Multinomial Theorem:

For any positive integer m and any nonnegative integer n, the multinomial formula tells us how a sum with m terms expands when raised to an arbitrary power n:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t},$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

is a **multinomial coefficient**. The sum is taken over all combinations of nonnegative integer indices k_1 through k_m such that the sum of all k_i is n. That is, for

$$a^2 b^0 c^1 \text{ has the coefficient } \binom{3}{2,0,1} = \frac{3!}{2! \cdot 0! \cdot 1!} = \frac{6}{2 \cdot 1 \cdot 1} = 3$$

$$a^1 b^1 c^1 \text{ has the coefficient } \binom{3}{1,1,1} = \frac{3!}{1! \cdot 1! \cdot 1!} = \frac{6}{1 \cdot 1 \cdot 1} = 6.$$

Multinomial coefficients [edit]

The numbers

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!},$$

which can also be written as

$$= \binom{k_1}{k_1} \binom{k_1 + k_2}{k_2} \cdots \binom{k_1 + k_2 + \cdots + k_m}{k_m} = \prod_{i=1}^{m} \binom{\sum_{j=1}^{i} k_j}{k_i}$$

are the multinomial coefficients. Just like "n choose k" are the coefficients when a *binomial* is raised to the n^{th} power (e.g., the coefficients are 1,3,3,1 for $(a + b)^3$, where $n = 3$), the multinomial coefficients appear when a *multinomial* is raised to the n^{th} power (e.g., $(a + b + c)^3$).

Sum of all multinomial coefficients [edit]

The substitution of $x_i = 1$ for all i into:

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = (x_1 + x_2 + \cdots + x_m)^n,$$

gives immediately that

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} = m^n.$$

Number of multinomial coefficients [edit]

The number of terms in a multinomial sum, $\#_{n,m}$, is equal to the number of monomials of degree n on the variables $x_1, \ldots, x_m$:

$$\#_{n,m} = \binom{n+m-1}{m-1}.$$

The count can be performed easily using the method of stars and bars.

to obtain the parity of a Stirling number of the second kind in $O(1)$ time. In pseudocode:

$$\left\{ {n \atop k} \right\} \bmod 2 := [((n-k)\ \&\ ((k-1) \operatorname{div} 2)) = 0];$$

Vandermonde's identity

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For the expression for a special determinant, see Vandermonde matrix.

In combinatorics, **Vandermonde's identity**, or **Vandermonde's convolution**, named after Alexandre-Théophile Vandermonde (1772), states that

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}, \qquad m, n, r \in \mathbb{N}_0,$$

for binomial coefficients. This identity was given already in 1303 by the Chinese mathematician Zhu Shijie (Chu Shi-Chieh). See Askey 1975, pp. 59–60 for the history.

There is a q-analog to this theorem called the q-Vandermonde identity.

The general form of Vandermonde's identity is

$$\binom{n_1 + \cdots + n_p}{m} = \sum_{k_1+\cdots+k_p=m} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_p}{k_p}.$$

- Let's prove that this number is equal to $\binom{x+y}{x} = \frac{(x+y)!}{x!\cdot y!}$. It's easy to observe that this formula also means the number of ways to match the string with the sequence of zeros and ones of the same length.