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Toward Robust Revenue Management: Competitive Analysis of Online Booking

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Abstract

In this paper, we consider the revenue management problem from the perspective of online algorithms. This approach eliminates the need for both demand forecasts and a risk neutrality assumption. The competitive ratio of a policy relative to a given input sequence is the ratio of the policy's performance to the offline optimal. Under the online algorithm approach, revenue management policies are evaluated based on highest competitive ratio they can guarantee. We are able to define lower bounds on the best possible performance and describe policies that achieve these lower bounds. We address the 2-fare problem in greatest detail but also treat the general multi-fare problem and the bid-price control problem.

The practice of revenue management grew up in the 1980's within the airline industry. The underlying assumptions of most revenue management models are based on characteristics of the airline problem. The success within the airline industry has led to present-day attempts at broader application of RM techniques across a wide range of industries. These other applications, in general, can have significantly different underlying structure than the airline case. Moreover, the airline industry itself is undergoing substantial upheaval, which is leading toward new approaches to defining the airline product. Both RM practice and RM models have largely employed two basic assumptions: (i) that demand can be forecast with reasonable accuracy; and, because individual booking decisions are extremely frequent and have low stakes, (ii) that a risk-neutral approach is justified; see, e.g., [7, 21, 15].

In this paper, we consider the RM problem from the perspective of *online algorithms* (see [1] for a recent survey). This approach, in fact, eliminates the need for either of these assumptions. An online algorithm addresses a sequential decision-making problem, where individual model inputs are received in sequence and a decision on each successive input must be made at the time the input is received. Once a decision is made regarding an individual input it cannot be reversed. An offline algorithm considers the entire input sequence simultaneously and makes a decision on the entire set of inputs. An offline optimal solution would be a solution obtained by an offline algorithm that optimized the objective function of interest. Such a solution is also called a "clairvoyant" optimal since it can be viewed as being based on the (unrealistic) assumption that the entire input stream is known in advance.

A popular and informative approach to the analysis of online algorithms employs a simulated competitive analysis in which an imaginary adversary is postulated that controls the input sequence. The goal of the adversary is to generate the worst possible algorithmic performance. Specifically, after each step of the online algorithm, the adversary selects the input so as to create the worst possible algorithmic performance. Here we measure performance as the ratio of the objective value achieved by the online algorithm divided by the objective value that could be achieved by an optimal offline algorithm operating on the same input sequence. We now precisely define this performance measure, which we call the *competitive ratio*, by considering an alternative definition based on a worst-case performance measure, under the assumption of a maximization problem. Let Ω_Υ be the set of all possible input sequences to an online algorithm Υ and, for any $I \in \Omega_\Upsilon$, let $v'_\Upsilon(I)$ be the objective value achieved by the online algorithm for input I and let $v^*(I)$ be the objective value achieved by an optimal offline algorithm. Then we define:

$$\text{competitive ratio of } \Upsilon \quad = \quad \inf_{I \in \Omega_\Upsilon} \frac{v'_\Upsilon(I)}{v^*(I)}$$

Thus, the competitive ratio of an online algorithm is a guarantee on a certain level of performance.

The competitive ratio of an online algorithm can sometimes be improved by allowing the use of randomization strategies within the algorithm. Specifically, at certain steps within the algorithm choices are made based on the outcome of the draw of a random number. In such cases, the performance of the application of an online algorithm to a given input sequence I is not deterministic and we consider the expected performance $E[v'_\Upsilon(I)]$ The competitive ratio is then defined relative to expected performance:

$$\text{competitive ratio of } \Upsilon \quad = \quad \inf_{I \in \Omega_\Upsilon} \frac{E[v'_\Upsilon(I)]}{v^*(I)}$$

Note that the competitive ratio measures algorithmic performance based on a *relative regret* criteria. Specifically, if $\Upsilon = 0$, then there would be an input sequence where some positive value could be achieved and the algorithm achieved a value of 0. Thus, on a relative scale the outcome would have a maximum level of regret. On the other hand, if $\Upsilon = 1$, then, under all input sequences, the online algorithm would achieve the maximum performance level possible and there would be no regret. We can see that by maximizing Υ, we minimize the maximum relative level of regret. This approach can, thus, be contrasted with approaches based on "absolute" regret such as the standard Minimax Regret approach used in Decision Analysis. We note that this relative regret approach is equivalent to applying the Minimax Regret approach to a utility function that is the logarithm of the total revenue. We refer the reader to Chapter 15 in [6] for an extensive discussion of decision theoretic foundations for the competitive ratio.

We study what is commonly referred to as the *single-leg* revenue management problem (see the recent book by Talluri and Van Ryzin [21] for background on the concepts discussed here). That is, we allocate inventory of a *single* resource or product to a demand stream, where each order in the stream has a requested fare drawn from a set of fare classes. Our approach is general in that we do not restrict ourselves to any class of algorithms but rather seek the best possible policy. Nonetheless, our results show that the best policies come from a well-known class, namely *nested protection level policies.* These policies define reservation buckets, where the 1st bucket is reserved for the highest fare orders, the 2nd bucket for orders with the highest two fares, the 3rd bucket for orders with the highest three fares, etc. Thus, for example, when an order in the 2nd highest fare class arrives, the policy first tries to assign it to the 2nd bucket, if it is full, it tries the 3rd bucket, and keeps proceeding. If there is no space in any of the 2nd, 3rd, etc. buckets, the order is rejected. Such policies are said to employ *standard nesting* and we show that with the proper setting of the protection level limits they achieve the best possible performance. We also show that the corresponding *order quantity control* policies achieve this performance level. Such policies are more commonly referred to as *theft nesting* in the literature, but we use this alternate terminology because it fits better within our development. These policies set a minimum accepted fare based on the *number* of order accepted thus far. Thus, all orders are accepted until a certain quantity threshold is met, then all orders except those in the lowest fare class are accepted until a second threshold is met, etc.

The following example illustrates a typical result based on the analysis presented in this paper.

Example: *Consider a flight with 95 available seats, and three fare classes. The highest fare is \$1,000; the intermediate fare \$750; and the lowest fare \$500. Which booking policy guarantees the largest possible percentage of the optimum revenue, for* any *demands and request sequence?*

Answer: *The following nested booking policy guarantees a revenue at least 63% of the maximum possible revenue, for any demands and request sequence: protect 15 seats for high-fare, and (a total of) 35 seats for the two higher-fare requests; hence, sell at most 60 seats at the lowest fare. Details are found in Section 2.1.* □

Note that there are no required assumptions on the distribution of demand. We also feel that the policy given is quite reasonable and would appear to be practically attractive. For the example given above, the corresponding order quantity control policy would be:

> *Until 60 orders have been accepted, accept requests at all three fare levels; after accepting 60 orders and until 80 orders have been accepted, accept requests at the two higher fare levels; after accepting 85 orders, only accept requests at the highest fare level.*

Note that order quantity control policies are more aggressive than nested protection level policies in the sense that they cut off lower fare requests more quickly than nested protection level policies. Nested protection level policies and order quantity control policies are identical when faced with streams of orders sequenced according to a low-before-high regime, but for general order sequences their outcomes can be different.

For the 2-fare case, we compare nested protection level policies with first-come, first-served policies and partitioned protection level policies. Partitioned policies create exclusive buckets for each fare class and do not allow higher fare orders to "overflow" into the buckets of lower fare classes as nested policies do. The best nested protection level policies strictly dominate these other classes of policies. Since the nested policies achieve the best possible competitive ratio, and since they strictly dominate the partitioned policies, based on this performance measure, nesting is optimal.

We also analyze the *bid-price control problem*. Here, fares do not come from a set of discrete classes but rather each order may "propose" an arbitrary fare from within a range. We are able to produce strong results for this problem by showing that it can be viewed as the limiting case for the multi-fare problem when the number of fare classes becomes large.

The standard analysis of online algorithms assumes that the "adversary" always uses an optimal policy. We briefly introduce, at the end of this paper, the concept of dynamic policies. These policies adjust the behavior of the online algorithm to the partial input sequence seen at each step of the algorithm. If the input sequence does not represent an optimal adversary strategy, then the online algorithm can be adjusted so as to achieve a better worst case performance. The a-priori worst case performance is not improved but, from a practical perspective, performance could be significantly improved.

We are not advocating the present approach as a stand-alone panacea. But it may useful in high-uncertainty or high-risk situations, possibly in combination with, or as a safeguard for, more traditional RM approaches. It can also be used to initialize adaptive approaches, e.g., as in [22], when there is little prior knowledge about demand. We also feel that this approach provides a new perspective on RM, for example, leading to consideration of new classes of policies and new approaches to analyzing historical demand data.

The single leg problem in RM has long been studied in the literature. Littlewood [14] considered two fare classes and assumed product is sold in a low-before-high (LBH) manner; i.e. demand in lowest fare class arrives first. Belobaba [2], [3] presented heuristic extensions of Littlewood's rule to multiple fare classes, again assuming LBH. Brumelle and McGill [8] formulated a stochastic dynamic program for the multiple fare class problem assuming LBH and independent demand. They proved the optimality of nested booking control policies for LBH. Later, Lee and Hersh [13] and Lautenbacher and Stidham [12] relaxed the LBH assumption and provided Markov Decision Process (MDP) formulations of the multi-fare problem and proved the optimality of nested policies for the dynamic case. We note in this case optimality is relative to maximizing expected revenue within the context of a stochastic model (in contrast to the online optimality discussed in the preceding paragraphs). The book by Talluri and van Ryzin, [21] provides a unified treatment of the literature on the single-leg problem as well as RM in general. All of the above papers assume that each customer belongs to a single fare class for which the probability distribution of demand or the arrival process is known. To our knowledge there is very limited RM literature for cases with limited or no demand information. Two papers [15], [20] have taken some steps in this direction. Talluri and van Ryzin [20] analyzed the multiple fare class problem where the demand is characterized by customer choice that depends on the product types available at the time of a customer's arrival. While their approach does not require the probability distribution of demand of each class to be known, it requires that customer choice behavior and arrival processes are known. McGill and van Ryzin [15] start with the assumption of a known distribution with unknown parameter values. Parameters are estimated over time using stochastic optimization techniques.

We now briefly discuss somewhat related work on the Economic Order Quantity (EOQ) problem and other inventory models where the demand is partially unknown. Starting with Scarf [18] in 1958, a number of authors have considered single-product inventory systems in which only partial information (in most cases the mean, often also the variance, higher moments, or fractiles) is known about future demand; see [9], [16] and [17] for discussion and references. There is some work that considers minimizing measures of regret for use with very limited (or non-existent) demand information. Kasugai and Kasegai [10, 11] only assumes demand to be in a known interval (see [23] for recent related work). Whereas this body of work generally consider measures of absolute regret, Yu, in [25], analyzes a robust EOQ model, in which he considers models that i) minimize total cost under all scenarios and ii) maximize the worst case performance ratio over all scenarios. The latter measure can be viewed as a relative regret measure similar to the one we consider. However, in addition to dealing with revenue maximization instead of cost minimization, the present work differs from such previous work on minimax distribution-free inventory models in two important respects. (i) This earlier work only considers single-product inventory systems, whereas the booking problems we consider originate with the existence of multiple products. (ii) We use a competitive analysis, relative to the offline optimum, while such earlier work is generally concerned with optimizing an absolute measure of performance.

While not directly related, we should note that the rapidly developing area of robust optimization (see for example, [5], [4]) has a similar philosophical underpinning to ours of optimizing in the presence of uncertainty where little or no distributional information is available.

Throughout this paper, let n denote the total number of units of capacity (seats, rooms, etc.) available. In the *discrete problem*, the demands are integer and each order must be entirely accepted or rejected. Thus, the protection levels may be restricted to being integer too. In the *continuous problem*, the demands may be any nonnegative real numbers and partial orders may be accepted. Thus, the protection levels may be any nonnegative real numbers as well. Although the continuous problem is less realistic, its analysis is simpler. Generally speaking, we are able to obtain policies for the discrete problem that closely approximate the continuous results.

In Section 1, we consider the case of two fare classes. Our exposition is detailed and aimed at a reader who is not necessarily familiar with the concepts and techniques of competitive analysis of online algorithms. We provide results in terms of r, the discount factor, or ratio of lower fare to higher fare. We define protection levels and show that the associated policies guarantee the best possible competitive ratio. Results are given for the continuous and discrete, deterministic and random cases.

In Section 2.1 we consider the case of m fare classes, with $m \geq 3$. All of the results of Section 1 for the continuous booking problem are extended to this case. We also treat order quantity controls in this section. In Section 2.2, we treat the bid-price control problem. For this problem requests arrive with arbitrary fare levels. We present a analysis of this case assuming fare requests are bounded by min and max fares; both the continuous and discrete cases are addressed. Finally, Section 3 considers dynamic policies. In this paper, we only analyze dynamic policies for the 2-fare case and suggest further pursuit of such approaches as a fertile area for future research.

1 Two fare classes

1.1 Derivation of Results

In the case of two fare classes, let $r = f_2/f_1$ denote the *discount ratio*, that is, the ratio of the discounted fare relative to the full fare. We naturally assume that $0 < r < 1$.

The following quantity will play an essential role in our analysis: let

$$b(r) = \frac{1}{2-r} \ . \qquad (1)$$

We first consider the continuous case.

Proposition 1 *For the continuous two-fare booking problem, the booking policy with protection level $\theta_1 = (1 - b(r))\, n$ has competitive ratio $b(r)$.*

Proof.

Consider any demand instance I; after applying the booking policy, let let q' be the total number of orders accepted, ℓ' the total number of low fare orders accepted and v' the total value of all orders accepted. Let v^* be the value achieved by the application of an optimal off-line policy.

Case I: $q' = n$. In this case,

$$\begin{aligned} v' &= \ell' f_2 + (n - \ell') f_1 \\ &\geq nb(r) f_2 + n(1 - b(r)) f_1 \end{aligned}$$

It is always the case that $v^* \leq n f_1$ so we have,

$$\begin{aligned} \frac{v'}{v^*} &\geq \frac{nb(r)f_2 + n(1-b(r))f_1}{nf_1} \\ &= \frac{\frac{f_1}{2-r} + \frac{2f_1 - rf_1 - f_1}{2-r}}{f_1} \\ &= \frac{\frac{f_1}{2-r}}{f_1} = b(r) \end{aligned}$$

Case IIa: $q' < n, \ell' < nb(r)$. In this case all orders offered are accepted and $v' = v^*$

Case IIb: $q' < n, \ell' = nb(r)$. In this case, $v' = nb(r)f_2 + (q' - \ell')f_1$. Now since no high value orders could be rejected we have: $v^* \leq (q' - \ell')f_1 + (n - (q' - \ell'))f_2$. Thus we have,

$$\frac{v'}{v^*} \geq \frac{nb(r)f_2 + (q' - \ell')f_1}{(q' - \ell')f_1 + (n - (q' - \ell'))f_2} \geq \frac{nb(r)f_2}{(n - (q' - \ell'))f_2} \geq \frac{nb(r)f_2}{nf_2} = b(r)$$

$\square$

We now show that the competitive ratio $b(r)$ of this booking policy is best possible. The following two "extreme instances" will play an essential role in obtaining upper bounds on competitive ratios for the 2-fare problem. Instance I_1 has n low-fare requests followed by n high-fare requests. Instance I_2 also has n low-fare requests but no high-fare request. The optimum off-line revenues are

$$R^*(I_1) = f_1\, n \qquad \text{and} \qquad R^*(I_2) = f_2\, n = f_1\, r\, n \ .$$

We first consider all deterministic booking policies.

Proposition 2 *For the continuous two-fare problem, no deterministic online booking policy, has a competitive ratio larger than $b(r)$.*

Proof.
Consider any deterministic algorithm A. Let x^A denote the number of low-fare requests accepted by A when first presented with a sequence of n low-fare requests. Then $0 \leq x^A \leq n$. The resulting revenue is $R(A, I_1) = f_2\, x^A + f_1(n - x^A)$ when applied to instance I_1, and $R(A, I_2) = f_2\, x^A$ when applied to instance I_2. If $x^A \geq b(r)\, n$ then

$$R(A, I_1) \leq f_1\big(n - (1-r)\, b(r)\, n\big) = f_1\, \frac{1}{2-r}\, n = b(r)\, R^*(I_1).$$

Else, $x^A < b(r)\, n$ and $R(A, I_2) < f_2\, b(r)\, n = b(r)\, R^*(I_2)$. This shows that no deterministic online algorithm can guarantee, for all instances, a revenue larger than $b(r)$ times the off-line optimum. $\square$

We now show that the competitive ratio $b(r)$ is best possible among *all* online booking policies, randomized as well as deterministic.

Proposition 3 *For the continuous two-fare problem, no booking policy, deterministic or randomized, has a competitive ratio larger than $b(r)$. This is true even when the set of instances is restricted to I_1 and I_2.*

Proof. Let $\mathcal{I}$ denote a set of instances of the continuous two-fare booking problem. For any $I \in \mathcal{I}$, let $R^*(I)$ denote the maximum revenue achievable by an off-line algorithm for this instance. Let $\mathcal{A}$ denote the set of all deterministic online algorithms for this instance class. For any deterministic algorithm $A \in \mathcal{A}$ let, as in proof of Proposition 2, $R(A, I)$ denote the revenue obtained when applying A to instance $I \in \mathcal{I}$.

Let $\mathcal{P}$ be the set of all probability distributions on $\mathcal{A}$. Any randomized algorithm may be viewed as a random choice $A(P)$ among deterministic algorithms, defined by some probability distribution $P \in \mathcal{P}$. Let $E_P[R(A(P), I)]$ denote its *expected* revenue when applied to instance I. We may interpret this situation as a zero-sum two-person game between a player choosing a randomized algorithm to maximize her expected competitive ratio and an *adversary* choosing a distribution of instances to minimize this expected ratio. The von Neuman/Yao principle (e.g., [19]) implies that the best possible competitive ratio c^* of any randomized algorithm satisfies

$$c^* = \sup_{P \in \mathcal{P}} \inf_{I \in \mathcal{I}} \frac{E_P[R(A(P), I)]}{R^*(I)} = \inf_{Q \in \mathcal{Q}} \sup_{A \in \mathcal{A}} E_Q\left[\frac{R(A, I(Q))}{R^*(I(Q))}\right]$$

where $\mathcal{Q}$ denotes the the set of all probability distributions on the instance set $\mathcal{I}$, and $I(Q)$ is a random instance chosen according to probability distribution $Q \in \mathcal{Q}$. The right hand side of this equality may be interpreted as as the adversary's problem of choosing a probability distribution of problem instances to force every deterministic algorithm to experience an expected competitive ratio at most c^*. In particular, for any given $Q \in \mathcal{Q}$ we have

$$c^* \leq \sup_{A \in \mathcal{A}} E_Q \left[\frac{R(A, I(Q))}{R^*(I(Q))} \right]. \tag{2}$$

For the continuous two-fare booking problem, we define such Q as follows: choose the two instances I_1 and I_2 with probabilities $b(r)$ and $1 - b(r)$, respectively. As in the proof of Proposition 2, let x^A denote the number of low-fare requests accepted by a deterministic algorithm A when presented a sequence of n low-fare requests. We have

$$c^* \leq \sup_{A \in \mathcal{A}} \left(b(r) \, \frac{R(A, I_1)}{R^*(I_1)} + (1 - b(r)) \frac{R(A, I_2)}{R^*(I_2)} \right) \leq \sup_{0 \leq x^A \leq n} \, \frac{1}{2 - r} = b(r).$$

This completes the proof. □

Putting Propositions 1 and 3 together, we obtain:

Theorem 1 *For the continuous two-fare booking problem, the booking policy with protection level $\theta_1 = (1 - b(r))\, n$ has competitive ratio $b(r)$. This is best possible among all online booking policies.*

We now turn to the discrete case, where the demands are restricted to integer values. If the protection level $(1 - b(r))n$ is integer, then all the preceding results apply. Otherwise, we may consider using the fractional value $(1 - b(r))n$ as protection level; namely, accepting at most $b(r)\, n$, or equivalently, at most $\lfloor b(r)\, n \rfloor$, low-fare requests. However (and in contrast with Littlewood's formula), this simple rounding does not necessarily define an optimum booking policy for the discrete case. To see this, note that Proposition 1 does not apply any more, since the revenue from accepted low-fare requests is only $f_2 \lfloor b(r)\, n \rfloor < f_2\, b(r)\, n$ when the protection level is attained. Then, as in the proof of Proposition 2, let x^A denote the maximum number of low-fare requests that are accepted by an algorithm A. We seek an $x^A \in \{0, 1, ..., n\}$ which maximizes the function

$$c(x^A) = \min \left\{ \frac{R(A, I_1)}{R^*(I_1)}, \; \frac{R(A, I_2)}{R^*(I_2)} \right\} = \min \left\{ \frac{n - (1 - r)\, x^A}{n}, \; \frac{x^A}{n} \right\}. \tag{3}$$

Since this single-variable function $c(x^A)$ is concave and attains its (continuous) maximum when $x^A = b(r)\, n$, let

$$\theta_D(r, n) = \begin{cases} n - \lfloor b(r)\, n \rfloor & \text{if } c\left(\lfloor b(r)\, n \rfloor\right) \geq c\left(\lceil b(r)\, n \rceil\right); \\ n - \lceil b(r)\, n \rceil & \text{otherwise.} \end{cases} \tag{4}$$

(The subscript D refers to the Discrete case.) Note that $c(n - \theta_D(r, n)) \leq b(r)$, with equality if and only if $\theta_D(r, n) = (1 - b(r))\, n$, corresponding to the continuous maximum. The following theorem shows that the *optimum rounding* in (4) defines an optimum deterministic booking policy for the discrete 2-fare problem.

Theorem 2 *For the discrete two-fare booking problem, the booking policy with protection level $\theta_1 = \theta_D(r,n)$ defined above has competitive ratio $c(n-\theta_D(r,n))$. This is best possible among all deterministic online booking policies.*

The proof of this theorem uses the same ideas as above, but it is more technical. It is found in the Appendix.

In contrast with the continuous problem, there *is* some advantage in using a randomized policy for the discrete problem, when the continuous protection level $(1-b(r))\,n$ is fractional:

Proposition 4 *For the discrete two-fare problem, there exists a randomized booking policy with competitive ratio $b(r)$, and this is best possible.*

Proof. Consider the following randomized booking policy $A^{0,n}$: choose a protection level $\theta = 0$ with probability $b(r)$, and $\theta = n$ with probability $1-b(r)$. As in earlier proofs, consider any instance I with ℓ low-fare requests and h high-fare requests, and assume w.l.o.g. that $0 \le \ell \le n$ and $0 \le h \le n$. Applying policy $A^{0,n}$ to this instance, the resulting revenue is $R(A^{0,n}, I|\theta = 0) = f_2\ell + f_1 \min\{h, n-\ell\}$ when $\theta = 0$, and $R(A^{0,n}, I|\theta = n) = f_1 h$ when $\theta = n$. If $\ell + h \ge n$ then the optimum revenue is $R^*(I) = f_1 h + f_2\,(n-h)$; that from booking policy $A^{0,n}$ is $R(A^{0,n}, I|\theta = 0) = f_2\ell + f_1\,(n-\ell)$ when $\theta = 0$, and its expected revenue is thus

$$
\begin{aligned}
E[R(A^{0,n}, I)] &= b(r)\,R(A^{0,n}, I|\theta = 0) + (1-b(r))R(A^{0,n}, I|\theta = n) \\
&= f_1\,\frac{n + (1-r)h - (1-r)\ell}{2-r} \\
&\ge f_1\,\frac{n + (1-r)h - (1-r)n}{2-r} \quad = \quad b(r)\,R^*(I)\ .
\end{aligned}
$$

Otherwise, $\ell + h < n$ and the optimum revenue is $R^*(I) = f_1 h + f_2\ell$; that from booking policy $A^{0,n}$ is $R(A^{0,n}, I|\theta = 0) = f_2\ell + f_1 h = R*(I)$ when $\theta = 0$, and its expected revenue is thus

$$
\begin{aligned}
E[R(A^{0,n}, I)] &= f_1\,\left(b(r)(r\,\ell + h) + (1-b(r))h\right) \;=\; f_1\,\left(h + b(r)\,r\,\ell\right) \\
&\ge f_1\,\left(b(r)\,h + b(r)\,r\,\ell\right) \;=\; b(r)\,R^*(I).
\end{aligned}
$$

Thus $E[R(A^{0,n}, I)] \ge b(r)\,R^*(I)$ for all instances I. Proposition 3 implies that this is best possible for the discrete 2-fare problem. $\square$

The preceding proof is based on one particular randomized policy which is easy to analyze. The next theorem gives a complete characterization of *all* optimal randomized booking policies for the discrete 2-fare problem.

Theorem 3 *For the discrete 2-fare booking problem, a randomized booking policy has optimal competitive ratio $b(r)$ if and only if the expected value of its protection level is $(1-b(r))\,n$.*

The proof of this Theorem is technical and is found in the Appendix.

1.2 Comparison with Other Policies

We now wish to put these results in perspective by comparing them with some alternative policies: first-come, first-served (FCFS) and partitioned protection levels (PPL). We will refer to the policies analyzed in the previous section as nested protection levels (NPL). FCFS is the policy of simply accepting orders in their arrival sequence until the capacity has been reached. PPL creates protection levels, like NPL, however, the buckets set up for each fare class are exclusively devoted to that fare class. Thus, in the 2-fare case, under PPL, if we protect θ_1 high fare seats, then the number protected (exclusively) for low fare orders is $\theta_2 = n - \theta_1$. When confronted with a stream of exactly n high fare orders, PPL would accept θ_1 high fare orders, while NPL would accept all n orders.

The performance of FCFS can easily be derived:

Proposition 5 *For the continuous 2-fare booking problem, FCFS has a competitive ratio of r.*

Proof. It is clear that the worst case performance of FCFS is achieved by a stream of n low-fare orders followed by n high-fare orders. □

While a stream of n high-fare orders did not play a role in the analysis of NPL, as was illustrated above it plays a key role in the analysis and worst case performance of PPL.

Proposition 6 *For the continuous 2-fare booking problem, the PPL policy with protection level $\theta_1 = \frac{1}{2}n$ achieves a competitive ratio of $\frac{1}{2}$ and this is the best possible for a PPL policy.*

Proof. Consider the input streams I_1, consisting of n high-fare orders, and I_2, consisting of n low-fare orders; their off-line optimal revenues are $R^*(I_1) = f_1 n$ and $R^*(I_2) = f_2 n$, respectively. Now, consider an arbitrary PPL policy A_θ, with $\theta = (\theta_1, \theta_2)$. The resulting revenue, when applied to I_1, is $R(A, I_1) = f_1\theta_1$ and, when applied to I_2, is $R(A, I_2) = f_2\theta_2$. For $\theta_2 \leq \frac{1}{2}n$, $R_2(A, I_2) \leq \frac{1}{2}nf_2 = \frac{1}{2}R^*(I_2)$ and for $\theta_2 > \frac{1}{2}n$, $R_1(A, I_1) < \frac{1}{2}nf_1 = \frac{1}{2}R^*(I_1)$. This shows that no PPL can achieve a performance level greater than $\frac{1}{2}$.

Now it is also clear that the PPL with $\theta_2 = \theta_1 = \frac{1}{2}n$, achieves a competitive ratio of $\frac{1}{2}$ on I_1 and I_2 and in fact achieves at least this performance level over all possible inputs. □

Figure 1 illustrates the performance of the three policies we have discussed, i.e. FCFS, the best NPL, ($\theta_1 = (1 - b(r))n$) and the best PPL ($\theta_1 = \frac{1}{2}n$). As is implied by the results in the previous section, NPL is uniformly better than either FCFS or PPL. This graph also suggests a hybrid policy based on FCFS and PPL, namely to use PPL for $r \leq \frac{1}{2}$ and to use FCFS for $r > \frac{1}{2}$. This dominates each one individually and can be viewed as a piecewise linear envelope of NPL.

It is also interesting to note that NPL and, as will be shown in the next section, its order quantity control version, both achieve the best possible competitive ratio, whereas the best PPL does not. This indicates that relative to a competitive ratio measure nesting is optimal. Of course, nested booking limits are widely used in practice and can be shown to be optimal under certain other models as well [13], [12]. On the other hand, there are consumer behavior regimes when nesting by fare order might not be optimal. This issue is explored in [20].

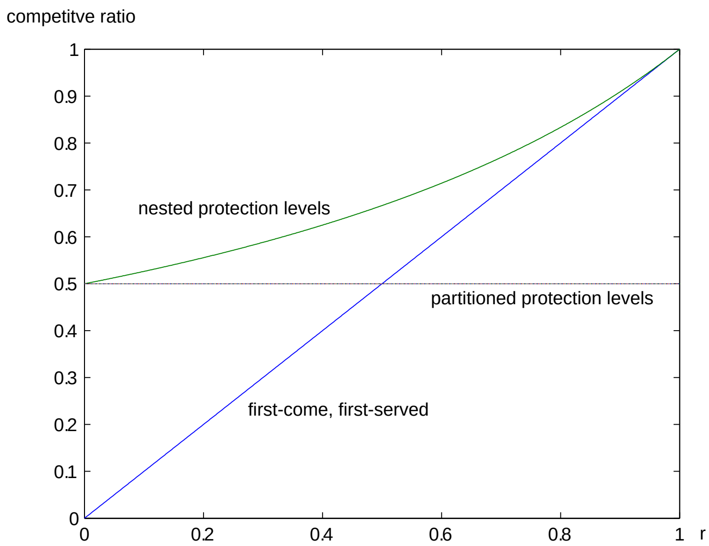

Figure 1: Comparison of first-come, first-served, nested booking limits and partitioned booking limits

2 Multiple Fare Classes and Bid Price Controls

We now consider the case of m fare classes, with $m \geq 3$ and *bid price control* policies. Under bid price controls requests are no longer selected from among a set of pre-determined fares. Instead, each request comes with a proposed fare, which may be determined, for example, as part of a package of multiple goods (e.g., the different flights on an itinerary), or as the result of decisions by the customer or a third party (e.g., a travel agent). We assume that each request specifies a fare f which can be any (real) number between a maximum fare $f_{\max}$ (usually known as the "regular fare") and a minimum fare $f_{\min}$ (the "most discounted fare") below which no request will even be considered. We refer to [7], [21], for discussions of bid price versus fare class control.

For either the bid price case or multiple discrete fare case, we define protection levels using a function $\theta(f)$, which is non-increasing in f. A policy that implements protection levels, $\theta(f)$, will insure, for any f, that the total quantity of all orders accepted with price f or lower is no more than $Q(f) = n - \theta(f)$. We call $Q(f)$ a *protection level control function*. Note that $Q(f)$ is non-decreasing in f. We also will apply our analysis for order quantity control policies and define an associated *order quantity control function*, $P(q)$. Under an order quantity control policy, an order with fare f that brings the total quantity of accepted orders to q, can only be accepted if $f \geq P(q)$. In either case for the continuous version of the booking problems, in order to enforce these conditions, in addition to rejecting orders outright, we may be forced to accept partial orders.

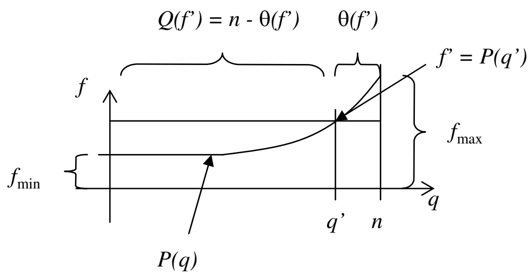

Figure 2: Relationship among $\theta(f), Q(f)$ and $P(q)$ for the bid price case

Given $Q(f)$, there will always be a corresponding $P(q)$, which informally should be its inverse. However, since $Q(f)$ is not necessarily a one-to-one function, we define the associated $P(q)$ for $0 \leq q \leq n$, as:

$$P(q) = \min\{f : Q(f) \geq q\}.$$

Note that the following property follows directly from the definition:

$$P(Q(f)) \leq f \quad \text{for all } f_{\min} \leq f \leq f_{\max} \tag{5}$$

Figure 2 illustrates these definitions for the bid price control case and Figure 3 for the multiple discrete fare class case.

Given a $\theta(f)$, we will now specifically define A_θ, the *protection level control* policy based on the associated $Q(f)$, and A'_θ, the *order quantity control* policy based on the associated P. Each policy takes as input s orders, which are characterized by a sequence, $O_1, O_2, ..., O_s$, of offered prices. A continuous policy sets values to variables $0 \leq x_i \leq 1$, which indicate the fraction of order i accepted. A discrete policy either accepts or rejects each order by setting values to the variables $x_i \in \{0, 1\}$. An order acceptance policy proceeds through s iterations, where at iteration $k+1$, the value of x_{k+1} is set. When considering order $k+1$, k orders have been processed and so we may define:

$$\begin{aligned} \hat{q} &= \sum_{i=1}^{k} x_i \\ \hat{q}(f) &= \sum_{i:1 \leq i \leq k; O_i \leq f} x_i \end{aligned}$$

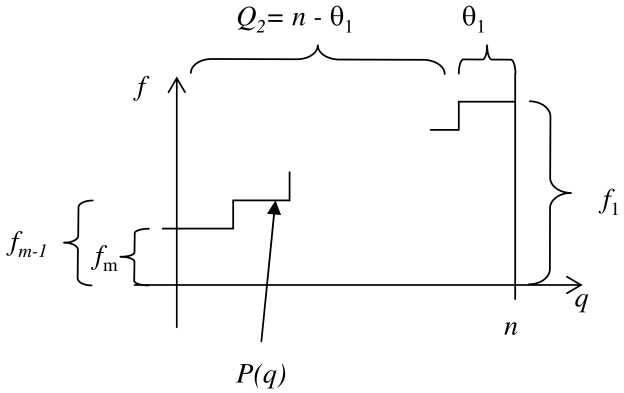

Figure 3: Relationship among $\theta(f), Q(f)$ and $P(q)$ for multiple fare class case case

i.e. $\hat{q}$ is the total quantity of orders accepted so far and $\hat{q}$ is the total quantity of orders accepted with fare $\leq f$. We can now define:

A_θ – **Protection Level Control:**

$$x_{k+1} = \sup\{\alpha : \alpha + \hat{q}(f) \leq Q(f) \text{ for all } f \geq O_{k+1}; 0 \leq \alpha \leq 1\}$$

A'_θ – **Order Quantity Control:** If $O_{k+1} < P(\hat{q})$ then set $x_{k+1} = 0$; otherwise,

$$x_{k+1} = \sup\{\alpha : P\left(\alpha + \hat{q}\right) \leq O_{k+1}; 0 \leq \alpha \leq 1\}$$

For the discrete versions we replace $0 \leq \alpha \leq 1$ in each of these expressions with $\alpha \in \{0, 1\}$. We note that the protection level control policy, as stated, cannot be directly implemented for the continuous case, as a constraint must be checked over a continuum of values. However, efficient implementations of such policies exist.

In the next two sections we treat the multiple discrete fare class and bid price control cases respectively.

2.1 Multiple fare classes

In considering the case of multiple fare classes, we let f_i denote the fare for class i ($i = 1, \ldots, m$), where $f_1 > f_2 > \cdots > f_m > 0$. Because of the complexities of the case for $m \geq 3$, we only consider the continuous case. Define the following quantity

$$\Delta = m - \sum_{i=2}^{m} \frac{f_i}{f_{i-1}} \ . \tag{6}$$

In deriving a bound on the best possible performance for an online algorithm, we use the following m "extreme instances" I_i $(i = 1, ..., m)$: in instance I_i a sequence of $(m - i + 1)\, n$ requests, namely, n requests for each fare class $m, m-1, \ldots, i$, arrive in this order. The optimum off-line revenues are $R^*(I_i) = f_i\, n$, for all $i = 1, ..., m$.

Theorem 4 *For the continuous m-fare problem, no booking policy, deterministic or randomized, has a competitive ratio larger than* $1/\Delta$.

Proof. Using the m instances I_i $(i = 1, ..., m)$ described above, we construct a random instance $\hat{I}$ by choosing instance I_1 with probability $p_1 = 1/\Delta$, and each other instance I_i with probability $p_i = (1 - f_i/f_{i-1})/\Delta$. Note that, by definition of Δ we have $\sum_{i=1}^{m} p_i = 1$.

Let $\mathcal{A}$ denote the set of all deterministic algorithms for this m-fare problem. For any $A \in \mathcal{A}$ and $i = 1, ..., m$, let x_i^A denote the number of requests in class i accepted by algorithm A when presented with instance I_1. Letting $X = \{x \in \mathbb{R}^m : x \geq 0 \text{ and } \sum_{i=1}^{m} x_i \leq n\}$, we have $x^A \in X$. Note that A has no way of knowing whether it is facing instance I_i or some I_j with $j < i$ (or some other instance) before it has seen the first $(m - i + 1)\, n + 1$ requests in the stated sequence. Therefore, for all $i \leq j \leq m$, A will accept x_j^A class-j requests when presented with instance I_i . Thus the revenue earned by applying algorithm A to these instances is $R(A, I_i) = \sum_{j=i}^{m} f_j x_j^A$, for all $i = 1, ..., m$.

By inequality (2), the competitive ratio c^* of any (deterministic or randomized) booking policy satisfies

$$c^* \leq \sup_{A \in \mathcal{A}} \sum_{i=1}^{m} p_i \frac{\sum_{j=i}^{m} f_j x_j^A}{f_i\, n} \leq \sup_{x^A \in P} \sum_{j=1}^{m} \left(\sum_{i=1}^{j} p_i \frac{f_j}{f_i} \right) \frac{x_j^A}{n} = \sup_{x^A \in P} \sum_{j=1}^{m} \frac{1}{\Delta} \frac{x_j^A}{n} = \frac{1}{\Delta} \ .$$

This completes the proof. □

We now construct a deterministic booking policy that achieves the optimum competitive ratio $1/\Delta$. Let $\theta_i = \theta(f_i)$ be the (nested) protection level for class $i = 1, \ldots, m$. Thus we let $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{m-1} \leq \theta_m = n$. We determine the θ_i with reference to the m instances $I_1, \ldots, I_m$ defined above, aiming to maximize the minimum, over these m instances, of the ratios $R(A_\theta, I_i)/R^*(I_i)$, where $R(A_\theta, I_i) = \sum_{j=i}^{m} f_j(\theta_j - \theta_{j+1})$. As suggested by the fact that all the corresponding probabilities in the proof of Theorem 4 are positive, we let all these ratios be equal. The resulting system of $m - 1$ linear equations in $m - 1$ unknowns has determinant equal to Δ defined in equation (6). Its solution is

$$\theta_i = \frac{n}{\Delta} \left(i - \sum_{j=1}^{i} \frac{f_{j+1}}{f_j} \right) \quad \text{for } i = 1, \ldots, m-1 \tag{7}$$

with, as indicated above, $\theta_m = n$. The associated protection level controls are:

$$Q_i = n - \theta_{i-1} \text{ for } i = 1, \ldots, m+1 \quad (8)$$
$$Q_1 = n \quad (9)$$

The associated order quantity control function is given by:

$$P(q) = f_i \text{ for } Q_{i+1} < q \leq Q_i \text{ and } 1 \leq i \leq m$$
$$0 \text{ for } q = Q_{m+1} = 0$$

Before proving that these policies achieve the best possible bound, we need the following Lemma.

Lemma 1 *If θ and Q are defined by (7),(8) and (9) then*

$$f_i(Q_i - Q_{i+1}) = \frac{n}{\Delta}(f_i - f_{i+1}) \text{ for } i = 1, \ldots, m-1 \quad (10)$$
$$f_m(Q_m - Q_{m+1}) = \frac{f_m n}{\Delta} \quad (11)$$
$$\sum_{i=i'}^{m} f_i(Q_i - Q_{i+1}) = f_{i'}\frac{n}{\Delta} \text{ for } 1 \leq i' \leq m \quad (12)$$

The proof of this Lemma is provided in the Appendix.

We note in particular Equation (12), which expresses a fundamental relationship critical to the structure of the policies and their performance guarantees (see Figure (4)). Consider an "adversary" policy of sending Q_m orders with fare f_m, $Q_{m-1} - Q_m$ orders with fare f_{m-1}, ..., $Q_{i'+2} - Q_{i'+1}$ orders with fare $f_{i'+1}$ and finally n orders of fare $Q_{i'}$. Both policies A_θ and A'_θ yield a total revenue of $v' = \int_{q=0}^{Q_{i'}} P(q) = \sum_{i=i'}^{m} f_i(Q_i - Q_{i+1})$ and the off-line optimal is $v^* = nf_{i'}$. Equation (12) states that *for any* i', the performance achieved by the online policies is $v'/v^* = 1/\Delta$. The following theorem shows that both A_θ and A'_θ achieve this bound, which also implies these adversary policies are the best possible.

Theorem 5 *For the continuous m-fare problem, the booking policy A_θ and A'_θ with protection levels defined by equation (7) have competitive ratio* $1/\Delta$.

Proof.
Case I: Protection level policy A_θ

Consider an arbitrary demand instance I; after applying the booking policy, let let q'_i be the total number of orders accepted with fare less than or equal to f_i for $i = 1, 2, ..., m$ and let $q'_{m+1} = 0$. Define $i' = \min\{i : q'_i = Q_i, 1 \leq i \leq m\}$. Note that $q'_i < Q_i$ for $1 < i'$ so that no orders are rejected with whose fare $> f_{i'}$. Let v' be the total value of all orders accepted and v^* be the value achieved by the application of an optimal off-line policy.
Case Ia: $i' = m + 1$. In this case, no orders are rejected and $v' = v^*$.
Case Ib: $1 < i' \leq m$. Let Γ be the cumulative value of all orders accepted whose fares are greater than $q'_{i'}$ and let $\tilde{q}$ be the total quantity of such orders. Then we have that

$$\begin{aligned} v' &\geq \sum_{i=i'}^{m} f_i(Q_i - Q_{i+1}) + \Gamma \\ &= f_{i'}\frac{n}{\Delta} + \Gamma \end{aligned}$$

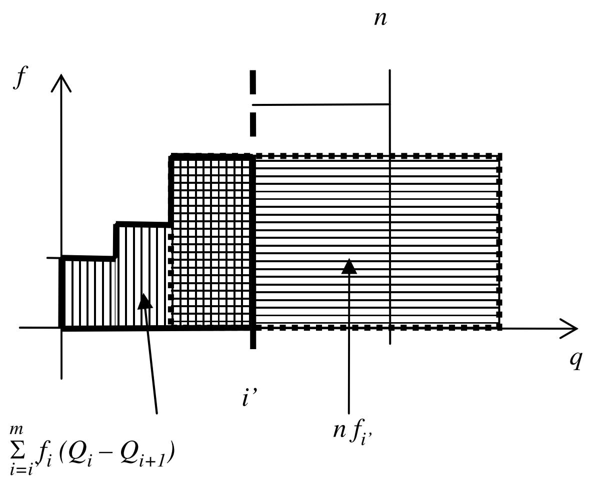

Figure 4: Quantities compared in Equation (12)

where the equation results from Lemma (1). Now since no fares with value greater than $f_{i'}$ could be rejected we have:

$$v^* \leq \Gamma + f_{i'}(n - \tilde{q})$$

Thus, we have,

$$\begin{aligned} v'/v^* &\geq \frac{f_{i'}\frac{n}{\Delta} + \Gamma}{\Gamma + f_{i'}(n - \tilde{q})} \\ &\geq \frac{f_{i'}\frac{n}{\Delta} + \Gamma}{\Gamma + f_{i'}n} \\ &\geq \frac{f_{i'}\frac{n}{\Delta}}{f_{i'}n} = \frac{1}{\Delta} \end{aligned}$$

Case Ic: $i' = 1$. Again using Lemma 1

$$v' \geq \sum_{i=1}^{m} f_i(Q_i - Q_{i+1}) = \sum_{i=1}^{m-1}(f_i - f_{i+1})\frac{n}{\Delta} + f_m\frac{n}{\Delta} = \frac{f_1 n}{\Delta}$$

Since $v^* \leq f_1 n$, we have,

$$v'/v^* \geq \frac{\frac{f_1 n}{\Delta}}{f_1 n} = 1/\Delta$$

Case II:

Order quantity control policy A'_θ. For this case, we define $\hat{q}$ to be the total quantity of all orders accepted and $i' = \min\{i : Q_i \leq \hat{q}\}$. As in Case I, we have that no orders were rejected whose fares are $> f_{i'}$. Further, as illustrated in Figure (5), we define Γ_1 to be the total value of all orders accepted after the total quantity accepted was $Q_{i'}$; for order accepted before the total quantity was $Q_{i'}$ we define Γ_2 to be the total value of the portion of those orders in excess of $f_{i'}$ (see Figure (5)). Now if we define $\Gamma = \Gamma_1 + \Gamma_2$, then we have a structure analogous to Case I, and we can apply the same arguments where i' and Γ are defined in this new manner. □

To illustrate the guaranteed performance levels provided by Theorem 5, we construct the following 3 m-fare scenarios. We let $f_1 = \$1,000$ in all cases. For the $m = 3$ case, we let $f_3 = r'f_1$, for the $m = 5$ case, we let $f_5 = r'f_1$ and for the $m = 10$ case, we let $f_{10} = r'f_1$. We then space the intermediate fares equally between the low and high fares, e.g. for $m = 5$, $f_3 = (f_1 + f_5)/2, f_2 = (f_1 + f_3)/2, f_4 = (f_3 + f_5)/2$. We then plot $1/\Delta$ as a function of r'. These graphs are given in Figure 6. Note that as m increases there is a degradation in the performance guarantee for lower values of r but for $r \geq 1/2$, the performance guarantee remains close in all cases.

2.2 Continuous Bid Price Control

We now treat the continuous case of bid price controls. In order to derive performance limits, we view such booking systems, as a limit of multiple fare class systems where the

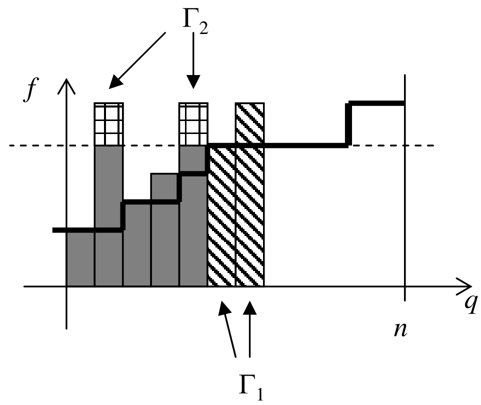

Figure 5: Definition of Γ_1 and Γ_2

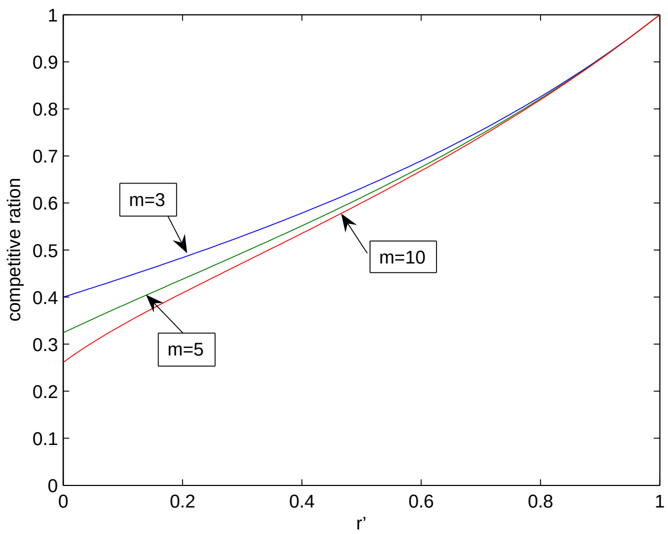

Figure 6: Illustration of $1/\Delta$ for evenly spaced fare classes for $m = 3, 5, 10$

number of classes goes to infinity and the fares f_i are dense (in the topological sense) in the $[f_{\min},\ f_{\max}]$ interval. In order to approximate such systems to arbitrarily high precision, it is convenient consider a sequence of m-fare systems ($m \geq 2$) with fares $f_{\max} = f_1^{(m)} > f_2^{(m)} > \cdots > f_m^{(m)} = f_{\min}$, such that the successive discount ratios $f_{i+1}^{(m)}/f_i^{(m)}$ are constant. Thus let $r = f_{\min}/f_{\max} < 1$ and, for $i = 1, \ldots, m$, let

$$f_i^{(m)} = f_{\max}\, r^{(i-1)/(m-1)} \qquad \text{for } i = 1, \ldots, m\,. \tag{13}$$

Applying Theorem 4 to such a problem, and taking the limit as m goes to infinity, we obtain:

Theorem 6 *For the bid price control problem with fare limits $f_{\max}$ and $f_{\min} = r\, f_{\max}$, no booking policy, deterministic or randomized, has a competitive ratio larger than $1/(1 - \ln r)$, where $\ln r$ is the natural logarithm of the maximum discount ratio r.*

Proof. By Theorem 4 applied to the m-fare problem with fares $f^{(m)}$ defined by equation (13), an upper bound on the competitive ratio of any algorithm for the m-fare problem is

$$c^{(m)} = \frac{1}{m - (m-1)\, r^{\frac{1}{m-1}}}\,.$$

Since every instance of this m-fare problem is an instance of the bid price control problem with fare interval $[f_{\min},\ f_{\max}]$, no bid price control policy can have a competitive ratio larger than $c^{(m)}$. Thus no bid price control policy can have a competitive ratio larger than

$$\liminf_{m\to\infty} c^{(m)} = \liminf_{m\to\infty} \frac{1}{m - (m-1)\, r^{\frac{1}{m-1}}} = \frac{1}{1 - \ln r}\,.$$

□

We use the same m-fare approximations to construct a bid price booking policy. By Theorem 5, the booking policy defined by protection levels

$$\theta^{(m)}(f_i) = i\, n\, c^{(m)} \left(1 - r^{\frac{1}{m-1}}\right) \qquad \text{for } i = 1, \ldots, m-1 \tag{14}$$

and $\theta^{(m)}(f_{\min}) = n$, achieves the competitive ratio $c^{(m)}$. By equation (13), the fare price f_i is an exponential function of the index i. Therefore the protection level $\theta^{(m)}(f_i)$ is a linear function of the logarithm $\ln f_i$ of the fare. Taking again the limit as m goes to infinity, we obtain the bid price booking policy determined by the (continuous) protection levels

$$\theta(f) = \frac{\ln f_{\max} - \ln f}{\ln f_{\max} - \ln f_{\min}} \left(1 - \frac{1}{1 - \ln r}\right) n \qquad \text{for all } f \text{ such that } f_{\min} \leq f \leq f_{\max}\,. \tag{15}$$

We now will show that, using this definition of $\theta(f)$, the two booking policies defined earlier achieve the best possible competitive performance ratio. First, we define a function that will be useful in developing our proofs. Given an order quantity control function, $P(q)$, the *value accumulation function* is defined for $0 \leq q \leq n$, as:

$$V(q) = \int_0^q P(s)ds$$

This functions tracks the minimum revenue accumulated by policies based on either $Q(f)$ or $P(q)$. Before proving the main result, we derive functional forms for $P(q)$ and $V(q)$ based on the definition of $\theta(f)$ given abvoe. First, define,

$$\rho = \frac{1}{1 - \ln r}$$

The required functional forms are given by:

Lemma 2 *If $\theta(f)$ is defined by (15), then,*

$$\begin{array}{rcl} P(q) = & f_{\min} & \textit{for } q < n\rho \\ & f_{\max} r^{(n-q)/(n(1-\rho))} & \textit{for } n\rho \leq q \leq n \\ V(q) = & q f_{\min} & \textit{for } q < n\rho \\ & \frac{nP(q)}{1-\ln r} & \textit{for } n\rho \leq q \leq n \end{array}$$

The proof is provided in the Appendix.

We can now derive the performance levels of the policies defined earlier.

Theorem 7 *For the continuous bid-price control problem, if $\theta(f)$ is defined by (15), then, the protection level control policy based on $Q(f)$ and the order quantity control policy based on $P(q)$ both achieve competitive ratios of at least $1/(1 - \ln r)$.*

Proof. We first prove the result for the order quantity control policy. Let q' be the total quantity accepted by the policy, v' the total value accumulated and v^*, the value that would be achieve by application of an optimal off-line algorithm. .
Case I: $q' < n\rho$. In this case, no orders are rejected so it must be the case that $v' = v^*$.
Case II: $n\rho \leq q' < n$. Figure 7 illustrates the construction we now describe. We associate with each order i, the interval $[\hat{q}, \hat{q} + x_i]$, where $\hat{q}$ is the total order quantity that had been accepted prior to the consideration of i. Note that $P(\hat{q} + x_i) \leq O_i$ and, since $P(q)$ is non-decreasing, O_i bounds $P(q)$ over the entire interval $[\hat{q}, \hat{q} + x_i]$. In fact, not only does O_i bound $P(q)$ over this interval but also $\min\{O_i, P(q')\}$ bounds $P(q)$ over this interval since $P(q)$ is non-decreasing. Using this association with intervals, we can integrate the function $\min\{O_i, P(q')\}$ over the interval $[0, q']$ and see that this integral dominates $\int_0^{q'} P(s)ds = V(q')$. If we define $\Gamma = \sum_i \max\{O_i - P(q'), 0\} x_i$, we have that,

$$v' = \sum_i O_i x_i = \sum_i \min\{O_i, P(q')\} x_i + \Gamma \geq V(q') + \Gamma$$

We now derive a bound on v^*. Since no order or portion of an order with $O_i > P(q')$ could be rejected, it follows that:

$$v^* \leq nP(q') + \Gamma$$

We now have that:

$$v'/v^* \geq \frac{V(q') + \Gamma}{nP(q') + \Gamma} \geq \frac{V(q')}{nP(q')} = \frac{1}{1 - \ln r}$$

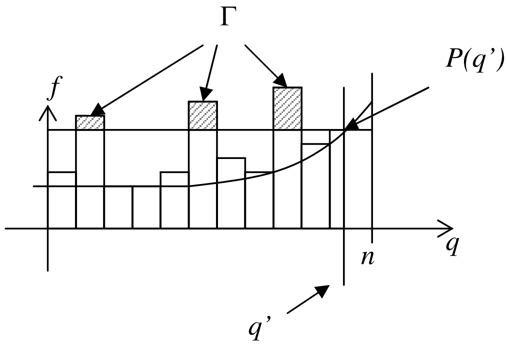

Figure 7: Illustration of Proof: each column represents and accepted order; the width equals x_i and the height O_i

Case III: $q' = n$. Using arguments similar to those used for Case II, it is easy to see that $v' \geq V(n)$. Clearly, $v^* \leq n f_{max}$ so we have:

$$v'/v^* \geq \frac{V(n)}{n f_{max}} = \frac{1}{1 - \ln r}$$

We now consider the protection level control case and define as before, q', v' and v^*. We also consider the same three cases.
Case I: $q' < n\rho$. In the case, no protection level control constraints can be tight so that no orders could have been rejected so $v' = v^*$.
Case II: $n\rho \leq q' < n$. We associate with each order i, the interval $[Q(O_i) - x_i, Q(O_i)]$. By (5), $P(Q(O_i)) \leq O_i$ and, since $P(q)$ is non-decreasing, O_i dominates $P(q)$ over the entire interval. Note that the total size of all intervals is q', that some may overlap and that they are not necessarily contiguous. We claim that there is a mapping from these intervals to a dense set of intervals that exactly covers $[0, q']$. The mapping can be obtained by iteratively finding an interval $[Q(O_{i1}) - x_{i1}, Q(O_{i1})]$ that has a non-zero intersection with another interval and that has the minimum value of $Q(O_{i1})$. Let $[Q(O_{i2}) - x_{i2}, Q(O_{i2})]$ be one of the intersecting intervals. It must be the case that $Q(O_{i2}) \geq Q(O_{i1})$. Now since the protection level control policy will insure that $\sum_{k:O_k \leq O_{i2}} x_k \leq Q(O_{i2})$, it must be the case that there is "room" to the left of $[Q(O_{i1}) - x_{i1}, Q(O_{i1})]$ to move the portion of the interval $[Q(O_{i1}) - x_{i1}, Q(O_{i1})]$ that intersects with the interval $[Q(O_{i2}) - x_{i2}, Q(O_{i2})]$. This movement represents the mapping that will remove the intersection. Furthermore, under such a mapping, O_{i1} will continue to

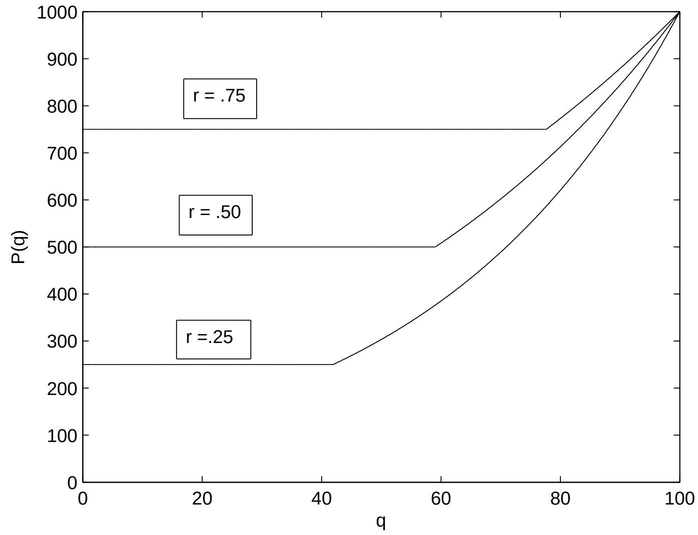

Figure 8: Graphs of $P(q)$ for $r = .25, .50, .75$

dominate $P(q)$. The procedure then iterates by finding the next "leftmost" interval with an intersection and again applies the mapping. Once such a complete mapping is constructed, the arguments used previously can be applied to prove this case.
Case III: $q' = n$. We can define a mapping as in Case II and use arguments similar to those used in the previous Case III to prove this result. $\square$

Figure 8 plots $P(q)$ for three values of r where $f_{max} = 1000$ and $n = 100$. Since, in the order quantity control case, this gives the minimum fare order accepted, it also can be viewed as a lower envelope of the distribution of orders accepted under either an order quantity control policy or a standard protection level control policy.

2.3 Discrete Bid Price Control

We now present, for the discrete bid price control problem, an order quantity policy which approaches the above continuous bid-price booking policy. In this case we start by defining a $P(q)$ and the associated order quantity control policy and then define $Q(f)$ and the associated protection level control. Given the maximum and minimum bid prices $f_{\max} > f_{\min} > 0$ and the total number n of seats for sale, this policy depends on a parameter ρ satisfying $0 < \rho < 1/(1 - \ln r)$, where, as above, $r = f_{\min}/f_{\max}$. In this case, we define $P_\rho(q)$, as:

$$P_\rho(q) = \begin{cases} f_{\min} & \text{for } q \le \lceil \rho\, n \rceil - 1 \quad (16) \\ f_{\min}(1 + \frac{1}{\rho n})^{\lceil q - \rho n - 1 \rceil} & \text{for } q \ge \rho\, n \quad (17) \end{cases}$$

Note that this policy unconditionally accepts the first $\lceil \rho\, n \rceil$ orders and thereafter accepts order i if $O_i \ge f_{\min}(1 + \frac{1}{\rho n})^{\lceil q - \rho n - 1 \rceil}$.

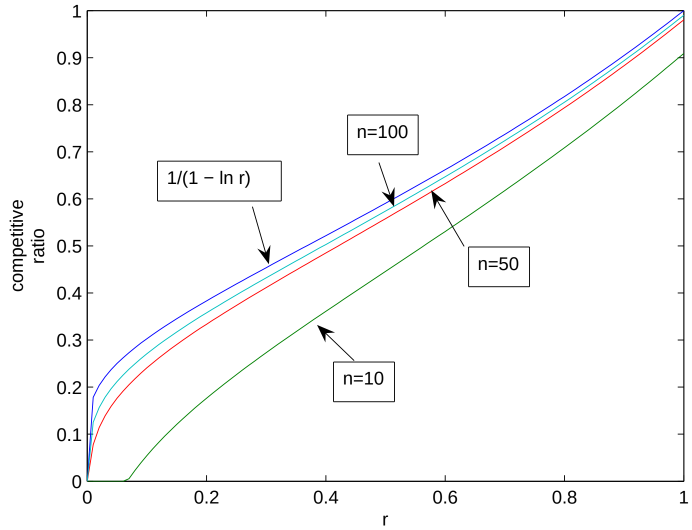

Figure 9: Comparison of Performance Guarantees for the Continuous Case (Theorem 7) and the Discrete Case (Theorem 8) for several values of n

In order to analyze the competitive ratio of the order control policy based on $P_\rho(q)$, we recall the following fact from elementary calculus (e.g. see [24]): the function $G : \mathbb{R}_+ \mapsto \mathbb{R}$ defined by $G(x) = (1 + 1/x)^x$ is increasing from $1 = \lim_{x\to 0} G(x)$ to $e = \exp(1) = \lim_{x\to\infty} G(x)$ as its variable x increases from 0 to $+\infty$. Thus let $G^{(-1)} : [1, e) \mapsto \mathbb{R}_+$ denote its functional inverse, that is, where $G^{(-1)}(1) = 0$ and, for $1 < a < e$, $x = G^{(-1)}(a)$ is the unique solution to the (transcendental) equation $G(x) = a$.

Theorem 8 *For any desired performance level ρ satisfying $0 < \rho < 1/(1 - \ln r)$ and such that the seat capacity*

$$n \geq \max\left\{\frac{1}{1-\rho},\ \frac{1}{\rho}\, G^{(-1)}\left(r^{\frac{-\rho}{1-\rho}}\right)\right\}, \tag{18}$$

the competitive ratio of the order control policy based on $P_\rho(q)$ is at least

$$\rho\left(1 - \frac{1}{\rho(\rho n + 1)}\right). \tag{19}$$

Figure 9 compares the bounds for the discrete case (based on Theorem 8) with the bounds for the continuous case (based on Theorem 7). Illustrated are the Theorem 7 bound, $1/(1 - \ln r)$ and the Theorem 8 bounds for several values of n. For the Theorem 8 bounds we use $\max\{0, \rho\left(1 - \frac{1}{\rho(\rho n+1)}\right)\}$ where $\rho = 1/(1 - \ln r)$. The max is taken because the bound given by expression in the Theorem can be negative for small values of r. Note that for moderate to higher values of n the Theorem 8 bounds are quite close to the continuous guarantee.

Given an order control policy based on a function P_ρ, we can define the corresponding protection level control policy by defining $Q_\rho(f)$ as the maximum quantity of orders with price $\leq f$ that can be accepted under the policy based on P_ρ. Applying this principle, it is easy to see that Q_ρ function corresponding to the $P(q)$ defined in (16),(17), is given by:

$$Q_\rho(f) = \lceil \rho n \rceil + i \quad \text{for } f_{\min}(1+\frac{1}{\rho n})^i \leq f < f_{\min}(1+\frac{1}{\rho n})^{i+1}; i = 0, ..., n - \lceil \rho n \rceil - 1 \quad (20)$$

$$n \quad \text{for } f \geq f_{\min}(1+\frac{1}{\rho n})^{n-\lceil \rho n \rceil} \quad (21)$$

Corollary 1 *If Q_ρ is defined by (20) and (21) and if ρ and n satisfy the conditions of Theorem 8, then the protection level control policy based on Q_ρ has competitive ratio at least*

$$\rho \left(1 - \frac{1}{\rho(\rho n + 1)}\right).$$

Proof. Suppose that the policy is applied to an arbitrary input sequence I and let q'_i be the number of orders, k, accepted with $O_k < f_{\min}(1+\frac{1}{\rho n})^{i+1}$. If $q'_i < \lceil \rho n \rceil + i$, for all i, then it must be the case that no orders have been rejected and the policy has generated the off-line optimum. If this is not the case, let i' be the maximum value of i such that $q'_i = \lceil \rho n \rceil + i$. Now associate each accepted order k with the interval $[i, i+1]$ where $f_{\min}(1+\frac{1}{\rho n})^i \leq O_k < f_{\min}(1+\frac{1}{\rho n})^{i+1}$. It must be the case that that total width of all such intervals with $i \leq i'$ must equal $q'_{i'}$. Using arguments similar to those used in the proof of Theorem 7, it can be shown that there is a mapping from these intervals onto the interval $[0, i'+1]$ that exactly covers $[0, i'+1]$ such that for each order k, O_k is greater than or equal to $P_\rho(q)$ over the interval to which order k has been mapped. The ordered sequence of these intervals represents a valid input to the corresponding order quantity control policy. Thus, the results of the theorem apply. □

3 Dynamic Policies

We now consider the development of an order control policy that dynamically adjusts to the set of orders previously offered at any point in time. Dynamic policies adjust to input order sequences that do not represent an optimal strategy for the adversary. In such cases, the policy's parameters may be adjusted so as to guarantee a higher performance level. Of course, these policies will not be able to achieve an a-priori worst case performance better than those described in the previous sections, however, in practice, when presented with actual order sequences their performance has the potential to be much greater.

We feel that this topic area is quite rich and will explore it more thoroughly in future papers. Here we only consider the simple two-fare case. Assume that n, f_1, f_2, r and $b(r)$ are defined as before. Suppose that a policy starts off as before accepting low fare requests. The static 2-fare policy was determined by a protection level, which could be converted into a maximum number of low fare orders to accept, $\hat{\ell}$, which was set to $b(r)n$. After each order is considered, a dynamic policy computes $\hat{\ell}'$, a revision of $\hat{\ell}$ based on the the entire order stream seen so far. Define:

$$\begin{aligned} h' &= \text{the number of high fare requests accepted so far} \\ \gamma &= h'/n \\ \alpha &= \frac{\gamma f_1 + (1-\gamma) f_2}{f_2} = \frac{\gamma}{r} + (1-\gamma) \end{aligned}$$

Using logic similar to that used in Proposition 1, the following expression gives the performance of a policy based on $\hat{\ell}'$ where all remaining requests are low fare:

$$\frac{n\hat{\ell}' f_2 + h' f_1}{h' f_1 + (n-h') f_2} = \frac{n\hat{\ell}' f_2 + n\gamma f_1}{n\alpha f_2}$$

The reduction in the denominator follows by noting that:

$$\begin{aligned} n\alpha f_2 &= \frac{\gamma f_1 + (1-\gamma) f_2}{f_2} n f_2 \\ &= n\left(\frac{h'}{n} f_1 + (1-\frac{h'}{n}) f_2\right) = h' f_1 + (n-h') f_2 \end{aligned}$$

On the other hand, the performance of a policy based on $\hat{\ell}'$ relative to a stream $n\hat{\ell}'$ low fare requests followed by at least n high fare requests would be:

$$\frac{n\hat{\ell}' f_2 + n(1-\hat{\ell}') f_1}{n f_1}$$

As was done earlier, we can maximize the competitive ratio by equating the performance levels for these two cases, i.e.

$$\begin{aligned} & \frac{n\hat{\ell}' f_2 + n\gamma f_1}{n\alpha f_2} = \frac{n\hat{\ell}' f_2 + n(1-\hat{\ell}') f_1}{n f_1} \\ \Rightarrow \quad & \frac{\hat{\ell}'}{\alpha} + \frac{\gamma}{\alpha r} = \hat{\ell}' r + (1-\hat{\ell}') \\ \Rightarrow \quad & \hat{\ell}' \left(1 + \frac{1}{\alpha} - r\right) = 1 - \frac{\gamma}{\alpha r} \\ \Rightarrow \quad & \hat{\ell}' = \frac{1-\frac{\gamma}{\alpha r}}{1+\frac{1}{\alpha}-r} \end{aligned}$$

We can now substitute this value into the expression for the performance level for the all-low case to get the performance level achieved by a $\hat{\ell}'$ policy:

$$\begin{aligned} & \frac{n\frac{1-\frac{\gamma}{\alpha r}}{1+\frac{1}{\alpha}-r} f_2 + n\gamma f_1}{n\alpha f_2} \\ = \quad & \frac{1-\frac{\gamma}{\alpha r}}{\alpha+1-\alpha r} + \frac{\gamma}{r\alpha} \\ = \quad & \frac{1-\frac{\gamma}{\alpha r}+\frac{\gamma}{r}+\frac{\gamma}{\alpha r}-\gamma}{1+\alpha(1-r)} \\ = \quad & \frac{1-\gamma+\frac{\gamma}{r}}{1+\alpha(1-r)} \\ = \quad & \frac{1+\gamma(\frac{1}{r}-1)}{1+\alpha(1-r)} \\ = \quad & \frac{1+\frac{\gamma}{r}(1-r)}{1+\alpha(1-r)} \end{aligned}$$

h'	$\hat{\ell}'$	dynamic perf guar	$\hat{\ell}$	static perf guar
1	65.78	.67	66.67	.67
5	62.30	.69	66.67	.67
10	58.06	.71	66.67	.67
20	50.00	.75	66.67	.67
30	42.42	.79	66.67	.67
40	35.29	.82	66.67	.67
50	28.57	.86	66.67	.67

Table 1: Comparison and Static and Dynamic Policies

We note that the $\hat{\ell}'$ produced by this analysis could be less than ℓ' the number of low fare requests already accepted. In this case the policy cannot be implemented; rather, no more low fare requests should be accepted. In such a case the worst case performance coincides with a final stream of high fare requests and the performance guarantee would be:

$$\frac{f_1(n-\ell')+f_2\ell'}{nf_1}$$

Dividing the numerator and denominator by f_1 yields:

$$\frac{(n-\ell')+f_2/f_1\ell'}{n}$$

$$= \frac{(n-\ell')+r\ell'}{n}$$

To illustrate the impact of a dynamic policy consider the case where $n = 100$ and $r = .5$. The Table 1 provides the values for $\hat{\ell}'$ and the performance guarantees for various values of h'. These are compared with the corresponding static values, which do not vary with h'. This table assume that $\hat{\ell}' > \ell'$.

Now note that the situation where $\hat{\ell}' \not> \ell'$ can only occur when ℓ' lies between the values of the 2nd and 4th column entries since ℓ' can never be greater than $\hat{\ell}$. For example for the case of $h' = 20$, $\ell' = 60$ would require the performance guarantee of .75 to be revised to .70. The case of $h' = 50$ and $\ell' = 40$ would lead to a revision of the performance guarantee from .86 to .8.

Clearly this approach can be extended to the other cases described in this paper. A more thorough analysis of dynamic policies should consider likely order distributions and a comparison with more traditional revenue management approaches.

4 Conclusions

Using the perspective of online algorithms and the competitive ratio, we have developed new revenue management policies and performance criteria. We believe our approach shows strong promise in providing an alternate way of viewing revenue management problems and their analysis. It might be appropriate to use the new policies directly in certain contexts, e.g. where there is no demand knowledge. On the other hand, we feel the best promise for practical use of our results could be in the design of hybrid approaches or by the development of extensions that take advantage of partial demand information. We intend to explore these topics in future papers.

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Appendix

Proof of Theorem 2.

We will use the two extreme input sequences defined earlier: I_1, a sequence of n lows followed by n highs, and I_2, a sequence of n lows. We proceed in a manner analogous to the proof of Proposition 1. Thus, we consider an arbitrary input sequence, I; after applying the booking policy, let q' be the total number of orders accepted, ℓ' the total number of low fare orders accepted and v' the total value of all orders accepted. Let v^* be the value achieved by the application of an optimal off-line policy.

Case I: $q' = n$.

$$\begin{aligned} v' &= \ell' f_2 + (n-\ell')f_1 \\ &\geq (n-\theta_D(r,n))f_2 + \theta_D(r,n)f_1 \end{aligned}$$

We know that $v^* \leq nf_1$ so that:

$$\begin{aligned} \frac{v'}{v^*} &\geq \frac{(n-\theta_D(r,n))f_2 + \theta_D(r,n)f_1}{nf_1} = \frac{R(A,I_1)}{R^*(I_1)} \\ &\geq c(\theta_D(r,n)) \end{aligned}$$

The last inequality follows directly from the definition of $c(*)$ and $\theta_D(r,n)$.

Case IIa: $q' < n, \ell' < n - \theta_D(r,n)$: In this case all orders offered are accepted and $v' = v^*$.

Case IIa: $q' < n, \ell' = n - \theta_D(r,n)$

As before, $v' = \ell' f_2 + (q'-\ell')f_1$ and we have:

$$\begin{aligned} \frac{v'}{v^*} &\geq \frac{(n-\theta_D(r,n))f_2 + (q'-\ell')f_1}{(n-(q'-\ell'))f_2 + (q'-\ell')f_1} \\ &\geq \frac{(n-\theta_D(r,n))f_2}{(n-(q'-\ell'))f_2} \\ &\geq \frac{(n-\theta_D(r,n))f_2}{nf_2} \\ &= \frac{R(A,I_2)}{R^*(I_2)} \geq c(\theta_D(r,n)) \end{aligned}$$

The last inequality following directly from the definitions of $c(*)$ and $\theta_D(r,n)$.

To show this is best possible among deterministic booking policies, we consider the two instances I_1 and I_2. Let again x^A denote the number of low-fare requests accepted by a deterministic algorithm A when first presented with a sequence of n low-fare requests. If $x^A \geq \lceil b(r)\,n \rceil$ then

$$\begin{aligned} R(A,I_1) &\leq f_1\left(n - (1-r)\ \lceil b(r)\,n \rceil\right) \ = \ f_1\, c(\lceil b(r)\,n \rceil)\, n \\ &\leq c(n-\theta_D(r,n))\, f_1\, n \ = \ c(n-\theta_D(r,n))\, R^*(I_1). \end{aligned}$$

Else, $x^A \leq \lfloor b(r)\, n \rfloor$ and

$$R(A, I_2) \leq f_2 \lfloor b(r)\, n \rfloor = f_2\, c(\lfloor b(r)\, n \rfloor)\, n \leq c(n - \theta_D(r,n))\, f_2\, n = c(n - \theta_D(r,n))\, R^*(I_2).$$

This shows that no deterministic online algorithm can guarantee, for all instances, a revenue larger than $c(n - \theta_D(r,n))$ times the off-line optimum. $\square$

Proof of Theorem 3. First, recall that Proposition 3 implies that $b(r)$ is the best possible competitive ratio for the discrete 2-fare problem. Next, consider a randomized booking policy A and its expected protection level $E[\theta_A]$. If $E[\theta_A] < (1 - b(r))\, n$, then consider instance $I_1:$ its optimum revenue is $R^*(I_1) = f_1\, n$; that from booking policy A is $R(A, I_1 | \theta = x) = f_2\,(n - x) + f_1\, x = f_1\ (n - (1-r)(n-x))$ when its protection level $\theta_A = x$; and thus the expected revenue

$$E[R(A, I_1)] = f_1\ (n - (1-r)(n - E[\theta_A])) < f_1\ (n - (1-r)\, b(r)\, n) = b(r) R^*(I_1).$$

If, on the other hand, $E[\theta_A] > (1 - b(r))\, n$, then consider instance $I_2:$ its optimum revenue is $R^*(I_2) = f_2\, n$; that from booking policy A is $R(A, I_2 | \theta_A = x) = f_2\,(n - x)$ when its protection level $\theta_A = x$; and thus the expected revenue

$$E[R(A, I_2)] = f_2\,(n - E[\theta_A]) < f_2\, b(r)\, n = b(r)\, R^*(I_2).$$

This shows that the condition $E[\theta_A] = b(r)\, n$ is necessary for any randomized algorithm A to have competitive ratio $b(r)$.

Conversely, let A be a booking policy with protection level θ_A satisfying $E[\theta_A] = (1 - b(r))\, n$. Let $p^A = (p_0^A, p_1^A, \ldots, p_n^A) \in \mathbb{R}^{n+1}$ be the vector of the corresponding probabilities $p_i^A = \text{Prob}\{\theta_A = i\}$. Then

$$p^A \in P = \left\{ p \in \mathbb{R}^{n+1} : p \geq 0,\ \sum_{i=0}^{n} p_i = 1 \text{ and } \sum_{i=0}^{n} i\, p_i = (1 - b(r))\, n \right\}.$$

Note that P is a polytope, and its extreme points $p^{u,v}$ satisfy $p_i^{u,v} = 0$ for all $i \notin \{u, v\}$ where $u, v \in \{0, 1, \ldots, n\}$ with $u < (1 - b(r))\, n < v$, or, in case $(1 - b(r))\, n$ is integer, $u = v = (1 - b(r))\, n$. Since $p \in P$ is a convex combination of these extreme points, say, $p = \sum_{u,v} \lambda_{u,v} p^{u,v}$, algorithm A may be viewed as a random choice from the randomized algorithms $A^{u,v}$ defined by the probability vectors $p^{u,v}$, each with probability $\lambda_{u,v}$. Thus it suffices to show that all these algorithms $A^{u,v}$ satisfy $E[R(A^{u,v}, I)] \geq b(r)\, R^*(I)$ for all instances I. This follows immediately from Proposition 1 if $u = v = (1 - b(r))\, n$. Thus assume $0 \leq u < (1 - b(r))\, n < v \leq n$ with $p^{u,v} \in P$. To simplify notations, let $p_u = p_u^{u,v}$ and $\theta^{u,v} = \theta^{A^{u,v}}$.

Fix an instance I with ℓ low fare requests and h high fare requests, where, as in the proof of Proposition 1, we may assume that $0 \leq \ell \leq n$ and $0 \leq h \leq n$. Recall that $R^*(I) = f_1\,(h + r\ \min\{n - h, \ell\})$. For any fixed $\theta \in \{0, 1, \ldots, n\}$, let

$$g(\theta) = R[A^\theta, I] = f_1\,(r\ \min\{\ell, n - \theta\} + \min\{h,\ n - \min\{\ell, n - \theta\}\})$$

denote the revenue from using the deterministic booking policy A^θ with protection level θ. Therefore

$$E[R(A^{u,v}, I)] = E[g(\theta^{u,v})] = p_u\, g(u) + (1 - p_u)\, g(v)\ .$$

If the function g is convex on the interval $[u, v]$, then it follows from Jensen's inequality and Proposition 1 that

$$E[R(A^{u,v}, I)] \geq g(E[\theta^{u,v}]) = g((1 - b(r))\, n) \geq b(r)\, R^*(I)\ , \tag{22}$$

as needed. However, as we shall see, g is not convex on the whole interval $0 \leq \theta \leq n$.

First, consider the case where $\ell + h \leq n$. In this case, $R^*(I) = r\,\ell + h$, and $g(\theta) = f_1(r \min\{\ell, n - \theta\} + h)$ is piecewise linear concave with breakpoint $\theta = n - \ell$. If $0 \leq u < v \leq n - \ell$, or if $n\ell \leq u < v \leq n$, then g is linear on the interval $[u, v]$ and the result follows from (22). Thus assume $0 \leq u < n - \ell < v \leq n$. These inequalities and the concavity of g imply that

$$E[g(\theta^{u,v})] \geq E[g(\theta^{0,n})] = q\, g(0) + (1 - q)\, g(n)$$

where $q = b(r)$ satisfies $q\, 0 + (1 - q)n = (1 - b(r))\, n = E[\theta^{u,v}]$. Thus it suffice to consider the case $u = 0$ and $v = n$. The required inequality $E[g(\theta^{u,v})] = E[R(A^{0,n}, I)] \geq b(r)\, R^*(I)$ was proved in Proposition 4.

The second case is where $\ell + h > n$. In this case, $R^*(I) = r\, n + (1 - r)\, h$, and $g(\theta)$ is piecewise linear, with (up to) three pieces: (i) $g(\theta) = r\ell + (n - \ell)$, a constant, for $0 \leq \theta \leq n - \ell$; (ii) $g(\theta)$ then increases at rate $1 - r$ for $n - \ell \leq \theta \leq h$; and (iii) it then decreases at rate r for $h \leq \theta \leq n$. Thus the needed inequality follows from (22), unless $0 \leq u < h < v \leq n$. Since g is concave on the interval $[n - \ell, h]$, let $w = \min\{u, n - \ell\}$, so it suffices to show that $E[g(\theta^{w,n})] \geq b(r)\, R^*(I)$ for all $w \in [0, n - \ell]$. Note that, for $w \in [0, n - \ell]$, we have $g(w) = r\ell + (n - \ell)$ and $E[g(\theta^{w,n})]$ is monotone, either increasing or decreasing, in w. Thus it suffices to consider the "extreme" cases $w = 0$ and $w = n - \ell$. The case $w = 0$ was already established in Proposition 4. If $w = n - \ell$ then $q\,(n - \ell) + (1 - q)n = (1 - b(r))n$ implies $q\ell = b(r)\, n$. Since $n - \ell > h$, we now have

$$\begin{aligned}
E[R(A^{n-\ell,n}, I)] &= f_1 \left[q \left(r\ell + (n - \ell)\right) + (1 - q)\, h\right] \\
&\geq f_1 \left[r\, q\, \ell + \left(q + (1 - q)\right) h\right] \\
&= f_1 \left[b(r)\, r\, n + h\right] \\
&\geq b(r)\, R^*(I)\ .
\end{aligned}$$

This implies $E[R(A^{u,v}, I)] \geq b(r)\, R^*(I)$ for all $A^{u,v}$, as required. □

Proof of Lemma 1:

For $i = 2, ..., m - 1$.

$$\begin{aligned}
f_i(Q_i - Q_{i+1}) &= f_i \left(\left(n - \frac{n}{\Delta} \left(i - 1 - \sum_{j=1}^{i-1} \frac{f_{j+1}}{f_j} \right) \right) - \left(n - \frac{n}{\Delta} \left(i - \sum_{j=1}^{i} \frac{f_{j+1}}{f_j} \right) \right) \right) \\
&= f_i \frac{n}{\Delta} \left(1 - \frac{f_{i+1}}{f_i} \right) \\
&= \frac{n}{\Delta} (f_i - f_{i+1})
\end{aligned}$$

and for $i = 1$, we have

$$\begin{aligned} f_1(Q_1 - Q_2) &= f_1 n - f_1 \left(n - \frac{n}{\Delta} \left(1 - \sum_{j=1}^{1} \frac{f_{j+1}}{f_j} \right) \right) \\ &= f_1 n - f_1 (n - \frac{n}{\Delta}(1 - \frac{f_2}{f_1})) \\ &= \frac{n}{\Delta}(f_1 - f_2) \end{aligned}$$

Thus, we have Derived equation (10)

For $i = m$, we have:

$$\begin{aligned} f_m(Q_m - Q_{m+1}) &= f_m \left(n - \frac{n}{\Delta} \left(m - \sum_{j=1}^{m-1} \frac{f_{j+1}}{f_j} \right) \right) \\ &= f_m \left(n - \frac{n}{m - \sum_{j=2}^{m} \frac{j}{j-1}} \left(m - 1 - \sum_{j=1}^{m-1} \frac{f_{j+1}}{f_j} \right) \right) \\ &= f_m \left(n - n \frac{m - \sum_{j=1}^{m-1} \frac{f_{j+1}}{f_j} - 1}{m - \sum_{j=2}^{m} \frac{j}{j-1}} \right) \\ &= f_m (n - n(1 - \frac{1}{\Delta})) \\ &= \frac{f_m n}{\Delta}, \end{aligned}$$

which gives equation (11).

Equation (12) now can easily be derived:

$$\begin{aligned} \sum_{i=i'}^{m} f_i(Q_i - Q_{i+1}) &= \sum_{i=i'}^{m} (f_i - f_{i+1}) \frac{n}{\Delta} \\ &= f_{i'} \frac{n}{\Delta} \end{aligned}$$

□

Proof of Lemma 2:

Proof for $P(q)$: Note that $Q(p)$ is monotone increasing on $[f_{min}, f_{max}]$ so that it takes on its minimum value at $f_{\min}$. Now since $Q(f_{\min}) = n\rho$, it follows that $P(q) = f_{\min}$ for $q < n\rho$. Now on the interval $[f_{\min}, f_{\max}]$, Q varies between $n\rho$ and n and is monotone so we can obtain $P(q)$ on the interval $[n\rho, n]$ by setting $q = Q(f)$ and solving for f as a function of q.

Starting with $q = n - \theta(f)$, we have

$$\begin{aligned}
n - q &= \theta(f) \\
&= n(1-\rho)\frac{\ln f_{\max} - \ln f}{\ln f_{\max} - \ln f_{\min}} \\
\ln f_{\max} - \ln f &= \frac{n-q}{n(1-\rho)}(\ln f_{\max} - \ln f_{\min}) \\
\ln f &= \ln f_{\max} - \frac{n-q}{n(1-\rho)}(\ln f_{\max} - \ln f_{\min}) \\
f &= \exp\left(\ln f_{\max} - \ln f_{\max}^{(n-q)/(n(1-\rho))} + \ln f_{\min}^{(n-q)/(n(1-\rho))}\right) \\
&= f_{\max} r^{(n-q)/n(1-\rho)}
\end{aligned}$$

This is the desired expression so that the result is proven.
Proof for V: The expression for the interval $0 \leq q < n\rho$ is clearly true and we now consider the case for $n\rho \leq q \leq n$.

$$\begin{aligned}
\int_0^q P(s)ds &= n\rho f_{\min} + f_{\max}\left(-\frac{n(1-\rho)}{\ln r}\right) r^{\frac{n-s}{n(1-\rho)}} \,|_{n\rho}^{q} \\
&= n\rho f_{\min} + f_{\max}\left(-\frac{n(1-\rho)}{\ln r}\right)(r^{\frac{n-q}{n(1-\rho)}} - r) \\
&= n\rho f_{\min} - \frac{n(1-\rho)f_{max}(-r)}{\ln r} - \frac{n(1-\rho)f_{\max}}{\ln r}(r^{\frac{n-q}{n(1-\rho)}}) \\
&= \frac{n\rho f_{\min}\ln r + n(1-\rho)f_{\min} - n(1-\rho)P(q)}{\ln r}
\end{aligned}$$

Substituting, $\rho = 1/(1-\ln r)$ and $1-\rho = -\ln r/(1-\ln r)$, we have:

$$\begin{aligned}
V(q) = \int_0^q P(s)ds &= \frac{\frac{nf_{\min}\ln r}{1-\ln r} - \frac{n\ln r f_{\min}}{1-\ln r} + \frac{n\ln r P(q)}{1-\ln r}}{\ln r} \\
&= \frac{nP(q)}{1-\ln r}
\end{aligned}$$

This is the desired expression and the proof is complete. □

Proof of Theorem 8: First, note that, since $0 < r < 1$ and $0 < \rho < 1/(1-\ln r) < 1$, we have $1 < r^{\frac{-\rho}{1-\rho}} < e$. Therefore $G^{(-1)}\left(r^{\frac{\rho}{1-\rho}}\right)$ is well defined.

For any request sequence $I = (O_1, O_2, \ldots, O_s)$, define v' as the value produced by the application of the stated policy, n', the number of order accepted and and v^* as the off-line optimum value. Thus we need to show that, under the assumptions of the theorem, $v'/v^* \geq \rho\left(1 - \frac{1}{\rho(\rho n+1)}\right)$. We have

$$\begin{aligned}
v' &\geq f_{\min}\,\rho\, n + f_{\min}\left(1+\frac{1}{\rho n}\right) + f_{\min}\left(1+\frac{1}{\rho n}\right)^2 + \ldots + f_{\min}\left(1+\frac{1}{\rho n}\right)^{\lceil n'-\rho n-1\rceil} \\
&= f_{\min}\left(\rho\, n\left(1+\frac{1}{\rho n}\right)^{\lceil n'-\rho n\rceil} - 1\right). \qquad (23)
\end{aligned}$$

We consider two cases.
Case I: $n' < n$. If $n' < \rho n + 1$ then no orders are rejected and $v' = v^*$, so clearly the result holds. Else $n' \geq \rho n + 1$. Since at least n' requests are accepted by the off-line optimum,

$$v^* \;\geq\; n' f_{\min} \;\geq\; (\rho n + 1) f_{\min}\,. \tag{24}$$

Note that any request i whose price O_i is strictly greater than $f_{\min}(1+\frac{1}{\rho n})^{\lceil n'-\rho n\rceil}$ is accepted by the stated policy. Since the number of such requests does not exceed $n' < n$, request O_i is also accepted by the off-line optimum. Let

$$\Gamma = \sum_{i=1}^{s} \max\left\{0,\ O_i - f_{\min}(1+\frac{1}{\rho n})^{\lceil n'-\rho n\rceil}\right\} \geq 0.$$

Inequality (23) can thus be strengthened as:

$$v' \;\geq\; f_{\min}\left(\rho\, n\left(1+\frac{1}{\rho n}\right)^{\lceil n'-\rho n\rceil} - 1\right) + \Gamma\,.$$

On the other hand, since the off-line optimum cannot accept more than n requests, we have

$$v^* \;\leq\; n\, f_{\min}\left(1+\frac{1}{\rho n}\right)^{\lceil n'-\rho n\rceil} + \Gamma\,.$$

Using $\rho \leq 1$ and (24), this implies

$$\begin{aligned}
v'/v^* \;&\geq\; \left(f_{\min}\,\rho\, n\left(1+\frac{1}{\rho n}\right)^{\lceil n'-\rho n\rceil} - f_{\min} + \Gamma\right)/v^* \\
&\geq\; \left(\rho\, n f_{\min}\left(1+\frac{1}{\rho n}\right)^{\lceil n'-\rho n\rceil} + \rho\Gamma - f_{\min}\right)/v^* \\
&\geq\; \rho - \frac{f_{\min}}{v^*} \\
&\geq\; \rho\left(1 - \frac{1}{\rho(\rho n+1)}\right).
\end{aligned}$$

This establishes the desired result (19) for Case I.
Case II: The number of accepted orders $n' = n$. Then by (23),

$$v' \;\geq\; f_{\min}\left(\rho\, n\left(1+\frac{1}{\rho n}\right)^{\lceil n-\rho n\rceil} - 1\right) \;\geq\; \rho\, n f_{\min}\left(1+\frac{1}{\rho n}\right)^{(1-\rho)n} - f_{\min}\,.$$

On the other hand, $v^* \leq n f_{\max}$ and therefore

$$\begin{aligned}
v'/v^* \;&\geq\; \left(\rho\, n f_{\min}\left(1+\frac{1}{\rho n}\right)^{(1-\rho)n} - f_{\min}\right)/\left(n f_{\max}\right) \\
&=\; \rho\, r\left(1+\frac{1}{\rho n}\right)^{(1-\rho)n} - \frac{r}{n}\,.
\end{aligned}$$

Since $n \geq \frac{1}{1-\rho}$, $\frac{1+\rho n}{\rho n} > 1$ and $0 < \rho < 1$, we have

$$\frac{1+\rho n}{n} \;\leq\; \frac{1+\rho n}{\rho n} \;\leq\; \left(1+\frac{1}{\rho n}\right)^{(1-\rho)n} \quad \text{implying} \quad \frac{r}{n} \;\leq\; r\left(1+\frac{1}{\rho n}\right)^{(1-\rho)n}\frac{1}{\rho n+1}$$

$$\text{and} \quad v'/v^* \;\geq\; \left(\rho\, r\left(1+\frac{1}{\rho n}\right)^{(1-\rho)n}\right)\left(1-\frac{1}{\rho(\rho n+1)}\right). \tag{25}$$

Since G is increasing and $\frac{1-\rho}{\rho} > 0$,

$$r\left(1+\frac{1}{\rho n}\right)^{(1-\rho)n} \;=\; r\left[G(\rho n)\right]^{\frac{1-\rho}{\rho}} \;\geq\; r\left[G\left(G^{(-1)}\left(r^{\frac{-\rho}{1-\rho}}\right)\right)\right]^{\frac{1-\rho}{\rho}} \;=\; 1.$$

Combining this inequality with (25) yields (19). $\square$