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ON A TRANSFORMATION OF CHARACTERISTIC FUNCTIONS *

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There seems to be an increasing interest in transformations of characteristic functions, more precisely in operations which transform a given characteristic function into a new characteristic function. Naturally one investigates also the properties of the transformed characteristic function.

As far as we know the first study of such a transformation is due to BRUNO DE FINETTI [2], somewhat later A. YA KHINTCHINE [5] introduced a transformation by means of an integral which converts an arbitrary characteristic function into the characteristic function of a unimodal distribution. This transformation was studied recently by M. GIRAULT [3], [4] and was generalized by H. LOEFFEL [6].

The present note is a supplement to an earlier paper [7] and deals with a transformation of an arbitrary characteristic function by means of two integrations.

We denote by $F(x)$ a distribution function, that is a never decreasing, rightcontinuous function such that $F(-\infty)=0$ and $F(+\infty)=1$. Its characteristic function

$$f(y)=\int_{-\infty}^{\infty} e^{iyx}\, d\,F(x) \tag{1}$$

is defined for all real y.

It is known that $f(t)$ is a continuous function such that $f(0)=1$, therefore $\varphi(t)=\ln f(t)$ is defined in an interval of t values which contains $t=0$ in its interior. The function $\varphi(t)$ is called the cumulant generating function of the distribution function $F(x)$.

A characteristic function $f(t)$ is said to be infinitely divisible, if for every positive integer n, it is the n-th power of some cha-

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racteristic function $f_n(t)$ which depends of course on n. Then $f_n(t)$ is uniquely determined by $f(t), f_n(t) = [f(t)]^{\frac{1}{n}}$, provided that one selects for the n-th root the principal branch. The characteristic functions of infinitely divisible distributions have no real zeros so that their cumulant generating function is defined for all t. Moreover they admit canonical representations. We will not need the most general representation but will use only the following theorem:

Kolmogorov's representation theorem. *The function* $f(t)$ *is an infinitely divisible characteristic function with finite variance if, and only if, it can be represented in the form*

$$\log f(t) = ict + \int_{-\infty}^{\infty} (e^{itu} - 1 - itu) \frac{dK(u)}{u^2} \tag{2}$$

where c *is real and constant and where* $K(u)$ *is a non-decreasing and bounded function such that* $K(-\infty) = 0$ *and* $\int_{-\infty}^{\infty} dK(u) = K(+\infty) < \infty$.

We form now the integral $h(u) = \int_0^u f(y)\,dy$ where $f(y)$ is given by (1) and where u is real. Then $h(u) = \int_0^u \left[\int_{-\infty}^{\infty} e^{iyx}\,dF(x)\right] dy$; from the boundedness of e^{iyx} we conclude that the order of the two integrations may be exchanged and we obtain

$$h(u) = \int_{-\infty}^{\infty} \frac{e^{iux} - 1}{ix}\,dF(x).$$

We form next the integral

$$\int_0^t h(u)\,du = \int_0^t \left[\int_{-\infty}^{\infty} \frac{e^{iux} - 1}{ix}\,dF(x)\right] du =$$

$$= \int_0^t \left[\int_{-\infty}^{-A} \frac{e^{iux} - 1}{ix}\,dF(x)\right] du + \int_0^t \left[\int_{-A}^{A} \frac{e^{iux} - 1}{ix}\,dF(x)\right] du +$$

$$+ \int_0^t \left[\int_{A}^{\infty} \frac{e^{iux} - 1}{ix}\,dF(x)\right] du$$

where $A > 0$ is an arbitrary constant. It is easily seen that the integrand $\frac{e^{iux} - 1}{ix}$ is bounded. The bound for the first and last integral is $\frac{2}{A}$, for the integral in the middle it equals A. The exchange of the order of the two integrations is then justified an we obtain

$$\int_0^t h(u)\,du = \int_{-\infty}^{\infty}\left[\int_0^t \frac{e^{iux}-1}{ix}\,du\right] d\,\mathrm{F}(x) = \int_{-\infty}^{\infty}\left[\frac{e^{itx}-1}{i^2x^2} - \frac{t}{ix}\right] d\mathrm{F}(x).$$

If we write

$$\psi(t) = -\int_0^t\left[\int_0^u f(y)\,dy\right]du \tag{3}$$

then we obtain from the last equation

$$\psi(t) = \int_{-\infty}^{\infty}(e^{itx}-1-itx)\frac{d\,\mathrm{F}(x)}{x^2}. \tag{4}$$

The function $\psi(t)$ has therefore the canonical representation (2) of KOLMOGOROV's theorem and we obtain the following result.

THEOREM *Let* f(y) *be an arbitrary characteristic function then*

$$\psi(\mathrm{t}) = -\int_0^{\mathrm{t}}\left[\int_0^{\mathrm{u}} \mathrm{f}(\mathrm{y})\,\mathrm{dy}\right]\mathrm{du}$$

is the cumulant generating function of an infinitely divisible law with finite variance.

We consider next a particular case and assume that $f(y)$ is the characteristic function of a distribution $\mathrm{F}(x)$ which has a finite second moment α_2. Then $f(t)$ may be differentiated twice and $f''(t) = -\int_{-\infty}^{\infty} x^2 e^{itx}\,d\,\mathrm{F}(x)$. The function

$$g(y) = \frac{1}{\alpha_2}\int_{-\infty}^{\infty} e^{itx}x^2\,d\,\mathrm{F}(x) \tag{5a}$$

is then a characteristic function, the corresponding distribution function is

$$\mathrm{G}(x) = \int_{-\infty}^{x}\frac{x^2}{\alpha_2}\,d\,\mathrm{F}(x). \tag{5b}$$

We apply the theorem and see that

$$\psi(t) = \frac{1}{\alpha_2}\int_0^t\left[\int_0^u f''(y)\,dy\right]du = \frac{1}{\alpha_2}[f(t) - 1 - i\alpha_1 t]$$

is the cumulant generating function of an infinitely divisible law. From this it follows immediately that $[f(t)-1]$ is the cumulant generating function of an infinitely divisible law. This is a particular case of a result of DE FINETTI [2], who showed that this is true even if the assumption concerning the existence of the second moment is not made.

We list also a few examples for the application of the theorem:

(*a*) Let $f(y) = e^{-|y|}$ be the characteristic function of CAUCHY's distribution. We obtain then $\psi(t) = 1 - e^{-|t|} - |t|$ and the corresponding characteristic function is $g(t) = e^{-|t|+1-e^{-|t|}}$. It follows from DE FINETTI's theorem that $e^{e^{-|t|}-1}$ is the characteristic function of an infinitely divisible distribution. We see therefore that

$$e^{-|t|} = g(t)\, e^{(e^{-|t|}-1)}. \tag{6}$$

This is a factorization of the CAUCHY distribution into two infinitely divisible factors which do not belong to stable distributions. Examples of factorizations of the CAUCHY distribution into two non-stable factors are not new. D. DUGUÉ [1] constructed already such examples by means of a theorem of PÓLYA.

(*b*) Let $f(y) = e^{iya}$ the characteristic function of a degenerate distribution. This is transformed into $\psi(t) = -\dfrac{it}{a} + \dfrac{1}{a^2}(e^{ita} - 1)$ which is clearly the cumulant generating function of an infinitely divisible law.

(*c*) If we use

$$f(t) = \begin{cases} 1 - |y| & \text{if } |y| \leqq 1 \\ 0 & \text{otherwise} \end{cases}$$

then we obtain after some computation

$$\psi(t) = \begin{cases} \dfrac{1}{6} - \dfrac{|t|}{2} & \text{if } |t| \geqq 1 \\[2ex] -\dfrac{t^2}{2} + \dfrac{|t|^3}{6} & \text{if } |t| \leqq 1 \end{cases}$$

which is the cumulant generating function of an infinitely divisible distribution.

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