Viscosity Solutions of Hamilton-Jacobi Equations and Optimal Control Problems

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1 Preliminaries: the method of characteristics

Given an open set $\Omega \subseteq \mathbb{R}^n$, consider the first order, scalar PDE

$$F(x, u, \nabla u) = 0, \qquad x \in \Omega. \tag{1.1}$$

It is convenient to introduce the variable $p \doteq \nabla u$, so that $(p_1, \ldots, p_n) = (u_{x_1}, \ldots, u_{x_n})$. We assume that the $F = F(x, u, p)$ is a $\mathcal{C}^1$ function[1] mapping $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}$.

Given the boundary data

$$u(x) = \bar{u}(x) \qquad x \in \partial\Omega, \tag{1.2}$$

a solution can be constructed (at least locally, in a neighborhood of the boundary) by the classical method of characteristics. We wish to determine the values $u(x)$ along a curve $s \mapsto x(s)$, starting from the boundary of Ω and solving a suitable ODE (Figure 1).
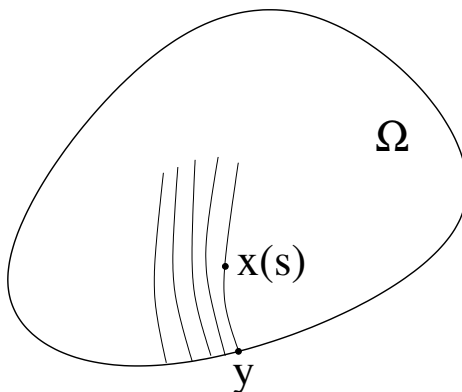

Figure 1: Computing a solution along characteristics.

Fix a boundary point $y \in \partial\Omega$ and consider a curve $s \mapsto x(s)$ with $x(0) = y$. We shall write

$$u(s) \doteq u(x(s)), \qquad p(s) \doteq p(x(s)) = \nabla u(x(s)),$$

and seek an ODE describing how u and $p = \nabla u$ change along the curve. Denoting by an upper dot the derivative w.r.t. the parameter s, one has

$$\dot{u} = \sum_i u_{x_i} \dot{x}_i = \sum_i p_i \dot{x}_i, \tag{1.3}$$

$$\dot{p}_j = \dot{u}_{x_j} = \sum_i u_{x_j x_i} \dot{x}_i. \tag{1.4}$$

In general, $\dot{p}_j$ thus depends on the second derivatives of u. The key idea in the method of characteristics is that, by a careful choice of the curve $s \mapsto x(s)$, the terms involving second derivatives disappear from the equations. Differentiating (1.1) w.r.t. x_j we obtain

$$\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial u} u_{x_j} + \sum_i \frac{\partial F}{\partial p_i} u_{x_i x_j} = 0. \tag{1.5}$$

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[1]Here and in the sequel, we denote by $\mathcal{C}(\Omega)$ the space of continuous functions defined on Ω, while $\mathcal{C}^k(\Omega)$ is the space of k times continuously differentiable functions.

Hence

$$\sum_i \frac{\partial F}{\partial p_i} u_{x_j x_i} = -\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial u} p_j. \tag{1.6}$$

If we now make the choice $\dot{x}_i = \partial F / \partial p_i$, the right hand side of (1.4) is computed by (1.6). We thus obtain a system of ODEs, which do not involve second order derivatives:

$$\begin{cases} \dot{x}_i = \dfrac{\partial F}{\partial p_i} & i = 1, \ldots, n, \\[2ex] \dot{u} = \displaystyle\sum_i p_i \dfrac{\partial F}{\partial p_i}, \\[2ex] \dot{p}_j = -\dfrac{\partial F}{\partial x_j} - \dfrac{\partial F}{\partial u} p_j & j = 1, \ldots, n. \end{cases} \tag{1.7}$$

This leads to a family of Cauchy problems, which in vector notation take the form

$$\begin{cases} \dot{x} = \dfrac{\partial F}{\partial p} \\[2ex] \dot{u} = p \cdot \dfrac{\partial F}{\partial p} \\[2ex] \dot{p} = -\dfrac{\partial F}{\partial x} - \dfrac{\partial F}{\partial u} \cdot p \end{cases} \qquad \begin{cases} x(0) = y \\[2ex] u(0) = u(y) \\[2ex] p(0) = \nabla u(y) \end{cases} \qquad y \in \partial\Omega. \tag{1.8}$$

The resolution of the first order boundary value problem (1.1)-(1.2) is thus reduced to the solution of a family of ODE's, depending on the initial point y. As y varies along the boundary of Ω, we expect that the union of the above curves $x(\cdot)$ will cover a neighborhood of $\partial\Omega$, where our solution u will be defined.

Remark 1.1 Consider the quasilinear equation

$$\sum_i \alpha_i(x, u) u_{x_i} + \beta(x, u) = 0,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. In this case $F(x, u, p) = p \cdot \alpha(x, u) + \beta(x, u)$ is affine w.r.t. p, and the partial derivatives $\partial F / \partial p_i = \alpha_i(x, u)$ do not depend on p. The first two equations in (1.7) can thus be solved independently, without computing p from the third equation:

$$\dot{x} = \alpha(x, u), \qquad \dot{u} = p \cdot \alpha(x, u) = -\beta(x, u).$$

Example 1.1 The equation

$$|\nabla u|^2 - 1 = 0 \qquad x \in \Omega \tag{1.9}$$

on $\mathbb{R}^2$ corresponds to (1.1) with $F(x, u, p) = p_1^2 + p_2^2 - 1$. Assigning the boundary data

$$u = 0 \qquad x \in \partial\Omega,$$

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a solution is provided by the distance function

$$u(x) = \text{dist}\,(x,\,\partial\Omega)\,.$$

The corresponding equations (1.8) are

$$\dot{x} = 2p\,, \qquad \dot{u} = p\cdot\dot{x} = 2\,, \qquad \dot{p} = 0\,.$$

Choosing the initial data at a point y, we have

$$x(0) = y, \qquad u(0) = 0, \qquad p(0) = \mathbf{n}\,,$$

where $\mathbf{n}$ is the interior unit normal to the set Ω at the point y. In this case, the solution is constructed along the ray $x(s) = y + 2s\mathbf{n}$, and along this ray one has $u(x) = |x - y|$. Even if the boundary $\partial\Omega$ is smooth, in general the distance function will be smooth only on a neighborhood of this boundary. If Ω is bounded, there will be a set γ of interior points $\bar{x}$ where the distance function is not differentiable (Figure 2). These are indeed the points such that

$$\text{dist}\,(\bar{x},\,\partial\Omega) = |\bar{x} - y_1| = |\bar{x} - y_2|$$

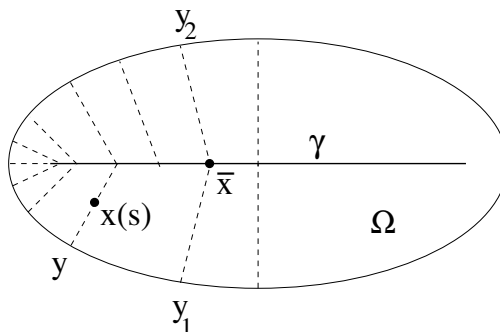for two distinct points $y_1, y_2 \in \partial\Omega$.

Figure 2: At points $\bar{x} \in \gamma$ the distance function is not differentiable.

The previous example shows that, in general, the boundary value problem for a first order PDE does not admit a global $\mathcal{C}^1$ solution. This suggests that we should relax our requirements, and consider solutions in a generalized sense. We recall that, by Rademacher's theorem, every Lipschitz continuous function $u : \Omega \mapsto I\!\!R$ is differentiable almost everywhere. It thus seems natural to introduce

Definition 1.1 *A function u is a generalized solution of (1.1)-(1.2) if u is Lipschitz continuous on the closure $\overline{\Omega}$, takes the prescribed boundary values, and satisfies the first order equation (1.1) at almost every point $x \in \Omega$.*

Unfortunately, this concept of solution is far too weak and does not lead to a useful uniqueness result.
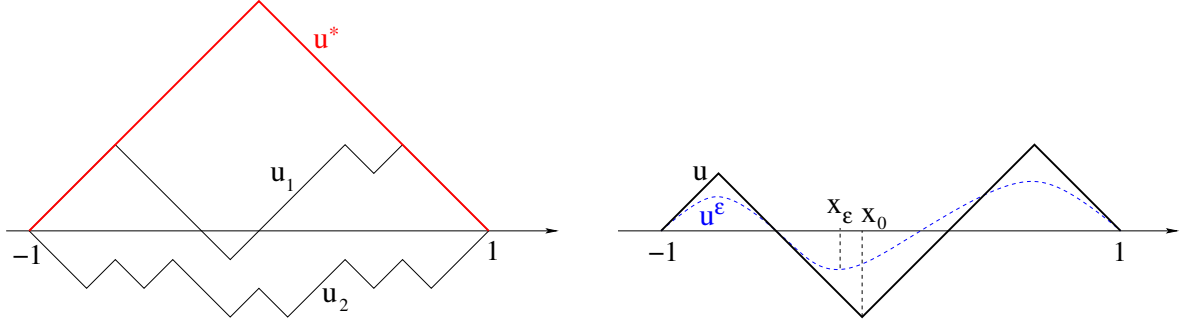
Figure 3: Left: the boundary value problem (1.10) has several generalized solutions. Among these, u^* is the only one that does not have local minima in the interior of the domain. Right: any generalized solution u having a strict local minimum at some interior point x_0 cannot be the limit of viscosity approximations u^ε.

Example 1.2 On the domain $\Omega = [-1, 1]$, the boundary value problem

$$|u_x| - 1 = 0 \qquad x(-1) = x(1) = 0, \tag{1.10}$$

admits infinitely many piecewise affine generalized solutions, as shown in Fig. 3, left.

However, it is a remarkable fact that, among all these solutions, the distance function

$$u^*(x) = 1 - |x|, \qquad x \in [-1, 1],$$

is the only one that can be obtained as a vanishing viscosity limit.

Indeed, any other generalized solution u with polygonal graph has at least one strict local minimum in the interior of the interval $[-1, 1]$, say at a point x_0 (see Fig. 3, right). Assume that there exists a family of $\mathcal{C}^2$ solutions to the viscous equations

$$|u_x^\varepsilon| - 1 = \varepsilon u_{xx}^\varepsilon,$$

such that $u^\varepsilon \to u$ uniformly on $[-1, 1]$, as $\varepsilon \to 0+$.

Then for every $\varepsilon > 0$ sufficiently small the function u^ε will have a local minimum at a nearby point x_ε. But this is impossible, because

$$|u_x^\varepsilon(x_\varepsilon)| - 1 = -1 \neq \varepsilon u_{xx}^\varepsilon(x_\varepsilon) \geq 0.$$

On the other hand, notice that if u^ε attains a local maximum at some interior point $x_0 \in]-1, 1[$, this does not lead to any contradiction.

In view of the previous example, one seeks a new concept of solution for the first order PDE (1.1), having the following properties:

(I) For every boundary data (1.2), a unique solution should exist, depending continuously on the boundary values and on the function F.

(II) This solution u should coincide with the limit of vanishing viscosity approximations. Namely, $u = \lim_{\varepsilon \to 0+} u^\varepsilon$, where the u^ε are solutions of

$$F(x, u^\varepsilon, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon.$$

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(III) In the case where (1.1) is the Hamilton-Jacobi equation describing the value function for some optimization problem, this new concept of solution should single out precisely this value function.

In the following sections we shall introduce the definition of *viscosity solution* and see how it fulfills the above requirements **(I)**–**(III)**.

2   One-sided differentials

Let $u : \Omega \mapsto I\!\!R$ be a scalar function, defined on an open set $\Omega \subseteq I\!\!R^n$. The set of *super-differentials* of u at a point x is defined as

$$D^+u(x) \;\doteq\; \left\{ p \in I\!\!R^n \,;\quad \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \;\leq\; 0 \right\}. \tag{2.1}$$

In other words, a vector $p \in I\!\!R^n$ is a super-differential iff the hyperplane $y \mapsto u(x) + p \cdot (y - x)$ is tangent from above to the graph of u at the point x (Fig. 4, left). Similarly, the set of *sub-differentials* of u at a point x is defined as

$$D^-u(x) \;\doteq\; \left\{ p \in I\!\!R^n \,;\quad \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \;\geq\; 0 \right\}, \tag{2.2}$$

so that a vector $p \in I\!\!R^n$ is a sub-differential iff the hyperplane $y \mapsto u(x) + p \cdot (y - x)$ is tangent from below to the graph of u at the point x (Fig. 4, right.).

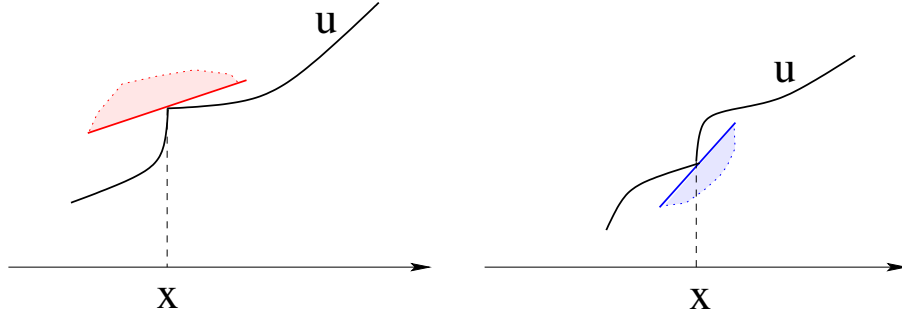Figure 4: Left: the geometric meaning of a super-differential of the function u at x. Right: a sub-differential.

Example 2.1 Consider the function (Fig. 5)

$$u(x) \doteq \begin{cases} 0 & \text{if} \quad x < 0, \\ \sqrt{x} & \text{if} \quad x \in [0, 1], \\ 1 & \text{if} \quad x > 1. \end{cases}$$
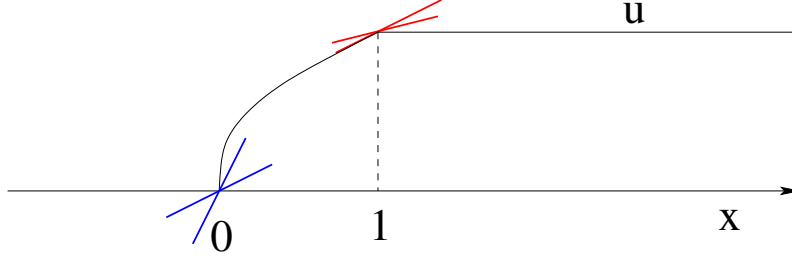
In this case we have

$$D^+u(0) \;=\; \emptyset, \qquad\qquad D^-u(0) \;=\; [0, \infty[,$$

Figure 5: Upper and lower differentials of the function u in Example 2.1.

$$D^+u(x) \;=\; D^-u(x) \;=\; \left\{\frac{1}{2\sqrt{x}}\right\} \qquad x \in \,]0,1[\,,$$

$$D^+u(1) \;=\; [0,\,1/2], \qquad D^-u(1) \;=\; \emptyset.$$

If $\varphi \in \mathcal{C}^1$, its differential at a point x is written as $\nabla\varphi(x)$. The following characterization of super- and sub-differentials is very useful.

Lemma 2.1 *Let* $u \in \mathcal{C}(\Omega)$. *Then*

(i) $p \in D^+u(x)$ *if and only if there exists a function* $\varphi \in \mathcal{C}^1(\Omega)$ *such that* $\nabla\varphi(x) = p$ *and* $u - \varphi$ *has a local maximum at* x.

(ii) $p \in D^-u(x)$ *if and only if there exists a function* $\varphi \in \mathcal{C}^1(\Omega)$ *such that* $\nabla\varphi(x) = p$ *and* $u - \varphi$ *has a local minimum at* x.

By adding a constant, it is not restrictive to assume that $\varphi(x) = u(x)$. In this case, we are saying that $p \in D^+u(x)$ iff there exists a smooth function $\varphi \geq u$ with $\nabla\varphi(x) = p$, $\varphi(x) = u(x)$. In other words, the graph of φ touches the graph of u from above at the point x (Fig. 6, left). A similar property holds for subdifferentials: $p \in D^-u(x)$ iff there exists a smooth function $\varphi \leq u$, with $\nabla\varphi(x) = p$, whose graph touches from below the graph of u at the point x (Fig. 6, right).
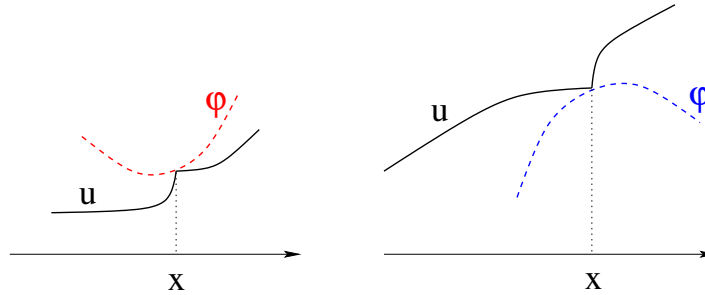

Figure 6: The characterization of super- and sub-differential given in Lemma 2.2.

Proof of Lemma 2.1. Assume that $p \in D^+u(x)$. Then we can find $\delta > 0$ and a continuous, increasing function $\sigma : [0, \infty[\, \mapsto I\!R$, with $\sigma(0) = 0$, such that

$$u(y) \;\leq\; u(x) + p \cdot (y - x) + \sigma(|y - x|)\,|y - x| \tag{2.3}$$

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for $|y - x| < \delta$. Indeed, for $r > 0$ one can take

$$\sigma(r) \;\doteq\; \max\left\{ 0 \,, \quad \sup_{0 < |y-x| \leq r} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \right\}. \tag{2.4}$$

Define

$$\rho(r) \;\doteq\; \int_0^r \sigma(t)\, dt$$

and observe that (see Fig. 7, left)

$$\rho(0) \;=\; \rho'(0) \;=\; 0, \qquad \rho(2r) \;\geq\; \int_r^{2r} \sigma(t)\, dt \;\geq\; \sigma(r)\, r\,. \tag{2.5}$$

By the above properties, the function

$$\varphi(y) \;\doteq\; u(x) + p \cdot (y - x) + \rho(2|y - x|) \tag{2.6}$$

is continuously differentiable and satisfies

$$\varphi(x) = u(x), \qquad \nabla\varphi(x) = p.$$

Moreover, for $|y - x| < \delta$, comparing (2.3) with (2.6) and using (2.5) we conclude

$$u(y) - \varphi(y) \;\leq\; \sigma(|y - x|)|y - x| - \rho(2|y - x|) \;\leq\; 0.$$

Hence, the difference $u - \varphi$ attains a local maximum at the point x.

To prove the opposite implication, assume that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at x. Then

$$\limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \;\leq\; \limsup_{y \to x} \frac{\varphi(y) - \varphi(x) - p \cdot (y - x)}{|y - x|} \;=\; 0\,. \tag{2.7}$$

This completes the proof of (i). The proof of (ii) is entirely similar. $\qquad \square$

Remark 2.1 By possibly replacing the function φ with $\widetilde{\varphi}(y) = \varphi(y) \pm |y - x|^2$, it is clear that in the above lemma we can require that $u - \varphi$ attains a *strict local minimum* or a *strict local maximum* at the point x. This is important in view of the following stability result.

Lemma 2.2 *Let $u : \Omega \mapsto \mathbb{R}$ be continuous. Assume that, for some $\phi \in \mathcal{C}^1$, the function $u - \phi$ has a strict local minimum (a strict local maximum) at a point $x \in \Omega$. If the convergence $u_m \to u$ holds uniformly on Ω, then there exists a sequence of points $x_m \to x$ with $u_m(x_m) \to u(x)$ and such that $u_m - \phi$ has a local minimum (a local maximum) at x_m, for all $m \geq 1$ sufficiently large.*

Proof. Assume that $u - \phi$ has a strict local minimum at x (Fig. 7, left). For every $\rho > 0$ sufficiently small, there exists $\varepsilon_\rho > 0$ such that

$$u(y) - \phi(y) \;>\; u(x) - \phi(x) + \varepsilon_\rho \qquad \text{whenever} \;\; |y - x| = \rho\,.$$

By the uniform convergence $u_m \to u$, for all $m \geq N_\rho$ sufficiently large one has $u_m(y) - u(y) < \varepsilon_\rho/4$ for $|y - x| \leq \rho$. Hence

$$u_m(y) - \phi(y) \;>\; u_m(x) - \phi(x) + \frac{\varepsilon_\rho}{2} \qquad |y - x| = \rho\,,$$

This shows that $u_m - \phi$ has a local minimum at some point x_m, with $|x_m - x| < \rho$. Letting $\rho, \varepsilon_\rho \to 0$, we obtain the desired sequence $\{x_m\}$. $\qquad \square$
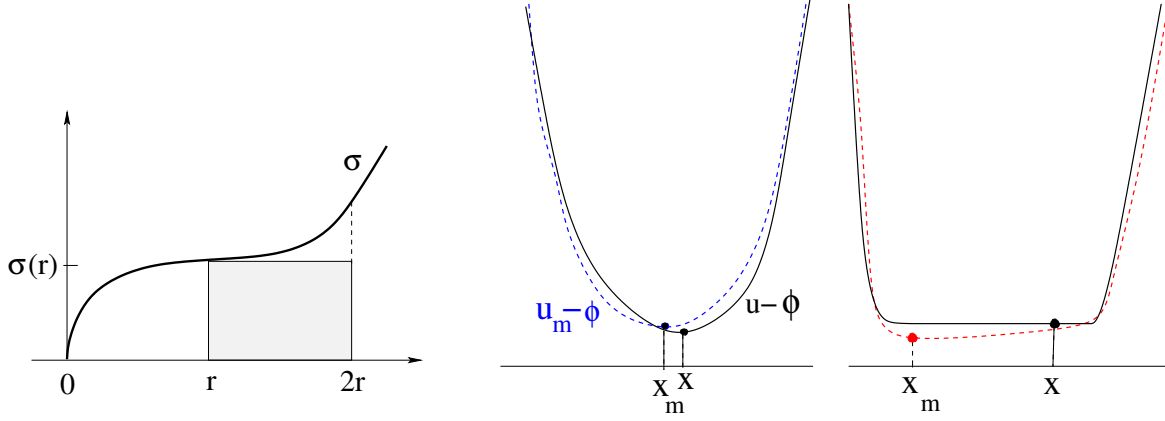
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Figure 7: Left: the function $\sigma(r)$ introduced at (2.4) and used in (2.5). Center and right: points of strict local minimum are stable. However, points of non-strict local minimum may be unstable.

Notice that, if x is a point of non-strict local minimum for $u - \phi$, the slightly perturbed functions $u_m - \phi$ may not have any local minimum x_m close to x (Fig. 7, right).

Some simple properties of super- and sub-differential are collected in the next lemma.

Lemma 2.3 *Let $u \in \mathcal{C}(\Omega)$. Then*

  (i) *If u is differentiable at x, then*
$$D^+u(x) \;=\; D^-u(x) \;=\; \{\nabla u(x)\}. \tag{2.8}$$

  (ii) *If the sets $D^+u(x)$ and $D^-u(x)$ are both non-empty, then u is differentiable at x, hence (2.8) holds.*

  (iii) *The sets of points where a one-sided differential exists:*
$$\Omega^+ \;\dot=\; \{x \in \Omega; \;\; D^+u(x) \;\neq\; \emptyset\}, \qquad \Omega^- \;\dot=\; \{x \in \Omega; \;\; D^-u(x) \;\neq\; \emptyset\} \tag{2.9}$$
are both non-empty. Indeed, they are dense in Ω.

Proof. To prove (i), assume that u is differentiable at x. Trivially, $\nabla u(x) \in D^\pm u(x)$. On the other hand, if $\varphi \in \mathcal{C}^1(\Omega)$ is such that $u - \varphi$ has a local maximum at x, then $\nabla \varphi(x) = \nabla u(x)$. Hence $D^+u(x)$ cannot contain any vector other than $\nabla u(x)$.

To prove (ii), assume that the sets $D^+u(x)$ and $D^-u(x)$ are both non-empty. Then we can find $\delta > 0$ and $\varphi_1, \varphi_2 \in \mathcal{C}^1(\Omega)$ such that (Fig. 8)
$$\varphi_1(x) \;=\; u(x) \;=\; \varphi_2(x), \qquad \varphi_1(y) \;\leq\; u(y) \;\leq\; \varphi_2(y) \quad \text{whenever} \quad |y - x| < \delta.$$

By a standard comparison argument, this implies that u is differentiable at x and $\nabla u(x) = \nabla \varphi_1(x) = \nabla \varphi_2(x)$.

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Concerning (iii), let $x_0 \in \Omega$ and $\varepsilon > 0$ be given. On the open ball $B(x_0, \varepsilon) \doteq \{x ; \ |x - x_0| < \varepsilon\}$ centered at x_0 with radius ε, consider the smooth function (Fig. 8)

$$\varphi(x) \ \doteq \ \frac{1}{\varepsilon^2 - |x - x_0|^2} \ .$$

Notice that $\varphi(x) \to +\infty$ as $|x - x_0| \to \varepsilon-$. Therefore, the function $u - \varphi$ attains a global maximum at some interior point $y \in B(x_0, \varepsilon)$. By Lemma 2.1, the super-differential of u at y is non-empty. Indeed, $\nabla\varphi(y) = \frac{2(y - x_0)}{(\varepsilon^2 - |y - x_0|^2)^2} \in D^+u(y)$. The previous argument shows that, for every $x_0 \in \Omega$ and $\varepsilon > 0$, the set Ω^+ contains a point y such that $|y - x_0| < \varepsilon$. Therefore Ω^+ is dense in Ω. The case of sub-differentials is entirely similar. $\qquad\square$
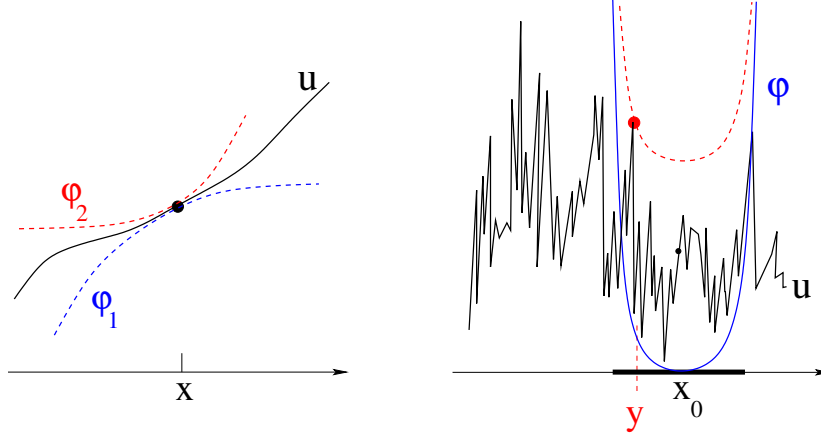

Figure 8: Left: if two functions $\varphi_1, \varphi_2 \in \mathcal{C}^1$ satisfy $\varphi_1(x) = \varphi_2(x)$ and $\varphi_1(y) \leq u(y) \leq \varphi_2(y)$ for all y, then $\nabla\varphi_1(x) = \nabla u(x) = \nabla\varphi_2(x)$. Right: on any open ball $B(x_0, \varepsilon)$ one can find a point y where the upper differential $D^+u(y)$ is not empty.

3   Viscosity solutions

In the following, we consider the first order PDE

$$F(x, u(x), \nabla u(x)) \ = \ 0, \qquad x \in \Omega. \tag{3.1}$$

Here $\Omega \in {I\!\!R}^n$ is an open set, and $F : \Omega \times {I\!\!R} \times {I\!\!R}^n \mapsto {I\!\!R}$ is a continuous (nonlinear) function.

Definition 3.1 *A function $u \in \mathcal{C}(\Omega)$ is a* **viscosity subsolution** *of (3.1) if*

$$F(x, u(x), p) \ \leq \ 0 \qquad \text{for every } x \in \Omega, \ p \in D^+u(x). \tag{3.2}$$

Similarly, $u \in \mathcal{C}(\Omega)$ is a **viscosity supersolution** *of (3.1) if*

$$F(x, u(x), p) \ \geq \ 0 \qquad \text{for every } x \in \Omega, \ p \in D^-u(x). \tag{3.3}$$

We say that u is a **viscosity solution** *of (3.1) if it is both a supersolution and a subsolution in the viscosity sense.*

Remark 3.1 In the classical theory of first order PDEs, the function F is assumed to be continuously differentiable, in order to make sense of the equations for characteristics (1.7). In the theory of viscosity solutions, on the other hand, we only need to assume that F is continuous.

Remark 3.2 In the definition of subsolution, we are imposing conditions on u only at points x where the super-differential is non-empty. Even if u is merely continuous, possibly nowhere differentiable, there are a lot of these points. Indeed, by Lemma 2.3, the set of points x where $D^+u(x) \neq \emptyset$ is dense on Ω. Similarly, for supersolutions we only impose conditions at points where $D^-u(x) \neq \emptyset$.

Remark 3.3 If u is a $\mathcal{C}^1$ function that satisfies (3.1) at every $x \in \Omega$, then u is also a solution in the viscosity sense. Viceversa, if u is a viscosity solution, then the equality (3.1) must hold at every point x where u is differentiable. In particular, if u is Lipschitz continuous, then by Rademacher's theorem it is a.e. differentiable. Hence (3.1) holds a.e. in Ω.

Remark 3.4 According to Definition 1.1, a Lipschitz continuous function is a *generalized solution* of (1.1) if the following implication holds true:

$$p \;=\; \nabla u(x) \qquad \Longrightarrow \qquad F(t, u(x), p) \;=\; 0 \,. \tag{3.4}$$

This can be written in an equivalent way, splitting each equality into two inequalities:

$$\Big[p \in D^+u(x) \;\; \text{and} \;\; p \in D^-u(x) \Big] \qquad \Longrightarrow \qquad \Big[F(t, u(x), p) \;\leq\; 0 \;\; \text{and} \;\; F(t, u(x), p) \;\geq\; 0 \Big]. \tag{3.5}$$

The definition of *viscosity solution*, on the other hand, requires two separate implications:

$$\begin{cases} p \in D^+u(x) & \Longrightarrow & F(t, u(x), p) \;\leq\; 0 & (u \text{ is a viscosity subsolution}) \,, \\ p \in D^-u(x) & \Longrightarrow & F(t, u(x), p) \;\geq\; 0 & (u \text{ is a viscosity supersolution}) \,. \end{cases} \tag{3.6}$$

Observe that, if $u(\cdot)$ satisfies the two implications in (3.6), then it also satisfies (3.5). In particular, if a Lipschitz function u is a viscosity solution, then u is also a generalized solution. However, the converse does not hold.

Example 3.1 Let $F(x, u, p) \doteq |p| - 1$. Observe that the function $u(x) = 1 - |x|$ is a viscosity solution of

$$|u_x| - 1 \;=\; 0 \tag{3.7}$$

on the open interval $]-1, 1[$. Indeed, u is differentiable and satisfies the equation (3.7) at all points $x \neq 0$. Moreover, we have

$$D^+u(0) \;=\; [-1, \, 1] \,, \qquad D^-u(0) \;=\; \emptyset. \tag{3.8}$$

To show that u is a supersolution, at the point $x = 0$ there is nothing else to check. To show that u is a subsolution, take any $p \in [-1, 1]$. Then $|p| - 1 \leq 0$, as required.

It is interesting to observe that the same function $u(x) = 1 - |x|$ is NOT a viscosity solution of the equation

$$1 - |u_x| \;=\; 0 \,. \tag{3.9}$$

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Indeed, at $x = 0$, taking $p = 0 \in D^+u(0)$ we find $1 - |0| = 1 > 0$. Since $D^-u(0) = \emptyset$, we conclude that the function $u(x) = 1 - |x|$ is a viscosity supersolution of (3.9), but not a subsolution.

Similar definitions also apply to evolution equations of the form

$$u_t + H(t, x, u, \nabla u) \ = \ 0, \qquad\qquad (t, x) \in]0, T[\times \Omega, \qquad\qquad (3.10)$$

where $\nabla u = (u_{x_1}, \ldots, u_{x_n})$ denotes the gradient of u w.r.t. x. Recalling Lemma 2.1, we can reformulate these definitions in an equivalent form:

Definition 3.2 *A continuous function* $u :]0, T[\times \Omega \mapsto I\!\!R$ *is a* **viscosity subsolution** *of (3.10) if, for every* $\mathcal{C}^1$ *function* $\varphi = \varphi(t, x)$ *such that* $u - \varphi$ *has a local maximum at* (t, x), *one has*

$$\varphi_t(t, x) + H(t, x, u, \nabla\varphi) \ \leq \ 0. \qquad\qquad (3.11)$$

Similarly, a continuous function $u :]0, T[\times \Omega \mapsto I\!\!R$ *is a* **viscosity supersolution** *of (3.10) if, for every* $\mathcal{C}^1$ *function* $\varphi = \varphi(t, x)$ *such that* $u - \varphi$ *has a local minimum at* (t, x), *one has*

$$\varphi_t(t, x) + H(t, x, u, \nabla\varphi) \ \geq \ 0. \qquad\qquad (3.12)$$

Remark 3.5 If $u - \varphi$ has a local maximum at the point (t, x), then $(\varphi_t, \nabla\varphi) = (\varphi_t, \varphi_{x_1}, \ldots, \varphi_{x_n})$ is an upper gradient for u at (t, x). To achieve the definition of lower solution, in (3.10) we replace u_t and ∇u with φ_t and $\nabla\varphi$ respectively, and ask that this expression be ≤ 0. A similar idea motivates the definition of upper solutions.

4 Stability properties

For the nonlinear first order PDE (1.1), the set of solutions may not be closed w.r.t. the topology of uniform convergence. Indeed, consider a sequence of solutions $(u_m)_{m \geq 1}$, with $u_m \to u$ uniformly on a domain Ω. In general, to conclude that u is itself a solution of (1.1) one should also check that the gradients ∇u_m converge to the corresponding gradient ∇u. In many cases this does not happen.

Example 4.1 A sequence of generalized solutions to the equation

$$|u_x| - 1 \ = \ 0, \qquad\qquad u(0) \ = \ u(1) \ = \ 0, \qquad\qquad (4.1)$$

is provided by the saw-tooth functions (Fig. 9)

$$u_m(x) \doteq \begin{cases} x - \frac{k-1}{m} & \text{if} \quad x \in \left[\frac{k-1}{m}, \ \frac{k-1}{m} + \frac{1}{2m} \right], \\[2mm] \frac{k}{m} - x & \text{if} \quad x \in \left[\frac{k}{m} - \frac{1}{2m}, \ \frac{k}{m} \right], \end{cases} \qquad k = 1, \ldots, m. \qquad (4.2)$$

Clearly $u_m \to 0$ uniformly on $[0, 1]$, but the zero function is not a solution of (4.1). In this case, the convergence of the functions u_m is not accompanied by the convergence of their derivatives.
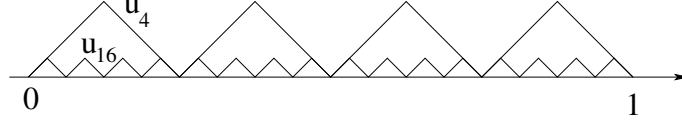
Figure 9: The functions u_m in Example 4.1 converge uniformly to the zero function, which is not a solution of the equation (4.1).

The next lemma shows that, in the case of viscosity solutions, a general stability theorem holds, without any requirement about the convergence of derivatives.

Lemma 4.1 *Consider a sequence of continuous functions u_m, which provide viscosity sub-solutions (super-solutions) to*

$$F_m(x, u_m, \nabla u_m) = 0 \qquad x \in \Omega. \tag{4.3}$$

As $m \to \infty$, assume that $F_m \to F$ uniformly on compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and $u_m \to u$ uniformly on Ω. Then u is a subsolution (a supersolution) of (3.1).

Proof. To prove that u is a subsolution, let $\phi \in \mathcal{C}^1$ be such that $u - \phi$ has a strict local maximum at a point x. We need to show that

$$F(x, u(x), \nabla \phi(x)) \leq 0. \tag{4.4}$$

By Lemma 2.2, there exists a sequence $x_m \to x$ such that $u_m - \phi$ has a local maximum at x_m, and $u_m(x_m) \to u(x)$ as $m \to \infty$. Since u_m is a subsolution,

$$F_m(x_m, u_m(x_m), \nabla \phi(x_m)) \leq 0. \tag{4.5}$$

Letting $m \to \infty$ in (4.5), by continuity we obtain (4.4). $\qquad\square$

The above result should be compared with Example 4.1. Notice the functions u_m in (4.2) are NOT viscosity solutions.

The definition of viscosity solution is naturally motivated by the properties of vanishing viscosity limits.

Theorem 4.1 *For each $\varepsilon > 0$, let u_ε be a $\mathcal{C}^2$ solution to the viscous equation*

$$F(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) = \varepsilon \Delta u_\varepsilon. \tag{4.6}$$

Assume that, as $\varepsilon \to 0+$, we have the convergence $u_\varepsilon \to u$ uniformly on an open set $\Omega \subseteq \mathbb{R}^n$. Then u is a viscosity solution of (3.1).

Proof. By the uniform convergence it follows that the limit function u is continuous. Fix $x \in \Omega$ and assume $p \in D^+u(x)$. To prove that u is a subsolution we need to show that $F(x, u(x), p) \leq 0$.

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1. By Lemma 2.1 and Remark 2.1, there exists $\varphi \in \mathcal{C}^1$ with $\nabla\varphi(x) = p$, such that $u - \varphi$ has a strict local maximum at x. For any given $\delta > 0$, we can then find $0 < \rho \le \delta$ and a function $\psi \in \mathcal{C}^2$ such that

$$|\nabla\varphi(y) - \nabla\varphi(x)| \; \le \; \delta \qquad \text{if} \quad |y - x| \le \rho\,, \tag{4.7}$$

$$\|\psi - \varphi\|_{\mathcal{C}^1} \; \le \; \delta \tag{4.8}$$

and such that the function $u_\varepsilon - \psi$ has a local maximum inside the closed ball $B(x;\rho)$ centered at x with radius ρ, for all $\varepsilon > 0$ small enough.

2. Let $x_\varepsilon \in \Omega$ be the location of this local maximum of $u_\varepsilon - \psi$. Since $u_\varepsilon \in \mathcal{C}^2(\Omega)$, this implies

$$\nabla\psi(x_\varepsilon) \; = \; \nabla u_\varepsilon(x_\varepsilon), \qquad\qquad \Delta u_\varepsilon(x_\varepsilon) \; \le \; \Delta\psi(x_\varepsilon). \tag{4.9}$$

Hence from (4.6) it follows

$$F(x_\varepsilon,\, u_\varepsilon(x_\varepsilon),\, \nabla\psi(x_\varepsilon)) \; \le \; \varepsilon\,\Delta\psi(x_\varepsilon). \tag{4.10}$$

3. We can now select a sequence $\varepsilon_\nu \to 0+$ such that $\lim_{\nu\to\infty} x_{\varepsilon_\nu} = \tilde{x}$ for some limit point $\tilde{x} \in B(x,\rho)$. Since $\psi \in \mathcal{C}^2$, we can pass to the limit in (4.10) and conclude

$$F(x,\, u(\tilde{x}),\, \nabla\psi(\tilde{x})) \; \le \; 0. \tag{4.11}$$

By (4.7)-(4.8) we have

$$|\nabla\psi(\tilde{x}) - p| \; \le \; |\nabla\psi(\tilde{x}) - \nabla\varphi(\tilde{x})| + |\nabla\varphi(\tilde{x}) - \nabla\varphi(x)|$$
$$\le \; \delta + \delta\,. \tag{4.12}$$

Since $\rho, \delta > 0$ can be taken arbitrarily small, (4.11) and the continuity of F and u imply $F(x, u(x), p) \le 0$. Hence u is a subsolution. The fact that u is a supersolution is proved in an entirely similar way. $\square$

Remark 4.1 In the light of the above result, it should not come as a surprise that the two equations

$$F(x, u, \nabla u) \; = \; 0 \qquad \text{and} \qquad -F(x, u, \nabla u) \; = \; 0$$

may have different viscosity solutions. Indeed, solutions to the first equation are obtained as limits of (4.6) as $\varepsilon \to 0+$, while solutions to the second equation are obtained as limits of (4.6) as $\varepsilon \to 0-$. These two limits can be substantially different.

5 Comparison theorems

A remarkable feature of the notion of viscosity solutions is that on one hand it requires a minimum amount of regularity (just continuity), and on the other hand it is stringent enough to yield general comparison and uniqueness theorems.

The uniqueness proofs are based on a technique of doubling of variables, which reminds of Kruzhkov's uniqueness theorem for conservation laws [11]. We shall illustrate this basic technique in a simple setting.

Theorem 5.1 (comparison). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u_1, u_2 \in \mathcal{C}(\overline{\Omega})$ be, respectively, viscosity sub- and supersolutions of*

$$u + H(x, \nabla u) = 0 \qquad x \in \Omega. \qquad (5.1)$$

Assume that

$$u_1(x) \leq u_2(x) \qquad for\ all\ \ x \in \partial\Omega. \qquad (5.2)$$

Moreover, assume that $H : \Omega \times \mathbb{R}^n \mapsto \mathbb{R}$ is uniformly continuous in the x-variable:

$$|H(x, p) - H(y, p)| \leq \omega\Big(|x - y|(1 + |p|)\Big), \qquad (5.3)$$

for some continuous and non-decreasing function $\omega : [0, \infty[\, \mapsto [0, \infty[$ with $\omega(0) = 0$. Then

$$u_1(x) \leq u_2(x) \qquad for\ all\ \ x \in \overline{\Omega}. \qquad (5.4)$$

Proof. To appreciate the main idea of the proof, consider first the case where u_1, u_2 are smooth (Fig. 10, left). If the conclusion (5.4) fails, then the difference $u_1 - u_2$ attains a positive maximum at some interior point $x_0 \in \Omega$. This implies $p \doteq \nabla u_1(x_0) = \nabla u_2(x_0)$. By definition of sub- and supersolution, we now have

$$\begin{aligned} u_1(x_0) + H(x_0, p) &\leq 0, \\ u_2(x_0) + H(x_0, p) &\geq 0. \end{aligned} \qquad (5.5)$$

Subtracting the second from the first inequality in (5.5) we conclude $u_1(x_0) - u_2(x_0) \leq 0$, reaching a contradiction.
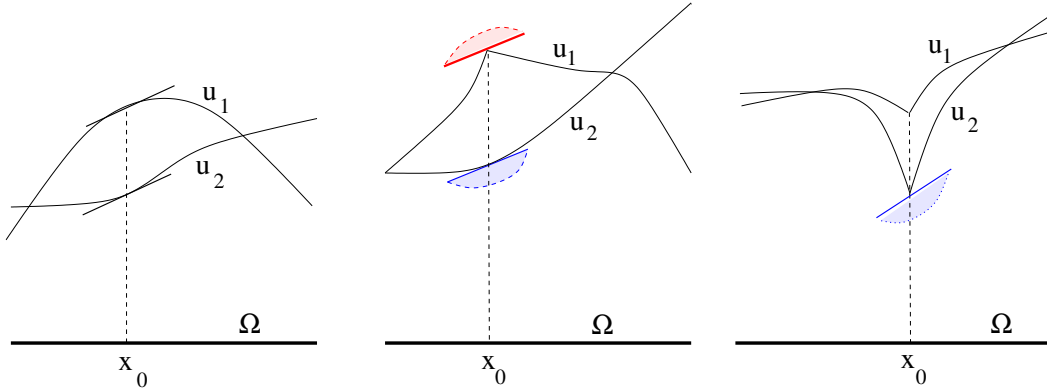

Figure 10: Proving Theorem 5.1. Let $u_1 - u_2$ attain a positive maximum at the point x_0. If u_1, u_2 are both differentiable at x_0 (left), one immediately obtains a contradiction. The same holds if there exists some vector $p \in D^- u_2(x_0) \cap D^+ u_1(x_0)$ (center). However, if $D^- u_2(x_0) = \emptyset$ or $D^+ u_1(x_0) = \emptyset$ (right), a different approach is needed.

Next, consider the non-smooth case. We can repeat the above argument and reach again a contradiction provided that we can find a point x_0 such that (Fig. 10, center)

(i) $u_1(x_0) > u_2(x_0)$,

(ii) some vector p lies at the same time in the upper differential $D^+ u_1(x_0)$ and in the lower differential $D^- u_2(x_0)$.

A natural candidate for x_0 is a point where $u_1 - u_2$ attains a global maximum. Unfortunately, at such point one of the sets $D^+ u_1(x_0)$ or $D^- u_2(x_0)$ may be empty, and the argument breaks down (Fig. 10, right). To proceed further, the key observation is that we do not need to compare values of u_1 and u_2 at exactly the same point. Indeed, to reach a contradiction, it suffices to find nearby points x_ε and y_ε such that (Fig. 11)

(i′) $u_1(x_\varepsilon) > u_2(y_\varepsilon)$,

(ii′) some vector p lies at the same time in the upper differential $D^+ u_1(x_\varepsilon)$ and in the lower differential $D^- u_2(y_\varepsilon)$.
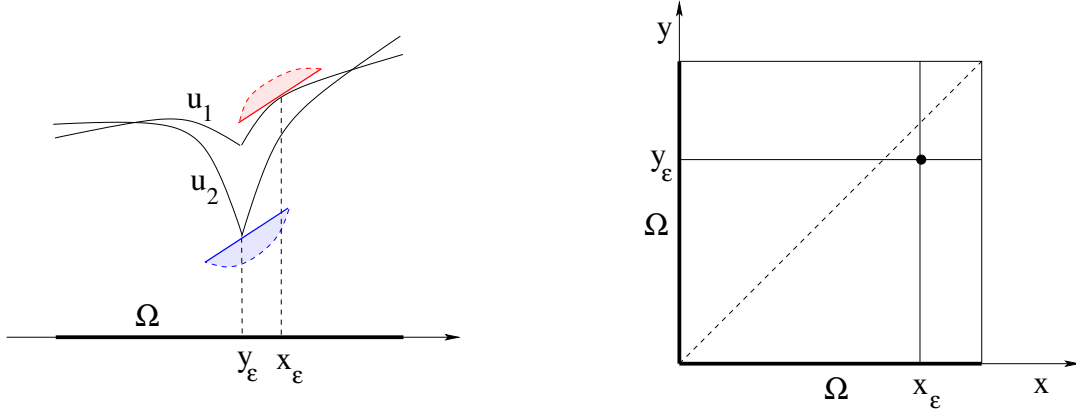

Figure 11: The main idea in the proof of Theorem 5.1. Left: if the conclusion fails, a contradiction is obtained by finding nearby points $x_\varepsilon, y_\varepsilon$ such that $u_1(x_\varepsilon) > u_2(y_\varepsilon)$ and $D^+ u_1(x_\varepsilon) \cap D^- u_2(y_\varepsilon) \neq \emptyset$. Right: the point $(x_\varepsilon, y_\varepsilon)$ is found by maximizing the function Φ_ε in (5.6) in the product space $\Omega \times \Omega$. Since the difference $|x_\varepsilon - y_\varepsilon|$ is heavily penalized, one must have $x_\varepsilon \approx y_\varepsilon$.

Can we always find such points? It is here that the variable-doubling technique comes in. The key idea is to look at the function of two variables

$$\Phi_\varepsilon(x, y) \doteq u_1(x) - u_2(y) - \frac{|x - y|^2}{2\varepsilon} . \tag{5.6}$$

This is a continuous function, which admits a global maximum over the compact set $\overline{\Omega} \times \overline{\Omega}$. If $u_1 > u_2$ at some point x_0, this maximum will be strictly positive. Moreover, taking $\varepsilon > 0$ sufficiently small, the boundary conditions imply that the maximum is attained at some interior point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$. Notice that the points $x_\varepsilon, y_\varepsilon$ must be close to each other, otherwise the penalization term in (5.6) will be very large and negative.

We now observe that the function of a single variable

$$x \; \mapsto \; u_1(x) - \left(u_2(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} \right) \; = \; u_1(x) - \varphi_1(x) \tag{5.7}$$

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attains its maximum at the point x_ε. Hence by Lemma 2.1

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \;=\; \nabla\varphi_1(x_\varepsilon) \;\in\; D^+u_1(x_\varepsilon).$$

Moreover, the function of a single variable

$$y \;\mapsto\; u_2(y) - \left(u_1(x_\varepsilon) - \frac{|x_\varepsilon - y|^2}{2\varepsilon}\right) \;=\; u_2(y) - \varphi_2(y) \qquad (5.8)$$

attains its minimum at the point y_ε. Hence

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \;=\; \nabla\varphi_2(y_\varepsilon) \;\in\; D^-u_2(y_\varepsilon).$$

We have thus discovered two points x_ε, y_ε and a vector $p = (x_\varepsilon - y_\varepsilon)/\varepsilon$ which satisfy the conditions (i′)-(ii′).

We now work out the details of the proof, in several steps.

1. If the conclusion fails, then there exists $x_0 \in \Omega$ such that

$$u_1(x_0) - u_2(x_0) \;=\; \max_{x\in\overline\Omega}\{u_1(x) - u_2(x)\} \;\doteq\; \delta \;>\; 0. \qquad (5.9)$$

For $\varepsilon > 0$, call $(x_\varepsilon, y_\varepsilon)$ a point where the function Φ_ε in (5.6) attains its global maximum on the compact set $\overline\Omega \times \overline\Omega$. By (5.9) one has

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \;\geq\; \delta \;>\; 0. \qquad (5.10)$$

2. Call M an upper bound for all values $|u_1(x)|$, $|u_2(x)|$, as $x \in \overline\Omega$. Then

$$\Phi_\varepsilon(x, y) \;\leq\; 2M - \frac{|x-y|^2}{2\varepsilon},$$

$$\Phi_\varepsilon(x, y) \;\leq\; 0 \qquad \text{if} \quad |x - y|^2 \;\geq\; 4M\varepsilon.$$

Hence (5.10) implies

$$|x_\varepsilon - y_\varepsilon| \;\leq\; 2\sqrt{M\varepsilon}. \qquad (5.11)$$

3. By the uniform continuity of the functions u_1, u_2 on the compact set $\overline\Omega$, for $\varepsilon' > 0$ sufficiently small we have

$$|u_i(x) - u_i(y)| \;<\; \frac{\delta}{2} \qquad \text{whenever} \quad |x - y| \;\leq\; \sqrt{M\varepsilon'}, \quad i = 1, 2. \qquad (5.12)$$

We now show that, choosing $\varepsilon < \varepsilon'$, the points $x_\varepsilon, y_\varepsilon$ cannot lie on the boundary of Ω. For example, if $x_\varepsilon \in \partial\Omega$, then by (5.2) and (5.11)-(5.12) it follows

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \quad\leq\quad \left(u_1(x_\varepsilon) - u_2(x_\varepsilon)\right) + \left|u_2(x_\varepsilon) - u_2(y_\varepsilon)\right| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}$$

$$\leq\quad 0 + \delta/2 + 0,$$

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in contradiction with (5.10).

4. Having shown that $x_\varepsilon, y_\varepsilon$ are interior points, we consider the functions of one single variable φ_1, φ_2 defined at (5.7)-(5.8). Since x_ε provides a local maximum for $u_1 - \varphi_1$ and y_ε provides a local minimum for $u_2 - \varphi_2$, we conclude that

$$p_\varepsilon \doteq \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \in D^+ u_1(x_\varepsilon) \cap D^- u_2(y_\varepsilon). \tag{5.13}$$

From the definition of viscosity sub- and supersolution we now obtain

$$\begin{aligned} u_1(x_\varepsilon) + H(x_\varepsilon, p_\varepsilon) &\leq 0, \\ u_2(y_\varepsilon) + H(y_\varepsilon, p_\varepsilon) &\geq 0. \end{aligned} \tag{5.14}$$

5. We now refine the estimate (5.11), showing that $\varepsilon^{-1/2}|x_\varepsilon - y_\varepsilon| \to 0$ as $\varepsilon \to 0$. Indeed, from

$$u_1(x_\varepsilon) - u_2(x_\varepsilon) = \Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \leq u_1(x_\varepsilon) - u_2(x_\varepsilon) + |u_2(x_\varepsilon) - u_2(y_\varepsilon)| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}$$

it follows

$$|u_2(x_\varepsilon) - u_2(y_\varepsilon)| - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \geq 0.$$

Using (5.11) and the uniform continuity of u_2, we thus obtain

$$\limsup_{\varepsilon \to 0+} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \leq \limsup_{\varepsilon \to 0+} |u_2(x_\varepsilon) - u_2(y_\varepsilon)| = 0. \tag{5.15}$$

6. By (5.10) and (5.13), subtracting the second from the first inequality in (5.14) and using (5.3), we obtain

$$\begin{aligned} \delta &\leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \\ &\leq u_1(x_\varepsilon) - u_2(y_\varepsilon) \\ &\leq |H(x_\varepsilon, p_\varepsilon) - H(y_\varepsilon, p_\varepsilon)| \\ &\leq \omega\Big((|x_\varepsilon - y_\varepsilon| \cdot (1 + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|)\Big). \end{aligned} \tag{5.16}$$

This yields a contradiction, Indeed, by (5.15) the right hand side of (5.16) becomes arbitrarily small as $\varepsilon \to 0$. $\qquad\square$

An easy consequence of the above result is the uniqueness of solutions to the boundary value problem

$$\begin{aligned} u + H(x, \nabla u) &= 0 & x \in \Omega, \tag{5.17} \\ u &= \psi & x \in \partial\Omega. \tag{5.18} \end{aligned}$$

Corollary 5.2 (uniqueness). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let the Hamiltonian function H satisfy the equicontinuity assumption (5.3). Then the boundary value problem (5.17)-(5.18) admits at most one viscosity solution.*

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Proof. Let u_1, u_2 be viscosity solutions. Since u_1 is a subsolution and u_2 is a supersolution, and $u_1 = u_2$ on $\partial\Omega$, by Theorem 5.1 we conclude $u_1 \leq u_2$ on $\overline{\Omega}$. Interchanging the roles of u_1 and u_2 one obtains $u_2 \leq u_1$, completing the proof. $\qquad\square$

5.1 Time dependent problems.

By similar techniques, comparison and uniqueness results can be proved also for Hamilton-Jacobi equations of evolutionary type. Consider the Cauchy problem

$$u_t + H(t, x, \nabla u) \;=\; 0 \qquad\qquad (t, x) \in\,]0, T[\, \times I\!\!R^n, \qquad\qquad (5.19)$$

$$u(0, x) \;=\; \bar{u}(x) \qquad\qquad x \in I\!\!R^n. \qquad\qquad (5.20)$$

Here and in the sequel, it is understood that $\nabla u \doteq (u_{x_1}, \ldots, u_{x_n})$ always refers to the gradient of u w.r.t. the space variables.

Theorem 5.2 (comparison). *Let the function $H : [0, T] \times I\!\!R^n \times I\!\!R^n$ satisfy the Lipschitz continuity assumptions*

$$|H(t, x, p) - H(s, y, p)| \;\leq\; C(|t - s| + |x - y|)(1 + |p|), \qquad\qquad (5.21)$$

$$|H(t, x, p) - H(t, x, q)| \;\leq\; C\,|p - q|. \qquad\qquad (5.22)$$

Let u, v be bounded, uniformly continuous sub- and super-solutions of (5.19) respectively. If $u(0, x) \leq v(0, x)$ for all $x \in I\!\!R^n$, then

$$u(t, x) \;\leq\; v(t, x) \qquad\qquad \text{for all } \ (t, x) \in [0, T] \times I\!\!R^n. \qquad\qquad (5.23)$$

Toward this result, as a preliminary we prove

Lemma 5.1 *Let u be a continuous function on $[0, T] \times I\!\!R^n$, which provides a subsolution of (5.19) for $t \in\,]0, T[$. If $\phi \in \mathcal{C}^1$ is such that $u - \phi$ attains a local maximum at a point (T, x_0), then*

$$\phi_t(T, x_0) + H(T, x_0, \nabla\phi(T, x_0)) \;\leq\; 0. \qquad\qquad (5.24)$$

Proof. We can assume that (T, x_0) is a point of strict local maximum for $u - \phi$. For each $\varepsilon > 0$ consider the function

$$\phi_\varepsilon(t, x) \;=\; \phi(t, x) + \frac{\varepsilon}{T - t}.$$

Each function $u - \phi_\varepsilon$ will then have a local maximum at a point $(t_\varepsilon, x_\varepsilon)$, with

$$t_\varepsilon \;<\; T, \qquad\qquad (t_\varepsilon, x_\varepsilon) \to (T, x_0) \quad \text{as} \ \ \varepsilon \to 0+\,.$$

Since u is a subsolution, one has

$$\phi_{\varepsilon, t}(t_\varepsilon, x_\varepsilon) + H(t_\varepsilon, x_\varepsilon, \nabla\phi_\varepsilon(t_\varepsilon, x_\varepsilon)) \;\leq\; 0\,,$$

$$\phi_t(t_\varepsilon, x_\varepsilon) + H(t_\varepsilon, x_\varepsilon, \nabla\phi(t_\varepsilon, x_\varepsilon)) \;\leq\; -\frac{\varepsilon}{(T - t_\varepsilon)^2}\,. \qquad\qquad (5.25)$$

Letting $\varepsilon \to 0+$, from (5.25) we obtain (5.24). $\qquad\square$

Proof of Theorem 5.2.

1. If (5.23) fails, then we can find $\lambda > 0$ such that

$$\sup_{t,x} \left\{ u(t,x) - v(t,x) - 2\lambda t \right\} \; \doteq \; \sigma \; > \; 0. \tag{5.26}$$

Assume that the supremum in (5.26) is actually attained at a point (t_0, x_0), possibly with $t_0 = T$. If both u and u are differentiable at such point, we easily obtain a contradiction. Indeed,

$$\nabla u(t_0, x_0) \; = \; \nabla v(t_0, x_0), \qquad u_t(t_0, x_0) - v_t(t_0, x_0) - 2\lambda \; \geq \; 0 \,,$$

while

$$u_t(t_0, x_0) + H(t_0, x_0, \nabla u) \; \leq \; 0 \,,$$
$$v_t(t_0, x_0) + H(t_0, x_0, \nabla v) \; \geq \; 0 \,.$$

2. To extend the above argument to the general case, we face two technical difficulties. First, the function in (5.26) may not attain its global maximum over the unbounded set $[0, T] \times I\!\!R^n$. Moreover, at this point of maximum the functions u, v may not be differentiable. These problems are overcome by inserting a penalization term, and doubling the variables. As in the proof of Theorem 5.1, we introduce the function

$$\Phi_\varepsilon(t, x, s, y) \; \doteq \; u(t,x) - v(s,y) - \lambda(t+s) - \varepsilon(|x|^2 + |y|^2) - \frac{1}{\varepsilon^2}(|t-s|^2 + |x-y|^2) \,.$$

Thanks to the penalization terms, the function Φ_ε clearly admits a global maximum at a point $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \in (]0, T] \times I\!\!R^n)^2$. Choosing $\varepsilon > 0$ sufficiently small, one has

$$\Phi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \; \geq \; \max_{t,x} \Phi_\varepsilon(t, x, t, x) \; \geq \; \frac{\sigma}{2} \,.$$

3. We now observe that the function

$$(t,x) \; \mapsto \; u(t,x) - \left[v(s_\varepsilon, y_\varepsilon) + \lambda(t+s_\varepsilon) + \varepsilon(|x|^2 + |y_\varepsilon|^2) + \frac{1}{\varepsilon^2}(|t-s_\varepsilon|^2 + |x-y_\varepsilon|^2) \right] \; \doteq \; u(t,x) - \phi(t,x)$$

attains its maximum at the point $(t_\varepsilon, x_\varepsilon)$. Since u is a subsolution and ϕ is smooth, this implies

$$\phi_t(t_\varepsilon, x_\varepsilon) + H(t_\varepsilon, x_\varepsilon, \nabla\phi(t_\varepsilon, x_\varepsilon)) \; = \; \lambda + \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + H\left(t_\varepsilon, \, x_\varepsilon, \, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon \right) \; \leq \; 0 \,. \tag{5.27}$$

Notice that, in the case where $t_\varepsilon = T$, (5.27) follows from Lemma 5.1.

Similarly, the function

$$(s,y) \; \mapsto \; v(s,y) - \left[u(t_\varepsilon, x_\varepsilon) - \lambda(t_\varepsilon + s) - \varepsilon(|x_\varepsilon|^2 + |y|^2) - \frac{1}{\varepsilon^2}(|t_\varepsilon - s|^2 + |x_\varepsilon - y|^2) \right] \; \doteq \; v(s,y) - \psi(s,y)$$

attains its minimum at the point $(s_\varepsilon, y_\varepsilon)$. Since v is a supersolution and ψ is smooth, this implies

$$\psi_t(s_\varepsilon, y_\varepsilon) + H(t_\varepsilon, x_\varepsilon, \nabla\psi(s_\varepsilon, y_\varepsilon)) \; = \; -\lambda + \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + H\left(s_\varepsilon, \, y_\varepsilon, \, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon \right) \; \geq \; 0 \,. \tag{5.28}$$

4. Subtracting (5.28) from (5.27) and using (5.21)-(5.22) we obtain

$$2\lambda \leq H\left(s_\varepsilon, y_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} - 2\varepsilon y_\varepsilon\right) - H\left(t_\varepsilon, x_\varepsilon, \frac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2} + 2\varepsilon x_\varepsilon\right)$$

$$\leq C\varepsilon(|x_\varepsilon| + |y_\varepsilon|) + C(|t_\varepsilon - s_\varepsilon| + |x_\varepsilon - y_\varepsilon|)\left(1 + \frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon^2} + \varepsilon(|x_\varepsilon| + |y_\varepsilon|)\right).$$

(5.29)

To reach a contradiction we need to show that the right hand side of (5.29) approaches zero as $\varepsilon \to 0$.

5. Since u, v are globally bounded, the penalization terms must satisfy uniform bounds, independent of ε. Hence

$$|x_\varepsilon|, \; |y_\varepsilon| \; \leq \; \frac{C'}{\sqrt{\varepsilon}}, \qquad |t_\varepsilon - s_\varepsilon|, \; |x_\varepsilon - y_\varepsilon| \; \leq \; C'\varepsilon \tag{5.30}$$

for some constant C'. This implies

$$\varepsilon(|x_\varepsilon| + |y_\varepsilon|) \; \leq \; 2C'\sqrt{\varepsilon}. \tag{5.31}$$

To obtain a sharper estimate, we now observe that $\Phi_\varepsilon(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \geq \Phi_\varepsilon(t_\varepsilon, x_\varepsilon, t_\varepsilon, x_\varepsilon)$, hence

$$u(t_\varepsilon, x_\varepsilon) - v(s_\varepsilon, y_\varepsilon) - \lambda(t_\varepsilon + s_\varepsilon) - \varepsilon\left(|x_\varepsilon|^2 + |y_\varepsilon|^2\right) - \frac{1}{\varepsilon^2}\left(|t_\varepsilon - s_\varepsilon|^2 + |x_\varepsilon - y_\varepsilon|^2\right)$$

$$\geq \; u(t_\varepsilon, x_\varepsilon) - v(t_\varepsilon, x_\varepsilon) - 2\lambda t_\varepsilon - 2\varepsilon|x_\varepsilon|^2,$$

$$\frac{1}{\varepsilon^2}\left(|t_\varepsilon - s_\varepsilon|^2 + |x_\varepsilon - y_\varepsilon|^2\right) \; \leq \; v(t_\varepsilon, x_\varepsilon) - v(s_\varepsilon, y_\varepsilon) + \lambda(t_\varepsilon - s_\varepsilon) + \varepsilon\left(|x_\varepsilon|^2 - |y_\varepsilon|^2\right). \tag{5.32}$$

By the uniform continuity of v, the right hand side of (5.32) tends to zero as $\varepsilon \to 0$. Therefore

$$\frac{|t_\varepsilon - s_\varepsilon|^2 + |x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{5.33}$$

By (5.30), (5.31) and (5.33), the right hand side of (5.29) also approaches zero, This yields the desired contradiction. $\qquad\square$

Corollary 5.1 (uniqueness). *Let the function H satisfy the assumptions (5.21)-(5.22). Then the Cauchy problem (5.19)-(5.20) admits at most one bounded, uniformly continuous viscosity solution $u : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$.*

6   Control systems

The time evolution of a system, whose state is described by a finite number of parameters, can be usually modelled by an O.D.E.

$$\dot{x} \; = \; f(x) \qquad x \in \mathbb{R}^n.$$

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Here and in the sequel the upper dot denotes a derivative w.r.t. time. In some cases, the system can be influenced also by the external input of a controller. An appropriate model is then provided by a **control system**, having the form

$$\dot{x} = f(x, u). \tag{6.1}$$

Here $x \in \mathbb{R}^n$, while the control $u : [0, T] \mapsto U$ is required to take values inside a given set $U \subseteq \mathbb{R}^m$. We denote by

$$\mathcal{U} \doteq \left\{ u : \mathbb{R} \mapsto \mathbb{R}^m \text{ measurable, } u(t) \in U \text{ for a.e. } t \right\}$$

the set of **admissible control functions**. To guarantee local existence and uniqueness of solutions, it is natural to assume that the map $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is Lipschitz continuous w.r.t. x and continuous w.r.t. u. The solution of the Cauchy problem (6.1) with initial condition

$$x(t_0) = x_0 \tag{6.2}$$

will be denoted as $t \mapsto x(t; t_0, x_0, u)$. It is clear that, as u ranges over the whole set of control functions, one obtains a family of possible trajectories for the system. Equivalently, these trajectories can be characterized as the solutions to the **differential inclusion**

$$\dot{x} \in F(x), \qquad F(x) \doteq \left\{ f(x, \omega); \ \omega \in U \right\}. \tag{6.3}$$

There are two main ways to assign a control u:

(i) As a function of time: $t \mapsto u(t)$. We then say that u is an **open loop control**.

(ii) As a function of the state: $x \mapsto u(x)$. We then say that u is a **feedback control**.

Example 6.1 (boat on a river). Call $x(t) \in \mathbb{R}^2$ the position of a boat on a river, and let $\mathbf{v}(x)$ be the velocity of the water at the point x. If the boat simply drifts along with the currrent, its position is described by the differential equation

$$\dot{x} = \mathbf{v}(x).$$

If we assume that the boat is powered by an engine, and can move in any direction with speed $\leq \rho$ (relative to the water), the evolution can be modelled by the control system

$$\dot{x} = f(x, u) = \mathbf{v}(x) + u, \qquad |u| \leq \rho.$$

This is equivalent to a differential inclusion where the sets of velocities are balls with radius ρ (Fig. 12):

$$\dot{x} \in F(x) = B(\mathbf{v}(x); \rho).$$

Example 6.2 (cart on a rail). Consider a cart which can move without friction along a straight rail (Fig. 13). For simplicity, assume that it has unit mass. Let $y(0) = \bar{y}$ be its initial
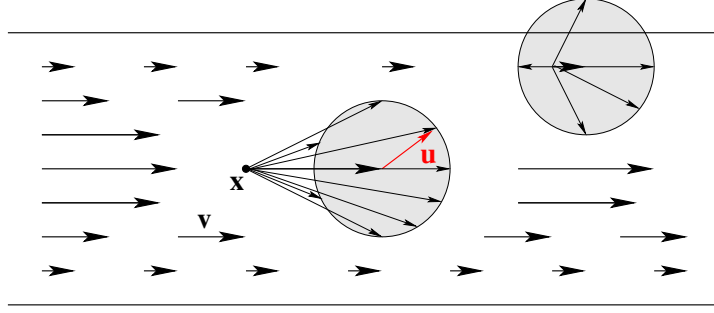
Figure 12: Velocity of the water and the set of possible velocities of a boat, in Example 6.1.

position and $\dot{y}(0) = \bar{v}$ be its initial velocity. If no forces are present, its future position is simply given by

$$y(t) = \bar{y} + t\,\bar{v}\,.$$

Next, assume that a controller is able to push the cart, with an external force $u = u(t)$. The evolution of the system is then determined by the second order equation

$$\ddot{y}(t) = u(t)\,. \tag{6.4}$$

Calling $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$ respectively the position and the velocity of the cart at time t, the equation (6.4) can be written as a first order control system:

$$(\dot{x}_1, \dot{x}_2) = (x_2, u). \tag{6.5}$$

Given the initial condition $x_1(0) = \bar{y}$, $x_2(0) = \bar{v}$, the solution of (6.5) is provided by

$$\begin{cases} x_1(t) & = \bar{y} + \bar{v}t + \displaystyle\int_0^t (t-s)u(s)\,ds, \\[2mm] x_2(t) & = \bar{v} + \displaystyle\int_0^t u(s)\,ds. \end{cases}$$

Assuming that the force satisfies the constraint

$$|u(t)| \leq 1\,,$$

the control system (6.5) is equivalent to the differential inclusion

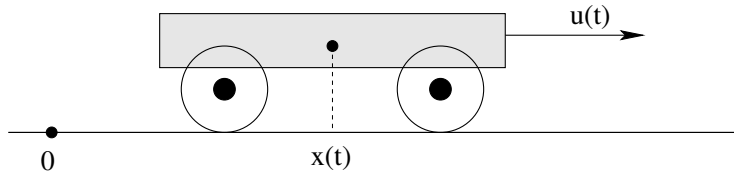$$(\dot{x}_1, \dot{x}_2) \in F(x_1, x_2) = \{(x_2, \omega)\,;\quad -1 \leq \omega \leq 1\}\,.$$

Figure 13: A cart moving on a rail, in Example 6.2.

We now consider the problem of steering the system to the origin $(0,0) \in I\!R^2$. In other words, we want the cart to be eventually at the origin with zero speed. For example, if the initial condition is $(\bar{y}, \bar{v}) = (2,2)$, this goal is achieved by the open-loop control

$$\tilde{u}(t) = \begin{cases} -1 & \text{if } 0 \le t < 4, \\ 1 & \text{if } 4 \le t < 6, \\ 0 & \text{if } t \ge 6. \end{cases}$$

A direct computation shows that $(x_1(t), x_2(t)) = (0,0)$ for $t \ge 6$. Notice, however, that the above control would not accomplish the same task in connection with any other initial data $(\bar{y}, \bar{v})$ different from $(2,2)$. This is a consequence of the backward uniqueness of solutions to the differential equation (6.5).

A related problem is that of asymptotic stabilization. In this case, we seek a feedback control function $u = u(x_1, x_2)$ such that, for every initial data $(\bar{y}, \bar{v})$, the corresponding solution of the Cauchy problem

$$(\dot{x}_1, \dot{x}_2) = (x_2, u(x_1, x_2)), \qquad (x_1, x_2)(0) = (\bar{y}, \bar{v})$$

approaches the origin as $t \to \infty$, i.e.

$$\lim_{t \to \infty} (x_1, x_2)(t) = (0,0).$$

There are several feedback controls which accomplish this task. For example, one can take $u(x_1, x_2) = -x_1 - x_2$.

Example 6.3 (fish harvesting). Let $x(t) \ge 0$ describe the total amount of fish in a lake, at time t. Denote by $u(t) \in [0, u^{max}]$ the instantaneous harvesting effort. Then the evolution of the fish population can be described by

$$\dot{x} = a\, x(M - x) - xu. \tag{6.6}$$

Here M is the maximum population supported by the habitat, and the constant a is a growth rate.

An important class of control systems have the form

$$\dot{x} = f(x) + g(x)\, u, \qquad u \in [-1, 1], \tag{6.7}$$

where f, g are vector fields on $I\!R^n$. This is equivalent to a differential inclusion

$$\dot{x} \in F(x) = \{f(x) + g(x)\, u; \quad u \in [-1, 1]\}, \tag{6.8}$$

where each set $F(x)$ of possible velocities is a segment (Fig. 14). Systems of this type, linear w.r.t. the control variable u, have been extensively studied using techniques from differential geometry.

Example 6.4 (car steering). The position of a car can be described by three scalar parameters: the coordinates (x, y) of its barycenter $B \in I\!R^2$ and the angle θ giving its orientation,
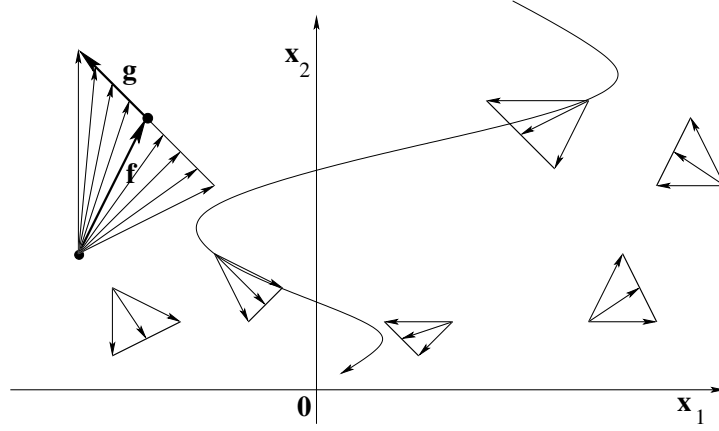
Figure 14: The sets of velocities for the planar system (6.7), where the control enters linearly.

as in Figure 15. The driver controls the motion of the car by acting on the gas pedal and on the steering wheel. The control function thus has two components: speed $u(t)$ of the car and the turning angle $\alpha(t)$. The motion is described by the control system

$$
\begin{cases}
\dot{x} & = u\,\cos\theta\,, \\
\dot{y} & = u\,\sin\theta\,, \\
\dot{\theta} & = \alpha\,u\,.
\end{cases}
\tag{6.9}
$$

It is reasonable here to impose bounds on speed of the car and on the steering angle, say

$$
u(t) \in [-m, M]\,, \qquad\qquad \alpha(t) \in [-\bar{\alpha},\, \bar{\alpha}]\,.
$$

A frequently encountered problem is the following: given the initial position, steer the car into a parking spot. The typical maneuver needed for parallel parking is illustrated in Figure 15.
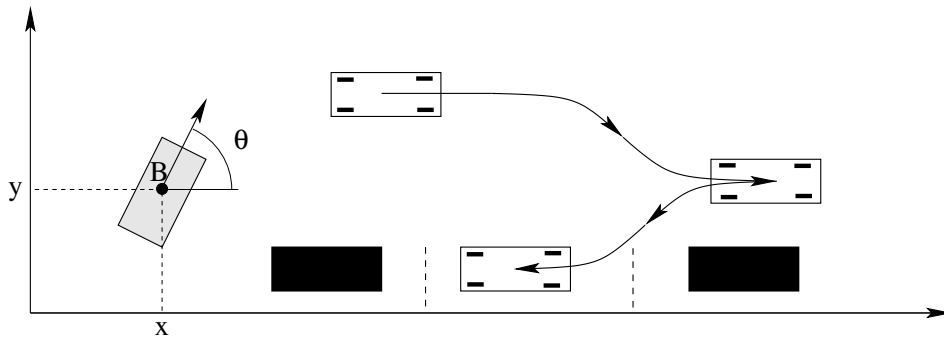

Figure 15: Car parking.

Given a control system in the general form (6.1), the **reachable set** at time T starting from x_0 at time t_0 (Fig. 16) will be denoted by

$$
R(T) \;=\; R(T; t_0, x_0) \;\doteq\; \left\{ x(T; t_0, x_0, u)\,; \quad u \in \mathcal{U} \right\}.
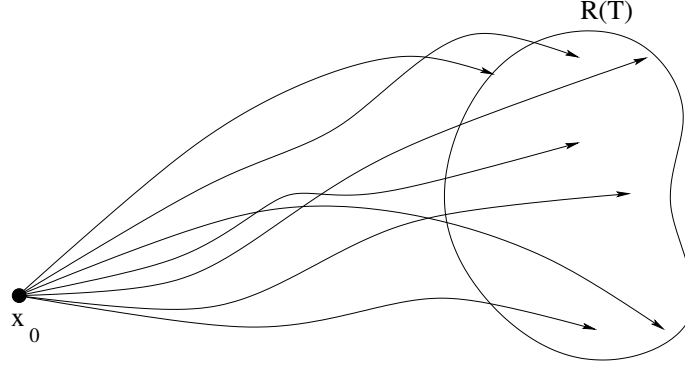\tag{6.10}
$$

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Figure 16: The reachable set at time T, starting from x_0.

Given a target set $\Omega \subset I\!\!R^n$, the **minimum time function** $T(\cdot)$ is defined as

$$T(x_0) \;=\; \inf \; \{t \geq 0\,; \quad R(t;0,x_0) \cap \Omega \neq \emptyset\}. \tag{6.11}$$

For any given $x_0 \in I\!\!R^n$, $T(x_0)$ is thus the time needed to steer the system from x_0 to some point in the target set Ω. In most applications, the target can be a single point, or a closed set.

The function $T(\cdot)$ provides a basic link between control and PDE theory. Indeed:

- The minimum time function can be characterized as the unique solution to a PDE of Hamilton-Jacobi type.

- If the minimum time function is known, then the optimal controls (reaching the target in minimum time) can be found.

In connection with the control system (6.1), various mathematical issues are of relevance:

1 - Dynamics. Starting from a point x_0 at time $t_0 = 0$, consider the set of all possible trajectories. Several questions arise:

- **Properties of the reachable set:** For any $T > 0$, is the reachable set $R(T)$ closed, bounded, or convex? Does it have non-empty interior?

- **Local controllability:** Can one reach a whole neighborhood of x_0, for every $T > 0$? In other words, is it true that $x_0 \in \text{int} R(T)$ for every $T > 0$?

- **Global controllability:** Given enough time, can one reach every point $x \in I\!\!R^n$? In other words, is it true that $\bigcup_{T>0} R(T) = I\!\!R^n$?

2 - Feedback Stabilization. In many applications, an important goal is to find a feedback control that stabilizes the system at an equilibrium point. For example, find a control $u = u(x)$ such that all trajectories of the system

$$\dot{x} \;=\; f(x,\,u(x))$$

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approach the origin asymptotically as $t \to \infty$.

3 - Optimal Control. Within a set $\mathcal{U}$ of admissible control functions, we seek a control $u(\cdot) \in \mathcal{U}$ which is optimal w.r.t. a given cost criterion. For example, given the initial condition $x(t_0) = x_0$, we want to minimize the cost

$$J(u) \doteq \int_{t_0}^{T} L(x(t),\, u(t))\, dt + \psi(x(T))$$

over all control functions $u \in \mathcal{U}$. Here it is understood that $x(t) = x(t;\, t_0, x_0, u)$, while

$$L : I\!R^n \times U \mapsto I\!R, \qquad \psi : I\!R^n \mapsto I\!R$$

are continuous functions. We call L the **running cost** and ψ the **terminal cost**.

Given an initial condition $x(0) = x_0$, one can also consider a minimization problem over all positive times $t > 0$. This leads to the infinite horizon problem

$$\text{minimize:} \qquad J(u) \;=\; \int_0^{+\infty} e^{-\gamma t} L(x(t),\, u(t))\, dt\,. \qquad (6.12)$$

Here L is the running cost while $e^{-\gamma t}$, with $\gamma \geq 0$, is a discount factor which assigns lower cost to what will happen far into the future.

6.1 Asymptotic Stabilization.

Consider the control system

$$\dot{x} \;=\; f(x, u), \qquad\qquad x \in I\!R^n, \quad u \in I\!R^m. \qquad (6.13)$$

To fix the ideas, assume that $f(0,0) = 0$, so that the origin $x = 0 \in I\!R^n$ is an equilibrium point. In general, however, if we take $u(x) \equiv 0$, this equilibrium can be unstable. How can we construct a feedback control function $u = u(x)$ that stabilizes the system?

To answer this question, consider first a general ODE on $I\!R^n$:

$$\dot{x} \;=\; g(x) \qquad\qquad (6.14)$$

with $g(0) = 0$. Let $M = (M_{ij}) = (\partial g_i / \partial x_j)$ be the $n \times n$ Jacobian matrix of partial derivatives of g at the origin. A classical theorem in ODE theory states that

- *If all the eigenvalues of the Jacobian matrix M have strictly negative real part, then the origin is an asymptotically stable equilibrium.*

We shall apply the above theorem to the ODE

$$\dot{x} \;=\; g(x) \;=\; f(x, u(x)), \qquad\qquad (6.15)$$

for a suitable feedback control $u(\cdot)$. Consider

- the $n \times n$ matrix $A = (A_{ij}) = (\partial f_i / \partial x_j)$,

- the $n \times m$ matrix $B = (B_{ik}) = (\partial f_i / \partial u_k)$,

- the $m \times n$ matrix $C = (C_{kj}) = (\partial u_k / \partial x_j)$,

where all derivatives are computed at the origin. By the chain rule, the Jacobian matrix of the function g at the origin is given by

$$M \;=\; (M_{ij}) \;=\; \left(A_{ij} + \sum_k B_{ik} C_{kj} \right) \;=\; A + BC.$$

Note that here the matrices A, B depend on f, hence are given a priori, while the matrix C can be chosen as we like. The asymptotic stabilization problem can thus be reduced to:

- *Find an $m \times n$ matrix C such that the resulting matrix $A + BC$ has all of its eigenvalues with negative real part.*

If such a matrix can be found, then the feedback control $u(x) = Cx$ (linear w.r.t. the state x of the system) is asymptotically stabilizing. Conditions which ensure the existence of such a matrix C are well known in the literature. See for example [2, 4].

6.2 Existence of optimal controls.

Consider the optimal control problem

$$\text{maximize:} \qquad \psi(x(T)), \tag{6.16}$$

$$\text{subject to:} \quad \dot{x} \;=\; f(x, u), \qquad x(t_0) = x_0, \qquad u(t) \in U. \tag{6.17}$$

As usual, we assume that f is differentable w.r.t. x and continuous in u. We state here a basic result on the existence of optimal controls.

Theorem 6.1 *Assume that*

(i) *All trajectories of (6.17) remain within a bounded set.*

(ii) *The set U is closed, bounded.*

(iii) *For each $x \in \mathbb{R}^n$, the set of possible velocities $F(x) = \{ f(x, u) ; \; u \in U \}$ is **convex**.*

(iv) *The payoff function ψ is continuous.*

Then the problem (6.16)-(6.17) admits an optimal solution.

Remark 6.1 In order to guarantee that all trajectories remain bounded (instead of going to infinity in finite time), one can require that the function f has sub-linear growth. Namely, the assumption (i) is satisfied provided that

$$|f(x, u)| \;\leq\; C(1 + |x|) \tag{6.18}$$

for some constant C and all $x \in \mathbb{R}^n$, $u \in U$.

Remark 6.2 In the above theorem, the crucial assumption is (iii). If the velocity sets are not convex, an optimal control may not exist, as shown by the following example:

$$\text{minimize:} \quad x_2(T)$$

$$\text{subject to:} \quad \begin{cases} \dot{x}_1 &= u, \\ \dot{x}_2 &= x_1^2, \end{cases} \qquad \begin{cases} x_1(0) &= 0, \\ x_2(0) &= 0, \end{cases} \qquad u(t) \in \{-1, 1\}.$$

The above equations yield

$$x_1(t) = \int_0^t u(s)\, ds, \qquad x_2(T) = \int_0^T \left(\int_0^t u(s)\, ds \right)^2 dt \geq 0.$$
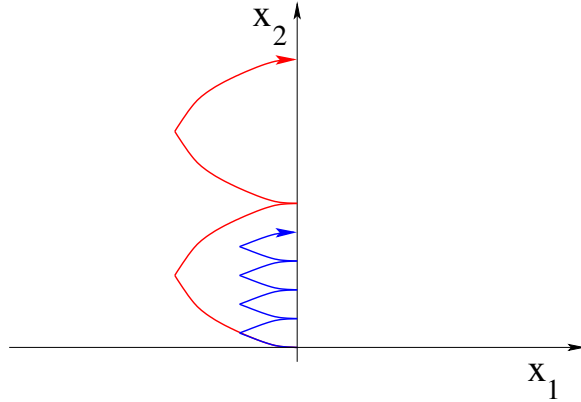

Figure 17: The trajectories corresponding to the oscillating controls $u_n(\cdot)$ in (6.19).

In connection with the highly oscillatory control functions

$$u_n(t) = \begin{cases} +1 & \text{if } t \in \left] (k-1)\frac{T}{n}, k\frac{T}{n} \right], \qquad k \text{ even}, \\ -1 & \text{if } t \in \left] (k-1)\frac{T}{n}, k\frac{T}{n} \right], \qquad k \text{ odd}, \end{cases} \tag{6.19}$$

the corresponding solutions (shown in Fig. 17) satisfy

$$x_{1,n}(t) \to 0, \qquad x_{2,n}(t) \to 0,$$

uniformly for $t \in [0, T]$. However, there is no optimal control $u : [0, T] \mapsto \{-1, 1\}$ which achieves $x_2(T) = 0$. Indeed, one has

$$x_2(T) = \int_0^T \left(\int_0^t u(s)\, ds \right)^2 dt = 0$$

only if $u(t) \equiv 0$. But such a control is not admissible.

7   The Pontryagin Maximum Principle

In connection with the system

$$\dot{x} = f(x, u), \qquad u(t) \in U, \qquad t \in [0, T], \qquad x(0) = x_0, \tag{7.1}$$

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we consider the **Mayer problem:**

$$\max_{u \in \mathcal{U}} \ \psi(x(T, u)) \,. \tag{7.2}$$

Here there is no running cost, but we have a terminal payoff to be maximized over all admissible controls. Let $t \mapsto u^*(t)$ be an optimal control function, and let $t \mapsto x^*(t) = x(t; 0, x_0, u^*)$ be the corresponding optimal trajectory (Fig. 18). We seek necessary conditions that will be satisfied by the control $u^*(\cdot)$.
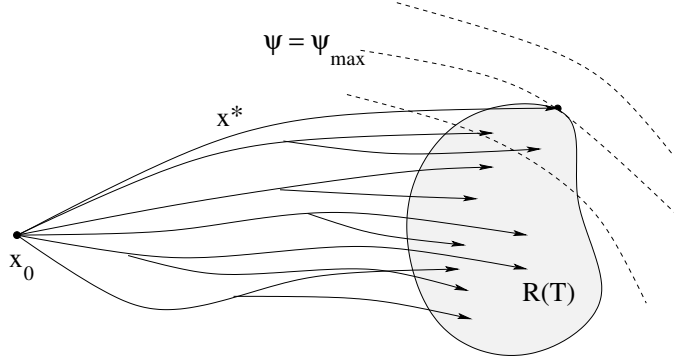

Figure 18: The optimal solution $x^*(\cdot)$ of the Mayer problem (7.2).

As a preliminary, we recall some basic facts from O.D.E. theory. Let $t \mapsto x(t)$ be a solution of the O.D.E.

$$\dot{x} \ = \ g(t, x) \,. \tag{7.3}$$

Assume that $g : [0, T] \times I\!\!R^n \mapsto I\!\!R^n$ is measurable w.r.t. t and continuously differentiable w.r.t. x. Consider a family of nearby solutions (Fig. 19), say $t \mapsto x_\varepsilon(t)$. Assume that at a given time s one has

$$\lim_{\varepsilon \to 0} \frac{x_\varepsilon(s) - x(s)}{\varepsilon} \ = \ v(s) \,.$$

Then the first order tangent vector

$$\mathbf{v}(t) \ \doteq \ \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t) - x(t)}{\varepsilon}$$

is well defined for every $t \in [0, T]$, and satisfies the linearized evolution equation

$$\dot{\mathbf{v}}(t) \ = \ A(t) \, \mathbf{v}(t) \,, \tag{7.4}$$

where

$$A(t) \ \doteq \ D_x g(t, \, x(t)) \tag{7.5}$$

is the $n \times n$ Jacobian matrix of first order partial derivatives of g w.r.t. x. Its entries are $A_{ij} = \partial g_i / \partial x_j$. Using the Landau notation, we can write $x_\varepsilon(t) = x(t) + \varepsilon \mathbf{v}(t) + o(\varepsilon)$, where $o(\varepsilon)$ denotes an infinitesimal of higher order w.r.t. ε. The relations (7.4)-(7.5) are formally derived by equating terms of order ε in the first order approximation

$$\dot{x}_\varepsilon(t) \ = \ \dot{x}(t) + \varepsilon \dot{\mathbf{v}}(t) + o(\varepsilon) \ = \ g(t, x_\varepsilon(t)) \ = \ \dot{x}(t) + D_x g(t, x(t)) \, \varepsilon \, \mathbf{v}(t) + o(\varepsilon) \,.$$
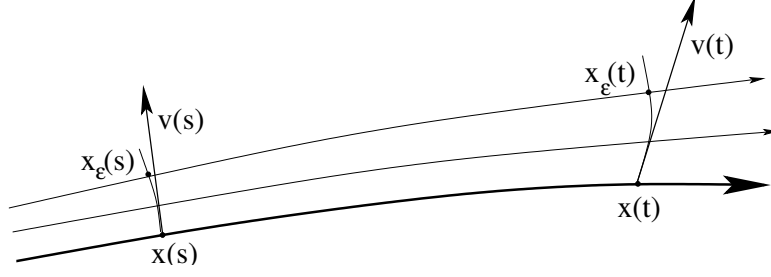
Figure 19: The evolution of a first order perturbation $v(t)$.

Together with (7.4), it is useful to consider the *adjoint system*

$$\dot{p}(t) \;=\; -\,p(t)A(t) \tag{7.6}$$

We regard $p \in \mathbb{R}^n$ as a row vector while $\mathbf{v} \in \mathbb{R}^n$ is a column vector. Notice that, if $t \mapsto p(t)$ and $t \mapsto \mathbf{v}(t)$ are any solutions of (7.6) and of (7.4) respectively, then the product $p(t)\mathbf{v}(t)$ is constant in time. Indeed

$$\frac{d}{dt}(p(t)\,\mathbf{v}(t)) \;=\; \dot{p}(t)\mathbf{v}(t) + p(t)\dot{\mathbf{v}}(t) \;=\; [-p(t)A(t)]\mathbf{v}(t) + p(t)[A(t)\mathbf{v}(t)] \;=\; 0. \tag{7.7}$$

After these preliminaries, we can now derive some necessary conditions for optimality. Since u^* is optimal, the payoff $\psi(x(T, u^*))$ cannot be further increased by any perturbation of the control $u^*(\cdot)$. Fix a time $\tau \in\,]0, T]$ and a control value $\omega \in U$. For $\varepsilon > 0$ small, consider the **needle variation** $u_\varepsilon \in \mathcal{U}$ (Fig. 20):

$$u_\varepsilon(t) \;=\; \begin{cases} \omega & \text{if} \quad t \in [\tau - \varepsilon,\, \tau], \\ u^*(t) & \text{if} \quad t \notin [\tau - \varepsilon,\, \tau]. \end{cases} \tag{7.8}$$
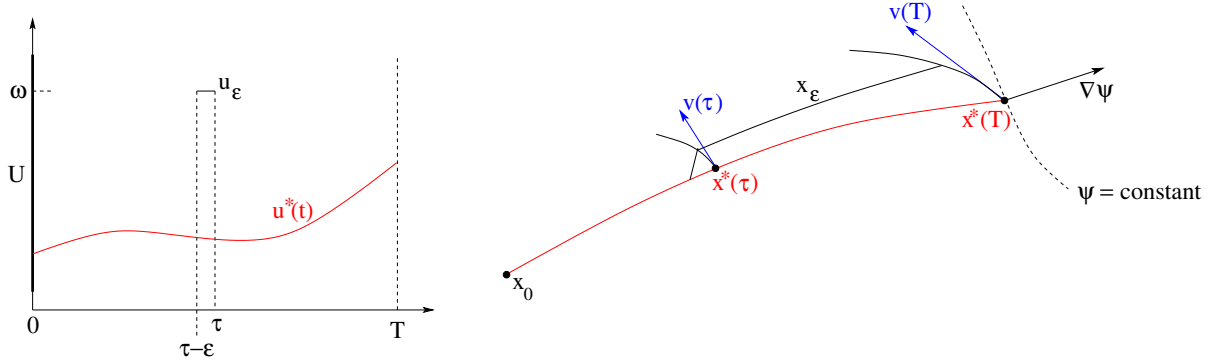

Figure 20: Left: a needle variation of the optimal control $u^*(\cdot)$. Right: the change in the corresponding trajectory $x^*(\cdot)$.

Call $t \mapsto x_\varepsilon(t) = x(t, u_\varepsilon)$ the perturbed trajectory. We shall compute the terminal point $x_\varepsilon(T) = x(T, u_\varepsilon)$ and check that the value of ψ is not increased by this perturbation.

Assuming that the optimal control u^* is continuous at time $t = \tau$, we have

$$\mathbf{v}(\tau) \;\doteq\; \lim_{\varepsilon \to 0} \frac{x_\varepsilon(\tau) - x^*(\tau)}{\varepsilon} \;=\; f(x^*(\tau),\, \omega) - f(x^*(\tau),\, u^*(\tau)). \tag{7.9}$$

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Indeed, $x_\varepsilon(\tau - \varepsilon) = x^*(\tau - \varepsilon)$ and on the small interval $[\tau - \varepsilon,\ \tau]$ we have

$$\dot{x}_\varepsilon \ \approx \ f(x^*(\tau),\ \omega), \qquad\qquad \dot{x}^* \ \approx \ f(x^*(\tau),\ u^*(\tau)).$$

Since $u_\varepsilon = u^*$ on the remaining interval $t \in [\tau,\ T]$, as in (7.4) the evolution of the tangent vector

$$\mathbf{v}(t) \ \doteq \ \lim_{\varepsilon \to 0} \frac{x_\varepsilon(t) - x^*(t)}{\varepsilon} \qquad\qquad t \in [\tau,\ T]$$

is governed by the linear equation

$$\dot{\mathbf{v}}(t) \ = \ A(t)\,\mathbf{v}(t) \tag{7.10}$$

with $A(t) \doteq D_x f(x^*(t),\ u^*(t))$. By maximality, $\psi(x_\varepsilon(T)) \le \psi(x^*(T))$, therefore (Fig. 21)

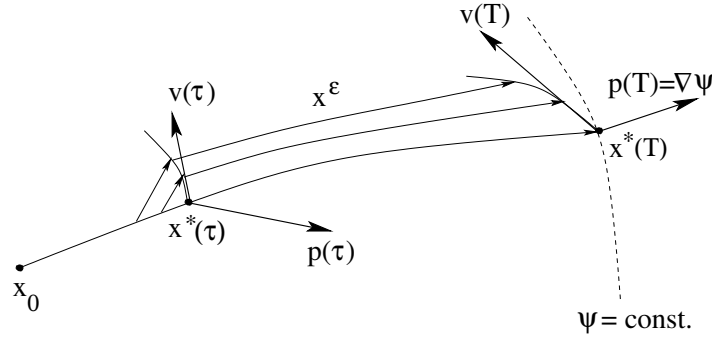$$\nabla\psi(x^*(T)) \cdot \mathbf{v}(T) \ \le \ 0. \tag{7.11}$$

Figure 21: Transporting the vector $p(T)$ backward in time, along the optimal trajectory.

Summing up, the previous analysis has established the following:

For every time $\tau \in\ ]0, T]$ where u^* is continuous and for every admissible control value $\omega \in U$, we can generate the vector

$$\mathbf{v}(\tau) \doteq f(x^*(\tau),\ \omega) - f(x^*(\tau),\ u^*(\tau))$$

and propagate it forward in time, by solving the linearized equation (7.10). The inequality (7.11) is then a necessary condition for optimality.

Instead of propagating the (infinitely many) vectors $v(\tau)$ forward in time, it is more convenient to propagate the single vector $\nabla\psi(x^*(T))$ backward. We thus define the row vector $t \mapsto p(t)$ as the solution of terminal value problem

$$\dot{p}(t) \ = \ -p(t)\,A(t), \qquad\qquad p(T) \ = \ \nabla\psi(x^*(T)). \tag{7.12}$$

By (7.7) one has $p(t)v(t) = p(T)v(T)$ for every t. In particular, (7.11) implies that

$$p(\tau) \cdot \Big[f(x^*(\tau),\ \omega) - f(x^*(\tau),\ u^*(\tau)) \Big] \ = \ \nabla\psi(x^*(T)) \cdot v(T) \ \le \ 0$$

for every $\omega \in U$. Therefore (Fig. 22)

$$p(\tau) \cdot \dot{x}^*(\tau) \ = \ p(\tau) \cdot f(x^*(\tau),\ u^*(\tau)) \ = \ \max_{\omega \in U}\ \Big\{ p(\tau) \cdot f(x^*(\tau),\ \omega) \Big\}. \tag{7.13}$$
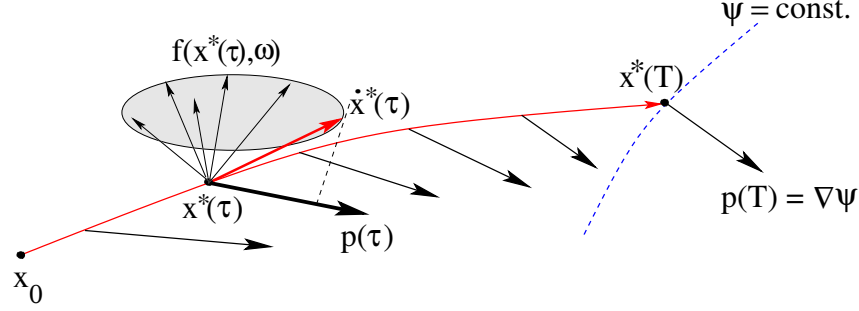
Figure 22: For a.e. time $\tau \in \,]0, T]$, the speed $\dot{x}^*(\tau)$ corresponding to the optimal control $u^*(\tau)$ is the one having inner product with $p(\tau)$ as large as possible.

With some additional care, one can show that the maximality condition (7.13) holds at every time τ which is a Lebesgue point of $u^*(\cdot)$, hence almost everywhere. The above result can be restated as

Theorem 7.1 (Pontryagin Maximum Principle for the Mayer Problem with free terminal point). *Consider the control system*

$$\dot{x} \;=\; f(x, u), \qquad u(t) \in U, \qquad t \in [0, T], \qquad x(0) = x_0 \,.$$

Let $t \mapsto u^*(t)$ be an optimal control and $t \mapsto x^*(t) = x(t, u^*)$ be the corresponding optimal trajectory for the maximization problem

$$\max_{u \in \mathcal{U}} \; \psi(x(T, u)) \,.$$

Define the vector $t \mapsto p(t)$ as the solution to the linear adjoint system

$$\dot{p}(t) \;=\; -\,p(t)\,A(t), \qquad\qquad A(t) \doteq D_x f(x^*(t),\, u^*(t)), \qquad\qquad (7.14)$$

with terminal condition

$$p(T) = \nabla \psi(x^*(T)). \qquad\qquad (7.15)$$

Then, for almost every $\tau \in [0, T]$ the following maximality condition holds:

$$p(\tau) \cdot f(x^*(\tau),\, u^*(\tau)) \;=\; \max_{\omega \in U}\; \Big\{ p(\tau) \cdot f(x^*(\tau),\, \omega) \Big\}. \qquad\qquad (7.16)$$

In the above theorem, x, f, v represent column vectors, $D_x f$ is the $n \times n$ Jacobian matrix of first order partial derivatives of f w.r.t. x, while p is a row vector. In coordinates, the equations (7.14)-(7.15) can be rewritten as

$$\dot{p}_i(t) \;=\; -\sum_{j=1}^{n} p_j(t) \frac{\partial f_j}{\partial x_i}(t, x^*(t), u^*(t)), \qquad\qquad p_i(T) \;=\; \frac{\partial \psi}{\partial x_i}(x^*(T)),$$

while (7.16) takes the form

$$\sum_{i=1}^{n} p_i(t) \cdot f_i(t, x^*(t), u^*(t)) \;=\; \max_{\omega \in U}\; \left\{ \sum_{i=1}^{n} p_i(t) \cdot f_i(t, x^*(t), \omega) \right\}.$$

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Relying on the Maximum Principle, the computation of the optimal control requires two steps:

STEP 1: solve the pointwise maximixation problem (7.16), obtaining the optimal control u^* as a function of p, x, i.e.

$$u^*(x, p) \; = \; \underset{\omega \in U}{\operatorname{argmax}} \, \{p \cdot f(x, \omega)\} \,. \tag{7.17}$$

STEP 2: solve the two-point boundary value problem

$$\begin{cases} \dot{x} & = \; f(x, \; u^*(x, p)), \\[2mm] \dot{p} & = \; -p \cdot D_x f(x, \; u^*(x, p)), \end{cases} \qquad \begin{cases} x(0) & = \; x_0 \,, \\[2mm] p(T) & = \; \nabla \psi(x(T)). \end{cases} \tag{7.18}$$

- In general, the function $u^* = u^*(p, x)$ in (7.17) is highly nonlinear. It may be multivalued or discontinuous.

- The two-point boundary value problem (7.18) can be solved by a **shooting method**: guess an initial value $p(0) = p_0$ and solve the corresponding Cauchy problem. Try to adjust the value of p_0 so that the terminal values $x(T)$, $p(T)$ satisfy the given conditions.

Remark 7.1 In the above theorem, the components of the vector $p(\tau) = (p_1(\tau), \ldots, p_n(\tau))$ can be interpreted as **shadow prices**. Namely, assume that, at time τ, an external agent offers to change our state from $x(\tau)$ to

$$x(\tau) + \varepsilon \mathbf{v}(\tau) \; = \; (x_1 + \varepsilon v_1, \; \ldots, x_n + \varepsilon v_n).$$

Neglecting higher order infinitesimals $o(\varepsilon)$, at the terminal time T the change in our payoff is computed by the inner product

$$\nabla \psi(T, x(T)) \cdot \varepsilon \mathbf{v}(T) \; = \; p(T) \cdot \varepsilon \mathbf{v}(T) \; = \; p(\tau) \cdot \varepsilon \mathbf{v}(\tau).$$

Therefore, if $p_i(\tau)$ is the price for a unit increase in the component x_i, it will indifferent to accept or decline the offer.

Example 7.1 (linear pendulum). Let $q(t) = $ be the position of a linearized pendulum with unit mass, controlled by an external force with magnitude $u(t) \in [-1, 1]$. Then $q(\cdot)$ satisfies the second order ODE

$$\ddot{q}(t) + q(t) \; = \; u(t), \qquad q(0) = \dot{q}(0) = 0, \qquad u(t) \in [-1, 1] \,.$$

We wish to maximize the terminal displacement $q(T)$.

Introducing the variables $x_1 = q$, $x_2 = \dot{q}$, we thus seek

$$\max_{u \in \mathcal{U}} \; x_1(T, u),$$

among all trajectories of the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - x_1 \end{cases} \qquad \begin{cases} x_1(0) &= 0 \\ x_2(0) &= 0. \end{cases}$$

Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory. The linearized equation for a tangent vector is

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The corresponding adjoint vector $p = (p_1, p_2)$ satisfies

$$(\dot{p}_1, \dot{p}_2) = -(p_1, p_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (p_1, p_2)(T) = \nabla\psi(x^*(T)) = (1, 0) \qquad (7.19)$$

because $\psi(x) \doteq x_1$. In this special linear case, we can explicitly solve (7.19) without needing to know x^*, u^*. An easy computation yields

$$(p_1, p_2)(t) = (\cos(T-t), \sin(T-t)). \qquad (7.20)$$

For each t, we must now choose the value $u^*(t) \in [-1, 1]$ so that

$$p_1 x_2 + p_2(-x_1 + u^*) = \max_{\omega \in [-1,1]} \{p_1 x_2 + p_2(-x_1 + \omega)\}.$$

By (7.20), the optimal control is

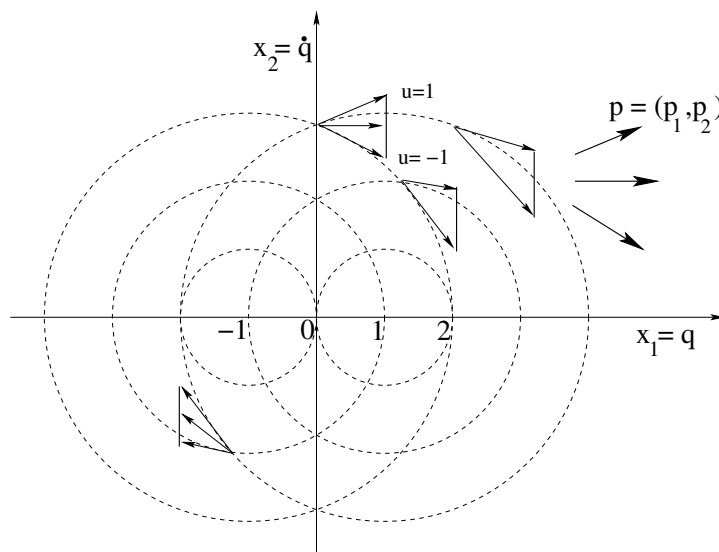$$u^*(t) = \operatorname{sign} p_2(t) = \operatorname{sign}(\sin(T-t)).$$

Figure 23: The trajectories determined by the constant control $u \equiv 1$ and $u \equiv -1$, for the equations of the linear pendulum, Example 7.2.

Example 7.2 Consider the problem on $I\!\!R^3$

$$\text{maximize } x_3(T) \qquad \text{over all controls } u : [0, T] \mapsto [-1, 1]$$

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for the system

$$\begin{cases} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 \\ \dot{x}_3 &= x_2 - x_1^2 \end{cases} \qquad \begin{cases} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_3(0) &= 0. \end{cases} \qquad (7.21)$$

The adjoint equations take the form

$$(\dot{p}_1, \dot{p}_2, \dot{p}_3) \;=\; (p_2 + 2x_1 p_3, \;\; -p_3, \;\; 0), \qquad (p_1, p_2, p_3)(T) \;=\; (0, 0, 1). \qquad (7.22)$$

Maximizing the inner product $p \cdot \dot{x}$ we obtain the optimality conditions for the control u^*

$$p_1 u^* + p_2 (-x_1) + p_3 (x_2 - x_1^2) \;=\; \max_{\omega \in [-1,1]} \left\{ p_1 \omega + p_2 (-x_1) + p_3 (x_2 - x_1^2) \right\}, \qquad (7.23)$$

$$u^*(p) \;\in\; \operatorname{sign}(p_1) \;=\; \begin{cases} 1 & \text{if} \quad p_1 > 0, \\ [-1, 1] & \text{if} \quad p_1 = 0, \\ -1 & \text{if} \quad p_1 < 0. \end{cases}$$

Solving the terminal value problem (7.22) for p_2, p_3 we find

$$p_3(t) \equiv 1, \qquad p_2(t) = T - t.$$

The function p_1 can now be found from the equations

$$\ddot{p}_1 \;=\; -1 + 2u^* \;=\; -1 + 2\operatorname{sign}(p_1), \qquad p_1(T) = 0, \quad \dot{p}_1(0) = p_2(0) = T.$$

The unique solution is found to be

$$p_1(t) \;=\; \begin{cases} -\dfrac{3}{2}\left(\dfrac{T}{3} - t\right)^2 & \text{if} \quad 0 \le t \le T/3, \\ 0 & \text{if} \quad T/3 \le t \le T. \end{cases}$$

The optimal control is

$$u^*(t) = \begin{cases} -1 & \text{if} \quad 0 \le t \le T/3, \\ 1/2 & \text{if} \quad T/3 \le t \le T. \end{cases}$$

Observe that on the interval $[T/3,\ T]$ the optimal control is derived not from the maximality condition (7.23) but from the equation $\ddot{p}_1 = (-1 + 2u) \equiv 0$. An optimal control with this property is called **singular**.

One should be aware that the Pontryagin Maximum Principle is a necessary condition, not sufficient for optimality.

Example 7.3 Consider the problem

$$\text{maximize:} \qquad x_2(T),$$

for the system with dynamics

$$\begin{cases} \dot{x}_1 &= u, \\ \dot{x}_2 &= x_1^2, \end{cases} \qquad \begin{cases} \dot{x}_1(0) &= 0, \\ x_2(0) &= 0, \end{cases} \qquad u(t) \in [-1, 1].$$
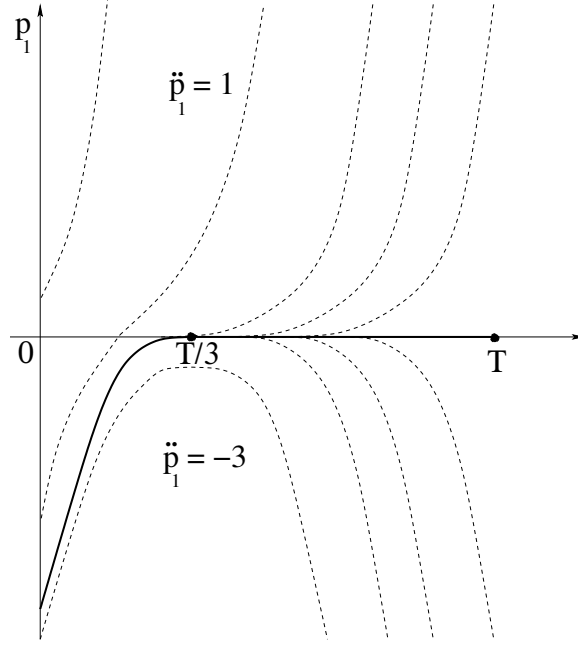
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Figure 24: The extremal trajectories, in Example 7.2.

The control $u^*(t) \equiv 0$ yields the trajectory $(x_1^*(t),\, x_2^*(t)) \equiv (0, 0)$. This solution satisfies the PMP. Indeed, the adjoint vector $(p_1(t), p_2(t)) \equiv (0, 1)$ satisfies

$$\begin{cases} \dot{p}_1 &= -p_2\, x_2^* \equiv 0\,, \\ \dot{p}_2 &= 0\,, \end{cases} \qquad \begin{cases} p_1(T) &= 0\,, \\ p_2(T) &= 1\,, \end{cases}$$

$$p_1(t)u(t) + p_2(t)x_1^*(t) \;=\; 0 \;=\; \max_{\omega \in [-1,1]} \Big\{ p_1(t)\,\omega + p_2(t)\,x_1^*(t) \Big\}.$$

However, in this solution the terminal value $x_2^*(T) = 0$ provides the global minimum, not the global maximum! Any control $t \mapsto u(t) \in [-1, 1]$ which is not identically zero yields a terminal value $x_2(T) > 0$. In this example, there are two optimal controls: $u_1(t) \equiv 1$ and $u_2(t) \equiv -1$.

Example 7.4 (Linear-quadratic optimal control). Consider the linear control system

$$\dot{x} \;=\; A\,x + B\,u, \qquad\qquad x(0) = \bar{x}\,. \tag{7.24}$$

Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, while A is an $n \times n$ matrix and B is an $n \times m$ matrix. The optimal control problem, with fixed terminal time T and quadratic running cost, takes the form

$$\min_{u(\cdot)} \int_0^T \left[x^\dagger Q x + u^\dagger R u \right]\, dt\,. \tag{7.25}$$

Here Q is a symmetric $n \times n$ matrix, R is a symmetric $m \times m$ matrix, and $\dagger$ denotes the transpose. The minimum is sought among all measurable controls $u : [0, T] \mapsto \mathbb{R}^m$.

We assume that Q is positive semi-definite and that R positive definite, so that

$$x^\dagger Q x \;\geq\; 0\,, \qquad\qquad u^\dagger R u \;\geq\; \theta |u|^2$$

for some $\theta > 0$ and all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. In particular, this implies that the symmetric matrix R is invertible.

Notice that, by minimizing the integral in (7.25), we are trying to reduce the distance of x from the origin without spending too much energy to control the system.

Introducing the additional variable x_{n+1}, the above optimization problem can be written in the standard form (7.1)-(7.2) as follows.

$$\min_{u(\cdot)} \; x_{n+1}(T) \, , \tag{7.26}$$

for the system with dynamics

$$\begin{cases} \dot{x}_i & = \; \sum_{j=1}^{n} A_{ij} x_j + \sum_{j=1}^{m} B_{ij} u_j \, , \qquad\qquad i = 1, \ldots, n \, , \\[4mm] \dot{x}_{n+1} & = \; \sum_{i,j=1}^{n} Q_{ij} x_i x_j + \sum_{i,j=1}^{m} R_{ij} u_i u_j \, , \end{cases} \tag{7.27}$$

and initial data

$$x_i(0) \; = \; \bar{x}_i \, , \qquad i = 1, \ldots, n, \qquad x_{n+1}(0) \; = \; 0.$$

Let $x^*(\cdot)$ be an optimal trajectory, generated by the optimal control $u^*(\cdot)$. Then there exists a vector $t \mapsto p(t) = (p_1, \ldots, p_n, p_{n+1})(t) \in \mathbb{R}^{n+1}$ satisfying the conditions stated in Theorem 7.1, with "max" replaced by "min" in (7.16).

In the present case, the terminal condition (7.15) is

$$p_i(T) \; = \; 0, \qquad i = 1, \ldots, n, \qquad p_{n+1}(T) \; = \; 1 \, .$$

By (7.27), the adjoint equations (7.14) take the form

$$\begin{cases} \dot{p}_i & = \; -\sum_{j=1}^{n} p_j A_{ji} - 2 \sum_{ij=1}^{n} Q_{ij} x_j \, p_{n+1} \qquad i = 1, \ldots, n, \\[4mm] \dot{p}_{n+1} & = \; 0 \, . \end{cases} \tag{7.28}$$

We thus have $p_{n+1}(t) \equiv 1$, while the other components of the adjoint vector satisfy

$$\dot{p}_i \; = \; -\sum_{j=1}^{n} p_j A_{ji} - 2 \sum_{ij=1}^{n} Q_{ij} x_j \, , \qquad p_i(T) \; = \; 0, \qquad i = 1, \ldots, n \, .$$

For a.e. time t, the optimality condition (7.16) takes the form

$$\sum_{ij} B_{ij} p_i(t) u_j^*(t) + \sum_{j,k} R_{jk} u_j^*(t) u_k^*(t) \; = \; \min_{\omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m} \left\{ \sum_{ij} B_{ij} p_i(t) \omega_j + \sum_{j,k} R_{jk} \omega_j \omega_k \right\} . \tag{7.29}$$

Since u can vary in the whole space $\mathbb{R}^m$, at any given time t the necessary conditions for (7.29) yield

$$\sum_i B_{ij} p_i(t) + 2 \sum_{j,k} R_{jk} u_k^*(t) \;=\; 0 \qquad\qquad j = 1, \ldots, n\,. \tag{7.30}$$

In vector notation this can be written as

$$p(t) \cdot B + 2 u^*(t)^\dagger R \;=\; 0.$$

Hence the optimal control satisfies

$$u^*(t) \;=\; -\frac{1}{2} R^{-1} B^\dagger p^\dagger(t). \tag{7.31}$$

The optimal trajectory for the linear-quadratic optimal control problem is thus found by solving the two-point boundary value problem on $\mathbb{R}^{n+n}$

$$\left\{ \begin{aligned} \dot{x} &= Ax - \tfrac{1}{2} B R^{-1} B^\dagger p^\dagger \,, \\[2mm] \dot{p} &= -pA - 2x^\dagger Q \,, \end{aligned} \right. \qquad\qquad \left\{ \begin{aligned} x(0) &= \bar{x}\,, \\[2mm] p(T) &= 0\,. \end{aligned} \right.$$

The optimal control $u^*(\cdot)$ is then provided by (7.31). Once again we recall that $x \in \mathbb{R}^n$ is a column vector, $p \in \mathbb{R}^n$ is a row vector, while † denotes transposition.

8 Extensions of the P.M.P.

8.1 Bolza problems, with running cost.

In connection with the control system

$$\dot{x} \;=\; f(t, x, u) \qquad u(t) \in U, \qquad\qquad x(0) = x_0 \in \mathbb{R}^n\,, \tag{8.1}$$

consider the more general optimization problem with terminal payoff and running cost

$$\max_{u \in \mathcal{U}} \left\{ \psi(x(T, u)) - \int_0^T L(t, x(t),\, u(t))\, dt \right\}. \tag{8.2}$$

To reduce (8.2) to a problem where only the terminal payoff is present, it suffices to introduce an additional variable x_{n+1} which evolves according to

$$\dot{x}_{n+1} \;=\; L(t, x(t),\, u(t))\,, \qquad\qquad x_{n+1}(0) = 0\,.$$

We now consider the maximization problem

$$\max_{u \in \mathcal{U}} \left\{ \psi(x(T, u)) - x_{n+1}(T, u) \right\},$$

where the cost depends only on the terminal state of the system. An application of Theorem 7.1 yields

Theorem 8.1 (PMP, Bolza problem). *Let $u^* : [0, T] \mapsto U$ be the optimal control for the problem (8.1)-(8.2), and let $x^*(\cdot)$ be the corresponding optimal trajectory. Let $p = (p_1, \ldots, p_n)$ be the vector which satisfies the adjoint linear system*

$$\dot{p}_i(t) = -\sum_{j=1}^{n} p_j(t) \frac{\partial f_j}{\partial x_i}(t, x^*(t), u^*(t)) + \frac{\partial L}{\partial x_i}(t, x^*(t), u^*(t)) \qquad i = 1, \ldots, n, \qquad (8.3)$$

$$p(T) = \nabla \psi(x^*(T)). \qquad (8.4)$$

Then, for a.e. time $t \in [0, T]$ the maximality condition holds:

$$p(t) \cdot f(t, x^*(t), u^*(t)) - L(t, x^*(t), u^*(t)) = \max_{\omega \in U} \left\{ p(t) \cdot f(t, x^*(t), \omega) - L(t, x^*(t), \omega) \right\}. \quad (8.5)$$

8.2 Problems with constraints.

Another important extension deals with the case where terminal constraints are given, say $x(T) \in S$, where the set S is defined as

$$S \doteq \left\{ x \in \mathbb{R}^n ; \ \phi_i(x) = 0, \qquad i = 1, \ldots, m \right\}.$$

Assume that, at a given point $x^* \in S$, the $m + 1$ gradients $\nabla \psi, \nabla \phi_1, \ldots, \nabla \phi_m$ are linearly independent. Then the tangent space to S at x^* is

$$T_S = \left\{ \mathbf{v} \in \mathbb{R}^n ; \quad \nabla \phi_i(x^*) \cdot \mathbf{v} = 0 \qquad i = 1, \ldots, m \right\}, \qquad (8.6)$$

while the tangent cone to the set

$$S^+ = \left\{ x \in S ; \ \psi(x) \geq \psi(x^*) \right\}$$

is

$$T_{S^+} = \left\{ \mathbf{v} \in \mathbb{R}^n ; \ \nabla \psi(x^*) \cdot \mathbf{v} \geq 0, \qquad \nabla \phi_i(x^*) \cdot \mathbf{v} = 0 \qquad i = 1, \ldots, m \right\}. \qquad (8.7)$$

When $x^* = x^*(T)$ is the terminal point of an admissible trajectory, we think of T_{S^+} as the **cone of profitable directions**, i.e. those directions in which we should move the terminal point, in order to increase the value of ψ and still satisfy the constraint $x(T) \in S$ (Fig. 25).

Lemma 8.1 *A vector $p \in \mathbb{R}^n$ satisfies*

$$p \cdot \mathbf{v} \geq 0 \qquad \text{for all } \ \mathbf{v} \in T_{S^+} \qquad (8.8)$$

if and only if it can be written as a linear combination

$$p = \lambda_0 \nabla \psi(x^*) + \sum_{i=1}^{m} \lambda_i \nabla \phi_i(x^*) \qquad (8.9)$$
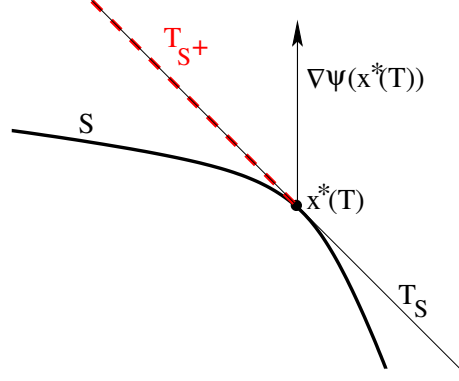
with $\lambda_0 \geq 0$.

Figure 25: The tangent cone T_S and the cone of profitable directions T_{S+}.

Proof. Define the vectors

$$\mathbf{w}_0 \doteq \nabla \psi(x^*), \qquad \mathbf{w}_i \doteq \nabla \phi_i(x^*) \qquad i = 1, \ldots, m.$$

By our previous assumption, these vectors are linearly independent. We can thus find additional vectors $\mathbf{w}_j$, $j = m+1, \ldots, n-1$ so that

$$\{\mathbf{w}_0, \mathbf{w}_1, \cdots, \mathbf{w}_m, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_{n-1}\}$$

is a basis of $I\!\!R^n$. Let

$$\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_{n-1}\}$$

be the dual basis, so that

$$\mathbf{v}_i \cdot \mathbf{w}_j = \left\{ \begin{array}{ll} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{array} \right.$$

We observe that

$$\mathbf{v} \in T_{S+} \qquad \text{if and only if} \qquad \mathbf{v} = c_0 \mathbf{v}_0 + \sum_{i=m+1}^{n-1} c_i \mathbf{v}_i \qquad (8.10)$$

for some $c_0 \geq 0$, $c_i \in I\!\!R$. An arbitrary vector $p \in I\!\!R^n$ can now be written as

$$p = \lambda_0 \mathbf{w}_0 + \sum_{i=1}^{m} \lambda_i \mathbf{w}_i + \sum_{i=m+1}^{n-1} \lambda_i \mathbf{w}_j.$$

Moreover, every vector $\mathbf{v} \in T_{S+}$ can be decomposed as in (8.10). Therefore

$$p \cdot \mathbf{v} = \lambda_0 c_0 + \sum_{i=m+1}^{n-1} \lambda_i c_i.$$

It is now clear that (8.8) holds if and only if $\lambda_0 \geq 0$ and $\lambda_i = 0$ for all $i = m+1, \ldots, n-1$. □
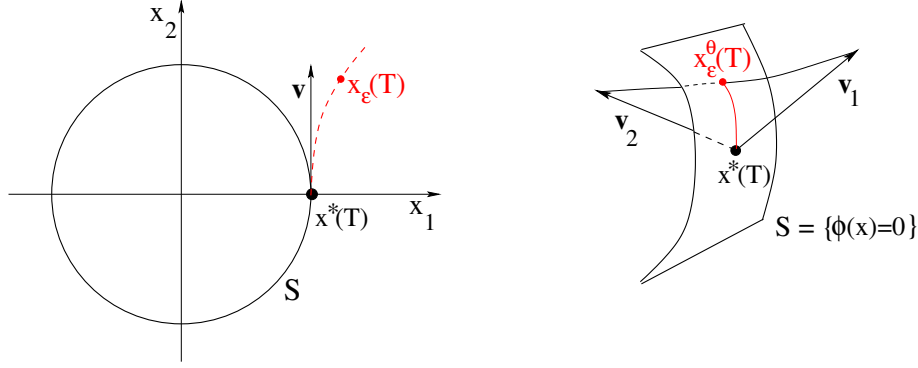
Figure 26: Left: in the presence of constraints on the terminal point, constructing a single tangent vector $\mathbf{v} \in T_{S+}$ does not rule out optimality of the trajectory $x^*(\cdot)$. Right: by continuously interpolating between two or more tangent vectors $\mathbf{v}_1, \mathbf{v}_2$, we can construct perturbed trajectories satisfying the terminal constraint $x_\varepsilon^\theta(T) \in S$.

Next, consider a trajectory $t \mapsto x^*(t) = x(t, u^*)$, generated by the control $u^*(\cdot)$. To test its optimality, we need to perturb u^* in such a way that the terminal point $x(T)$ still lies in the admissible set S.

Example 8.1 Consider the problem

$$\text{maximize:} \quad x_2(T)$$

for a control system of the general form (8.1), with terminal constraint

$$x(T) \ \in \ S \ \doteq \ \left\{ x = (x_1, x_2); \ \ x_1^2 + x_2^2 = 1 \right\}.$$

Let $u^*(\cdot)$ be an admissible control, such that the trajectory $t \mapsto x^*(t) = x(t, u^*)$ reaches the terminal point $x^*(T) = (1, 0) \in S$.

Assume that we can construct a family of needle variations $u_\varepsilon(\cdot)$ generating the vector

$$\mathbf{v}(T) \ = \ \lim_{\varepsilon \to 0} \frac{x_\varepsilon(T) - x^*(T)}{\varepsilon} \ = \ (0, 1) \ \in \ T_{S+}.$$

This does not necessarily rule out the optimality of u^*, because none of the points $x_\varepsilon(T)$ for $\varepsilon > 0$ may lie in the target set S. (Fig. 26, left).

As in the previous section, for any choice of $\tau \in \,]0, T]$ and $\omega \in U$, consider the family of needle variations

$$u_\varepsilon(t) \ = \ \begin{cases} \omega & \text{if} \quad t \in [\tau - \varepsilon, \tau], \\ u^*(t) & \text{if} \quad t \notin [\tau - \varepsilon, \tau]. \end{cases}$$

Call

$$\mathbf{v}^{\tau, \omega}(T) \ \doteq \ \lim_{\varepsilon \to 0+} \frac{x(T, u_\varepsilon) - x(T, u^*)}{\varepsilon} \tag{8.11}$$

the first order variation of the terminal point of the corresponding trajectory. At this stage, a key observation is that these needle variations can be combined together. As shown in Fig. 27,
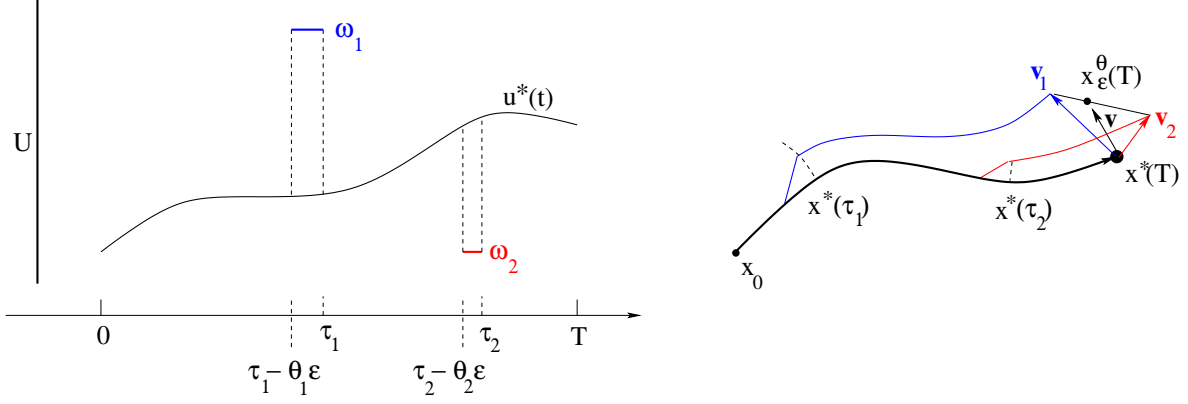
Figure 27: By perturbing the control u^* at time τ_1, the terminal point $x^*(T)$ can be shifted in the direction of the vector $\mathbf{v}_1 = \mathbf{v}^{\tau_1,\omega_1}(T)$. By perturbing the control u^* at time τ_2, we can achieve a shift in the direction of the vector $\mathbf{v}_2 = \mathbf{v}^{\tau_2,\omega_2}(T)$. By combining these two needle variations, we can shift the terminal point in the direction of the vector $\mathbf{v} = \theta_1\mathbf{v}_1 + \theta_2\mathbf{v}_2$, for any $\theta_1, \theta_2 \geq 0$.

for any $\theta_1, \theta_2 \geq 0$, by superimposing two perturbations we can generate a shift in the terminal point of the trajectory, given by

$$\lim_{\varepsilon \to 0} \frac{x(T, u_\varepsilon) - x(T, u^*)}{\varepsilon} = \theta_1\mathbf{v}^{\tau_1,\omega_1}(T) + \theta_2\mathbf{v}^{\tau_2,\omega_2}(T).$$

Define Γ to be the smallest convex cone containing all vectors $v^{\tau,\omega}(T)$. In other words, Γ is the set of all finite linear combination of the vectors $v^{\tau,\omega}(T)$ with positive coefficients.

We think of Γ as a **cone of feasible directions**, i.e. directions in which we can move the terminal point $x(T, u^*)$ by suitably perturbing the control u^* (Fig. 28).
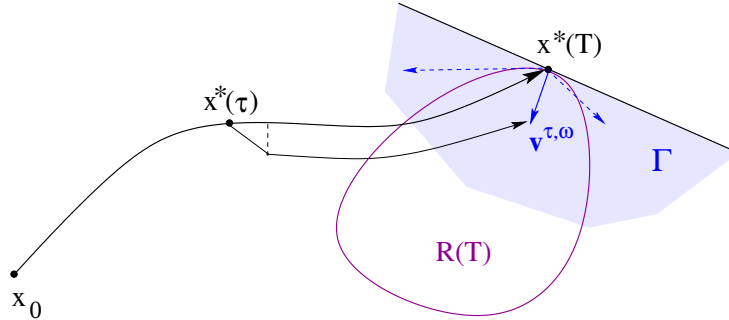

Figure 28: The cone Γ of feasible directions (shaded area). This is the set of all linear combinations of the vectors $\mathbf{v}^{\tau,\omega}(T)$, with positive coefficients.

We can now state necessary conditions for optimality for the **Mayer Problem with terminal constraints:**

$$\text{maximize:} \quad \psi(x(T, u)), \tag{8.12}$$

for the control system

$$\dot{x} = f(t, x, u), \qquad u(t) \in U, \qquad t \in [0, T], \tag{8.13}$$

with initial and terminal constraints

$$x(0) = x_0, \qquad \phi_i(x(T)) = 0, \qquad i = 1, \ldots, m. \tag{8.14}$$

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Theorem 8.2 (PMP, geometric version). *Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory for the problem (8.12)–(8.14), corresponding to the control $u^*(\cdot)$. Assume that, at the terminal point $x^*(T)$, the $m+1$ gradients $\nabla\psi, \nabla\phi_1, \ldots, \nabla\phi_m$ are linearly independent.*

Then the cones Γ and T_{S^+} are **weakly separated**. *Namely, there exists a non-zero vector $p(T) \in \mathbb{R}^n$ such that*

$$p(T) \cdot \mathbf{v} \geq 0 \qquad \text{for all } \mathbf{v} \in T_{S^+}, \tag{8.15}$$

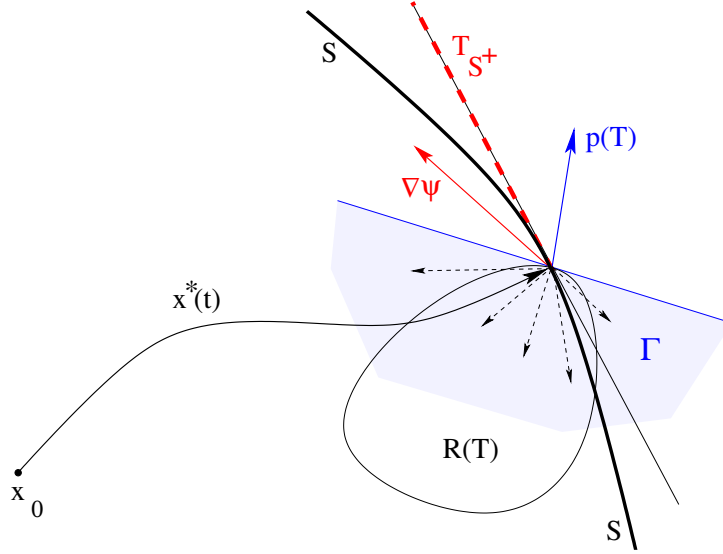$$p(T) \cdot \mathbf{v} \leq 0 \qquad \text{for all } \mathbf{v} \in \Gamma. \tag{8.16}$$

Figure 29: The adjoint vector $p(T)$ weakly separates the cones Γ and T_{S^+}. Namely, the inequalities (8.15)-(8.16) hold.

This separation property is illustrated in Fig. 29. An equivalent statement is:

Theorem 8.3 (PMP, analytic version). *Let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory for the problem (8.12)–(8.14), corresponding to the control $u^*(\cdot)$. Assume that, at the terminal point $x^*(T)$, the $m+1$ gradients $\nabla\psi, \nabla\phi_1, \ldots, \nabla\phi_m$ are linearly independent.*

Then there exists a non-zero vector function $t \mapsto p(t)$ such that

$$p(T) = \lambda_0 \nabla\psi(x^*(T)) + \sum_{i=1}^{m} \lambda_i \nabla\phi_i(x^*(T)) \qquad \text{with } \lambda_0 \geq 0, \tag{8.17}$$

$$\dot{p}(t) = -p(t) D_x f(t, x^*(t), u^*(t)) \qquad \text{for a.e. } t \in [0, T], \tag{8.18}$$

$$p(\tau) \cdot f(\tau, x^*(\tau), u^*(\tau)) = \max_{\omega \in U} \left\{ p(\tau) \cdot f(\tau, x^*(\tau), \omega) \right\} \qquad \text{for a.e. } \tau \in [0, T]. \tag{8.19}$$

We show here the equivalence of the two formulations.

By Lemma 8.1, (8.15) is equivalent to (8.17).

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Next, we recall that, for every choice of $\tau \in \,]0, T]$ and $\omega \in U$, the vector $\mathbf{v}^{\tau, \omega}$ in (8.11) satisfies the linear evolution equation

$$\dot{\mathbf{v}}^{\tau, \omega}(t) \;=\; D_x f(t,\, x^*(t),\, u^*(t)) \, \mathbf{v}^{\tau, \omega}(t).$$

Hence, if $t \mapsto p(t)$ satisfies (8.18), then the product $p(t) \cdot \mathbf{v}^{\tau, \omega}(t)$ is constant. Therefore

$$p(T) \cdot \mathbf{v}^{\tau, \omega}(T) \;\leq\; 0$$

if and only if

$$p(\tau) \cdot \mathbf{v}^{\tau, \omega}(\tau) \;=\; p(\tau) \cdot \Big[f(\tau,\, x^*(\tau),\, \omega) - f(\tau,\, x^*(\tau),\, u^*(\tau)) \Big] \;\leq\; 0.$$

This is true for every τ, ω if and only if (8.19) holds.

Remark 8.1 (Normal and abnormal extremals). The adjoint vector p enters linearly in the equations (8.18)-(8.19). Therefore, we can replace $p(t)$ by $c\, p(t)$, for any constant $c > 0$, without changing the solution $u^*(t)$, $x^*(t)$.

If $\lambda_0 > 0$, in (8.17) we can thus replace each λ_i by λ_i / λ_0. Therefore, without loss of generality, only two cases must be considered: $\lambda_0 = 1$ or $\lambda_0 = 0$.

- **Normal extremals.** This is the most common case, where the necessary conditions (8.17)-(8.19) are satisfied with $\lambda_0 = 1$.

- **Abnormal extremals.** This is the case where $\lambda_0 = 0$, hence the payoff function ψ plays no role in determining the optimal solution. An example where this happens is shown in Fig. 30. Here Q is the only point on the target set which is reachable at time T. Regardless of the function ψ we want to maximize, any trajectory such that $x(T) = Q$ will be optimal.

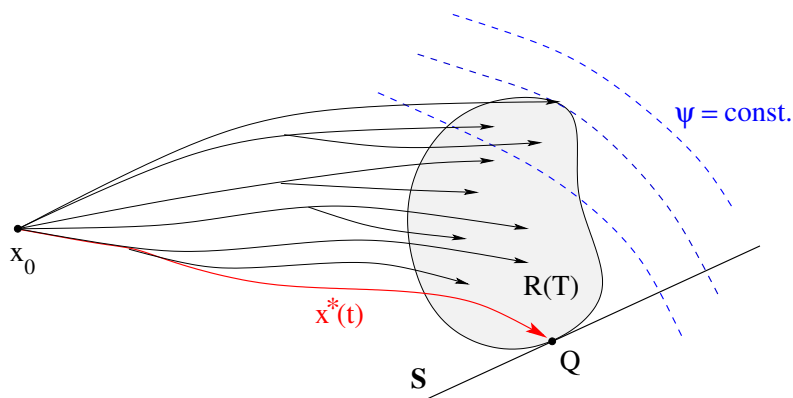Figure 30: An abnormal extremal. Since Q is the only point on the target set S which is reachable at time T, the trajectory reaching Q is trivially optimal, regardless of the function ψ.

Example 8.2 Consider a cart moving on a rail, as in Example 6.2. Starting at the origin with zero speed, we want to move the cart so that, at a given time $T > 0$, it comes back to the origin with maximum speed.

Calling $x_1(t)$ the position of the cart and $x_2(t)$ its speed at time $t \in [0, T]$, this problem can be formulated as

$$\text{maximize:} \quad \psi(x(T)) \doteq x_2(T),$$

$$\text{subject to:} \quad \begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u(t), \end{cases} \quad u(t) \in [-1, 1], \quad \begin{cases} x_1(0) &= 0, \\ x_2(0) &= 0, \end{cases}$$

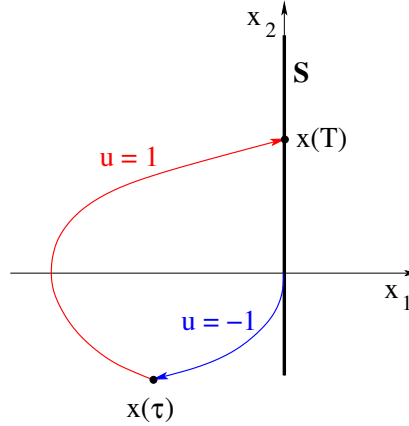$$\text{with terminal constraint:} \quad \phi_1(x(T)) \doteq x_1(T) = 0.$$

Figure 31: The solution to the optimal control problem with terminal constrains, in Example 8.2. Here the target set is the line $S = \{x_1 = 0\}$.

Taking $\lambda_0 = 1$ (i.e., guessing that we have a normal extremal), the necessary conditions (8.17)–(8.19) yield

$$p(T) = \nabla \psi(x(T)) + \lambda_1 \nabla \phi_1(x(T)) = (0, 1) + (\lambda_1, 0),$$

$$\begin{cases} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \end{cases}$$

$$u^*(t) = \arg \max_{\omega \in [-1,1]} p(t) \cdot f(x(t), \omega) = \arg \max_{\omega \in [-1,1]} \left\{ p_1(t) x_2(t) + p_2(t) \omega \right\} = \text{sign}(p_2(t)).$$

To find the optimal control, we thus have to solve the two-point boundary value problem

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \text{sign}(p_2) \end{cases} \quad \begin{cases} \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 \end{cases} \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \end{cases} \quad \begin{cases} x_1(T) = 0 \\ p_2(T) = 1 \end{cases}$$

Notice that, compared with the example 7.1, we now have one more equation for $x(T)$ and one less equation for $p(T)$. Solving the equations for p we obtain

$$p_1(t) \equiv \lambda_1, \qquad p_2(t) = 1 + \lambda_1(T - t),$$

hence p_2 is an affine function that can change sign only once. Therefore (taking $\lambda_1 < 0$), the optimal control must have the form (Fig. 31)

$$u^*(t) = \text{sign}(p_2(t)) = \begin{cases} -1 & \text{if} \quad t \in [0, \tau], \\ +1 & \text{if} \quad t \in]\tau, T]. \end{cases}$$

Imposing the constraint $x_1(T) = 0$, we find $\tau = \left(1 - \frac{1}{\sqrt{2}}\right) T$.

8.3   Lagrange Minimization Problem with fixed initial and terminal points.

As a special case, consider the problem

$$\min_{u \in \mathcal{U}} \ \int_0^T L(t, x, u) \, dt \,, \tag{8.20}$$

for the control system on $\mathbb{R}^n$

$$\dot{x} \ = \ f(t, x, u) \,, \qquad\qquad u(t) \in U \,, \tag{8.21}$$

with initial and terminal constraints

$$x(0) \ = \ a, \qquad\qquad x(T) \ = \ b \,. \tag{8.22}$$

An adaptation of the previous analysis yields

Theorem 8.4 (PMP, Lagrange problem). *For the problem (8.20)-(8.22), let $t \mapsto x^*(t) = x(t, u^*)$ be an optimal trajectory, corresponding to the optimal control $u^*(\cdot)$. Then there exist a constant $\lambda \geq 0$ and a row vector $t \mapsto p(t)$ (not both $= 0$) such that*

$$\dot{p}(t) \ = \ -p(t) \, D_x f(t, \, x^*(t), \, u^*(t)) - \lambda \, D_x L(t, \, x^*(t), \, u^*(t)) \,, \tag{8.23}$$

$$p(t) \cdot f(t, \, x^*(t), \, u^*(t)) + \lambda \, L(t, \, x^*(t), \, u^*(t))$$

$$= \ \min_{\omega \in U} \Big\{ p(t) \cdot f(t, \, x^*(t), \, \omega) + \lambda \, L(t, \, x^*(t), \, \omega) \Big\} \,. \tag{8.24}$$

This follows by applying the previous results to the Mayer problem

$$\min_{u \in \mathcal{U}} \ x_{n+1}(T, u)$$

with

$$\dot{x} \ = \ f(t, x, u), \qquad\qquad \dot{x}_{n+1} \ = \ L(t, x, u), \qquad\qquad x_{n+1}(0) = 0 \,.$$

Observe that the evolution of the adjoint vector $(p, p_{n+1}) = (p_1, \ldots, p_n, p_{n+1})$ is governed by the linear system

$$(\dot{p}_1, \ldots, \dot{p}_n, \dot{p}_{n+1}) \ = \ -(p_1, \ldots, p_n, p_{n+1}) \begin{pmatrix} \partial f_1 / \partial x_1 & \cdots & \partial f_1 / \partial x_n & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \partial f_n / \partial x_1 & \cdots & \partial f_n / \partial x_n & 0 \\ \partial L / \partial x_1 & \cdots & \partial L / \partial x_n & 0 \end{pmatrix} .$$

Because of the terminal constraints $(x_1, \ldots, x_n)(T) = (b_1, \ldots, b_n)$, the only requirement on the terminal value $(p_1, \ldots, p_n, p_{n+1})(T)$ is

$$p_{n+1}(T) \geq 0.$$

Since $\dot{p}_{n+1} = 0$, we have $p_{n+1}(t) \equiv \lambda$ for some constant $\lambda \geq 0$.
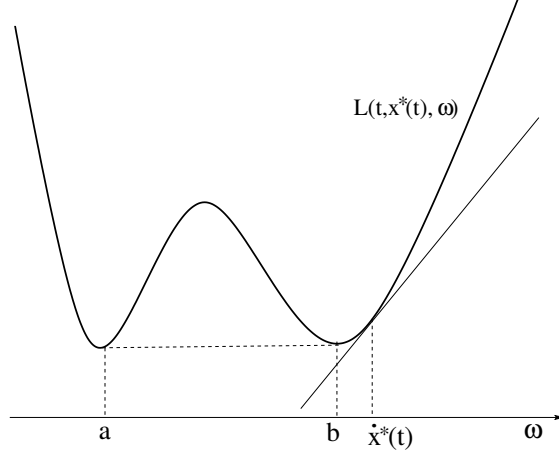
Figure 32: The Weierstrass necessary condition: for an optimal trajectory of (8.25), the derivative $\dot{x}^*(t)$ must take values on the domain where L coincides with its convex envelope, i.e., outside the open interval $]a, b[$.

Theorem 8.4 can be further specialized to the **Standard Problem of the Calculus of Variations:**

$$\text{minimize} \quad \int_0^T L(t, x(t), \dot{x}(t)) \, dt \tag{8.25}$$

over all absolutely continuous functions $x : [0, T] \mapsto I\!R^n$ such that

$$x(0) = a, \qquad x(T) = b. \tag{8.26}$$

This corresponds to the optimal control problem (8.20), for the trivial control system

$$\dot{x} = u, \qquad u(t) \in U \doteq I\!R^n. \tag{8.27}$$

We assume that L is smooth, and that $x^*(\cdot)$ is an optimal solution. By Theorem 8.4 there exist a constant $\lambda \geq 0$ and a row vector $t \mapsto p(t)$ (not both $= 0$) such that

$$\dot{p}(t) = -\lambda \frac{\partial}{\partial x} L(t, x^*(t), \dot{x}^*(t)), \tag{8.28}$$

$$p(t) \cdot \dot{x}^*(t) + \lambda L(t, x^*(t), \dot{x}^*(t)) = \min_{\omega \in I\!R^n} \left\{ p(t) \cdot \omega + \lambda L(t, x^*(t), \omega) \right\}. \tag{8.29}$$

If $\lambda = 0$, then $p(t) \neq 0$. But in this case $\dot{x}^*$ cannot provide a minimum over the whole space $I\!R^n$. This contradiction shows that we must have $\lambda > 0$.

Since λ, p are determined up to a positive scalar multiple, we can assume $\lambda = 1$. According to (8.29), the global minimum on the right hand side is attained when $\omega = \dot{x}^*(t)$. Differentiating w.r.t. ω, a necessary condition is found to be

$$p(t) = -\frac{\partial}{\partial \dot{x}} L(t, x^*(t), \dot{x}^*(t)). \tag{8.30}$$

Inserting (8.30) in the evolution equation (8.28) (with $\lambda = 1$), one obtains the famous **Euler-Lagrange equations**

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} L(t, x^*(t), \dot{x}^*(t)) \right] = \frac{\partial}{\partial x} L(t, x^*(t), \dot{x}^*(t)). \tag{8.31}$$

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In addition, the minimality condition (8.29) (always with $\lambda = 1$) implies

$$p(t) \cdot \dot{x}^*(t) + L(t, x^*(t), \dot{x}^*(t)) \ \leq \ p(t) \cdot \omega + L(t, x^*(t), \omega) \qquad \text{for every} \ \ \omega \in I\!R^n.$$

Replacing $p(t)$ by its expression given at (8.30), one obtains the **Weierstrass necessary conditions**

$$L(t, x^*(t), \omega) \ \geq \ L(t, x^*(t), \dot{x}^*(t)) + \frac{\partial L(t, x^*(t), \dot{x}^*(t))}{\partial \dot{x}} \cdot (\omega - \dot{x}^*(t)), \qquad (8.32)$$

valid for every $\omega \in I\!R^n$. In other words (Fig. 32), for every time t, the graph of $\omega \mapsto L(t, x^*(t), \omega)$ lies entirely above its tangent hyperplane at the point $(t, x^*(t), \dot{x}^*(t))$.

9 Dynamic programming

Consider again a control system of the form

$$\dot{x}(t) \ = \ f(x(t), u(t)), \qquad u(t) \in U, \qquad t \in [0, T]. \qquad (9.1)$$

We now assume that the set $U \subset I\!R^m$ of admissible control values is compact, while $f : I\!R^n \times U \mapsto I\!R^n$ is a continuous function such that

$$|f(x, u)| \ \leq \ C, \qquad |f(x, u) - f(y, u)| \ \leq \ C\,|x - y| \qquad \text{for all } x, y \in I\!R^n, \ u \in U, \quad (9.2)$$

for some constant C. As in the previous sections, we call

$$\mathcal{U} \ \doteq \ \left\{ u : I\!R \mapsto I\!R^m \ \ \text{measurable}, \ \ u(t) \in U \ \ \text{for a.e. } t \right\} \qquad (9.3)$$

the family of admissible control functions. Given an initial data

$$x(s) \ = \ y \in I\!R^n, \qquad (9.4)$$

under the assumptions (9.2), for every choice of the measurable control function $u(\cdot) \in \mathcal{U}$ the Cauchy problem (9.1)-(9.2) has a unique solution, which we denote as $t \mapsto x(t; s, y, u)$ or sometimes simply as $t \mapsto x(t)$. Given initial data (s, y), we consider the optimization problem

$$\text{minimize:} \qquad J(s, y, u) \ \doteq \ \int_s^T L(x(t), u(t))\, dt + \psi(x(T)). \qquad (9.5)$$

Here it is understood that $x(t) = x(t; s, y, u)$. The minimum is sought over all measurable functions $u : [s, T] \mapsto \mathcal{U}$. We shall assume that the cost functions L, ψ satisfy the bounds

$$|L(x, u)| \ \leq \ C, \qquad |\psi(x)| \ \leq \ C, \qquad (9.6)$$

$$|L(x, u) - L(y, u)| \ \leq \ C\,|x - y|, \qquad |\psi(x) - \psi(y)| \ \leq \ C\,|x - y|, \qquad (9.7)$$

for all $x, y \in I\!R^n, \ u \in U$.

Remark 9.1 The global bounds assumed in (9.2), (9.6), and (9.7) appear to be very restrictive. In practice, one can often obtain an a-priori bound on all trajectories of the control system which start from a bounded set S. Say, $|x(t)| \leq M$ for all $t \in [0, T]$. For all subsequent applications, it then suffices to assume that the bounds (9.2) and (9.6)-(9.7) hold as long as $|x|, |y| \leq M$.

Following the method of dynamic programming, an optimal control problem can be studied by looking at the **value function**:

$$V(s, y) \doteq \inf_{u(\cdot) \in \mathcal{U}} J(s, y, u). \tag{9.8}$$

We consider here a whole family of optimal control problem, all with the same dynamics (9.1) and cost functional (9.5). We are interested in how the minimum cost varies, depending on the initial data (s, y) in (9.4). As a preliminary, we prove

Lemma 9.1 *Let the functions f, ψ, L satisfy the assumptions (9.2), (9.6) and (9.7). Then the value function V in (9.8) is bounded and Lipschitz continuous. Namely, there exists a constant C' such that*

$$|V(s, y)| \leq C', \tag{9.9}$$

$$|V(s, y) - V(s', y')| \leq C'(|s - s'| + |y - y'|). \tag{9.10}$$

Proof. Let $(\bar{s}, \bar{y})$ and $\varepsilon > 0$ be given. Choose a measurable control $u_\varepsilon : [0, T] \mapsto \mathcal{U}$ which is almost optimal for the optimization problem with initial data $x(\bar{s}) = \bar{y}$, namely

$$J(\bar{s}, \bar{y}, u_\varepsilon) \leq V(\bar{s}, \bar{y}) + \varepsilon.$$

Call $t \mapsto x(t) = x(t; \bar{s}, \bar{y}, u_\varepsilon)$ the corresponding trajectory. Using the same control $u_\varepsilon(\cdot)$ in connection with a different initial data, say $x(s) = y$, we obtain a new trajectory $t \mapsto z(t) = x(t; s, y, u_\varepsilon)$. By the boundedness assumptions (9.2) it follows

$$|x(s) - z(s)| \leq |x(s) - x(\bar{s})| + |x(\bar{s}) - z(s)| \leq C|s - \bar{s}| + |\bar{y} - y|.$$

Since f is Lipschitz continuous, Gronwall's lemma yields

$$|x(t) - z(t)| \leq e^{C|t - s|} |x(s) - z(s)| \leq e^{CT} \left(C|s - \bar{s}| + |\bar{y} - y| \right).$$

Using the bounds (9.6)-(9.7) we thus obtain

$$J(s, y, u_\varepsilon) = J(\bar{s}, \bar{y}, u_\varepsilon) + \int_s^{\bar{s}} L(z, u_\varepsilon) \, dt + \int_{\bar{s}}^T \left(L(z, u_\varepsilon) - L(x, u_\varepsilon) \right) dt + \psi(z(T)) - \psi(x(T))$$

$$\leq J(\bar{s}, \bar{y}, u_\varepsilon) + C|s - \bar{s}| + \int_{\bar{s}}^T C|z(t) - x(t)| \, dt + C|z(T) - x(T)|$$

$$\leq J(\bar{s}, \bar{y}, u_\varepsilon) + C'(|s - \bar{s}| + |y - \bar{y}|)$$

for some constant C'. This implies

$$V(s, y) \leq J(s, y, u_\varepsilon) \leq V(\bar{s}, \bar{y}) + \varepsilon + C'(|s - \bar{s}| + |y - \bar{y}|).$$

Letting $\varepsilon \to 0$ and interchanging the roles of (s, y) and $(\bar{s}, \bar{y})$, one obtains the Lipschitz continuity of the value function V. $\qquad \square$

We will show that the value function V can be characterized as the unique viscosity solution to a Hamilton-Jacobi equation. Toward this goal, a basic step is provided by Bellman's principle of dynamic programming.

Theorem 9.1 (Dynamic Programming Principle). *For every* $\tau \in [s,T]$ *and* $y \in \mathbb{R}^n$, *one has*

$$V(s,y) \;=\; \inf_{u(\cdot)} \left\{ \int_s^\tau L(x(t;s,y,u),\; u(t))\, dt \;+\; V(\tau,\; x(\tau;s,y,u)) \right\}. \tag{9.11}$$

In other words (Fig. 33), the optimization problem on the time interval $[s,T]$ can be split into two separate problems:

- As a first step, we solve the optimization problem on the sub-interval $[\tau,T]$, with running cost L and terminal cost ψ. In this way, we determine the value function $V(\tau, \cdot)$, at the intermediate time τ.

- As a second step, we solve the optimization problem on the sub-interval $[s,\tau]$, with running cost L and terminal cost $V(\tau, \cdot)$, determined by the first step.

At the initial time s, by (9.11) we claim that the value function $V(s, \cdot)$ obtained in step 2 is the same as the value function corresponding to the global optimization problem over the whole interval $[s,T]$.
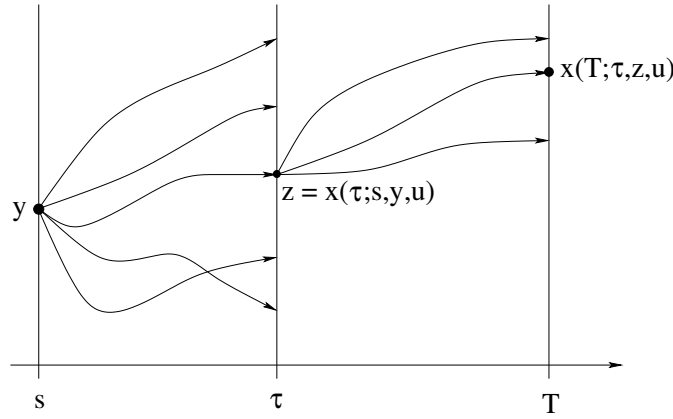

Figure 33: The optimization problem on $[s,T]$ can be decomposed into two sub-problems, on the time intervals $[s,\tau]$ and on $[\tau,T]$, respectively.

Proof. Call J^τ the right hand side of (9.11).

1. To prove that $J^\tau \leq V(s,y)$, fix $\varepsilon > 0$ and choose a control $u : [s,T] \mapsto U$ such that

$$J(s,y,u) \;\leq\; V(s,y) + \varepsilon.$$

Observing that

$$V(\tau, x(\tau; s, y, u)) \ \leq \ \int_\tau^T L(x(t; s, y, u), \ u(t)) \, dt + \psi(x(T; s, y, u)),$$

we conclude

$$J^\tau \ \leq \ \int_s^\tau L(x(t; s, y, u), \ u(t)) \, dt + V(\tau, x(\tau; s, y, u))$$

$$\leq \ J(s, y, u) \ \leq \ V(s, y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this first inequality is proved.

2. To prove that $V(s, y) \leq J^\tau$, fix $\varepsilon > 0$. Then there exists a control $u_1 : [s, \tau] \mapsto U$ such that

$$\int_s^\tau L(x(t; s, y, u_1), \ u_1(t)) \, dt + V(\tau, \ x(\tau; s, y, u_1)) \ \leq \ J^\tau + \varepsilon. \tag{9.12}$$

Moreover, there exists a control $u_2 : [\tau, T] \mapsto U$ such that

$$J(\tau, \ x(\tau; s, y, u_1), \ u_2) \ \leq \ V(\tau, \ x(\tau; s, y, u_1)) + \varepsilon. \tag{9.13}$$

We now define a new control $u : [s, T] \mapsto U$ as the concatenation of u_1 and u_2:

$$u(t) \doteq \begin{cases} u_1(t) & \text{if} \quad t \in [s, \tau], \\ u_2(t) & \text{if} \quad t \in \,]\tau, T]. \end{cases}$$

By (9.12) and (9.13) it now follows

$$V(s, y) \ \leq \ J(s, y, u) \ \leq \ J^\tau + 2\varepsilon.$$

Since $\varepsilon > 0$ can be taken arbitrarily small, this second inequality is also proved. $\qquad\square$

9.1 Recovering the optimal control from the value function.

Given the initial data $x(s) = y$, consider any control $u(\cdot)$ and let $x(\cdot)$ be the corresponding trajectory. Observe that the map

$$t \ \mapsto \ \Phi^u(t) \ \doteq \ \int_s^t L(x(t), u(t)) \, dt + V(t, x(t)), \qquad t \in [s, T], \tag{9.14}$$

is non-decreasing. Indeed, for $t_1 < t_2$ we have

$$\Phi^u(t_1) - \Phi^u(t_2) \ = \ V(t_1, x(t_1)) - \left[\int_{t_1}^{t_2} L(x(t), u(t)) \, dt + V(t_2, x(t_2)) \right] \ \leq \ 0.$$

Here the last inequality follows from Theorem 9.1, with (s, y) replaced by $(t_1, x(t_1))$. Notice that the control $u(\cdot)$ is optimal if and only if it achieves the minimum cost $V(s, y)$, namely

$$\Phi^u(s) \ = \ V(s, y) \ = \ \int_s^T L(x(t), u(t)) \, dt + \psi(x(T)) \ = \ \Phi^u(T).$$

From the previous analysis it follows:

the map $t \mapsto \Phi^u(t)$ in (9.14) is constant if and only if the control u is optimal.

If the value function V is known, the optimal control can be determined as follows.

1. Fix (s, y) and assume that $u : [s, T] \mapsto U$ is an optimal control for the minimization problem (9.5) with dynamics (9.1) and initial data (9.4). Since the map in (9.14) is constant, at every point where u is continuous and V is differentiable, we have

$$
\begin{aligned}
0 &= \frac{d}{d\tau}\left[\int_s^\tau L(x(t), u(t))\, dt + V(\tau, x(\tau))\right] \\
&= L(x(\tau), u(\tau)) + V_t(\tau, x(\tau)) + \nabla V(\tau, x(\tau)) \cdot f(x(\tau), u(\tau)).
\end{aligned}
$$

In particular, at $\tau = s$ we have

$$
L(y, u(s)) + V_t(s, y) + \nabla V(s, y) \cdot f(y, u(s)) = 0. \tag{9.15}
$$

2. On the other hand, if we choose any constant control $u(t) \equiv \omega \in U$, the corresponding map $t \mapsto \Phi^\omega(t)$ will be nondecreasing, hence

$$
\begin{aligned}
0 &\leq \frac{d}{d\tau}\left[\int_s^\tau L(x(t), \omega)\, dt + V(\tau, x(\tau))\right] \\
&= L(x(\tau), \omega) + V_t(\tau, x(\tau)) + \nabla V(\tau, x(\tau)) \cdot f(x(\tau), \omega).
\end{aligned}
$$

In particular, at $\tau = s$ we have

$$
L(y, \omega) + V_t(s, y) + \nabla V(s, y) \cdot f(y, \omega) \geq 0. \tag{9.16}
$$

3. Since $u(s) \in U$, comparing (9.15) with (9.16) we conclude

$$
\min_{\omega \in U}\left\{L(y, \omega) + V_t(s, y) + \nabla V(s, y) \cdot f(y, \omega)\right\} = 0. \tag{9.17}
$$

$$
u(s) = u^*(s, y) \in \operatorname*{argmin}_{\omega \in U}\left\{L(y, \omega) + \nabla V(s, y) \cdot f(y, \omega)\right\}. \tag{9.18}
$$

If the minimum is attained at a unique point, this uniquely determines the optimal control, in feedback form, as a function of (s, y).

4. Introducing the function

$$
H(y, p) \doteq \min_{\omega \in U}\left\{L(y, \omega) + p \cdot f(y, \omega)\right\}, \tag{9.19}
$$

by (9.17) we see that, at point (s, y) where it is differentiable, the value function V satisfies the equation

$$
V_t(s, y) + H(y, \nabla V(s, y)) = 0. \tag{9.20}
$$

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10 The Hamilton-Jacobi-Bellman Equation

In the previous section, the PDE (9.20) was derived under two additional assumptions:

(i) at the point (s, y), the partial derivatives $\partial_t V$, $\partial_{x_i} V$ exist, and

(ii) an optimal control $u : [s, T] \mapsto U$ exists, which is right continuous at the initial time $t = s$.

Our present goal is to remove these regularity assumptions, and characterize the value function as the unique solution of a first order P.D.E., in the viscosity sense. As before, we assume that the set U is compact and that the functions f, L, ψ satisfy the bounds (9.2), (9.6) and (9.7).

Theorem 10.1 (PDE satisfied by the value function). *In connection with the control system (9.1), consider the value function $V = V(s, y)$ defined by (9.8) and (9.5). Then V is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$-\Big[V_t + H(x, \nabla V)\Big] \;=\; 0 \qquad\qquad (t, x) \in\,]0, T[\, \times I\!R^n, \tag{10.1}$$

with terminal condition

$$V(T, x) \;=\; \psi(x) \qquad\qquad x \in I\!R^n, \tag{10.2}$$

and Hamiltonian function

$$H(x, p) \;\doteq\; \min_{\omega \in U} \Big\{ L(x, \omega) + p \cdot f(x, \omega) \Big\}. \tag{10.3}$$

Proof. By Lemma 9.1, the value function is bounded and uniformly Lipschitz continuous on $[0, T] \times I\!R^n$. The terminal condition (10.2) is obvious. To show that V is a viscosity solution, let $\varphi \in \mathcal{C}^1(]0, T[\, \times I\!R^n)$. Two separate statements need to be proved:

(P1) If $V - \varphi$ attains a local maximum at a point $(t_0, x_0) \in\,]0, T[\, \times I\!R^n$, then

$$\varphi_t(t_0, x_0) + \min_{\omega \in U} \Big\{ \nabla\varphi(t_0, x_0) \cdot f(x_0, \omega) + L(x_0, \omega) \Big\} \;\geq\; 0. \tag{10.4}$$

(P2) If $V - \varphi$ attains a local minimum at a point $(t_0, x_0) \in\,]0, T[\, \times I\!R^n$, then

$$\varphi_t(t_0, x_0) + \min_{\omega \in U} \Big\{ \nabla\varphi(t_0, x_0) \cdot f(x_0, \omega) + L(x_0, \omega) \Big\} \;\leq\; 0. \tag{10.5}$$

Notice that the minus sign on the left hand side of (5.19) is important, as explained in Remark 4.1.

1. To prove (P1), we can assume that

$$V(t_0, x_0) \;=\; \varphi(t_0, x_0), \qquad\qquad V(t, x) \;\leq\; \varphi(t, x) \quad \text{for all } t, x\,.$$

If (10.4) does not hold, then there exists $\omega \in U$ and $\theta > 0$ such that

$$\varphi_t(t_0, x_0) + \nabla\varphi(t_0, x_0) \cdot f(x_0, \omega) + L(x_0, \omega) \; < \; -\theta. \tag{10.6}$$

We shall derive a contradiction by showing that this control value ω is "too good to be true". Namely, by choosing a control function $u(\cdot)$ with $u(t) \equiv \omega$ for $t \in [t_0, \; t_0 + \delta]$ and such that u is nearly optimal on the remaining interval $[t_0 + \delta, \; T]$, we obtain a total cost $J(t_0, x_0, u)$ strictly smaller than $V(t_0, x_0)$. Indeed, by continuity (10.6) implies

$$\varphi_t(t, x) + \nabla\varphi(t, x) \cdot f(x, \omega) \; < \; -L(x, \omega) - \theta. \tag{10.7}$$

whenever

$$|t - t_0| \; < \; \delta, \qquad |x - x_0| \; \leq \; C\delta, \tag{10.8}$$

for some $\delta > 0$ small enough and C the constant in (9.2). Let $x(t) \doteq x(t; t_0, x_0, \omega)$ be the solution of

$$\dot{x}(t) \; = \; f(x(t), \omega), \qquad x(t_0) \; = \; x_0,$$

i.e. the trajectory corresponding to the constant control $u(t) \equiv \omega$. We then have

$$\begin{aligned} V(t_0 + \delta, \; x(t_0 + \delta)) - V(t_0, x_0) \; &\leq \; \varphi(t_0 + \delta, \; x(t_0 + \delta)) - \varphi(t_0, x_0) \\[2mm] &= \; \int_{t_0}^{t_0 + \delta} \left[\frac{d}{dt}\varphi(t, \; x(t)) \right] dt \\[2mm] &= \; \int_{t_0}^{t_0 + \delta} \left[\varphi_t(t, \; x(t)) + \nabla\varphi(t, \; x(t)) \cdot f(x(t), \omega) \right] dt \\[2mm] &\leq \; -\int_{t_0}^{t_0 + \delta} L(x(t), \; \omega) \, dt - \delta\theta, \end{aligned} \tag{10.9}$$

because of (10.7). On the other hand, the Dynamic Programming Principle (9.11) yields

$$V(t_0, x_0) \; \leq \; \int_{t_0}^{t_0 + \delta} L(x(t), \omega) \, dt + V(t_0 + \delta, \; x(t_0 + \delta)) \,. \tag{10.10}$$

Together, (10.9) and (10.10) yield a contradiction, hence (P1) must hold.

2. To prove (P2), we can assume that

$$V(t_0, x_0) \; = \; \varphi(t_0, x_0), \qquad V(t, x) \; \geq \; \varphi(t, x) \quad \text{for all } t, x \,.$$

If (P2) fails, then there exists $\theta > 0$ such that

$$\varphi_t(t_0, x_0) + \nabla\varphi(t_0, x_0) \cdot f(x_0, \omega) + L(x_0, \omega) \; > \; \theta \qquad \text{for all } \omega \in U. \tag{10.11}$$

In this case, we shall reach a contradiction by showing that no control function $u(\cdot)$ is good enough. Namely, whatever control function $u(\cdot)$ we choose on the initial interval $[t_0, \; t_0 + \delta]$, even if during the remaining time $[t_0 + \delta, \; T]$ our control is optimal, the total cost will still be considerably larger than $V(t_0, x_0)$. Indeed, by continuity, (10.11) implies

$$\varphi_t(t, x) + \nabla\varphi(t, x) \cdot f(x, \omega) \; > \; \theta - L(x, \omega) \qquad \text{for all } \omega \in U, \tag{10.12}$$

for all t, x close to t_0, x_0, i.e. such that (10.8) holds. Choose an arbitrary control function $u : [t_0, \, t_0 + \delta] \mapsto U$, and call $t \mapsto x(t) = x(t; t_0, x_0, u)$ the corresponding trajectory. We now have

$$V(t_0 + \delta, \ x(t_0 + \delta)) - V(t_0, x_0) \ \geq \ \varphi(t_0 + \delta, \ x(t_0 + \delta)) - \varphi(t_0, x_0)$$

$$= \ \int_{t_0}^{t_0+\delta} \left[\frac{d}{dt} \varphi(t, \ x(t)) \right] \, dt \ = \ \int_{t_0}^{t_0+\delta} \varphi_t(t, \ x(t)) + \nabla \varphi(t, \ x(t)) \cdot f(x(t), u(t)) \, dt$$

$$\geq \ \int_{t_0}^{t_0+\delta} \theta - L(x(t), u(t)) \, dt,$$

(10.13)

because of (10.12). Therefore, for every control function $u(\cdot)$ we have

$$V(t_0 + \delta, \ x(t_0 + \delta)) + \int_{t_0}^{t_0+\delta} L(x(t), u(t)) \, dt \ \geq \ V(t_0, x_0) + \delta\theta. \tag{10.14}$$

Taking the infimum of the left hand side of (10.14) over all control functions u, we see that this infimum is still $\geq V(t_0, x_0) + \delta\theta$. On the other hand, by the Dynamic Programming Principle (9.11), the infimum should be exactly $V(t_0, x_0)$. This contradiction shows that (P2) must hold, completing the proof. $\qquad\square$

Remark 10.1 (sufficient conditions for optimality). One can combine Theorems 5.2 and 10.1, and obtain sufficient conditions for the optimality of a control $u(\cdot)$. The standard setting is as follows. Consider the problem of minimizing the cost functional (9.5). Assume that, for each initial condition (s, y), we can guess a (hopefully optimal) control $u^{s,y} : [s, T] \mapsto U$. We then call

$$\widetilde{V}(s, y) \ \doteq \ J(s, y, u^{s,y}) \tag{10.15}$$

the corresponding cost. Typically, these control functions $u^{s,y}$ are found by applying the Pontryagin Maximum Principle, which provides a necessary condition for optimality. On the other hand, consider the true value function V, defined at (9.8) as the infimum of the cost over all admissible control functions $u(\cdot) \in \mathcal{U}$. By Theorem 10.1, V provides a viscosity solution to the Hamilton-Jacobi equation (10.1) with terminal condition $V(T, y) = \psi(y)$. If our function $\widetilde{V}$ at (10.15) also provides a viscosity solution to the same equations (10.1)-(10.2), then by the uniqueness of the viscosity solution stated in Theorem 5.2 we can conclude that $\widetilde{V} = V$. This guarantees that all controls $u^{s,y}$ are optimal.

10.1 Control problems in infinite time horizon.

We consider here an optimization problem on an unbounded time interval, with exponentially discounted payoff.

$$\text{Maximize:} \quad \int_0^{+\infty} h(x(t), u(t)) \, e^{-\gamma t} \, dt, \tag{10.16}$$

for a system with dynamics

$$\dot{x}(t) \ = \ f(x(t), u(t)), \qquad x(0) \ = \ x_0, \qquad u(t) \in U. \tag{10.17}$$

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Here h is a running payoff and $\gamma > 0$ is a discount rate. We assume that f, h are bounded, Lipschitz continuous functions, while $U \subset \mathbb{R}^m$ is a compact set. Let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be the value function, namely

$$V(x_0) \ \dot{=} \ \sup_{u(\cdot)} \ \int_0^{+\infty} h\Big(x(t, x_0, u), u(t)\Big) e^{-\gamma t} \, dt \,, \tag{10.18}$$

where $t \mapsto x(t, x_0, u)$ denotes the solution to the Cauchy problem (10.17) determined by the control $u(\cdot)$. Notice that in this case the function V depends only on the initial position x_0, not on time. Using the dynamic programming principle, one obtains a PDE satisfied by V.

Theorem 10.2 (PDE satisfied by the value function, for an infinite horizon problem). *The value function V for the problem (10.16)-(10.17) is a viscosity solution to the Hamilton-Jacobi equation*

$$\gamma V(x) - H(x, \nabla V(x)) \ = \ 0, \tag{10.19}$$

where the Hamiltonian function is

$$H(x, p) \ \dot{=} \ \sup_{\omega \in U} \Big\{ h(x, \omega) + p \cdot f(x, \omega) \Big\}. \tag{10.20}$$

For a detailed proof, similar to that of Theorem 10.1, we refer to [8], p. 104. Here we shall only give an intuitive argument, explaining why the equation (10.19) should hold.

1. For the infinite horizon, exponentially discounted problem, the dynamic programming principle takes the form

$$V(y) \ = \ \sup_{u(\cdot)} \left\{ \int_0^\tau h(x(t, y, u), u(t)) e^{-\gamma t} \, dt + e^{-\gamma \tau} V(x(\tau, y, u)) \right\}. \tag{10.21}$$

As a consequence, for every measurable control $u : [0, \infty[\mapsto U$ the following map is non-increasing:

$$\tau \ \mapsto \ \Psi^u(\tau) \ \dot{=} \ \int_0^\tau h(x(t, y, u), u(t)) e^{-\gamma t} \, dt + e^{-\gamma \tau} V(x(\tau, y, u)). \tag{10.22}$$

Moreover, the map $\tau \mapsto \Psi^u(\tau)$ is constant if and only if the control $u(\cdot)$ is optimal.

2. Assume that $u : [0, \infty[\mapsto U$ is an optimal control, and let $x(\cdot)$ be the corresponding trajectory. Since the map in (10.22) is constant, if u is continuous at time τ and and V is differentiable at the point $x(\tau)$, then

$$\begin{aligned} 0 \ &= \ \frac{d}{d\tau} \left[\int_0^\tau h(x(t), u(t)) e^{-\gamma t} \, dt + e^{-\gamma \tau} V(x(\tau)) \right] \\ &= \ e^{-\gamma \tau} \Big[h(x(\tau), u(\tau)) - \gamma V(x(\tau)) + \nabla V(x(\tau)) \cdot f(x(\tau), u(\tau)) \Big]. \end{aligned}$$

In particular, if these regularity assumptions are true at $\tau = 0$, we have

$$h(y, u(0)) - \gamma V(y) + \nabla V(y) \cdot f(y, u(0)) \ = \ 0. \tag{10.23}$$

3. On the other hand, if we choose any constant control $u(t) \equiv \omega \in U$, the corresponding map $t \mapsto \Phi^\omega(t)$ will still be nonincreasing. Hence

$$
\begin{aligned}
0 \ \geq \ & \frac{d}{d\tau} \left[\int_0^\tau h(x(t), \omega) e^{-\gamma t} \, dt + e^{-\gamma \tau} V(x(\tau)) \right] \\
= \ & e^{-\gamma \tau} \Big[h(x(\tau), \omega) - \gamma V(\tau, x(\tau)) + \nabla V(x(\tau)) \cdot f(x(\tau), \omega) \Big].
\end{aligned}
$$

In particular, if V is differentiable at the point y, then for every $\omega \in U$ we have

$$
h(y, \omega) - \gamma V(y) + \nabla V(y) \cdot f(y, \omega) \ \leq \ 0. \tag{10.24}
$$

4. Since the initial value of the optimal control satisfies $u(0) \in U$, comparing (10.23) with (10.24) we conclude

$$
\max_{\omega \in U} \Big\{ h(y, \omega) - \gamma W(y) + \nabla W(y) \cdot f(y, \omega) \Big\} \ = \ 0. \tag{10.25}
$$

$$
u(0) \ = \ u^*(y) \ \in \ \operatorname*{argmax}_{\omega \in U} \Big\{ h(y, \omega) + \nabla V(y) \cdot f(y, \omega) \Big\}. \tag{10.26}
$$

If the maximum is attained at a unique point, this uniquely determines the optimal control, in feedback form as a function of (y).

Defining the function $H(y, p)$ as in (10.20), the above equation (10.25) takes the form (10.19).

Remark 10.2 For smooth solutions, the sign of the left hand side in the equations (10.19) does not make a difference. However, this sign becomes important at points where the solution is not differentiable. Indeed, the definition of viscosity solution puts restrictions on the type of kinks that are admissible. Referring to Fig. 34, a useful guideline to keep in mind is:

- For a minimization problem, upward kinks are admissible, downward kinks are not.

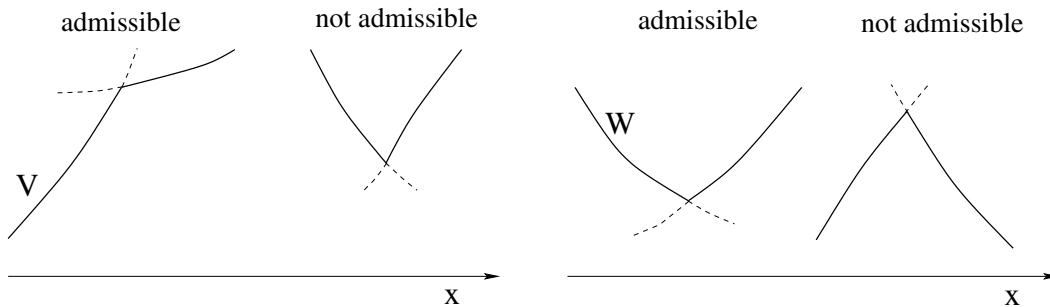- For a maximization problem, downward kinks are admissible, upward kinks are not.

Figure 34: Left: the value function V for a minimization problem. Right: the value function W for a maximization problem.

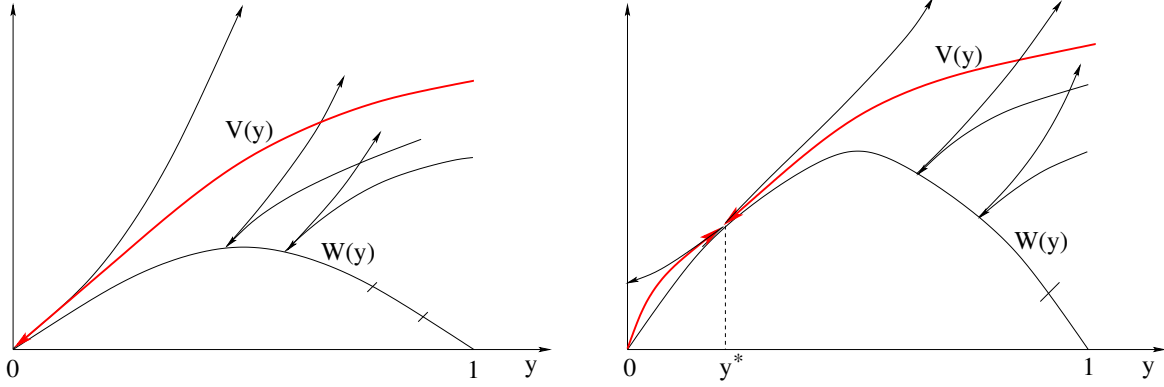Figure 35: The value function V for the optimal harvesting problem. Left: $\alpha < \gamma$, the optimal strategy leads to extinction. Right: if $\alpha > \gamma$, the optimal strategy eventually brings the population to the state $y^* = (\alpha - \gamma)/2\alpha$.

Example 10.1 Consider the optimization problem

$$\text{maximize:} \quad \int_0^\infty e^{-\gamma t}\left(2x^2 - x^4 - \frac{u^2}{2}\right)dt, \tag{10.27}$$

for a system with dynamics

$$\dot{x} = u \qquad u \in \mathbb{R}. \tag{10.28}$$

The

$$\gamma V(x) - H(x, V'(x)) = 0, \tag{10.29}$$

where

$$H(x,p) \doteq \max_{\omega \in [0,1]}\left\{\omega + p\,\alpha x(1-x) - p\omega\right\} = p\,\alpha x(1-x) + \max\{1-p,\ 0\}.$$

Solutions of the implicit ODE (10.29) can be constructed by concatenating trajectories of

$$V'(x) = \frac{\gamma V(x)}{\alpha x(1-x)} \qquad \text{if } V'(x) > 1,$$

$$V'(x) = \frac{1 - \gamma V(x)}{1 - \alpha x(1-x)} \qquad \text{if } V'(x) < 1.$$

The first case corresponds to a control $u(x) = 0$, the second case to $u(x) = 1$. On the other hand, if $V'(x) = 1$, then any value $u(x) \in [0,1]$ may occur.

We observe that

$$V(y) \geq W(y) \doteq \frac{\alpha y(1-y)}{\gamma}$$

for every $y \in [0,1]$. Indeed, starting at $x(0) = y$, this is the payoff achieved by the constant control $u(t) \equiv \alpha y(1-y)$, so that $x(t) \equiv y$. However, such a control can be optimal only if $W'(y) = 1$. To construct the function V, we consider two cases.

CASE 1: $\gamma \geq \alpha$. In this case $u \equiv 1$ is always the optimal control and all solutions approach zero in finite time. This leads to extinction. The value function is the solution to

$$V'(x) = \frac{1 - \gamma V(x)}{1 - \alpha x(1-x)}, \qquad\qquad V(0) = 0.$$

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CASE 2: $\gamma < \alpha$. In this case there is a value $y^* = (\alpha - \gamma)/2\alpha$ at which $W'(y^*) = 1$. The value function is obtained by solving the ODEs

$$V'(y) = \frac{\gamma V(y)}{\alpha y(1-y)} \qquad \text{for} \ \ y \leq y^*, \tag{10.30}$$

$$V'(y) = \frac{1 - \gamma V(y)}{1 - \alpha y(1-y)} \qquad \text{for} \ \ y \geq y^*, \tag{10.31}$$

with initial data .

$$V(y^*) = W(y^*) = \frac{\alpha}{\gamma} y^*(1 - y^*) \tag{10.32}$$

Notice that, as $t \to \infty$, all trajectories converge to the stable equilibrium $y^* = (\alpha - \gamma)/2\alpha$. As time approaches infinity, the average payoff over a unit interval of time is

$$\lim_{\tau \to \infty} \int_\tau^{\tau+1} u(t)\, dt = \alpha y^*(1 - y^*) = \frac{\alpha}{4}\left(1 - \frac{\gamma^2}{\alpha^2}\right).$$

This quantity approaches the maximum value $\alpha/4$ as $\gamma \to 0$, i.e. when planners strive for long term goal. On the other hand, as $\gamma \to \alpha$, this long term average payoff approaches zero.

10.2 Pontryagin's Maximum Principle and dynamic programming.

There is a fundamental relation between the O.D.E. satisfied by optimal trajectories of an optimal control problem and the P.D.E. of dynamic programming (10.1). Namely:

- *The trajectories which satisfy the Pontryagin Maximum Principle provide characteristic curves for the Hamilton-Jacobi equation of Dynamic Programming.*

We shall justify the above claim, assuming that all functions involved are sufficiently smooth. As a first step, we derive the equations of characteristics, in connection with the evolution equation

$$V_t + H(x, \nabla V) = 0. \tag{10.33}$$

Call $p \doteq \nabla V$ the spatial gradient of V, so that $p = (p_1, \ldots, p_n) = (V_{x_1}, \ \ldots, V_{x_n})$. Observe that

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}.$$

Differentiating (10.33) w.r.t. x_i one obtains

$$\frac{\partial p_i}{\partial t} = \frac{\partial^2 V}{\partial x_i \partial t} = -\frac{\partial H}{\partial x_i} - \sum_j \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial x_j}. \tag{10.34}$$

If now $t \mapsto x(t)$ is any smooth curve, the total derivative of p_i along x is computed by

$$\frac{d}{dt} p_i(t, x(t)) = \frac{\partial p_i}{\partial t} + \sum_j \dot{x}_j \frac{\partial p_i}{\partial x_j}$$

$$= -\frac{\partial H}{\partial x_i} + \sum_j \left(\dot{x}_j - \frac{\partial H}{\partial p_j}\right) \frac{\partial p_i}{\partial x_j}. \tag{10.35}$$

In general, the right hand side of (10.35) contains the partial derivatives $\partial p_i / \partial x_j$. However, if we choose the curve $t \mapsto x(t)$ so that $\dot{x} = \partial H / \partial p$, the last term will disappear. This observation lies at the heart of the classical method of characteristics. To construct a smooth solution of the equation (10.33) with terminal data

$$V(T, x) = \psi(x), \tag{10.36}$$

we proceed as follows. For each point $\bar{x}$, we find the solution to the Hamiltonian system of O.D.E's

$$\begin{cases} \dot{x}_i &= \dfrac{\partial H}{\partial p_i}(x, p), \\[2mm] \dot{p}_i &= -\dfrac{\partial H}{\partial x_i}(x, p), \end{cases} \qquad\qquad \begin{cases} x_i(T) &= \bar{x}_i, \\[2mm] p_i(T) &= \dfrac{\partial \psi}{\partial x_i}(\bar{x}). \end{cases} \tag{10.37}$$

This solution will be denoted as

$$t \mapsto x(t, \bar{x}), \qquad\qquad t \mapsto p(t, \bar{x}). \tag{10.38}$$

For every t we have $\nabla V(t, x(t, \bar{x})) = p(t, \bar{x})$. To recover the function V, we observe that along each solution of (10.37) one has

$$\frac{d}{dt} V(t, x(t, \bar{x})) = V_t + \dot{x} \cdot \nabla V = -H(x, p) + p \cdot \frac{\partial H}{\partial p}. \tag{10.39}$$

Therefore

$$V(t, x(t, \bar{x})) = \psi(\bar{x}) + \int_t^T \left(H(x, p) - p \cdot \frac{\partial H}{\partial p} \right) ds, \tag{10.40}$$

where the integral is computed along the solution (10.38).

Next, assume that the hamiltonian function H comes from a minimization problem, and is thus given by (10.3). To simplify our derivation, in the following we shall assume that optimal controls exist, and take values in the interior of the admissible set U. This last assumption is certainly true if $U = I\!\!R^m$. By (10.3) we now have

$$H(x, p) = \min_\omega \{ p \cdot f(x, \omega) + L(x, \omega) \} = p \cdot f(x, u^*(x, p)) + L(x, u^*(x, p)), \tag{10.41}$$

where

$$u^*(x, p) = \arg\min_\omega \{ p \cdot f(x, \omega) + L(x, \omega) \}. \tag{10.42}$$

At the point u^* where the minimum is attained, since u^* lies in the interior of U one has

$$p \cdot \frac{\partial f}{\partial u}(x, u^*(x, p)) + \frac{\partial L}{\partial u}(x, u^*(x, p)) = 0. \tag{10.43}$$

Differentiating the right hand side of (10.41) and using (10.43) we obtain

$$\begin{cases} \dfrac{\partial H}{\partial p}(x, u^*(x, p)) &= f(x, u^*(x, p)), \\[3mm] \dfrac{\partial H}{\partial x}(x, u^*(x, p)) &= p \cdot \dfrac{\partial f}{\partial x}(x, u^*(x, p)) + \dfrac{\partial L}{\partial x}(x, u^*(x, p)). \end{cases}$$

The Hamiltonian system (10.37) thus takes the form

$$\begin{cases} \dot{x} &= f(x,\, u^*(x,p))\,, \\[2mm] \dot{p} &= -\,p \cdot \dfrac{\partial f}{\partial x}(x,\, u^*(x,p)) - \dfrac{\partial L}{\partial x}(x,\, u^*(x,p))\,, \end{cases} \qquad \begin{cases} x(T) &= \bar{x}\,, \\[2mm] p(T) &= \nabla\psi(\bar{x})\,. \end{cases} \qquad (10.44)$$

We observe that the evolution equations in (10.44) and the optimality conditions (10.42) are precisely those given in the Pontryagin Maximum Principle. In other words, let $t \mapsto u^*(t)$ be a control for which the Pontryagin Maximum Principle is satisfied. Then the corresponding trajectory $x(\cdot)$ and the adjoint vector $p(\cdot)$ provide a solution to the equations of characteristics for the corresponding hamiltonian system (10.33).

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