

HAMILTONCITY OF SPARSE PSEUDORANDOM GRAPHS

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ABSTRACT. We show that every (n, d, λ) -graph contains a Hamilton cycle for sufficiently large n , assuming that $d \geq \log^{10} n$ and $\lambda \leq cd$, where $c = \frac{1}{9000}$. This significantly improves a recent result of Glock, Correia and Sudakov, who obtain a similar result for d that grows polynomially with n . The proof is based on the absorption technique combined with a new result regarding the second largest eigenvalue of the adjacency matrix of a subgraph induced by a random subset of vertices. We believe that the latter result is of an independent interest and will have further applications.

1. INTRODUCTION

A *Hamilton cycle* in a graph is a cycle that passes through all the vertices of the graph exactly once, and a graph containing a Hamilton cycle is called *Hamiltonian*. Even though a Hamilton cycle is a relatively simple structure, determining whether a certain graph is Hamiltonian was included in the list of 21 NP-hard problems by Karp [21]. Thus, there is a large interest in deriving conditions that ensure Hamiltonicity in a given graph. For instance, the celebrated Dirac's theorem [8] states that every graph on $n \geq 3$ vertices with minimum degree $n/2$ is Hamiltonian. For more results on Hamiltonicity, readers can refer to the surveys [13, 28, 29].

Most classical sufficient conditions for a graph to be Hamiltonian are only available for relatively dense graphs, such as the graphs considered in Dirac's Theorem. Establishing sufficient conditions for Hamiltonicity in sparse graphs is known to be much more challenging. Sparse random graphs are natural objects to consider as starting points, and they have attracted a lot of attention in the past few decades. In 1976, Pósa [37] proved that for some large constant C , the binomial random graph model $G(n, p)$ with $p \geq C \log n/n$ is typically Hamiltonian. In the following few years, Korshunov [24] refined Pósa's result and in 1983, Bollobás [5], and independently Komlós and Szemerédi [23] showed a more precise threshold for Hamiltonicity. Their results demonstrate that if $p = (\log n + \log \log n + \omega(1))/n$, then the probability of the random graph $G(n, p)$ being Hamiltonian tends to 1 (we say such an event happens *with high probability*, or *whp* for brevity).

Following the fruitful study of random graphs, it is natural to study families of deterministic graphs that behave in some ways like random graphs; these are sometimes called *pseudorandom graphs*. A natural candidate to begin with is the following: suppose that we sample a random graph $G \sim G(n, p)$, and then allow an adversary to delete some constant fraction of the edges incident to each vertex. The resulting subgraph $H \subseteq G$ loses all its randomness. Thus, we cannot use, for example, a multiple exposure trick and concentration inequalities, which were heavily used in the proof of Hamiltonicity of a typical $G \sim G(n, p)$. Under such a model, one of the central problem to consider is quantifying the *local resilience* of the random graph G with respect to Hamiltonicity. In [41], Sudakov and Vu initiated the study of local resilience of random graphs, and they showed that for any $\varepsilon > 0$, if p is somewhat greater than $\log^4 n/n$, then $G(n, p)$ typically has the property that every spanning subgraph with minimum degree at least $(1 + \varepsilon)np/2$ contains a Hamilton cycle.

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They also conjectured that this remains true as long as $p = (\log n + \omega(1))/n$, which was solved by Lee and Sudakov [30]. Later, an even stronger result, the so-called “hitting-time” statement, was shown by Nenadov, Steger and Trujić [35], and Montgomery [32], independently.

It transpires that exploring the properties of pseudorandom graphs, which has been attracted many researchers in the area, is much more challenging than in random graphs. The first quantitative notion of pseudorandom graphs was introduced by Thomason [42, 43]. He initiated the study of pseudorandom graphs by introducing the so-called (p, λ) -jumbled graphs, which satisfy $|e(U) - p\binom{|U|}{2}| \leq \lambda|U|$ for every vertex subset $U \subseteq V$. Since then, there has been a great deal of investigation into different types and various properties of pseudorandom graphs, for example, [1, 7, 16, 17, 22, 34], and this is still a very active area of research in graph theory.

One special class of pseudorandom graphs, which has been studied extensively, is the class of the so-called *spectral expander graphs*, also known as (n, d, λ) -graphs. Given a graph G on vertex set $V = \{v_1, \dots, v_n\}$, its *adjacency matrix* $A := A(G)$ is an $n \times n$, 0/1 matrix, defined by $A_{ij} = 1$ if and only if $v_i v_j \in E(G)$. Let $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$ be the *singular values* of A (see Definition 3.3). Observe that for a d -regular graph G we always have $s_1(G) := s_1(A(G)) = d$, so the largest singular value is not a very interesting quantity. We say that G is an (n, d, λ) -graph if it is a d -regular graph on n vertices with $s_2(G) \leq \lambda$.

The celebrated Expander Mixing Lemma (see, e.g. Chapter 9 in [3]) provides a powerful formula to estimate the edge distribution of an (n, d, λ) -graph, which suggests that (n, d, λ) -graphs are indeed special cases of jumbled graphs, and that G has stronger expansion properties for smaller values of λ . Thus, it is natural to seek for the best possible condition on the *spectral gap* (defined as the ratio λ/d) which guarantees certain properties; examples of such results can be found e.g. in [2, 4, 18, 36]. For much more on (n, d, λ) -graphs and their many applications, we refer the reader to the surveys of Hoory, Linial and Wigderson [20], Krivelevich and Sudakov [27], the book of Brouwer and Haemers [6], and the references therein.

Hamiltonicity of (n, d, λ) -graphs was first studied by Krivelevich and Sudakov [26], who proved a sufficient condition on the spectral gap forcing Hamiltonicity. More precisely, they showed that for sufficiently large n , any (n, d, λ) -graph with

$$\lambda/d \leq \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)}$$

has a Hamilton cycle. In the same paper, Krivelevich and Sudakov made the following conjecture.

Conjecture 1.1. *There exists an absolute constant $c > 0$ such that for any sufficiently large integer n , any (n, d, λ) -graph with $\lambda/d \leq c$ contains a Hamilton cycle.*

Although there are numerous related results on this direction, there has been no improvement on the original bound until the recent result given by Glock, Correia and Sudakov [15]. In their paper, they improved the above result in two different ways: (i) they demonstrated that the spectral gap $\lambda/d \leq c/(\log n)^{1/3}$ already guarantees Hamiltonicity; (ii) they confirmed Conjecture 1.1 in the case where $d \geq n^\alpha$ for every fixed constant $\alpha > 0$.

In this paper, we improve the second result in [15].

Theorem 1.2. *There exists an absolute constant $c > 0$ such that for any sufficiently large integer n , any (n, d, λ) -graph with $\lambda/d \leq c$ and $d \geq \log^{10} n$ contains a Hamilton cycle.*

Our proof works for $c = \frac{1}{9000}$, although we made no serious attempt to optimize this constant.

In order to find Hamilton cycles in graphs, an *absorption method* usually helps. It was introduced as a general method by Rödl, Ruciński and Szemerédi [38], [39], though similar ideas had appeared earlier, for example by Erdős, Gyárfás and Pyber [9] and by Krivelevich [25]. Our proof adopts an approach similar to that of Montgomery [33]: we find an *absorbing path* formed by a path and

a bunch of *absorbers*, and partition the remaining part into some long paths. Then we use some of the absorbers to connect the endpoints of the long paths using a *connecting lemma*, and thus obtain a desired Hamilton cycle. A similar technique can be also found in, for example, [11], [12], [15].

The main new trick that we use in our proof is the “inherence” of the spectral gap of a random induced subgraph of an (n, d, λ) -graph. Indeed, we use the results on norms of principal submatrices, e.g. the Rudelson-Vershynin Theorem in [40] (see Section 5), and obtain that with probability $1 - n^{-\Theta(1)}$, for a not too small random subset, the spectral gap of the induced subgraph is still $O(\lambda/d)$. We believe that our result will have further applications.

The paper is organized as follows. In Section 2, we give an outline of our proof. In Section 3, we prove the expander mixing lemma for matrices and we then study the special case for *almost* (n, d, λ) -graphs in Section 4. Our key lemma regarding the second singular value of a random induced subgraph of an (n, d, λ) -graph is in Section 5, and the main technical lemma, the Connecting Lemma, is demonstrated in Section 7 based on some arguments related to matchings in Section 6. Finally in Section 8, we prove our main result Theorem 1.2 together with a general version, Theorem 8.1, which is stated for almost (n, d, λ) -graphs. We also put some standard tools from linear algebra and some technical proofs in the Appendix for readers’ convenience.

1.1. Notation.

Graphs. For a graph $G = (V, E)$, let $e(G) := |E(G)|$. We mostly simply assume that $V = [n]$. For a subset $A \subseteq V$ of size m , we simply call it an m -set, and we denote the family of all m -sets of V by $\binom{V}{m}$. For two vertex sets $A, B \subseteq V(G)$, we define $E_G(A, B)$ to be the set of all edges $xy \in E(G)$ with $x \in A$ and $y \in B$, and set $e_G(A, B) := |E_G(A, B)|$. For two disjoint subsets $X, Y \subseteq V$, we write $G[X, Y]$ to denote the induced bipartite subgraph of G with parts X and Y . Moreover, we define $N_G(v)$ to be the neighborhood of a vertex v , and define $N_G(A) := \bigcup_{v \in A} N_G(v) \setminus A$ for a subset $A \subseteq V$. We write $N_G(A, B) = N_G(A) \cap B$ and for a vertex v , let $N_G(v, B) = N_G(v) \cap B$. We also write $\deg_G(v) := |N_G(v)|$ and $\deg_G(v, B) := |N_G(v, B)|$. Finally, let $\delta(G)$ be the minimum degree of G and let $\Delta(G)$ be the maximum degree of G .

The *adjacency matrix* of G , denoted by $A := A(G)$, is a 0/1, $n \times n$ matrix such that $A_{i,j} = 1$ if and only if $ij \in E(G)$. Moreover, given any subset $X \subseteq V$, its characteristic vector $\mathbb{1}_X \in \mathbb{R}^n$ is defined by

$$\mathbb{1}_X(i) = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{otherwise} \end{cases}.$$

Digraphs. The notations of digraphs are similar to those of graphs. Given a digraph $D = (V, E)$ and a vertex v in V , we define $\deg^+(v)$ as the *out-degree* of v , and define $\deg^-(v)$ as the *in-degree* of v . We write $\deg^\pm(v)$ for $\deg^+(v)$ and $\deg^-(v)$. For two disjoint subsets $X, Y \subseteq V$, we define $\vec{E}_D(X, Y)$ to be the set of directed edges $xy \in E(D)$ with $x \in X$ and $y \in Y$, and set $\vec{e}_D(X, Y) := |\vec{E}_D(X, Y)|$. Also, we write $D[X, Y]$ to denote the induced bipartite subgraph of D with edge set $\vec{E}_D(X, Y)$. The *adjacency matrix* of D , denoted by $A := A(D)$, is a 0/1, $n \times n$ matrix such that $A_{i,j} = 1$ if and only if $ij \in E(D)$.

We will often omit the subscript to ease the notation, unless otherwise stated. Since all of our calculations are asymptotic, we will often omit floor and ceiling functions whenever they are not crucial.

2. PROOF OUTLINE

Our strategy for finding a Hamilton cycle in an (n, d, λ) -graph $G = (V, E)$ follows ideas from [12] and [33]. The proof consists of two main phases. First, we construct a family of not too many

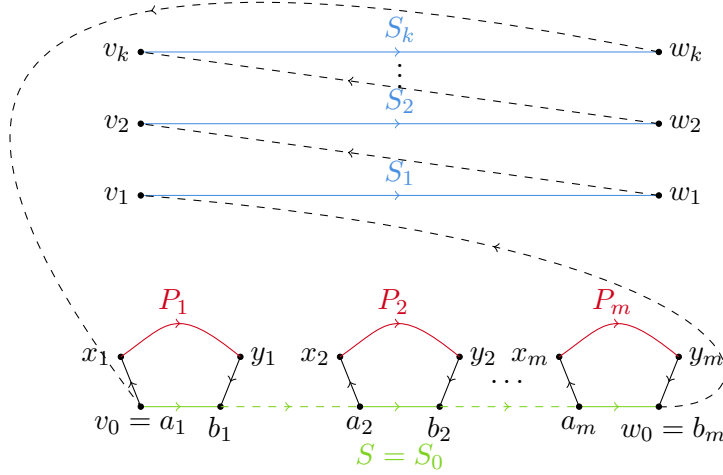


FIGURE 1. All vertices of the graph, but not all edges, are shown on this figure. Our goal is to connect vertices $\{v_i\}$ to vertices $\{w_i\}$ by some vertex-disjoint paths shown by the black dashed lines. The building blocks for these connecting paths are absorbers – the cycles that look like cupcakes at the bottom of the figure. The connecting paths will be built by concatenating some of the red paths P_1, \dots, P_m – the frosting of the pancakes. The green path $S = S_{k+1}$ can be used to bypass the paths P_i that were not utilized.

vertex-disjoint paths that together cover all the vertices of G , where one of the paths is called the *absorbing path* and contains a family of *absorbers*. The family of absorbers has the property that one can remove any subfamily of it from the absorbing path and obtain another path spanning on the remaining vertices while keeping the same end vertices. Then, we can use a subset of the absorbers to connect all these paths into a Hamilton cycle as described below. This proof strategy falls into the framework of the so-called *absorption method* as mentioned in Section 1, and so far has had numerous applications.

We now explain our method in more details. First, we find $m := n/\log^3 n$ random vertex-disjoint cycles $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$, each of which covers $O(\log n)$ vertices and has 4 designated (consecutive) vertices x_i, a_i, b_i, y_i in some fixed order (e.g. counterclockwise). For convenience, denote the longer path that connects x_i and y_i by P_i . These cycles, together with the designated vertices, serve as our *absorbers* and they will play a crucial role in the proof described below.

Second, we aim to construct our *absorbing path* P with certain absorbing property that will be described in details later. In order to find such a path, we first find a path S that contains all the edges $a_i b_i$, $i = 1, \dots, m$, and no other vertices from the C_i s. Now our absorbing path P can be formed by the union of the path S and the cycles C_i s and then omitting all the edges $a_i b_i$ s. Next, we cover the remaining vertices (that is, the vertices in $V \setminus (V(P) \cup V(\mathcal{C}))$) by $k := n/\log^5 n$ vertex-disjoint paths $\{S_1, \dots, S_k\}$. Finally, we wish to tailor all the paths $\{S_1, \dots, S_k, S, P_1, \dots, P_m\}$ into a Hamilton cycle. For convenience, we write $S_0 := S$. As in Figure 1, we let v_i be the starting vertex of S_i , and let w_i be the terminal vertex of S_i for each $0 \leq i \leq k$, where $v_0 := a_1$ and $w_0 := b_m$. We also write

$$X_0 := \{v_0, \dots, v_k\} \quad \text{and} \quad Y_0 := \{w_0, \dots, w_k\}.$$

The idea to find the Hamilton cycle using these paths is based on the absorbing property of the absorbing path P ; that is, for any $I \subseteq [m]$, there is a path with the same endpoints as S , covering exactly $V(S) \cup \bigcup_{i \in I} V(P_i)$. To demonstrate the absorbing property, let us move along the path S to “absorb” some paths P_i s. Whenever we reach a vertex a_i , we have two options as in Figure 1:

- (1) proceed from a_i to b_i directly, or
- (2) move from a_i to x_i , then walk along the path P_i to vertex y_i , and finally arrive at b_i from y_i .

With this key observation, it is feasible to remove some paths P_i for later use, and construct a path with the same endpoints as S covering exactly S and the remaining P_i s. Thus, the major problem is to connect paths S_0, \dots, S_k with the aid of some paths P_i s, that is, to connect w_j and v_{j+1} for each $0 \leq j \leq k$, where $v_{k+1} := v_0$. If so, we are able to construct the Hamilton cycle in G as desired.

Let us now explain how we find paths P_i s to connect the paths S_0, \dots, S_k . First, notice that the internal vertices of P_i s are never touched during the connecting procedure. Consequently, by contracting each P_i into a single vertex z_i , whose out-degree depends only on y_i and in-degree depends only on x_i , we obtain an auxiliary digraph D (see Subsection 8.5) on vertex set $X_0 \cup Y_0 \cup \{z_1, \dots, z_m\}$. If the digraph D is a good expander (see (P1)–(P4)), we are then allowed to apply the Connecting Lemma (Lemma 7.1) to connect X_0 and Y_0 in D , such that the pairs of vertices w_j and v_{j+1} are connected by vertex-disjoint directed paths with internal vertices in $\{z_1, \dots, z_m\}$. Thus, by returning each contracted vertex z_i to the path P_i , the Hamilton cycle in G can be found.

It thus remains to demonstrate that the digraph D is a good expander. By our construction, the spectral properties of D depend only on the endpoints of the paths S_j s and P_i s, which were randomly selected at the very beginning. Thus, it suffices to study the spectral properties of random induced subgraphs of G and this is the main contribution of this paper. Using results on norms of principal matrices, e.g. Rudelson-Vershynin theorem in [40], we show that with probability at least $1 - n^{-\Theta(1)}$, the spectral gap of a random induced subgraph of (n, d, λ) -graph is still $O(\lambda/d)$ (see Theorem 5.2).

3. EXPANDER MIXING LEMMA FOR MATRICES

One of the most useful tools in spectral graph theory is the expander mixing lemma, which asserts that an (n, d, λ) -graph is an expander (see, e.g., [20]).

Theorem 3.1 (Expander mixing lemma). *Let $G = (V, E)$ be an (n, d, λ) -graph. Then, for any two subsets $S, T \subseteq V$, we have*

$$\left| e(S, T) - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S| \left(1 - \frac{|S|}{n}\right) |T| \left(1 - \frac{|T|}{n}\right)}.$$

We will need a more general version of the expander mixing lemma, which can be applied to non-regular graphs, digraphs, and even to general $m \times n$ matrices A . To state such a general result, it is convenient to normalize A in the following way:

Definition 3.2 (Normalized matrix). *Let A be an $m \times n$ matrix. Let $L = L(A)$ be the $m \times m$ diagonal matrix with $L_{i,i} = \sum_j A_{i,j}$ for all i (that is, the sum of entries in the i th row), and $R = R(A)$ be the $n \times n$ diagonal matrix with $R_{j,j} = \sum_i A_{i,j}$ (that is, the sum of entries in the j th column). The normalized matrix of the matrix A is defined as*

$$\bar{A} := L^{-1/2} A R^{-1/2}.$$

In particular, if A is a symmetric $n \times n$ matrix, then the diagonal matrix $L(A) = R(A) =: D(A)$ is called the degree matrix of A .

Since the notion of eigenvalues is undefined for non-square matrices, it would be convenient for us to work with *singular values* which are defined as follows for all matrices.

Definition 3.3 (Singular values). *Let A be a real $m \times n$ matrix. The singular values of A are the nonnegative square roots of the eigenvalues of the symmetric positive semidefinite matrix $A^\top A$. We will always assume that $s_k(A)$ is the k th singular value of A in nonincreasing order. In particular, the singular values and the eigenvalues of a symmetric positive semidefinite matrix A coincide.*

We are now ready to state a more general version of the expander mixing lemma.

Theorem 3.4 (Expander mixing lemma for matrices). *Let A be an $m \times n$ matrix with nonnegative entries, and let \bar{A} be the normalized matrix of A . Then, for any two subsets $S \subseteq [m]$ and $T \subseteq [n]$, we have*

$$\left| A(S, T) - \frac{A(S, n) A(m, T)}{A(m, n)} \right| \leq s_2(\bar{A}) \sqrt{A(S, n) \left(1 - \frac{A(S, n)}{A(m, n)} \right) A(m, T) \left(1 - \frac{A(m, T)}{A(m, n)} \right)},$$

where we adopt the notation $A(S, T) := \sum_{i \in S, j \in T} A_{i,j}$, and we abbreviate $A(S, n) := A(S, [n])$, $A(m, T) := A([m], T)$, and $A(m, n) := A([m], [n])$.

Observe that Theorem 3.4 trivially implies Theorem 3.1, since the adjacency matrix of a d -regular graph satisfies

$$\bar{A} = \frac{1}{d} A.$$

The proof of Theorem 3.4 is almost identical to the standard proof of Theorem 3.1 found e.g. Proposition 4.3.2 in [6]. Since we could not find a reference for this specific statement and proof, we include the proof of Theorem 3.4 for the convenience of the reader, and without claiming any originality. It is based on the following crucial observation.

Observation 3.5. *Let A be an $m \times n$ matrix. Let $a := A(m, n)$ and let $\mathbb{1}_n$ denote the vector in \mathbb{R}^n whose all coordinates are equal to 1. Consider the vectors $\mathbf{u}_1 := a^{-1/2} L^{1/2} \mathbb{1}_m$ and $\mathbf{v}_1 := a^{-1/2} R^{1/2} \mathbb{1}_n$. Then:*

- (1) both \mathbf{u}_1 and \mathbf{v}_1 are unit vectors;
- (2) $\bar{A} \mathbf{v}_1 = \mathbf{u}_1$;
- (3) $s_1(\bar{A}) = \|\bar{A}\| = \mathbf{u}_1^\top \bar{A} \mathbf{v}_1 = 1$.

Proof. The first two parts readily follow from the definitions of a , L and R , and \bar{A} . As for the third part, the equation $s_1(\bar{A}) = \|\bar{A}\|$ holds for any matrix. Let us show that $\|\bar{A}\| \leq 1$. For every $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, we have

$$0 \leq \sum_{i \in [m], j \in [n]} A_{i,j} \left(\frac{x_i}{\sqrt{L_{i,i}}} - \frac{y_j}{\sqrt{R_{j,j}}} \right)^2 = 2 - 2 \sum_{i \in [m], j \in [n]} \frac{A_{i,j} x_i y_j}{\sqrt{L_{i,i} R_{j,j}}} = 2 - 2 \mathbf{x}^\top \bar{A} \mathbf{y}.$$

This implies that $\mathbf{x}^\top \bar{A} \mathbf{y} \leq 1$ for all unit vectors \mathbf{x} and \mathbf{y} , which yields $\|\bar{A}\| \leq 1$.

Moreover, by definition of \bar{A} we have $\mathbf{u}_1^\top \bar{A} \mathbf{v}_1 = 1$. Therefore, by definition of the operator norm, it follows that $\|\bar{A}\| \geq 1$. The observation is proved. \square

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. Let $r = \text{rank}(\bar{A})$, and let $1 = s_1 \geq s_2 \geq \dots \geq s_r > 0$ be all the positive singular values of \bar{A} in nonincreasing order. Applying the singular value decomposition theorem

(Theorem A.3) combined with Observation 3.5, we can find orthonormal bases $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of \mathbb{R}^m and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of \mathbb{R}^n with vectors \mathbf{v}_1 and \mathbf{u}_1 defined in Observation 3.5, and such that

$$\bar{A} = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^\top.$$

In particular, $\bar{A}\mathbf{v}_j = s_j \mathbf{u}_j$ for $j = 1, \dots, r$ and $\bar{A}\mathbf{v}_j = \mathbf{0}$ for $j > r$. Now, let $S \subseteq [m]$ and $T \subseteq [n]$ be two arbitrary subsets. Then

$$A(S, T) = \mathbb{1}_S^\top A \mathbb{1}_T = \chi_S^\top \bar{A} \chi_T, \quad \text{where} \quad \chi_S := L^{1/2} \mathbb{1}_S, \quad \chi_T := R^{1/2} \mathbb{1}_T.$$

Expanding both vectors as

$$\chi_S = \sum_{j=1}^m a_j \mathbf{u}_j, \quad \text{and} \quad \chi_T = \sum_{j=1}^n b_j \mathbf{v}_j,$$

we obtain

$$A(S, T) = \sum_{j=1}^r s_j a_j b_j = a_1 b_1 + \sum_{j=2}^r s_j a_j b_j.$$

Recall from Observation 3.5 that all singular values of \bar{A} are bounded by 1, and $r = \text{rank}(\bar{A}) \leq \min(m, n)$. Thus, by Cauchy-Schwarz inequality, we have

$$|A(S, T) - a_1 b_1| \leq \sum_{j=2}^r |a_j b_j| \leq \left(\sum_{j=2}^m a_j^2 \right)^{1/2} \left(\sum_{j=2}^n b_j^2 \right)^{1/2}. \quad (1)$$

Now observe that $a_1 = \langle \chi_S, \mathbf{u}_1 \rangle = a^{-1/2} A(S, n)$ and $b_1 = \langle \chi_T, \mathbf{v}_1 \rangle = a^{-1/2} A(m, T)$, so

$$a_1 b_1 = \frac{A(S, n) A(m, T)}{a}.$$

Moreover,

$$\sum_{j=2}^m a_j^2 = \|\chi_S\|_2^2 - a_1^2 = A(S, n) - \frac{A(S, n)^2}{a} = A(S, n) \left(1 - \frac{A(S, n)}{a} \right),$$

and similarly

$$\sum_{j=2}^n b_j^2 = A(m, T) \left(1 - \frac{A(m, T)}{a} \right).$$

Substitute the last three identities into (1) to complete the proof. \square

4. ALMOST REGULAR EXPANDERS

Our argument relies on some spectral properties of random subgraphs of (n, d, λ) -graphs. Since random subgraphs are not expected to be *exactly* regular, we extend the definition of (n, d, λ) -graphs as follows:

Definition 4.1 (Almost (n, d, λ) -graphs). *Let $d, \lambda > 0$ and $\gamma \in [0, 1)$. We say that a graph G is an $(n, (1 \pm \gamma)d, \lambda)$ -graph if¹ G is a graph on n vertices whose all degrees are $(1 \pm \gamma)d$ and the second singular value of the adjacency matrix of G satisfies $s_2(A) \leq \lambda$.*

¹In this definition and elsewhere in the paper, we write $a = b \pm c$ as a shorthand for the double-sided inequality $b - c \leq a \leq b + c$. We use other similar abbreviations, whose exact meaning should be clear from context.

Almost (n, d, λ) -graphs behave similar to exact (n, d, λ) -graphs in many ways. If G is an (*exactly*) d -regular graph with adjacency matrix A , its normalized adjacency matrix is obviously

$$\bar{A} = \frac{1}{d}A$$

according to Definition 3.2. If G is an *almost* d -regular graph, its degree matrix $D = \text{diag}(d_1, \dots, d_n)$ is close to dI , and we can expect that

$$\bar{A} = D^{-1/2}AD^{-1/2} \approx \frac{1}{d}A$$

in some sense. Below we show that such an approximation indeed holds in the sense of the closeness of all singular values.

Corollary 4.2 (Singular values of almost regular graphs). *Let $\gamma \in [0, 1)$ and $d > 0$. Let G be a graph whose all vertices have degrees $(1 \pm \gamma)d$. Then the adjacency matrix A and the normalized adjacency matrix \bar{A} of the graph G satisfy*

$$\frac{s_k(A)}{(1 + \gamma)d} \leq s_k(\bar{A}) \leq \frac{s_k(A)}{(1 - \gamma)d} \quad \text{for all } k \in [n].$$

Proof. Using the chain rule for singular values (Lemma A.5), we obtain

$$s_k(A) = s_k\left(D^{1/2}\bar{A}D^{1/2}\right) \leq \|D^{1/2}\|^2 s_k(\bar{A}).$$

Since $\|D^{1/2}\|^2 = \|D\| = \max_i d_i \leq (1 + \gamma)d$, the lower bound in Corollary 4.2 follows. The upper bound can be proved similarly. \square

4.1. Expander mixing lemma for almost regular expanders. Let us specialize Theorem 3.4 for almost (n, d, λ) -graphs.

Corollary 4.3 (Expander mixing lemma for almost (n, d, λ) -graphs). *Let G be an $(n, (1 \pm \gamma)d, \lambda)$ -graph. Then, for any two subsets $S, T \subseteq V(G)$, we have*

$$\frac{(1 - \gamma)^2 d |S||T|}{(1 + \gamma)n} - \varepsilon \leq e(S, T) \leq \frac{(1 + \gamma)^2 d |S||T|}{(1 - \gamma)n} + \varepsilon, \quad (2)$$

where

$$\varepsilon = \frac{1 + \gamma}{1 - \gamma} \cdot \lambda \sqrt{|S||T|}.$$

Proof. Let A and \bar{A} be the adjacency and the normalized adjacency matrices of G , respectively. Theorem 3.4 yields

$$\left| A(S, T) - \frac{A(S, n) A(n, T)}{A(n, n)} \right| \leq s_2(\bar{A}) \sqrt{A(S, n) A(n, T)}. \quad (3)$$

By Corollary 4.2 and assumption, we have

$$s_2(\bar{A}) \leq \frac{s_2(A)}{(1 - \gamma)d} \leq \frac{\lambda}{(1 - \gamma)d}.$$

Moreover, since A is an adjacency matrix, we have $A(S, T) = e(S, T)$, $A(S, n) = \sum_{v \in S} \deg(v) = (1 \pm \gamma)d|S|$, $A(n, T) = \sum_{v \in T} \deg(v) = (1 \pm \gamma)d|T|$, $A(n, n) = \sum_{v \in V(G)} \deg(v) = (1 \pm \gamma)d|V(G)| = (1 \pm \gamma)dn$. Substitute all this into (3) and use triangle inequality to complete the proof. \square

Sometimes all we need is at least one edge between disjoint sets of vertices S and T . Corollary 4.3 provides a convenient sufficient condition for this:

Corollary 4.4 (At least one edge). *Let G be an $(n, (1 \pm \gamma)d, \lambda)$ -graph. Let $S, T \subseteq V(G)$ be two disjoint subsets with*

$$\sqrt{|S||T|} > \frac{(1 + \gamma)^2}{(1 - \gamma)^3} \cdot \frac{\lambda n}{d}.$$

Then $e(S, T) > 0$.

Proof. Under our assumptions, the lower bound in (2) is strictly positive. \square

5. RANDOM SUBGRAPHS OF ALMOST REGULAR EXPANDERS

In this section, we show that a random induced subgraph of an almost (n, d, λ) -graph or a bipartite spectral expander is typically a spectral expander by itself. This serves as our main tool in the proof of our main result.

5.1. Chernoff's bounds. We extensively use the following well-known Chernoff's bounds for the upper and lower tails of the hypergeometric distribution throughout the paper. The following lemma was proved by Hoeffding [19] (also see Section 23.5 in [14]).

Lemma 5.1 (Chernoff's inequality for hypergeometric distribution). *Let $X \sim \text{Hypergeometric}(N, K, n)$ and let $\mathbb{E}[X] = \mu$. Then*

- $\mathbb{P}[X < (1 - a)\mu] < e^{-a^2\mu/2}$ for every $a > 0$;
- $\mathbb{P}[X > (1 + a)\mu] < e^{-a^2\mu/3}$ for every $a \in (0, \frac{3}{2})$.

5.2. Random induced subgraphs. The following theorem is the main result of this section. It asserts that with probability at least $1 - n^{-\Theta(1)}$, random (induced) subgraphs of spectral expanders are also spectral expanders.

Theorem 5.2 (Random subgraphs of spectral expanders). *Let $\gamma \in (0, \frac{1}{200}]$ be a constant. There exists an absolute constant C such that the following holds for sufficiently large n . Let $d, \lambda > 0$, let $\sigma \in [\frac{1}{n}, 1)$, and let G be an $(n, (1 \pm \gamma)d, \lambda)$ -graph. Let $X \subseteq V(G)$ with $|X| = \sigma n$ be a subset chosen uniformly at random, and let $H := G[X]$ be the subgraph of G induced by X . Assume that*

$$\sigma d \geq C\gamma^{-2} \log n \quad \text{and} \quad \sigma \lambda \geq C\sqrt{\sigma d \log n}.$$

Then with probability at least $1 - n^{-1/13}$, H is a $(\sigma n, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda)$ -graph.

Let us comment on the two conditions appearing in Theorem 5.2. The first condition allows the random subgraph to be quite sparse – with degrees of the order of $\log n$ – but not sparser than that. Below that level, the degrees of the random subgraph will become unstable and it will not be approximately regular. The second condition is, up to a logarithmic factor, the Alon-Boppana bound, which dictates that the second singular value of an approximately σd -regular graph must always be $\Omega(\sqrt{\sigma d})$. In other words, the conditions of Theorem 5.2 are almost necessary for a subgraph H to be an almost regular expander.

The proof of Theorem 5.2 is based on bounds of the spectral norm of a *random submatrix*, which is obtained from a given $n \times n$ matrix B by choosing a uniformly random subset of rows and a uniformly random subset of columns of B .

There are two natural ways to choose a random subset of the set $[n]$. We can make a random subset I by selecting every element of $[n]$ independently at random with probability $\sigma \in (0, 1)$. In this case, we write

$$I \sim \text{Subset}(n, \sigma).$$

Alternatively, we can choose any m -set J of $[n]$ with the same probability $1/\binom{n}{m}$. In this case, we write

$$J \sim \text{Subset}(n, m).$$

Note that if $m = \sigma n$, the models $\text{Subset}(n, \sigma)$ and $\text{Subset}(n, m)$ are closely related but not identical. It should be clear from the context which one we consider.

For a given subset $I \subset [n]$, we denote by P_I the orthogonal projection in \mathbb{R}^n onto \mathbb{R}^I . In other words, P_I is the diagonal matrix with $P_{ii} = 1$ if $i \in I$ and $P_{ii} = 0$ if $i \notin I$.

The main tool of this section is the following bound. It is worth mentioning that several similar results have been proved before, for example, in [40] and [46].

Theorem 5.3 (Norms of random submatrices). *Let B be an $n \times n$ matrix. Let $I, I' \sim \text{Subset}(n, \sigma)$ be two independent subsets, where $\sigma \in (0, 1)$. Let $p \geq 2$ and let $q = \max\{p, 2 \log n\}$. Then*

$$\mathbb{E}_p \|P_I B P_{I'}\| \leq \sigma \|B\| + 3\sqrt{q\sigma} \left(\|B\|_{1 \rightarrow 2} + \|B^\top\|_{1 \rightarrow 2} \right) + 8q \|B\|_\infty.$$

Here $\mathbb{E}_p[X] = (\mathbb{E}|X|^p)^{1/p}$ is the L_p norm of the random variable X ; the norm $\|\cdot\|_{1 \rightarrow 2}$ denotes the norm of a matrix as an $\ell_1 \rightarrow \ell_2$ linear operator, which also equals to the maximum ℓ_2 norm of a column; and $\|\cdot\|_\infty$ denotes the maximum absolute entry of a matrix.

We use the following result to derive Theorem 5.3.

Lemma 5.4 (Rudelson-Vershynin [40]). *Let A be an $m \times n$ matrix with rank r . Let $I \sim \text{Subset}(n, \sigma)$, where $\sigma \in (0, 1)$. Let $p \geq 2$ and let $q = \max\{p, 2 \log r\}$. Then*

$$\mathbb{E}_p \|A P_I\| \leq \sqrt{\sigma} \|A\| + 3\sqrt{q} \mathbb{E}_p \|A P_I\|_{1 \rightarrow 2}.$$

Proof of Theorem 5.3. By applying Lemma 5.4 twice (in the same manner as in [46]), we obtain

$$\begin{aligned} \mathbb{E}_p \|P_I B P_{I'}\| &\leq \sqrt{\sigma} \mathbb{E}_p \|P_I B\| + 3\sqrt{q} \mathbb{E}_p \|P_I B P_{I'}\|_{1 \rightarrow 2} \\ &\leq \sigma \|B\| + 3\sqrt{q\sigma} \mathbb{E}_p \|B^\top P_I\|_{1 \rightarrow 2} + 3\sqrt{q} \mathbb{E}_p \|P_I B P_{I'}\|_{1 \rightarrow 2} \\ &\leq \sigma \|B\| + 3\sqrt{q\sigma} \|B^\top\|_{1 \rightarrow 2} + 3\sqrt{q} \mathbb{E}_p \|P_I B\|_{1 \rightarrow 2}, \end{aligned}$$

where the last inequality follows since the $1 \rightarrow 2$ norm of a submatrix is bounded by the $1 \rightarrow 2$ norm of a matrix. To complete the proof, we use the following bound due to Tropp [46]:

$$\begin{aligned} \mathbb{E}_p \|P_I B\|_{1 \rightarrow 2} &\leq \sqrt{\sigma} \|B\|_{1 \rightarrow 2} + 2^{1.25} \sqrt{q} \|P_I B\|_\infty \\ &\leq \sqrt{\sigma} \|B\|_{1 \rightarrow 2} + 2^{1.25} \sqrt{q} \|B\|_\infty. \end{aligned} \quad \square$$

Now we would like to make $I = I'$ and change the model of sampling because our goal is to study random subsets of fixed size of a given set. The following tools make this possible.

Lemma 5.5 (Decoupling [46]). *Let B be a diagonal-free symmetric $n \times n$ matrix. Let $I, I' \sim \text{Subset}(n, \sigma)$ be two independent subsets, where $\sigma \in (0, 1)$. Then for every $p \geq 2$, we have*

$$\mathbb{E}_p \|P_I B P_I\| \leq 2 \mathbb{E}_p \|P_I B P_{I'}\|.$$

Lemma 5.6 (Random subset models [45]). *Let B be an $n \times n$ matrix. Let $I \sim \text{Subset}(n, \sigma)$ and $J \sim \text{Subset}(n, m)$ be two independent subsets, where $\sigma \in (0, 1)$ and $m = \sigma n \geq 1$. Then for every $p \geq 2$, we have*

$$\mathbb{E}_p \|P_J B P_J\| \leq 2^{1/p} \mathbb{E}_p \|P_I B P_I\|.$$

By combining the two lemmas above and Theorem 5.3, we can obtain a corollary as follows:

Corollary 5.7 (Norms of random submatrices). *Let B be a symmetric real $n \times n$ matrix. Let $J \sim \text{Subset}(n, m)$, where $\sigma \in (0, 1)$ and $m = \sigma n \geq 1$. Let $p \geq 2$ and let $q = \max\{p, 2 \log n\}$. Then*

$$\mathbb{E}_p \|P_J B P_J\| \leq 4\sigma \|B\| + 24\sqrt{q\sigma} \|B\|_{1 \rightarrow 2} + 35q \|B\|_\infty.$$

Proof. Consider the symmetric, diagonal-free matrix $B_0 = B - D$ where $D := \text{diag}(B_{1,1}, \dots, B_{n,n})$. Combining Theorem 5.3 with Lemmas 5.5 and 5.6, we obtain the following:

$$\mathbb{E}_p \|P_J B_0 P_J\| \leq 4\sigma \|B_0\| + 24\sqrt{q\sigma} \|B_0\|_{1 \rightarrow 2} + 32q \|B_0\|_\infty.$$

Note that $\|B_0\| \leq \|B\| + \|D\|$, $\|B_0\|_{1 \rightarrow 2} \leq \|B\|_{1 \rightarrow 2}$, $\|B_0\|_\infty \leq \|B\|_\infty$, and $\|P_J B P_J\| \leq \|P_J B_0 P_J\| + \|P_J D P_J\| \leq \|P_J B_0 P_J\| + \|D\|$. This implies

$$\mathbb{E}_p \|P_J B P_J\| \leq \mathbb{E}_p \|P_J B_0 P_J\| + \|D\| \leq 4\sigma (\|B\| + \|D\|) + 24\sqrt{q\sigma} \|B\|_{1 \rightarrow 2} + 32q \|B\|_\infty + \|D\|.$$

Notice that $\|D\| = \max_i |B_{i,i}| \leq \|B\|_\infty$ to complete the proof. \square

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. Let $C > 0$ be a sufficiently large absolute constant. To see that the random induced subgraph $H := G[X]$ is almost regular whp, we can apply Lemma 5.1 with $n, (1 \pm \gamma)d, \sigma n, \gamma/(1 + \gamma)$ in place of N, K, n, α . Since $\sigma d \geq C\gamma^{-2} \log n$ for sufficiently large absolute constant $C > 0$, it follows that, with probability at least $1 - n^{-1}$, all degrees of H are $(1 \pm 2\gamma)\sigma d$. Thus, it remains to bound the second singular value of A_H whp, where A_H is the adjacency matrix of H .

It is convenient to first work with normalized matrices. So let us consider the normalized adjacency matrix

$$\bar{A}_G = D^{-1/2} A_G D^{-1/2}, \quad \text{where } D = \text{diag}(d_1, \dots, d_n) \quad (4)$$

is the degree matrix of G . Note that for any real $m \times n$ matrix A , we have that $s_2(A) = \min_B \|A - B\|$, where the minimum is over all rank-one $m \times n$ matrices B (see Lemma A.4). Thus, by applying Observation 3.5, we have

$$s_2(\bar{A}_G) = \|B\|, \quad \text{where } B = \bar{A}_G - \frac{1}{a} D^{1/2} \mathbb{1}_n \mathbb{1}_n^\top D^{1/2} \quad \text{and } a = \sum_{i=1}^n d_i. \quad (5)$$

Applying Corollary 5.7 for any $p \geq 2$ and $q = \max\{p, 2 \log n\}$, we obtain

$$\mathbb{E}_p \|P_X B P_X\| \leq 4\sigma \|B\| + 24\sqrt{q\sigma} \|B\|_{1 \rightarrow 2} + 35q \|B\|_\infty. \quad (6)$$

Let us bound each of the three terms on the right hand side.

Bounding $\|B\|$. First, by (5), Corollary 4.2 and the assumptions, we have

$$\|B\| = s_2(\bar{A}_G) \leq \frac{s_2(A_G)}{(1 - \gamma)d} \leq \frac{1.1\lambda}{d}. \quad (7)$$

Bounding $\|B\|_{1 \rightarrow 2}$. Triangle inequality yields

$$\|B\|_{1 \rightarrow 2} \leq \|\bar{A}_G\|_{1 \rightarrow 2} + \frac{1}{a} \|D^{1/2} \mathbb{1}_n \mathbb{1}_n^\top D^{1/2}\|_{1 \rightarrow 2}. \quad (8)$$

Let us bound each of the terms appearing on the right hand side. First,

$$\|\bar{A}_G\|_{1 \rightarrow 2} = \|D^{-1/2} A_G D^{-1/2}\|_{1 \rightarrow 2} \leq \|D^{-1}\| \|A_G\|_{1 \rightarrow 2}.$$

We have $\|D^{-1}\| = \max_i (1/d_i) \leq 1.1/d$ and $\|A_G\|_{1 \rightarrow 2} = \max_j \sqrt{d_j} \leq 1.1\sqrt{d}$. Thus,

$$\|\bar{A}_G\|_{1 \rightarrow 2} \leq \frac{1.3}{\sqrt{d}}. \quad (9)$$

Next,

$$a = \sum_{i=1}^n d_i \geq 0.9dn. \quad (10)$$

Moreover,

$$\|D^{1/2}\mathbb{1}_n\mathbb{1}_n^\top D^{1/2}\|_{1 \rightarrow 2} \leq \|D\| \cdot \|\mathbb{1}_n\mathbb{1}_n^\top\|_{1 \rightarrow 2} \leq (1+\gamma)d \cdot \sqrt{n} \leq 1.1d\sqrt{n}. \quad (11)$$

Putting (9), (10) and (11) into (8), we get

$$\|B\|_{1 \rightarrow 2} \leq \frac{1.3}{\sqrt{d}} + \frac{1}{0.9dn} \cdot 1.1d\sqrt{n} \leq \frac{2.6}{\sqrt{d}}. \quad (12)$$

Bounding $\|B\|_\infty$. Triangle inequality yields

$$\|B\|_\infty \leq \|\bar{A}_G\|_\infty + \frac{1}{a} \|D^{1/2}\mathbb{1}_n\mathbb{1}_n^\top D^{1/2}\|_\infty. \quad (13)$$

All entries of \bar{A}_G are $1/\sqrt{d_i d_j} \leq 1.1/d$, and all entries of $D^{1/2}\mathbb{1}_n\mathbb{1}_n^\top D^{1/2}$ are $\sqrt{d_i d_j} \leq 1.1d$. Also, recall that $a \geq 0.9dn$ by (10). Thus, plugging them into (13), we obtain

$$\|B\|_\infty \leq \frac{1.1}{d} + \frac{1}{0.9dn} \cdot 1.1d \leq \frac{2.4}{d}. \quad (14)$$

Putting (7), (12) and (14) into (6), we obtain

$$\mathbb{E}_p \|P_X B P_X\| \leq \frac{4.4\sigma\lambda}{d} + 63\sqrt{\frac{q\sigma}{d}} + \frac{84q}{d}.$$

Multiplying on the left and right by $D^{1/2}$ inside the norm, we conclude that

$$\mathbb{E}_p \|D^{1/2} P_X B P_X D^{1/2}\| \leq \|D\| \mathbb{E}_p \|P_X B P_X\| \leq 5\sigma\lambda + 70\sqrt{q\sigma d} + 93q =: \lambda_0,$$

where we used that $\|D\| = \max_i d_i \leq 1.1d$. Since diagonal matrices commute, we can express the matrix above as follows:

$$D^{1/2} P_X B P_X D^{1/2} = P_X D^{1/2} B D^{1/2} P_X = P_X A_G P_X - \frac{1}{a} P_X D \mathbb{1}_n \mathbb{1}_n^\top D P_X,$$

where in the last step we used (4) and (5). Note that $\frac{1}{a} P_X D \mathbb{1}_n \mathbb{1}_n^\top D P_X$ is a rank one matrix. Thus, by Lemma A.4, we have $s_2(P_X A_G P_X) \leq \|D^{1/2} P_X B P_X D^{1/2}\|$, and thus

$$\mathbb{E}_p s_2(P_X A_G P_X) \leq \lambda_0.$$

Since the adjacency matrix A_H of the induced subgraph H is a $\sigma n \times \sigma n$ submatrix of the $n \times n$ matrix $P_X A_G P_X$, by the Interlacing Theorem for singular values (Theorem A.2), it follows that

$$\mathbb{E}_p s_2(A_H) \leq \lambda_0.$$

Now choose $p = 2 \log n$ and thus $q = p = 2 \log n$. Applying Markov's inequality, we obtain

$$\begin{aligned} \mathbb{P} [s_2(A_H) \geq 1.1\lambda_0] &= \mathbb{P} [s_2(A_H)^p \geq (1.1\lambda_0)^p] \leq \left(\frac{\mathbb{E}_p s_2(A_H)}{1.1\lambda_0} \right)^p \\ &\leq (1.1)^{-p} = (1.1)^{-2 \log n} \leq n^{-0.08}. \end{aligned}$$

In other words, with probability at least $1 - n^{-0.08}$, we have

$$s_2(A_H) < 1.1\lambda_0 \leq 5.5\sigma\lambda + 109\sqrt{\sigma d \log n} + 205 \log n.$$

To complete the proof, we show that the first term dominates the right hand side. Indeed, since the absolute constant C is sufficiently large, the first condition in Theorem 5.2 implies that $205 \log n \leq \sqrt{\sigma d \log n}$. Similarly, the second condition in the theorem implies that $110\sqrt{\sigma d \log n} \leq 0.5\sigma\lambda$. Then it follows that

$$s_2(A_H) \leq 5.5\sigma\lambda + 0.5\sigma\lambda = 6\sigma\lambda.$$

Therefore, with probability at least $(1 - n^{-1})(1 - n^{-0.08}) \geq 1 - n^{-1/13}$, H is a $(\sigma n, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda)$ -graph, which completes the proof of Theorem 5.2. \square

5.3. Bipartite spectral expanders. We next consider *bipartite spectral expanders* with partition $V = V_1 \cup V_2$ which is defined as below:

Definition 5.8. We say that a bipartite graph $H = (V_1 \cup V_2, E)$ is an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander if H is an induced bipartite subgraph of some $(n, (1 \pm \gamma)d, \lambda)$ -expander G with $V(G) = V_1 \cup V_2$, and for each $i = 1, 2$, and for every $v \in V_i$, we have $\deg_H(v) = (1 \pm \gamma) \frac{d|V_3-i|}{n}$.

By a similar approach, we can also prove that for randomly chosen $X \subseteq V_1$ and $Y \subseteq V_2$, the induced bipartite subgraph $G[X, Y]$ is typically also a bipartite spectral expander. We describe this property precisely as follows.

Theorem 5.9 (Random subgraphs of bipartite spectral expanders). *Let $\gamma \in (0, \frac{1}{200}]$ be a constant. There exists an absolute constant C such that the following holds for sufficiently large n . Let $d, \lambda > 0$, let $\sigma_1, \sigma_2 \in [\frac{1}{n}, 1)$, and let $G = (V_1 \cup V_2, E)$ be an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander. Let $X \subseteq V_1$ with $|X| = \sigma_1|V_1|$ be a subset chosen uniformly at random, and independently let $Y \subseteq V_2$ with $|Y| = \sigma_2|V_2|$ be a subset chosen uniformly at random. Let $H := G[X, Y]$ be the bipartite subgraph of G induced by X and Y . Assume that for each $i = 1, 2$,*

$$\sigma_i d \geq C\gamma^{-2} \log n \quad \text{and} \quad \sigma_i \lambda \geq C\sqrt{\sigma_i d \log n}.$$

Then with probability at least $1 - n^{-1/13}$, H is an $(m, (1 \pm 2\gamma) \frac{dm}{n}, 6\sigma\lambda)$ -bipartite expander, where $m = \sigma_1|V_1| + \sigma_2|V_2|$ and $\sigma = \max\{\sigma_1, \sigma_2\}$.

The proof of Theorem 5.9 is almost the same as Theorem 5.2, and is included in Appendix B.

Sometimes we will work on the bipartite subgraph $G[X, Y]$ induced by random disjoint subsets $X, Y \subseteq V$. We also have $G[X, Y]$ is a bipartite spectral expander whp, which is a direct corollary of Theorem 5.2.

Corollary 5.10. *Let $\gamma \in (0, \frac{1}{200}]$ be a constant. There exists an absolute constant C such that the following holds for sufficiently large n . Let $d, \lambda > 0$, let $\sigma_1, \sigma_2 \in [\frac{1}{n}, 1)$, and let G be an $(n, (1 \pm \gamma)d, \lambda)$ -graph. Let $X, Y \subseteq V(G)$ with $|X| = \sigma_1 n$ and $|Y| = \sigma_2 n$ be two disjoint subsets chosen uniformly at random, and let $H := G[X, Y]$ be the bipartite subgraph of G induced by X and Y . Let $\sigma := \sigma_1 + \sigma_2$. Assume that*

$$\sigma_1 d, \sigma_2 d \geq C\gamma^{-2} \log n \quad \text{and} \quad \sigma \lambda \geq C\sqrt{\sigma d \log n}.$$

Then with probability at least $1 - n^{-1/14}$, H is a $(\sigma n, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda)$ -bipartite expander.

Proof. Let $C > 0$ be a sufficiently large absolute constant. Since $\sigma_1 n, \sigma_2 n \geq C\gamma^{-2} \log n$, by Chernoff's bounds, we have that

$$\mathbb{P}[\exists v \in V, \deg(v, X) \neq (1 \pm 2\gamma)\sigma_1 n] \leq n^{-1}$$

and

$$\mathbb{P}[\exists v \in V, \deg(v, Y) \neq (1 \pm 2\gamma)\sigma_2 n] \leq n^{-1}.$$

Next, note that $X \cup Y$ is a random subset of size $|X| + |Y| = \sigma_1 n + \sigma_2 n = \sigma n$. Now, since $\sigma d = \sigma_1 d + \sigma_2 d \geq 2C\gamma^{-2} \log n$ and $\sigma \lambda \geq C\sqrt{\sigma d \log n}$, Theorem 5.2 implies that with probability at least $1 - n^{-1/13}$,

$$G[X \cup Y] \text{ is a } (\sigma n, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-graph.}$$

Therefore, we have that with probability at least $(1 - 2n^{-1})(1 - n^{-1/13}) \geq 1 - n^{-1/14}$,

$$H := G[X, Y] \text{ is a } (\sigma n, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-bipartite expander,}$$

which completes the proof. \square

6. MATCHINGS IN BIPARTITE SPECTRAL EXPANDERS

In this section we prove some auxiliary results related to the existence of matchings in bipartite spectral expanders. These results will be extensively used in the proof of the Connecting Lemma (Lemma 7.1) and also in the proof of the main theorem.

First, we prove the existence of perfect matchings in a bipartite spectral expander with a balanced bipartition.

Lemma 6.1. *Let $\gamma \in [0, \frac{1}{200}]$ be a constant, let $d > 0$ and let $\lambda \leq d/5$. Let $G = (V, E)$ be an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander with parts $V = V_1 \cup V_2$ such that $|V_1| = |V_2|$. Then G contains a perfect matching.*

Proof. It is enough to verify the following condition which is equivalent to Hall's condition (see Theorem 3.1.11 in [47]): For all $i \in [2]$ and $S \subseteq V_i$ of size $|S| \leq |V_i|/2$, we have $|N(S)| \geq |S|$.

Suppose to the contrary that there exists $i \in [2]$ and an $S \subseteq V_i$, such that the set $T := N(S)$ is of size less than $|S|$. Since G is an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander and since $|V_1| = |V_2| = n/2$, we have that

$$e(S, T) \geq \frac{(1 - \gamma)d}{2} \cdot |S|.$$

On the other hand, using the assumption that $\gamma \leq 1/200$ and the expander mixing lemma for almost regular expanders (Corollary 4.3), we obtain that

$$e(S, T) \leq \frac{(1 + \gamma)^2 d |S| |T|}{(1 - \gamma)n} + \frac{1 + \gamma}{1 - \gamma} \cdot \lambda \sqrt{|S| |T|} < \frac{1.1d|S|}{4} + 0.21d|S|,$$

where we also used $|T| < |S| \leq n/4$ and $\lambda \leq d/5$.

Combining these two estimates we obtain a contradiction. This completes the proof. \square

Next, we extend the definition of a matching to a *star-matching*, which plays a crucial role in the proof of the Connecting Lemma (Lemma 7.1).

Definition 6.2. *Let $G = (V, E)$ be a graph and let $V = S \cup T$ be a partition. Let $X \subseteq S$ and $Y \subseteq T$ be subsets with $|X| \leq |Y|/k$. A collection of $|X|$ vertex-disjoint copies of $K_{1,k}$, each of which is centered at some vertex in X and with leaves in Y , is called a k -matching from X to Y . In particular, a 1-matching from S to T is a perfect matching assuming $|S| = |T|$.*

We state the following lemma which provides us a sufficient condition for the existence of a star-matching. We show that for an induced subgraph of a bipartite spectral expander, if all the vertices in one of its parts have large degrees, then this subgraph contains a star-matching from this part to the other.

Lemma 6.3. *Let $\gamma \in [0, \frac{1}{200}]$ be a constant, let $d > 0$ and let $\lambda \leq d/25$. Let $G = (V, E)$ be an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander with parts $V = V_1 \cup V_2$ such that $|V_1| \leq |V_2|$. Then for every $X \subseteq V_1$ and $Y \subseteq V_2$ of size $|Y| \geq 6|X|$, if $\deg(x, Y) \geq \frac{2d|V_2|}{3n}$ for every $x \in X$, then there exists a 3-matching from X to Y .*

Proof. We need to verify Hall's condition. That is, we need to show that for all $S \subseteq X$, we have $|N(S) \cap Y| \geq 3|S|$. Suppose to the contrary that there exists such an S with $T := N(S) \cap Y$ of size less than $3|S|$. By the assumption on the degrees, we obtain that

$$e(S, T) \geq \frac{2d|V_2|}{3n} \cdot |S|.$$

On the other hand, using the expander mixing lemma for almost regular expanders (Corollary 4.3) and the assumptions that $\gamma \leq 1/200$, $\lambda \leq d/25$, and $|T| < 3|S| \leq |V_2|/2$, we obtain that

$$\begin{aligned} e(S, T) &\leq \frac{(1+\gamma)^2 d|S||T|}{(1-\gamma)n} + \frac{1+\gamma}{1-\gamma} \cdot \lambda \sqrt{|S||T|} \\ &< \frac{1.02d|S||V_2|}{2n} + 0.041d\sqrt{3|S|^2} \\ &\leq \frac{d|S||V_2|(1.02 + 0.164 \cdot \sqrt{3})}{2n} \\ &< \frac{2d|S||V_2|}{3n}, \end{aligned}$$

which is a contradiction. This completes the proof. \square

Lastly, we prove that even if a small proportion of vertices in V_2 are prohibited to be matched, one can still find a large subset of V_1 such that the star-matching from this subset to the remaining part of V_2 exists. We will show this by proving the set of vertices with low degrees is small.

Lemma 6.4. *Let $\gamma \in [0, \frac{1}{200}]$ be a constant, $d > 0$, and $\lambda \leq d/250$. Let $G = (V, E)$ be an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander with parts $V = V_1 \cup V_2$ such that $|V_1| \leq |V_2| \leq 10|V_1|$. Then for every $X \subseteq V_1$ and for every $Y \subseteq V_2$ such that $|Y| \geq 9|V_2|/10$ and $|Y| \geq 6|X|$, there exists a 3-matching from some subset $Z \subseteq X$ to Y , such that $|Z| \geq |X| - |V_1|/260$.*

Proof. By assumptions, we have that $|V_2 \setminus Y| \leq |V_2|/10 \leq |V_1|$ and that

$$\frac{1}{2} \leq \frac{|V_2|}{n} = \frac{1}{|V_1|/|V_2| + 1} \leq \frac{10}{11}.$$

First, we claim that the set

$$D := \left\{ x \in X \mid \deg(x, Y) < \frac{2d|V_2|}{3n} \right\}$$

is of size at most $|V_1|/40$. Recall that by Definition 5.8, for every $v \in V_1$, $\deg(v) \geq (1-\gamma) \frac{d|V_2|}{n}$. So we obtain that

$$e(D, V_2 \setminus Y) \geq (1-\gamma) \frac{d|V_2|}{n} |D| - \frac{2d|V_2|}{3n} |D| \geq \left(\frac{1}{6} - \frac{\gamma}{2} \right) d|D|.$$

On the other hand, using the assumption that $\gamma \leq 1/200$ and the expander mixing lemma for almost regular expanders (Corollary 4.3), we have that

$$\begin{aligned} e(D, V_2 \setminus Y) &\leq \frac{(1+\gamma)^2 d|D||V_2 \setminus Y|}{(1-\gamma)n} + \frac{1+\gamma}{1-\gamma} \cdot \lambda \sqrt{|D||V_2 \setminus Y|} \\ &\leq \frac{1.02d|D|}{11} + 0.0041d\sqrt{|D||V_1|}, \end{aligned}$$

where in the last inequality we used the assumption $\lambda \leq d/250$.

By combining these two estimates and rearranging, and using the assumption that $\gamma \leq 1/200$, we obtain

$$\frac{1}{15} d|D| \leq \left(\frac{1}{6} - \frac{1}{400} - \frac{1.02}{11} \right) d|D| \leq 0.0041d\sqrt{|D||V_1|},$$

which in turn implies that $|D| \leq |V_1|/260$ as desired.

Now we are ready to prove the lemma. Let $Z := X \setminus D$ and we wish to show that there exists a 3-matching from Z to Y . By the definition of D , observe that $\deg(z, Y) \geq \frac{2d|V_2|}{3n}$ for all $z \in Z$. Moreover, we have $|Y| \geq 6|X| \geq 6|Z|$. Therefore, by applying Lemma 6.3 to Z and Y , we conclude that there exists a 3-matching from Z to Y as desired. This completes the proof. \square

7. CONNECTING LEMMA

In this section we state and prove the Connecting Lemma. The proof is a slight modification of an ingenious argument by Montgomery (see [31]). Before we state the lemma, recall that for a digraph $D = (V, E)$ and for two disjoint subsets $X, Y \subseteq V$, $D[X, Y]$ is the induced bipartite subgraph of D with edge set $\vec{E}(X, Y)$. It would also be convenient for us to call $D[X, Y]$ a *directed* $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander if by keeping only the edges oriented from X to Y and by ignoring their directions, we obtain an (undirected) $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander.

Lemma 7.1 (Connecting Lemma). *Let $\gamma \in [0, \frac{1}{200}]$ and $\alpha \in (0, 1)$ be constants, and let $d_1, d_2, d_3 > 0$. Let $D = (V, E)$ be a digraph on n vertices for sufficiently large integer n . Let $\ell, k \in \mathbb{N}$ be such that $\log n \leq \ell < \frac{1}{7}n^\alpha$ and $k \leq \left(\frac{3}{2}\right)^\ell$. Let $X_0, Y_0 \subseteq V$ be two disjoint subsets with $X_0 = \{x_1, \dots, x_k\}$ and $Y_0 = \{y_1, \dots, y_k\}$, and let $W \subseteq V \setminus (X_0 \cup Y_0)$ be a subset of size $|W| \geq 22k\ell$. Suppose that $\lambda_i \leq d_i/250$ for all i and that the following properties hold:*

- (P1) *For at least $1 - n^{-\alpha}$ proportion of pairs of disjoint subsets $W_1, W_2 \subseteq W$ of equal size k , the induced bipartite subgraphs $D[X_0, W_1]$, $D[W_2, Y_0]$ and $D[W_1, W_2]$ are directed $(2k, (1 \pm \gamma)d_1, \lambda_1)$ -bipartite expanders.*
- (P2) *For at least $1 - n^{-\alpha}$ proportion of subsets $W_1 \subseteq W$ of size $10k$, the induced bipartite subgraphs $D[X_0, W_1]$ and $D[W_1, Y_0]$ are directed $(11k, (1 \pm \gamma)d_2, \lambda_2)$ -bipartite expanders.*
- (P3) *For at least $1 - n^{-\alpha}$ proportion of pairs of disjoint subsets $W_1, W_2 \subseteq W$ with $|W_1| = k$ and $|W_2| = 10k$, the induced bipartite subgraphs $D[W_1, W_2]$ and $D[W_2, W_1]$ are directed $(11k, (1 \pm \gamma)d_2, \lambda_2)$ -bipartite expanders.*
- (P4) *For at least $1 - n^{-\alpha}$ proportion of pairs of disjoint subsets $W_1, W_2 \subseteq W$ of equal size $10k$, the induced bipartite subgraph $D[W_1, W_2]$ is a directed $(20k, (1 \pm \gamma)d_3, \lambda_3)$ -bipartite expander.*

Then, there exist vertex-disjoint directed paths P_1, \dots, P_k , such that for each $i \in [k]$, the directed path P_i is from x_i to y_i , and $V(P_i) \setminus \{x_i, y_i\} \subseteq W$.

The proof consists of three steps. First, we show that W can be partitioned in a convenient way.

Lemma 7.2. *Under the assumptions of Lemma 7.1, there exists a partition*

$$W = \bigcup_{i=1}^{\ell} (X_i \cup Y_i \cup R_{X,i} \cup R_{Y,i}) \cup U \quad (15)$$

with $|X_i| = |Y_i| = k$ and $|R_{X,i}| = |R_{Y,i}| = 10k$ for all $1 \leq i \leq \ell$, such that:

- (Q1) *$D[X_i, X_{i+1}]$ and $D[Y_{i+1}, Y_i]$ are directed $(2k, (1 \pm \gamma)d_1, \lambda_1)$ -bipartite expanders for all $0 \leq i \leq \ell - 1$;*
- (Q2) *$D[X_i, R_{X,1}]$ and $D[R_{Y,1}, Y_i]$ are directed $(11k, (1 \pm \gamma)d_2, \lambda_2)$ -bipartite expanders for all $0 \leq i \leq \ell$;*
- (Q3) *$D[R_{X,i}, R_{X,i+1}]$, $D[R_{Y,i+1}, R_{Y,i}]$ and $D[R_{X,\ell}, R_{Y,\ell}]$ are directed $(20k, (1 \pm \gamma)d_3, \lambda_3)$ -bipartite expanders for all $1 \leq i \leq \ell - 1$;*
- (Q4) *$\vec{e}(A, B) > 0$ for every subsets $A \subseteq R_{X,\ell}$ and $B \subseteq R_{Y,\ell}$ of size at least $k/12$.*

Proof. Take a uniformly random partition as in (15) with $|X_i| = |Y_i| = k$ and $|R_{X,i}| = |R_{Y,i}| = 10k$. By assumptions (P1)–(P4) and the union bound, the probability that at least one of (Q1)–(Q3) does not hold is at most $7\ell n^{-\alpha} < 1$, given that $\ell < \frac{1}{7}n^\alpha$ and that there are at most 7ℓ events to consider. Moreover, (Q4) follows from (Q3) by Corollary 4.4 since

$$\frac{k}{12} > \frac{(1 + \gamma)^2}{(1 - \gamma)^3} \cdot \frac{20k\lambda_3}{d_3}.$$

Therefore, with positive probability we obtain the desired partition. □

In the rest of this section, we will extensively use the following simple lemma, which is a slightly stronger version of Lemma B.1 in [10]. We include its (almost) trivial proof for completeness.

Lemma 7.3. *Let X and Y be disjoint sets of vertices such that for every $y \in Y$, there exists a directed path from some $x \in X$ to y . Then for every $\alpha \in (0, 1]$, there exist a non-empty subset $X' \subseteq X$ of size at most $\lceil \alpha |X| \rceil$ and a subset $Y' \subseteq Y$ of size at least $\lfloor \alpha |Y| \rfloor$, so that for every $y \in Y'$ there is a directed path from some $x \in X'$ to y .*

Proof. Suppose to the contrary that for every subset $W \subseteq X$ of size $|W| = \lceil \alpha |X| \rceil$, the subset $Y_W \subseteq Y$, where each $y \in Y_W$ is connected with some $x \in W$ by a directed path, is of size $|Y_W| < \lfloor \alpha |Y| \rfloor$.

We prove the statement by double counting. On one hand, by definition of Y_W , we have that

$$\sum_{|W|=\lceil \alpha |X| \rceil} |Y_W| < \binom{|X|}{\lceil \alpha |X| \rceil} \lfloor \alpha |Y| \rfloor.$$

On the other hand, each $x \in X$ is contained in $\binom{|X|-1}{\lceil \alpha |X| \rceil - 1}$ subsets $W \subseteq X$ of size $|W| = \lceil \alpha |X| \rceil$. Since each $y \in Y$ is connected to at least one vertex $x \in X$ by a directed path, we have that

$$\begin{aligned} \sum_{|W|=\lceil \alpha |X| \rceil} |Y_W| &\geq \binom{|X|-1}{\lceil \alpha |X| \rceil - 1} |Y| \\ &= \binom{|X|}{\lceil \alpha |X| \rceil} \frac{|Y| \lceil \alpha |X| \rceil}{|X|} \\ &\geq \binom{|X|}{\lceil \alpha |X| \rceil} \alpha |Y|, \end{aligned}$$

which is a contradiction. This completes the proof. \square

The second, and main, step in the proof of the Connecting Lemma is summarized in the following lemma that enables us to connect a single pair of vertices with the same indices.

Lemma 7.4. *Using the same notations as in Lemma 7.1, consider a partition of W as obtained by Lemma 7.2. For each $1 \leq i \leq \ell$, let $R'_{X,i} \subseteq R_{X,i}$ and $R'_{Y,i} \subseteq R_{Y,i}$ be subsets of size at least $9k$. Let $A := \{u_1, \dots, u_t\} \subseteq X_h$ and $B := \{v_1, \dots, v_t\} \subseteq Y_h$ for some $0 \leq h \leq \ell$ and $t \geq k/6$. Then, there exists an index j and a directed path P_j of length $2\ell + 1$ such that:*

- (1) P_j is from u_j to v_j ,
- (2) $V(P_j) \setminus \{u_j, v_j\} \subseteq \bigcup_{i=1}^{\ell} (R'_{X,i} \cup R'_{Y,i})$, and
- (3) $|P_j \cap R'_{X,i}|, |P_j \cap R'_{Y,i}| \leq 1$ for all $1 \leq i \leq \ell$.

Proof. We will repeatedly apply the following claim.

Claim 7.5. *For every $A' \subseteq A$ of size at least $|A|/2$, there exist a vertex $u_i \in A'$ and a subset $Z_\ell \subseteq R'_{X,\ell}$ of size $|A'|$ such that there exist directed paths of length ℓ from u_i to every vertex in Z_ℓ .*

Assume for a moment that we have proved the claim above. Then after applying it $\frac{t}{2} + 1$ times, we obtain a sequence of indices

$$I = \left\{ i_1, \dots, i_{\frac{t}{2}+1} \right\} \subseteq [t]$$

and $\frac{t}{2} + 1$ not necessarily disjoint subsets

$$\left\{ W_{i_1}, \dots, W_{i_{\frac{t}{2}+1}} \right\},$$

such that for all $1 \leq j \leq \frac{t}{2} + 1$, $W_{i_j} \subseteq R'_{X,\ell}$ has size $|W_{i_j}| = |A|/2$ and there are directed paths of length ℓ from u_{i_j} to every vertex in W_{i_j} .

Applying similar arguments to Y_i and $R'_{Y,i}$ (here the “directed bipartite expander” is the digraph obtained by keeping the edges directed towards Y), we can find a set of $\frac{t}{2} + 1$ indices

$$I' = \left\{ i'_1, \dots, i'_{\frac{t}{2}+1} \right\} \subseteq [t]$$

and $\frac{t}{2} + 1$ not necessarily disjoint subsets

$$\left\{ W'_{i'_1}, \dots, W'_{i'_{\frac{t}{2}+1}} \right\},$$

such that for all $1 \leq j \leq \frac{t}{2} + 1$, $W'_{i'_j} \subseteq R'_{Y,\ell}$ has size $|W'_{i'_j}| = |A|/2$ and there are directed paths of length ℓ from every vertex in $W'_{i'_j}$ to $v_{i'_j}$.

Since $|I| + |I'| > t$, there must exist some $i \in I \cap I'$. Since $|W_i| = |W'_i| = |A|/2 \geq k/12$, by (Q4), there exists a directed edge from some $w \in W_i$ to some $w' \in W'_i$. Concatenating the directed path of length ℓ from u_i to w , the directed edge ww' , and the directed path of length ℓ from w' to v_i , this yields a directed path of length $2\ell + 1$ from u_i to v_i as desired. Thus, it remains to prove Claim 7.5.

Proof of Claim 7.5. We start by expanding A' to $R'_{X,1}$ using a 3-matching as follows:

Since by (Q2) we have that $D[X_h, R_{X,1}]$ is a directed $(11k, (1 \pm \gamma)d_2, \lambda_2)$ -bipartite expander, it follows by Lemma 6.4 with $X = A'$ and $Y = R'_{X,1}$ that there exists a 3-matching from A' to $R'_{X,1}$ in $D[A', R'_{X,1}]$ of size at least $|A'| - k/260 \geq |A'|/2$. Let $A'_1 \subseteq X_h$ and $Z'_1 \subseteq R'_{X,1}$ be the centers and the leaves of such a 3-matching, respectively, and observe that $|Z'_1| = 3|A'_1| \geq 3|A'|/2$.

Note that for every $z \in Z'_1$, there exists a directed path (in fact, a directed edge) from some $u \in A'_1$ to z . Thus, by applying Lemma 7.3 (with $\alpha = |A'|/|Z'_1| \leq 2/3$), we obtain $Z_1 \subseteq Z'_1$ of size $|Z_1| = |A'|$ and $A_1 \subseteq A'_1$ of size at most $(|A'|/|Z'_1|) \cdot |A'_1| \leq \frac{2|A'|}{3}$, such that for every $z \in Z_1$ there exists a directed path from some vertex $u \in A_1$ to z .

Next, we expand Z_1 into $R'_{X,2}$ using a 3-matching as follows (note that this step is similar to the previous, and the only reason that we describe it again is just because here the sets $R'_{X,1}$ and $R'_{X,2}$ are of equal size so the parameters are a little different):

Since by (Q3) we have that $D[R_{X,1}, R_{X,2}]$ is a directed $(20k, (1 \pm \gamma)d_3, \lambda_3)$ -bipartite expander, it follows by Lemma 6.4 with $X = Z_1$ and $Y = R'_{X,2}$ that there exists a subset $A'_2 \subseteq Z_1$ of size at least $|Z_1| - 10k/260 \geq |Z_1|/2 = |A'|/2$, such that there exists a 3-matching from A'_2 to $R'_{X,2}$. Let $Z'_2 \subseteq R'_{X,2}$ be the set of matched vertices and observe that $|Z'_2| = 3|A'_2| \geq 3|A'|/2$. Since for every $z \in Z'_2$ there exists a directed path from some $u \in A_1$ to z , by applying Lemma 7.3 (to a family of directed paths of length 2), we obtain $Z_2 \subseteq Z'_2$ of size $|Z_2| = |A'|$ and $A_2 \subseteq A_1$ of size at most $(|A'|/|Z'_2|) \cdot |A_1| \leq \frac{2|A_1|}{3} \leq \frac{4|A'|}{9}$, such that for every $z \in Z_2$, there exists a directed path from some $u \in A_2$ to z .

Now, by iterating the above procedure, we conclude that after the i th iteration we are left with subsets $A_i \subseteq A'$ of size at most $\left(\frac{2}{3}\right)^i |A'|$ and $Z_i \subseteq R'_{X,i}$ of size $|Z_i| = |A'|$, such that for every vertex $z \in Z_i$ there exists a directed path from some vertex in A_i to z . Moreover, since after the ℓ th iteration we have $|A_\ell| = \left(\frac{2}{3}\right)^\ell |A'| \leq \left(\frac{2}{3}\right)^\ell k \leq 1$ and since A_ℓ is nonempty, we must have $|A_\ell| = 1$ which means that $A_\ell = \{u_i\}$ for some $i \in [t]$. Furthermore, there exist directed paths of length ℓ from u_i to every vertex in Z_ℓ as desired.

This completes the proof the Claim 7.5. \square

The proof of Lemma 7.4 is completed. \square

Finally, we are ready to prove the Connecting Lemma.

Proof of Lemma 7.1. By Lemma 7.2, there exists a partition of W as (15) such that properties (Q1)–(Q4) hold. We begin by applying Lemma 7.4 as many times as possible to generate directed paths from X_0 to Y_0 using vertices in $W \setminus U$. In each iteration, we ignore the vertices in X_0 that have directed paths to their corresponding vertices in Y_0 and remove the vertices. After i iterations, the subsets of remaining vertices $R'_{X,i} \subseteq R_{X,i}$ and $R'_{Y,i} \subseteq R_{Y,i}$ have size

$$|R'_{X,i}| = |R'_{Y,i}| \geq 10k - k = 9k.$$

By Lemma 7.4, we stop when $5k/6$ pairs of vertices are linked by vertex-disjoint directed paths of length $2\ell + 1$.

Denote the sets of remaining vertices in X_0 and Y_0 by A_1 and B_1 . Since by (Q1) we have that $D[X_0, X_1]$ is a directed $(2k, (1 \pm \gamma)d_1, \lambda_1)$ -bipartite expander, it follows by Lemma 6.4 with $X = A_1$ and $Y = X_1$ that there exists a 3-matching M_{A_1} from A_1 to X_1 , and we orient the edges directed from A_1 to X_1 . Similarly, there exists a 3-matching M_{B_1} from B_1 to Y_1 by Lemma 6.4, and we orient the edges directed from Y_1 to B_1 . Let $W_1 = X_1 \cap V(M_{A_1})$ and $Z_1 = Y_1 \cap V(M_{B_1})$. We now apply Lemma 7.4 to connect the corresponding vertices in W_1 and Z_1 by directed paths of length $2\ell + 1$, yielding directed paths of length $2\ell + 3$ from vertex $x_i \in A_1$ to $y_i \in B_1$ for some $i \in [k]$. Here by *corresponding vertices* we mean a pair of vertices $(w, z) \in W_1 \times Z_1$ such that for some $i \in [k]$, w is matched to $x_i \in A_1 \subseteq X_0$ in M_{A_1} and z is matched to $y_i \in B_1 \subseteq Y_0$ in M_{B_1} . We can continue applying Lemma 7.4 to the vertices of W_1 and Z_1 while avoiding the vertices that have been used in the paths found in previous steps, and stop when there are $k/6$ unconnected vertices in W_1 and Z_1 . This corresponds to $k/12$ unconnected vertices in the original sets X_0 and Y_0 . Continuing in this fashion, we use $\lceil \log k \rceil$ iterations to connect all the vertices. Note that the number of available vertices in each $R_{X,i}$ or $R_{Y,i}$ at each step is at least $10k - k = 9k$, which guarantees that Lemma 7.4 can be applied at each iteration. This completes the proof. \square

8. PROOF OF THEOREM 1.2

In this section, we will prove our main theorem, Theorem 1.2. Since regular spectral expanders can be regarded as $(n, (1 \pm \gamma)d, \lambda)$ -graphs, the following, slightly stronger, statement will immediately imply Theorem 1.2.

Theorem 8.1. *For any constant $\gamma \in (0, \frac{1}{1200}]$ and sufficiently large integer n , any $(n, (1 \pm \gamma)d, \lambda)$ -graph with $\lambda \leq d/9000$ and $d \geq \log^{10} n$ contains a Hamilton cycle.*

In order to find a Hamilton cycle, we will first randomly partition the vertex set $V(G)$ into several parts, and then apply the Connecting Lemma (Lemma 7.1) to build the absorbing path. Subsequently, we will find a family of not too many vertex-disjoint paths covering all the remaining vertices. Finally, we will introduce an auxiliary digraph which will be used to connect the paths into a Hamilton cycle.

We need the following results which are proved by simple averaging.

Proposition 8.2. *Let $\alpha \in [0, 1]$, $0 < m \leq h \leq n$ be integers, and V be an n -element set. Let $\mathcal{F} \subseteq \binom{V}{m}$ with $|\mathcal{F}| \geq (1 - \alpha) \binom{n}{m}$. Then for a uniformly random h -set $Y \subseteq V$, with probability at least $1 - \alpha^{1/2}$, we have that for at least $1 - \alpha^{1/2}$ proportion of m -sets $B \subseteq Y$, $B \in \mathcal{F}$.*

Proof. Suppose to the contrary that there are greater than $\alpha^{1/2} \binom{n}{h}$ h -sets $Y \subseteq V$ each containing at least $\alpha^{1/2} \binom{h}{m}$ m -sets B such that $B \notin \mathcal{F}$. Since each such subset B is counted at most $\binom{n-m}{h-m}$

times, there are in total greater than

$$\frac{1}{\binom{n-m}{h-m}} \cdot \alpha^{1/2} \binom{n}{h} \cdot \alpha^{1/2} \binom{h}{m} = \alpha \binom{n}{m}$$

m -sets B such that $B \notin \mathcal{F}$. This contradicts the assumption. \square

Next, we show that in a graph $G = (V, E)$, given a graph property \mathcal{P} (e.g. “being a bipartite expander” with certain parameters), if for many pairs of disjoint subsets $A, B \subseteq V$ we have $G[A, B] \in \mathcal{P}$, then for random disjoint subsets A and $S \subseteq V \setminus A$ with size larger than B , whp many subsets $B \subseteq S$ satisfy $G[A, B] \in \mathcal{P}$.

Proposition 8.3. *Let $\alpha \in [0, 1]$, let $0 < m \leq h \leq n/2$ be integers, and let \mathcal{P} be a graph property. Let $G = (V, E)$ be a graph on n vertices. Suppose that there are $1 - \alpha$ proportion of pairs of disjoint m -sets $A, B \subseteq V$ such that $G[A, B] \in \mathcal{P}$. Then for $A \in \binom{V}{m}$ and $S \in \binom{V}{h}$ chosen uniformly at random such that $A \cap S = \emptyset$, with probability at least $1 - \alpha^{1/2} - \alpha^{1/4}$, we have that for at least $1 - \alpha^{1/4}$ proportion of m -sets $B \subseteq S$, $G[A, B] \in \mathcal{P}$.*

Proof. Let $\mathcal{F} \subseteq \binom{V}{m}$ be the family of m -sets A such that for at least $1 - \alpha^{1/2}$ proportion of m -sets $B \subseteq V \setminus A$, $G[A, B] \in \mathcal{P}$. We claim that $|\mathcal{F}| \geq (1 - \alpha^{1/2}) \binom{n}{m}$. Suppose to the contrary that there are at least $\alpha^{1/2} \binom{n}{m}$ m -sets $A \notin \mathcal{F}$. Then by definition, each such A contributes at least $\alpha^{1/2} \binom{n-m}{m}$ pairs (A, B) such that $G[A, B] \notin \mathcal{P}$. Then there are at least $\alpha^{1/2} \binom{n}{m} \cdot \alpha^{1/2} \binom{n-m}{m} = \alpha \binom{n}{m} \binom{n-m}{m}$ pairs (A, B) such that $G[A, B] \notin \mathcal{P}$, contradicting the assumption.

Now, for $A \in \mathcal{F}$, let $\mathcal{F}_A \subseteq \binom{V \setminus A}{m}$ be the m -sets $B \subseteq V \setminus A$ such that $G[A, B] \in \mathcal{P}$. Note that $|\mathcal{F}_A| \geq (1 - \alpha^{1/2}) \binom{n-m}{m}$. Then Proposition 8.2 applied to \mathcal{F}_A implies that for a uniformly random h -set $S \subseteq V \setminus A$, with probability at least $1 - \alpha^{1/4}$, $|\mathcal{F}_A \cap \binom{S}{m}| \geq (1 - \alpha^{1/4}) \binom{h}{m}$. By union bound, for disjoint uniformly randomly chosen $A \in \binom{V}{m}$ and $S \in \binom{V}{h}$, with probability at least $1 - \alpha^{1/2} - \alpha^{1/4}$, we have that $A \in \mathcal{F}$ and $|\mathcal{F}_A \cap \binom{S}{m}| \geq (1 - \alpha^{1/4}) \binom{h}{m}$. That is, for at least $1 - \alpha^{1/4}$ proportion of m -sets $B \subseteq S$, $G[A, B] \in \mathcal{P}$. This completes the proof. \square

We also prove a similar proposition for a graph property \mathcal{P} regarding induced bipartite subgraphs. In fact, we prove that in a graph G , for two disjoint subsets $X, Y \subseteq V(G)$, if for many pairs $A \subseteq X$ and $B \subseteq Y$ we have $G[A, B] \in \mathcal{P}$, then for a random $S \subseteq X$ of larger size than A , whp many pairs $A \subseteq S$ and $B \subseteq Y$ satisfy $G[A, B] \in \mathcal{P}$.

Proposition 8.4. *Let $\alpha \in [0, 1]$, let $0 < m \leq h \leq n/2$ be integers, and let \mathcal{P} be a graph property. Let $G = (V, E)$ be a graph on n vertices, and let $X, Y \subseteq V$ such that $X \cap Y = \emptyset$, $|X| \geq h$ and $|Y| \geq m$. Suppose that there are $1 - \alpha$ proportion of pairs of disjoint m -sets $A \subseteq X, B \subseteq Y$ such that $G[A, B] \in \mathcal{P}$. Then for a uniformly random h -set $S \subseteq X$, with probability at least $1 - \alpha^{1/4}$, we have that for at least $(1 - \alpha^{1/2})(1 - \alpha^{1/4})$ proportion of pairs of m -sets $A \subseteq S$ and $B \subseteq Y$, $G[A, B] \in \mathcal{P}$.*

Proof. Let $\mathcal{F} \subseteq \binom{X}{m}$ be the family of m -sets $A \subseteq X$ such that for at least $1 - \alpha^{1/2}$ proportion of m -sets $B \subseteq Y$, $G[A, B] \in \mathcal{P}$. Note that $|\mathcal{F}| \geq (1 - \alpha^{1/2}) \binom{|X|}{m}$. Indeed, if there are more than $\alpha^{1/2} \binom{|X|}{m}$ m -sets $A \notin \mathcal{F}$, then by definition of \mathcal{F} , there are more than $\alpha^{1/2} \binom{|X|}{m} \cdot \alpha^{1/2} \binom{|Y|}{m} = \alpha \binom{|X|}{m} \binom{|Y|}{m}$ pairs (A, B) such that $G[A, B] \notin \mathcal{P}$. This is a contradiction.

Now, Proposition 8.2 applied to \mathcal{F} implies that for a uniformly random h -set $S \subseteq X$, with probability at least $1 - \alpha^{1/4}$, $|\mathcal{F} \cap \binom{S}{m}| \geq (1 - \alpha^{1/4}) \binom{h}{m}$. By the definition of \mathcal{F} , there are at least $(1 - \alpha^{1/2})(1 - \alpha^{1/4})$ proportion of pairs of m -sets $A \subseteq S$ and $B \subseteq Y$ such that $G[A, B] \in \mathcal{P}$.

This completes the proof. \square

In the rest of this section, we will always assume that $\gamma \leq 1/1200$, $\lambda \leq d/9000$, and that $G = (V, E)$ is an $(n, (1 \pm \gamma)d, \lambda)$ -graph with sufficiently large integer n such that $d \geq \log^{10} n$. Since an $(n, (1 \pm \gamma)d, \lambda)$ -graph is also an $(n, (1 \pm \gamma)d, \lambda')$ -graph when $\lambda \leq \lambda'$, we may assume that $\sqrt{d} \log^4 n \leq \lambda \leq d/9000$. Also, let

$$h := \frac{n}{\log n}, \quad m := \frac{n}{\log^3 n}, \quad k := \frac{n}{\log^5 n}, \quad \sigma := \frac{2m}{n} \quad \text{and} \quad \rho := \frac{2k}{n}.$$

8.1. Partitioning the graph. First, we find a partition of the vertex set of the $(n, (1 \pm \gamma)d, \lambda)$ -graph G with some nice properties.

Lemma 8.5. *There exists a partition $V = \bigcup_{i=1}^8 U_i$ with $|U_1| = |U_2| = |U_3| = |U_4| = m$, $|U_5| = |U_6| = h$, $|U_7| = \frac{n}{2}$, and $U_8 = V \setminus \left(\bigcup_{i=1}^7 U_i\right)$ such that the following properties hold:*

(R1) *for each $1 \leq i \leq 8$ we have*

$$\deg(v, U_i) = (1 \pm 2\gamma) \frac{d|U_i|}{n} \quad \text{for every vertex } v \in V;$$

(R2) *for each $1 \leq i < j \leq 4$ we have*

$$G[U_i, U_j] \text{ is a } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-bipartite expander};$$

(R3) *for each $1 \leq i \leq 4$ and $j = 5, 6$, for at least $1 - n^{-1/56}$ proportion of subsets $S \subseteq U_j$ of size m ,*

$$G[U_i, S] \text{ is a } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-bipartite expander};$$

also, for at least $1 - n^{-1/56}$ proportion of subsets $S \subseteq U_j$ of size $10m$,

$$G[U_i, S] \text{ is an } \left(11m, (1 \pm 2\gamma) \frac{11\sigma d}{2}, 33\sigma\lambda\right) \text{-bipartite expander};$$

(R4) *for each $1 \leq i \leq 4$, for at least $1 - n^{-1/53}$ proportion of subsets $S \subseteq U_i$ and $T \subseteq U_7$ of equal size k ,*

$$G[S, T] \text{ is a } (2k, (1 \pm 2\gamma)\rho d, 6\rho\lambda) \text{-bipartite expander};$$

also, for at least $1 - n^{-1/53}$ proportion of subsets $S \subseteq U_i$ and $T \subseteq U_7$ with $|S| = 10k$ and $|T| = k$,

$$G[S, T] \text{ is an } \left(11k, (1 \pm 2\gamma) \frac{11\rho d}{2}, 60\rho\lambda\right) \text{-bipartite expander};$$

(R5) *$G[U_5]$ and $G[U_6]$ are $\left(h, (1 \pm 2\gamma) \frac{dh}{n}, \frac{6\lambda h}{n}\right)$ -graphs, and $G[U_7]$ is an $\left(\frac{n}{2}, (1 \pm 2\gamma) \frac{d}{2}, \lambda\right)$ -graph.*

Proof of Lemma 8.5. Let $V = \bigcup_{i=1}^8 U_i$ be a uniformly random partition with $|U_1| = |U_2| = |U_3| = |U_4| = m$, $|U_5| = |U_6| = h$, and $|U_7| = \frac{n}{2}$. We wish to prove that each property among (R1)–(R5) holds whp.

Property (R1). First, note that since $\frac{d|U_i|}{n} = \omega(\log n)$ for each $1 \leq i \leq 8$, it follows by Chernoff's bounds and a union bound that

$$\mathbb{P} \left[\exists v \in V, \deg(v, U_i) \neq (1 \pm 2\gamma) \frac{d|U_i|}{n} \text{ for some } 1 \leq i \leq 8 \right] \leq ne^{-\Theta(d|U_i|/n)} = o(1).$$

Thus, property (R1) holds with probability $1 - o(1)$. In the rest of the proof, it is enough to condition on property (R1).

Property (R2). For each $1 \leq i < j \leq 4$, note that U_i and U_j are uniformly random disjoint subsets of equal size m . Since $\sigma d = \omega(\gamma^{-2} \log n)$ and $\sigma \lambda = \omega(\sqrt{\sigma d \log n})$, Corollary 5.10 implies that with probability at least $1 - n^{-1/14}$,

$$G[U_i, U_j] \text{ is a } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-bipartite expander.}$$

Thus, by a union bound, (R2) holds with probability at least $1 - 6n^{-1/14} = 1 - o(1)$.

Property (R3). With another application of Corollary 5.10, one can obtain that for at least $1 - n^{-1/14}$ proportion of disjoint subsets $U_0, U'_0 \subseteq V$ of equal size m ,

$$G[U_0, U'_0] \text{ is a } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-bipartite expander.}$$

Note that for each $1 \leq i \leq 4$ and $j = 5, 6$, U_i is a uniformly random m -set and $U_j \subseteq V \setminus U_i$ is a uniformly random h -set. Thus, by Proposition 8.3 applied with $U_i, U_j, m, h, n^{-1/14}$ in place of A, S, m, h, α , we have that with probability at least $1 - n^{-1/28} - n^{-1/56} \geq 1 - n^{-1/57}$, for at least $1 - n^{-1/56}$ proportion of m -sets $S \subseteq U_j$,

$$G[U_i, S] \text{ is a } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda) \text{-bipartite expander.}$$

Therefore, by a union bound, the first part of (R3) holds with probability at least $1 - 8n^{-1/57} = 1 - o(1)$. The same argument also implies that the second part of (R3) holds with probability at least $1 - 8n^{-1/57} = 1 - o(1)$.

Property (R4). Since $G[U_7 \cup U_7^c] = G$ is an $(n, (1 \pm \gamma)d, \lambda)$ -graph by assumption, it follows that $G[U_7, U_7^c]$ is an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander by (R1). Thus, since $\rho d = \omega(\gamma^{-2} \log n)$ and $\rho \lambda = \omega(\sqrt{\rho d \log n})$, Theorem 5.9 implies that for at least $1 - n^{-1/13}$ proportion of pairs of subsets $S \subseteq U_7^c$ and $T \subseteq U_7$ of equal size k ,

$$G[S, T] \text{ is a } (2k, (1 \pm 2\gamma)\rho d, 6\rho\lambda) \text{-bipartite expander.}$$

Note that for each $1 \leq i \leq 4$, $U_i \subseteq U_7^c$ is a uniformly random m -set. Thus, by Proposition 8.4 applied with $U_7^c, U_7, U_i, k, m, n^{-1/13}$ in place of X, Y, S, m, h, α , we have that the following holds with probability at least $1 - n^{-1/52}$: for at least $(1 - n^{-1/26})(1 - n^{-1/52}) \geq 1 - n^{-1/53}$ proportion of pairs of subsets $S \subseteq U_i$ and $T \subseteq U_7$ of equal size k , $G[S, T]$ is a $(2k, (1 \pm 2\gamma)\rho d, 6\rho\lambda)$ -bipartite expander.

Therefore, by a union bound, the first part of (R4) holds with probability at least $1 - 4n^{-1/52} = 1 - o(1)$. The same argument also implies that the second part of (R4) holds with probability at least $1 - 4n^{-1/52} = 1 - o(1)$.

Property (R5). First, for each $i = 5, 6$, Theorem 5.2 implies that with probability at least $1 - n^{-1/13}$, $G[U_i]$ is an $\left(h, (1 \pm \gamma)\frac{dh}{n}, \frac{6\lambda h}{n}\right)$ -graph. Next, by (R1), we have that for every vertex $v \in V$, $\deg(v, U_7) = (1 \pm 2\gamma)\frac{d|U_7|}{n} = (1 \pm 2\gamma)\frac{n}{2}$. Now, note that $G[U_7]$ is an induced subgraph of G , so the adjacency matrix of $G[U_7]$ is a submatrix of the adjacency matrix of G . Thus, by Interlacing Theorem for singular values (Theorem A.2), $s_2(G[U_7]) \leq s_2(G) \leq \lambda$. Therefore, $G[U_7]$ is an $\left(\frac{n}{2}, (1 \pm 2\gamma)\frac{d}{2}, \lambda\right)$ -graph.

All in all, with positive probability all properties (R1)–(R5) hold, which guarantees a partition $V = \bigcup_{i=1}^8 U_i$ satisfying all desired properties. This completes the proof. \square

8.2. Constructing Absorbers. Next, we pick a partition $V = \bigcup_{i=1}^8 U_i$ as in Lemma 8.5. In order to be consistent with the proof outline, we will use the following notation:

$$X := U_1, \quad A := U_2, \quad B := U_3, \quad Y := U_4, \quad R_1 := U_5, \quad R_2 := U_6, \quad W_1 := U_7, \quad W_2 := U_8.$$

By (R2), the bipartite induced subgraph $G[X, A]$ is a $(2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda)$ -bipartite expander, so $G[X, A]$ has a perfect matching by Lemma 6.1 since $6\sigma\lambda \leq \sigma d/5$. Similarly, we can ensure perfect matchings in $G[A, B]$ and $G[B, Y]$, respectively. Thus, there are labels $X = \{x_1, \dots, x_m\}$, $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m\}$, and $Y = \{y_1, \dots, y_m\}$ such that for each $1 \leq i \leq m$, we have

$$x_i a_i, a_i b_i, b_i y_i \in E(G).$$

Next, we show how to complete the construction of our absorbers (which are just cycles with designated vertices x, a, b, y) and additionally the absorbing path using the Connecting Lemma (Lemma 7.1). Recall that $0 < \gamma \leq 1/1200$, $m = \frac{n}{\log^3 n}$, $d \geq \log^{10} n$ and $\sqrt{d} \log^4 n \leq \lambda \leq d/9000$.

Lemma 8.6. *There exist vertex-disjoint paths $\mathcal{P} := \{P_i\}_{i \in [m]}$ and $\mathcal{Q} := \{Q_i\}_{i \in [m-1]}$, such that P_i connects x_i and y_i , Q_i connects b_i and a_{i+1} , $V(P_i) \setminus \{x_i, y_i\} \subseteq R_1$ and $V(Q_i) \setminus \{b_i, a_{i+1}\} \subseteq R_2$.*

Proof. We only prove the existence of \mathcal{P} as the existence of \mathcal{Q} follows a similar application of Lemma 7.1. We shall verify (P1)–(P4) of Lemma 7.1 with parameters $4\gamma, 1/57, \log n, m$ in place of γ, α, ℓ, k , and vertex sets X, Y, R_1 playing the role of X_0, Y_0, W , respectively. Note that Lemma 7.1 can also be applied to undirected graphs (one can simply treat each edge as being directed in both directions).

Property (P2). By (R3), for at least $1 - 2n^{-1/56}$ proportion of subsets $S \subseteq R_1$ of size $10m$,

$$\text{both } G[X, S] \text{ and } G[S, Y] \text{ are } \left(11m, (1 \pm 2\gamma)\frac{11\sigma d}{2}, 33\sigma\lambda\right)\text{-bipartite expanders.}$$

Since $33\sigma\lambda \leq \frac{1}{250} \cdot \frac{11\sigma d}{2}$, this proves (P2) with a union bound.

Properties (P1), (P3) and (P4). First, by (R3), for at least $1 - 2n^{-1/56}$ proportion of subsets $S \subseteq R_1$ of size m ,

$$\text{both } G[X, S] \text{ and } G[S, Y] \text{ are } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda)\text{-bipartite expanders.}$$

Next, consider two disjoint subsets $S, S' \subseteq R_1$ of size $|S| = am$ and $|S'| = bm$, where $a, b \in [10]$. Since $\sigma = \frac{2m}{n}$, $\sigma d = \omega(\gamma^{-2} \log n)$ and $\sigma\lambda = \omega(\sqrt{\sigma d} \log n)$, it follows by Corollary 5.10 that for at least $1 - n^{-1/14}$ proportion of such subsets S and S' ,

$$G[S, S'] \text{ is an } \left((a+b)m, (1 \pm 4\gamma)\frac{(a+b)\sigma d}{2}, 18(a+b)\sigma\lambda\right)\text{-bipartite expander.}$$

Substituting (a, b) by $(1, 1), (1, 10), (10, 10)$ and using $36\sigma\lambda \leq \sigma d/250$, we obtain (P1), (P3) and (P4).

Therefore, by the Connecting Lemma (Lemma 7.1), we conclude the existence of vertex-disjoint paths $\mathcal{P} := \{P_i\}_{i \in [m]}$, such that P_i connects x_i and y_i , and $V(P_i) \setminus \{x_i, y_i\} \subseteq R_1$. \square

Now we have our absorbing path: let P be the path obtained by concatenating the paths in \mathcal{P} and \mathcal{Q} , the perfect matching between X and A , and the perfect matching between B and Y (see Figure 1). Note that the end vertices of P are a_1 and b_m . Also, recall that the key property of the absorbing path is that we can remove any subfamily of \mathcal{P} from P and still obtain a path with ends a_1 and b_m (by using the matching edges between A and B).

8.3. Spectral properties of $G[Y, X]$. After building the absorbing path, we aim to define an auxiliary digraph which will later be used to tailor a system of paths into a Hamilton cycle using the Connecting Lemma. To this end, we need to make sure that this auxiliary digraph satisfies some “nice” properties (see (P1)–(P4)) so that we can apply the Connecting Lemma to it. Since a large proportion of the digraph will be obtained by contracting carefully chosen pairs of vertices from X and Y , and orienting some edges between the contracted pairs, it will be useful to first describe some spectral properties of $G[Y, X]$.

Lemma 8.7. *The following properties hold:*

- (1) *For at least $1 - n^{-1/13}$ proportion of pairs of disjoint subsets $X' \subseteq X$ and $Y' \subseteq Y$ of equal size k , the induced bipartite subgraph*

$$G[Y', X'] \text{ is a } (2k, (1 \pm 4\gamma)\rho d, 36\rho\lambda)\text{-bipartite expander.}$$

- (2) *For at least $1 - n^{-1/13}$ proportion of pairs of disjoint subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|Y'| = k$ and $|X'| = 10k$, the induced bipartite subgraph*

$$G[Y', X'] \text{ is an } \left(11k, (1 \pm 4\gamma)\frac{11\rho d}{2}, 198\rho\lambda\right)\text{-bipartite expander.}$$

- (3) *For at least $1 - n^{-1/13}$ proportion of pairs of disjoint subsets $X' \subseteq X$ and $Y' \subseteq Y$ of equal size $10k$, the induced bipartite subgraph*

$$G[Y', X'] \text{ is a } (20k, 10(1 \pm 4\gamma)\rho d, 360\rho\lambda)\text{-bipartite expander.}$$

Proof. First recall that by (R2) (with $X := U_1$ and $Y := U_4$) we have that

$$G[Y, X] \text{ is a } (2m, (1 \pm 2\gamma)\sigma d, 6\sigma\lambda)\text{-bipartite expander.}$$

Consider two disjoint subsets $Y' \subseteq Y$ and $X' \subseteq X$ of size $|Y'| = ak$ and $|X'| = bk$, where $a, b \in [10]$. Since $\rho = \frac{2k}{n}$, $\rho d = \omega(\gamma^{-2} \log n)$ and $\rho\lambda = \omega(\sqrt{\rho d \log n})$, it follows by Theorem 5.9 that for at least $1 - n^{-1/13}$ proportion of such subsets Y' and X' ,

$$G[Y', X'] \text{ is a } \left((a+b)k, (1 \pm 4\gamma)\frac{(a+b)\rho d}{2}, 18(a+b)\rho\lambda\right)\text{-bipartite expander.}$$

Substituting (a, b) by $(1, 1), (1, 10), (10, 10)$, we obtain properties (1)–(3). \square

8.4. Covering the vertex-set with paths. We now show how to cover all the vertices which are not covered by the absorbing path P by a small number of paths. For reasons that will become clear in the next subsection, we will also need to show that the set of endpoints of these paths satisfy some expansion properties.

In order to do so, note that all the work we have done so far (constructing the absorbing path P) took place within a subset of size $O(n/\log n)$. Therefore, at least intuitively, we should be able to move the unused vertices around without ruining spectral properties of the other sets too much. More specifically, we move all the vertices in $R_1 \cup R_2$ that were not used in our absorbing path P to the set W_2 , and let

$$W'_2 := W_2 \cup R_1 \cup R_2 \setminus (V(\mathcal{P}) \cup V(\mathcal{Q})).$$

First, we claim that both $G[W'_2]$ and $G[W_1, W'_2]$ are still spectral expanders.

Lemma 8.8. *The following properties hold:*

- (1) $G[W'_2]$ is a $(|W'_2|, (1 \pm 3\gamma)\frac{d}{2}, \lambda)$ -graph, and
(2) $G[W_1, W'_2]$ is a $(|W_1| + |W'_2|, (1 \pm 3\gamma)d, \lambda)$ -bipartite expander.

Proof. We start by estimating the degrees of vertices into W'_2 and $W_1 \cup W'_2$. Recall that W'_2 is obtained by removing $O(\frac{n}{\log n})$ vertices from $W_2 \cup R_1 \cup R_2$, that $|W_2| = \frac{n}{2} - O(\frac{n}{\log n})$, and that $|R_1 \cup R_2| = 2h = O(\frac{n}{\log n})$. Moreover, by (R1), for every $v \in V$ we have

$$\deg(v, W_i) = (1 \pm 2\gamma) \frac{d|W_i|}{n} \text{ for each } i = 1, 2,$$

and

$$\deg(v, R_1 \cup R_2) = (1 \pm 2\gamma) \frac{2dh}{n} = o\left(\frac{d|W_2|}{n}\right).$$

Therefore, it follows that

$$\deg(v, W'_2) = (1 \pm 3\gamma) \frac{d}{2} \quad \text{and} \quad \deg(v, W_1 \cup W'_2) = (1 \pm 3\gamma)d.$$

Next, we prove (1). Note that $G[W'_2]$ is an induced subgraph of G , so the adjacency matrix of $G[W'_2]$ is a submatrix of the adjacency matrix of G . Therefore, by the Interlacing Theorem for singular values (Theorem A.2), we obtain that the second singular value $s_2(G[W'_2]) \leq s_2(G) \leq \lambda$. Thus, by combining it with the above estimate on the degrees, it follows that

$$G[W'_2] \text{ is a } \left(|W'_2|, (1 \pm 3\gamma) \frac{d}{2}, \lambda\right)\text{-graph}$$

as desired.

Similarly, we prove (2). Once again, by applying the Interlacing Theorem for singular values to the adjacency matrix of $G[W_1 \cup W'_2]$, we obtain that $s_2(G[W_1 \cup W'_2]) \leq s_2(G) \leq \lambda$. Thus, by combining it with the above estimates on the degrees, it follows that

$$G[W_1, W'_2] \text{ is a } (|W_1| + |W'_2|, (1 \pm 3\gamma)d, \lambda)\text{-bipartite expander}$$

as desired. This completes the proof. \square

Next, we construct a collection of not too many vertex-disjoint paths covering all the vertices in $W_1 \cup W'_2$. The basic idea here is to partition W_1 and W'_2 at random into several (labeled) parts of size k or $k - 1$, and then to find maximal matchings between each two consecutive parts. It is important that we will later need to connect all these paths together. In order to achieve it, we need to make sure that the subsets of endpoints of the paths, denoted by $V_1, V_t \subseteq W_1 \cup W'_2$, are such that $G[X, V_1]$ and $G[V_t, Y]$ have some “nice” spectral properties (see (P1)–(P4)). Since W_1 is still untouched, if we insist on choosing the (disjoint) sets V_1, V_t uniformly at random from W_1 , the desired properties will be implied by the pseudorandomness of $G[X, W_1]$ and $G[Y, W_1]$ (as guaranteed in (R4)) and the machinery that we introduced before. The formal details are summarized in the following lemma. Recall that $\rho = \frac{2k}{n}$.

Lemma 8.9. *Let $t = \lceil \frac{|W_1| + |W'_2|}{k} \rceil$ and $t_1 = \lceil \frac{|W_1| - k}{k} \rceil$. Then there exist partitions*

$$W_1 = V_1 \cup V_t \cup \bigcup_{i=2}^{t_1} V_i \quad \text{and} \quad W'_2 = \bigcup_{i=t_1+1}^{t-1} V_i, \quad (16)$$

where each part is of size either k or $k - 1$, and in particular $|V_1| = |V_t| = k$, such that the following properties hold:

- (S1) for every vertex $v \in W_1 \cup W'_2$, $\deg(v, V_i) = (1 \pm 4\gamma) \frac{\rho d}{2}$ for each $1 \leq i \leq t$;
- (S2) for each $1 \leq i \leq t - 1$,

$$G[V_i, V_{i+1}] \text{ is a } (|V_i| + |V_{i+1}|, (1 \pm 6\gamma)\rho d, 13\rho\lambda)\text{-bipartite expander;}$$

(S3) for at least $1 - n^{-1/213}$ proportion of pairs of disjoint subsets $X' \subseteq X$ and $Y' \subseteq Y$ of equal size k , the induced bipartite subgraphs

$G[V_1, X']$ and $G[Y', V_t]$ are $(2k, (1 \pm 2\gamma)\rho d, 6\rho\lambda)$ -bipartite expanders;

(S4) for at least $1 - n^{-1/213}$ proportion of pairs of disjoint subsets $X' \subseteq X$ and $Y' \subseteq Y$ of equal size $10k$, the induced bipartite subgraphs

$G[V_1, X']$ and $G[Y', V_t]$ are $\left(11k, (1 \pm 2\gamma)\frac{11\rho d}{2}, 60\rho\lambda\right)$ -bipartite expanders.

Proof. Let W_1 and W'_2 be randomly partitioned uniformly and independently into sets of size either k or $k - 1$ each, so that

$$W_1 = V_1 \cup V_t \cup \bigcup_{i=2}^{t_1} V_i \quad \text{and} \quad W'_2 = \bigcup_{i=t_1+1}^{t-1} V_i,$$

and $|V_1| = |V_t| = k$. Recall that $G[W_1]$ is an $\left(\frac{n}{2}, (1 \pm 2\gamma)\frac{d}{2}, \lambda\right)$ -graph by (R5) and $G[W'_2]$ is a $\left(|W'_2|, (1 \pm 3\gamma)\frac{d}{2}, \lambda\right)$ -graph by Lemma 8.8. Since V_i is a uniformly random subset of size k or $k - 1$, Chernoff's bounds imply that with probability at least $1 - n^{-1} = 1 - o(1)$, $\deg(v, V_i) = (1 \pm 4\gamma)\frac{dk}{n} = (1 \pm 4\gamma)\frac{\rho d}{2}$ for every $v \in V$ and $1 \leq i \leq t$, which verifies property (S1).

We now show that property (S2) holds whp. We consider pairs of subsets (V_i, V_{i+1}) in each of the graphs $G[W_1]$, $G[W'_2]$ and $G[W_1, W'_2]$ separately, as each of these graphs is an expander with slightly different parameters:

Case 1: $1 \leq i \leq t_1 - 1$.

Recall that $G[W_1]$ is an $\left(\frac{n}{2}, (1 \pm 2\gamma)\frac{d}{2}, \lambda\right)$ -graph and note that $V_i, V_{i+1} \subseteq W_1$ are disjoint subsets of size k or $k - 1$ chosen uniformly at random. Since $\rho d = \omega(\gamma^{-2} \log n)$ and $\rho\lambda = \omega(\sqrt{\rho d \log n})$, by Corollary 5.10, we have that with probability at least $1 - n^{-1/14}$,

$G[V_i, V_{i+1}]$ is a $(|V_i| + |V_{i+1}|, (1 \pm 4\gamma)\rho d, 12\rho\lambda)$ -bipartite expander.

Case 2: $t_1 + 1 \leq i \leq t - 2$.

Recall that $G[W'_2]$ is a $\left(|W'_2|, (1 \pm 3\gamma)\frac{d}{2}, \lambda\right)$ -graph and note that $V_i, V_{i+1} \subseteq W'_2$ are disjoint subsets of size k or $k - 1$ chosen uniformly at random. Since $|W'_2| = \frac{n}{2} - O(\frac{n}{\log n})$, $\rho d = \omega(\gamma^{-2} \log n)$ and $\rho\lambda = \omega(\sqrt{\rho d \log n})$, by Corollary 5.10, we have that with probability at least $1 - n^{-1/14}$,

$G[V_i, V_{i+1}]$ is a $(|V_i| + |V_{i+1}|, (1 \pm 6\gamma)\rho d, 13\rho\lambda)$ -bipartite expander.

Case 3: $i = t_1$ or $t - 1$.

Recall that by Lemma 8.8, $G[W_1, W'_2]$ is a $(|W_1| + |W'_2|, (1 \pm 3\gamma)d, \lambda)$ -bipartite expander and note that V_i and V_{i+1} are uniformly random subsets of different parts of size k or $k - 1$. Since $|W_1 \cup W'_2| = n - O(\frac{n}{\log n})$, $\rho d = \omega(\gamma^{-2} \log n)$ and $\rho\lambda = \omega(\sqrt{\rho d \log n})$, by Theorem 5.9, we have that with probability at least $1 - n^{-1/13}$,

$G[V_i, V_{i+1}]$ is a $(|V_i| + |V_{i+1}|, (1 \pm 6\gamma)\rho d, 7\rho\lambda)$ -bipartite expander.

Therefore, by a union bound, we have that with probability at least $1 - 2n^{-1/13} - (t-3)n^{-1/14} = 1 - o(1)$, $G[V_i, V_{i+1}]$ is a $(|V_i| + |V_{i+1}|, (1 \pm 6\gamma)\rho d, 13\rho\lambda)$ -bipartite expander for each $1 \leq i \leq t-1$, which verifies property (S2).

We now show that (S3) holds whp. Recall that by (R4), for at least $1 - n^{-1/53}$ proportion of subsets $X' \subseteq X$ and $S \subseteq W_1$ of equal size k ,

$$G[X', S] \text{ is a } (2k, (1 \pm 2\gamma)\rho d, 6\rho\lambda) \text{-bipartite expander.}$$

Since $V_1 \subseteq W_1$ is a uniformly random subset of size k , by Proposition 8.4 applied with $W_1, X, V_1, k, k, n^{-1/53}$ in place of X, Y, S, m, h, α , we obtain that with probability at least $1 - n^{-1/212}$, for at least $(1 - n^{-1/106})(1 - n^{-1/212}) \geq 1 - n^{-1/213}$ proportion of k -sets $X' \subseteq X$, the induced bipartite subgraph

$$G[V_1, X'] \text{ is a } (2k, (1 \pm 2\gamma)\rho d, 6\rho\lambda) \text{-bipartite expander.}$$

By the same argument and union bound, we have that each (S3) and (S4) holds with probability at least $1 - 2n^{-1/212}$.

By union bound we conclude the existence of desired partitions of W_1 and W'_2 . \square

Now we can obtain the paths covering exactly $W_1 \cup W'_2$ that we need. First, we find one path that makes the remaining parts of equal size. Indeed, by (S1), we are allowed to find one edge from an arbitrary vertex of a part of size k to the next part of size k . Doing this from V_1 to V_t in sequence, we are able to find a path passing through exactly all parts of size k .

Next, we cover the rest vertices by vertex-disjoint paths obtained by perfect matchings. For simplicity, let us denote the remaining subsets of V_i by V'_i . By applying (S1), (S2) and Interlacing Theorem for singular values (Theorem A.2), we have that

(S2') for each $1 \leq i \leq t-1$,

$$G[V'_i, V'_{i+1}] \text{ is a } (2(k-1), (1 \pm 7\gamma)\rho d, 13\rho\lambda) \text{-bipartite expander.}$$

Thus, Lemma 6.1 implies that $G[V'_i, V'_{i+1}]$ has a perfect matching for each $1 \leq i \leq t-1$ since $13\rho\lambda \leq \rho d/5$. By concatenating all perfect matchings in each subgraph induced by two consecutive remaining parts, together with the path above, we obtain k vertex-disjoint paths S_1, \dots, S_k whose union covers $W_1 \cup W'_2$, such that for each $1 \leq i \leq k$, one of the endpoints of S_i is in V_1 and the other is in V_t .

8.5. Obtaining the Hamilton cycle. With all the ingredients in previous sections, we are now ready to complete the construction of a Hamilton cycle.

First, we aim to use Lemma 7.1 again to connect X_0 and Y_0 by the set of paths $\{x_i P_i y_i \mid i \in [m]\}$. Recall that a_1 and b_m are the endpoints of the absorbing path P . Let $v_0 := a_1$, $w_0 := b_m$, $\{v_j\} := V(S_j) \cap V_1$ and $\{w_j\} := V(S_j) \cap V_t$ for each $1 \leq j \leq k$. Let $X_0 := \{v_0, \dots, v_k\}$ and $Y_0 := \{w_0, \dots, w_k\}$.

In order to complete the connection, we define an auxiliary digraph D as follows: let $V(D) = X_0 \cup Y_0 \cup Z$, where $Z := \{z_1, \dots, z_m\}$, and D contains the following directed edges:

- (i) $z_i z_j$ if $y_i x_j \in E(G)$;
- (ii) $v_\alpha z_i$ if $v_\alpha x_i \in E(G)$;
- (iii) $z_i w_\alpha$ if $y_i w_\alpha \in E(G)$.

Observe that a uniformly random subset of Z naturally corresponds to a uniformly random subset of indices, which then leads to a uniformly random subset of X and a uniformly random subset of Y , where their indices are exactly the same. Thus, since $\lambda \leq d/9000$, properties (P1)–(P4) can be verified by Lemma 8.7, (S3), (S4) and the definition of the auxiliary digraph D . Therefore, by applying Lemma 7.1 to X_0 , Y_0 and Z we obtain a family of directed paths $w_i T_i v_{i+1}$, $0 \leq i \leq k$ in D , where $v_{k+1} := v_0$ and the inner vertices of the paths are from Z . For each path $w_i T_i v_{i+1}$, if we replace each inner vertex z_j by the path $x_j P_j y_j$, then by the definition of D , this gives rise to

a path $w_i T'_i v_{i+1}$ in G . Thus, by the absorbing property, we can remove a family of paths in \mathcal{P} and use them to construct the paths $w_i T'_i v_{i+1}$, $0 \leq i \leq k$.

Formally, let

$$P_0 := b_m T'_0 v_1 S_1 w_1 \dots S_k w_k T'_k a_1$$

be a path in G . For each $1 \leq i \leq m$, if $x_i P_i y_i$ is contained in P_0 , then let Z_i be the empty path on no vertices, and otherwise let Z_i be $x_i P_i y_i$. Let

$$P'_0 := a_1 Z_1 b_1 Q_1 a_2 Z_2 b_2 \dots a_m Z_m b_m.$$

Therefore, $P_0 \cup P'_0$ is a Hamilton cycle of G . This completes the proof of both Theorems 8.1 and 1.2.

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APPENDIX A. LINEAR ALGEBRA BACKGROUND

In this section we collect some standard tools from linear algebra.

The following theorem provides a convenient tool for computing/bounding eigenvalues of a real symmetric matrix (see for example Theorem 2.4.1 in [6]).

Theorem A.1 (Courant-Fischer Minimax Theorem). *Let A be a symmetric real $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Then,*

$$\lambda_k = \max_{\dim(U)=k} \min_{\mathbf{x} \in U \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \min_{\dim(U)=n-k+1} \max_{\mathbf{x} \in U \setminus \{\mathbf{0}\}} \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Since the notion of eigenvalues is undefined for non-square matrices, it would be convenient for us to work with singular values which are defined for all matrices (see Definition 3.3). The following theorem proved by Thompson [44] is useful when one wants to obtain non-trivial bounds on the singular values of submatrices.

Theorem A.2 (Interlacing Theorem for singular values). *Let A be an $m \times n$ matrix and let*

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\min\{m,n\}}$$

be its singular values. Let B be any $p \times q$ submatrix of A and let

$$\beta_1 \geq \beta_2 \geq \dots \geq \beta_{\min\{p,q\}}$$

be its singular values. Then

$$\begin{aligned} \alpha_i &\geq \beta_i, & \text{for } i = 1, 2, \dots, \min\{p, q\}, \\ \beta_i &\geq \alpha_{i+(m-p)+(n-q)}, & \text{for } i \leq \min\{p + q - m, p + q - n\}. \end{aligned}$$

One of the most commonly used tools in linear algebra is singular value decomposition. We need a slightly stronger version of it, which almost immediately follows from the standard proof:

Theorem A.3 (Singular value decomposition). *Let M be a real $m \times n$ matrix with rank r . Let $s_1 \geq s_2 \geq \dots \geq s_r$ be all the positive singular values of M . Let $\mathbf{u}_1 \in \mathbb{R}^m$ and $\mathbf{v}_1 \in \mathbb{R}^n$ be unit vectors such that $M\mathbf{v}_1 = s_1\mathbf{u}_1$. Then we can find an orthonormal bases $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of \mathbb{R}^m and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of \mathbb{R}^n with \mathbf{u}_1 and \mathbf{v}_1 as above, and such that*

$$M = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^T.$$

In particular, this equality implies that $M\mathbf{v}_j = s_j\mathbf{u}_j$ for $j = 1, \dots, r$ and $M\mathbf{v}_j = \mathbf{0}$ for $j > r$.

We will also make use of the following simple corollary of the above theorem, which proof is included for completion.

Lemma A.4 (Best low-rank approximation). *Let A be a real $m \times n$ matrix. Then*

$$s_2(A) = \min_B \|A - B\|,$$

where the minimum is over all rank-one $m \times n$ matrices B , and $\|\cdot\|$ denotes the operator norm.

Moreover, the minimum is attained by $B = s_1(A)\mathbf{u}_1\mathbf{v}_1^T$, where $\mathbf{v}_1 \in \mathbb{R}^n$ and $\mathbf{u}_1 \in \mathbb{R}^m$ are any unit vectors such that $A\mathbf{v}_1 = s_1(A)\mathbf{u}_1$.

Proof. Let A be a real $m \times n$ matrix with rank r . Let $s_1 \geq s_2 \geq \dots \geq s_r$ be all positive singular values of A , and let $\mathbf{v}_1 \in \mathbb{R}^n$ and $\mathbf{u}_1 \in \mathbb{R}^m$ be unit vectors such that $A\mathbf{v}_1 = s_1\mathbf{u}_1$. By Theorem A.3, there exist orthonormal bases $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of \mathbb{R}^n and $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of \mathbb{R}^m , such that

$$A = \sum_{j=1}^r s_j \mathbf{u}_j \mathbf{v}_j^T.$$

First, note that $B = s_1\mathbf{u}_1\mathbf{v}_1^T$ is a rank-one matrix that satisfies

$$\|A - B\| = \left\| \sum_{j=2}^r s_j \mathbf{u}_j \mathbf{v}_j^T \right\| = s_2.$$

Therefore, to finish the proof it suffices to show $s_2 \leq \|A - B\|$ for every rank-one $m \times n$ matrix B . We can express such a matrix as $B = \mathbf{x}\mathbf{y}^T$ for some nonzero vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Next, we can find a nontrivial linear combination $\mathbf{w} = a\mathbf{v}_1 + b\mathbf{v}_2$ such that $\langle \mathbf{y}, \mathbf{w} \rangle = 0$; this implies $B\mathbf{w} = \mathbf{x}(\mathbf{y}^T\mathbf{w}) = 0$. Without loss of generality, we can scale \mathbf{w} so that $\|\mathbf{w}\| = 1$, or equivalently, $a^2 + b^2 = 1$. Therefore,

$$\|A - B\|^2 \geq \|(A - B)\mathbf{w}\|_2^2 = \|A\mathbf{w}\|_2^2 = a^2 s_1^2 + b^2 s_2^2 \geq s_2^2.$$

This completes the proof. □

Finally, we state the chain rule for singular values, which is used in the proof of Corollary 4.2.

Lemma A.5 (Chain rule for singular values). *Let A, B, C be $n \times n$ matrices. Then*

$$s_k(ABC) \leq \|A\| \|B\| s_k(C) \quad \text{for all } k \in [n].$$

Proof. First assume that $C = I$. By the Minimax Theorem A.1, we have

$$s_k(AB) = \min_{\dim(U)=n-k+1} \max_{\mathbf{x} \in S(U)} \|AB\mathbf{x}\|_2,$$

where $S(U)$ denotes the set of all unit vectors in U . Since $\|AB\mathbf{x}\|_2 \leq \|A\| \|B\mathbf{x}\|_2$, it follows that $s_k(AB) \leq \|A\| s_k(B)$. This argument also yields $s_k(BC) \leq s_k(B) \|C\|$ once we notice that $s_k(BC) = s_k(C^\top B^\top)$. Combining these two bounds, we complete the proof. \square

APPENDIX B. PROOF OF THEOREM 5.9

In this section, we prove Theorem 5.9 which states that a random induced subgraph of a bipartite expander is still a bipartite expander with high probability.

The proof of Theorem 5.9 is very similar to Theorem 5.2. For a given partition $[n] = V_1 \cup V_2$ and subsets $I \subset V_1$ and $J \subset V_2$, we denote by $Q_{I,J}$ the orthogonal projection in \mathbb{R}^n onto $\mathbb{R}^{I \cup J}$. In other words, $Q_{I,J}$ is the diagonal matrix with $Q_{ii} = 1$ if $i \in I \cup J$ and $Q_{ii} = 0$ if $i \notin I \cup J$.

Our main tool is to change the model of sampling in order to use the results in Section 5, which is described as below. The proof follows from [45] and we include its proof for completion.

Lemma B.1 (Random subset models). *Let $[n] = V_1 \cup V_2$ be a partition. Let B be an $n \times n$ matrix. Let $I \sim \text{Subset}(|V_1|, \sigma_1 |V_1|)$, $J \sim \text{Subset}(|V_2|, \sigma_2 |V_2|)$ and $K \sim \text{Subset}(n, \sigma)$ be independent subsets, where for each $i \in [2]$, $\sigma_i \in (0, 1)$, $\sigma_i |V_i| \geq 1$ and $\sigma := \max\{\sigma_1, \sigma_2\}$. Then for every $p \geq 2$, we have*

$$\mathbb{E}_p \|Q_{I,J} B Q_{I,J}\| \leq 4^{1/p} \mathbb{E}_p \|P_K B P_K\|.$$

Proof. Let $m := |V_1|$, and let $k_i := \sigma_i |V_i|$ for each $i \in [2]$. Let $Q_{I,J} = Q_1 + Q_2$, where Q_1 and Q_2 are both 0/1 diagonal matrices, the last $n - m$ entries of Q_1 are zeros, and the first m entries of Q_2 are zeros. Moreover, let $P_K = P_1 + P_2$, where P_1 and P_2 are both 0/1 diagonal matrices, the last $n - m$ entries of P_1 are zeros, and the first m entries of P_2 are zeros. We can make the computations as follows:

$$\begin{aligned} \mathbb{P} [\|P_K B P_K\|^p > t] &\geq \sum_{j=k_1}^m \sum_{j'=k_2}^{n-m} \mathbb{P} [\|(P_1 + P_2) B (P_1 + P_2)\|^p > t \mid \|P_1\| = j, \|P_2\| = j'] \\ &\quad \cdot \mathbb{P} [\|P_1\| = j] \cdot \mathbb{P} [\|P_2\| = j'] \\ &\geq \mathbb{P} [\|(P_1 + P_2) B (P_1 + P_2)\|^p > t \mid \|P_1\| = k_1, \|P_2\| = k_2] \\ &\quad \cdot \sum_{j=k_1}^m \mathbb{P} [\|P_1\| = j] \cdot \sum_{j'=k_2}^{n-m} \mathbb{P} [\|P_2\| = j'] \\ &\geq \frac{1}{4} \mathbb{P} [\|(Q_1 + Q_2) B (Q_1 + Q_2)\|^p > t] \\ &= \frac{1}{4} \mathbb{P} [\|Q_{I,J} B Q_{I,J}\|^p > t]. \end{aligned}$$

The second inequality holds because the norm of a submatrix is at most the norm of the matrix itself, and the last inequality relies on the fact that the median of the binomial distribution $\text{Bin}(\sigma, m)$ lies between $\sigma m - 1$ and σm , and $k_i \leq \sigma |V_i|$ for each $i \in [2]$. Integrate with respect to t to complete the proof. \square

By combining Lemma B.1 and the tools in Section 5, we can obtain a corollary as follows:

Corollary B.2 (Norms of random submatrices). *Let B be a symmetric real $n \times n$ matrix. Let $I \sim \text{Subset}(|V_1|, \sigma_1|V_1|)$ and $J \sim \text{Subset}(|V_2|, \sigma_2|V_2|)$ be independent subsets where for each $i \in [2]$, $\sigma_i \in (0, 1)$ and $\sigma_i|V_i| \geq 1$. Let $\sigma := \max\{\sigma_1, \sigma_2\}$, let $p \geq 2$ and let $q = \max\{p, 2 \log n\}$. Then*

$$\mathbb{E}_p \|Q_{I,J} B Q_{I,J}\| \leq 4\sigma \|B\| + 24\sqrt{q\sigma} \|B\|_{1 \rightarrow 2} + 35q \|B\|_\infty.$$

Proof. Consider the symmetric, diagonal-free matrix $B_0 = B - D$ where $D := \text{diag}(B_{1,1}, \dots, B_{n,n})$. Combining Theorem 5.3 with Lemmas 5.5 and B.1, we obtain the following:

$$\mathbb{E}_p \|Q_{I,J} B_0 Q_{I,J}\| \leq 4\sigma \|B_0\| + 24\sqrt{q\sigma} \|B_0\|_{1 \rightarrow 2} + 32q \|B_0\|_\infty.$$

Note that $\|B_0\| \leq \|B\| + \|D\|$, $\|B_0\|_{1 \rightarrow 2} \leq \|B\|_{1 \rightarrow 2}$, $\|B_0\|_\infty \leq \|B\|_\infty$, and $\|Q_{I,J} B Q_{I,J}\| \leq \|Q_{I,J} B_0 Q_{I,J}\| + \|Q_{I,J} D Q_{I,J}\| \leq \|Q_{I,J} B_0 Q_{I,J}\| + \|D\|$. This implies

$$\mathbb{E}_p \|Q_{I,J} B Q_{I,J}\| \leq \mathbb{E}_p \|Q_{I,J} B_0 Q_{I,J}\| + \|D\| \leq 4\sigma (\|B\| + \|D\|) + 24\sqrt{q\sigma} \|B\|_{1 \rightarrow 2} + 32q \|B\|_\infty + \|D\|.$$

Notice that $\|D\| = \max_i |B_{i,i}| \leq \|B\|_\infty$ to complete the proof. \square

We are now ready to prove Theorem 5.9.

Proof of Theorem 5.9. We want to show that whp, $H := G[X, Y]$ is an $(m, (1 \pm 2\gamma)\frac{dm}{n}, 6\sigma\lambda)$ -bipartite expander, where $m = \sigma_1|V_1| + \sigma_2|V_2|$ and $\sigma = \max\{\sigma_1, \sigma_2\}$. Since $G = (V_1 \cup V_2, E)$ is an $(n, (1 \pm \gamma)d, \lambda)$ -bipartite expander, there is an $(n, (1 \pm \gamma)d, \lambda)$ -graph F with $V(F) = V_1 \cup V_2$, such that G is an induced bipartite subgraph of F , and moreover

$$\deg_F(v, V_2) = (1 \pm \gamma) \frac{d|V_2|}{n} \quad \text{for every } v \in V_1, \quad (17)$$

and

$$\deg_F(v, V_1) = (1 \pm \gamma) \frac{d|V_1|}{n} \quad \text{for every } v \in V_2. \quad (18)$$

So it suffices to show that whp, the induced subgraph $F_0 := F[X \cup Y]$ is an $(m, (1 \pm 2\gamma)\frac{dm}{n}, 6\sigma\lambda)$ -graph, and moreover $\deg_{F_0}(v, Y) = (1 \pm 2\gamma) \frac{d|Y|}{n}$ for every $v \in X$ and $\deg_{F_0}(v, X) = (1 \pm 2\gamma) \frac{d|X|}{n}$ for every $v \in Y$.

Let $C > 0$ be a sufficiently large absolute constant. To see that the random induced subgraph $F_0 = F[X \cup Y]$ is almost regular whp, we can apply Lemma 5.1 with parameters $|V_i|, (1 \pm \gamma)d|V_i|/n, \sigma_i|V_i|, \gamma/(1 + \gamma)$ in place of N, K, n, α , where $i = 1, 2$. Since $\sigma_i d \geq C\gamma^{-2} \log n$ for each $i = 1, 2$ and sufficiently large absolute constant $C > 0$, one can obtain from (17) and (18) that with probability at least $1 - n^{-1}$,

$$\deg_{F_0}(v, Y) = (1 \pm 2\gamma) \frac{d|Y|}{n} \quad \text{for every } v \in X, \quad (19)$$

and

$$\deg_{F_0}(v, X) = (1 \pm 2\gamma) \frac{d|X|}{n} \quad \text{for every } v \in Y. \quad (20)$$

So for every vertex $v \in X \cup Y$, $\deg_{F_0}(v) = \deg_{F_0}(v, X) + \deg_{F_0}(v, Y) = (1 \pm 2\gamma) \frac{d(|X| + |Y|)}{n} = (1 \pm 2\gamma) \frac{dm}{n}$. Thus, it remains to bound the second singular value of A_{F_0} whp, where A_{F_0} is the adjacency matrix of F_0 .

It is convenient to first work with normalized matrices. So let us consider the normalized adjacency matrix

$$\bar{A}_F = D^{-1/2} A_F D^{-1/2}, \quad \text{where } D = \text{diag}(d_1, \dots, d_n) \quad (21)$$

is the degree matrix of F . According to Lemma A.4 and Observation 3.5, we have

$$s_2(\bar{A}_F) = \|B\|, \quad \text{where } B = \bar{A}_F - \frac{1}{a} D^{1/2} \mathbb{1}_n \mathbb{1}_n^\top D^{1/2} \quad \text{and } a = \sum_{i=1}^n d_i. \quad (22)$$

Applying Corollary B.2 for any $p \geq 2$ and $q = \max\{p, 2 \log n\}$, we obtain

$$\mathbb{E}_p \|Q_{X,Y} B Q_{X,Y}\| \leq 4\sigma \|B\| + 24\sqrt{q\sigma} \|B\|_{1 \rightarrow 2} + 35q \|B\|_\infty. \quad (23)$$

Recall that $\|B\| \leq 1.1\lambda/d$ in (7), $\|B\|_{1 \rightarrow 2} \leq 2.6/\sqrt{d}$ in (12), and $\|B\|_\infty \leq 2.4/d$ in (14). Plugging them into (23), we obtain

$$\mathbb{E}_p \|Q_{X,Y} B Q_{X,Y}\| \leq \frac{4.4\sigma\lambda}{d} + 63\sqrt{\frac{q\sigma}{d}} + \frac{84q}{d}.$$

Multiplying on the left and right by $D^{1/2}$ inside the norm, we conclude that

$$\mathbb{E}_p \left\| D^{1/2} Q_{X,Y} B Q_{X,Y} D^{1/2} \right\| \leq \|D\| \mathbb{E}_p \|Q_{X,Y} B Q_{X,Y}\| \leq 5\sigma\lambda + 70\sqrt{q\sigma d} + 93q =: \lambda_0,$$

where we used that $\|D\| = \max_i d_i \leq 1.1d$. Since diagonal matrices commute, we can express the matrix above as follows:

$$D^{1/2} Q_{X,Y} B Q_{X,Y} D^{1/2} = Q_{X,Y} D^{1/2} B D^{1/2} Q_{X,Y} = Q_{X,Y} A_F Q_{X,Y} - \frac{1}{a} Q_{X,Y} D \mathbb{1}_n \mathbb{1}_n^\top D Q_{X,Y},$$

where in the last step we used (21) and (22). Note that $\frac{1}{a} Q_{X,Y} D \mathbb{1}_n \mathbb{1}_n^\top D Q_{X,Y}$ is a rank one matrix. Thus, by Lemma A.4, we have $s_2(Q_{X,Y} A_F Q_{X,Y}) \leq \left\| D^{1/2} Q_{X,Y} B Q_{X,Y} D^{1/2} \right\|$, and thus

$$\mathbb{E}_p s_2(Q_{X,Y} A_F Q_{X,Y}) \leq \lambda_0.$$

Since the adjacency matrix A_{F_0} of the induced subgraph F_0 is a $m \times m$ submatrix of the $n \times n$ matrix $Q_{X,Y} A_F Q_{X,Y}$, by the Interlacing Theorem for singular values (Theorem A.2), it follows that

$$\mathbb{E}_p s_2(A_{F_0}) \leq \lambda_0.$$

Now choose $p = 2 \log n$ and thus $q = p = 2 \log n$. Applying Markov's inequality, we obtain

$$\begin{aligned} \mathbb{P} [s_2(A_{F_0}) \geq 1.1\lambda_0] &= \mathbb{P} [s_2(A_{F_0})^p \geq (1.1\lambda_0)^p] \leq \left(\frac{\mathbb{E}_p s_2(A_{F_0})}{1.1\lambda_0} \right)^p \\ &\leq (1.1)^{-p} = (1.1)^{-2 \log n} \leq n^{-0.08}. \end{aligned}$$

In other words, with probability at least $1 - n^{-0.08}$, we have

$$s_2(A_{F_0}) < 1.1\lambda_0 \leq 5.5\sigma\lambda + 109\sqrt{\sigma d \log n} + 205 \log n.$$

To complete the proof, we show that the first term dominates the right hand side. Indeed, since the absolute constant C is sufficiently large, the first condition in Theorem 5.9 implies that $205 \log n \leq \sqrt{\sigma d \log n}$. Similarly, the second condition in the theorem implies that $110\sqrt{\sigma d \log n} \leq 0.5\sigma\lambda$. Then it follows that

$$s_2(A_{F_0}) \leq 5.5\sigma\lambda + 0.5\sigma\lambda = 6\sigma\lambda.$$

Therefore, with probability at least $(1 - n^{-1})(1 - n^{-0.08}) \geq 1 - n^{-1/13}$, H is an $(m, (1 \pm 2\gamma)\frac{dm}{n}, 6\sigma\lambda)$ -bipartite expander, which completes the proof of Theorem 5.9. \square

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