mathcal{G}$ be a graph. The *boundary* of a subset of vertices $A \subset V(\mathcal{G})$, denoted ∂A, is the set of vertices connected to A by an edge but not in A itself. The *isoperimetric constant* of $\mathcal{G}$ is given by

$$h_0(\mathcal{G}) = \inf \frac{|\partial A}{|A|}$$

where the infimum is over all finite subsets $A \subset V(\mathcal{G})$. It is the best constant $\lambda \geq 0$ such that

$$|\partial A| \geq \lambda \cdot |A|$$

for all finite A.

By making its edges have length 1, the Cayley graph induces a metric on $V(G)$. A positive isoperimetric inequality forces exponential growth of balls in this metric; we have:

$$|B(x,r)| \geq (1+\lambda)^r$$

for any integer $r \geq 0$.

The Cayley graph. Given a *finitely generated* group G, together with a choice of generating set S, we can build the *Cayley graph* $\mathcal{G}$ whose vertices $V = G$ and whose edges connect elements differing by a generator. We will usually assume $S^{-1} = S$, and take the edges to be undirected.

The distance $d(e, g)$ is just the length of the shortest word in the generators S that expresses g.

Examples. In $\mathbb{Z}^n$ with its standard generating set, the size of the ball $B_r = B(e,r)$ satisfies $|B_r| \asymp r^n$ and $|\partial B_r| \asymp r^{n-1}$. Letting $r \to \infty$, we see that $h_0(\mathbb{Z}^n) = 0$.

On the other hand, for $\mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$, we have

$$|B_r| = 1 + 4 + 4\dot{3} + \cdots + 4\dot{3}^{r-1} = 1 + 4 \cdot \frac{3^r - 1)}{3-1} = 2 \cdot 3^j - 1.$$

Thus $|\partial B_r|/|B_r|$ behaves like $(3^{r+1} - 3^r)/3^r = 2$, and in fact

$$h_0(\mathcal{G}(\mathbb{Z} * \mathbb{Z}, \{a, b\})) = 2.$$

Følner sets. Here is a very useful geometric formulation of amenability.

Theorem 15.3 (Følner's condition) *Let G be a finitely generated group. Then G is amenable iff the isoperimetric constant of its Cayley graph is zero.*

Corollary 15.4 *Any finitely generated nonamenable group has exponential growth.*

This property does not characterize nonamenable groups. In fact, there are also solvable (and hence amenable) groups of exponential growth; e.g. $\langle a, b \,:\, ab = b^2 a\rangle$. For such a group the Følner sets cannot be chosen to be balls.

Functional analysis and types of means. The proof of Theorem 15.3 will combine ideas from functional analysis and combinatorics.

To set up the proof, recall that

$$L^1(G)^* = L^\infty(G) \quad \text{and} \quad L^\infty(G)^* \supset M(G),$$

where $M(G)$ is the space of means on G. By Alaoglu's theorem, $M(G)$ is compact in the weak* topology. In this topology, a net $m_\alpha \to m$ if and only if $m_\alpha(f) \to m(f)$ for every $f \in L^\infty(G)$.

Let $W(G) \subset L^1(G)$ be the space of finite *weights*, by which we mean functons $w(x)$ with finite support such that $w \geq 0$ and $\sum w(x) = 1$. We have a natural inclusion $W(G) \subset M(G)$, where the mean of a weight is given by

$$m_w(f) = \sum f(x)w(x).$$

Clearly the weak* topology that $W(G)$ inherits from $M(G)$ is the same as the weak topology it inherits from $L^1(G)$.

The first part of the proof depends on two important facts: (i) $W(G)$ is convex; and (ii) in any Banach space, the weak closure and norm closure of a convex set agree.

Lemma 15.5 *The weights $W(G)$ are dense in $M(G)$.*

Proof. Let m be a mean on G. A neighborhood U of m in the weak* topology is given by

$$U = \{m' \,:\, \max_i |m(f_i) - m'(f_i)| < \epsilon\},$$

where $\epsilon > 0$ and $f_1, \ldots, f_n \in L^\infty(G)$. We can assume the functions f_i are simple (they take on only finite many values). Thus there is a finite partition $G = \bigcup G_j$ such that $f_i|G_j$ is constant for all i, j. Pick a point $x_j \in G_j$ and define a weight by $w(x_j) = m(G_j)$ and $w(x) = 0$ elsewhere. Then $w(f_i) = m(f_i)$ for all i, so $w \in U$. This shows the closure of $W(G)$ contains m. ■

Now let us assume that G is generated by a finite set S. We define, for any weight w,

$$\mathrm{Var}_S(w) = \sum_{g \in S} \|g \cdot w - w\|_1.$$

Lemma 15.6 *If G is amenable, there exists a sequence of weights w_n such that* $\mathrm{Var}_S(w_n) \to 0$.

Proof. Let $m \in M(G)$ be a G–invariant mean. Let $w_\alpha \in W(G)$ be a net of weights converging to m. Then for any $g \in S$, we have $g \cdot w_\alpha - w_\alpha \to 0$ in the weak* topology. This shows that 0 lies in the weak closure of the convex set $(g - I)(W(G))$. By convexity, it also lies in the strong closure. Thus we can find weights with $\|g \cdot w_n - w_n\|_1 \to 0$. The argument for the full set S is similar, using the map

$$\prod_{g \in S}(g - I) : L^1(G) \to \prod_S L^1(G).$$

■

Proof of Theorem 15.3. Given $\epsilon > 0$, choose w such that $\mathrm{Var}_S(w) < \epsilon$.

$$w = \sum a_i \chi_{G_i}/|G_i|,$$

where $a_i > 0$, $\sum a_i = 1$ and $G_1 \subset G_2 \subset \cdots G_n$ are finite sets. Because of nesting, for any $x, g \in G$ the quantities $\chi_{G_i}(x) - \chi_{G_i}(xg)$ all have the same sign as i varies. Thus

$$\mathrm{Var}_S(w) = \sum_S \|gw - w\|_1 = \sum a_i \sum_S \frac{|G_i \triangle gG_i|}{|G_i|} \geq \sum a_i \frac{|\partial G_i|}{|G_i|} \cdot .$$

It follows that $|\partial G_i|/|G_i| < \epsilon$ for some i. Since ϵ was arbitrary, this shows $h_0(G) = 0$. ■

Exponential growth. By the Følner property, any nonamenable group has exponential growth, since the boundary of any ball has size comparable to the ball itself.

Representations and almost invariance. Next we relate amenability to representations of G.

Let G be a locally compact topological group. A unitary representation of G on a Hilbert space H has *almost invariant vectors* if for any compact set $K \subset G$ and $\epsilon > 0$, there is an $f \in H$ with $\|f\| = 1$ and

$$\|g \cdot f - f\| < \epsilon$$

for all $g \in K$. Equivalently, $\langle g \cdot f, f \rangle \approx 1$ for all $g \in K$.

Theorem 15.7 *The group G is amenable iff $L^2(G)$ has almost invariant vectors.*

Proof. If G is amenable, then by Følner there are finite sets G_i that are almost invariant. Set $f_i = \chi_{G_i}/|G_i|^{1/2}$ so $\|f_i\| = 1$. Then as $i \to \infty$,

$$\langle gf_i, f_i\rangle = \frac{|(g \cdot G_i) \cap G_i|}{|G_i|} \to 1,$$

so $L^2(G)$ has almost invariant vectors.

Conversely, if f_i is almost invariant, then $m_i(h) = \langle hf_i, f_i\rangle$ for $h \in L^\infty(G)$ is an almost-invariant mean. Indeed, we have:

$$\begin{aligned} \langle (g^{-1} \cdot h) f_i, f_i\rangle &= \langle h(g \cdot f_i, g \cdot f_i\rangle \approx \langle h(g \cdot f_i, f_i\rangle \\ &= \langle (g \cdot f_i, hf_i\rangle \approx \langle f_i, hf_i\rangle = \langle hf_i, hf_i\rangle. \end{aligned}$$

Taking a weak* limit we conclude that G is amenable. ■

Appendix: Amenability and Poincaré series.

Theorem 15.8 *Let $Y \to X$ be a covering of a compact Riemann surface X of genus $g \geq 2$. Then the Poincaré operator $\Theta_{Y/X} : Q(Y) \to Q(X)$ satisfies $\|\Theta_{Y/X}\| < 1$ if and only if the covering is nonamenable. In particular, $\|\Theta\| < 1$ for the universal covering.*

[Mc1], [Mc2].

This result has applications to the construction of hyperbolic structures on 3-manifolds, following Thurston.

16 Expanding graphs

In this section we define expanding graphs and discuss some explicit constructions of them. The fact that these construction yield expanders will be verified in later sections, using the Laplacian and property T.

Expanders. Let $\mathcal{G} = (V, E)$ be a finite graph. For such a graph we define a modified isoperimetric constant by:

$$h_1(\mathcal{G}) = \inf_A \frac{|\partial A|}{|A|},$$

where $A \subset V(\mathcal{G})$ ranges over all sets with $0 < |A| \leq |V(\mathcal{G})|/2$. Note that $\mathcal{G}$ is disconnected if and only if $h_1(\mathcal{G}) = 0$.

Recall that for infinite graphs we have similarly defined $h_0(\mathcal{G})$, allowing all finite sets A. For finite graphs we have $h_0(\mathcal{G}) = 0$ and $h_1(\mathcal{G})$ is a natural substitute. (The notation is also related to the eigenvalues λ_0 and λ_1 of the Laplacian, as we will see below.)

The *degree* of a graph, $\deg(\mathcal{G})$, is the maximum number of edges incident to a single vertex.

An *expanding graph* is not a single graph but rather a sequence of finite graphs $\mathcal{G}_n$ such that:

1. $|V(\mathcal{G}_n)| \to \infty$;

2. The degree $\deg(\mathcal{G}_n)$ is bounded, independent of n; and

3. The isoperimetric constant is bounded below: we have $\inf h_1(\mathcal{G}_n) > 0$.

Properties of expanders. Expanding graphs are well connected and serve as finite models for negatively curved spaces. More concretely, if $\mathcal{G}_n$ is a sequence of expanders then:

1. Balls in $\mathcal{G}_n$ have exponential growth, until they reach a size comparable to the whole space.

2. As a consequence, the diameter of $\mathcal{G}_n$ is $O(\log |G_n|)$.

3. Put differently, each $\mathcal{G}_n$ is *small world.* Although each vertex has only a bounded number of neighbors, any two vertices are connected by a small sequence of mutual neighbors. It is thus a model for social networks and the spread of rumors or infections.

4. The graph $\mathcal{G}_n$ is hard to disconnect: if one removes a set of vertices B, $|B| \ll |V(\mathcal{G}_n)|$, then the complemenent A of the largest remaining component satisfies $|A| = O(|B|)$. This is because $\partial A \subset B$.

5. Consequently, networks based on $\mathcal{G}_n$ are robust: the failure of k nodes affects only $O(k)$ clients.

The Laplacian on a graph. A finite undirected graph $\mathcal{G}$ with vertices V can be formally described by its symmetric, integral *adjacency matrix* A, whose entry $A_{ij} \geq 0$ gives the number of edges connecting $i, j \in V$. The diagonal entries describe loops, and parallel edges arise when $A_{ij} > 1$.

The vector $d_i = \sum_j A_{ij}$ gives the *degree* of the vertex i. When this degree is a constant d, we say $\mathcal{G}$ is a d–regular graph. Under this assumption, we define the *Laplacian*

$$\Delta : L^2(V) \to L^2(V)$$

by $\Delta = dI - A$. This operator can be considered as the sum of the derivatives of f in every direction: we have

$$(\Delta f)(i) = \sum f(i) - f(j),$$

where there is one term in the sum for each edge of G incident to i (and this edge joins i to j).

Since A_{ij} is a symmetric, the Laplacian is diagonalizable over $\mathbb{R}$ and its spectrum is real. In fact its least eigenvalue is given by $\lambda_0 = 0$; the harmonic functions on $\mathcal{G}$ are constant, as they would be on a compact manifold. Provided $\mathcal{G}$ is connected, the smallest positive eigenvalue is given by

$$\lambda_1(\mathcal{G}) = \inf \left\{ \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} \; : \; 0 \neq f \in L^2_0(V) \right\}.$$

Lemma 16.1 *For any connected d–regular graph $\mathcal{G}$, we have*

$$\lambda_1(\mathcal{G}) \leq 2d(d+1)h_1(\mathcal{G}).$$

Proof. Let $V = A \cup B$ be a partition of V into two sets, with $0 < |A| \leq |B|$. Let $f(i) = |B|$ on A and let $f(i) = -|A|$ on B. Then $f \in L^2_0(V)$ and

$$\langle f, f \rangle = |B|^2|A| + |A|^2|B| = |A||B||V| \geq |A||V|^2/2.$$

We have $(\Delta f)(i) = 0$ unless i belongs to an edge joining A to B, in which case $|(\Delta f)(i)| \leq d|V|$. The number of such vertices is no more than $(d+1)|\partial A|$, and at each such vertex we also have $|f(i)| \leq |V|$. This shows

$$\langle \Delta f, f \rangle \leq d|V|^2 \cdot (d+1)|\partial A|.$$

Taking the ratio we find

$$\lambda_1(\mathcal{G}) \leq \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} \leq \frac{2d(d+1)|\partial A|}{|A|}.$$

Taking the inf over A yields the Lemma. ∎

Corollary 16.2 *Let $\mathcal{G}_n$ be a sequence finite d–regular graphs with $|V(\mathcal{G}_n)| \to \infty$ and $\inf \lambda_1(\mathcal{G}_n) > 0$. Then $\mathcal{G}_n$ is a sequence of expandeers.*

In fact one can give a bound in the other direction, so this necessary condition is also sufficient.

Groups and graphs. Now let G be a finitely generated group, acting transitively on a set V. Let $S \subset G$ by a *symmetric* generating set for G, meaning $S^{-1} = S$ and $\langle S \rangle = G$. We then turn V into the vertices of a graph $\mathcal{G}(V, S)$ with adjacency matrix

$$A_{ij} = |\{s \in S \;:\; s \cdot i = j\}|.$$

The assumption that S is symmetric implies A_{ij} is a symmetric matrix; the corresponding Laplacian on $L^2(V)$ is given by

$$(\Delta f)(i) = \sum_S f(i) - f(s \cdot i).$$

To obtain a sequence of graphs, we can consider a sequence of finite quotients G_n of G, with $|G_n| \to \infty$, and set $\mathcal{G}_n = \mathcal{G}(G_n, S)$. Since $|S|$ is fixed, these graphs have finite degree. If $\inf h_1(\mathcal{G}_n) > 0$ then we have obtained a sequence of expanders. Changing the generating set S for G only changes $h_1(\mathcal{G}_n)$ be a bounded factor, thus the expansion property depends only on G and its sequence of quotients G_n.

Examples of expanders. It is now easy to describe examples of expanding graphs.

Theorem 16.3 *Fix $r \geq 2$, and fix a finite generating set S for $G = \mathrm{SL}_r(\mathbb{Z})$. Then the graphs associated to the quotient groups $G_p = \mathrm{SL}_r(\mathbb{F}_p)$, with p prime, form a sequence of expanders.*

(This result also holds with the primes p replaced by the integers $n \geq 2$.)

The proof will be developed in the sequel. For $r = 2$ it uses Selberg's 3/16th and eigenvalues of the Laplacian. For $r \geq 3$ it uses Kazhdan's property T.

The expansion of $\mathrm{SL}_2(\mathbb{F}_p)$ passes immediately to any set on which it acts transitively; thus we have:

Corollary 16.4 *The graphs $\mathcal{G}_p$ with vertices $\mathbb{P}^1(\mathbb{F}_p)$ and edges from x to $x+1$ and $-1/x$ form a sequence of expanders.*

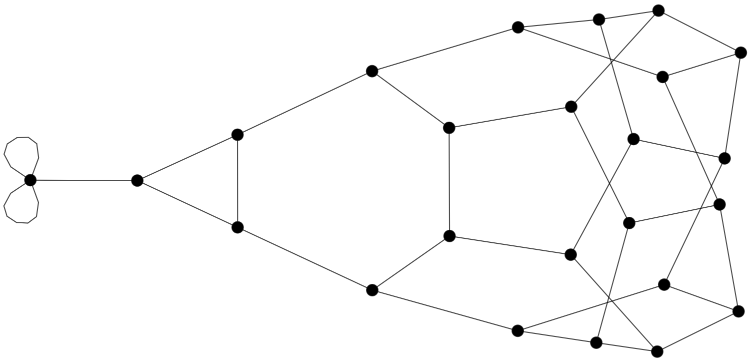


Figure 12. An expanding graph with vertices $\mathbb{P}^1(\mathbb{F}_{23})$.

p	$\lambda_1(\mathcal{G}_p)$	$\mathrm{diam}(\mathcal{G}_p)$
17	0.298549	5
23	0.258809	5
541	0.0779798	14
7919	0.0727643	22
17389	0.0724445	24
27449	0.0714854	26

Table 13. Eigenvalues and diameter estimates for expanders on $\mathbb{P}^1(\mathbb{F}_p)$.

For an example see Figure 12.

Eigenvalues of the Laplacian. Let us now connected nonamenability and expansion to coverings of manifolds.

Let M be a Riemannian manifold. We *define* the smallest eigenvalue of the Laplacian of M by:

$$\lambda_0(M) = \inf \frac{\int_M |\nabla f|^2}{\int_M |f|^2}.$$

If M has finite volume, then we can take f to be a constant and hence $\lambda_0(M) = 1$. In the finite volume case we similarly define $\lambda_1(M)$ as above by adding the condition that $\int_M f = 0$.

When M is compact, these eigenvalues have associated smooth eigenvectors satisfying $\Delta f_i = \lambda_i f_i$. But the notions make sense even for noncompact manifolds; for example, it is easy to see that

$$\lambda_0(\mathbb{R}^n) = 0,$$

even though there are no L^2 harmonic functions on $\mathbb{R}^n$.

Coverings of manifolds. We can now formulate:

Theorem 16.5 (Brooks) *The universal cover of a compact manifold M satisfies $\lambda_0(\widetilde{M}) = 0$ if and only if $\pi_1(M)$ is an amenable group.*

Sketch of the proof. Suppose $G = \pi_1(M)$ is amenable.

Let $D \subset \widetilde{M}$ be a smooth polyhedron forming a fundamental domain for the action of $\pi_1(M)$. Choose a finite set of generators S for G such that sD, $s \in S$ enumerate the tiles adjacent to D. By amenability, we can find a sequence of finite sets $G_n \subset G$ such that $|\partial G_n|/|G_n| \to 0$, where $\partial G_n = SG_n \backslash G_n$. Now define a smooth function f_n on $\widetilde{M}$ such that $f_n = 1$ on $G_n \cdot D$, f_n gradually decreases to zero along the tiles adjacent to $G_n \cdot D$, and f_n vanishes elsewhere. We then have

$$\frac{\int |\nabla f_n|^2}{\int |f_n|^2} \asymp \frac{|\partial G_n|}{|G_n|} \to 0$$

as $n \to \infty$. Thus $\lambda_0(\widetilde{M}) = 0$.

The reverse implication pivots on Cheeger's inequality, to be discussed later. ■

What about expanders? For these a similar argument shows the following.

Theorem 16.6 *Let $M_n \to M$ be a sequence of finite Galois covers of a compact Riemann manifold M, corresponding to a sequence of finite quotients $G = \pi_1(M) \to G_n$ with $|G_n| \to \infty$. Fix a finite generating set S for $\pi_1(M)$, and let $\mathcal{G}_n = \mathcal{G}(G_n, S)$. Then*

$$\inf h_1(\mathcal{G}_n) > 0 \iff \inf \lambda_1(M_n) > 0.$$

The examples for $\mathrm{SL}_2(\mathbb{Z})$ will arise from Selberg's theorem, which states that $\lambda_1(\Gamma(n)\backslash\mathbb{H}) \geq 3/16$ for all n. Some additional argument is required, because $\Gamma(n)$ is not cocompact.

Expansion and property *T*. Finally we connect expansion to a strong form of non–amenability called Property T.

To state this result, let G be a group with finite generating set S. Given a unitary representation $\rho : G \to U(H)$, let

$$\mathrm{Var}_S(\rho) = \inf_{f \neq 0} \sup_S \frac{\|f - s \cdot f\|}{\|f\|}.$$

We say G has Kazhdan's *property* T if there exists an $\epsilon > 0$ such that

$$\mathrm{Var}_S(\rho) < \epsilon \implies \rho \text{ has a fixed vector.}$$

Theorem 16.7 *Let G be a group with finite generating set S, acting transitively on a finite set V with associated graph $\mathcal{G} = \mathcal{G}(V, S)$. Then we have*

$$\mathrm{Var}_S(\rho)^2 \leq 2h_1(\mathcal{G}(G_n, S)),$$

where ρ gives the action of G on $L^2_0(V)$.

Proof. Suppose we have a partition of V into two sets A and B, with $|A| \leq |B|$. (Note that $|A| + |B| = |V|$ and $|B| \geq |V|/2$).

Define $f : V \to \mathbb{R}$ so that $f(x) = |B|$ on A and $f(x) = -|A|$ on B. Then f has mean zero, and

$$\int |f|^2 = |B|^2|A| + |A|^2|B| = |A||B||V| \geq |A||V|^2/2.$$

On the other hand, we have $f(s \cdot i) = f(i)$ unless $i \in A$ and $s \cdot i \in B$, or vice–versa, in which case $|f(s \cdot i) - f(i)| = |V|$. Thus:

$$\int |f - s \cdot f|^2 \leq 2|\partial A||V|^2.$$

Thus

$$\frac{\|s \cdot f - f\|}{\|f\|} \leq \left(\frac{2|\partial A|}{|A|}\right)^{1/2}.$$

By definition the left hand side is an upper bound for $\mathrm{Var}_S(\rho)$; taking the inf over A on the right, and squaring, we find that

$$\mathrm{Var}_S(\rho)^2 \leq 2h_1(\mathcal{G}(G_n, S)).$$

■

Corollary 16.8 *If G has property T, then any sequence of finite quotient groups G_n of G with $|G_n| \to \infty$ determines a sequence of expanding graphs.*

In the sequel we will show that $G = \mathrm{SL}_r(\mathbb{Z})$ has property T for $r \geq 3$. The fact that we get expanders from the quotients $G_p = \mathrm{SL}_r(\mathbb{F}_p)$ will then follow.

Averaging and property *T*. Here is another formulation of the remarkable property enjoyed by a discrete group G with property T. Let S be a set of generators for G, *including* the identity. Defining an averaging operator by

$$A_S(f) = \frac{1}{|S|}\sum_{s \in S} s \cdot f.$$

Property T can be formulated as follows:

> *The group G has property T if and only if, for any unitary representation $G \to U(H)$ without fixed vectors, we have $\|A_S\| < \lambda(s) < 1$.*

One can show, for example, that SO(5) contains a subgroup with property T, and thus there exist a finite set of rotations S such that *any* $f \in L^2_0(S^4)$ is strictly contracted by averaging over S.

Expansion and natural selection. Embedding the nonamenable tree into the finite universe of the Earth, according to Darwin (*Origin of Species*, Chapter 4):

> *If during the long course of ages and under varying conditions of life, organic beings vary at all in the several parts of their organization, and I think this cannot be disputed; if there be, owing to the high geometrical powers of increase of each species, at some age, season, or year, a severe struggle for life, and this certainly cannot be disputed; then, considering the infinite complexity of the relations of all organic beings to each other and to their conditions of existence, causing an infinite diversity in structure, constitutions, and habits, to be advantageous to them, I think it would be a most extraordinary fact if no variation ever had occurred useful to each being's own welfare, in the same way as so many variations have occurred useful to man.*

Maybe the small world of an expanding graph, with many local but truncated trees, is a good model for the ecological interconnectivity of the life on earth.

17 The Laplacian

References for this section: [Bus], [Lub], [Sar].

Riemannian manifolds. The Laplacian on functions is defined by

$$\Delta f = - * d * df.$$

It is the self-adjoint operator associated to the quadratic form $\langle \nabla f, \nabla f \rangle$, in the sense that

$$\int_M |\nabla f|^2 = \int_M f \Delta f$$

for any compactly supported function.

For a compact manifold M the space $L^2(M)$ has a basis of eigenfunctions of the Laplacian.

For general manifolds we can define the least eigenvalue of the Laplacian by minimizing the *Ritz-Rayleigh quotient*

$$\lambda_0(M) = \inf \frac{\int |\nabla f|^2}{\int |f|^2}.$$

If M has finite volume then $\lambda_0 = 0$ and we define $\lambda_1(M)$ by minimizing the above subject to $\int f = 0$.

These definitions agree with the usual eigenvalues on a compact manifold but give only the discrete spectrum in general (even in the case of finite volume).

Behavior under coverings. If $Y \to X$ is a covering map, where X and Y have *finite volume*, then $\lambda_1(Y) \leq \lambda_1(X)$, since any test function on X can be pulled back to Y. It is a challenge to find an infinite tower of coverings with $\lambda_1(Y_i)$ bounded away from zero.

Note that λ_1 can drop precipitously: if X is a hyperbolic surface with a very short non-separating geodesic γ, it can still happen that $\lambda_1(X)$ is large; but there is a $\mathbb{Z}/2$ covering space $\pi : Y \to X$ such that $\pi^{-1}(\gamma)$ cuts Y into two equal pieces, so $\lambda_1(Y)$ is small.

The Euclidean Laplacian. On $\mathbb{R}^n$ we have

$$\Delta f = - * d * df = - \sum \frac{\partial^2 f}{\partial^2 x_i}.$$

It is easy to see $\lambda_0(\mathbb{R}^n) = 0$ for all n. In fact the Laplacian has continuous spectrum going down to zero.

On the other hand, on a circle C_r of radius r we have $\lambda_1(C_r) = 1/r^2$ by considering the function $\sin(x/r)$, and the full spectrum is of the form n^2/r^2, $n \geq 0$.

The continuous spectrum of $\mathbb{R}$ can be thought of as the limit of the discrete spectra of C_r as $r \to \infty$.

The hyperbolic Laplacian. On $\mathbb{H}$ with the metric $ds^2 = (dx^2 + dy^2)/y^2$ of constant curvature -1, we have

$$\Delta f = - * d * df = -y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = (z - \overline{z})^2 \frac{\partial^2 f}{\partial z \partial \overline{z}}.$$

This is the positive operator associated to the positive definite quadratic form $\int |\nabla f|^2 = \langle \Delta f, f \rangle$. The sign comes from Stokes' theorem.

Basic examples: $\Delta y^s = s(1-s) y^s$.

Bounded primitive for volume: for $\omega = -dx/y$ on $\mathbb{H}$ we have $|\omega(v)| \leq 1$ for any unit vector v, and $d\omega = dx\, dy/y^2$ is the hyperbolic area element. From this follows that hyperbolic space is nonamenable in a continuous sense. More precisely:

Theorem 17.1 *We have*

$\lambda_0(\mathbb{H}) \geq 1/4$, *and*
$\ell(\partial\Omega) \geq \operatorname{area}(\Omega)$ *for any compact smooth region* $\Omega \subset \mathbb{H}$.

Proof. For the isoperimetric inequality just note that

$$\operatorname{area}(\Omega) = \int_\Omega d\omega = \int_{\partial\Omega} \omega \leq \ell(\partial\Omega) \sup|\omega| = \ell(\partial\Omega).$$

For the bound on λ_0, let $f : \mathbb{H} \to \mathbb{R}$ be compactly supported; then

$$d(f^2\omega) = f^2\, d\omega + 2f\, df\, \omega,$$

so by Stokes' theorem and Cauchy-Schwarz we have

$$\left(\int f^2 d\omega\right)^2 = \left(\int 2f\, df\, \omega\right)^2 \leq 4\left(\int f^2\right)^2 \left(\int |df|^2\right)^2,$$

from which $\int |df|^2 \geq (1/4)\int f^2$. ∎

Remark. It can be shown that $\lambda_0(\mathbb{H}) = 1/4$.

Pyramid schemes. On an infinite regular tree of degree $d \geq 3$, it is similarly possible to define a unit speed flow with definite divergence at every vertex. This is directly linked to nonamenability of the graph. Indeed, on an amenable graph, such a pyramid scheme would increase the average wealth per vertex (defined using the mean).

Cheeger's constant. We will now see that an isoperimetric inequality and a lower bound on λ_0 always go hand in hand.

Given a Riemannian n-manifold (M, g) its *Cheeger constant* is defined by

$$h(M) = \inf_X \frac{\operatorname{area}(X)}{\min(\operatorname{vol}(A), \operatorname{vol}(B))},$$

where the infimum is over all co-oriented compact separating hypersurfaces $X \subset M$, and where $M - X = A \sqcup B$. (Although X may cut M into more than 2 pieces, the orientation of the normal bundle of X puts them into 2 classes.)

If $\operatorname{vol}(M) = \infty$ then the Cheeger constant reduces to the isoperimetric constant

$$h(M) = \inf_A \frac{\operatorname{area}(\partial A)}{\operatorname{vol}(A)},$$

where the inf is over all *compact* submanifolds A.

The constant $h(M)$ is not a conformal invariant; it is homogeneous of degree -1 under scaling the metric.

We will prove the following:

Theorem 17.2 (Cheeger) *For any Riemannian manifold M, we have*

$$\lambda_i(M) \geq h(M)^2/4,$$

where $i = 0$ for infinite volume and $i = 1$ for finite volume.

A continuous isoperimetric inequality.

Theorem 17.3 *Suppose M is infinite volume and $f : M \to [0, \infty)$ is a compactly supported smooth function. Then we have:*

$$\int |\nabla f| \, dV \geq h(M) \int f \, dV.$$

Proof. Let $A_t = f^{-1}[t, \infty)$, let $X_t = \partial A_t = f^{-1}(t)$, and let $V(t) = \operatorname{vol}(A_t)$.

The *coarea formula* says that for a proper function,

$$\int_{f^{-1}[a,b]} |\nabla f| \, dV \;=\; \int_a^b \operatorname{area}(X_t) \, dt.$$

To understand this formula, just note that the width of the region between X_t and $X_{t+\epsilon}$ is approximately $\epsilon/|\nabla f|$.

Since f is compactly supported we have

$$\begin{aligned} \int |\nabla f| \, dV &= \int_0^\infty \operatorname{area}(X_t) \, dt \geq h(M) \int_0^\infty \operatorname{vol}(A_t) \, dt \\ &= h(M) \int_0^\infty -tV'(t) \, dt = h(M) \int |f| \, dV, \end{aligned}$$

where we have integrated by parts and used the fact that $-V'(t) \, dt$ is the push-forward of the volume form from M to $\mathbb{R}$. ■

Proof of Cheeger's theorem for infinite volume and $\lambda_0(M)$.

Let $F : M \to \mathbb{R}$ be a compactly supported smooth function on M; we need to bound $\inf |\nabla F|^2$ from below. To this end, let $f = F^2$, so $\nabla f = 2F\nabla F$. Then we have

$$\begin{aligned} h(M)^2 \left(\int |F|^2 \right)^2 &= \left(h(M) \int |f| \right)^2 \leq \left(\int |\nabla f| \right)^2 \\ &= \left(2 \int |F|\,|\nabla F| \right)^2 \leq 4 \int |F|^2 \int |\nabla F|^2, \end{aligned}$$

and thus

$$\frac{h(M)^2}{4} \leq \frac{\int |\nabla F|^2}{\int |F|^2}.$$

■

The case of finite volume.

Theorem 17.4 *Let $f : M \to \mathbb{R}$ be a smooth function, and assume the level set of 0 cuts M into 2 pieces of equal volume. Then*

$$\int_M |\nabla f|\, dV \geq h(M) \int_M |f|\, dV.$$

Proof. Let $X_t = f^{-1}(t)$. Let $M - X_t = A_t \sqcup B_t$ where $A_t = f^{-1}(t, \infty)$. Then for $t > 0$, A_t is the smaller piece, so we have

$$V(t) = \mathrm{vol}(A_t) \leq h(M)\, \mathrm{area}(X_t).$$

The proof then follows the same lines as for the case of infinite volume, by applying the coarea formula to f. That is, we find

$$\int_{A_0} |\nabla f|\, dV \geq h(M) \int_0^\infty V(t)\, dt = \int_0^\infty -tV'(t)\, dt = \int_{A_0} |f|,$$

and similar for B_0. ■

Proof of Cheeger's inequality for finite volume. Let $f : M \to \mathbb{R}$ satisfy $\int f = 0$ and $\int f^2 = 1$; we need to bound $\int |\nabla f|$ from below. Since $\int (f+c)^2 \geq \int f^2$, we can change f by a constant and trade the assumption $\int f = 0$ for the assumption that $f^{-1}(0)$ cuts M into two pieces of equal size. Rescaling so $\int f^2 = 1$ only reduces $\int |\nabla f|$.

Now let $F(x) = f(x)|f(x)|$. Then $\int |F| = 1$ and $F^{-1}(0)$ also cuts M into 2 pieces of equal size. Thus

$$\int |\nabla F| \geq h(M) \int |F| = h(M).$$

But $|\nabla F| = 2|f \nabla f|$, so by Cauchy-Schwarz we have

$$\begin{aligned} h(M)^2 &\leq \left(\int |\nabla F| \right)^2 = \left(\int 2|f||\nabla f| \right)^2 \\ &\leq 4 \int |f|^2 \int |\nabla f|^2 = 4 \int |\nabla f|^2, \end{aligned}$$

hence the Theorem. ■

Theorem 17.5 *For a closed hyperbolic surface of* fixed *genus g, $\lambda_1(X)$ is small iff X has a collection of disjoint simple geodesics $\gamma_1, \ldots, \gamma_k$, $k \leq 3g-3$, such that the length of $S = \bigcup \gamma_i$ is small and S separates X.*

Proof. If $\lambda_1(X)$ is small then by Cheeger there is a separating 1-manifold S such that $\text{length}(S)/\,\text{area}(A)$ is small, where A is the smaller piece of $X - S$. Some of the curves have to be essential, else we would have $\text{area}(A) \ll \text{length}(S)$. These essential pieces are homotopic to geodesics of the required form.

Conversely, a collection of short geodesics has a large collar, which makes $\lambda_1(X)$ small if it is separating. ■

The combinatorial Laplacian. On a *regular* graph $\mathcal{G} = (V, E)$ of degree d, we define the Laplacian for functions $f : V \to \mathbb{R}$ by

$$(\Delta f)(x) = f(x) - \frac{1}{d} \sum_{y - x} f(y).$$

The sum is over the d vertices y adjacent to x. (A different formula is more natural for graphs with variable degree, and this is why our normalization differs from that in [Lub].)

The combinatorial *gradient* $|\nabla f|$ is a function on the edges of $\mathcal{G}$ defined by $|\nabla f|(e) = |f(x) - f(y)|$ if $e = \{x, y\}$.

Theorem 17.6 *For a regular graph $\mathcal{G} = (V, E)$ of degree d,*

$$\frac{1}{2d}\sum_E |\nabla f|^2 \quad = \quad \langle f, \Delta f\rangle = \sum_V f(x)\Delta f(x).$$

Proof. We have

$$\begin{aligned}\sum |\nabla f|^2 \quad &= \quad \sum_{x-y} |f(x)-f(y)|^2 = 2d\sum_x |f(x)|^2 - 2\sum_{x-y} f(x)f(y) \\ &= \quad 2d\sum_x f(x)\left(f(x) - \frac{1}{d}\sum_{x-y} f(x)f(y)\right) = 2d\langle f, \Delta f\rangle.\end{aligned}$$

■

The maximum principle. On a finite connected graph, the only harmonic functions are constant; just note that $f(x)$ must be equal to $f(y)$ on the neighbors of any point x where the maximum of f is achieved.

Random walks. Let x_n be a random walk on a regular graph $\mathcal{G}$. Then the expected value of a function f_0 after n steps is given by

$$f_n(x) = E(f(x_n)) = (I - \Delta)f_{n-1}(x) = (I - \Delta)^n f_0(x).$$

If $\mathcal{G} = (V, E)$ is finite and connected, we can find a basis ϕ_i for $L^2(V)$ with $\Delta\phi_i = \lambda_i\phi_i$, $0 = \lambda_0 < \lambda_1 < \ldots$. If $f_0 = \sum a_i\phi_i$ then

$$f_n = \sum a_i(1 - \lambda_i)^n\phi_i.$$

Thus for f_0 orthogonal to the constants we have:

$$\|f_n\|_2 \le (1 - \lambda^*)^n\|f_0\|_2,$$

where $(1 - \lambda^*) = \sup_{i\neq 0}|1 - \lambda_i|$; frequently $\lambda^* = \lambda_1$ (a counterexample is when $\mathcal{G}$ is bipartite).

In other words, λ^* measures the rate at which a random walk diffuses towards the uniform distribution on the vertices of $\mathcal{G}$.

The heat equation. In the continuum limit we have Brownian motion x_t on a compact Riemannian manifold, and the random walk equation for $f_t(x) = E(f_0(x_t))$ becomes

$$\frac{df_t}{dt} = -\Delta f_t.$$

If $f_0 = \sum a_i \phi_i$ then

$$f_t = \sum a_i \exp(-\lambda_i t)\phi_i$$

and we see λ_1 gives exactly the rate of decay of f_t towards its mean value.

The Poisson random walk on a graph. To mimic the heat equation more closely, we can consider the equation $df_t/dt = -\Delta f_t$ on a graph. Since $\exp(-t\Delta) = \lim(I - \Delta/n)^n$, we have a dual diffusion process x_t that can be thought of as a limit of random walks. In this process, one waits at a given vertex until a Poisson event occurs, then one takes a random step to a new vertex.

For this process, λ_1 is exactly the rate of diffusion.

The Cheeger constant for $\mathcal{G}$ is defined by

$$h(\mathcal{G}) = \inf_{A,B} \frac{|E(A,B)|}{\min(|A|,|B|)}$$

where $V = A \sqcup B$ is a partition of V, and $E(A,B)$ is the set of edges joining A to B.

Theorem 17.7 *For a regular graph of degree d,*

$$\lambda_i(\mathcal{G}) \le \frac{h(\mathcal{G})}{2d},$$

where $i = 0$ when $\mathcal{G}$ is infinite and $i = 1$ when $\mathcal{G}$ is finite.

Proof. Suppose $\mathcal{G}$ is infinite. Consider a partition of the vertices $V = A \sqcup B$ with $|A|$ finite, and let $f = \chi_A$. Then $\sum f^2 = |A|$, $\sum |\nabla f|^2 = |E(A,B)|$, and thus

$$\lambda_0(\mathcal{G}) \le \frac{\langle f, \Delta f\rangle}{\langle f, f\rangle} = \frac{\sum |\nabla f|^2}{2d\langle f, f\rangle} = \frac{|E(A,B)|}{2d|A|}.$$

Taking the inf over all partitions yields the theorem. The case of $\mathcal{G}$ finite is similar. ■

Cf. [Ch, Chap. 2] or [Lub, Prop. 4.2.4]. Note that the former has a different normalization for Cheeger's constant, while the latter has a different normalization for the Laplacian!

For an infinite regular graph, $h(\mathcal{G})$ is the same (to within a factor depending on d) as the *expansion constant* γ introduced earlier in the definition of amenability.

Theorem 17.8 *A finitely-generated group G is amenable iff $\lambda_0(\mathcal{G}) = 0$, where $\mathcal{G}$ is its Cayley graph.*

Proof. If G is amenable then the indicator functions of Følner sets show $\lambda_0 = 0$. Conversely, if $\lambda_0(\mathcal{G}) = 0$ then $h(\mathcal{G}) = 0$ and thus G has Følner sets. ∎

Theorem 17.9 (Brooks, [Br]) *Let $\widetilde{X}$ be a Galois covering of a compact Riemannian manifold X, with deck group G. Then $\lambda_0(\widetilde{X}) = 0$ iff G is amenable.*

Expanders. Let $\mathcal{G}_n$ be a collection of finite regular graphs of bounded degree, such that the number of vertices tends to infinity, but $h(\mathcal{G}_n) > H > 0$. Then $\mathcal{G}_n$ are *expanders.* By Cheeger's inequality going both ways we have:

Theorem 17.10 *The graphs $\mathcal{G}_n$ are expanders iff* $\inf \lambda_1(\mathcal{G}_n) > 0$.

Selberg's theorem: $\lambda_1(\mathbb{H}/\Gamma(n)) \geq 3/16$ for all n. Selberg's conjecture replaces 3/16 by 1/4 and has the more natural significance that the complementary series do not occur in the decomposition of $L^2(\Gamma\backslash G(n))$ into irreducible representations of G.

Examples of expanding graphs. (a) Fix generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ for $\mathrm{SL}_2(\mathbb{Z})$ and use them to construct the Cayley graphs for $G_n = \mathrm{SL}_2(\mathbb{Z}/n)$.

(b) For p a prime, consider the projective line $P_p = \mathbb{Z}/p \cup \{\infty\}$, and make it into a trivalent graph by joining x to $x+1$ and $-1/x$.

Theorem 17.11 *Both these families of graphs are expanders.*

Proof. (a) Suppose to the contrary that G_n has a low eigenvalue; we will show this implies $X(n) = \mathbb{H}/\Gamma(n)$ has a low eigenvalue $\lambda_1(X(n))$, contradicting Selberg's 3/16 theorem.

To $\lambda_1(X(n))$ is small, it suffices to construct a function $f \in L^2(X(n))$ with a small Ritz–Rayleigh quotient whose support occupies at most half the measure of $X(n)$. Then correcting f by a constant gives a function of mean zero with a small Ritz–Rayleigh quotient, and the latter bounds λ_1.

Let

$$F = \{z \in \mathbb{H} \,:\, |\operatorname{Re} z \ \le 1/2 \quad \text{and} \quad |z| \ge 1\}$$

be the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}$. Tile $X(n)$ by copies of F, and identify these copies with the vertices of G_n. The edges of G_n then correspond to edges where two tiles of $X(n)$ meet.

Fix a small constant $L > 0$. Let F' denote F made into a compact set by truncating along a horocycle of length L, i.e. by cutting off the region where $y > 1/L$. The long sides of F' then have length $\log(1/L) + O(1)$. Then a truncated version $X'(n)$ of $X(n)$ is tiled by copies of F'.

Let $V \subset G_n$ be a set of vertices such $|\partial V|/|V|$ is small and $|V| < |G_n|/2$. Let f be a function on $X'(n)$ with $f(x) = 1$ on the tiles corresponding to V, and $f(x) = 0$ on the other tiles. We call these the 0-tiles and the 1-tiles.

On each edge where a 0-tile and 1-tile meet, we modify f along a strip of width L and then make the transition from 0 to 1 linearly. Note that the hyperbolic length of ∇f is then $1/L$.

The result is a function on $X'(n)$ with

$$\int |f|^2 \asymp |V|.$$

Along one of the long strips S, of length $\log(1/L)+O(1)$, we get a contribution to $\int_{X'(n)} |\nabla f|^2$ of size

$$|\nabla f|^2 \times \operatorname{area}(S) = (1/L)^2 \times L \log(1/L) = (1/L)\log(1/L).$$

The contributions along the other strips are smaller. So all together, we have

$$\int_{X'(n)} |\nabla f|^2 \le (1/L)\log(1/L)|\partial V|.$$

It now remains to extend f to $X(n)$. This can done by scaling f linearly down to zero between the horocycles of length L and $L/2$ in each tile, and then extending by zero to the rest of the tile. This only increases $\int |f|^2$, but every 1-tile then makes a contribution of $O(L)$ to $\int |\nabla|^2$. So we finally get

$$\int_{X(n)} |\nabla f|^2 \leq C_1|V| + C_2|\partial V|,$$

where $C_1 = O(L)$ and $C_2 = O((1/L)\log(1/L))$. So finally we get

$$\lambda_1(X(n)) \leq \frac{\int |\nabla f|^2}{\int |f|^2} \leq C_1 + C_2 \frac{|bdryV|}{|V|}.$$

Choose L small enough that $C_1 < 1/100$. Then once the expansion constant of G_n is small enough, we find $\lambda_1(X(n)) < 1/50$, contradicting the lower bound $\lambda_1(X(n)) \geq 3/16$.

Thus the graphs G_n are expanders.

(b) The projective plane P_p is covered by G_p, so it is at least as expanding as G_p. ■

Appendix: Ramanujan graphs. The constant $\lambda_0(\mathbb{H}) = 1/4$ plays an important role in the theory of automorphic forms and in the spectral theory of the Laplacian on hyperbolic surfaces in general. For example, whenever a hyperbolic surface X contains a large ball, we can transport an approximate eigenfunction from $\mathbb{H}$ to X to show $\lambda_1(X) \leq 1/4 + \epsilon$. Similarly, when X has a cusp its Laplacian has continuous spectrum starting at 1/4.

For d–regular graphs, the role of the eigenvalue 1/4 for Δ on $\mathbb{H}$ taken over by the eigenvalue $2\sqrt{d-1}$ for the adjacency matrix A of the (infinite) d–regular tree T_d.

A finite graph d–regular $\mathcal{G}$ is a *Ramanujan graph* if its adjacency matrix has eigenvalues

$$d = \alpha_1 \geq \ldots \geq \alpha_n$$

and $\alpha_2 \leq 2\sqrt{d-1}$. (Some authors exclude the bipartite case where $\alpha_n = -d$.) We have labeled these eigenvalues by α_i to avoid confusion with the eigenvalues of the Laplacian,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots$$

We have $\Delta = I - (1/d)A$ and hence

$$\alpha_i = d(1 - \lambda_{i-1}).$$

Lines and intervals. There are several ways to relate $2\sqrt{d-1}$ to the regular tree T_d. To begin the discussion, first consider the tree $T_2 \cong \mathbb{Z}$ with vertices labeled by i.

The adjacency operator on $\ell^2(\mathbb{Z})$ satisfies

$$(Af)(i) = f(i-1) + f(i+1).$$

For any $\mu \in \mathbb{C}$, $f(i) = \mu^i$ is an eigenfunction with eigenvalue

$$\alpha = \mu + 1/\mu.$$

For $\mu < 1$, we have $\sum_{i>0} |f(i)|^2 < \infty$, and thus $f(i)$ is in L^2 when restricted to the part of the graph with $i \gg 0$.

It is also instructive to consider the eigenfunction

$$f(i) = \sin(\pi i/n) = (\mu^i + \mu^{-i})/2,$$

where $\mu = \exp(\pi i/n)$. This function satisfies

$$\Delta f = 2\cos(\pi/n) f.$$

It vanishes for $i \in n\mathbb{Z}$. In particular, when restricted to the interval $[0, n]$, it gives a non–negative function vanishing at the endpoints. This shows that the truncated finite graph $T_{2,n}$, which has vertices $\{1, 2, \ldots, n-1\}$, has

$$\alpha_1(T_{2,n}) = 2\cos(\pi/n).$$

The fact that $\alpha_1(T_{2,n}) \to 2$ is one way of making precise the fact that $\alpha_1(T_2) = 2$.

Note that $T_{2,n}$ is *not* a regular graph; its endpoints have degree one.

Finite and infinite trees. To discuss regular trees of degree $d > 2$, we introduce a *height function* $i : T_d \to \mathbb{Z}$. Here the height is a horofunction, analogous to the function $\log y$ on $\mathbb{H}$. It has the property that every vertex at height i is connected to $d-1$ vertices at height $i+1$ and a unique vertex at height $i-1$.

Thus if $f : V(T_d) \to \mathbb{R}$ depends only on the height, the same is true of its image under the adjacency operator, and we have

$$(Af)(i) = f(i-1) + (d-1)f(i+1).$$

Again, $f(i) = \mu^i$ is an eigenfunction for any μ, this time with eigenvalue

$$\alpha = \mu + (d-1)/\mu.$$

Note that the involution $\mu \mapsto (d-1)/\mu$ leaves α unchanged. Its fixed point $\mu = \sqrt{d-1}$, and the corresponding eigenvalue

$$\alpha = 2\sqrt{d-1},$$

again play a special role. One way to see this is to consider the eigenfunction

$$f(i) = \sin(\pi i/n)(d-1)^{-i/2}.$$

This function vanishes for $i \in n\mathbb{Z}$. Let $T_{d,n}$ be any component of the part of T_d with $1 \le i \le n-1$. By considering the restrction of this eigenfunction to $T_{d,n}$, we see that

$$\alpha_1(T_{d,n}) = 2\sqrt{d-1}\cos(\pi/n).$$

This quantity converges to $2\sqrt{d-1}$ as $n \to \infty$.

Note that $T_{d,n}$ has $(d-1)^i$ vertices at height $1 \le i \le n-1$. The root vertex at height $i = 1$ has degree $d-1$, and the leaves of the tree at height $i = n-1$ have degree 1; the rest have degree d. Letting $n \to \infty$ gives a natural sense in which

$$\alpha_1(T_d) = 2\sqrt{d-1}.$$

One should compare the eigenfunctions $f(i) = \mu^i$ on T_d to the eigenfunctions

$$f(y) = y^s = \mu^{\log y}$$

on $\mathbb{H}$, where $\mu = \exp(s)$. Note that $\Delta f = s(1-s)f$ and the symmetry $s \mapsto 1-s$, which preserves the eigenvalue, corresponds to $\mu \mapsto e/\mu$. The fixed–point of this symmetry gives $s = 1/2$ and the eigenvalue $\lambda = 1/4$.

Constructions of Ramanujan graphs. Clearly any sequence of finite, d–regular Ramanujan graphs $\mathcal{G}_n$ gives a sequence of expanders, with concrete expansion constants. Such graphs can be constructed using quaternion algebras, upon replacing $\mathrm{SL}_2(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{Q}_p)$. For details see [Lub].

18 All unitary representations of $\mathrm{PSL}_2(\mathbb{R})$

Classification of all irreducible unitary representations of $G = \mathrm{PSL}_2(\mathbb{R})$ [GGP].

Theorem 18.1 *Every such representation occurs in the principal, complementary or discrete series.*

First consider the representations that might occur in the regular representation on $L^2(\mathbb{H})$. As is the case for $\mathbb{R}$, the irreducible representations should correspond to eigenfunctions of the Laplacian (like $\exp(it)$ on $\mathbb{R}$), since Δ commutes with the action of G. Also we should not expect the eigenfunctions themselves to be in L^2. Finally the eigenvalues should be real and positive since Δ is a positive operator on L^2. (Note $\Delta e^{it} = t^2 e^{it}$ by our sign convention.)

To produce eigenfunctions of Δ, consider a conformal density $\mu = \mu(x)|dx|^s$ on $S^1 = \partial\mathbb{H}$. A point z in $\mathbb{H}$ determines a visual metric on S^1, which we can use to convert μ into a function and take its average, resulting in a function $f(z)$. Put differently, if we normalize so $z = 0$ in the disk model, then $\mu = \mu(\theta)|d\theta|^s$ and $f(0) = \int \mu(\theta)\, d\theta$. Since this operator is natural, it commutes with the action of G.

For $\mu = \delta_\infty(x)|dx|^s$ on $\mathbb{R}$, where δ_∞ is the delta function at the point at infinity, we find $f(z) = Cy^s$ by naturality. We have thus shown:

> *conformal densities* of dimension s on S^1_∞ determine
>
> *eigenfunctions of* Δ on $\mathbb{H}$ with eigenvalue $\lambda = s(1-s)$.

The principal series. These are the representations that arise in the decomposition of $L^2(\mathbb{H})$. Since $\lambda_0(\mathbb{H}) = 1/4$ we only expect $\lambda \geq 1/4$ to arise, and this corresponds to $s = 1/2 + it$, $t \in \mathbb{R}$. Then for any s-dimensional density μ, the quantity $|\mu|^2$ is naturally a measure (since $|g'z|^{it}$ has modulus 1 for any diffeomorphism g of S^1.) These generalize the half-densities.

For any such $s = 1/2 + it$, let $H(s)$ be the Hilbert space of conformal densities such that

$$\|\mu\|^2 = \int_{S^1} |\mu|^2 < \infty.$$

Then G acts unitarily on $H(s)$, and the set of all such representations (as t ranges in $\mathbb{R}$) is the *principal series.*

We have

$$L^2(\mathbb{H}) = \int_{1/2-i\infty}^{1/2+i\infty} H(s)\, ds.$$

(The explicit form of this isomorphism is given by the *Plancheral formula*, which is in turn related to the *Mellin transform*, the analogue of the Fourier transform for $\mathbb{R}^+$.)

Note: the ergodic action of G on S^1 gives rise, by pure measure theory, to a unitary action on half-densities; this is $H(1/2)$.

Note: $H(1/2+it)$ and $H(1/2-it)$ are equivalent; otherwise these representations are distinct.

The complementary series. Next consider conformal densities μ of dimension s in the range $0 < s < 1$, $s \neq 1/2$. Then μ also determines a density $\mu \times \mu$ on the space of geodesics. In this range we define the Hilbert space $H(s)$ by requiring that the norm

$$\|\mu\|^2 = \int_{S^1 \times S^1} (\mu \times \overline{\mu}) \left(\frac{dx\, dy}{|x-y|^2} \right)^{1-s} = \int_{\mathbb{R}\times\mathbb{R}} \frac{\mu(x)\overline{\mu(y)}}{|x-y|^{2-2s}}\, dx\, dy < \infty.$$

Note that $dx\, dy/|x-y|^2$ is the invariant measure on the space of geodesics. The integral requires some regularization when $s > 1/2$.

As $s \to 0$, the density $|dx|^s$ becomes more and more nearly invariant, and thus $H(s)$ tends to the trivial representation. These representations are the *complementary series*; no two are equivalent.

The discrete series. The remaining representations have no K-invariant vectors. This reflects the fact that they come from sections of nontrivial line-bundles over $\mathbb{H}$ rather than functions on $\mathbb{H}$.

For any integer $n > 0$ we let $H(n)$ be the Hilbert space of holomorphic n-forms $f(z)\, dz^n$ on $\mathbb{H}$, with norm

$$\|f\| = \int \rho^{1-n} |f|^2 = \int_{\mathbb{H}} |f(z)|^2 y^{2n-2} |dz|^2.$$

Here $\rho = |dz|^2/y^2$ is the hyperbolic area form. For example $H(1)$ is the space of holomorphic 1-forms (whose unitary structure requires no choice of metric).

These representations, together with their antiholomorphic analogues $H(-n)$, form the discrete series. Since they have no K-invariant vectors they are isolated from the trivial representation.

The complete picture. The space of all unitary representations of $\mathrm{PSL}_2\, \mathbb{R}$ can be organized as a collection $H(s)$ where $s = 1/2+it$ (the principal series), $0 < s < 1$, $s \neq 1/2$ (the complementary series), or $s = n$, an integer different from zero (the discrete series).

All these representations consist of eigenspaces of the *Casimir operator*, an invariant differential operator on $C^\infty(G)$, which combines the Laplace operator with $d^2/d\theta^2$ in the K-direction. Since Casimir is the same as the Laplacian on K-invariant functions, we also denote it by Δ. Now the holomorphic forms that make up the discrete series are sections of line bundles over $\mathbb{H}$; that is, they occur from induced representations of K, so they can also be considered as functions on G transforming by a given character $\chi : K \to \mathbb{C}^*$. Then the functions in $H(n)$ satisfy $\Delta f = n(1-n)f$. In other words, elements of the discrete series give *negative* eigenfunctions of the Casimir operator.

19 Expanding graphs and property T

In this section we study Kazhdan's property T; prove it holds for $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, and provide several applications, including to the construction of expander graphcs.

Property *T*. Let $\rho : G \to U(H)$ be a unitary representation of a locally compact topological group G. Suppose that for each compact set $K \subset G$ and $\epsilon > 0$, there exists a unit vector $f \in H$ such that

$$\|g \cdot f - f\| < \epsilon \; \forall g \in K.$$

In this case we say G has *almost invariant vectors.* Equivalently, one says that ρ *weakly contains* the trivial representation.

We say G has *property T* if any representation that has almost invariant vectors also has invariant vectors.

A better feel for the definition may be conveyed by formulating the contrapositive.

Proposition 19.1 *Suppose G has property T. Then there exists a compact set $K \subset G$, and an $\epsilon > 0$, such that for every ergodic unitary representation of G, and every unit vector f, we have $\|g \cdot f - f\| > \epsilon$ for some $g \in K$.*

Examples.

1. The group $\mathbb{Z}$ does not have property T. Acting on $\ell^2(\mathbb{Z})$, $\mathbb{Z}$ has almost invariant vectors but no invariant vectors.

2. Any finite or compact group has property T. If v is almost invariant, then $\int_G gv\,dg$ *is* invariant.

3. An infinite discrete group G with property T is nonamenable. Otherwise the regular representation of G on $L^2(G)$ has almost invariant vectors.

4. Any quotient of a group with property T also has property T. Thus any quotient is nonamenable or finite.

   In particular, the abelianization $G/[G,G]$ is finite.

5. The free group F_n on n generators does *not* have property T. Although it is nonamenable, it admits the amenable group $\mathbb{Z}$ as a quotient.

What groups *do* have propert T? We have:

Theorem 19.2 (Kazhdan) *Let G be a connected semisimple Lie group with finite center, all of whose factors have $\mathbb{R}$-rank at least 2. Then G has property T.*

Corollary 19.3 $\mathrm{SL}_n \mathbb{R}$ *has property T for any $n \geq 3$.*

Lemma 19.4 *If $\mathrm{SL}_2 \mathbb{R} \ltimes \mathbb{R}^2$ has almost invariant vectors, then $\mathbb{R}^2$ has invariant vectors.*

Proof. Decompose the representation over $\mathbb{R}^2$ so

$$H = \int_{\widehat{\mathbb{R}^2}} H_t \, d\sigma(t).$$

Then any $f \in H$ determines a measure μ_f on $\widehat{\mathbb{R}^2}$ by

$$\mu_f(E) = \int_E f^2 \, d\sigma = \|\pi_E(f)\|^2.$$

The total mass of this measure is $\|f\|^2$. Also we have

$$|\mu_f(E) - \mu_g(E)| = \left| \int_E (f-g)(f+g) \right| \leq \|f-g\| \cdot \|f+g\|,$$

so $f \mapsto \mu_f$ is continuous in the weak topology.

Now suppose f_n is a sequence of unit vectors, more and more nearly invariant under the action of a compact generating set K for $\mathrm{SL}_2 \mathbb{R}$. Then

$\mu_n = \mu_{f_n}$ is nearly invariant under the action of K, in the sense that $\mu_n(gE) \approx \mu_n(E)$ for all $g \in K$.

Now if $\mu_n(0) > 0$ then $\mathbb{R}^2$ has an invariant vector, so we are done. Otherwise we can push μ_n forward by the projection map

$$\mathbb{R}^2 - \{0\} \to \mathbb{RP}^1 \cong S^1,$$

to obtain a sequence of probability measure ν_n on the circle. But then there is a convergent subsequence, which must be invariant under the action of $\mathrm{SL}_2 \mathbb{R}$ by Möbius transformations. Clearly no such measure exists; hence there is an $\mathbb{R}^2$-invariant vector. ■

The same argument shows a relative form of property T for $\mathrm{SL}_n \mathbb{R} \ltimes \mathbb{R}^n$ for any $n \geq 2$. For the proof one uses the fact that there is no $\mathrm{SL}_n(\mathbb{R})$ invariant measure on $\mathbb{RP}^n$ for $n \geq 2$.

This fact fails for $n = 1$ and in fact the theorem is false in that case. When $n = 1$ the group $AN = \mathbb{R}_+ \ltimes \mathbb{R}$ is amenable, and hence it has almost invariant vectors that are not invariant.

Theorem 19.5 $\mathrm{SL}_n \mathbb{R}$ *has property T for all* $n \geq 3$.

Proof. Embed $\mathrm{SL}_{n-1} \mathbb{R} \ltimes \mathbb{R}^{n-1}$ into $\mathrm{SL}_n \mathbb{R}$, so $\mathbb{R}^{n-1} = P$. Then if $\mathrm{SL}_n(\mathbb{R})$ has almost invariant vectors, so does this semidirect product, and hence P has an invariant vector. But by the Howe–Moore Theorem 10.1, this implies that $\mathrm{SL}_n \mathbb{R}$ has a fixed vector. ■

Induced representations. Let Γ be a lattice in G, and suppose a represenation $\rho : \Gamma \to U(H)$ is given. Then we can form the corresponding flat bundle of Hilbert spaces over $\Gamma\backslash G$ and consider the Hilbert space of sections thereof.

Equivalently, we can consider the space $H' = L^2_\rho(G, H)$ of maps $f : G \to H$ such that $f(gx) = g \cdot f(x)$ for all $g \in \Gamma$, normed by:

$$\|f\|^2_{H'} = \int_{\Gamma\backslash G} \|f(g)\|^2_H \, dg,$$

where dg is normalized to give a probability measure on $\Gamma\backslash G$.

Then G acts unitarily on H'. This action is the *induced representation* of G, denoted by $\mathrm{Ind}^G_\Gamma(\rho)$.

Lemma 19.6 *A representation of* Γ *has a fixed vector iff the induced representation of* G *has a fixed vector.*

Proof. If $F \in H$ is fixed by Γ, then the constant map $f(g) = F$ is fixed by G. Conversely, any G–fixed vector must assume a single value in $F \in H$ which is fixed by Γ. ■

Lifting. Now let $D \subset G$ be a measurable fundamental domain for the action of Γ. We then have a tiling

$$G = \bigcup_{\Gamma} gD,$$

which gives rise to a natural isometry

$$\phi_D : H \to L^2_\rho(G, H)$$

which respects the action of Γ. This lifting is characterized by the property that $\phi_D(F)|D = F$. Its image consists of functions that are constant on each tile, the value on gD being $\rho(g)F$.

Lemma 19.7 *If a representation of* Γ *has almost fixed vectors, then so does the induced representation of* G.

Proof. Let $K \subset G$ be a compact set. Let $D \subset G$ be a fundamental domain as above. Then $K \cdot D$ has finite measure, so given $\epsilon > 0$ we can cover all but measure ϵ of $K \cdot D$ with *finitely many* tiles $(g_i D)$, $i = 1, \ldots, n$, $g_i \in \Gamma$. (This set of small measure is not required when Γ is cocompact.) Choose a unit vector $F \in H$ that is nearly fixed by $(g_1, \ldots, g_n)$, in the sense that $\|g_i \cdot F - F\| < \epsilon$. Then the unit vector $\phi_D(F)$ is moved at most distance 2ϵ by any $g \in K$. ■

Theorem 19.8 *Let* $\Gamma \subset G$ *be a lattice in a Lie group with property T. Then* Γ *also has property T.*

Proof. Apply the previous two Lemmas. Suppose Γ has almost invariant vectors. Then the induced representation of G does too. Then by property T, G has an invariant vector, which furnishes an invariant vector for Γ as well. ■

Corollary 19.9 $\mathrm{SL}_n \mathbb{Z}$ *has property T for all $n \geq 3$.*

Proof. This subgroup is a lattice (by a general result of Borel and Harish-Chandra; see [Rag, Cor 10.5]). ■

Corollary 19.10 $\mathrm{SL}_2 \mathbb{R}$ *does not have property T.*

Proof. Consider any torsion-free lattice Γ in $\mathrm{SL}_2 \mathbb{R}$. Then $\mathbb{H}/\Gamma$ is a surface whose first homology $H^1(\Gamma, \mathbb{Z})$ is nontrivial; thus Γ maps surjectively to $\mathbb{Z}$ and so it does not have property T, so neither does $\mathrm{SL}_2 \mathbb{R}$. ■

Another proof can be given using the fact that the complementary series of unitary representations of $\mathrm{SL}_2 \mathbb{R}$ accumulate at the trivial representation in the Fell topology. (These correspond to eigenfunctions whose least eigenvalue is tending to zero.) A direct sum of such representations has almost invariant vectors but no invariant vector.

Theorem 19.11 *Fix a generating set for $\mathrm{SL}_3(\mathbb{Z})$, and let $\langle G_p \rangle$ denote the corresponding Cayley graphs for $\mathrm{SL}_3(\mathbb{Z}/p)$. Then $\langle G_p \rangle$ is a family of expanders.*

Proof. The action of $\mathrm{SL}_3(\mathbb{Z})$ on $L^2_0(G_p)$ has no invariant vectors, so it has no almost invariant vectors, and thus $h_1(G_p) > \epsilon > 0$ where ϵ does not depend on p. ■

Cohomological characterization.

Theorem 19.12 *G has property T if and only if every unitary (or orthogonal) affine action of G on a Hilbert space H has a fixed-point.*

Equivalently, $H^1(G, H) = 0$ for every unitary G-module H.

Group cohomology. The group $H^1(G, H_\pi) = Z^1(G, H)/B^1(G, H)$ classifies affine actions with a given linear part, modulo conjugation by translations.

An affine action of G on H can be written as

$$g(x) = \pi(g)x + \alpha(g).$$

The linear part $\pi : G \to \operatorname{Aut} H$ is a group homomorphism; the translation part, $\alpha : G \to H$, is a *1-cocycle* on G. This means α is a crossed-homomorphism:

$$\alpha(gh) = \alpha(g) + \pi(g)\alpha(h).$$

For example, when π is the trivial representation, then α is just a homomorphism. The *coboundary* of $y \in H$ is the cocycle given by

$$\alpha(g) = gy - y;$$

it corresponds to the trivial action conjugated by $x \mapsto x + y$.

The groups $Z^1(G, H) \supset B^1(G, H)$ denote the cocycles and coboundaries respectively. A cocycle α is a coboundary if and only if the affine action of G corresponding to α has a fixed-point $y \in H$.

Theorem 19.13 *$H^1(G, H_\pi) = 0$ implies property T for π.*

Proof. Suppose π has no fixed vectors. Then the coboundary map $\delta : H \to B^1 = Z^1(G, H)$ is bijective. But $Z^1(G, H)$ is a Banach space. For example, if G is finitely generated, then a cocycle α is determined by its values of generators $g_1, \ldots, g_n$ and we can define

$$\|\alpha\| = \max \|\alpha(g_i)\| = \max \|g_i x - x\|$$

if $\alpha = \delta(x)$. Since δ is bijective, by the open mapping theorem it is bounded below: that is, there is an $\epsilon > 0$ such that $\|\delta(x)\| > \epsilon\|x\|$. This says exactly that π has no almost-invariant vectors. ■

Application of property T to means on the sphere. Cf. [Lub], [Sar].

Theorem 19.14 (Margulis, Sullivan, Drinfeld) *Lebesgue measure is the only finitely-additive rotationally invariant measure defined on all Borel subsets of S^n, $n \geq 2$.*

The result is false for $n = 1$ (Banach). We will sketch the proof of $n \geq 4$ due to Margulis and Sullivan (independently).

The *Banach-Tarski paradox* states that for $n \geq 2$, S^n can be partitioned into a finite number of (non-measurable) pieces, which can be reassembled by rigid motions to form 2 copies of the sphere.

Corollary 19.15 *There is no finitely-additive rotation invariant probability measure defined on all subsets of S^n.*

Lemma 19.16 (Cantor-Bernstein) *Given injections $\alpha : A \to B$ and $\beta : B \to A$, we can construct a bijection*

$$\phi : A \to B$$

by $\phi(x) = \beta^{-1}(x)$ or $\alpha(x)$ according to whether $x \in \beta(C)$ or not. Here

$$C = \bigcup_0^\infty (\alpha\beta)^n (B - \alpha(A)).$$

Proof. Classify points in A according to their inverse orbits under the combined dynamical system $\alpha \cup \beta : (A \cup B) \to (A \cup B)$. On infinite orbits, α is a bijection. Finite orbits terminate in either A or B. If they terminate in A, then α gives a bijection; if they terminate in B, then β^{-1} is a bijection. The set $\beta(C)$ consists of those that terminate in B. ■

Lemma 19.17 *The free group $G = \langle a, b \rangle$ can be decomposed into a finite number of sets that can be rearranged by left translation to form $2G$, two copies of G.*

Proof. Letting A and A' denote the words beginning with a and a^{-1}, B and B' similarly for b and b^{-1}, and E for the identity element, we have $G = A \sqcup A' \sqcup B \sqcup B' \sqcup E$, $G = A \sqcup aA'$, $G = B \sqcup bB'$, and thus G is congruent to $2G \cup E$. Now applying the Cantor-Bernstein argument to get a bijection. ■

More explicitly, shifting the set $\{e, a, a^2, \ldots\}$ into itself by multiplication by a, we see G is congruent to $G - E$, and thus to $2G$.

Lemma 19.18 *For $n \geq 2$, $\mathrm{SO}(n+1)$ contains a free group on 2 generators.*

Proof. The form

$$x_0^2 - \sqrt{2} \sum_1^n x_i^2$$

is Galois equivalent to a definite form, so we get an isomorphism $\mathrm{SO}(n, 1, \mathcal{O}) \cong \mathrm{SO}(n+1, \mathcal{O})$, where $\mathcal{O}$ denotes the ring of integers in $\mathbb{Q}(\sqrt{2})$. The group $\mathrm{SO}(n, 1, \mathcal{O})$ contains a pair of hyperbolic elements that generate a free group. ■

Lemma 19.19 *For any countable set E in S^2, $S^2 - E$ is scissors congruent with S^2.*

Proof. Find a rotation R such that $R^n(E) \cap E = \emptyset$ if $n \neq 0$. Then by applying R, we see $S^2 - \bigcup_0^\infty R^n(E)$ is congruent to $S^2 - \bigcup_1^\infty R^n(E)$. Adding $\bigcup_1^\infty R^n(E)$ to both sides we see $S^2 - E$ is congruent to S^2. ■

Proof of the Banach-Tarski paradox. Let $G = Z * Z$ act faithfully on S^2. Then after deleting a countable set E, the action is free. Thus $S^2 - E = G \times F$ for some $F \subset S^2$, where the bijection sends (g, f) to $g \cdot f$. Now the congruence $G = 2G$ gives $(S^2 - E) = 2(S^2 - E)$. On the other hand, $S^2 - E$ is congruent to S^2 on both sides, so we are done.

To prove the paradox for S^n, $n > 2$, use induction on n and suspension to get a decomposition for S^{n+1} from one for S^n. ■

Lemma 19.20 *If ν is a finitely-additive rotation invariant measure on S^n, $n \geq 2$, then ν is absolutely continuous with respect to Lebesgue measure.*

Proof. If D_i are disks whose diameters tend to zero, then $\nu(D_i) \to 0$ since many of these disks can be packed in S^n. On the other hand, an extension of Banach-Tarski shows that D_i is congruent to S^n. So a set of Lebesgue measure zero is congruent to a subset of D_i for any i and thus its ν-measure must also be zero. ■

Lemma 19.21 *For $n \geq 4$, $\mathrm{SO}(n+1)$ contains a countable subgroup with property T.*

Proof. Construct an arithmetic Kazhdan group $\mathrm{SO}(2, n-1, \mathcal{O})$ using the form

$$x_1^2 + x_2^2 - \sqrt{2} \sum_3^{n+1} x_i^2.$$

Then apply a Galois automorphism to map it into $\mathrm{SO}(n+1, \mathcal{O})$.

■

Note: $\mathrm{SO}(p,q)$ has real rank $r = \min(p,q)$ since the form

$$x_1y_1 + \ldots + x_ry_r + \sum_{1}^{p+q-2r} z_i^2$$

has type (p,q). Also $\mathrm{SO}(2,p)$ is simple for $p \geq 3$; $\mathrm{SO}(2,2)$, on the other hand, is locally isomorphic to $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$ just as $\mathrm{SO}(4) \cong \mathrm{SO}(3) \times \mathrm{SO}(3)$. That is why we need to go up to at least $\mathrm{SO}(5)$.

Theorem 19.22 *Lebesgue measure is the only rotation invariant mean on $L^\infty(S^n)$, $n \geq 4$.*

Proof. Let $G \subset \mathrm{SO}(n+1)$ be a finitely-generated group with property T. Then there is an $\epsilon > 0$ such that for any $f \in L^2_0(S^n)$, with $\|f\| = 1$, we have $\|f - g_if\| > \epsilon$ for one of the generators g_i of G.

Now suppose there is an invariant mean that is not proportional to Lebesgue measure. The space $L^1(S^n)$ is dense in the space of means, so we can find $F_n \in L^1(S^n)$ converging weakly to the invariant mean. Setting $f_n = \sqrt{F_n}$, we have $\|f_n\|_2 = 1$ and $\|g_if_n - f_n\| \to 0$ for each generator of G. Thus the projection of f_n to the mean-zero functions must tend to zero, which implies that F_n converges to Lebesgue measure on S^n. ■

20 The circle at infinity and Mostow rigidity

In this section we study the action of a surface group on $S^1_\infty = \partial\mathbb{H}$, relate it to quasigeodesics, and then briefly sketch the proof of:

Theorem 20.1 (Mostow) *Let $f : M_1 \to M_2$ be a homotopy equivalence between compact hyperbolic n–manifolds, $n \geq 3$. Then f is homotopic to an isometry.*

Put differently, this result says that $M = \Gamma\backslash\mathbb{H}^n$ is determined, up to isometry, by the abstract group $\pi_1(M)$.

Topology of the $\boldsymbol{\Gamma}$-action on $\boldsymbol{S^1_\infty}$. There is a great deal of information packaged in the action of $\Gamma \cong \pi_1(X)$ on the circle at infinity. To unpack it, we will look at the action on not just the circle, but also various powers of the circle as well. These actions become less chaotic as the power increases.

Theorem 20.2 *For any finite volume surface $Y = \mathbb{H}/\Gamma$,*

> *(a) Every Γ orbit on S^1_∞ is dense;*
> *(b) fixed-points are dense in S^1_∞;*
> *(c) Γ acts ergodically with respect to the Lebesgue measure class on S^1_∞.*

Proof. More generally consider any closed Γ-invariant set $E \subset S^1_\infty$ Then the closed convex hull of E in $\mathbb{H}$ must be all of $\mathbb{H}$, since it descends to give a convex surface of Y carrying the fundamental group. Therefore $E = S^1_\infty$. This prove (a) and (b).

For (c), given an invariant set $A \subset S^1_\infty$ of positive measure, we can extend χ_A to a bounded, Γ–invariant harmonic function $u(z)$ on $\mathbb{H}$, which descends then to Y. When Y is compact we can apply the maximum principle to conclude that u is constant and hence A has full measure. For the case of finite volume we first apply removable singularities to fill in u over the punctures of Y. ∎

Action on $S^1_\infty \times S^1_\infty$. We remark that there is no σ-finite Γ–invariant measure in the Lebesgue measure class on S^1_∞. However, once we go the two points on the circle, there is a natural G–invariant measure on

$$S^1_\infty \times S^1_\infty \cong \widehat{\mathbb{R}} \times \widehat{\mathbb{R}} \approx \mathcal{G} = G/A,$$

given by $dx\, dy/(x-y)^2$. Note that we must consider pairs of *distinct* points to get a correspondence to oriented geodesics.

Theorem 20.3 *Let $X = \mathbb{H}/\Gamma$ be a finite area surface.*

> *(a) The action of Γ on $\mathcal{G}$ is transitive (there exists a dense orbit).*
> *(b) A point $x \in \mathcal{G}$ has a discrete Γ-orbit iff the geodesic $\gamma_x \subset X$ is closed or joins a pair of cusps.*
> *(c) Closed geodesics are dense in $T_1 X$.*
> *(d) The action of Γ is ergodic for the (infinite) invariant measure on $\mathcal{G}$.*

Proof. (a) and (d) follow from ergodicity of the geodesic flow on $\Gamma\backslash G$.

For (b), we note that a discrete Γ orbit corresponds to a *locally finite* configuration of geodesics in $\mathbb{H}$, and hence a properly immersed geodesic in X, which must either be closed or divergent in both directions.

For (c), we use the a dense geodesic $\gamma : \mathbb{R} \to \mathrm{T}_1 X$ as provided by (a). Choose $s_n \to -\infty$, $t_n \to +\infty$ such that $d(\gamma(s_n), \gamma(t_n)) \to 0$ in $T_1 X$. Then for n large we can *close* γ by perturbing it slightly so $\gamma(s_n) = \gamma(t_n)$, and then taking the geodesic representative δ_n of this loop. The closed geodesic is very close to $\gamma(t)$ for t small compared to s_n and t_n, so closed geodesics are dense.

Note: the same argument shows geodesics joining cusps are dense when X is noncompact.

■

Aside: The space of $Z = \Gamma \backslash \mathcal{G}$ of geodesics on X as a non-commutative space. Note that Z is a smooth manifold, with natural measure theory and so on, but not Hausdorff. Every continuous function on Z is constant (because there is a dense point); every measurable function on Z is constant (by ergodicity of the geodesic flow).

Traditional spaces such as manifolds can be reconstructed from commutative algebras such as $C(Y)$, $L^\infty(Y)$, etc. For Z these commutative algebras are trivial. To reconstruct Z, we need to associate to it *non-commutative* algebras.

To make an interesting algebra on Z, we consider sections of bundles over Z. The most basic bundle is the Hilbert space bundle $H \to Z$ whose fiber over $p = [\Gamma x]$ is $H_p = \ell^2(\Gamma x)$. We can then build the bundle $\mathcal{B}(H)$ of bounded operators on the fiber; this is a bundle of algebras.

Note that every element f of $L^\infty(\mathcal{G})$ gives a section of this bundle, by consider the multiplication operator $f(\gamma x)$ on $L^2(\Gamma x)$. This shows the bundle $H \to Z$ is *nontrivial*. Indeed, by ergodicity, any section of the *trivial* Hilbert space bundle over Z is constant almost everywhere.

Additional (unitary) sections of the bundle of algebras $\mathcal{B}(H)$ are obtained by considering the action of Γ on the fibers. Together these produce a non-commutative algebra $\mathcal{A}$ that in some sense records the space Z (Connes).

Action on $S^1_\infty \times S^1_\infty \times S^1_\infty$. Let T denote the set of *ordered* 3-tuples of distinct points (a, b, c) on the circle at infinity. This space has two components, depending on the orientation of the triangle with vertices (a, b, c). Let T_0 denote one of these components. Then G acts simply transitively on T_0. In other words, we can identify T_0 with $\mathrm{T}_1\mathbb{H}$.

The identification can be made geometric by taking the geodesic γ from a to c, and dropping a perpendicular from b to obtain a point $p \in \gamma$. Then the point p together with the direction of γ gives an element of $\mathrm{T}_1\mathbb{H}$.

Note that T_0 retracts onto the space of triples of the form $(a, \omega a, \omega^2 a)$ where $\omega^3 = 1$, i.e. it retracts onto the space of equilateral triangles, so evidently

$$\pi_1(G) = \pi_1(T_0) = \pi_1(T_1\mathbb{H}) = \mathbb{Z}.$$

Now T_0 comes equipped with a 1-dimensional foliation obtained by varying b while holding a and c fixed. This foliation covers the foliation of T_1X by geodesics. We have shown:

Theorem 20.4 *The space of positively oriented triples of distinct points on S^1_∞, modulo the action of $\pi_1(X)$, is homeomorphic to T_1X with its foliation by the orbits of the geodesic flow.*

What is the circle at infinity? Because of all these results, one is led to ask the question: does the action of $\pi_1(X)$ on the circle *depend* on the hyperbolic metric on X? We will see that, at least topological, it does *not*. Thus one would like to be able to construct the 'circle at infinity' canonically, in terms of topology rather than geometry. This can be done using the notion of quasi–geodesics, which tend to be stable under variation of the metric.

Quasi-geodesics. A path $\gamma(s)$ in $\mathbb{H}$, parameterized by arclength, is a *quasi-geodesic* if $d(\gamma(s), \gamma(t)) > \epsilon|s - t|$ for all s and t.

Theorem 20.5 *Any quasigeodesic is a bounded distance from a unique geodesic.*

Proof. Let δ_n be the hyperbolic geodesic joining $\gamma(-n)$ to $\gamma(n)$, and consider the r-neighborhood N of γ_n for $r \gg 0$. Then projection from ∂N to δ_n shrinks distance by a factor of about e^{-r}.

Suppose $\gamma([a, b])$ with $[a, b] \subset [-n, n]$ is a maximal segment outside of N, with endpoints in ∂N. Then we can join $\gamma(a)$ to $\gamma(b)$ by running distance r to δ_n, and then running along the projection of $\gamma([a, b])$. The total length so obtained is $r + e^{-r}|a - b| > \epsilon|a - b|$, so we see a and b must be close (if we choose r so $e^{-r} \ll \epsilon$.)

Therefore excursions outside of N have bounded length, so for another $R > r$ we have $\gamma[-n, n]$ contained in an R-neighborhood of δ_n.

Taking a limit of the geodesics δ_n as $n \to \infty$ we obtain the theorem. ■

Coarser versions. More generally, a map $f : X \to Y$ between a pair of metric spaces is a *quasi-isometry* if there exists a $K > 1$ and $R > 0$ such that

$$Kd(x, x') \geq d(fx, fx') \geq d(x, x')/K - R$$

for all $x, x' \in X$. The factor of R means that distances are preserved, up to a factor of K, whenever $x, x' \in X$ are far enough apart. The preceding argument also shows:

Theorem 20.6 *The image of any quasi-isometry $f : \mathbb{Z} \to \mathbb{H}$ lies within a bounded distance of a complete geodesic.*

Smoother versions. One can also control the behavior of a path by its curvature. Any path in $\mathbb{H}$ which is nearly straight is a quasi-geodesic.

Theorem 20.7 *Any curve $\gamma \subset \mathbb{H}$ with geodesic curvature bounded by $k < 1$ is a quasi-geodesic.*

Proof. Consider the line L_s orthogonal to $\gamma(s)$ at p. For $k < 1$ these lines are disjoint (the extreme case is a horocycle where $k = 1$). Also $(d/ds)d(L_0, L_s) > c(k)s$, where $c(k) \to 1$ as $k \to 0$. Thus $d(L_s, L_t) > c(k)|s - t|$ and this provides a lower bound for $d(\gamma(s), \gamma(t))$. ■

Theorem 20.8 *A polygonal path γ of segments of length at least L and bends at most $\theta < \pi$ is a quasi-geodesic when L is long enough compared to θ. Also as $(L, \theta) \to (\infty, 0)$ the distance from γ to its straightening tends to zero.*

Proof. As before L_s advances at a definite rate, indeed at a linear rate, except near the bends of γ. The size of the neighborhood of the bend that must be excluded tends to zero as $\theta \to 0$. Thus the geodesic representative of γ is very close to γ between the bends when L is long. ■

The circle at infinity. The next result shows that, as a topological space, the circle S^1_∞ is canonically attached to any compact surface. There are many ways to view this result. To begin with, we will show:

Theorem 20.9 *Let $h : X \to Y$ be a diffeomorphism between closed hyperbolic surfaces. Then $\widetilde{h} : \mathbb{H} \to \mathbb{H}$ extends to a homeomorphism $S^1_\infty \to S^1_\infty$ conjugating Γ_X to Γ_Y.*

Proof. The map h is bilipschitz, so $\widetilde{h}$ maps geodesics to quasigeodesics. Straightening these, we obtain a map $S^1_\infty \to S^1_\infty$. To check continuity, note that geodesics near infinity straighten to geodesics near infinity. ■

Topological construction of S^1_∞. The preceding result can be viewed more functorially as follows. First, associated to a closed surface S is a the fundamental group $\pi_1(S)$. Choosing a finite generating set, we obtain a metric on $\pi_1(S)$ that is well–defined up to isometry. We can then functorially construct

$$\partial\pi_1(S) = \{\text{quasi-geodesics } \mathbb{N} \to \pi_1(S)\}/\sim$$

as a topological space. Here we regard two rays in $\pi_1(S)$ as equivalent if they are a bounded distance apart.

Next, we observe that the universal cover of a pointed, connected, compact manifold (M, x) is quasi-isometric to $\pi_1(M, x)$. In particular, if $X = \mathbb{H}/\Gamma$ and we have an isomorphism $\pi_1(S) \cong \pi_1(X)$, then we get a natural homeomorphism

$$\partial\pi_1(S) \to S^1_\infty = \partial\mathbb{H}$$

by straightening quasigeodesics. This shows that $\partial\pi_1(S)$ is a circle and that this circle is naturally identified with $\partial\mathbb{H}$ once a marking of X is chosen.

Finally if we have two marked hyperbolic surfaces, then both their circles at infinity are marked by $\partial\pi_1(S)$ and hence we obtain a homeomorphism on the circle at infinity.

Flexibility and rigidity of foliations. We are now ready to study the geodesic and horocycle foliations of T_1X, and their dependence on the hyperbolic structure on X. Indeed, Theorem 20.9 immediately yields:

Theorem 20.10 *If X and Y are homeomorphic closed surfaces, then the foliations of T_1X and T_1Y by orbits of the geodesic flow are topologically equivalent.*

Proof. The actions of $\pi_1(X) \cong \pi_1(Y)$ on S^1_∞ are topologically the same, and T_1X together with its geodesic foliation can be reconstructed from the diagonal action on $(S^1_\infty)^3$. ■

Note however that we have:

Theorem 20.11 *Let X and Y be hyperbolic surfaces. Suppose the geodesic flows on T_1X and T_1Y are topologically conjugate. Then X is isometric to Y.*

Proof. The surfaces X and Y then have the same marked length spectrum, and this is enough to get an isometry. ■

A glimpse of Mostow rigidity. Here is the first inkling of rigidity for hyperbolic manifolds. The same argument will yield a much stronger conclusion in higher dimensions.

Theorem 20.12 *Let $X_i = \mathbb{H}/\Gamma_i$, $i = 1, 2$, be a pair of compact hyperbolic surfaces. Let $f : S^1_\infty \to S^1_\infty$ be a homeomorphism that sends the action of Γ_1 to the action of Γ_2.*

Suppose $0 < f'(x) < \infty$ exists at one point *$x \in S^1_\infty$. Then f is a Möbius transformation, and X_1 and X_2 are isometric.*

Proof. By assumption we have an isomorphism $\gamma \mapsto \gamma' \in \Gamma'$ such that $f(\gamma x) = \gamma' f(x)$ for all $x \in S^1_\infty$ and $\gamma \in \Gamma$. Choose coordinates on the upper halfplane such that $S^1_\infty = \widehat{\mathbb{R}}$ and $f(0) = 0$ and $f'(0) = 1$. Let $A_n(x) = nx$. Then we have

$$A_n f(A_n^{-1} x) \to x$$

uniformly on S^1_∞, by the definition of differentiability. Since $\Gamma \ G$ is compact, we can find $\gamma_n \in \Gamma$ such that $A_n \gamma_n'$ is bounded in G. We then have

$$A_n \gamma_n' f(\gamma_n^{-1} A_n^{-1} x) = A_n f(A_n^{-1} x) \to x.$$

Pass to a subsequence such that $A_n \gamma_n' \to A' \in G$. Since f is a homeomorphism, we then also have $A_n \gamma_n \to A \in G$, and hence

$$A' f(A^{-1} x) = x.$$

This shows that $f(x)$ is a Möbius transformation. ■

Corollary 20.13 *The map f is either a Möbius transformation or it is completely singular (it sends a set of full measure to a set of measure zero).*

Proof. A monotone function like f is differentiable a.e. We have just shown that $f'(x) = 0$ or ∞ whenever it exists, so f is singular. (In general $m(E) = \int_E f'(x)\, dx$ gives the absolutely continuous part of $f^*(dx)$.) ■

Hölder continuity and quasi-symmetry. Here is a direct proof that if f is Lipschitz, then it is a Möbius transformation. Namely, for $\lambda, \delta < 1$, the geometric sequences λ^n and δ^n are related by a Lipschitz map iff $\lambda = \delta$. This implies that the lengths of corresponding geodesics agree.

Similarly, if we know that $f, f^{-1} \in C^\alpha$, meaning

$$|f(x) - f(y)| \leq M|x - y|^\alpha,$$

then the lengths of geodesics change at most by a multiplicative factor controlled by α.

In fact, the mapping f is always *quasisymmetric*, which implies Hölder continuity.

Mostow rigidity. We conclude with a sketch of the proof of Theorem 20.1.

A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *quasiconformal* if there exists a $K \geq 1$ such that for any ball $B(x, r)$, there exists an $s > 0$ such that

$$B(f(x), s) \subset f(B(x, r)) \subset B(f(x), Ks).$$

Let us denote that best K that works here by $K(x, s)$, and let

$$K(x) = \limsup_{s \to 0} K(x, s).$$

This notion can be made local and then generalized so it applies to manifolds with a conformal structure. In particular one can talk about K–quasiconformal maps between Riemann surfaces and spheres.

In dimension $n \geq 2$, these maps enjoy unexpected regularity: e.g. they are absolutely continuous, they are differentiable *a.e.*, etc. Moreover, if $K(x) = 1$ a.e., then f is actually *conformal.* In dimension two, conformality implies f is analytic; in dimension 3 or more, it implies f is locally a Möbius transformation.

All these properties fail in dimension two, as we have seen above for conjugacies $f : S^1 \to S^1$.

Proof of Theorem 20.1. Let $f : M_1 \to M_2$ be a homotopy equivalence between hyperbolic manifolds $M_i = \Gamma_i \backslash \mathbb{H}^n$, $i = 1, 2$, $n \geq 3$. Lifting to the universal covers, we obtain a coarse quasi–isometry $\widetilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ intertwining the actions of Γ_1 and Γ_2. This map sends geodesics to quasi–geodesics, and in turn extends to a homeomorphism $F : S^{n-1} \to S^{n-1}$. The bounded distortion of metric geometric of $\widetilde{f}$ yields, in the limit, a bound on the conformal distortion of F; that is, F is quasiconformal. In particular, DF_p exists for almost every $p \in S^{n-1}$, since $n \geq 2$.

For simplicity, let us now assume that $n = 2$. Since Γ_i acts conformally, $K(x)$ is a Γ_1–invariant function on S^2. If $K(x) = 1$ a.e., then F extends to an isometry of $\mathbb{H}^3$ that descends to give the desired isometry between M_1 and M_2. Other words, $K(x)$ is a constant $K > 1$ almost everywhere.

This means that wherever DF_p is defined, the preimage of a circle in the tangent space $\mathrm{T}_{f(p)}(S^2)$ under DF_p is an ellipse of eccentricity K. The major axis of this ellipse defines a natural line $L_p \subset \mathrm{T}_p(S^2)$. This line field is invariant under the action of Γ_1 on $\mathrm{T}S^2$.

We now define a function $\theta(v)$ on $\mathrm{T}_1\mathbb{H}^3$ that measures the angle between the lines L_p and L_q at the two endpoints of the geodesic γ through v, measured using parallel transport along γ. This function is constant along any geodesic, so it descends to a function on $\mathrm{T}_1 M_1$ that is invariant under the geodesic flow. By ergodicity of the geodesic flow, $\theta(v)$ is constant almost everywhere. But it is easy to see that there is no line field on S^2 with this property. This contradiction shows $K = 1$. ■

Proof 2. Blow up a point of differentiability of F as we did in Theorem 20.12. For $n \geq 2$ the end result, suitably normalized, is a real linear map L on $\mathbb{R}^n \cup \{\infty\}$ conjugating Γ_1 to Γ_2. If L is not conformal, then one can show its behavior at ∞ is different from elsewhere, so the entire group Γ_1 must fix infinity, which is impossible. ■

21 Ergodic theory at infinity of hyperbolic manifolds

Function theory on hyperbolic manifolds.

Theorem 21.1 *$M = \mathbb{H}^n/\Gamma$ admits a non-constant bounded harmonic function h if and only if the action of Γ on S^{n-1}_∞ is ergodic.*

Note: in general there is no invariant measure on S^{n-1}_∞!

Proof. The boundary values of h give a Γ-invariant measurable function on S^{n-1}_∞, and conversely any bounded Γ-invariant function on S^{n-1}_∞ extends, by visual average, to a harmonic function lifted from M. ■

Theorem 21.2 *The geodesic flow on $T_1M = \mathbb{H}^n/\Gamma$ is ergodic if and only if the action of Γ on $S^{n-1}_\infty \times S^{n-1}_\infty$ is ergodic.*

Example. Let $M \to N$ be a Galois covering of a closed hyperbolic manifold with deck group $\mathbb{Z}^3$. Then Γ_M is ergodic on S^{n-1}_∞ but not on $S^{n-1}_\infty \times S^{n-1}_\infty$. This is because random walks in $\mathbb{Z}^3$ are transient (which implies the geodesic flow is *not* ergodic), but bounded harmonic functions are constant.

For more on function-theory classes of manifolds, see [SN], [LS], [Ly], [MS], [W], [Th, Ch. 9].

Mostow rigidity.

Theorem 21.3 *Let $\phi : M \to N$ be a homotopy equivalence between a pair of closed hyperbolic n-manifolds, $n \geq 3$. Then $\widetilde{\phi} : \mathbb{H}^n \to \mathbb{H}^n$ extends to a quasiconformal map on S^{n-1}.*

Proof. By compactness, $\widetilde{\phi}$ is a coarse K–quasi-isometry. Let us first treat the case of dimension $n = 2$. Let (a, b, c, d) be the cyclically ordered vertices of a regular ideal quadrilateral in S^1_∞, mapping to (a', b', c', d') under $\widetilde{\phi}$. Note that $d([a, b], [c, d])$ is a constant. Since quasigeodesics are a bounded distance from geodesics, we have $d([a', b'], [c', d']) \leq C_K$. Arranging that $a = a' = \infty$, this condition implies that $\widetilde{\phi}|\mathbb{R}$ is a quasi-symmetric map, i.e. there is a $k > 1$ depending only on K such that

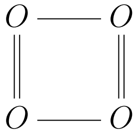
$$\frac{1}{k} \leq \frac{|b' - c'|}{|c' - d'|} \leq k.$$

The proof in higher dimensions is similar. Arrange that $\widetilde{\phi}$ fixes infinite and consider a sphere $S(x, r) \subset \mathbb{R}^{n-1} \subset S^{n-1}_\infty$. Let $x' = \widetilde{\phi}(x)$, and similarly for y and z. Suppose the image of the sphere contains points y', z'. Note that $d([\infty, y], [x, z]) = O(1)$. Thus $d([\infty, y'], [x', z']) = O(1)$. This implies that $d(x', y')/d(x', z') = O(1)$. ■

Theorem 21.4 *Let $\phi : M \to N$ be a homotopy equivalence between a pair of closed hyperbolic n-manifolds, $n \geq 3$. Then ϕ is homotopic to an isometry.*

Proof. Lift ϕ to an equivariant map $\widetilde{\phi} : \mathbb{H}^n \to \mathbb{H}^n$. Then $\widetilde{\phi}$ extends to a quasiconformal map $S^{n-1}_\infty \to S^{n-1}_\infty$. If $\widetilde{\phi}$ is not conformal, then by the theory of quasiconformal maps (when $n > 2$) the directions of maximal stretch define a Γ_M-invariant k-plane field on S^n_∞. When $n = 3$ the angle between the lines at the endpoints of a geodesic descends to a function on $T_1 M$ invariant under the geodesic flow. This angle must then be constant, since the geodesic flow is ergodic, and one is quickly lead to a contradiction. The argument in dimension $n > 3$ is similar. ■

Diversion: a cocompact Kleinian group. Perhaps the easiest way to describe a cocompact Kleinian group is to give the Coxeter diagram of a hyperbolic tetrahedron. A particular nice one is the tetrahedron T given by:



The group Γ generated by reflections in the sides of T is cocompact; indeed, T is a fundamental domain for the group action, so we obtain a tiling of $\mathbb{H}^3$ by tetrahedra.

All the faces of T are congruent. One can compute that the interior angles of a face are not all rational multiples of π! Nevertheless, if we take the plane P through a face of T, it will intersect each tile gT, $g \in \Gamma$ just in a face, and these intersections will tile H. The tiling that results is shown in Figure 14.

As a hint to resolving this apparent paradox, we note that the angles (α, β, γ) of F can be ordered so that $\alpha = \pi/4$ and $\beta + \gamma = \pi/2$.

22 Lattices: Dimension 1

Let $\mathcal{L}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ be the space of oriented unimodular lattices $L \subset \mathbb{R}^n$. Then $\mathcal{X}_n = \mathrm{SO}_n(\mathbb{R})\backslash\mathcal{L}_n$ is the moduli space of oriented flat tori $X = \mathbb{R}^n/L$ of volume one.

The unique torus $S^1 = \mathbb{R}/\mathbb{Z}$ in dimension one still has a rich structure when one examines its automorphisms and endomorphisms.

Automorphisms and endomorphism. For example, any irrational $t \in S^1$ determines an isometry $f : S^1 \to S^1$ by $f(x) = x + t \bmod 1$.

1. Every orbit of f is dense (consider it as a subgroup).

2. Thus f is ergodic (use Lebesgue density).

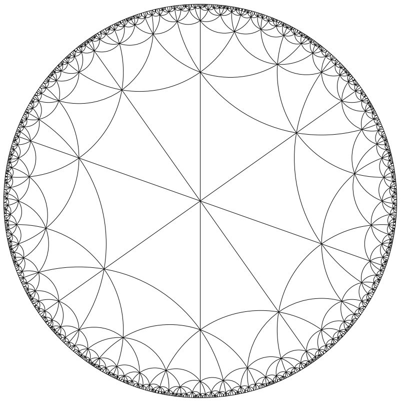


Figure 14. A strange tiling of $\mathbb{H}^2$ by congruent triangles.

3. In fact f is uniquely ergodic (consider the action on $L^2(S^1) \cong \ell^2(\mathbb{Z})$).
4. The suspension of f_t determines a foliated 2-torus X_t. Two such foliated tori are diffeomorphic iff $t_1 \sim t_2$ under the action of $\mathrm{GL}_2(\mathbb{Z})$ by Möbius transformations.
5. Given a homeomorphism $g : S^1 \to S^1$, define its rotation number by $\rho(g) = \lim \widetilde{g}^n(x)/n$. If $t = \rho(g)$ is irrational, then g is semiconjugate to rotation by t.
6. If g is C^2, then this semiconjugacy is actually a conjugacy. If g is analytic with a good rotation number, then the conjugacy is also analytic (Herman). When $g(z)$ is a rational function preserving S^1, this implies g has a 'Herman ring'. (Such rings were considered by Fatou and Julia in the 1920s but their existence had to wait till much later.)

Endomorphisms. Note that $\mathrm{End}(S^1) = \mathbb{Z}$ has the same rank as S^1 and corresponds to the unique order in the field $\mathbb{Q}$.

Now consider $f : S^1 \to S^1$ given by $f(x) = dx$, $|d| > 1$; for concreteness we consider the case $d = 2$. This map is very simple; for example, on the level of binary digits, it is just the shift. At the same time it is remarkably rich. Here are some of its properties.

1. Lebesgue measure is ergodic and invariant — hence almost every orbit is uniformly distributed. (Cf. normal numbers).

2. The map is mixing (e.g. it corresponds to $n \mapsto 2n$ on the level of the dual group $\widehat{S^1} = \mathbb{Z}$.

3. Periodic cycles are dense and correspond to rationals p/q with q odd. The other rationals are pre-periodic.

4. Consequently the map if far from being uniquely ergodic.

5. Theorem. Any degree two LEO covering map $g : S^1 \to S^1$ is topologically conjugate to f.

   Proof: the conjugacy is given by $h = \lim f^{-n} \circ g^n$, which is easily analyzed on the universal cover. The map h is a monotone semiconjugacy in general, and a homeomorphism when g is LEO.

   Note: if g is just continuous then it is still semiconjugate to f. If g is a covering map, the semiconjugacy is monotone. For a non-monotone example, see Figure 15.

6. Example: $B(z) = e^{i\theta} z(z-a)/(1-az)$, for $a \in \Delta$, leaves invariant Lebesgue measure (and is ergodic). It is therefore expanding! and hence LEO. We get *lots* of ergodic invariant measures on (S^1, f) this way.

7. Periodic cycles also give invariant measures.

8. Theorem. For any p/q there exists a unique periodic cycle with rotation number p/q.

9. Theorem. For any irrational t, there exists a monotone semiconjugacy from $f|K_t$ to rotation by t, where K is a Cantor set. This gives more ergodic invariant measures (these of entropy zero).

   The complementary intervals $(I_1, I_2, \ldots)$ of K_t have lengths $1/2, 1/4, 1/8, \ldots$ and thus K_t is covered by n intervals each of length at most 2^{-n} (those disjoint from $I_1, \ldots, I_n$). Thus $\mathrm{H.dim}(K_t) = 0$. Cf. simple geodesics.

Entropy. Let $f : X \to X$ be a continuous endomorphism of a compact metric space. A set $E \subset X$ is (r,n)-separated if for any $x \neq y$ in E, there exists $0 \le k \le n$ such that $d(f^k(x), f^k(y)) > r$. The entropy of $f : X \to X$ is given by

$$h(f) = \lim_{r \to 0} \limsup_{n \to \infty} \frac{\log N(n,r)}{\log n}$$

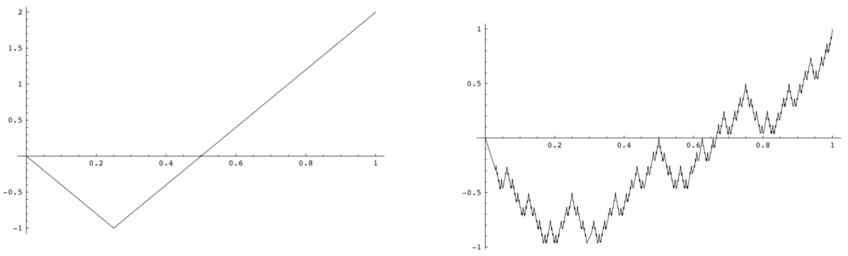


Figure 15. A degree 2 map $f : S^1 \to S^1$, and its conjugacy to $x \mapsto 2x$.

where $N(r, n)$ is the maximum number of points in (r, n)-separated set.

The entropy of an endomorphism of S^1 of degree d is given by $\log d$. For local similarities, like $f(x) = 3x \bmod 1$ on the traditional Cantor set, we have the rule of thumb:

$$\dim(X) = \frac{\log h(X)}{\log |Df|}.$$

This explain, in particular, why rotation set for $x \mapsto 2x$ have measure zero.

23 Dimension 2

We now turn to the discussion of lattice $L \subset \mathbb{R}^2$; collecting along the way some important ideas that hold in all dimensions:

1. Mahler's compactness criterion;

2. The well-rounded spine;

3. Action of A: the ring $\mathrm{End}_A(L)$;

4. Compact A-orbits and ideals;

5. Norm geometry.

6. Forms, discreteness and integrality.

Theorem 23.1 (Mahler) *The set of $L \in \mathcal{L}_n$ whose shortest vector is of length $\geq r > 0$ is compact. Similarly, the set of $X \in \mathcal{T}_n$ whose shortest closed geodesic is of length $\geq r > 0$ is compact.*

Proof. Let $v_1, \ldots, v_n \in L$ be chosen so v_1 is the shortest nonzero vector in $\mathcal{L}_n$ and $v_{i+1} \in L$ is the shortest vector in $L/(\mathbb{Z}v_1 \oplus \cdot \oplus \mathbb{Z}v_i)$ (with the quotient norm). Thus $0 < r = |v_1| \leq \cdots \leq |v_n|$. By construction, the vectors (v_i) form a basis for L.

We wish to show that there is a constant $C_n(r)$ such that $|v_i| \leq C_n(r)$ for $i = 1, \ldots, n$. For in this case, $(v_1, \ldots, v_n)$ range in a compact set, and span a lattice of bounded covolume, which must contain L with bounded index.

The proof will be by induction on n. We can take $C_1(r) = r$. For the inductive step, let $L' = L/\mathbb{Z}v_1$ with the quotient norm. We can think of L' as the projection of L to $v_1^\perp \subset \mathbb{R}^n$.

Let $(v_2', \ldots, v_n')$ be the basis for L' obtained by the procedure above. Then there is an *alpha* $\in [-1/2, 1/2]$ such that $v_2' + \alpha v_1 \in L$ and hence

$$r \leq |v_2'|^2 + \alpha^2 |v_1|^2 \leq |v_2'|^2 + r^2/4,$$

which gives $|v_2'| \geq (\sqrt{3}/2)r$. Since $\det(\mathbb{Z}v_1) = r$, we have $\det(L') \leq 1/r$. By induction, $|v_2'|, \ldots, |v_n'|$ are bounded above in terms of r. Just as for v_2', each v_i' has a lift to L with

$$|v_i|^2 \leq |v_i|^2 + r^2/4,$$

so we obtain a uniform bound $C_n(r)$ on $|v_i|$ for all $i \leq n$. ■

The greedy basis. One might attempt to construct a basis for L by choosing v_1 to be the shortest vector in L, and then choosing v_{i+1} to be the shortest vector outside the real subspace spanned by $(v_1, \ldots, v_i)$. For this algorithm one can again show $|v_i| \leq C_n'(r)$, but the vectors (v_i) need *not* form a basis for L! However they form a basis for a lattice of bounded index in L, which is enough to prove Mahler's compactness theorem.

Well-rounded lattices. Given a lattice $L \subset \mathbb{R}^n$, we let

$$|L| = \inf\{|y| \,:\, y \in L, y \neq 0\}.$$

A lattice is *well-rounded* if the set of vectors with $|y| = |L|$ span $\mathbb{R}^n$. This implies $|L| \geq 1$. Thus by Mahler's criterion, the set of well-rounded lattices $\mathcal{W}_n \subset \mathcal{L}_n$ is compact.

Example: shortest vectors need not span *L*. In high enough dimensions, the vectors $(v_1, \ldots, v_n)$ need *not* span L. For example, let $L \subset \mathbb{R}^n$ be the lattice spanned by $\mathbb{Z}^n$ and $v = (1/2, 1/2, \ldots, 1/2)$. (Thus $[L : \mathbb{Z}^n] = 2$.) For $n > 4$ we have $|v|^2 = n/4 > 1$, which easily implies that L is well-rounded, but its shortest vectors only span $\mathbb{Z}^n$.

Theorem 23.2 *The space of all lattices $\mathcal{L}_n$ retracts onto the compact spine $\mathcal{W}_n$.*

Sketch of the proof. Expand the subspace of $\mathbb{R}^n$ spanned by the shortest vectors, and contract the orthogonal subspace, until there is a new, linearly independent shortest vector. ■

Hyperbolic space. In dimension two, we can regard the space of lattices G/Γ as the unit tangent bundle $\mathrm{T}_1(\mathcal{M}_1)$ to the moduli space of elliptic curves. We can regard the space of marked lattices up to rotation, $K\backslash G$, as the hyperbolic plane. The point $\tau \in \mathbb{H}$ corresponds to $L = \mathbb{Z} \oplus \mathbb{Z}\tau$ (implicitly rescaled to have determinant one).

Retraction to the well-rounded spine. We have $|L| = 1$ iff τ lies on subarc of the circle $|\tau| = 1$ between 60° and 120°. Thus the well-rounded spine is the orbit of this arc under $\mathrm{SL}_2(\mathbb{Z})$; see Figure 16.

The equivariant retraction of $\mathcal{L} - 2$ to $\mathcal{W}_2$ is given in each horoball-like complementary region in $\mathbb{H}$ by flowing along geodesic rays from the center (at infinity) of the horoball to the spine. It is *not* given by retraction to the closest point, and indeed the closest point need not be unique because the spine is not convex.

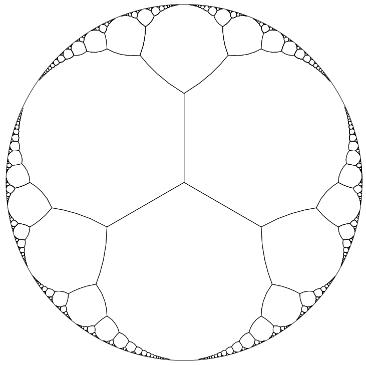


Figure 16. The well-rounded spine for $\mathrm{SL}_2(\mathbb{Z})$.

Closed geodesics.

Theorem 23.3 *Closed geodesics in $\mathcal{L}_2$ correspond to integral points on the 1-sheeted hyperboloid, or equivalently to rational points in the dual to $\mathbb{H}$.*

(Those with square discriminant connect cusps; the others are closed geodesics.)

Proof. The endpoints of a geodesic of either type are at worst exchanged under the Galois group, so the geodesic itself is defined over $\mathbb{Q}$. Conversely, an integral point has a discrete orbit in $\mathbb{R}^{2,1}$ so it defines a closed subset of $\mathbb{H}$. ■

Quadratic orders. A *quadratic order* is a commutative ring with identity, R, which is isomorphic to $\mathbb{Z}^2$ as an additive group. Any such ring is isomorphic to one of the form

$$\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c),$$

where $D = b^2 - 4c$. The *discriminant* D is an invariant of the ring. This ring has nontrivial nilpotent elements iff $D = 0$; otherwise, $K = \mathcal{O}_D \otimes \mathbb{Q}$ is isomorphic to the field $\mathbb{Q}(\sqrt{D})$, when D is not a square, and to $\mathbb{Q} \oplus \mathbb{Q}$, when D *is* a square. The latter case can of course only occur when $D > 0$.

For $D < 0$, there is a unique embedding of $\mathcal{O}_D$ into $\mathbb{R}[K] \cong \mathbb{C} \subset M_2(\mathbb{R})$, up to the Galois involution.

For $D > 0$, there is a unique embedding of $\mathcal{O}_D$ into $\mathbb{R}[A] \cong \mathbb{R} \oplus \mathbb{R} \subset M_2(\mathbb{R})$, up to the Galois involution.

Norm geometry and Euclidean geometry. The quadratic forms $|x|^2 = x_1^2 + x_2^2$ and $N(x) = x_1 x_2$ are preserved by the subgroups K and A of $\mathrm{SL}_2(\mathbb{R})$ respectively, where $K = \mathrm{SO}(2, \mathbb{R})$ is compact and $A \cong \mathbb{R}^*$ is the diagonal subgroup. Thus $\mathcal{X}_2 = K\backslash G/\Gamma$ classifies the lattices with respect to their Euclidean geometry; while the more exotic space $A\backslash G/\Gamma$ classifies the lattices with respect to their norm geometry.

Complex multiplication. The endomorphisms of a lattice as a group form a ring $\mathrm{End}(L) \cong M_2(\mathbb{Z})$. The subring $\mathrm{End}_K(L) = \mathrm{End}(L) \cap \mathbb{R} \cdot K \cong \mathbb{C}$ is an invariant of the Euclidean geometry of L; it is constant along the orbit $K \cdot L$. When this ring is bigger than $\mathbb{Z}$, it satisfies $End_K(L) = \mathcal{O}_D$ for a unique $D < 0$, and we say L admits *complex multiplication* by $\mathcal{O}_D$.

Theorem 23.4 *The set of lattices $\mathcal{L}_2[D]$ admitting complex multiplication by $\mathcal{O}_D$ is a finite union of K-orbits. The set of orbits corresponds naturally to the group* $\mathrm{Pic}\,\mathcal{O}_D$ *of (proper) ideal classes for $\mathcal{O}_D \subset \mathbb{C}$, as well as the set of elliptic curves $E \in \mathcal{M}_1$ admitting complex multiplication by $\mathcal{O}_D$.*

Proof. Any ideal $I \subset \mathcal{O}_D \subset \mathbb{C} \cong \mathbb{R}^2$ can be rescaled to become unimodular; conversely, any lattice L with CM by $\mathcal{O}_D$ can be rescaled to contain 1; it then contains $\mathcal{O}_D$ with finite index and can hence be regarded as a fractional ideal.

Let us write $\mathcal{O}_D = \mathbb{Z} \oplus \mathbb{Z}T$, where $T(z) = \lambda z$. It suffices to prove the Theorem for the set $\mathcal{L}_2[T]$ of lattices such that $T(L) \subset L$, which is obviously closed.

We claim $\mathcal{L}_2[T]$ is compact. To see this, consider the shortest vector v in a unimodular lattice $L \subset \mathbb{C}$ admitting complex multiplication by $\mathcal{O}_D$. Then L contains the sublattice $L' = \mathbb{Z}v \oplus \mathbb{Z}Tv$, and we have

$$1 \le \operatorname{area}(\mathbb{R}^2/L) \le \operatorname{area}(\mathbb{R}^2/L') \le \|T\| \cdot |v|^2.$$

Thus v cannot be too short, so by Mahler's theorem we have compactness.

Now suppose $L_n \to M \in \mathcal{L}_2[T]$; then there are $g_n \to \mathrm{id}$ in $\mathrm{SL}_2(\mathbb{R})$ such that $g_n : M \to L_n$. By assumption, $T_n = g_n^{-1} T g_n \in \operatorname{End}(M)$ for all n, and clearly $T_n \to T$. But $\operatorname{End}(M)$ is discrete, so $T_n = T$ for all n sufficiently large. Thus g_n commutes with T, which implies $g_n \in K$ and thus $L_n \in K \cdot M$ for all $n \gg 0$.

Coupled with compactness, this shows $\mathcal{L}_2[T]$ is a finite union of K-orbits.

■

Properness. An ideal I for $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})$ may also be an ideal for a larger order $\mathcal{O}_E$ in the same field. If not, we say I is a *proper* ideal, or a *proper* $\mathcal{O}_D$-module. These proper ideal classes correspond to the lattices with $\operatorname{End}_K(L) \cong \mathcal{O}_D$ and form a group $\operatorname{Pic} \mathcal{O}_D$ (every proper ideal is invertible, in the quadratic case).

Examples of CM. Note that any lattice $L \subset \mathbb{Q} \oplus i\mathbb{Q}$ admits complex multiplication by an order in $\mathbb{Q}(i)$; thus the lattices with CM are dense.

For $D < -1$ odd, consider the lattices $L_i = \mathbb{Z} \oplus \mathbb{Z}\tau_i$ where $\tau_1 = \sqrt{-D}$ and $\tau_2 = (1 + \sqrt{-D})/2$. These points both lie in the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}$, so they represent different lattices; and they both admit CM by $\mathbb{Z}[\sqrt{-D}] = \mathcal{O}_{4D}$. This shows $h > 1$ for $\mathbb{Z}[\sqrt{-3}], \mathbb{Z}[\sqrt{-5}]$, etc.

For the case of $\sqrt{-5}$, the fact $h > 1$ is related to the failure of unique factorization: $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}] = \mathcal{O}_{-20}$. Note that $\mathcal{O}_{-20}$ *is* the maximal order in $\mathbb{Q}(\sqrt{-5})$; since $-5 \neq 1 \bmod 4$, there is no quadratic order $\mathcal{O}_{-5}$.

Although factorization is unique in the Gaussian and Eisenstein integers ($\mathbb{Z}[i] \cong \mathcal{O}_{-4}$ and $\mathbb{Z}[\omega] \cong \mathcal{O}_{-3}$), the class number of the (non-maximal) order $\mathbb{Z}[\sqrt{-3}] \cong \mathcal{O}_{-12}$ is two. In general proper suborder have much larger class groups.

Discriminant of an order. For a complex quadratic order, the discriminant D can be alternatively described by:

$$D = 4\,\mathrm{area}(\mathbb{C}/\mathcal{O}_D)^2.$$

For example, $\mathcal{O}_{-20}$ has a fundamental domain with sides of length 1 and $\sqrt{5}$.

Real multiplication. Now let $\mathrm{End}_A(L) = \mathrm{End}(L) \cap \mathbb{R}[A]$. This ring is an invariant of the norm geometry of L. It is constant along the orbit $A \cdot L$.

Either $\mathrm{End}_A(L) = \mathbb{Z}$ or $\mathrm{End}_A(L) = \mathcal{O}_D \subset A$ for a unique $D > 0$. In the latter case we say L admits *real multiplication* by $\mathcal{O}_D$. (The terminology is *not* standard, and is borrowed from the theory of Abelian varieties.)

Suppose L lies on a closed geodesic in $\mathcal{L}_2$. Then there is a diagonal matrix $a \in A$ such that $a(L) = L$, so certainly $\mathrm{End}_A(L) = \mathcal{O}_D$ for some $D > 0$.

Theorem 23.5 *The set $\mathcal{L}_2[D]$ of closed geodesics coming from lattices with real multiplication by $\mathcal{O}_D$ is finite, and corresponds naturally to the group of proper ideal classes for $\mathcal{O}_D \subset \mathbb{R}^2$. Thus there are $h(D)$ such geodesics, all of the same length.*

Corollary 23.6 *The class number $h(D)$ of $\mathcal{O}_D$ is finite, and $\mathcal{O}_D^* \cong \mathbb{Z} \times (\mathbb{Z}/2)$.*

Regulator. A *fundamental unit* $\epsilon \in \mathcal{O}_D^*$ is a unit that generates $\mathcal{O}_D^*/(\pm 1)$. We can choose this unit so $\epsilon > 1$ for a given real embedding $\mathcal{O}_D \hookrightarrow \mathbb{R}$; then the *regulator* of $\mathcal{O}_D$ is given by

$$R_D = \mathrm{vol}(\mathbb{R}_+/\epsilon^{\mathbb{Z}}) = \log \epsilon > 0.$$

If $\epsilon' > 0$ (equivalently, if $N(\epsilon) = 1$) the fundamental unit is said to be positive. In this case ϵ generates A_L for any $L \in \mathcal{L}_2[D]$. Since the corresponding matrix $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{smallmatrix}\right)$ acts by $\tau \mapsto \epsilon^2 \tau$ on $\mathbb{H}$, we find that the closed geodesic $\gamma_L \subset \mathcal{M}_1$ has length

$$\ell(\gamma_L) = 2 \log \epsilon = 2R_D.$$

On the other hand, if $N(\epsilon) = -1$, then A_L is generated by ϵ^2, and hence

$$\ell(\gamma_L) = 4R_D.$$

Discriminant and area. For a real quadratic order, the discriminant D can be alternatively described by:

$$D = \mathrm{area}(\mathbb{R}^2/\mathcal{O}_D)^2.$$

For example, $\mathcal{O}_{20} \subset \mathbb{R}^2$ is spanned by $(1,1)$ and $(-\sqrt{5},\sqrt{5})$, and hence

$$\mathrm{area}(\mathbb{R}^2/\mathcal{O}_{20}) = \det\begin{pmatrix} 1 & 1 \\ -\sqrt{5} & \sqrt{5} \end{pmatrix} = 2\sqrt{5}.$$

Lorentz tori. A marked torus $E = \mathbb{C}/L$ lies on the closed geodesic corresponding to $g \in \mathrm{SL}_2(\mathbb{Z})$ iff the linear action of g on E has *orthogonal foliations.* These foliations are the zero sets of the form xy which can be regarded as a Lorentz metric on E. Thus E admits Lorentzian isometries.

Examples. Given $D = 4n$, the ring $\mathcal{O}_D = \mathbb{Z}[\sqrt{n}]$ embeds in $\mathbb{R}^2$ to give the lattice generated by the two orthogonal vectors

$$L = \mathbb{Z}(1,1) \oplus \mathbb{Z}(\sqrt{n}, -\sqrt{n}).$$

It is challenging to find a generator $g = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{smallmatrix}\right)$ of A_L! Such a number must satisfy

$$g \cdot (1,1) = (a + b\sqrt{n}, a - b\sqrt{n}) = (\epsilon, 1/\epsilon),$$

and thus $a^2 - nb^2 = 1$. That is, we want a solution to *Pell's equation* for a given n, with of course $b \neq 0$. Some minimal solutions are shown in Table 17.

We remark that the largest solution — for $n = 61$ — comes from ϵ^6, where $\epsilon = (39 + 5\sqrt{61})/2$ is a fundamental unit for $\mathcal{O}_{61}$. Although ϵ is fairly small in height, we must pass to ϵ^3 to get a unit in $\mathbb{Z}[\sqrt{61}]$, and then square this again to get a unit with norm 1.

Siegel's theorem. For any $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that

$$h(D) > C(\epsilon) D^{1/2-\epsilon}$$

for $D < 0$, and

$$h(D) R_D > C(\epsilon) D^{1/2-\epsilon}$$

for $D > 0$. Both cases measure the *total volume* of the K or A orbits for complex or real multiplication by $\mathcal{O}_D$. Upper bounds of $O(D^{1/2+\epsilon})$ are also known.

n	(a, b)
2	(3,2)
3	(2,1)
5	(9,4)
6	(5,2)
7	(8,3)
8	(3,1)
10	(19,6)
11	(10,3)
12	(7,2)
13	(649,180)
14	(15,4)
15	(4,1)
17	(33,8)
18	(17,4)
19	(170,39)
20	(9,2)
21	(55,12)
22	(197,42)
23	(24,5)
24	(5,1)
26	(51,10)
27	(26,5)
28	(127,24)
29	(9801,1820)
30	(11,2)
31	(1520,273)
32	(17,3)
33	(23,4)
34	(35,6)
35	(6,1)
37	(73,12)
38	(37,6)
39	(25,4)
40	(19,3)
41	(2049,320)
42	(13,2)
43	(3482,531)
44	(199,30)
45	(161,24)
46	(24335,3588)
47	(48,7)
48	(7,1)
50	(99,14)
51	(50,7)
52	(649,90)
53	(66249,9100)
54	(485,66)
55	(89,12)
56	(15,2)
57	(151,20)
58	(19603,2574)
59	(530,69)
60	(31,4)
61	(1766319049,226153980)
62	(63,8)
63	(8,1)
65	(129,16)
66	(65,8)
67	(48842,5967)
68	(33,4)
69	(7775,936)
70	(251,30)
71	(3480,413)

Table 17. Minimal solutions to Pell's equation $a^2 - nb^2 = 1$

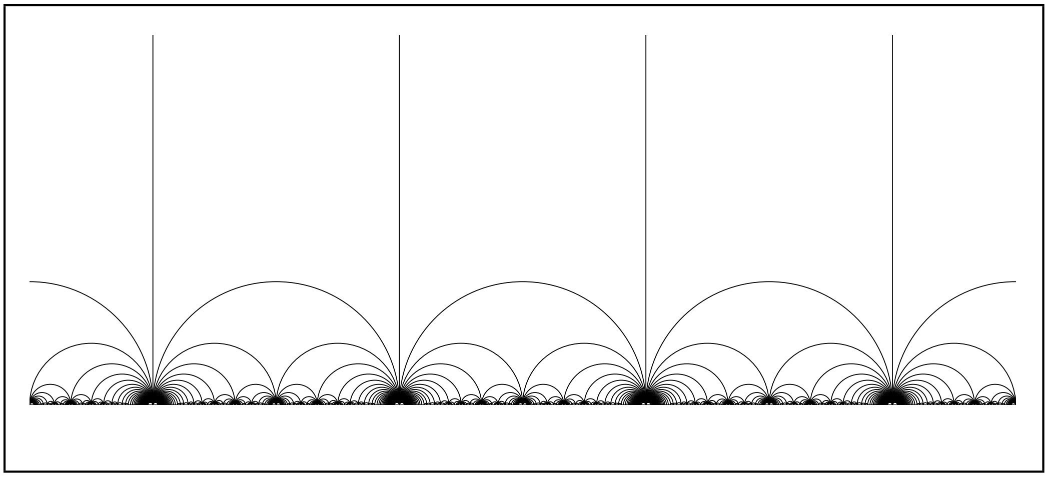


Figure 18. Tiling for continued fractions.

Continued fractions. Now consider the tiling of $\mathbb{H}$ generated by reflections in the sides of the ideal triangle T with vertices $\{0, 1, \infty\}$. Given $t \in [1, \infty)$ irrational, we can consider the geodesic ray $\gamma(i, t)$ joining i to t. (Any point iy will do as well as i). As this geodesic crosses the tiles, it makes a series of left or right turns, giving rise to a word

$$w = L^{a_1} R^{a_2} L^{a_3} \cdots .$$

We will need to use the map $R(t) = 1/t$. Notice that this map does not send $\mathbb{H}$ to itself; however it does if we compose with complex conjugation. Thus we define $R(z) = 1/\overline{z}$ on $\mathbb{H}$. The maps $R(t)$ and $T(t) = t + 1$ generate an action of $\mathrm{PGL}_2(\mathbb{Z})$ on $\mathbb{H}$, preserving the tiling. Moreover $R(T) = T$, reversing orientation, and $\mathrm{PGL}_2(\mathbb{Z})$ is the full group of isometries preserving the tiling. (In particular, the stabilizer of T itself is isomorphic to S_3).

One can think of $\mathrm{PGL}_2(\mathbb{R})$, since it commutes with complex conjugation, as acting naturally on

$$\widehat{\mathbb{C}}/(z \sim \overline{z}) = \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}.$$

Theorem 23.7 *We have $t = a_1 + 1/a_2 + 1/a_3 + \cdots$.*

Proof. We have $t \in (a_1, a_1 + 1)$ and thus the first integer in its continued fraction expansion agrees with the first exponent in the word w. Now let $t' =$

$G(t) = R(T^{-a_1}(t)) \in (1,\infty)$. Letting p denote the point where $\gamma(i,t)$ first crosses $\gamma(a_1, a_1+1)$ and begins its first right turn. Then G sends $\gamma(a_1, a_1+1)$ to $\gamma(0,\infty)$ and $g(p) = iy$ for some y. The map G reverses orientation, and thus it sends $\gamma(p,t) \subset \gamma(i,t)$ to $\gamma(iy, t')$ with associated word

$$w' = L^{a_2} R^{a_3} L^{n_4} \cdots$$

Thus $t \mapsto t' = 1/\{t\}$ acts as the shift on both its geodesic word and its continued fraction expansion, and hence all the exponents agree. ■

Corollary 23.8 *Two points $\alpha, \beta \in (0,\infty)$ are in the same orbit of* $\mathrm{PGL}_2(\mathbb{Z})$ *iff the tails of their continued fractions agree.*

We say $t \in \mathbb{R}$ is of *bounded type* if the continued fraction expansion $|t| = a_1 + 1/a_2 + \cdots$ has bounded a_i's. By convention, rational numbers do not have bounded type.

Theorem 23.9 *The geodesic $\gamma(iy,t)$ is bounded in $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ iff t has bounded type.*

Proof. Clearly a large value of a_1 implies a deep excursion into the cusp by the preceding picture. Conversely, a deep excursion into the cusp that ultimately returns must, near its deepest point, make many turns in the same directly (all L or all R). ■

Horoballs and Diophantine approximation. Noting that the horoball $(y \geq 1)/\mathbb{Z}$ embeds in $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, we see the ball of *diameter* $1/q^2$ resting on p/q also embeds, which leads to:

Corollary 23.10 *The real number t has bounded type iff t is Diophantine of exponent two: that is, there is a $C > 0$ such that*

$$\left| t - \frac{p}{q} \right| \geq \frac{C}{q^2}$$

for all $p/q \in \mathbb{Q}$.

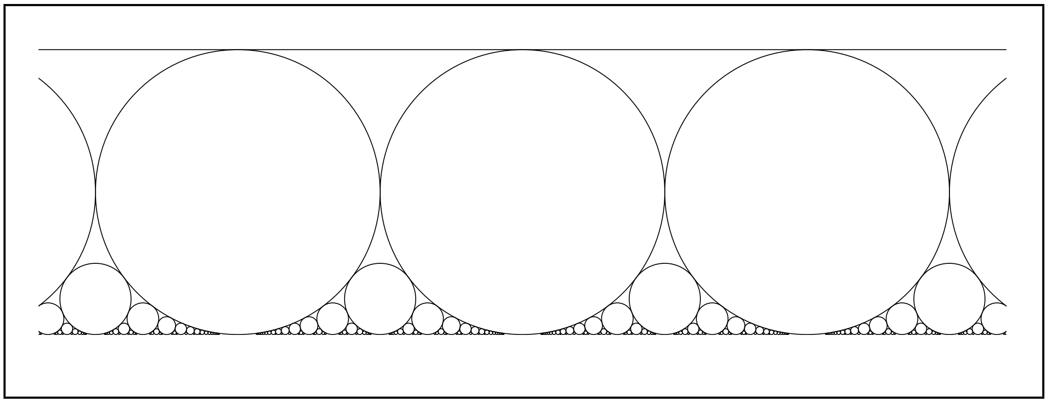


Figure 19. A neighborhood of the cusp: balls of diameter $1/q^2$ resting on p/q.

Examples. Given $u, v \in \mathbb{R}$, How can we find a lattice whose A-orbit gives the geodesic $\gamma(u, v)$? Answer: just take $L = L(u, v)$ to be the lattice spanned by $v_1 = (1, 1)$ and $v_2 = (u, v)$.

To see this works, notice that as a lattice with basis v_1 and v_2 degenerates, in the limit the basis elements v_1 and v_2 become parallel. The number $u \in \widehat{\mathbb{R}}$ satisfies $|u| = \lim |v_2|/|v_2|$, i.e. it records the ratio of the limiting positions of v_1 and v_2 on their 1-dimensional space. (This is clear in the case $v_1 = (1, 0)$ and $v_2 = (x, y)$, y small, corresponding to $\tau = x + iy \in \mathbb{H}$ close to the real point x.)

In the case at hand, as we apply the action of A, the lattice L is pushed towards either the x or y axis. Thus the limiting points in $\widehat{\mathbb{R}}$ for a basis $v_i = (x_i, y_i)$ are x_2/x_1 and y_2/y_1. In particular $L(u, v)$ degenerates to $u, v \in \widehat{\mathbb{R}}$.

Theorem 23.11 *The geodesic $\gamma(u, v)$ is periodic iff $u \neq v$ are conjugate real quadratic numbers.*

Proof. Periodicity is equivalent to $L(u, v) \otimes \mathbb{Q} = t\mathbb{Q}(\sqrt{D})$ with its usual embedding into $\mathbb{R}^2$, for some real t. Since $L(u, v)$ already contains $(1, 1)$, t is rational, and hence $(u, v) = (u, u')$ where the prime denotes Galois conjugation. ■

Corollary 23.12 *The continued fraction expansion of t is pre-periodic iff t is a quadratic irrational.*

Proof. If t is a quadratic irrational, then $\gamma(t', t)$ is periodic and hence the bi-infinite word in R and L representing this geodesic is periodic; hence its tail is preperiodic. Conversely, the preperiodic word for $\gamma(i, t)$ can be replaced by a periodic one by changing only one endpoint; thus $\gamma(u, t)$ is periodic for some u, and hence t is a quadratic irrational. ∎

Remark: complete separable metric spaces. The space of all irrationals in $(1, \infty)$ is homeomorphic, by the continued fraction expansion, to $\mathbb{N}^{\mathbb{N}}$. On the other hand, for any separable complete metric space X there is a continuous surjective map $f : \mathbb{N}^{\mathbb{N}} \to X$. This shows (cf. [Ku, §32, II]):

Theorem 23.13 *Every complete separable metric space is a quotient of the irrational numbers.*

Intersection with the well-rounded spine. We remark that the tiling of $\mathbb{H}$ by ideal triangles is precisely the dual of the well-rounded spine. Thus the continued fraction expansion of a geodesic also records its intersections with the well-rounded spine, and can be thought of as a record of the shortest vectors along $a_t \cdot L$ as t varies.

Higher Diophantine exponents. Khinchin's theorem, as presented by Sullivan, says that if the size of $a(q)$ only depends on the size of q, then almost every real number satisfies

$$\left| t - \frac{p}{q} \right| \le \frac{a(q)}{q^2}$$

infinitely often iff $\int a(x)\, dx/x$ diverges. (For example, $a(q) = 1/\log(q)$ works and strengthens the usual result where $a(q) = 1$, which holds for *all* x.)

One direction is obvious: the subset $E_q \subset [0, 1]$ where a given q works has measure $|E_q| = O(a(q)/q)$, and if $\sum |E_q| < \infty$ then $\limsup E_q$ has measure zero. The other direction uses a version of the Borel-Cantelli lemma (approximate independence of the events E_q.)

One can also consider the geodesic $\gamma(s)$ to the point t, parameterized by hyperbolic arc length. Then t is Diophantine of optimal exponent exactly α if and only if

$$\limsup \frac{d(\gamma(0), \gamma(s))}{s} = \frac{\alpha - 2}{\alpha} \cdot$$

(Almost every t satisfies $d(\gamma(0), \gamma(s)) = O((\log s)^{1+\epsilon})$ and thus the lim sup is zero and $\alpha = 2$.)

For more on this theorem and generalizations to approximating complex numbers by $p/q \in \mathbb{Q}(\sqrt{-D})$ (using the hyperbolic orbifold $\mathbb{H}^3 / \mathrm{SL}_2(\mathbb{Z}[\sqrt{-D}])$), see [Sul].

24 Lattices, norms and totally real fields.

Class numbers and units. From Mahler's Theorem we can deduce some important results in number theory.

Let K be a totally real field of degree n, and let $\mathcal{O}_K$ denote its ring of integers. Consider K as a subgroup of $\mathbb{R}^n$, using an ordering of its real places. Then $\mathcal{O}_K$ becomes a lattice, with

$$\mathrm{vol}(\mathbb{R}^n / \mathcal{O}_K)^2 = \mathrm{disc}(\mathcal{O}_K).$$

The multiplicative action of K on itself gives a natural embedding $K \hookrightarrow \mathbb{R}[A] \subset M_n(\mathbb{R})$.

We say $M \subset K$ is a *full module* if $M \cong \mathbb{Z}^n$ (and hence $\mathbb{Q} \cdot M = K$). This implies $M \cap \mathcal{O}_K$ has finite index in both, and thus M also becomes a lattice under the embedding $K \hookrightarrow \mathbb{R}^n$.

The set of $x \in K$ such that $xM \subset M$ form an order $R \subset \mathcal{O}_K$, naturally isomorphic to $\mathrm{End}_A(L)$. Let $\mathcal{L}_n(R)$ denote the set of all lattices with $\mathrm{End}_A(L) = R$.

Theorem 24.1 *The locus $\mathcal{L}_n(R)$ is a finite union of compact A-orbits.*

Proof. Pick an element $\theta \in \mathcal{O}_M$ that generates K over $\mathbb{Q}$. Let $\theta_1, \ldots, \theta_n$ be images of θ under the n distinct real embeddings of K. Let $A \subset \mathrm{SL}_n(\mathbb{R})$ be the diagonal subgroup, and let $T \in A$ be the matrix $\mathrm{diag}(\theta_1, \ldots, \theta_n)$. Let

$$X = \{L \in \mathcal{L}_n \,:\, T(L) \subset L\}.$$

Clearly $\mathcal{L}_n(R) \subset X$, and X is closed. But it is also compact, since the very short vectors of L must be permuted by T and span a lattice of rank smaller than n, contradicting the fact that T is of degree n over $\mathbb{Q}$.

Now suppose $L_n \to L \in X$. Then $L_n = g_n(L)$ where $g_n \to \mathrm{id} \in \mathrm{SL}_n(\mathbb{R})$. Thus $T_n = g_n^{-1} T g_n \to T \in \mathrm{End}(L)$. But $\mathrm{End}(L)$ is discrete, so $T_n = T$

for all $n \gg 0$. This implies g_n and T commute. Since $T \in A$ has distinct eigenvalues (i.e., it is a regular element), its connected centralizer in $\mathrm{SL}_n(\mathbb{R})$ is the group of diagonal matrices, and thus $g_n \in A$ for all $n \gg 0$. Thus X/A is finite, and hence $\mathcal{L}_n(R)$ is a finite union of compact A-orbits. ■

Corollary 24.2 *The group of units in R has rank $(n-1)$, and its set of ideal classes is finite.*

Corollary 24.3 *Any full module M in K gives rise to a lattice in $\mathbb{R}^n$ whose A-orbit is compact.*

Norm and compact A-orbits. We now define the norm $N : \mathbb{R}^n \to \mathbb{R}$ by

$$N(x) = |x_1 \cdot x_2 \cdots x_n|.$$

Clearly $N(x)$ is invariant under the action of A, and thus

$$N(L) = \inf\{N(x) \,:\, x \in L, x \neq 0\}$$

is an invariant of the A-orbit of a lattice L.

Theorem 24.4 *For any $x \in \mathbb{R}^n$, we have*

$$\sqrt{n} N(x)^{1/n} \leq |x|.$$

If $N(y) \neq 0$, then equality holds for some $x \in Ay$.

Theorem 24.5 *The orbit $A \cdot L$ is bounded iff $N(L) > 0$.*

Proof. If $N(L) = r > 0$ then $|aL| > \sqrt{n} r^{1/n}$ for all $a \in A$ and hence, by Mahler's criterion, $A \cdot L$ is bounded. Conversely, $A \cdot L$ is bounded then there is an r such that $|aL| > r$ for all $a \in A$; then given $x \neq 0$ in L we can find an a such that

$$N(x) = N(ax) = |ax|^n/\sqrt{n} \geq r^n/\sqrt{n} > 0,$$

so $N(L) > 0$. ■

Theorem 24.6 *Let N be the norm form on the ring of integers $\mathcal{O}_K$ in a totally real number field K. Then $(N, \mathcal{O}_K)$ is equivalent to $(f, \mathbb{Z}^n)$ where f is an integral form.*

Proof. Let ϵ_i, $i = 1, \ldots, n$ be an integral basis for $\mathcal{O}_K$. Then the coefficients of $N(\sum a_i \epsilon_i)$ are rational, as well as algebraic integers. ■

Theorem 24.7 *For any unimodular lattice $L \subset \mathbb{R}^n$, the following conditions are equivalent:*

1. *$A \cdot L$ is compact.*

2. *L arises from a full module M in a totally real field $K/\mathbb{Q}$.*

3. *We have $N(L) > 0$, and $\{N(y) : y \in L\}$ is a discrete subset of $\mathbb{R}$.*

4. *The pair (L, N) is equivalent to $(\mathbb{Z}^n, \alpha f)$ where $\alpha \in \mathbb{R}$ and f is an integral form that does not represent zero.*

Proof of Theorem 24.7. (1) $\implies$ (2). Suppose $A \cdot L$ is compact, and let A_L denote the stabilizer of L in A. Then $L \otimes \mathbb{Q}$ is a module over the commutative algebra $K = \mathbb{Q}[A_L] \subset M_n(\mathbb{R})$. The matrices in A have only real eigenvalues, so K is a direct sum of m totally real fields, and therefore the rank of $\mathcal{O}_K^*$ is $n - m$. But the matrix group $A_L \cong \mathbb{Z}^{n-1}$ embeds in the unit group $\mathcal{O}_K^*$, so $m = 1$ and K itself is a totally real field. Thus $L \otimes \mathbb{Q}$ is a 1-dimensional vector space over K, so the lattice L itself is obtained from a full module $M \subset K$ by the construction above.

The implication (2) $\implies$ (3) is immediate from discreteness of the norm $N_{\mathbb{Q}}^K(x)$ on M.

To see (3) $\implies$ (1), observe that the map $g \mapsto N(g^{-1}x) = \phi(x)$ gives a proper embedding of $A \backslash G$ into the space of degree n polynomials on $\mathbb{R}^n$. There is a finite set $E \subset \mathbb{Z}^n$ such that $\phi | E$ determines ϕ. Consequently, if the values of $N(x)$ on $L = g \cdot \mathbb{Z}^n$ are discrete, then $[g] \cdot \Gamma$ is closed in $A \backslash G$, and therefore $A \cdot L$ is closed in G/Γ. Since $N(L) > 0$, by Theorem 24.5 the orbit $A \cdot L$ is actually compact.

(2) and (4) are equivalent by the preceding result. ■

An formulation purely in terms of forms is:

Theorem 24.8 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a product of linear forms that does not represent zero. Then the following are equivalent:*

1. *$f(\mathbb{Z}^n)$ is discrete;*
2. *f is proportional to an integral form; and*
3. *$(\mathbb{Z}^n, f)$ is equivalent to the norm form on a proper ideal I for an order in a totally real number field.*

The case $n = 2$: bounded A-orbits that are not compact. Recall $L(u, v) = \mathbb{Z}(1, 1) \oplus \mathbb{Z}(u, v) \subset \mathbb{R}^2$. Theorem 23.9 implies:

Theorem 24.9 *We have $N(L(u, v)) > 0$ iff u and v are numbers of bounded type. In particular, there are plenty of lattice in $\mathbb{R}^2$ with $N(L) > 0$ that do not come from number fields/closed geodesics.*

We will shortly examine Margulis's conjecture which implies to the contrary:

Conjecture 24.10 *A lattice $L \subset \mathbb{R}^n$, $n \geq 3$, has $N(L) > 0$ iff L comes from an order in a number field. Equivalently, the set of L with $N(L) > 0$ is a countable union of compact A-orbits.*

Remark. In $\mathbb{R}^2$ the region $N(x) < 1$ has infinite area, since $\int_0^\infty dx/x = \infty$. For the same reason, the region $N(x) < \epsilon$ is a neighborhood of the coordinate planes of infinite volume. The conjecture says that in dimensions 3 or more, it is very hard to construct a lattice that avoids this region. The only possible construction uses arithmetic, i.e. the integrality of the norm on algebraic integers in a totally real field.

25 Dimension 3

The well-rounded spine. Although $\mathcal{X}_3$ has dimension 5, its spine $\mathcal{W}_3$ is only 3-dimensional.

A well-rounded lattice can be rescaled so its Gram matrix $m_{ij} = v_i \cdot v_j$ satisfies $m_{ii} = 2$. Then by consider the vectors $v_i \pm v_j$ we see $|m_{ij}| \leq 1$ for the three off-diagonal elements. Thus we describe $\mathcal{W}_3$ by the coordinates

$$m(x, y, z) = \begin{pmatrix} 2 & x & y \\ x & 2 & z \\ z & y & 2 \end{pmatrix}.$$

Since the off-diagonal elements are bounded by one, $\mathcal{W}_3$ is a subset of a cube. (Actually it is a quotient of this set by permutations of coordinates and changing the signs of any pair of coordinates.)

The lattice $m(1, 1, 1)$ gives the densest sphere packing $L \subset \mathbb{R}^3$, the *face-centered cubic* (fcc) packing with Voronoi cell the rhombic dodecahedron. Thus there are twelve vectors with $|x| = |L|$.

We can think of L as a laminated lattice: start with the hexagonal lattice, and center the second hexagonal over the deep holes of the first. (Alternatively, one can start with a square packing, then again add the next layer in the deep holes. In these coordinates, the lattice is the subgroup of index two in $\mathbb{Z}^3$ where the coordinates have even sum.)

As we roll the second lattice between two adjacent deep holes, the Gram matrix is given by $m(1, 1, t)$, with t going from 1 to 0. Thus $m(1, 1, 0)$ also represents the fcc lattice. The corresponding lattice L_t has determinant $d(t) = 2(1 + t)(2 - t)$, which achieves its minimum value 4 along $[0, 1]$ at the endpoints.

For $t \in [-1, 0)$ the matrix $m(1, 1, t)$ no longer corresponds to a well-rounded lattice. Indeed, the vector $-v_1 + v_2 + v_3$ has squared length $2(1 + 1 + 1 - x - y + z) = 2(3 - 2 - t) < 2$ in this range. Thus this segment must be removed from the edge of the cube; the twelve edges forming its orbit under the symmetry group must similarly be removed.

As shown by Soulé, in these coordinates $\mathcal{W}_3$ is the convex hull of the remaining 12 half-edges. That is $\mathcal{W}_3$ is obtained from the cube by trimming off four of its eight corners [So].

Dimension two. Similar considerations show we can identify the well-rounded spine for $\mathcal{L}_2$ with the set of matrices

$$m(x) = \begin{pmatrix} 2 & x \\ x & 2 \end{pmatrix}$$

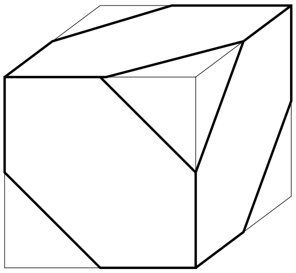


Figure 20. The well-rounded spine in dimension three.

where $x \in [-1, 1]$, modulo $x \equiv -x$.

Margulis and Littlewood. We can now relate the following two open problems.

Conjecture 25.1 (Littlewood) *For any $\alpha, \beta \in \mathbb{R}$ we have*

$$\inf_{n>0} n \cdot \|n\alpha\| \cdot \|n\beta\| = 0.$$

Conjecture 25.2 (Margulis) *Every bounded A-orbit in $\mathcal{L}_n$, $n \geq 3$ is closed (and hence comes from a totally real field).*

We will show that Margulis's conjecture for $n = 3$ implies Littlewood's conjecture. Cf. [Mg, §2].

To give some more context for Margulis's conjecture, we note that if we replace A by $H = \mathrm{SO}(2, 1, \mathbb{R})$ the corresponding result is known to be true. This is the Oppenheim conjecture (see Theorem 13.6).

Compact A-orbits: isolation. To study the case where $H = \mathrm{SO}(2, 1, \mathbb{R})$ is replaced by the abelian A, we begin with an 'isolation result' of Cassels and Swinnerton-Dyer [CaS]. This result shows, for example, that you cannot construct a bounded A orbit that spirals or oscillates between one or more compact A orbits.

Theorem 25.3 *Let $T \subset \mathcal{L}_3$ be a compact, A-invariant torus, and suppose $X = \overline{A \cdot L_0}$ meets T. Then either $X = T$, or $N(L_0)$ is dense in $\mathbb{R}_+$.*

Proof. Suppose $X \neq T$. We will show $N(L_0)$ is dense. It suffices, by semicontinuity, to show $N(L)$ is dense for some $L \in X$.

Let $V \subset G = \mathrm{SL}_3(\mathbb{R})$ denote the closure of the set of g such that $X \cap gT \neq \emptyset$. Since T is compact, V is closed. For $a, b \in A$ we have

$$X \cap agbT = a(X \cap gT),$$

so V is invariant under the action of $A \times A$.

Now suppose $X \neq T$. Then V contains elements of the form

$$v = (1+\delta)(I + \epsilon v_{ij})$$

where ϵ and δ are arbitrarily small, $|v_{ij}| \leq 1$, v_{ij} vanishes on the diagonal, and some off-diagonal element, say v_{12}, is equal to one.

Conjugating by $a = \mathrm{diag}(a_1, a_2, a_3)$ sends v_{ij} to $v_{ij} a_i / a_j$. In particular, conjugating by $\mathrm{diag}(a, 1/a, 1)$ multiplies v_{ij} by the matrix:

$$\begin{pmatrix} 1 & a^2 & a \\ a^{-2} & 1 & a^{-1} \\ a^{-1} & a & 1 \end{pmatrix}.$$

This conjugation makes v_{12} much larger than the remaining elements of v_{ij}. It can also be used, at the same time (by appropriate choice of a), to adjust ϵv_{12} so it is close to any prescribed value. Since V is closed, this implies V contains the 1-parameter subgroup

$$U = \left\{ u_t = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; : \; t \in \mathbb{R} \right\}.$$

Note that U is normalized by A, i.e. $aUa^{-1} = U$ for all $a \in A$.

Now let $I \subset \mathcal{O} \subset K$ an ideal for an order in a totally real cubic field, such that T contains a lattice L_1 proportional to the standard embedding $I \subset \mathbb{R}^3$. By assumption $u_t a(L_1) \in X$ for some $t \neq 0$. But U is normalized by A, and thus $u_t(L_1) \in X$ for some $t \neq 0$.

Thus to complete the proof, we need only show that $N(u_t L_1)$ is dense; or equivalently, that $N(u_t I)$ is dense. To see this, just note that

$$N(u_t x) = |(x_1 + t x_2) x_2 x_3| = |x_1 x_2 x_3 (1 + t x_2 / x_1)|.$$

Pick $x \neq 0$ in I, and let $s \in \mathcal{O}^*$ be a positive unit. Then $sx \in I$ as well, and we have:

$$N(u_t sx) = |x_1 x_2 x_3 (1 + t(s'/s)(x_2/x_1))|. \tag{25.1}$$

Now $\mathcal{O}^*_+ \cong \mathbb{Z}^2$ maps injectively into $\mathbb{R}_+$ via $s \mapsto s'/s$, so its image is dense. Thus $N(u_t x)$ contains a half line. By varying the choice of x we then see $N(u_t I)$ is dense in $\mathbb{R}$.

By semicontinuity of the norm, it follows that the values of N on L_0 are also dense in $\mathbb{R}_+$. ■

Theorem 25.4 *Margulis's conjecture implies Littlewood's conjecture.*

Proof. Suppose $(\alpha, \beta) \in \mathbb{R}^2$ is a counterexample to Littlewood's conjecture. Consider the unimodular lattice $L_0 \subset \mathbb{R}^3$ generated by

$$\{(e_1, e_2, e_3)\} = \{(1,0,0), (0,1,0), (\alpha, \beta, 1)\},$$

and let $M_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset L_0$. We then have, for any $(a, b, n) \in \mathbb{Z}^3$,

$$N(-ae_1, -be_2, ne_3) = |n\alpha - a| \cdot |n\beta - b| \cdot n \geq n \cdot \|n\alpha\| \cdot \|n\beta\|.$$

Thus the norm is bounded away from zero on $L_0 - M_0$, and vanishes exactly on M_0; in particular, $N(L_0)$ is not dense.

Now we get rid of the lattice M_0. Let $a_t = \operatorname{diag}(t, t, t^{-2})$, and let $L_t = a_t(L_0) \supset M_t = a_t(M_0)$. Then as $t \to \infty$, the null vectors $M_t = a_t(M) \subset L_t$ are pushed off to infinity. Thus the shortest vector in L_t must lie in $L_t - M_t$; but there the norm is bounded below, so the shortest vector also has length bounded below.

By Mahler's compactness criterion, there is a subsequence such that $L_t \to L_\infty$. The limit has no nontrivial null-vectors, and thus $N(L_\infty)$ is bounded below. By Margulis's conjecture, $L_\infty \in T$ for some compact torus orbit. But then $X = A \cdot L_0$ contains T. Since $N(L_0)$ is not dense, $X = T$ and thus $L_0 \in T$. This contradicts the fact that L_0 has null vectors. ■

Form version. The preceding result and conjecture can be recast in terms of cubic forms as follows. Let us say f is a *cubic norm form* if it is a multiple of the integral norm form on an ideal in a totally real cubic field.

Theorem 25.5 *Suppose $f = f_1 f_2 f_3 : \mathbb{R}^3 \to \mathbb{R}$ is a product of linear and $f(\mathbb{Z}^3)$ is not dense in $\mathbb{R}$. Then either f is a cubic norm form, or $\overline{\mathrm{SL}_3(\mathbb{Z}) \cdot f}$ contains* no *cubic norm forms.*

In fact: if f is an integral form that does not represent zero, and $f_n \to f$ are distinct from f, then $f_n(\mathbb{Z}^3)$ becomes denser and denser in $\mathbb{R}$. See [CaS].

Conjecture 25.6 *If f does not represent zero then either $f(\mathbb{Z}^3)$ is dense or f is a cubic norm form.*

Now suppose the conjecture above holds. If α and β are a counterexample to Littlewood's conjecture, then the form

$$f(x, y, z) = x(x\alpha - y)(x\beta - z)$$

is bounded away from zero; and it only represents zero on the plane $x = 0$. Thus means we can take $g_n \in \mathrm{SL}_n(\mathbb{Z})$ such that $g_n \cdot f \to h$ where h does not represent zero. But then h is a cubic norm form, contrary to the preceding theorem.

The case $n = 2$. The first part of the Cassels and Swinnerton-Dyer argument, that establishes the existence of a unipotent subgroup in V, works just as well in any dimension. (This is exactly the same as finding a single unipotent u such that $L \in X$ and $u(L) \in X$, since then $AuA \subset V$ and AuA contains a 1-parameter unipotent subgroup through u.)

In dimension two it shows that if $X \subset \mathcal{L}_2$ is a compact A-invariant set, then either X consists of a finite union of closed geodesics or X contains two geodesics with the same endpoint (since this is what it means for leaves to be related by the unipotent horocycle flow.)

When X comes from a simple lamination, these asymptotic leaves are clearly visible: they come from the boundary of a complementary region such as an ideal triangle.

Gaps of slopes. The reason one does *not* have an isolation result in dimension two is that the ratios x'/x as x ranges in the units of a real quadratic field form a *discrete* set.

In dimension three, one finds the following. Suppose $X = \overline{A \cdot L} \subset \mathcal{L}_3$ is compact, but $A \cdot L$ is not. Then the set of slopes:

$$S = \{x_1/x_2 \; : \; x \in L\}$$

is nowhere dense in $\mathbb{R}$. Otherwise (25.1) would give that $N(L) = 0$.

Forms, discreteness and integrality. A general setting that includes both the actions of A and $\mathrm{SO}(p, q, \mathbb{R})$ is the following.

Let $G = \mathrm{SL}_n(\mathbb{R})$, let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous form of degree d, and let $H \subset G$ be the largest connected subgroup of G leaving f invariant. Geometrically f determines a subvariety $V(f) \subset \mathbb{P}^{n-1}$, and H determines a continuous group of symmetries of $V(f)$.

Examples: $n = 2$ and $f(x, y) = x^2 + y^2$ we have $H = K$; for $f(x, y) = xy$ we have $H = A$; and for $f(x, y) = x$ we have $H = N$. For general n and $f(x) = x_1 \cdots x_n$, we have $H = A$; for

$$f(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots x_{p+q}^2,$$

$p + q = n$, we have $H = \mathrm{SO}(p, q, \mathbb{R})$. For a more exotic example, let $n = m^2$, so $\mathbb{R}^n = M_n(\mathbb{R}^m)$, and let $f(X) = \det(X)$. Then $H = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ acting by $X \mapsto h_1 X h_2^{-1}$.

A theorem of Jordan asserts that if a form f of degree $d \geq 3$ defines a smooth projective variety, then its stabilizer in $\mathrm{GL}_n(\mathbb{C})$ is finite. This means the irreducible factors of 'exotic examples' (forms which are not products of linear or quadratic forms) must have vanishing discriminant. See [Bor, §I.6.9]. For the classification of 'exotic examples', see [KS].

The space F can be thought of as the space of homogeneous forms on $\mathbb{R}^n$ of the same type as f: for example, quadratic forms of signature (p, q), or products of n linear forms.

We can study the action of H on G/Γ dynamically as we have for the geodesic and horocycle flows above. But we can also equivalently study the action of Γ on the space $F = H\backslash G$. It is easy to see, for example:

Theorem 25.7 *The action of H on G/Γ is ergodic iff the action of Γ on $F = H\backslash G$ is ergodic. The orbit $Hx \subset G/\Gamma$ is closed iff the orbit $x\Gamma \subset H\backslash G$ is closed.*

Let us now assume that:

1. H is a connected reductive group without compact factors; and
2. Any homogeneous form of the same degree as f that is stabilized by H is proportional to f.

This holds in all the examples considered above.

Note that Γ preserves the discrete subset $F(\mathbb{Z})$ of forms with integral coefficients. We say f *represents zero* if there is an $x \neq 0$ in $\mathbb{Z}^n$ such that $f(x) = 0$.

Theorem 25.8 (Integral Forms) *Suppose f does not represent zero. Then the following are equivalent.*

1. *The set $f(\mathbb{Z}^n) \subset \mathbb{R}$ is discrete.*

2. *The orbit $H \cdot \mathbb{Z}^n \subset G/\Gamma$ is compact.*

3. *A nonzero multiple of f lies in $F(\mathbb{Z})$, and hence takes only integral values on $\mathbb{Z}^n$.*

Proof. (1) implies (2): $X = H \cdot \mathbb{Z}^n$ is closed, and $|f(v)| > \epsilon > 0$ for all nonzero $v \in L \in X$. By continuity of f, this implies $|v| > \epsilon' > 0$ and hence X is compact by Mahler's criterion.

(2) implies (3): By assumption, $H(\mathbb{Z}) = H \cap \Gamma$ is a lattice in H. Invariance of f under each integral matrix $g \in H(\mathbb{Z})$ imposes a rational linear condition on the coefficients of f. These conditions uniquely determine the stabilizer H of f, by the Borel density theorem ($H(\mathbb{Z})$ is Zariski dense in H). Finally H determines f up to a constant multiple by assumption.

(3) implies (1): immediate. ■

26 Dimension 4, 5, 6

Suppose $K/\mathbb{Q}$ is a number field with r_1 real places and r_2 complex places. Then, a general version of the arguments presented before shows the rank of the unit group, $\mathcal{O}_K^*$, is $r_1 + r_2 - 1$.

Suppose $u \in K$ is a unit of degree $d = r_1 + 2r_2 = \deg(K/\mathbb{Q}) > 1$, and (under some embedding) we also have $u \in S^1$. Then u must be Galois conjugate to $\overline{u} = 1/u \neq u$, and thus its minimal polynomial is *reciprocal* and of *even degree.*

In dimension 2, such a u must be a root of unity, but in dimension 4 or more it can be a Salem number: its polynomial has two roots outside the circle and two on the circle. By irreducibility those on the circle have infinite multiplicative order.

By considering $L = \mathcal{O}_K \subset \mathbb{R}^4$, or an ideal, we obtain a Euclidean lattice with a self-adjoint automorphism $T : L \to L$ that acts hyperbolically on one $\mathbb{R}^2$ and by an irrational rotation on another $\mathbb{R}^2$. This action is *not* structurally stable!

Going to dimension 6, one can obtain a $\mathbb{Z}^2$ action on a torus with the same kind of partial hyperbolicity. Damjanović has shown such a $\mathbb{Z}^2$ action *is* structurally stable (e.g. in $\mathrm{Diff}^\infty(\mathbb{R}^6/\mathbb{Z}^6)$); see e.g. [DK].

To obtain such an action, following Damjanović, let K be a totally real field of degree d, and let L/K be a degree two extension with one complex place and $2d-2$ real places. We can use this place to regard L as a subfield of $\mathbb{C}$. The unit group of a field with r_1 real places and r_2 complex places has rank r_2+r_1-1. Thus K^* has rank $d-1$ and L^* has rank $2d-2$. Consequently the map $L^* \to K^*$ given by $u \mapsto u\overline{u}$ has a kernel of rank (at least) $d-1$. Since the rank of $\mathcal{O}_L$ is $2d$, this gives a partially hyperbolic action of $\mathbb{Z}^{d-1}$ on $(S^1)^{2d}$.

27 Higher rank dynamics on the circle

In 1967 Furstenberg showed [Fur1]:

Theorem 27.1 *Let $X \subset S^1 = \mathbb{R}/\mathbb{Z}$ be a closed set invariant under $x \mapsto 2x$ and $x \mapsto 3x$. Then either X is finite, or $X = S^1$.*

Lemma 27.2 *Let $S = \{0 = s_0 < s_1 < s_2 < s_3 \ldots\} \subset \mathbb{R}$ be an infinite discrete set satisfying $S + S \subset S$. Then either $S \subset \mathbb{Z}a$ for some $a > 0$, or $s_{n+1} - s_n \to 0$.*

Proof. After scaling we can assume $s_1 = 1$ and thus $\mathbb{N} \subset S$. Assume S is not contained in $(1/q)\mathbb{Z}$ for any q; then the projection then S to $S^1 = \mathbb{R}/\mathbb{Z}$ is evidently a dense semigroup G. Now if $[x] \in G$ then $x + n \in S$ for some integer $n \geq 0$, and thus all but finitely many elements of $x + \mathbb{N}$ belong to S. The same is true with x replaced by any finite subset $X \subset G$.

Given $\epsilon > 0$, choose a finite set $[X] \subset G \subset S^1$ so the complementary gaps have length less than ϵ. Then $X + \mathbb{N}$ also has gaps of length at most ϵ. Since $X + \mathbb{N}$ is eventually contained in S, we have $\limsup s_{n+1} - s_n < \epsilon$ and hence the gaps in S tend to zero. ■

Now let $A \subset \mathbb{N}$ be the multiplicative semigroup generated by 2 and 3. We can write $A = \{t_1 < t_2 < \cdots\}$. The preceding Lemma implies $t_{n+1}/t_n \to 1$ and thus:

Corollary 27.3 *As $x \to 0$, the rescaled sets $xA \cap [0,1]$ converge to $[0,1]$ in the Hausdorff topology.*

Isolation. We now prove an easy analogue of the 'isolation result' of Cassels and Swinnerton-Dyer. Note that the compact A-orbits on S^1 are just certain finite sets of rational numbers. We will see that any more exotic A-invariant sets must stay away from the rationals.

Lemma 27.4 *Let $X \subset S^1$ be a closed set such that $AX = X$, and suppose 0 is not an isolated point of X. Then $X = S^1$.*

Proof. Take $x_n \in X$ tending to zero, and observe that X contains the projection of $x_n A \cap [0,1]$ under $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$. By the preceding Corollary, these sets become denser and denser as $n \to \infty$, and thus $X = S^1$. ■

Corollary 27.5 *If X contains a non-isolated rational point p/q then $X = S^1$.*

Proof. Then qX accumulates at 0, so $qX = S^1$ which implies $X = S^1$. ■

Minimal sets. Let $X \subset S^1$ be a nonempty closed set that is (forward) invariant under the endomorphisms given by a semigroup $A \subset \mathbb{Z}$ (such as $A = \langle 2, 3\rangle$.) Then X contains, by the Axiom of Choice, a *minimal* such set F.

Note that for any $a \in A$, we have $A(aF) = a(AF) \subset A(F)$. Thus a minimal set satisfies $aF = F$ for all $a \in A$ (else aF would be a smaller invariant set).

We will begin by proving Fursternberg's theorem for A-minimal sets, $A = \langle 2, 3\rangle$.

Lemma 27.6 *Let $F \subset S^1$ be a minimal, A-invariant set and suppose $X = F - F = S^1$. Then $F = S^1$.*

Proof. This is the trickiest step in the proof. Let

$$\pi_n : A \to (\mathbb{Z}/5^n)^*$$

be reduction of the integers in A modulo 5^n. (Since 2 and 3 are relatively prime to 5, the image lies in the multiplicative group modulo 5^n.) Let $A_n = \operatorname{Ker} \pi_n \subset A$.

Now let $F_1 \supset F_2 \supset \cdots$ be a sequence of nonempty closed subset of F such that F_n is minimal for the action of A_n.

The key point is that if $x_1, x_2 \in A$ and $\pi_n(x_1) = \pi_n(x_2)$ then $x_1 F_n = x_2 F_n$. To see this, choose $z \in A$ (say a power of x_1) such that $zx_1 \in A_n$. Then $zx_2 \in A_n$ as well. Consequently $zx_1 F_n = zx_2 F_n = F_n$. But then

$$x_1 F_n = x_1(zx_2)F_n = x_2(zx_1)F_n = x_2 F_n$$

as desired.

Now let $x_1, \ldots, x_m \in A$ be a finite set such that $\pi_n(A) = \{\pi_n(x_1), \ldots, \pi_n(x_m)\}$. Then by what we have just shown, $\bigcup_1^m x_i F_n$ is invariant under A. By minimality, $\bigcup x_i F_n = F$, and thus $\bigcup x_i(F_n - F) = S^1$. Thus one of these closed sets has nonempty interior, which by A_n-invariance implies it is equal to S^1. That is, $F_n - F = S^1$ for all n. Consequently $F_\infty - F = S^1$, where $F_\infty = \bigcap F_n$.

Let r be any point in F_∞, and let $s = p/5^n \in S^1$. Then $s \in F_n - F$; say $s = s_n - f_n$. Now $\overline{A_n s_n} = F_n \supset F_\infty$, so we can find a sequence $x_i \in A_n$ such that $x_i s_n \to r$. But then $x_i s = s = x_i(s_n - f_n) \to r - f_n'$, where $f_n' \in F$ as well. Thus $s \in r - F$. Since s was an arbitrary rational of the form $p/5^n$, it follows that F is dense in S^1. ■

Corollary 27.7 *If F is a A-minimal set, then F is finite.*

Proof. If F is infinite then $X = F - F$ is a A-invariant set that accumulates at zero, so it is S^1; but then F itself is S^1 by the preceding result, and S^1 is not minimal. ■

Proof of Theorem 27.1. It suffices to treat the case where $X = \overline{Ax}$ and x is irrational. In this case X contains a minimal set and hence X contains a rational point p/q. Since x is irrational, p/q is not isolated. Hence 0 is a non-isolated point of qX, and hence $qX = S^1$. This implies X contains an open interval and hence $X = S^1$. ■

28 The discriminant–regulator paradox

We now turn to conjectures about the equidistribution of periodic orbits, following [ELMV].

Regulator and discriminant. There are three basic invariants that can be attached to a compact orbit $A \cdot L \subset \mathcal{L}_n$:

1. The *order* $\mathcal{O} = \operatorname{End}_A(L) = \operatorname{End}(L) \cap \mathbb{R}[A]$;
2. The *regulator* $R = \operatorname{vol}(A \cdot L) = \operatorname{vol}(A/A_L)$; and
3. The *discriminant* $D = \operatorname{vol}(\mathbb{R}[A]/\operatorname{End}_A(L))^2$.

Here the space of diagonal matrices $\mathbb{R}[A] \cong \mathbb{R}^n$ is given the usual Euclidean volume, which can also be described using the inner product $\langle X, Y\rangle = \operatorname{tr}(XY)$ on matrices.

Let $K = \mathcal{O} \otimes \mathbb{Q}$ be the totally real field associated to $\mathcal{O}$. Since $\mathcal{O}$ is embedded in $\mathbb{R}[A]$ the n real places of K, the trace on matrices in $\mathcal{O}$ coincides with the usual $\operatorname{tr}^K_{\mathbb{Q}} : K \to \mathbb{Q}$. Consequently if we choose a basis $(e^1, \ldots, e^n)$ for $\mathcal{O}$ we see that

$$D = \det(e^i_{jj})^2 = \det(\operatorname{tr}(e^i e^j)) = \operatorname{disc}(\mathcal{O})$$

coincides with the usual discriminant of $\mathcal{O}$ from number theory. In particular $\mathcal{O}$, as an abstract ring, determines D.

Similarly we use the exponential map $\exp : \mathbb{R}^n \to \mathbb{R}_+ \cdot A$ to identify A with the locus $\sum x_i = 0$ in $\mathbb{R}^n$, with the Euclidean measure normalized so projection to any coordinate plane $\mathbb{R}^{n-1}$ is volume-preserving. Thus if we take a basis $(u^1, \ldots, u^{n-1})$ for $A_L \cong \mathcal{O}^*_+$, we find

$$R = \operatorname{vol}(A/A_L) = |\det(\log u^i_{jj})| = 2^m \operatorname{reg}(\mathcal{O}),$$

where $\operatorname{reg}(\mathcal{O})$ is the usual regulator (using the absolute values of *all* the units) and $2^m = |\,\mathcal{O}^* / (\pm \mathcal{O}^*_+)|$.

In particular, $\mathcal{O}$ determines the invariants D and R.

Theorem 28.1 *For any order $\mathcal{O}$ we have:*

$$C_n \log(D) < R < C_{n,\epsilon} D^{1/2+\epsilon}.$$

The lower bound is easy to see: e.gin the quadratic case, if $\epsilon > 1$ is a fundamental positive unit for $\mathcal{O}$, then we have $R = \log|\epsilon|$; but since $\mathbb{Z}[\epsilon] \subset \mathcal{O}$ we also have

$$D \leq \left(\det\left(\begin{smallmatrix} 1 & \epsilon \\ 1 & \epsilon^{-1}\end{smallmatrix}\right)\right)^2 = (\epsilon - \epsilon^{-1})^2 \leq \epsilon^2$$

which gives $(1/2)\log D \leq R$. The upper bound in the quadratic case follows from the result of Siegel:

Theorem 28.2 *For any real quadratic order $\mathcal{O}_D$ with class number $h(D)$ and regulator $R(D)$, we have:*

$$h(D)R(D) \leq C_\epsilon D^{1/2+\epsilon}.$$

Recall that geometrically $h(D)$ is the number of closed geodesics in $\mathcal{L}_1[D]$ and $R(D)$ is the length of each.

Warning: Geodesics of the same length can be associated to different discriminants! Equivalently, $\mathbb{Z}[\epsilon]$ can be (and often is) a proper suborder of $\mathcal{O}_D$, so different orders can have the same group of units.

For example, the fundamental positive unit in $\mathbb{Z}[\sqrt{2}] = \mathcal{O}_8$ is $\epsilon = 3+2\sqrt{2}$, but $\mathbb{Z}[\epsilon] = \mathbb{Z}[2\sqrt{2}] = \mathcal{O}_{32}$. So $\mathcal{O}_8$ and $\mathcal{O}_{32}$ have the same group of positive units.

Things get worse: for $\mathcal{O} = \mathbb{Z}[\sqrt{13}]$, a fundamental positive unit is $\epsilon = 649 + 180\sqrt{13}$ which generates a subring of index 180.

Equidistribution. In the setting of unipotent orbits we have the following important result [MS]. Let Γ be a lattice in a Lie groups G. Let us say an algebraic probability measure μ on G/Γ is *unipotent* if it is ergodic for some one-parameter unipotent subgroup of G. Then we have:

Theorem 28.3 *The set of unipotent algebraic probability measures on G/Γ is closed.*

We have seen a special case of this phenomenon: a long closed horocycle on a Riemann surface of finite volume becomes equidistributed.

One might ask if the same result holds for the ergodic A-invariant measures on $\mathcal{L}_n = G/\Gamma = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. Of course it fails badly for $n = 2$: the limits need not be algebraic. In addition, the measure may 'escape to infinity': there are closed geodesics that spend most of their time in the cusp of $\mathcal{L}_2$.

Using a result of Duke [Du2], it is shown in [ELMV, §7.2] that measure can also escape to infinity in $\mathcal{L}_n$:

Theorem 28.4 *For any $n \geq 2$ there is a sequence of compact A-orbits whose associated probability measures μ_n converge to an A-invariant measure ν with total mass strictly less than one.*

It is still unknown if the limit of such μ_n can be 'exotic', e.g. if it can have non-algebraic support.

Large orbits. In these 'counterexamples', the compact A-orbits are small, in the sense that the regulator R is on the order of $(\log D)^{n-1}$ rather than $D^{1/2}$. Such an orbit represents only a small part of $\mathcal{L}_n[\mathcal{O}]$. Thus one is led (following [ELMV]) to:

Conjecture 28.5 *As the discriminant of $\mathcal{O}$ goes to infinity, the union $\mathcal{L}_n[\mathcal{O}]$ of all the associated compact A-orbits becomes equidistributed in $\mathcal{L}_n$.*

Conjecture 28.6 *If X_i is a sequence of compact A-orbits in $\mathcal{L}_n$ with regulators and discriminants satisfying*

$$R_i > D_i^{\epsilon} \to \infty$$

for a fixed $\epsilon > 0$, then X_i becomes uniformly distributed as $i \to \infty$.

The case $n = 2$. These conjectures have interesting content even for $\mathcal{L}_2$. For $n = 2$ the first was studied by Linnik and resolved by Duke at least in the case of fundamental discriminants [Du1]. The second is still open.

For $\mathcal{M}_1 = \mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ we can relate the second conjecture to a general discussion about hyperbolic surfaces X of finite volume. For any such X there is a sequence of closed geodesics γ_n which become uniformly distributed in $T_1(X)$ as $n \to \infty$. Equivalently, the length of any closed geodesic δ on X satisfies

$$L(\delta) = (\pi/2)\operatorname{area}(X)\lim \frac{i(\gamma_n, \delta)}{L_X(\gamma_n)} \cdot$$

How can we distinguish a sequence $g_n \in \operatorname{SL}_2(\mathbb{Z})$ whose associated equidistributed in this sense? The proposed answer is to take any sequence of geodesics that are 'long' relative to their discriminants.

Discriminant and regulator in $\mathbf{SL_2}(\mathbb{Z})$. Let us now consider a closed geodesic on the modular surface $\mathcal{M}_1$ corresponding to an element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}).$$

We will compute from g its invariants $\mathcal{O}$, R and D. Acting by the center of $\mathrm{SL}_2(\mathbb{Z})$, we can assume

$$t = \operatorname{tr}(g) = a + d > 2.$$

1. The order $\mathcal{O} \subset \mathrm{M}_2(\mathbb{Z})$ is just the subgroup of integral matrices commuting with g.

   Thus $\mathcal{O} \otimes \mathbb{Q} = \mathbb{Q}(g)$, and certainly $\mathcal{O} \supset \mathbb{Z}[g]$, but it may be much larger (just as $\mathcal{O}$ may be larger than $\mathbb{Z}[\epsilon]$).

2. The regulator is easy to compute. The element g corresponds to a fundamental positive unit $\epsilon \in \mathcal{O}$ with

   $$\epsilon + \epsilon^{-1} = e^R + e^{-R} = 2\cosh(R) = \operatorname{tr}(g) = t,$$

   and thus

   $$R = \log|\operatorname{tr}(g)| + O(1).$$

3. While the regulator is an invariant of the conjugacy class of g in $\mathrm{SL}_2(\mathbb{R})$, the order $\mathcal{O}$ and its discriminant D depend on the conjugacy class in $\mathrm{SL}_2(\mathbb{Z})$.

   Theorem 28.7 *We have* $D = (t^2 - 4)/(\gcd(a-d, b, c))^2$.

   More generally, the order in $M_2(\mathbb{Z})$ given by the commutator of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ has discriminant

   $$D = \frac{(a-d)^2 + 4bc}{\gcd(a-d, b, c)^2},$$

   and is generated by $(1/g)\left(\begin{smallmatrix} a-d & b \\ c & 0 \end{smallmatrix}\right)$. This agrees with the formula above when $\det(g) = ad - bc = 1$.

Lenstra's heuristics: the paradox. We now come to the paradox. Recall that for $n = 2$ the discriminant D determines $\mathcal{O}_D$, $h(D)$ and $R(D)$, which satisfy

$$h(D)R(D) \asymp D^{1/2 \pm \epsilon}.$$

Now according to heuristics of Cohen and Lenstra there should a definite percentage of D such that $h(D) = 1$ [CL], i.e. such that there is a unique geodesic on $\mathcal{M}_1$ of discriminant D. Thus it should be likely for $R(D)$ to be comparable to $D^{1/2}$, and hence with a unique associated geodesic.

(Even if $h(D) = 1$, there need not be a unique geodesic on $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ of length $R(D)$. If the fundamental unit doesn't generate the maximal order, there can be lots of other discriminants D with the same regulator, and each gives at least one geodesic of the same length.)

On the other hand, it is easy to construct sequences $g_n \in \operatorname{SL}_2(\mathbb{Z})$ such that:

1. g_n gives a primitive geodesic on $\mathcal{M}_1$,
2. $\operatorname{tr}(g_n) \to \infty$,
3. the geodesics g_n do not become equidistributed.

In fact, almost any construction works. For example, let

$$g_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1+n & 2 & n \\ n & 1 & \end{pmatrix};$$

this geodesic spends most of its time n the cusp. Or take $g_n = A^n B^n$ where A and B are hyperbolic; in the limit this geodesic spirals between the geodesics for A and B.

Let D_n be the corresponding discriminants. Since these geodesics do *not* become equidistributed, we should have $R(D_n) = O(D_n^\epsilon)$ for every $\epsilon > 0$. Thus:

$$h(D_n) \gg D_n^{1/2-\epsilon};$$

in particular, the class numbers for any such construction must go rapidly to infinity! How does this fit with the Cohen-Lenstra heuristic?

Resolution. An explanation is that the when D is chosen 'at random', the trace of its fundamental positive unit is not random at all. Namely we have:

Theorem 28.8 *Let $\epsilon \in \mathcal{O}_D$ be unit of norm 1 and trace $t \in \mathbb{Z}$. Then $(t^2-4)/D$ is a square.*

Proof. We can write $\epsilon = a + b(D+\sqrt{D})/2$; then $t = 2a + Db$ and

$$4 = 4N(\epsilon) = (2a+bD)^2 - b^2 D = t^2 - b^2 D$$

and so $(t^2-4)/D = b^2$. ∎

Now the good geodesic should have $\log t$ of size $D^{1/2}$, i.e. t should be enormous compared to D (as is often the case for a fundamental unit). It should then furthermore have the remarkable property that $t^2 - 4$ is 'almost a square'.

Thus the set of t that arise from 'typical D' are very unusual! In other words, we get a totally different 'measures' on the set of closed geodesics if we order them by length and if we order them by discriminant.

Example. Fundamental positive units ϵ for $\mathbb{Z}[\sqrt{n}]$ (with $D = 4n$) are computed in Table 17. For example, when $n = 29$ we have $\epsilon = 9801 + 1820\sqrt{29}$, with trace $t = 19602$ and

$$t^2 - 4 = 384238400 = 29 \cdot 3640^2.$$

Turning the resolution, we obtain theorems such as:

Theorem 28.9 *As $t \to \infty$, the class number $h(t^2 - 4) \to \infty$.*

Proof. The matrix $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & n \end{smallmatrix}\right)$ has trace t and discriminant $D = t^2 - 4$, so the class number must be comparable to $D^{1/2}$. ■

(The matrix g corresponds to the unit $\epsilon = (t + \sqrt{t^2 - 4})/2$.)

Indeed it can be shown that most fields have large class number, if they are organized by their regulators; see [Sp2], [Sp1].

Challenge. Find explicit $g_n \in \mathrm{SL}_2(\mathbb{Z})$ and $\epsilon > 0$ such that $R(g_n) > D(g_n)^\epsilon \to \infty$.

Circle maps. We conclude by discussing an analogous phenomenon for the doubling map $f : S^1 \to S^1$. (Note that again there are nontrivial conjectures and results without the need for a rank two action like $\times 2$ $\times 3$.)

In this setting a compact orbit corresponds to a periodic cycle. Now if x has period R, then $x = p/D$ where D is odd, and $2^R = 1 \bmod D$. In this setting we regard:

1. the denominator D as the *discriminant* of the orbit; and

2. the period R as the *regulator*.

Note that $R = R(D) =$ the order of 2 in $(\mathbb{Z}/D)^*$. It does not depend on x. As before we have

$$\log_2(D) < R < \phi(D) < D.$$

We can also let $h(D)$ denote the number of orbits of the doubling map on $(\mathbb{Z}/D)^*$; we then have:

$$R(D)h(D) = \phi(D).$$

Binary expansions. Note that $R(D)$ is the same as the period of the fraction

$$1/D = .x_1 \cdots x_R x_1 \cdots x_R \cdots$$

in base two. The fact that R can be quite long — as long as D itself — is like the fact that the fundamental unit can be very large. We will conjecture that the case where the period is long is dynamically well-behaved.

Examples. We have seen that there are periodic cycles on which f behaves like a rotation, and that these can only accumulate on subsets of S^1 of Hausdorff dimension one. In particular they are very badly distributed. We have also seen that there are infinite-dimensional families of smooth expanding mappings that are topologically conjugate to $f(x)$ and hence give rise to ergodic measures which can in turn be encoded as limits of periodic orbits. Thus there are many sources of orbits which are not uniformly distributed.

Equidistribution of long orbits. The discriminant D is highly sensitive to the arithmetic of $S^1 = \mathbb{R}/\mathbb{Z}$. The orbits mentioned above will tend to have large values of D, e.g. $D = 2^R - 1$. On the other hand we can formulate:

Conjecture 28.10 *If $X_n \subset S^1$ is a sequence of periodic cycles such that $R_n > D_n^\epsilon$, then X_n becomes uniformly distributed on the circle.*

This conjecture has now been resolved by [Bo]; see below.

The paradox. In this case the paradox assumes the following (more comprehensible) form.

Suppose we pick D at random. Then there is a good chance that $R(D)$ is comparable to D. Indeed, the Cohen-Lenstra heuristic is now similar to a conjecture of Artin's, which asserts that 2 is a generator of $(\mathbb{Z}/D)^*$ for infinitely many (prime) D. In other words, $R(D) = D - 1$ infinitely often.

On the other hand, suppose we construct a point x of period R at random. Then $x = p/(2^R - 1)$ for some p. Now it is likely that there is little cancellation in this fraction, and so D is comparable to 2^R.

The resolution in this case is again that only very special periodic points $x = p/D = s/(2^R - 1)$ arise for D with $h(D) = O(D^\epsilon)$. Namely for such D the number

$$s = x_1 \cdots x_R \asymp 2^R$$

must almost be a divisor of $2^R - 1$; more precisely we have:

$$sD = 0 \bmod 2^R - 1.$$

Thus the digits of s must be carefully chosen so s accounts for all the divisors of $2^R - 1$ not present in D, so the fraction $s/(2^R - 1)$ collapses to the much simpler fraction p/D. Such s are very rare in the range $[0, 2^R]$.

Positive results. From [BGK] we have:

Theorem 28.11 *Fix $\epsilon > 0$. Let $p_n \to \infty$ be a sequence of primes, and let $G_n \subset S^1$ be a sequence of multiplicative subgroups of the p_nth roots of unity such that $|G_n| > p_n^\epsilon$. Then G_n becomes uniformly distributed on S^1.*

Even better one has a bound on Gauss sums: for any $a \neq 0 \bmod p_n$,

$$\left| \sum_{G_n} z^a \right| \leq |G| p_n^{-\delta}.$$

Corollary 28.12 *Let p_n be a sequence of primes such that the order of 2 in $(\mathbb{Z}/p_n)^*$ is greater than p_n^ϵ. Then the orbits of $1/p_n$ under $x \mapsto 2x \bmod 1$ become equidistributed as $n \to \infty$.*

Note: $\log D \ll N \ll D$. Cases where $N(x) \asymp \log D(x)$ are easily constructed by taking $x = p/(2^n - 1)$.

The assumption that the denominators p_n are primes is completely removed in [Bo].

For arithmetic study of $x \mapsto 2x$, it may be useful to consider the ring $\mathcal{O} = \mathbb{Z}[1/2]$. Although this ring is infinitely-generated as an additive group, its unit group $(\pm 1) \times 2^{\mathbb{Z}}$ is essentially cyclic, so it behaves like a real quadratic field.

29 Problems

1. Consider the powers of 2, $x_n = 2^n$ for $n = 0, 1, 2, \ldots$, written in base 10. What proportion of these numbers begin with the digit 1?

2. Let T be an infinite tree with degree 3 at each vertex. Show that T is an expanding graph.

   More precisely, show that for any finite set of vertices V of T, we have $|\partial V| \geq |V|$. Here ∂V consists of those vertices of T that are outside of V but are joined to V by an edge.

3. Describe explicitly a measure on the bowtie X in Figure 1 that is invariant under the automorphism $f = \iota_x \circ \iota_y$. (Hint: find a nowhere–vanishing algebraic 1–form on this elliptic curve.)

4. Let $I = [0,1]$ and $J = [0,1] \times [0,1]$ denote the unit interval and unit square, each endowed with Lebesgue measure. (a) Show there exists a continuous map $f : I \to J$ that induces, by $\phi \mapsto \phi \circ f$, an isometric isomorphism between $L^2(J)$ and $L^2(I)$. (b) Show there is no continuous map $g : J \to I$ that induces an isomorphism $L^2(I) \cong L^2(J)$.

5. Let $x \in [0,1]$ be chosen at random, and let its continued fraction expansion be $x = 1/(a_1 + 1/(a_2 + \cdots))$. What proportion of the a_i's are 1?

6. Show that the spectrum of a unitary operator U is contained in S^1. (Hint: start by using a power series to invert $\lambda - U$ whenever $|\lambda| > 1$.)

7. Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be the torus, and let $T(x,y) = (x,y) + (a,b)$ be a translation on X.

   (i) Under what conditions on (a,b) is T ergodic? (ii) Assuming T is ergodic, what is its spectral measure and multiplicity function on S^1?

8. Give an example of an ergodic transformation such that T^n is not ergodic for some $n > 1$. Characterize such examples in terms of their spectral measures.

9. Let $T : S^1 \to S^1$ be an irrational rotation, and let $n, \epsilon > 0$ be given. Construct directly a measurable set $E \subset S^1$ such that the sets $E, T(E), \ldots, T^n(E)$ are disjoint and cover all of S^1 save a set of measure $< \epsilon$.

10. Show that for any measure–preserving transformation T without periodic points, the spectrum of $U = T|L^2(X)$ is the whole circle S^1. (Hint: use the Rohlin–Halmos theorem).

11. Let $T : S^1 \to S^1$ be a rotation. Show that there exists an $f \in L^2(S^1)$ with $\int f = 0$ that is not a L^2–coboundary, i.e. $f \not\in \mathrm{Im}(I - T)$.

12. True or false: Any rotation–invariant linear function $\phi : L^\infty(S^1) \to \mathbb{R}$ is a multiple of Lebesgue measure, i.e. $\phi(f) = C \int_{S^1} f(x)\, dx$.

13. Let $U((a_0, a_1, a_2, \ldots)) = (0, a_0, a_1, a_2, \ldots)$ be the right shift operator on $\ell^2(\mathbb{N})$. Show that U is an isometry. What is the spectrum of U? What is the adjoint U^*?

    Explain why the spectral theorem does not apply to this operator.

14. Let $T(x) = 2x$ acting on the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. Show that T is mixing, in the sense that $\int f \circ T^n g \to \int f \int g$ for any $f, g \in L^2(S^1)$. Here the circle is endowed with its rotation–invariant probability measure.

15. Show that $L^\infty[0,1]$ isometrically embeds in $\mathcal{B}(H)$, $H = L^2[0,1]$, by sending $f(x)$ to the operator $T_f(g) = fg$ for all $f \in L^\infty[0,1]$.

    Show that the image of $L^\infty[0,1]$ is a *maximal* commutative subalgebra of $\mathcal{B}(H)$.

16. The *commutant* of a set $A \subset \mathcal{B}(H)$ of bounded operators on a Hilbert space H is defined by $A' = \{T \in \mathcal{B}(H) \, : \, TS = ST \, \forall S \in A\}$.

    Let $H = L^2[0,1]$ and let $A = C[0,1]$ acting on H by multiplication; i.e. $T_f(g) = f(x)g(x)$ for all $f \in C[0,1]$. What is A'? What is A''?

17. Let $H = L^2[0,1]$, and let $T : [0,1] \to [0,1]$ be an invertible measure–preserving map with associated unitary operator U. Show that the subalgebra of $\mathcal{B}(H)$ generated by U and $L^\infty[0,1]$ has trivial center (only the constants) if and only if T is ergodic.

18. What is the spectral measure on S^1 and the multiplicity function for the parabolic automorphism $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of the torus $X = \mathbb{R}^2/\mathbb{Z}^2$?

    For each prime p, let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ be the additive topological group of p–adic integers.

19. Show that the dual of $G = \mathbb{Z}_p$ can be naturally identified with group

    $$\widehat{G} \cong \{x \in \mathbb{Q}/\mathbb{Z} \, : \, x = r/p^n \text{ for some } r, n \in \mathbb{Z}, n \geq 0\},$$

    with the *discrete* topology. (Hint: first show that the image H of any character $\chi : \mathbb{Z}_p \to S^1$ is a finite group, using the fact that H is closed and $p^n x \to 0$ for all $x \in H$.)

20. Let p, q be integers such that p is prime and $\gcd(p, q) = 1$. Let $T : \mathbb{Z}_p \to \mathbb{Z}_p$ be given by $T(x) = qx$.

    (i) Show that T_q preserves Haar measure.

    (ii) Show that T_q is never ergodic.

    (iii) Find the spectral measure and multiplicity function for T_q.

21. Find the spectral measure and multiplicity function for $T(x) = x + 1$ acting on the p–adic integers $\mathbb{Z}_p$ with respect to Haar measure.

22. Let $T : G \to G$ be an automorphism of a (discrete) abelian group. Suppose every orbit of T, other than the identity, is infinite. Show that T has infinitely many orbits, provided G is not the trivial group.

23. (Continuation.) Show that every ergodic automorphism of a compact abelian group G has Lebesgue spectrum of infinite multiplicity, provided G is not the trivial group.

24. Let $T : [0, 1] \to [0, 1]$ be an automorphism of the interval preserving Lebesgue measure. Suppose $m(A - T(A)) = 0$ for all measurable sets A. Prove that $T(x) = x$ for almost every $x \in [0, 1]$.

25. Let $T \in \mathcal{B}(H)$ be a unitary operator on H. Show that if λ is in the spectrum $\sigma(T)$, then for any $\epsilon > 0$ there exists an $f \in H$ with $\|f\| = 1$ and
$$\|Tf - \lambda f\| < \epsilon.$$
    In this case we say f is an *almost eigenvector* for T.

26. Consider the shift operator T acting on $\ell^2(\mathbb{Z})$. Given $\lambda \in S^1$ and $\epsilon > 0$, find an explicit almost eigenvector for T with eigenvalue λ.

27. Consider the action of the linear map $T = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ on $L^2(S^1 \times S^1)$. Given $\lambda \in S^1$ and $\epsilon > 0$, find an explicit almost eigenvector for T with eigenvalue λ.

28. Let T be an automorphism of a probability space (X, μ). Given $f \in L^1(X)$, let $F = \lim S_n(f)$. Show that F is the unique function in L^1 such that (a) F is T-invariant and (b) $\int_A F = \int_A f$ for any T-invariant set A.

29. The strong law of large numbers is easy to prove for *bounded*, independent, identically distributed random variables X_i.

    (i) Suppose $E(X_i) = 0$, $|X_i| \le 1$ and let $S_n = (X_1 + \ldots + X_n)/n$. Show that $E(S_n^2) = O(1/n)$.

    (ii) Show that if $\sum 1/n_k < \infty$, then $S_{n_k} \to 0$ almost surely. (Use the Borel–Cantelli lemma and Chebyshev's inequality.)

    (iii) Let $n_k = k^2$. Show S_n varies only a little for $k^2 < n < (k+1)^2$, and use (ii) to show $S_n \to 0$ almost surely.

30. (i) Give an example of a compact convex set $K \subset \mathbb{R}^3$ whose extreme points do not form a closed set. (ii) Show that no such example is possible in dimension two.

31. Let $T : X \to X$ be a continuous map on a compact metric space. Prove that the extreme points of the space of invariant probability measures $P(X)^T$ coincide with the measures for which T is ergodic. (Hint: use the Radon–Nikodym theorem.)

32. Let T be an irrational rotation of S^1. Show there exists an $f \in C(S^1)$ with $\int f = 0$ such that f is not a coboundary, i.e. such that f does not have the form $f = g - g \circ T$ for some $g \in C(S^1)$.

33. When is the sequence αn^2, $n \in \mathbb{Z}$, uniformly distributed on $S^1 = \mathbb{R}/\mathbb{Z}$?

34. Let $T(x, y) = (x + y, y)$ on the torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Describe all the ergodic probability measures for T, and show do *not* form a closed set (in the weak* topology).

35. Let $r_i > 0$ be a sequence of positive numbers converging to zero. Let $K(r) \subset \ell^2(\mathbb{N})$ be the set of all sequences (a_i) such that

$$\sum \frac{|a_i|^2}{r_i^2} \le 1.$$

    (i) Show that $K(r)$ is a compact, convex set in the norm topology.

    (ii) Show that the extreme points of $K(r)$ are dense in $K(r)$.

    (iii) Is such an example possible in $\mathbb{R}^n$?

36. Let $T : S^3 \to S^3$ be a rotation of infinite order. What are the possibilities for the closure of an orbit of T?

37. Let $T : (\mathbb{Z}/2)^{\mathbb{Z}} \to (\mathbb{Z}/2)^{\mathbb{Z}}$ be the full shift on 2 symbols, given by $T(a)_i = a_{i+1}$.

    (i) Find the two fixed points p and q for T.

    (ii) Show there is a sequence of ergodic measures μ_n, with finite support, such that μ_n converges to the nonergodic invariant measure $(\delta_p + \delta_q)/2$.

    (iii) More generally, show that ergodic measures with finite support are dense in the space of all invariant measures.

38. *(Continuation.) Let $T \in \mathrm{GL}_2(\mathbb{Z})$ define a hyperbolic automorphism of the torus $X = \mathbb{R}^2/\mathbb{Z}^2$. Show that ergodic invariant measures with finite support are dense among all invariant measures for T.

39. Show that $U = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ satisfies $U^{2m} = \pm I \bmod f_m$, where f_m is the mth Fibonacci number.

40. Let $\epsilon > 0$ be the golden ratio, satisfying $\epsilon^2 = \epsilon + 1$. Show that the discriminant of the ring $\mathbb{Z}[\epsilon^m]$ is $5f_m^2$, where f_m is the mth Fibonacci number. How does this relate to the previous problem?

41. Let $T : S^1 \to S^1$ be an irrational rotation, where $S^1 = \mathbb{R}/\mathbb{Z}$. Show that there exists a universal constant C (independent of T) such that $d(T^q(0), 0) \leq C/q$ for infinitely many $q > 0$. (Challenge: find the best value of C.)

42. (Challenge.) Show there exists a universal constant $C > 0$, independent of T, such that the balls the form $B(T^q(0), C/q)$, $q > 0$ fail to cover S^1. (The best value of C is not known; cf. Cassels, 1952, and [GL, p.577].)

43. (Challenge.) Let $T : S^1 \to S^1$ be an irrational, let $U \subset S^1$ be an open set, and let $S_n(x) = |\{i \ : \ 0 \leq i \leq n \quad \text{and} \quad f^i(x) \in U\}/n$.

    Give a direct proof that $S_n(x) \to |U|/|S^1|$ a.e., without using the ergodic theorem.

44. Let $f : G \to H$ be a measurable homomorphism between a pair of Lie groups. Show that f is smooth. (Hint: first assume $G = \mathbb{R}$).

45. Let $N = (0, 0, 1)$ be the north pole on $S^2 \subset \mathbb{R}^3$. *Stereographic projection* is the map $\sigma : S^2 \to \widehat{\mathbb{C}}$ characterized by $\pi(p) = z = x + iy$ if and only the line from N to p in $\mathbb{R}^3$ passes through $(x, y, 0)$. (We set $\pi(N) = \infty$).

    Show that this map is an isometry from the induced metric on S^2 to the conformal metric $2|dz|/(1 + |z|^2)$ on $\widehat{\mathbb{C}}$.

46. Characterize the circles $C \subset \widehat{\mathbb{C}}$ that correspond to great circles under stereographic projection.

47. Assume the circumference of the earth is 40,000 km. Compute the distance from Boston (42° N, 70° W) to Singapore (1° N, 104° E).

48. Give an explicit formula for the natural covering map / group homomorphism $f : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

49. The area of a lune of angle θ (the region between two great circles on the unit sphere S^2) is clearly 2θ. Use this fact to prove the Gauss-Bonnet theorem for a spherical triangle T: the area of T coincides with its excess angle (the sum of its interior angles, minus π).

50. Construct and draw, explicitly, a finite collection of circles in $\widehat{\mathbb{C}}$ that is invariant under a copy of A_5 in $\mathrm{Aut}(\widehat{\mathbb{C}})$. (Note: 6 circles suffice.)

51. Prove that for $0 < \alpha, \beta, \gamma < \pi$, we have

    $$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos(\alpha) \cos(\beta) \cos(\gamma) < 1$$

    if and only if $\alpha + \beta + \gamma > \pi$. Compare equation (4.2).

52. There is no canonical conformal metric on the Riemann sphere $\widehat{\mathbb{C}}$, since its group of automorphisms $\mathrm{SL}_2(\mathbb{C})$ is noncompact. However the $(2, 3, 5)$ orbifold, as a Riemann surface, carries a *unique* conformal metric of curvature $+1$. How is this possible? What happens with the (n, n) orbifold, $n > 1$?

53. (i) Let $P \subset \Delta$ be a regular hyperbolic hexagon with internal angles of 90°, centered at $z = 0$, with one side S orthogonal to the positive real axis. Find the center and radius of the Euclidean circle containing S.

    (ii) Show how a compact hyperbolic surface of genus 2 can be assembled from 4 copies of P. Can this be done in more than one way?

54. Give formulas for distances in S^2 and $\mathbb{H}^2$ in terms of the Hermitian inner products on $\mathbb{C}^2$ of signatures $(2,0)$ and $(1,1)$.

55. Find a natural definition for the cross product of two vectors in Minkowski space $\mathbb{R}^{2,1}$, with the property that $\langle (p \times q), r\rangle$ is the determinant.

56. Let $[g] \in G/A \cong \mathcal{G}$ be an oriented geodesic in $\mathbb{H}$. Describe, in terms of group theory, its two endpoints in $S^1_\infty = G/AN$. When does a point $[h] \in \mathbb{H} \cong G/K$ lie on the geodesic $[g]$?

57. A Lie group is *unimodular* if it carries a smooth measure that is both right and left invariant.

    (i) Show that $\mathrm{SL}_2(\mathbb{R})$ is unimodular.

    (ii) Show that AN, the upper–triangular group in $\mathrm{SL}_2(\mathbb{R})$, is not unimodular.

    (iii) Show that $S^1 = G/AN$ carries no G–invariant measure. Relate this fact to (i) and (ii).

58. Let $B = \{(x,y) : x^2+y^2 < 1\} \subset \mathbb{RP}^2$ be the Klein model for the hyperbolic plane. Give a formula for the hyperbolic metric (as a Riemannian metric) on B in these coordinates.

59. Relate the hyperbolic distance function $d(p,q)$ on B to the cross-ratio of the ordered points (p', p, q, q'), where p', q' are the points where the line through p and q meets $S^1 = \partial B$.

60. Prove that triangles T in $\mathbb{H}^2$ are thin: there exists an $R > 0$ such that if S_1, S_2, S_3 are the edges of T, then $B(S_1 \cup S_2, R) \supset S_3$.

61. (Continuation.) What is best value of R when T is an ideal triangle? Show that this value of R works for all triangles.

62. Construct an explicit map

    $$\phi : \mathbb{R}^2 - \{(0,0)\} \to (\text{the space of horocycles in } \mathbb{H}),$$

    sending the linear action of $\mathrm{SL}_2(\mathbb{R})$ to its isometric action on $\mathbb{H}$.

    Note: a horocycle η in the upper halfplane $\mathbb{H}$ is uniquely determined by its *center* $c \in \partial\mathbb{H} = \widehat{\mathbb{R}}$, and its *height* $h = \sup_{z\in\eta} \mathrm{Im}(z)$. Thus one can regard the target of ϕ as the product $\widehat{\mathbb{R}} \times \mathbb{R}_+$, and write the map as $\phi(x,y) = (c(x,y), h(x,y))$.

63. Prove that in the Minkowski model, if $\langle p, p\rangle = -1$ and $\langle q, q\rangle = 1$, then $\langle p, q\rangle = \pm \sinh d(p, \gamma_q)$, where γ_p is the oriented geodesic determined by q. Explain how the sign is related to the orientation of γ_q and to the choice of one of the two sheets of the hyperboloid defined by $\langle p, p\rangle = -1$.

64. Characterize horocycles in the Klein model for $\mathbb{H}^2$.

65. Draw some pictures with `lim`, available at

    `math.harvard.edu/~ctm/programs`.

66. Let S be an affine 2-dimensional subspace of $\mathbb{R}^{2,1}$, and let $L_S = \mathcal{H} \cap S$, thought of as a subset of $\mathcal{H} \cong \mathbb{H}$. Show that S is either the empty set, a point, a hyperbolic circle, a horocycle, a geodesic, or a parallel of a geodesic. In particular, S has constant curvature.

67. Show that every conic tangent to $S^1 = \partial B^2 \subset \mathbb{RP}^2$, and meeting B^2, arises as L_S for some S.

68. Let S be an affine 2-dimensional subspace of $\mathbb{PR}^{2,1}$, and let $G_S = \mathcal{G} \cap S$, thought of as a family of geodesics in $\mathbb{H}$. Describe the families G_S that arise in this way, geometrically. (For example, G_S might be all the geodesics through a given point in $\mathbb{H}$.)

69. Let $X = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. To show the horocycle flow on T_1X is minimal, following Hedlund, prove the following assertions. We say a horocycle $H \subset \mathbb{H}$ is *transitive* if ΓH is dense in the space of all horocycles; equivalently, if H gives a dense horocycle orbit in T_1X.

    (i) There exists a transitive horocycle. (Use Baire category).

    (ii) If horocycles H_1 and H_2 rest on the same point $Q \in S^1_\infty$, and H_1 is transitive, then so is H_2.

    (iii) If H rests on a fixed point Q of a hyperbolic element in Γ, then H is transitive. (Consider the closure of ΓH and use (i) and (ii).

    (iv) Suppose ΓH contains a sequence of horocycles $H_1, H_2, \ldots$ that tend to infinity, in the sense that they eventually enclose every compact subset of $\mathbb{H}$. Then H is transitive. (Show that $\overline{\Gamma H}$ contains a horocycle of the type considered in (iii).)

(v) Let F be a compact fundamental domain for Γ. Since X is compact, for any horocycle H and $R > 0$, there exists a $\gamma \in \Gamma$ such that H encloses γF and $d(\gamma F, H) > R$. Using this, show every horocycle is of the type considered in (iv).

70. (i) Show that for any two disjoint circles C_1 and C_2 in the complex plane, there exists a Möbius transformation A such that $A(C_1)$ and $A(C_2)$ are both centered at $z = 0$. (ii) Explain how to construct $A^{-1}(0)$ in terms of the planes in hyperbolic 3-space bounded by C_1 and C_2.

71. Let $[a, b] \cup [b, c]$ be the union of two geodesic segments in $\mathbb{H}^2$, each of length L, with bending angle $\pi > \beta > 0$ at b. (We have $\beta = 0$ if the segments lie on a straight line.) Let γ_a be the geodesic orthogonal to $[a, b]$ at a, and similarly for γ_c. Determine the greatest angle $B(L)$ such that $\gamma_a \cap \gamma_c = \emptyset$ for all $\beta \le B(L)$.

    Now let $\gamma = \bigcup_{i=-\infty}^{i=\infty} [a_i, a_{i+1}]$ be an infinite broken geodesic, comprised of segments of length L with all bending angles less than $B' < B(L)$. Show that γ is a quasigeodesic; more precisely, show that γ is a $K(B', L)$–quasigeodesic, for an explicit function $K(B', L)$.

72. What is the length of the shortest closed geodesic(s) on the triply–punctured sphere, $X = \mathbb{H}/\Gamma(2)$? Draw a picture of the homotopy class of this geodesic.

73. Prove that $G = \mathrm{SL}_2(\mathbb{R}) \neq N^t A N$. What elements are missing? Then prove that $G = BWB$, where $B = AN$ and $W = \langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$. This is the Bruhat decomposition.

74. (i) Consider the statement 'given a two horocycles H_1 and H_2, there is a third horocycle H_3 tangent to both'. Show this statement is false; make a small correction so it becomes true; and then prove the corrected statement.

    (ii) Show that $G \neq N N^t N$. Which elements of G do not occur?

75. A rocket undertakes an intergalactic voyage [And], starting from rest, by accelerating at $g = 9.8\, m/s^2$, so its passengers feel an apparent gravitational force that is the same as on Earth. Let $c = 3 \times 10^8\, m/s$ be the speed of light.

(i) Show that the path of the rocket can be described by a hyperbola in space–time coordinates of the form:

$$(x(s), t(s)) = (Ac\cosh(s/A), A\sinh(s/A)),$$

where $A = c/g$.

(ii) Show that s measures the passage of time as seen from the perspective of the passengers of the rocket.

(iii) Estimate how far, in light years, the rocket has traveled by the time its passengers have aged 1 year; and by the time they have aged 10 years.

76. Let $p : \mathbb{R} \rightarrow \mathbb{H}$ be a horocycle parameterized by arclength. What is the approximate behavior of $d(p(0), p(s))$ as $s \rightarrow \infty$?

77. Let $p_i : \mathbb{R} \rightarrow \mathbb{H}$ be a pair of geodesics parameterized by arclength. What is the approximate behavior of $d(p_1(s), p_2(s))$ as $s \rightarrow \infty$? (There are two regimes.)

78. Give a formula, in coordinates, for the G–invariant measure on the group $G = \mathrm{SL}_2(\mathbb{R}) \subset \mathbb{R}^4$ defined by $ad - bc = 1$.

79. Let $H = \mathrm{SL}_2(\mathbb{R}) \subset G = \mathrm{SL}_2(\mathbb{C})$. Discuss the homogeneous space $\mathcal{C} = G/H$. Show X can be identified with the space of oriented circles $C \subset \widehat{\mathbb{C}}$, give coordinates on this space, and describe its G–invariant measure μ in those coordinates. (Hint: use the one–sheeted hyperboloid in $\mathbb{R}^{3,1}$.)

80. (Continuation.) Given $r < R$, show that the measure of the set of circles C nested between $|z| = r$ and $|z| = R$ is proportional to $\log(R/r)$. (We do not require that C is centered at the origin.)

81. Fix a G–invariant measure on the light cone $\mathcal{L} = G/N$. Estimate the measure $m(r)$ of the set of horocycles in Δ with Euclidean diameter $\geq r > 0$.

82. Let Λ be the limit set of a nonelementary discrete group $\Gamma \subset G = \mathrm{SL}_2(\mathbb{R})/(\pm I)$.

(i) Show that the fixed points of hyperbolic element of Γ are dense in Λ.

(ii) Suppose $\Lambda = S^1_\infty$. Show there exists a dense geodesic on $X = \Gamma\backslash\mathbb{H}$. (Hint: show that for each open ball $B \subset \mathbb{H}$, the set of $x \in S^1_\infty$ such that the geodesic ray $[i, x]$ enters ΓB is an open and dense, and apply the Baire category theorem.)

83. Let X be a finite volume hyperbolic surface. Show that closed geodesics are dense in $\mathrm{T}_1 X$.

84. Let $H \subset G = \mathrm{SL}_2(\mathbb{R})/(\pm I)$ be a closed, connected subgroup.

    (i) Prove that if $K \subset H$ then $H = K$ or $H = G$. (Hint: start by considering the action of H on G/K.)

    (ii) Prove that if $N \subset H$ then $H = N$, AN or G.

    (iii) Suppose $A \subset H$. What are the possibilities for H?

85. Let $X = \Gamma\backslash\mathbb{H}$ be a finite volume hyperbolic surface.

    (i) Show there is a compact subset $K \subset X$ such that every horocycle $\eta \subset X$ that does not meet K is a closed loop around a cusp.

    (ii) Let $Z \subset \mathrm{T}_1 X = \Gamma\backslash G$ be a closed, N–invariant set containing no closed N–orbit. Show that Z contains a minimal set Y. This means Y is closed, nonempty, and every N–orbit in Y is dense.

86. Discuss the moduli space $M = A\backslash G/N$ of pairs (γ, η), where γ is an oriented geodesic and η is a horocycle.

    In particular, construct a continuous map $D : M \to \mathbb{R}$ that is generically one–to–one; relate the values of D to the Minkowski inner product; and show M is not Hausdorff.

    For what values of $D(\gamma, \eta)$ is γ tangent to η? For what values do γ and η cross? When does one endpoint of γ coincide with the center (at infinity) of η?

    Let $\iota : M \to M$ be the map that reverses the orientation of γ. Show that M/ι is Hausdorff and describe it as a topological space.

87. Show that for any hyperbolic element $g \in \mathrm{SL}_2(\mathbb{Z})$, there exists a quadratic irrational number $x \in \mathbb{R}$ such that the fixed points of g are simply (x, x'). Here x' is the Galois conjugate of x.

88. (Continuation.) Show that, conversely, if $x \in \mathbb{R}$ is a quadratic irrational then there exists a $g \in \mathrm{SL}_2(\mathbb{Z})$ whose fixed points are x and its Galois conjugate x'. (You may use the theory of Pell's equation.)

89. Find a hyperbolic element $g \in \mathrm{SL}_2(\mathbb{Z})$ that fixes $\sqrt{5} \in \mathbb{R}$.

90. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Show that for any $g \in \mathrm{GL}_2(\mathbb{Q})$, the group

$$\Delta = \Gamma \cap (g\Gamma g^{-1})$$

has finite index in $\mathrm{SL}_2(\mathbb{Z})$. (Hint: use the fact that for any $d > 0$, the matrices congruent to the identity mod d form a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$.)

91. Using the solution to problem 88 or problem 90, show that the pairs of fixed points (x, y) for hyperbolic elements of $\mathrm{SL}_2(\mathbb{Z})$ form a dense subset of $\mathbb{R}^2$.

Conclude from this that closed geodesics are dense in $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

92. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a lattice. Prove that Γ contains a basis for the vector space $\mathbb{R}^4 \cong \mathrm{M}_2(\mathbb{R}) \supset \mathrm{SL}_2(\mathbb{R})$. (This is a special case of the fact that Γ is Zariski dense.)

93. Let $L \subset \mathbb{R}^n$ be a lattice and let $T = \mathbb{R}^n/L$ be the corresponding compact torus. Let $G = \mathbb{R}^n$ act on $V = L^2(T)$ by $t \cdot f(x) = f(x+t)$. Show there is a lattice $\widehat{L} \subset \widehat{G}$ such that for all $f \in V$, the support of μ_f is contained in $\widehat{L}$. How is $\widehat{L}$ related to L?

94. Let $x \mapsto x'$ denote the Galois involution on $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$, and let $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$.

(i) Show that Γ is dense in $G = \mathrm{SL}_2(\mathbb{R})$.

(ii) Show that any measurable function $f : G \to \mathbb{R}$ that is invariant under the left action of Γ is constant a.e.

(iii) Show that the image of Δ of Γ under the map $g \mapsto (g, g')$ is discrete in $G \times G$.

(iv) Show that the action of each factor of $G \times G$ on $\Delta\backslash(G \times G)$ is ergodic. You may use the fact that Δ is a lattice in $G \times G$.

[It then follows, from the Howe–Moore theorem in the semisimple case, that the action of $G \times G$ is mixing.]

95. Given $n \geq 2$ let $G_i \subset \mathrm{SL}_n(\mathbb{R})$ denote the subgroup isomorphic to $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{R}e_i \oplus \mathbb{R}e_n$ and the identity on the remaining coordinates. Prove that $G_1, \ldots, G_{n-1}$ generate $\mathrm{SL}_n(\mathbb{R})$.

96. For $1 \leq j \neq k \leq n$, let e_{jk} be the $n \times n$ matrix with 1 in position (j,k) and 0s elsewhere. Let $U_{jk} \subset \mathrm{SL}_n(\mathbb{R})$ be the set of matrices of the form $I + te_{jk}$, $t \in \mathbb{R}$.

    (i) Prove that U_{jk} is a subgroup of $\mathrm{SL}_n(\mathbb{R})$.

    (ii) Prove that these groups generate $\mathrm{SL}_n(\mathbb{R})$.

97. Let $\rho : AN \to U(H)$ be a unitary representation of the upper triangular group $AN \subset \mathrm{SL}_2(\mathbb{R})$.

    (i) Prove that any A–invariant vector is actually AN–invariant.

    (ii) Show by example that an N–invariant need not be A–invariant.

    (iii) Prove directly that your example does not extend to a unitary representation of $\mathrm{SL}_2(\mathbb{R})$.

98. Explain the fallacy in the following 'proof' that the horocycle flow for a compact hyperbolic surface X is uniquely ergodic: (i) by mixing of the geodesic flow, any large circle $S^1(x,r)$ in nearly equidistributed in T_1X; and (ii) horocycles are limits of spheres as $r \to \infty$, so they too are uniformly distributed. (Hint: (i) is true even when X has finite volume, but in that case the horocycle flow is *not* uniquely ergodic.)

99. Prove that the solvable Lie group $AN \subset G = \mathrm{SL}_2(\mathbb{R})$ contains no lattice.

100. Let $\alpha \in \mathbb{R}$ be a quadratic irrational. Show there exists a $C > 0$ such that

$$|\alpha - p/q| \geq C/q^2$$

    for all rationals p/q. (Hint: use the fact that $(\alpha - p/q)(\alpha' - p/q) \in \mathbb{Q} - \{0\}$, where α' is the Galois conjugate of α.)

101. Show there exists an irrational quadratic form, $Q(x,y) = x^2 - \beta y^2$, such that $|Q(x,y)| > \epsilon > 0$ for all nonzero $(x,y) \in \mathbb{Z}^2$. (Hint: apply the preceding problem with $\beta = \alpha^2$; e.g. $\beta = (1+\sqrt{2})^2$ will work.)

102. Prove that the quadratic form $Q(x) = x_1^2 + x_2^2 - D(x_3^2 + x_4^2)$ does not represent zero when $D = 3 \bmod 4$. (Hint: work mod 8.)

103. Fix $r > 0$. Let $X_n = \Gamma_n \backslash \mathbb{H}$ be a sequence of a compact hyperbolic surfaces of genus g, with the length of the shortest closed geodesic on X_n bounded below by r.

Show that after passing to a subsequence, there is a discrete group Γ such that $\Gamma_n \to \Gamma$ in the Hausdorff topology on closed subsets of $\mathrm{SL}_2(\mathbb{R})$; and that $X = \Gamma \backslash \mathbb{H}$ is also a compact Riemann surface of genus g.

104. Find the stabilizer $\mathrm{SO}(Q, \mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{Z})$ of the following quadratic forms on $\mathbb{Z}^2$:

(i) $Q(x, y) = x^2 - y^2$;

(ii) $Q(x, y) = x^2 - 4y^2$;

(iii) $Q(x, y) = x^2 - 3xy - y^2$.

Describe the corresponding geodesics on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, paying attention to which orbifold points they intersect.

105. Let $\Lambda \in \mathcal{L}_2 = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ be a unimodular lattice in $\mathbb{R}^2$, and let $U \subset \mathrm{SL}_2(\mathbb{Z})$ be the cyclic group generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

(i) Show that $\overline{\Lambda \cdot U}$ is either a finite set, a circle, or all of $\mathcal{L}_2$. Which types of lattices correspond to each alternative?

(ii) Show the $\overline{\Lambda \cdot \mathrm{SL}_2(\mathbb{Z})}$ is either a finite set, or all of $\mathcal{L}_2$. Which types of lattices correspond to each alternative?

(Hint: use Ratner's theorem to answer (i) and then (i) to answer (ii).)

106. Let

$$G = \mathrm{ASL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \ : \ ad - bc = 1 \right\} \subset \mathrm{SL}_3(\mathbb{R}).$$

Let $\Gamma = \mathrm{ASL}_2(\mathbb{Z})$, and let H be the subgroup of G where $x = y = 0$.

(i) Show that G can be interpreted as the space of affine maps $g : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $g(v) = Av + b$, where $\det(A) = 1$.

(ii) Show that $\Gamma\backslash G$ can be interpreted as a space of translates of lattices in $\mathbb{R}^2$, by setting $\Lambda = g^{-1}(\mathbb{Z}^2)$.

(iii) By Ratner's theorem, for every $x \in \Gamma\backslash G$ there is a closed subgroup $J \subset G$ such that $\overline{xH} = xJ$. What are the possibilities for J?

(iv) Which types of lattice translates, as in (ii), correspond to the different possible groups J?

107. Using Ratner's theorem, find all the possibilities for $\overline{xU}$ in $\Gamma\backslash G$, where

$$\Gamma_0 = \mathrm{SL}_2(\mathbb{Z}) \subset G_0 = \mathrm{PSL}_2(\mathbb{R}),$$

$G = G_0 \times G_0$, $\Gamma = \Gamma_0 \times \Gamma_0$, and U is the image of G_0 in $G_0 \times G_0$ under the diagonal embedding.

Interpret your answer in terms of pairs of unimodular lattices of elliptic curves.

(Hint: what happens with pairs of lattices such that $\Lambda_1 \cap \Lambda_2$ has finite index in both?)

108. Let $\Lambda \subset \mathbb{R}^n$ be a lattice, let $A \subset \mathrm{SL}_n(\mathbb{R})$ be the group of diagonal matrices, and let $A_\Lambda \subset A$ be the stabilizer of Λ. Suppose that A/A_Λ is compact.

(i) Show that $K = \mathbb{Q}(A_\Lambda)$ is a *subfield* of the ring of matrices $\mathrm{M}_n(\mathbb{R})$.

(ii) Show that A_Λ is a subgroup of the group of units, $\mathcal{O}_K^\times$.

(iii) Show that there exists a K–linear isomorphism between $\Lambda \otimes \mathbb{Q}$ and K, sending Λ to an additive subgroup of finite index in $\mathcal{O}_K$.

109. A $\mathbb{Z}$–invariant mean $m : L^\infty(\mathbb{Z}) \to \mathbb{R}$ is sometimes called a *Banach limit*.

(i) Let $a = (a_n)$ be the sequence that is 1 when $|n|$ is prime, and otherwise zero. Let m be a Banach limit. What can you say about $m(a)$?

(ii) Let $S_N(a)$ be the average of a_n over all indices n with $|n| \leq N$. Give an example of a sequence $a \in L^\infty(\mathbb{Z})$ such that $\lim_{N\to\infty} S_N(a)$ does not converge.

(iii) Prove there exists more than one Banach limit.

(iv) Prove that if $a \in L^\infty(\mathbb{Z})$ and $m(a) = L$ for all Banach limits m, then $S_N(a) \to L$ as $N \to \infty$.

110. Let $G \subset \mathrm{Aff}(\mathbb{R})$ be the group generated by $a(x) = 2x$ and $b(x) = x+1$.

(i) Show that G is amenable.

(ii) Find explicit finite sets $G_i \subset G$ such that $|\partial G_i|/|G_i| \to 0$. Here the boundary is computed using the Cayley graph of G with generating set $S = \langle a^{\pm 1}, b^{\pm 1}\rangle$.

111. Let $\mathbb{Z}^2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and let G be the group of maps $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ of the form

$$f(x) = Ax + B,$$

with $(A, B) \in \mathrm{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$.

Given $A \in \mathrm{SL}_2(\mathbb{Z})$, let G_A be the subgroup of of G generated by $a(x) = Ax$ and $b_i(x) = x + e_i$, $i = 1, 2$. Show that:

(i) G_A is a solvable group.

(ii) When A has finite order, G_A contains $\mathbb{Z}^2$ with finite index.

(iii) When $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, G_A is nilpotent.

(iv) When $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, G_A has exponential growth with respect to the generators (a, b_1, b_2).

112. Let A be a subset of an Abelian group G. Prove that if $a + A$ and $b + A$ are disjoint, and both are contained in A, then A is empty. (This explains why the proof that $\mathbb{Z} * \mathbb{Z}$ is nonamenable cannot work for an Abelian group.)

113. Let Λ be a lattice in $\mathbb{R}^n$, and let $X = \mathbb{R}^n/\Lambda$. The Laplacian on X is given by $\Delta = -\sum_1^n d^2/dx_i^2$. Let $X_i \to X$ be a sequence of covering spaces of degree $d_i \to \infty$.

(i) Determine the spectrum $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$ of the Laplacian on X. (Hint: the characters of X, thought of as a group, form an L^2 basis of eigenfunctions.)

(ii) Give a concrete upper bound on $\lambda_1(X_i)$ in terms of d_i and constants depending only on X.

(iii) Show that $\lambda_1(X_i) \to 0$ as $i \to \infty$.

114. Show that there is no probability measure μ on $\mathbb{RP}^n$, $n \ge 1$, that is invariant under the action of $\mathrm{SL}_{n+1}(\mathbb{R})$. (Hint: use translations to show μ assigns zero mass to every affine open set $\mathbb{R}^n \subset \mathbb{RP}^n$.)

115. Let G_n be the double of a bifurcating tree of depth n across its endpoints. Do the graphs G_n form a sequence of expanders?

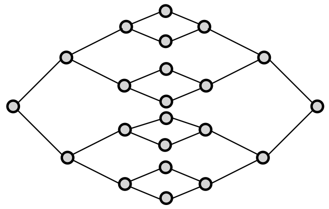


Figure 21. The double of a bifurcating tree of depth 3.

116. *Show that if G_n is sequence of expanding graphs, then only finitely many of these graphs are planar. (Hint: any planar graph on n vertices can be cut into two pieces of substantial size by removing $O(\sqrt{n})$ vertices.)

117. Let $\Delta = dI - A$ be the Laplacian for a finite d-regular graph with vertices V and adjacency matrix A. Show that Δ is a non-negative operator, by proving that

$$\langle \Delta f, f\rangle = \frac{1}{2}\sum_{i,j} A_{ij}|f(i) - f(j)|^2 \geq 0$$

for all $f \in L^2(V)$. Explain how the sum above can be interpreted as $\langle \nabla f, \nabla f\rangle$.

118. Let G be a finite group, with symmetric generating set S. Let $\mathcal{G} = \mathcal{G}(G,S)$ and let $\mathcal{H} = \mathcal{G}(G/H, S)$, where H is a subgroup of G.

(i) Construction an injective map from $L^2(G/H)$ to $L^2(G)$ that respects the action of G.

(ii) Show that $\lambda_1(\mathcal{G}) \leq \lambda_1(\mathcal{H})$.

(iii) Similarly, give a lower bound on $h_1(\mathcal{H})$ in terms of $h_1(\mathcal{G})$.

(iv) Fix a generating set S for $\mathrm{SL}_2(\mathbb{Z})$. Assume the graphs $\mathcal{G}(\mathrm{SL}_2(\mathbb{F}_p), S)$, for p prime, form a sequence of expanders. Show the same is true for the smaller graphs $\mathcal{G}(\mathbb{P}^1(\mathbb{F}_p), S)$.

119. Find a finite set of generators S for $\mathrm{SL}_3(\mathbb{Z})$.

120. Let $\rho : K \to U(H)$ be a unitary representation of a compact group with almost invariant vectors. Prove that K has an invariant vector.

121. Let H be the space of holomorphic 1–forms $\omega = \omega(z)\,dz$ on the upper half plane such that

$$\|\omega\|^2 = \int_{\mathbb{H}} |\omega(z)|^2\,|dz|^2 = \frac{i}{2}\int_{\mathbb{H}} \omega \wedge \overline{\omega}$$

is finite.

(i) Show that H is a Hilbert space with respect to the norm above.

(ii) Show there is a natural unitary action of $\mathrm{SL}_2(\mathbb{R})$ on H.

(iii) Show that H has no K–invariant vectors. (Hint: this is equivalent to showing there is no holomorphic 1–form on the unit disk that is invariant under all rotations.)

(iv) Conclude from (iii) that there are no almost invariant vectors for the action of $\mathrm{SL}_2(\mathbb{R})$ on H.

122. Let Γ be a torsion-free lattice in $G = \mathrm{SL}_3(\mathbb{R})$, let $K = \mathrm{SO}(3,\mathbb{R})$, and let $M = \Gamma\backslash G/K$. Show that $H^1(M,\mathbb{R}) = 0$.

123. Let $\Gamma \subset \mathrm{SO}(n,1)$ be a discrete group. Its *limit set* is defined by

$$\Lambda = \overline{\Gamma p} \cap S^{n-1}_{\infty},$$

for any $p \in \mathbb{H}^n$.

(i) Show that the definition of Λ does not depend on p, and that Λ is invariant under Γ.

(ii) Show that $|\Lambda| \leq 2$ iff Γ is elementary (meaning Γ contains an abelian subgroup with finite index).

(iii) Suppose from now on the $|\Lambda| > 2$. Show that the action of Γ on Λ is minimal (every orbit is dense).

(iv) Show that Λ is perfect (it has no isolated points).

(v) Show that Λ is the smallest closed, nonempty, Γ-invariant subset of the sphere.

(vi) Show that fixed points of elements of Γ are dense in Λ.

(vii) Show that the action of Γ on $\Omega = S^{n-1}_{\infty} - \Lambda$ is properly discontinuous; more precisely, show that Ω/Γ is an orbifold of dimension $(n-1)$.

124. Let $f : X \to Y$ be a homeomorphism between a pair of compact hyperbolic surfaces, let $\widetilde{f} : \Delta \to \Delta$ be its lift to the universal cover, and let $F : S^1 \to S^1$ be the continuous extension of $\widetilde{f}$ to the circle.

(i) Show that F is Hölder continuous ($|f(x) - f(y)| \le C|x-y|^{\alpha}$, some $\alpha > 0$).

(ii) Show that the Hölder exponent of F controls the ratios of lengths of corresponding geodesics on X and Y.

(iii) Show that F is *quasi-symmetric.* This means there exists a $k > 1$ such that for any pair of adjacent arcs on the circle of the same length, $I = [a, b]$ and $I' = [b, c]$, the length ratio of their images satisfies

$$\frac{1}{k} \le \frac{|f(I)|}{|f(I')|} \le k.$$

(iv) Show that if F is Lipschitz, then all lengths agree and therefore F is a Möbius transformation.

125. Let $\Gamma, \Gamma' \subset \mathrm{PSL}_2(\mathbb{R})$ be Fuchsian groups, and suppose $f : S^1_{\infty} \to S^1_{\infty}$ is a homeomorphism conjugating Γ to Γ'. Prove that f is differentiable at every parabolic fixed point of Γ.

126. Let $f : \mathbb{R} \to \mathbb{H}$ be a path parameterized by arclength, satisfying

$$d(f(s), f(t)) \ge D(|s-t|),$$

where $D(r) \to \infty$ as $r \to \infty$. Suppose the image of f lies within a bounded distance of a geodesic γ. Prove that f is a quasigeodesic, in the sense that there is a $k > 1$ such that

$$d(f(s), f(t)) \ge |s-t|/k - 1$$

for all $s, t \in \mathbb{R}$.

127. Consider f as above such that $D(r) = r^{\alpha} - C$, $0 < \alpha < 1$. Prove that the image of f lies within a bounded distance of a geodesic (and hence f is a quasigeodesic).

128. Show there is an f as above with $D(r) \approx \log(r)$ (for larger r) such that the image of f does *not* lie within a bounded distance of a geodesic.

129. Let Γ be a countable discrete group acting by homeomorphisms on a complete metric space X. Suppose the orbit Γp is closed. Prove that Γp is discrete.

130. Let d and ρ denote the hyperbolic distance and metric on the unit disk Δ. Let $\rho'(z)|dz| = \rho(z)/d(0,z)^2|dz|$.

(i) Show that the metric completion of (Δ, ρ') is naturally homeomorphic to $\Delta \cup S^1$.

(ii) Show that the Hausdorff dimension of the metric completion is infinite.

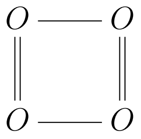
131. Let $N(X, L)$ denote the number of primitive, oriented closed geodesics on a compact hyperbolic surface $X = \mathbb{H}/\Gamma$ with length $\leq L$. Show that $\lim_{L\to\infty}(\log N(L))/L = 1$.

132. Let $Y \to X$ be a degree two covering of a hyperbolic surface X. Then by the prime number theorem in hyperbolic geometry, $N(Y, L)$ and $N(X, L)$ are both asymptotic to $L/\log(L)$. But only 'half' of the closed geodesics on X lift to Y. How are these two assertions compatible? Can you explain the case of a general finite covering $Y \to X$?

133. Characterize, in terms of their dihedral angles, when three planes A, B, C in $\mathbb{H}^3$ have a common perpendicular plane D.

134. Let A, B, C be 3 planes in hyperbolic 3-space, passing through a single point p. Then the intersections AC and BC determine 2 lines in the plane C, meeting at p with angle θ. Give a formula for $\cos\theta$ in terms of the dihedral angles between the three planes A, B and C.

135. Using the solution to the preceding problem, show that the cosines of the interior angles of one of the triangular faces of the arithmetic hyperbolic tetrahedron T with Coxeter diagram:



are given by $\sqrt{1/2}$, $\sqrt{1/3}$ and $\sqrt{2/3}$. In particular, two of the angles are not rational multiples of π. How can this be consistent with the fact that this tetrahedron tiles $\mathbb{H}^3$?

136. Let $L \subset \mathbb{R}^n$ be a unimodular lattice. A *greedy basis* is obtained by first picking a shortest vector $v_1 \in L$, then a shortest vector v_2 linearly independent from v_1, up to v_n.

(i) For what values of n can we insure that (v_i) forms a basis for L over $\mathbb{Z}$?

(ii) Show there is a function $C_n(r)$ such that $|v_n| \le C_n(|v_1|)$.

137. Let X be a compact hyperbolic surface of genus g. Let $\gamma_1, \ldots, \gamma_{3g-3}$ be a maximal set of disjoint simple closed geodesic, chosen by the greedy algorithm: γ_1 is the shortest simple closed geodesic, and γ_i is the shortest among those disjoint from $\gamma_1, \ldots, \gamma_{i-1}$.

(i) Given an explicit upper bound $L_{g,1}$ on the length $L(\gamma_1)$.

(ii) Give an explicit upper bound $L_{g,2}$ on $L(\gamma_2)$.

*(iii) Given an explicit upper bound $L_{g,3g-3}$ for $L(\gamma_{3g-3})$.

*(iv) Show that we can find X such that $L(\gamma_{3g-3})$ is on the order of $\sqrt{g}$ (or more).

(See Buser's book.)

138. *(Winkler.) The next season of Survivor will be played as follows. At each tribal council, each person on the island writes down the name of another person the island, *possibly themselves*, chosen at random. Then *all* the named people are voted off the island. If and when only one person is left, they receive a million dollars. Let p_n be the probability that someone wins, starting with n people on the island.

(i) Plot, using a computer, the values of p_n for $N \le n \le Ne$, with $N = 50$ and $N = 200$.

(ii) Explain why these two plots are almost the same.

**(iii) Prove that $\lim p_n$ does not exist. (This represents a failure of the 'Law of Large Numbers'.) For a hint, see Prodinger's article *How to select a loser.*

A Appendix: The spectral theorem

For reference this section provides the statement of the spectral theorem and a detailed sketch of the proof. We treat both the case of a single unitary operator and the case of a locally compact Abelian group.

Hilbert space. Let H be a nontrivial separable Hilbert space over the complex numbers. We remark that such a Hilbert spaces space is determined up to isomorphism by its dimension, which can be $1, 2, 3 \ldots, \infty$. For example, if $\dim H = \infty$, then it admits an orthonormal basis $(e_i)_{i=0}^{\infty}$; then we have an isomorphism

$$\iota : \ell^2(\mathbb{N}) \cong H$$

defined by

$$\iota(a_i) = \sum_0^\infty a_i e_i.$$

Here $\|a_i\|^2 = \sum |a_i|^2 \|\iota(a_i)\|$.

Two other useful models for the same Hilbert space are given by $L^2(S^1)$ and $\ell^2(\mathbb{Z})$. Here the norm on $L^2(S^1)$ is given by

$$\|f\|^2 = \frac{1}{2\pi} \int_{S^1} |f(z)|^2 \, |dz|.$$

With this norm, the functions $e_i(z) = z^i$, $i \in \mathbb{Z}$, for an orthonormal basis, and give the isomorphism

$$L^2(S^1) \cong \ell^2(\mathbb{Z}).$$

This isomorphism can also be regarded as an instance of Pontryagin duality, i.e. the 'Fourier transform' isomorphism $L^2(G) \cong L^2(\widehat{G})$ between L^2 of a locally compact group and its dual.

Operators and algebras. Let $\mathcal{B}(H)$ denote the space of bounded linear operators $A : H \to H$, with the usual operator norm:

$$\|A\| = \sup_{|x|=1} \|Ax\| = \sup_{|x|=|y|=1} |\langle Ax, y\rangle|.$$

The space $\mathcal{B}(H)$ is an Banach algebra: that is, we have $\|AB\| \leq \|A\| \cdot \|B\|$. More importantly, $\mathcal{B}(H)$ is an example of a C^*-algebra. That is, the natural map $A \mapsto A^*$ sending an operator to its transpose satisfies the important identity:

$$\|A^*A\| = \|A\|^2.$$

To see this identity, note that $\|A^*\| = \|A|$ and thus:

$$\|A\|^2 \geq \|A^*A\| = \sup_{|x|=|y|=1} |\langle A{*}Ax, y\rangle| = \sup_{x,y} |\langle Ax, Ay\rangle| \geq \sup_{x} |\langle Ax, Ax\rangle| = \|A\|^2.$$

Spectra and $\boldsymbol{C^*}$-algebras. The spectrum of $A \in \mathcal{B}(H)$ is defined by

$$\sigma(A) = \{\lambda \in \mathbb{C} \, : \, (\lambda I - A) \text{ is not invertible in } \mathcal{B}(H)\}.$$

(Note: we require a 2-sided inverse.) It is easy to see that the spectrum is a bounded, closed subset of the complex plane. The *spectral radius* of A is defined by

$$\rho(A) = \sup\{|\lambda| \, : \, \lambda \in \sigma(A)\} = \lim \|A^n\|^{1/n}.$$

Now suppose A and A^* commute, i.e. A is a *normal operator*. The most important cases arise when A is *self–adjoint* ($A = A^*$), and when A is *unitary* ($A^* = A^{-1}$). Then the norm closure of the polynomial algebra generated by A and A^* gives a *commutative* $\mathbb{C}^*$–algebra,

$$\mathcal{A} = \overline{\mathbb{C}[A]} \subset \mathcal{B}(H).$$

By the general theory of $\mathbb{C}^*$ algebras, the algebra $\mathcal{A}$ is isomorphic, as a normed $*$-algebra, to the full algebra of continuous functions on its space of maximal ideals. Moreover a maximal ideal m corresponds to the kernel of a multiplicative linear functional $\phi : \mathcal{A} \to \mathbb{C}$. Clearly ϕ is uniquely determined by the value $\lambda = \phi(A)$, and moreover

$$A - \lambda I \in \operatorname{Ker} \phi = m,$$

so $\lambda \in \sigma(A)$. The converse is also true — every element of the spectrum gives a unique maximal ideal m. The ideal m corresponding to λ can be constructed from starting with the principal ideal $(\lambda I - A)$ and extending it to a maximal ideal; by the preceding argument, the result is unique.

Hence we have an isomorphism

$$\mathcal{A} = \overline{\mathbb{C}[A]} \cong C(\sigma(A)).$$

This isomorphism sends the L^2–norm to the sup–norm, and the adjoint operation to complex conjugation.

To justify these statements, one uses the key fact that $\|A\| = \rho(A)$ for a normal operator. For example, when A is self–adjoint, this follows from the equation $\|A * A\| = \|A\|^2 = \|A^2\|$, which implies $\|A\| = \|A^n\|^{1/n} \to \rho(A)$.

Statements of the spectral theorem. We are now in a position to formulate the spectral theorem. We wish to give a *model* for a general normal operator A. The simplest such model is the following. Let K be a compact subset of the complex plane, let μ be a probability measure on K of full support, and let $H = L^2(K, \mu)$. Then we have a natural map from $C(K)$ into $\mathcal{B}(H)$, given by

$$T_f(g) = f(x)g(x).$$

This map is a C^*–algebra isomorphism to its image; in particular, $\|T_f\|_2 = \sup \|f\|_\infty$. (Here we use the fact that μ has full support.) Moreover, f is invertible iff it does not vanish on K; thus the spectrum of T_f is given by $\sigma(A_f) = f(K)$. In particular, if $f(z) = z$, then $A = T_f$ satisfies

$$\sigma(A) = K.$$

Note that the *eigenvectors* for A correspond to the *atoms* of μ; aside from these, A has a *continuous spectrum.* In particular, if μ has no atoms then A has no eigenvalues, properly speaking; but for any measurable subset $E \subset K$, one can regard the subspace

$$L^2(E, \mu|E) \subset L^2(K, \mu)$$

as the subspace where the eigenvalues of A are, morally, in E.

We consider the eigenvalues of A to have *multiplicity one*, since the elements of $L^2(\mu|E)$ take values in a 1-dimensional space (namely $\mathbb{C}$).

Multiplicity. It is apparent that we need to incorporate the possibility of *multiple eigenvalues* into our model. To this end, for $n = 1, 2, \ldots, \infty$ we let

$$H_n = \ell^2(\mathbb{Z}/n),$$

where $\mathbb{Z}/\infty = \mathbb{Z}$. These are model Hilbert spaces of dimension n. Continuing with the space (K, μ) above, we let $L^2(K, \mu, H_n)$ denote the space of functions $f : K \to H_n$ with the norm:

$$\|f\|^2 = \int_K \|f(x)\|^2 \, d\mu(x). \tag{A.1}$$

Now the atoms of μ give n–dimensional eigenspaces for the operator $A(f) = zf(z)$.

Finally we must allow the multiplicity of eigenvalues to vary. Thus we introduce a measurable multiplicity function

$$m : K \to \mathbb{Z}_+ \cup \{\infty\},$$

and use it to form a *Hilbert space bundle* $\mathcal{H}(m) \to K$, such that the fiber over x is $H_{m(x)}$. Finally we let $L^2(K, \mu, m)$ denote the space of sections $f : K \to \mathcal{H}(m)$, with the norm again given by (A.1).

Since we are working in the Borel category, the bundle $\mathcal{H}(m)$ be can be trivialized. More precisely, we can partition K into measure sets $E_n = m^{-1}(n)$; then $\mathcal{H}|E_n \cong E_n \times H_n$, and we have

$$L^2(K, \mu, m) = \bigoplus_{n=1}^{\infty} L^2(E_n, \mu|E_n, H_n).$$

We may now finally state:

Theorem A.1 (The spectral theorem) *Let $A \in \mathcal{B}(H)$ be a bounded operator on a separable Hilbert space, such that $[A, A^*] = 0$. Then there exists a Borel measure μ of full support on $\sigma(A)$, and a multiplicity function m on K, and an isomorphism*

$$H \cong L^2(\sigma(A), \mu, m)$$

sending the action of $A \in \mathcal{B}(H)$ to the operator $f(z) \mapsto zf(z)$ on $L^2(\sigma(A), \mu, m)$.

Complement. *The measure class of μ, and the function m (defined a.e.), are complete invariants of A.*

(Measures μ and ν on $\mathbb{C}$ are in the same *measure class* if they have the same sets of measure zero.)

Self–adjoint operators and quantum theory. To describe the proof as well as some of the physical intuition behind it, consider the case of a self–adjoint operator A on a separable Hilbert space H. As above we let $\mathcal{A} \subset \mathcal{B}(H)$ denote the norm–closed, commutative C^*–algebra generated by A. We then have an isomorphism

$$\mathcal{A} \cong C(\sigma(A)),$$

sending A to the function $f(z) = z$. Since $A = A^*$, we have $f(z) = \overline{f}(z)$, and thus $\sigma(A) \subset \mathbb{R}$. It is natural to write $f(A)$ for the element of $\mathcal{B}(\mathbb{H})$ corresponding to $f \in C(\sigma(A))$. Indeed, this isomorphism is part of the 'functional calculus', which allows one to compose A with various functions such as f on its spectrum.

Now any vector $\psi \in H$ of norm one determines a *state* on the algebra $\mathcal{A}$, i.e. a positive linear functional, by

$$\phi : f(A) \mapsto \langle f(A)\psi, \psi\rangle.$$

Here *positive* means that $\phi(aa^*) \geq 0$ for all $a \in A$.

This functional extends by continuity to a bounded operator on $\mathcal{A} \cong C(\sigma(A))$. If $f \in C(\sigma(A))$ is non-negative, then it can be written as $f = g^2$ with $g = \overline{g}$. Then on the level of operators we have $f(A) = g(A)g(A)^*$, so $\phi(f) \geq 0$. This implies that ϕ can be regarded as a measure μ on the spectrum $\sigma(A)$, i.e. there is a unique measure μ such that

$$\phi(f) = \int_{\sigma(A)} f(x)\, d\mu.$$

Since $\phi(1) = \|\psi\|^2 = 1$, μ is a probability measure.

Quantum interpretation. In quantum mechanics, self–adjoint operators correspond to observables, which take *random values*. The *expected value* of the observable A for a system in the state ψ is given by

$$\langle A\rangle = \langle A\psi, \psi\rangle = \int_{\sigma(A)} d\mu.$$

The spectral measure μ gives much more complete information: it determines the distribution of the random variable A.

Using the spectral measure coming from the state ψ, we obtain an isometric injection

$$\mathcal{A} \cdot \psi \to L^2(\sigma(A), \mu),$$

which extends by continuity to an isomorphism on the norm closure $H_1 \subset H$ of left–hand side. Moreover, under this isomorphism, the action of A becomes $A(f) = xf(x)$.

If $H_1 = H$, we are done with the proof of the spectral theorem; otherwise, we apply the same analysis starting with a suitable new stable in $H_1^{\perp}$. Using separability, we can insure that the process terminates after countably many steps. We then get a spectral decomposition

$$H \cong \oplus_1^\infty L^2(\sigma(A), \mu_n),$$

for a finite or countable sequence of probability measures μ_n.

This countable sum can be easily converted to the form given in the spectral theorem as stated above. Namely we let $\mu = \sum_1^\infty \mu_n/2^n$ (in the case of an infinite sum), we let

$$E_n = \{x \ : \ d\mu_i/d\mu > 0\}$$

and we let $m(x) = \sum \chi_{E_n}(x)$. It is then easy to construct an isomorphism

$$\bigoplus_1^\infty L^2(\sigma(A), \mu_n) \cong L^2(\sigma(A), \mu, m),$$

compatible with the action of A.

The construction makes no real use of the fact that A is self–adjoint, so it applies to a general normal operator.

Positive definite sequences and functions. We now turn to a more systematic discussion of unitary representations of abelian groups; cf. [Ka].

A finite matrix a_{ij} is *positive definite* if

$$\sum a_{ij} b_i \overline{b_j} \geq 0$$

for all b_i.

A continuous complex function $f(x)$ on an abelian group G is *positive definite* if the matrix $a_{ij} = f(x_i - x_j)$ is positive definite for all finite sets $x_i \in G$. The main cases of interest for us will be the cases $G = \mathbb{Z}$ and $G = \mathbb{R}$.

By considering a 2-point set (x_1, x_2) we find $f(-x) = \overline{f(x)}$ and $|f(x)| \leq f(0)$.

Examples.

Any unitary character $f : G \to S^1$ is positive definite. The positive definite functions form a convex cone. (We will see all positive definite functions are positive combinations of characters.)

If U is a unitary operator, then

$$a_n = \langle U^n \xi, \xi \rangle$$

is a positive definite sequence. Proof:

$$\sum a_{i-j} b_i \overline{b_j} \;\; = \;\; \| \sum b_i U^i \xi \|^2 \geq 0.$$

Similarly if U^t is a unitary representation of $\mathbb{R}$, then $f(t) = \langle U^t \xi, \xi \rangle$ is positive definite.

It turns out *every* positive definite function $f(g)$ on G is a linear combination of characters. What would that mean? It means there is a positive measure μ on $\widehat{G}$ such that

$$f(g) = \int_{\widehat{G}} \chi(g) \, d\mu(\chi).$$

But this says exactly that f is the Fourier transform of a measure!

We will prove this result for the cases $G = \mathbb{Z}$ and $G = \mathbb{R}$.

Theorem A.2 (Herglotz) *A sequence a_n is the Fourier series of a positive measure on S^1 if and only if it is positive definite.*

Proof. Let $\mu = \sum a_n z^n$ formally represent the desired measure. Then if $f(z) = \sum b_n z^n$ has a finite Fourier series, we can formally integrate:

$$\frac{1}{2\pi} \int f(z) \, d\mu(z) = \sum a_n b_{-n}.$$

It will suffice to show $\int f \, d\mu \geq 0$ whenever $f \geq 0$, since we have $\int 1 \, d\mu = 2\pi a_0$, and thus we will have $|\mu(f)| = O(\|f\|_\infty)$ and μ will extend to a bounded linear functional on $C(S^1)$.

Now the point of positive definiteness is that

$$\frac{1}{2\pi} \int |f(z)|^2 \, d\mu(z) = \sum_{j-i=n} a_n b_i \overline{b_j} = \sum_{i,j} a_{j-i} b_i \overline{b_j} \geq 0.$$

(We could now finish as before, using the fact that every positive f is of the form $|g(z)|^2$. We will take an alternate route).

Next note that there are trigonometric polynomials $k^N(z)$ such that:

- $\delta^N(z) = |k^N(z)|^2 \geq 0$, $\int_{S^1} \delta^N(z) \, |dz| = 1$, and $f * \delta^N \to f$ uniformly for any trigonometric polynomial f.

In other words δ^N form an approximate identity in the algebra $L^1(S^1)$ with respect to convolution, and δ^N tends to the distributional delta-function (which satisfies $f * \delta = \delta$).

To construct $\delta^N(z) = \sum d_n^N z^n$, note that $\widehat{f * \delta^N} = \hat{f}\widehat{\delta^N}$, so we want $d_n^N \to 1$ as $N \to \infty$. The first thing to try is $k^N(z) = \sum_{-N}^{N} z^n$. Then $k^N * f \to f$ but unfortunately k^N is not positive. To remedy this, set $\delta^N(z) = |k^N(z)|^2/(2N+1)$. Then

$$d_n^N = \frac{\max(0, 2N+1-|n|)}{2N+1},$$

and the desired properties are easily verified. (Note:

$$\delta_N(z) \;\; = \;\; \frac{1}{2N+1}\left|\frac{\sin\frac{N+1}{2}\theta}{\sin\theta/2}\right|^2$$

is known as the Fejér kernel.)

Now since $\delta^N = |k_N|^2$ by positivity of a_n we have $\int \delta^N(x+y)\,d\mu(x) \geq 0$ for any y. Thus

$$0 \leq \lim_N \int (f * \delta^N)\,d\mu = \int f\,d\mu$$

as desired. Here the limit is evaluated using the fact that $\widehat{f * \delta^N}$ is a trigonometric polynomial of the same degree as f, whose coefficients converge to those of f. ∎

Bochner's theorem and unitary representations of $\mathbb{R}$.

Theorem A.3 (Bochner) *The positive definite functions $f : \mathbb{R} \to \mathbb{C}$ are exactly the Fourier transforms of finite measures on $\mathbb{R}$.*

This means $f(t) = \int_{\mathbb{R}} e^{itx}\,d\mu(x)$.

Corollary A.4 *Let U^t be a unitary action of $\mathbb{R}$ on H with cyclic vector ξ. Then there is:*

- *a probability measure μ on $\mathbb{R}$, and*
- *an isomorphism $H \to L^2(\mathbb{R}, \mu)$, such that*

- *ξ corresponds to the constant function* 1*, and*
- *U^t is sent to the action of multiplication by e^{ixt}.*

Proof. First let $f(t) = \langle U^t \xi, \xi \rangle$. Then

$$\sum f(t_i - t_j) a_i \overline{a_j} \;\; = \;\; \| \sum a_i U^{t_i} \xi \|^2 \geq 0$$

so f is positive definite.

Let $A \subset L^2(\mathbb{R}, \mu)$ be the algebra spanned by the characters, i.e. those functions of the form $g(x) = \sum_1^N a_i e^{ixt_i}$. (Note that the product of two characters is another character.) Map A to H by sending g to $\sum_1^N a_i U^{t_i} \xi$. It is easily verified that this map is an isometry (by the definition of μ), and that A is dense in $L^2(\mathbb{R}, \mu)$ (by the Stone-Weierstrass theorem). The completion of this map gives the required isomorphism. ■

Proof of Bochner's theorem. The argument parallels the proof for $\mathbb{Z}$. Let x and t denote coordinates on $\mathbb{R}$ and $\widehat{\mathbb{R}}$. On the real line, the Fourier transform and its inverse are given by:

$$\hat{f}(t) = \int f(x) e^{-ixt} \, dx$$

and

$$f(x) = \frac{1}{2\pi} \int \hat{f}(t) e^{ixt} \, dt.$$

Consider first the space A of functions $f(x) \in L^1(\mathbb{R})$ such that $\hat{f}(t)$ is compactly supported. These are the analogues of trigonometric polynomials; they are analytic functions of x.

Let $\phi(t)$ be a positive definite function. Define a linear functional on A formally by

$$\int f(x) \, d\mu = \frac{1}{2\pi} \int \hat{f}(-t) \phi(t) \, dt.$$

We claim $f \geq 0$ implies $\int f \, d\mu \geq 0$. To see this we use approximate identities again. That is, define on the level of Fourier transforms,

$$\widehat{k^T} = 1 \text{ for } |t| \leq T,$$

and

$$\widehat{\delta^T} = \frac{\widehat{k^T} * \widehat{k^T}}{2T} = 1 - \frac{|t|}{2T} \text{ for } |t| \leq 2T.$$

Then

$$k^T(x) = \frac{1}{2\pi} \frac{\sin(Tx)}{x/2}$$

and

$$\delta^T(x) = \frac{|k^T(x)|^2}{2T}$$

satisfies $\int \delta^T(x)\, dx = 1$.

By positive definiteness of $\phi(t)$, for $f \in A$ we have

$$\int |f(x)|^2 \, d\mu \geq 0;$$

thus $\int \delta^T(x+y)\, d\mu(x) \geq 0$ for any y. As before, for $f \geq 0$ in A we then have

$$0 \leq \int (\delta^T * f)(x)\, d\mu \to \int f(x)\, d\mu$$

and we have verified positivity.

Now enlarge A to the class S of $f(x)$ such that $\hat{f}$ decays rapidly at infinity. Then $\hat{f} \in L^1$, so the definition of $\int f\, d\mu$ extends to S, and the above argument generalizes to show positivity on S.

The important difference between S and A is that S includes all C^∞ functions of compact support. We now show $\int f\, d\mu = O(\|f\|_\infty)$ for such functions.

Let $f \in S$ have compact support and satisfy $0 \leq f \leq 1 - \epsilon$. Observe that for T small, $\delta^T(x)$ is almost constant near the origin, and

$$\int \delta^T(x)\, d\mu = \frac{1}{2\pi} \int_{-2T}^{2T} \left(1 - \frac{|t|}{2T}\right) \phi(t)\, dt \sim \frac{2T}{2\pi} \phi(0)$$

as $T \to 0$. Letting $f_T(x) = (\pi/T)\delta^T(x)$ we have $f_T(0) = 1$ and $\int f_T(x)\, d\mu \to \phi(0)$. If T is sufficiently small, then $f \leq f_T$ and thus $\int f\, d\mu \leq \phi(0)$. It follows that for any $f \in S$ with compact support we have $|\int f\, d\mu| \leq \phi(0)\|f\|_\infty$, and from this we find μ is a measure of finite total mass. ■

More generally, given a set X, a function $f : X \times X \to \mathbb{C}$ is positive definite (or of positive type) if

$$\sum \lambda_i \overline{\lambda_j} f(x_i, x_j) \geq 0$$

for any finite sequence $x_i \in X$ and $\lambda_i \in \mathbb{C}$. Cf. [HV, §5].

Theorem A.5 *The kernel f is positive definite if and only if there exists a Hilbert space H and a map $F : X \to H$ such that the linear span of $F(X)$ is dense in H and*

$$f(x, y) = \langle F(x), F(y) \rangle.$$

The pair (F, H) is unique up to isometry.

Proof. Consider the vector space $V = \mathbb{C}^X$ (each element of X gives a basis vector); use f to define an inner product on this space, take the completion, and mod out by vectors of length zero to obtain H. ■

The spectral measure for actions of $\mathbb{R}^n$. Let G be an n–dimensional vector space over $\mathbb{R}$. Then the dual group $\widehat{G} = \mathrm{Hom}(G, S^1)$ is isomorphic to the dual vector space G^*. Each $x^* \in G^*$ gives a character on G by

$$\chi(x) = \exp(2\pi i \langle x, x^* \rangle).$$

Now let $\rho : G \to U(H)$ be a unitary representation of G. Then each Borel set $A \subset \widehat{G}$ determines an orthogonal projection $\pi_A : H \to H_A$. Roughly speaking, H_A is the part of H where G acts by the characters in A.

Each unit vector $\xi \in H$ then naturally determines a Borel measure on $\widehat{G}$, by

$$\mu(A) = \|\pi_A(\xi)\|^2.$$

To see countable additivity of μ, note that of A_i are disjoint Borel sets, $\bigcup A_i = A$, and $\pi_i : H \to H_i$ are the corresponding projections, we then have

$$H_A = \oplus H_i$$

and hence

$$\mu(A) = \|\pi_A(\xi)\|^2 = \sum \|\pi_i(\xi)\|^2 = \sum \mu(A_i).$$

The transformation from ξ to μ has many useful functorial properties. For example, if H is a closed subgroup of G, then we have a natural projection $p : \widehat{G} \to \widehat{H}$, and the measure ν that ξ determines on $\widehat{H}$ is simply given by $p_*(\mu)$.

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