HW

Find zeros of $\boxed{y = x^3 - 2x^2 + \varepsilon x = 0}$ (I) for $\varepsilon \to 0$

Assume $x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ ← This is our Ansatz

Zero-th order approximation: ε^0: $y \sim y_0 = x^3 - 2x^2 \to$ find solutions.

↳ Here we consider only the terms that do not contain any ε. → This we suppress the (εx) term

$x_1 = 0$, $x_2 = 0$, $x_3 = 2$

a] Assume $x = 2 + a\varepsilon$ (*), find a.

Here in essence we want to approximate the solution near $x_3 = 2$.

The zero-th order approximation gave us a coarse estimation about the position of the root/zero, but we'd like to have a more precise estimation.

Thus, we plug-in $x = 2 + a\varepsilon$ into $y = x^3 - 2x^2 + \varepsilon x = 0$ and we try to calculate the coefficient a by balancing/equating the terms that contain ε^1 at both sides of the equation.

$$\Rightarrow y = (2 + a\varepsilon)^3 - 2(2 + a\varepsilon)^2 + \varepsilon(2 + a\varepsilon) = 0$$

$$= 8 + 12a\varepsilon + \dots - 8 - 8a\varepsilon - \dots + 2\varepsilon = 0$$ → Balance red terms.

↳ here I neglected the terms ε^2 and higher.

1st order approx: ε^1: $12a - 8a + 2 = 0 \to \boxed{a = -\frac{1}{2}}$

Thus, $\boxed{x \sim 2 - \frac{1}{2}\varepsilon \quad \text{as } \varepsilon \to 0}$ This is the approximation up to 1st order for the solution of the initial eq (I) near $x = 2$

(*) This is the Ansatz after suppressing the terms ε^2 and higher, and after incorporating the information that we gained from the zeroth-order approx, namely that $x_0 = 2$.

b] But why this doesn't work for the other roots near $x=0$?

Looking at eq (I) again $y=x^3-2x^2+\varepsilon x$, we might notice that in the vicinity of $x=0$, the cubiq term is negligible compared to the (εx) term.

For $x\to 0$, $x^3 \ll \varepsilon x$, thus it is not sensible to drop the εx term as we did previously, since the term εx dominates the equation near the origin.

[$\rightsquigarrow$ The <u>dominant balance</u> depends on the question/on the solution we try to approximate]

The x^3 term is also negligible compared to the $-2x^2$ term as $x\to 0$.

So essentially we want to balance the terms εx and $-2x^2$ $\rightsquigarrow$ The balance is correct when $-2x^2$ and εx are about the same size, therefore $(x\approx\varepsilon)$ x is of the order of ε

x=εX, y=(ε^2)Y

c] Find a rescaling ~~$X=\varepsilon x$, $Y=\varepsilon^2 y$~~ to approximately represent zeros near $x=0$ and solve eq. with standard expansion.

Since the less dominant term is x^3 as $x\to 0$, $\varepsilon x = O(-2x^2)=O(x^2)$, and $x=O(\varepsilon)$, then $y\sim\varepsilon^2$ as $\varepsilon\to 0$.

By plugging in the new variables we obtain:

$$Y=\varepsilon X^3-2X^2+X=0$$ $\to$ indeed here the cubiq term will be considered as negligible

Plug in the Ansatz: $X=X_0+\varepsilon X_1$:

$$Y=\varepsilon(X_0+\varepsilon X_1)^3-2(X_0+\varepsilon X_1)^2+(X_0+\varepsilon X_1)=$$

$$=\ldots=\underbrace{X_0-2X_0^2}+\varepsilon\underbrace{\left[X_0^3-4X_0X_1+X_1\right]}+\varepsilon^2[\ldots]=\underbrace{0}$$ $\to$ neglect.

Now balance both sides

- Zeroth order: ε^0: $X_0-2X_0^2=0 \Rightarrow X_0=0$ or $X_0=1/2$
- First order: ε^1: $X_0^3-4X_0X_1+X_1=0 \Rightarrow X_1=1/8$

We approximate the solution that is near zero but different than 0.

Thus, $X \sim \frac{1}{2} + \frac{1}{8}\varepsilon$

We return back in the original variables, small x and y:

$x = \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \dots$ or $x \sim \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$

Altogether we have the approximations for the 3 roots:

$\boxed{\begin{aligned} &x \sim 2 - \frac{1}{2}\varepsilon \\ &x \sim \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 \\ &x = 0 \end{aligned}}$

$x \sim 2 - \frac{1}{2}\varepsilon$ → from question a]

$x \sim \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$ → just calculated above

$x = 0$ → also calculated above, but we didn't focus on this since the zeroth order parameter is 0 and the successive parameters also.

To summarise the steps we follow are:

- Propose an Ansatz, usually $x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots \quad , \varepsilon \to 0$
- Plug in the Ansatz to the eq. we want to approximately solve
- Balance the zeroth order coefficients to obtain x_0
- Balance the 1st order/ε^1 coefficients to obtain x_1

  ⋮

  → Stop at some order and obtain approx solution for $\varepsilon \to 0$ by introducing calculated coeffs in Ansatz

→ If we need to rescale the variables, because the ε term of the initial equation dominates in the vicinity of the solution we'd want to approximate.

- We find an appropriate rescaling/transformation
- Rewrite the eq. in terms of the newly defined variables.
- Follow the above procedure, but now by proposing an Ansatz in terms of the new variables.
- Finally express the approximation in terms of the initial variables.