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Citation: Chaos: An Interdisciplinary Journal of Nonlinear Science **9**, 841 (1999); doi: 10.1063/1.166467
View online: http://dx.doi.org/10.1063/1.166467
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/9/4?ver=pdfcov
Published by the AIP Publishing

Dynamics of a gravitational billiard with a hyperbolic lower boundary

M. L. Ferguson, B. N. Miller,[a] and M. A. Thompson
Department of Physics, Texas Christian University, Fort Worth, Texas 76129

(Received 15 January 1999; accepted for publication 28 July 1999)

Gravitational billiards provide a simple method for the illustration of the dynamics of Hamiltonian systems. Here we examine a new billiard system with two parameters, which exhibits, in two limiting cases, the behaviors of two previously studied one-parameter systems, namely the wedge and parabolic billiard. The billiard consists of a point mass moving in two dimensions under the influence of a constant gravitational field with a hyperbolic lower boundary. An iterative mapping between successive collisions with the lower boundary is derived analytically. The behavior of the system during transformation from the wedge to the parabola is investigated for a few specific cases. It is surprising that the nature of the transformation depends strongly on the parameter values. © *1999 American Institute of Physics.* [S1054-1500(99)00904-0]

Two themes in the study of Hamiltonian systems concern the causes of instability and the features of a chaotic system that distinguish it from an integrable one. In the classical literature (i.e., before the computer's arrival on the scene) these questions were addressed by studying small perturbations of the equations of motion for integrable systems, such as the simple harmonic oscillator, with a small additional, nonlinear force. Since the development of the computer we have been able to study strongly nonintegrable systems. For conservative systems, we typically find that the amount of chaos increases with the energy. In this study we have found a system that, by smooth geometric changes in the lower boundary, or equivalent changes in the energy, cycles from near-integrability to chaos and back to integrability. We study two such cycles that show qualitatively different types of transformations between integrable systems.

I. INTRODUCTION

Many of the seminal studies of nonlinear, Hamiltonian, dynamical systems were carried out for billiards, i.e., unaccelerated particles free to move within a restricted boundary from which the particle is elastically scattered. Both the boundary shape and the dimension determine the ergodic properties of the system. It is possible to construct either completely stable systems (e.g., a circular or rectangular boundary) or chaotic systems (nonfocusing boundary, or an oval such as the ''stadium'') by design.[1]

Perhaps the simplest means of generalizing the classic billiard is to introduce a constant acceleration while still retaining the elastic boundaries. Such a system was studied by one of us about ten years ago when, for a particular value of the external parameter, it was proven to be dynamically isomorphic to the autonomous system consisting of three parallel planar mass sheets moving under their mutual gravitational attraction.[2,3] The system, known today as the wedge billiard, consists of a point mass moving in the plane under a constant acceleration. Two straight, intersecting, elastic barriers are inclined by equal angles to the direction of acceleration, forming a two-dimensional bowl (wedge) that confines the motion. Since energy conservation restricts the ''height'' of the particle, it is not necessary to completely enclose the boundary to achieve a compact phase space and bound the motion.

The only unscalable parameter of the system is the vertex angle of the wedge, conveniently expressed as 2θ. The remarkable property of the system is that any type of behavior can be obtained simply by varying θ. Numerical simulations demonstrate that for $0<\theta<\pi/4$, the phase space consists of coexisting regions characterized by stable and unstable motion. When $\theta=\pi/4$, and in the limit of 0 and $\pi/2$ radians, the motion is completely integrable. For $\pi/2>\theta>\pi/4$ the motion is chaotic.

Since its introduction, the system has been used extensively to investigate many attributes and theories of chaotic dynamics. Both classical[4–9] and quantum mechanical[10–14] versions of the wedge billiard have been studied by a number of investigators. In the first group, using Pessin theory, Wojtkowski has proven[4] that when the half-angle is in the range $\pi/2>\theta>\pi/4$ the dynamics is hyperbolic and the phase space consists of a single ergodic component. Whelan *et al.*[5] studied the system using the isomorphism with two gravitating balls in one dimension derived by Wojtkowski.[4] Bouchaud *et al.* demonstrated the universal scaling of the average, finite time, Lyapunov exponent;[6] Richter *et al.* showed a ''breathing'' oscillation in the relative measure of the chaotic component of the energy surface as the wedge angle is varied below $\pi/4$.[7] Dagaeff *et al.* proved that the preimages of the discontinuity at the wedge vertex ''fashion'' the phase space[8] and Hansen used symbolic dynamics to construct periodic orbits of the system.[9] In the second group, Szeredi, working with Goodings and Lefebvre, has extensively studied the quantum version using semiclassical methods.[10–13] One of the strongest and most interesting results was the confirmation of the Gutzwiller trace formula for the distribution of the energy eigenvalues.[10] In other work, Rouvinez

[a]Electronic mail: b.miller@tcu.edu

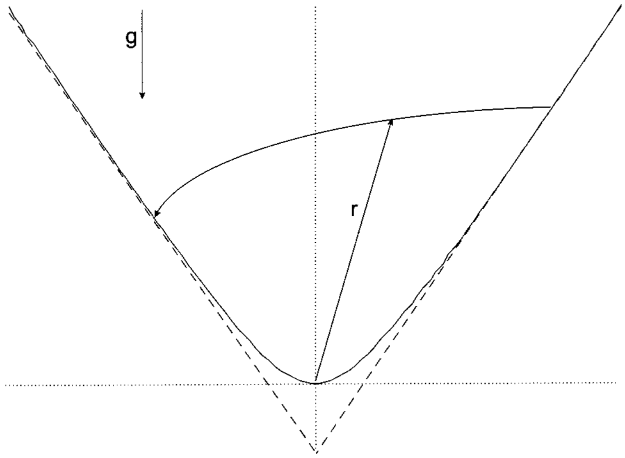

FIG. 1. Configuration space of the hyperbolic gravitational billiard.

and Smilansky investigated a scattering approach to the quantization of the Hamiltonian.[14] From the fact that the stability of orbits depends sensitively on the vertex angle, it is clear that, as for a normal billiard, the boundary shape of an accelerated billiard is important in determining its ergodic properties. Independently Chaplygin[15] and Korsch and Lang[16] showed that when the shape of the bowl is parabolic, rather than wedge shaped, the system is completely integrable for all values of the energy.

An important question concerns the role of the discontinuity in the velocity in the wedge billiard, which occurs between trajectories that collide close to, but on different sides of, the sharp vertex. It is observed that chaotic regions of the phase space always contain at least one orbit that intersects the vertex, so it is natural to wonder if the discontinuity is the chief ''source'' of chaos, or hyperbolicity. Certainly there is no discontinuity in the ''parabolic'' billiard that has a complete lack of unstable orbits. To address this question we reshaped the boundary into a smooth hyperbola with asymptotes that correspond to the original wedge boundaries (see Fig. 1). The direction of acceleration is along the symmetry axis of the hyperbola. In this paper we report our findings for this ''hyperbolic'' accelerating billiard.

In contrast with the wedge billiard, the hyperbolic system is less amenable to scaling, and the phase space geometry now depends on the energy. Intuition tells us that a low-energy billiard will be relatively stable as it only experiences the rounded tip of the hyperbola, which is nearly parabolic in shape. Conversely, an energetic billiard will spend most of its time bouncing off the nearly straight portions of the boundary, so we (correctly) anticipate that its motion is close to that of the wedge. It is in the transition region that intuition is inadequate, and we shall see that the results are surprising.

The dynamics of the system is not only dependent on the energy and the slope of the asymptote. There is a third parameter that arises from the equation for the lower boundary, which we will call b. This parameter defines the position of the directrix of the hyperbola. Changing the placement of the directrix will cause the hyperbola to become more wedge-like or more parabola-like in shape. If the energy is held constant, then the area of the lower boundary within which the billiard is free to collide is bounded. When the parameter b is varied, the overall shape of the lower boundary changes. After some study of the system, it became clear that the variation of this parameter, while holding the energy fixed, was equivalent to changing the energy of the system while holding the parameter fixed. An exact relation between the energy and the parameter can be derived and is described in Sec. II.

Section II contains technical information about the system and could be skipped by the reader so inclined. In it, we will describe the system and provide a simple proof showing the relation between the parabolic, hyperbolic, and wedge billiard. In Sec. III we will use a Poincaré map to explore the stability of the system for different boundary configurations and energies. In Sec. IV we will summarize our results and give suggestions for further investigations.

II. SYSTEM DESCRIPTION

In this paper we are concerned with the motion of a uniformly accelerated point mass in $\mathfrak{R}^2$. The particle undergoes elastic collision with a hyperbolic lower boundary. The motivation for studying the system arises from two previously studied billiard systems with different lower boundaries. One is typically called the wedge billiard studied by Lehtihet and Miller.[2] In this system the lower boundary consists of two intersecting lines. The other is a billiard with a parabolic lower boundary studied by Chaplygin[15] and Korsch and Lang.[16]

The hyperbolic lower boundary has two parameters. One is the slope of the asymptotes and the other determines the position of the focal point within the asymptotes. It can be shown that given the slope of the asymptotes and the energy of the particle per unit mass, the dynamics of the hyperbolic billiard will approach that of the wedge as the focal point approaches the origin and that of the parabolic billiard as the focal point goes to infinity on the vertical axis.

This system presents a unique opportunity for study. The dynamics of the wedge billiard is highly complex, exhibiting integrable, KAM, and K-system behavior that is dependent upon the slope of the lines of the lower boundary. For three cases in the wedge the motion is integrable. The most obvious of these is a slope of zero. In this case the horizontal velocity is conserved. Another case is the limit as the slope goes to infinity. In this case the system closely approximates a central force system in which the square of the angular momentum is a conserved quantity. The last case is for a slope of one. Here, new orthogonal coordinates can be defined perpendicular to each side of the boundary. In this coordinate system the motion separates into that of two completely independent one-dimensional falling bodies. For slopes larger than one, the system displays coexisting stable and chaotic behavior. For slopes less than one, the motion is completely chaotic, suggesting K-system behavior.[2]

The parabola is purely integrable for all values of its one parameter. There exist two different cases, however, that are characterized by the stability of the central fixed point. Here the mass is bouncing vertically in the center. It is stable when the focal point of the parabola is above the maximum height of the billiards trajectory. When the focal point passes below this height the orbit loses its stability and a stable period two orbit develops.[16]

In this study we will be concerned with a point mass in a uniform gravitational field undergoing elastic collisions with a lower boundary $f(x)$. The lower boundary is of the form

$$f(x)=m\sqrt{x^2+b^2}, \tag{1}$$

where m is the slope of the asymptotes and b is a positive parameter that specifies the shape of the hyperbola within the asymptotes. Thus, the lower boundary for this system has two parameters. This is a deviation from two previously studied gravitational billiards that both have only one parameter. One has a wedge-shaped lower boundary described by the equation

$$y=m|x|, \tag{2}$$

and one has a parabolic lower boundary described by the equation

$$y=\tfrac{1}{2}\alpha x^2. \tag{3}$$

It can be shown that our system approaches each of these two previously studied systems for certain limits of parameter values for b.

Recall that in Sec. I we stated that variation of the parameter b, while holding the energy constant, was equivalent to holding b constant and varying the energy. By use of the lower boundary equation, we can derive a relation between the two parameters as follows: Since the energy of the system is simply $E=y+|v|^2$ in appropriate units, it is easy to show that

$$\frac{x_{\max}}{b}=\sqrt{\left(\frac{E}{mb}\right)^2-1}, \tag{4}$$

where $x_{\max}$ is the maximum available value of x on the energy hypersurface within the phase space. Since the dynamic behavior of the system depends on the shape of the hyperbola seen by the billiard particle with a given energy, it is clear that this is fixed by the dimensionless ratio $x_{\max}/b$. Therefore, from Eq. (4), it is clear that the system dynamics can be controlled by either fixing the parameter b and varying the energy, or fixing the energy and varying the parameter. In this paper we have selected the latter approach.

For $x\neq 0$ we can expand $f(x)$ in a Taylor series with respect to b/x. Thus, in the limit as b/x goes to zero we have that

$$\lim_{b/x\to 0} f(x)=m|x|. \tag{5}$$

Implicit in this approximation is the fact that we are only concerned with a finite scaling of x. That is, in order for us to say that b/x is going to zero, we must be able to argue that x cannot go to zero with b in the formal limit. We can show that given any numbers x and $\epsilon>0$ there exists a number $1>\delta>0$ such that for all b where $|b/x|<\delta$ the absolute value of the remaining terms in the Taylor series expansion is less than ϵ. This is a simple proof, and it will not be shown here.

Similarly, we can expand $f(x)$ in a Taylor series about x/b. Thus, in the limit as x/b goes to zero:

$$\lim_{x/b\to 0} f(x)=mb\left(1+\frac{1}{2}\frac{x^2}{b^2}\right). \tag{6}$$

For the Taylor series expansion in x/b, we must be able to argue that x stays finite as b goes to infinity. This argument can be made with an epsilon–delta argument as above or we can simply utilize the fact that conservation of energy and the lower boundary bound the values of x. Thus, as b goes to zero or infinity, the hyperbolic lower boundary approaches the two previously studied systems with lower boundaries in the shape of a wedge and a parabola.

In order to describe the system, we will use a mapping of the form

$$\mathbf{B}:(x,y,v_x,v_y)\to(x',y',v_x',v_y'), \tag{7}$$

where (v_x,v_y) is the momentum vector at the point of collision with the lower boundary and (x,y) are the coordinates of the collision point. The primes denote coordinates at a subsequent collision point.

Let τ be the time interval between collisions. Then by the equations of motion for a point mass in a gravitational field and the equation for the lower boundary, we arrive at a fourth degree polynomial equation in τ:

$$\tau^4+\frac{4}{g}v_y\tau^3+\frac{4(yg+v_y^2m^2v_x^2)}{g^2}\tau^2+\frac{8(yv_y-m^2xv_x)}{g^2}\tau=0. \tag{8}$$

The four roots of this equation correspond to the intersection of the parabolic trajectory of the billiard with *both the upper and lower parts of the hyperbola*. One root is negative, one is zero, and two are positive. The negative root and the largest positive root correspond to intersections of the trajectory with the reflection of the lower boundary in the half-plane $y<0$. Thus, they make no physical sense. The zero root is the intersection we already know and the remaining lesser positive root is the one we want.

The equation for this root is extremely long and cumbersome and provides no insight into the system, therefore it will not be shown here. In addition to the analytical solution, we also found the roots by means of the Newton–Raphson method. Both techniques gave the same values for τ.

The magnitudes for total energy and gravitational field strength can be arbitrarily chosen by an appropriate linear transformation of the time and coordinates. For convenience we have chosen units where the magnitudes of the gravitational field and the energy per unit mass are $\frac{1}{2}$. In these units conservation of energy is expressed by the equation

$$1=\|\mathbf{v}\|^2+y. \tag{9}$$

Given the energy, initial conditions, and τ one can easily determine the coordinates and velocity components of the next boundary collision. The equation for this mapping is shown in the Appendix. Using the principle of conservation of energy and the equation for the lower boundary, one may reduce the number of phase space coordinates necessary to describe the system from four to two. In this study we have chosen the coordinates (x,v_t), where x denotes the horizon-

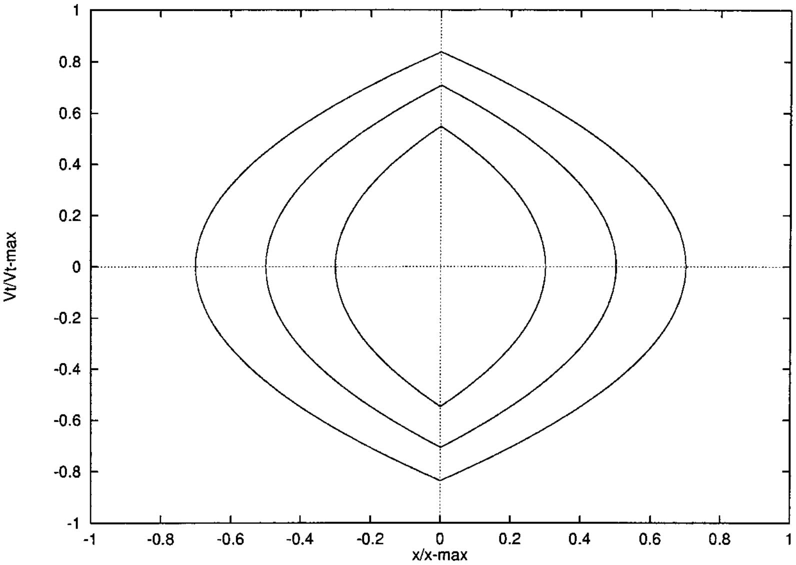

FIG. 2. A characteristic surface of section of the hyperbolic billiard for a small value of the parameter b ($b=10^{-5}$) displaying three orbits. In this range the system closely approximates an integrable case of the wedge billiard. Note the discontinuity in slope on the V_t axis.

tal positional coordinate and v_t denotes the velocity components tangent to the lower boundary. Note that the points (x,v_t), because of the conservation of energy and our choices for the gravitational field and energy per unit mass, are restricted to the area described by the equation

$$v_t^2+f(x)\leqslant 1. \tag{10}$$

III. RESULTS AND APPLICATIONS (STUDY OF STABILITY)

In order keep things simple we initially studied the behavior of our system between the limiting systems of the parabola and the wedge while keeping the parameter m at a constant value. This allowed us to choose a value of the parameter ($m=1$) such that both of the limiting surfaces of section displayed only integrable behavior.[2] Any behavior other than integrable behavior would be a result of the variation of the parameter b. Figures 2 and 3 display the general wedge and parabola cross sections, respectively, for $m=1$.

In transforming the hyperbolic billiard from the parabolic limit to the wedge limit, the tori of the parabola system had to be either transformed into those of the wedge or destroyed and then recreated in the form of the wedge system's tori. It appeared that the tori were all destroyed, yielding to a region of growing global chaos. Another interesting feature was that for a finite range of values of the parameter b, the single stable period two orbit, shared by both the wedge and the parabola systems, bifurcated into two distinct period two orbits (see Fig. 4). The region of global chaos grew until what appeared to be unstable manifolds developed about the periodic points of the system starting with low period orbits, going to higher and higher period orbits (see Fig. 5). The stable manifolds squeezed the region of chaos into smaller and smaller areas in the phase space. These areas finally became so small that they seemed to closely approximate the integrable trajectories of the wedge billiard system.

For the wedge billiard the discontinuity in the lower

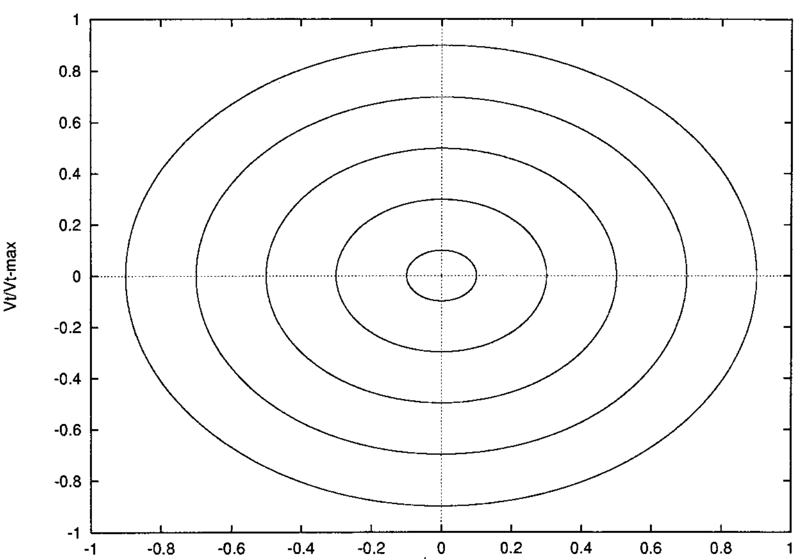

FIG. 3. A characteristic surface of section for a large value of the parameter b ($b=100$) displaying five orbits. In this range the system closely approximates a parabolic billiard with a boundary coefficient of $m/2b$.

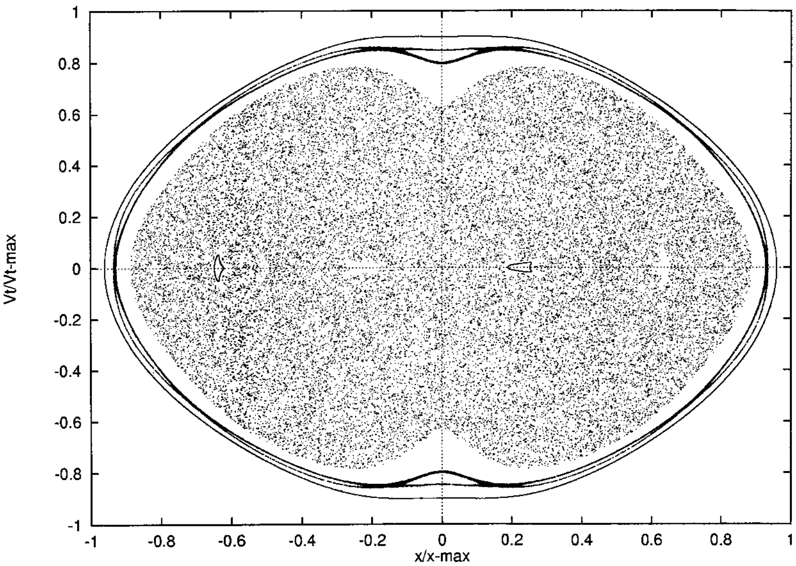

FIG. 4. A characteristic surface of section for an intermediate value of the parameter b ($b=0.5$). Note (1) the growing chaotic region; (2) the small tori within the chaotic region associated with a stable period two orbit. The islands associated with a second period two are not shown. Their presence is apparent from the pair of gaps in the chaotic sea.

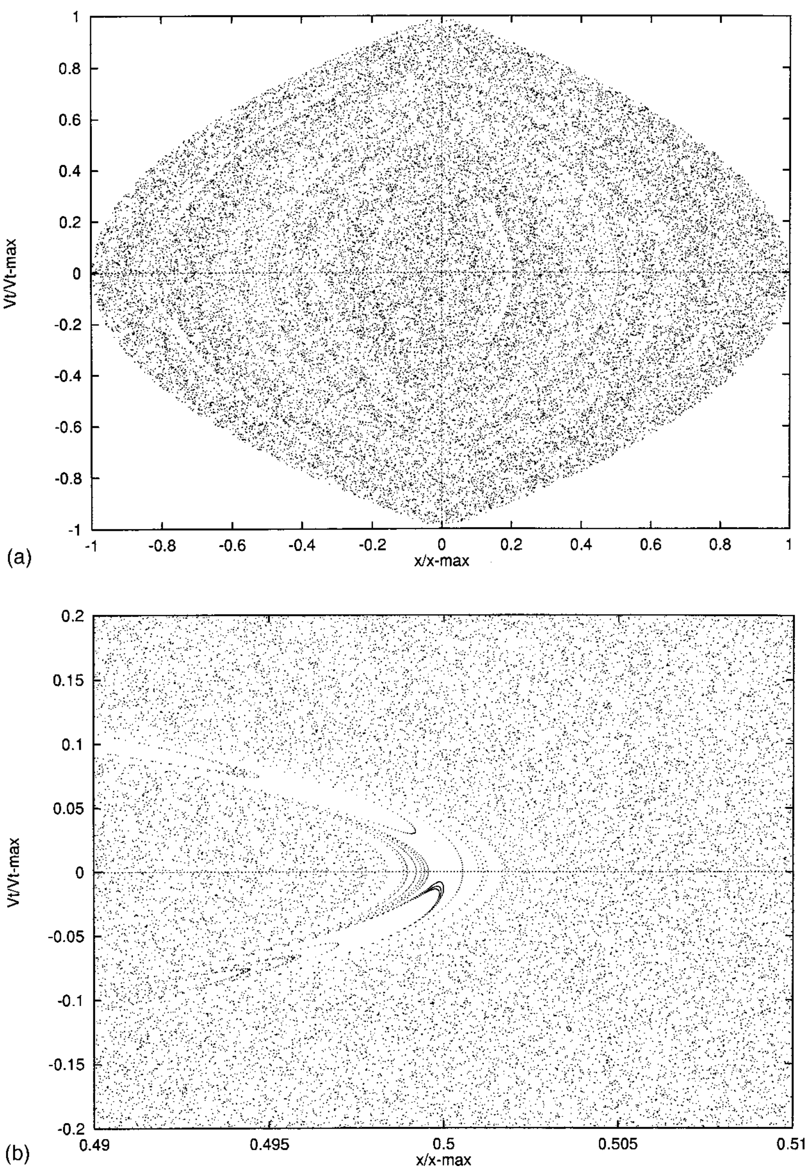

FIG. 5. (a) A Poincaré surface for $b=0.015$. Here the periodic points have become unstable and the unstable manifolds have begun to shape the chaotic region. As the value of b is decreased and the system begins to approximate the wedge billiard the Poincaré surface will begin to approach that of Fig. 2. (b) A close-up of 5(a) centered on an unstable periodic point. The influence of the unstable manifolds are clearly evident.

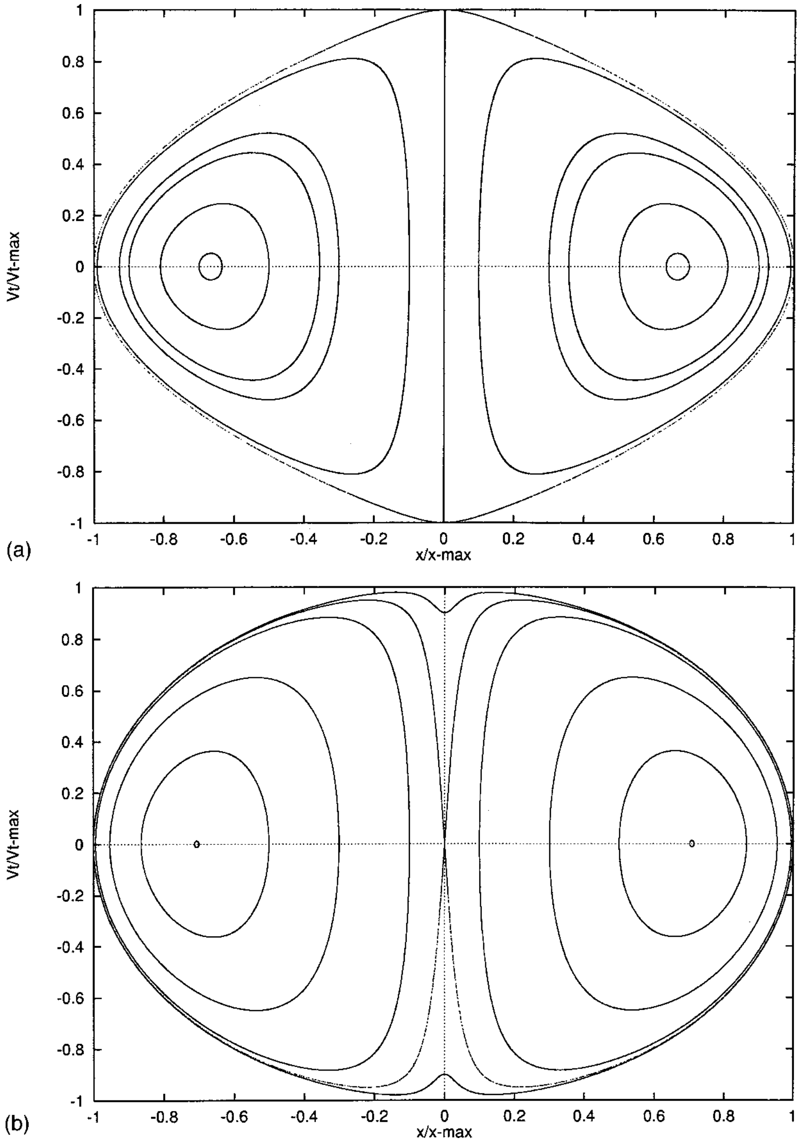

FIG. 6. The two qualitatively different types of cross section that occur in transition from the wedge to the parabola when the parameter m is held constant at 10^4. (a) $b=10^{-7}$; (b) $b=50.0$. These figures show very little evidence of chaos.

boundary at the origin results in a discontinuity in the phase space mapping. Since the derivative of the boundary does not exist at the origin, the tangent vector is undefined. Consequently, reflection of the mass point at the origin is undefined. The wedge billiard displays KAM behavior along with K-system behavior with continuous variation of the wedge half angle up to 45°. For half-angles greater than 45° the system displays only K-system behavior. Variation of the half-angle in the wedge corresponds to the variation of the parameter m in our system. Two more values of the parameter m are of particular interest to us: $m=\tan(30)$, which corresponds to a wedge half-angle of 60°, where the wedge system has the largest Lyapunov exponent, and large m, which corresponds to a small wedge half-angle. For large m the wedge billiard approximates a central force system.[2]

In observing the transformation of the system for large m again we go from approximate integrability to approximate integrability in the limiting cases. Recall the previous case where it was stated that the tori of one limiting system must either transform into those of the other limiting system or they must be destroyed through the KAM mechanism and then new tori must be recreated. For $m=1$, the tori were destroyed. For large m, the tori appear to transform smoothly. Figures 6(a) and 6(b) show the two qualitatively different general cross sections that occur in transition from the wedge [Fig. 6(a)] to the parabola (Fig. 3). There does not appear to be any chaos at all on large scales. This seems to agree with intuition since the system never stops approximating a central field system. There is, however, the possibility of chaos on very fine scales that does occur in the form of generic KAM behavior. Figure 7 is a magnified version of Fig. 6(b). This chaos occurs because the tori of the parabolic limit are forced to intersect at the origin as the parameter b approaches zero and the system approaches the wedge.

The last m value examined was for $m=\tan(30)$ i.e., a wedge half-angle of 60°. Here the system went from the limiting integrability of the parabola[16] to the pure K-system behavior for the wedge.[2] The transition is made through KAM torus destruction and a corresponding increase in the region of global chaos. We see the elliptical fixed point become unstable yielding to a pair of elliptical period two periodic points. Again this pair splits giving rise to two pair of

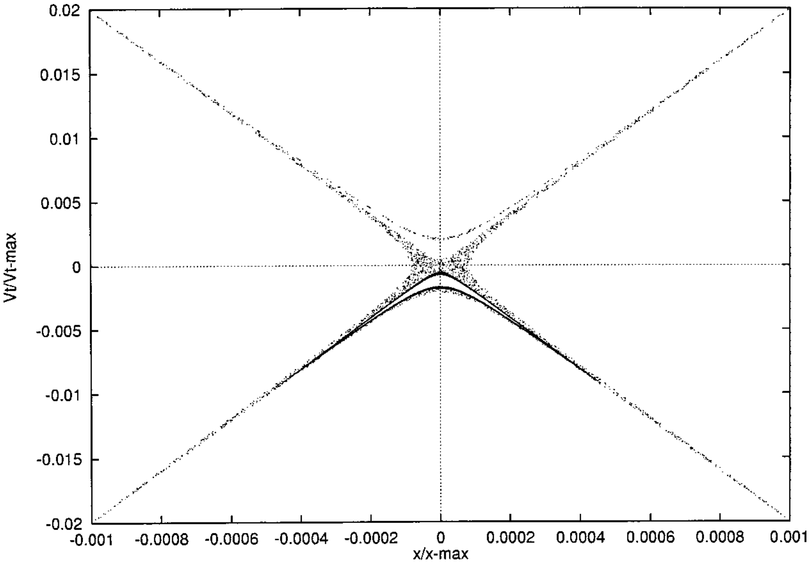

FIG. 7. A close-up of Fig. 6(b) centered at the origin. This figure displays generic KAM behavior as the torii are squeezed closer together.

elliptical period two points whose neighboring tori seem to be the last tori to be destroyed just before the points become unstable. Figure 8 displays a general cross section of the system for an intermediate value of b.

IV. DISCUSSION

Three gravitational billiard systems have now been brought under the umbrella of one. All are simple systems having a conic section for a lower boundary (if you will grant the wedge to be a conic section). The hyperbolic gravitational billiard is a rich system, and offers an opportunity to study a plethora of behaviors characteristic of nonlinear dynamics by the simple means of parameter variation. This system provides an infinite number of examples of transformation from integrability to chaos in a restricted area of phase space through different paths in the parameter space (m,b). We have hopes that it may, after further study, provide some insight into a new class of Hamiltonian systems.

In most previously studied nonlinear Hamiltonian systems the measure of chaotic trajectories on the energy surface in the phase space increases with the amount of energy in the system. Two seminal examples are the Henon–Heiles Hamiltonian[17] and the Fermi Piston,[18] while a more recent example is the system of gravitating shells studied by Youngkins and Miller.[19] In this system we have found a case in which the amount of chaos can actually increase and then decrease with an increase in energy ($m=1$ and $m \gg 1$). This shows that in a system where energy changes the structure of the phase space, chaos is not always a direct result of increased energy.

Recall the two qualitatively different methods of transformation discussed in Sec. IV. In one case the tori of the integrable systems were destroyed and then new tori were generated. In the another case the tori seemed to undergo a topologically invariant transformation. Keep in mind that under the topologically invariant transformation the wedge billiard is not entirely integrable except in the limit that $m \to \infty$. This means that some tori from the parabolic billiard system are, in fact, destroyed while others appear be preserved. In future work we would like to include the identifi-

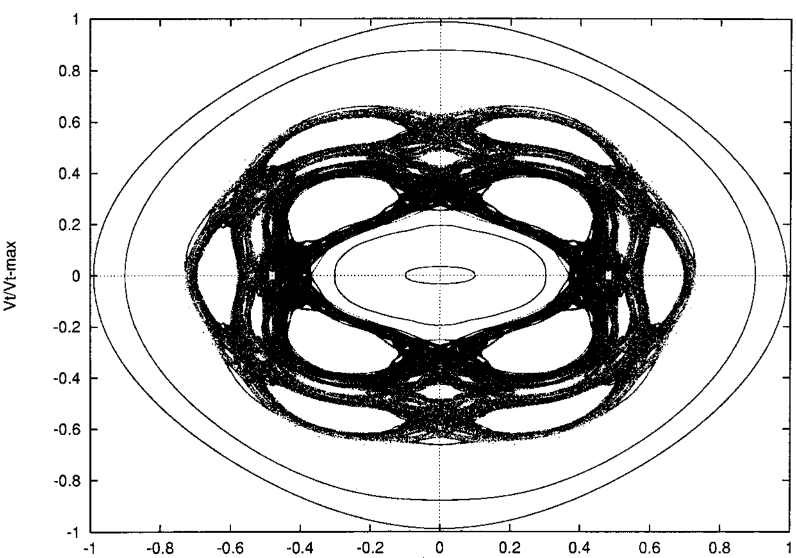

FIG. 8. General cross sections of the system as m is held constant at tan(30) and $b=1.2$. As expected, for this value of m the system goes from integrability (Fig. 3) to the pure K-system behavior of the wedge.

cation of torus winding numbers and show how they transform with changes in the parameter b. This will allow us to show how tori are destroyed in the first case and how they are preserved in the second case.

ACKNOWLEDGMENTS

We are grateful for the support of the McNair Foundation and The Research Foundation of Texas Christian University.

APPENDIX:

In order to describe the mapping, it becomes necessary to define two parameters:

$$\phi(v_t, v_n) = \pm \arctan\left(\frac{m\sqrt{(1-v_t^2-v_n^2)^2 - m^2 b^2}}{1-v_t^2-v_n^2} \right), \quad \text{(A1)}$$

and

$$\phi'(v_t, v_n) = \pm \arctan\left(\frac{m\sqrt{(1-[v_t\cos(\phi)+v_n\sin(\phi)]^2-[v_n\cos(\phi)-v_t\sin(\phi)+g\tau]^2)^2 - m^2 b^2}}{1-[v_t\cos(\phi)+v_n\sin(\phi)]^2-[v_n\cos(\phi)-v_t\sin(\phi)+g\tau]^2} \right), \quad \text{(A2)}$$

where v_t and v_n are the components of the billiard's velocity tangent to and normal to the lower boundary immediately after a collision. Then we can write the mapping for the Poincaré surface of section as

$$\begin{pmatrix} v_t' \\ v_n' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\phi') & -\sin(\phi') \\ \sin(\phi') & \cos(\phi') \end{pmatrix} \times \left[\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} v_t \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ g\tau \end{pmatrix} \right] \quad \text{(A3)}$$

where primes indicate the values of v_t and v_n after the next boundary collision. Using energy conservation we transform the Poincaré surface to the coordinates (x, vt) which are displayed in Sec. III and Figs. 2–8.

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