forthright48

/cpps

Bell Numbers

In combinatorial mathematics, the Bell numbers count the number of partitions of a set.

In mathematics, a partition of a set is a grouping of the set's elements into non-empty subsets, in such a way that every element is included in one and only one of the subsets.

Starting with $B_0 = B_1 = 1$, the first few Bell numbers are:

1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804, 682076806159, 5832742205057

Application

1. **Set Partition**: B_n ounts the number of different ways to partition a set that has exactly n elements.

Calculation

Recurrence

$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k$$

Imagine we have $n + 1$ object. Then the $(n+1)_{th}$ $object will be in some partition along with other objects. Let that have size k+1$ $. Then, there are \binom{n}{k}$ $ways of choosing the partition and remaining objects gets further partitioned in B_{n-k}$ ways.

Bell Triangle

Similar to Pascal Triangle, but generating rule is different. Let it be a two dimensional array `B[][]`.

1. Start with 1 in first row.
2. For subsequent rows, the first element is the last element of previous row.
3. `B[i][j] = B[i][j-1] + B[i-1][j-1]`

The top element of the Bell's triangle is indexed as $B[1][1]$. Here, $B[n][k]$ means number of way we can partition $(n+1)$ object such that $k+1$ is the largest singleton. Therefore, $B[n][n] = B_n$.

Augmented Bell Triangle

Similar to Bell Triangle, but the first element of row grater than 1 is calculated as difference of rightmost and leftmost elemnt of previous row.

Basically, $B[n][0] = B[n-1][n-1] - B[n-1][0]$.

Modular Arithmetic

If p is any prime number, then

$$B_{p^m+n} \equiv mB*n* + *B*{n+1} \text{ (mod p) }$$

Property of Bell Triangle

Sum of Row

The sum of n_{th} row of Bell triangle counts the number of partitions of n elements into subsets, where one of the subsets is distinguished from the others.

For example, there are 10 ways of partitioning three items into subsets and then choosing one of the subsets.

The sum of $n*{th}*$ row is same as difference between $B*{n+1}* and B_n$.

Suppose we have to distinguish a subset in partition of n elements. Then simply partition set of $n+1$ element and the subset that contains $n+1$ is the chosen subset. But, sometimes $n+1$ is a singleton. There are exactly $B*n* number of such cases. So subtract those. Therefore, result is *B*{n+1} - B_n$.