PROOF BY INDUCTION PROBLEMS

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* (1) Some Sum Facts

Prove the following identities using induction.

a) Sum of consecutive odd natural numbers:
$1 + 3 + 5 + \ldots + 2((n-1)-1) + (2n-1) = n^2$

What is the base case to prove? Prove it.

What is the inductive hypothesis to assume?

What is the inductive case to prove? Prove it.

b) Sum of consecutive even natural numbers:
$2 + 4 + 6 + \ldots + 2n = n(n+1)$

What is the base case to prove? Prove it.

What is the inductive hypothesis to assume?

What is the inductive case to prove? Prove it.
(Hint: Write the inductive case to explicitly show the 2nd-to-last term, after the "…")

** (2) Famous Fibonacci

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … is defined by:

$F_1 = 1$
$F_2 = 1$
$F_n = F_{n-2} + F_{n-1}$

Using induction, prove that F_{3n} (that is, every third Fibonacci number – F_1, F_3, F_6, F_9, …) is even for every integer $n \geq 1$. Recall that an integer x is called even if x = 2y for some other integer y.

** (3) Some Somewhat Sneakier Sum Facts

Prove the following sum facts. If you use induction, remember to state and prove the base case, and to state and prove the inductive case.

a) Sum of squares of consecutive natural numbers:
$\mathbf{1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = n(n+1)(2n+1)/6}$ for all integers $n \geq 1$

b) Sum of squares of consecutive multiples of three:
$\mathbf{3^2 + 6^2 + 9^2 + \ldots + (3n)^2 = (3/2) \cdot n \cdot (n+1) \cdot (2n+1)}$ for all integers $n \geq 1$

(continued)

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c) Sum of powers:
$x^0 + x^1 + x^2 + x^3 + x^4 + \ldots + x^n = (x^{n+1} - 1) / (x - 1)$ for all integers $n \geq 1$

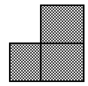
** (4) Fun False Proof: I Want a (Colored) Pony!

A way to say that something is surprisingly different from usual is to exclaim "Now, *that's* a horse of a different color!" The well-known mathematician George Pólya posed the following false "proof" showing through mathematical induction that actually, *all horses are of the same color*.

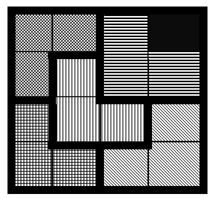
> **Base case:** If there's only *one* horse, there's only one color, so of course it's the same color as itself.
>
> **Inductive case:** Suppose within any set of n horses, there is only one color. Now look at any set of $n + 1$ horses. Number them: 1, 2, 3, ..., n, $n + 1$. Consider the sets $\{1, 2, 3, ..., n\}$ and $\{2, 3, 4, ..., n + 1\}$. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n + 1$ horses.

**** (5) Triminoes

Given a 2^n by 2^n checkerboard with *any* one square deleted, use induction to prove that it is possible cover the board with rotatable L-shaped pieces each covering three squares (called triminoes):



For example, for a 4 x 4 (n = 2) board with a corner removed:



Note that you do not necessarily have to show what the covering is, just that it exists.

**** (6) Sums of Combinations

Recall that a combination ${}_nC_k$ is the number of ways to choose k elements from n elements ($n \geq k$), disregarding order. A nifty formula is ${}_nC_k = \dfrac{n!}{k! \cdot (n-k)!}$

a) Prove $\mathbf{{}_nC_0 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \ldots + {}_nC_{n-1} + {}_nC_n = 2^n}$

b) Prove $\mathbf{0 \cdot {}_nC_0 + 1 \cdot {}_nC_1 + 2 \cdot {}_nC_2 + 3 \cdot {}_nC_3 + \ldots + n \cdot {}_nC_n = n \cdot 2^{n-1}}$