Logical symbols and some basic equivalences (incomplete list, note that some trivial rules have been omitted, and that some dual rules have been listed only once, e.g. the absorption rule):

Propositional logic	
$\land$	and
$\lor$	or
$\neg$	not
$\implies$	implies
$\iff$	if and only if
Predicate logic	
$\exists$	exists
$\forall$	for all

Propositional logic	
$A \land (B \lor C) \equiv (A \land B) \lor (A \land C)$	(Distributivity)
$A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$	(Distributivity)
$A \implies B \equiv \neg A \lor B$	(Implication)
$A \iff B \equiv (A \implies B) \land (B \implies A)$	(Biconditional)
$A \land (A \lor B) \equiv A$	(Absorption)
$\neg(A \land B) \equiv \neg A \lor \neg B$	(De Morgan)
$\neg(A \lor B) \equiv \neg A \land \neg B$	(De Morgan)
$\text{true} \land A \equiv A \quad \text{false} \lor A \equiv A$	(Identity)
$\text{true} \lor A \equiv \text{true} \quad \text{false} \land A \equiv \text{false}$	(Domination)
Predicate logic	
$\neg(\exists x.P) \equiv \forall x.\neg P$	
$\neg(\forall x.P) \equiv \exists x.\neg P$	
$(\exists x.P) \lor (\exists x.Q) \equiv \exists x.(P \lor Q)$	
$(\forall x.P) \land (\forall x.P) \equiv \forall x.(P \land Q)$	

Some basic rules for logic with arithmetic in $\mathbb{Z}$:

$s \neq t$	$\equiv \neg(s = t)$
$s \leq t$	$\equiv \neg(s > t)$
$s < t$	$\equiv (s \leq t) \land (s \neq t)$
$s = t$	$\equiv (s \leq t) \land (s \geq t)$

$\boxed{1.1}$ Common rules

The following rules are often useful when dealing with weakest preconditions:

(a) $\neg(A \implies B) \equiv A \land \neg B$
(b) $A \implies B \equiv \neg B \implies \neg A$
(c) $A \iff B \equiv (A \land B) \lor (\neg A \land \neg B)$
(d) If $A \implies B$ holds, then $A \lor B \equiv B$ and $A \land B \equiv A$.
(e) $(A \implies B) \land (\neg A \implies C) \equiv (A \land B) \lor (\neg A \land C)$

Prove them.

Hint: In order to prove the first part of (d), it suffices to show $(A \implies B) \implies ((A \lor B) \iff B) \equiv \text{true}$. *Note that this is a generalization of the Absorption rule.*

Answer of exercise $\boxed{1.1}$

For clarity, all used rules are annotated, except Associativity and Commutativity.

(a)

$$\begin{aligned} \neg(A \implies B) &\equiv \neg(\neg A \lor B) && \text{(Implication)} \\ &\equiv \neg\neg A \land \neg B && \text{(De Morgan)} \\ &\equiv A \land \neg B && \text{(Double Negation)} \end{aligned}$$

(b)

$$\begin{aligned} A \implies B &\equiv \neg A \lor B && \text{(Implication)} \\ &\equiv \neg A \lor \neg\neg B && \text{(Double Negation)} \\ &\equiv \neg B \implies \neg A && \text{(Implication)} \end{aligned}$$

(c)

$$\begin{aligned} A \iff B &\equiv (A \implies B) \land (B \implies A) && \text{(Biconditional)} \\ &\equiv (\neg A \lor B) \land (\neg B \lor A) && \text{(Implication)} \\ &\equiv (\neg A \land \neg B) \lor (\neg A \land A) \lor (B \land \neg B) \lor (B \land A) && \text{(Distributivity)} \\ &\equiv (\neg A \land \neg B) \lor \text{false} \lor \text{false} \lor (B \land A) && \text{(Contradiction)} \\ &\equiv (\neg A \land \neg B) \lor (B \land A) && \text{(Identity)} \end{aligned}$$

(d) For the first part, show the hint:

$$\begin{aligned}
&(A \implies B) \implies ((A \lor B) \iff B) \\
&\equiv \neg(A \implies B) \lor ((A \lor B) \iff B) && \text{(Implication)} \\
&\equiv (A \land \neg B) \lor ((A \lor B) \iff B) && \text{(by (a))} \\
&\equiv (A \land \neg B) \lor ((A \lor B) \land B) \lor (\neg(A \lor B) \land \neg B) && \text{(by (c))} \\
&\equiv (A \land \neg B) \lor B \lor (\neg(A \lor B) \land \neg B) && \text{(Absorption)} \\
&\equiv (A \land \neg B) \lor B \lor (\neg A \land \neg B \land \neg B) && \text{(De Morgan)} \\
&\equiv (A \land \neg B) \lor B \lor (\neg A \land \neg B) && \text{(Idempotence)} \\
&\equiv ((A \lor \neg A) \land \neg B) \lor B && \text{(Distributivity)} \\
&\equiv (\text{true} \land \neg B) \lor B && \text{(Tautology)} \\
&\equiv \neg B \lor B && \text{(Identity)} \\
&\equiv \text{true} && \text{(Tautology)}
\end{aligned}$$

The second part is analogous.

(e)

$$\begin{aligned}
&(A \implies B) \land (\neg A \implies C) \\
&\equiv (\neg A \lor B) \land (\neg\neg A \lor C) && \text{(Implication)} \\
&\equiv (\neg A \lor B) \land (A \lor C) && \text{(Double Negation)} \\
&\equiv (\neg A \land A) \lor (B \land A) \lor (\neg A \land C) \lor (B \land C) && \text{(Distributivity)} \\
&\equiv \text{false} \lor (B \land A) \lor (\neg A \land C) \lor (B \land C) && \text{(Contradiction)} \\
&\equiv (B \land A) \lor (\neg A \land C) \lor (B \land C) && \text{(Identity)} \\
&\equiv (B \land A) \lor (\neg A \land C) \lor (B \land C) \lor (B \land C) && \text{(Idempotence)} \\
&\equiv (B \land A \land B \land C) \lor (\neg A \land C \land B \land C) \lor (B \land C) && \text{(Distributivity)} \\
&\equiv (B \land C \land A) \lor (B \land C \land \neg A) \lor (B \land C) && \text{(Idempotence)}
\end{aligned}$$

If we now show that $B \land C \implies (B \land C \land A) \lor (B \land C \land \neg A)$ holds, we are done by using part (d). So, continue with:

$$\begin{aligned}
\text{true} &\equiv \neg(B \land C) \lor (B \land C) && \text{(Tautology)} \\
&\equiv (B \land C) \implies (B \land C) && \text{(Implication)} \\
&\equiv (B \land C) \implies (B \land C \land \text{true}) && \text{(Identity)} \\
&\equiv (B \land C) \implies (B \land C \land (A \lor \neg A)) && \text{(Tautology)} \\
&\equiv (B \land C) \implies (B \land C \land A) \lor (B \land C \land \neg A) && \text{(Distributivity)}
\end{aligned}$$

1.2 More equivalence transformations (*2017, H1.6*)

Show the following equivalences by using equivalence transformations.

(a) $((B \land \neg C) \lor (B \land C)) \implies (A \lor D) \equiv A \lor (B \implies D)$

(b) $\neg C \equiv (\text{false} \impliedby (A \land C \land \neg A)) \land (C \implies (\neg C \land B))$

(c) $(A \lor B) \iff ((A \implies D) \land \neg B) \equiv A \land \neg B \land D$

Answer of exercise 1.2

(a)

$$\begin{aligned}
&(\underline{(B \land \neg C) \lor (B \land C)}) \implies A \lor D \\
&\equiv B \land (\underline{C \lor \neg C}) \implies A \lor D && \text{(Distributivity)} \\
&\equiv \underline{B \land \text{true}} \implies A \lor D && \text{(Tautology)} \\
&\equiv \underline{B \implies A \lor D} && \text{(Identity)} \\
&\equiv \underline{\neg B} \lor A \underline{\lor D} && \text{(Implication)} \\
&\equiv A \lor (B \implies D) && \text{(Implication)}
\end{aligned}$$

(b)

$$
\begin{aligned}
&(\text{false} \impliedby \underline{A} \wedge C \underline{\wedge \neg A}) \wedge (C \implies \neg C \wedge B) \\
&\equiv (\text{false} \impliedby \underline{\text{false} \wedge C}) \wedge (C \implies \neg C \wedge B) && \text{(Contradiction)} \\
&\equiv (\underline{\text{false} \impliedby \text{false}}) \wedge (C \implies \neg C \wedge B) && \text{(Identity)} \\
&\equiv \underline{\text{true} \wedge (C \implies \neg C \wedge B)} && \text{(Implication, Domination)} \\
&\equiv \underline{C \implies \neg C \wedge B} && \text{(Identity)} \\
&\equiv \underline{\neg C \vee \neg C \wedge B} && \text{(Implication)} \\
&\equiv \neg C && \text{(Absorption)}
\end{aligned}
$$

(c) **Lemma:** $X \wedge (\neg X \vee Y) \equiv X \wedge Y$.
Proof of the Lemma:

$$
\begin{aligned}
&X \wedge (\neg X \vee Y) \\
&\equiv (\underline{X \wedge \neg X}) \vee (X \wedge Y) && \text{(Distributivity)} \\
&\equiv X \wedge Y && \text{(Contradiction, Identity)}
\end{aligned}
$$

Now, let's prove the actual statement:

$$
\begin{aligned}
&A \vee B \iff (\underline{A \implies D}) \wedge \neg B \\
&\equiv A \vee B \iff (\neg A \vee D) \wedge \neg B && \text{(Implication)} \\
&\equiv ((\underline{A \vee B}) \wedge (\neg A \vee D) \underline{\wedge \neg B}) \vee (\neg(A \vee B) \wedge \neg((\neg A \vee D) \wedge \neg B)) && \text{(by 1 (c))} \\
&\equiv (\neg B \wedge \underline{A \wedge (\neg A \vee D)}) \vee (\neg(A \vee B) \wedge \neg((\neg A \vee D) \wedge \neg B)) && \text{(by Lemma)} \\
&\equiv (\neg B \wedge A \wedge D) \vee (\underline{\neg(A \vee B)} \wedge \underline{\neg((\neg A \vee D) \wedge \neg B)}) && \text{(by Lemma)} \\
&\equiv (\neg B \wedge A \wedge D) \vee (\neg A \wedge \underline{\neg B \wedge ((A \wedge \neg D) \vee B)}) && \text{(De Morgan, Double Negation)} \\
&\equiv (\neg B \wedge A \wedge D) \vee (\underline{\neg A \wedge A} \wedge \neg D) && \text{(by Lemma)} \\
&\equiv (\neg B \wedge A \wedge D) \vee (\underline{\text{false} \wedge \neg D}) && \text{(Contradiction)} \\
&\equiv \neg B \wedge A \wedge D && \text{(Dominance, Identity)}
\end{aligned}
$$

$\boxed{\textbf{1.3}}$ Comparing assertions (*2017, H1.7*)

For the following pairs (A and B) of assertions, decide whether $A \implies B$, $B \implies A$, both or none of them hold. Assume that x, y, z are of type *integer*, i.e. can only take values in $\mathbb{Z}$.

(a) $x > 0$ and $x > -2$
(b) $z \neq 1$ and $z < -1$
(c) $y \geq 3$ and $3 = y \wedge x < 0$
(d) $x < y$ and $x < 12$
(e) $x \neq 0$ and $x \leq 1 \wedge -x \leq -1$
(f) $x = 42 \wedge z = 42$ and $x = z$
(g) false and $x < y \wedge z > y \wedge z < x$
(h) $y = 2 \vee z = -1$ and $z \leq y$
(i) $x^2 \geq 1$ and $x \geq 1$
(j) $x < 0 \wedge y > 3$ and $x < -1 \wedge y > 2$
(k) $|x| > 1 \vee x = 0$ and $z = 1 \wedge x \neq z$
(l) $x! = 120$ and $x = 3 \vee x = 5$

Answer of exercise $\boxed{\textbf{1.3}}$

(a) $A \implies B$
(b) $A \impliedby B$
(c) $A \impliedby B$
(d) no implication holds
(e) $A \impliedby B$
(f) $A \implies B$

(g) $A \iff B$
(h) no implication holds
(i) $A \impliedby B$
(j) no implication holds
(k) no implication holds
(l) $A \implies B$

$\boxed{\textbf{1.4}}$ Comparing assertions II (*2021, T1.1*)

This exercise is similar to exercise $\boxed{\textbf{1.3}}$ and can be found on `https://artemis.ase.in.tum.de/courses/147/exercises/5279`.

Answer of exercise $\boxed{\textbf{1.4}}$

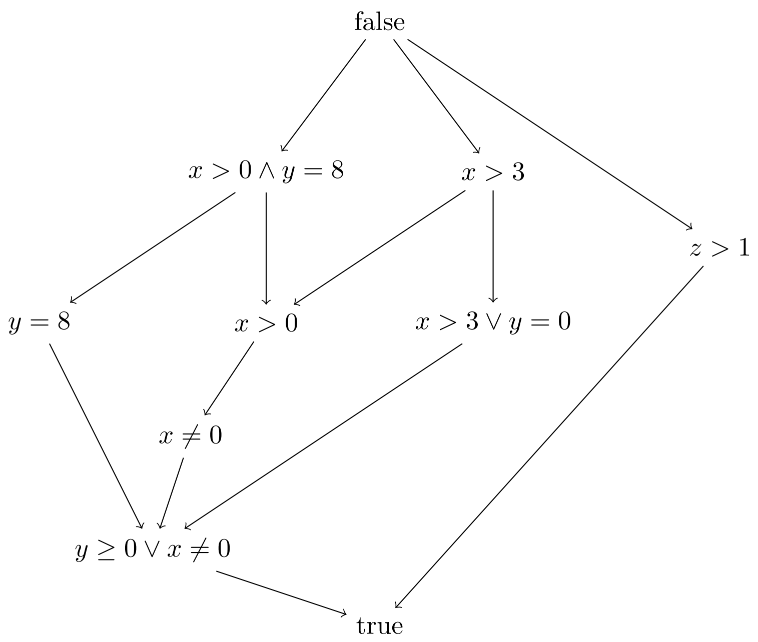
See `https://artemis.ase.in.tum.de/courses/147/exercises/5279`.

$\boxed{\textbf{1.5}}$ Comparing assertions III (*2017, H1.8*)

Order the following assertions in a Hasse diagram[1] w.r.t. the relation $F \leq_{\mathcal{A}} G :\equiv F \implies G$. Put the strongest assertion on top, and the weakest assertion at the bottom.

false	true	$x > 3 \lor y = 0$	$z > 1$	$x > 3$
$x > 0$	$y = 8$	$y \geq 0 \lor x \neq 0$	$x \neq 0$	$x > 0 \land y = 8$

Answer of exercise $\boxed{\textbf{1.5}}$



$\boxed{\textbf{1.6}}$ Simplification (*2017, H3.5*)

Simplify the following statements as much as possible.

(a) $(x > 4 \implies z = 3) \land (z \geq 3 \land z \leq 3 \impliedby x = -3)$
$\land ((x > 12 \implies \text{false}) \implies z = 3)$
(b) $(y + 5x < 30 - 5(y - x) \lor (y \geq 5 \land 3(17x + 1) - 5y \geq 3 - 5y))$
$\land (y < 5 \lor 3(z + 2) \leq 51 \land 44 - 2z \leq 14)$
(c) $(x > y \implies (z = 2x - 3 \implies 0 = x - (1 + y)))$
$\lor (y > x - 2 \implies (2z + 6 = 4x \implies x = y + 1))$

Answer of exercise $\boxed{\textbf{1.6}}$

In the following simplifications, we will use the following equivalences, for which we omit the trivial proofs.

1. Lemma 1: $A \implies \text{false} \equiv \neg A$
2. Lemma 2: $\text{true} \implies A \equiv A$

[1] `https://en.wikipedia.org/wiki/Hasse_diagram`

3. Lemma 3: $(A \implies C) \land (B \implies C) \equiv (A \lor B) \implies C$
4. Lemma 4: $(A \implies C) \lor (B \implies C) \equiv (A \land B) \implies C$
5. Lemma 5: $A \implies (B \implies A) \equiv \text{true}$

(a)

$$\begin{aligned}
&(x > 4 \implies z = 3) \land (z \geq 3 \land z \leq 3 \impliedby x = -3) \\
&\quad \land ((x > 12 \implies \text{false}) \implies z = 3) \\
&\equiv (x > 4 \implies z = 3) \land (z = 3 \impliedby x = -3) \land (x \leq 12 \implies z = 3) && \text{(Lemma 1)} \\
&\equiv (x > 4 \lor x = -3 \lor x \leq 12 \implies z = 3) && \text{(Lemma 3)} \\
&\equiv z = 3 && \text{(Lemma 2)}
\end{aligned}$$

(b)

$$\begin{aligned}
&(y + 5x < 30 - 5(y - x) \lor (y \geq 5 \land 3(17x + 1) - 5y \geq 3 - 5y)) \\
&\quad \land (y < 5 \lor 3(z + 2) \leq 51 \land 44 - 2z \leq 14) \\
&\equiv (y < 5 \lor (y \geq 5 \land x \geq 0)) \land (y < 5 \lor (z \leq 15 \land 15 \leq z)) && \text{(Simplification)} \\
&\equiv (y < 5 \lor x \geq 0) \land (y < 5 \lor z = 15) && \text{(Distributivity)} \\
&\equiv y < 5 \lor (x \geq 0 \land z = 15) && \text{(Distributivity)}
\end{aligned}$$

(c)

$$\begin{aligned}
&(x > y \implies (z = 2x - 3 \implies 0 = x - (1 + y))) \\
&\quad \lor (y > x - 2 \implies (2z + 6 = 4x \implies x = y + 1)) \\
&\equiv (x > y \implies (z = 2x - 3 \implies x = y + 1)) \\
&\quad \lor (y + 2 > x \implies (z = 2x - 3 \implies x = y + 1)) \\
&\equiv x > y \land y + 2 > x \implies (z = 2x - 3 \implies x = y + 1) && \text{(Lemma 4)} \\
&\equiv x = y + 1 \implies (z = 2x - 3 \implies x = y + 1) \\
&\equiv \text{true} && \text{(Lemma 5)}
\end{aligned}$$

$\boxed{\textbf{1.7}}$ A realistic example (*2017, H3.6*)

For the following propositions A and B, show that $B \impliedby A$ holds.

(a) $B :\equiv (x = i! \land i \geq 0 \land z > x - 1) \lor (x = i! \land z \leq x + 1)$

$A :\equiv x = i! \land i \geq 3$

(b) $B :\equiv \left(x > y \land n = \dfrac{x^2}{y^2}\right) \lor (n = 1 \land x \leq 1) \lor (n < x \land n = 1)$

$A :\equiv \left(x \geq y \land n = \dfrac{x^2}{y^2}\right)$

(c) $B :\equiv y = 0 \lor \left(x > 0 \land n = -1 \land y = \displaystyle\sum_{k=-x}^{0} k\right) \lor \left(y = \displaystyle\sum_{k=1}^{x} nk \land n > 0\right)$

$A :\equiv y = \displaystyle\sum_{k=0}^{x} nk \land n \geq -1$

Hint: This task is a realistic example of simplifications that will occur during WP-calculations.

Answer of exercise $\boxed{\textbf{1.7}}$

(a)

$$\begin{aligned}
B &\equiv (x = i! \land i \geq 0 \land z > x - 1) \lor (x = i! \land z \leq x + 1) \\
&\impliedby (x = i! \land i \geq 3 \land z > x - 1) \lor (x = i! \land i \geq 3 \land z \leq x + 1) \\
&\equiv x = i! \land i \geq 3 \land (z > x - 1 \lor z \leq x + 1) \\
&\equiv x = i! \land i \geq 3 \equiv A
\end{aligned}$$

(b)

$$\begin{aligned}
B &\equiv \left(x > y \land n = \frac{x^2}{y^2}\right) \lor (n = 1 \land x \leq 1) \lor (n < x \land n = 1) \\
&\equiv \left(x > y \land n = \frac{x^2}{y^2}\right) \lor (n = 1 \land x \leq n) \lor (n < x \land n = 1) && (1 \rightsquigarrow n) \\
&\equiv \left(x > y \land n = \frac{x^2}{y^2}\right) \lor (n = 1 \land (x \leq n \lor n < x)) && \text{(Distributivity)} \\
&\equiv \left(x > y \land n = \frac{x^2}{y^2}\right) \lor n = 1 && \text{(Tautology)} \\
&\equiv \left(x > y \land n = \frac{x^2}{y^2}\right) \lor n = 1 \lor (n = 1 \land x = y) && \text{(Absorption)} \\
&\equiv \left(x > y \land n = \frac{x^2}{y^2}\right) \lor n = 1 \lor \left(n = \frac{x^2}{y^2} \land x = y\right) && \left(1 \rightsquigarrow \tfrac{x^2}{y^2}\right) \\
&\equiv \left((x > y \lor x = y) \land n = \frac{x^2}{y^2}\right) \lor n = 1 && \text{(Distributivity)} \\
&\equiv \left(x \geq y \land n = \frac{x^2}{y^2}\right) \lor n = 1 \\
&\Longleftarrow\ x \geq y \land n = \frac{x^2}{y^2} \equiv A
\end{aligned}$$

(c) This solution is very detailed, so it can easily be understood. Some transformations could be done at once, making the proof shorter. We will use two lemmas in the following:

- Lemma 1: For $x \geq 0$, we have $\sum_{k=-x}^{0} k = \sum_{k=0}^{x} -1k$.
- Lemma 2: $\sum_{k=1}^{x} k = \sum_{k=0}^{x} k$

The proof of Lemma 2 is trivial. Lemma 1 can be proven by induction on x. However, we will omit the proof here.

$$\begin{aligned}
B &\equiv y = 0 \lor \left(x > 0 \land n = -1 \land y = \sum_{k=-x}^{0} k\right) \lor \left(y = \sum_{k=1}^{x} nk \land n > 0\right) \\
&\equiv y = 0 \lor \left(x > 0 \land n = -1 \land y = \sum_{k=0}^{x} -1k\right) \lor \left(y = \sum_{k=0}^{x} nk \land n > 0\right) && \text{(Lemma 1 and 2)} \\
&\equiv y = 0 \lor \left(x > 0 \land n = -1 \land y = \sum_{k=0}^{x} nk\right) \lor \left(y = \sum_{k=0}^{x} nk \land n > 0\right) && (-1 \rightsquigarrow n) \\
&\Longleftarrow\ (y = 0 \land (n = 0 \lor x \leq 0)) \lor \left(x > 0 \land n = -1 \land y = \sum_{k=0}^{x} nk\right) \\
&\qquad \lor \left(y = \sum_{k=0}^{x} nk \land n > 0\right) \\
&\equiv (y = \sum_{k=0}^{x} nk \land (n = 0 \lor x \leq 0)) \lor \left(x > 0 \land n = -1 \land y = \sum_{k=0}^{x} nk\right) \\
&\qquad \lor \left(y = \sum_{k=0}^{x} nk \land n > 0\right) && (0 \rightsquigarrow \Sigma_{k=0}^{x} nk) \\
&\equiv y = \sum_{k=0}^{x} nk \land (n = 0 \lor x \leq 0 \lor (x > 0 \land n = -1) \lor n > 0) && \text{(Distributivity)}
\end{aligned}$$

$$\Longleftarrow\; y = \sum_{k=0}^{x} nk \wedge ((n = 0 \wedge x > 0) \vee x \leq 0 \vee (x > 0 \wedge n = -1) \vee (n > 0 \wedge x > 0))$$

$$\equiv y = \sum_{k=0}^{x} nk \wedge ((x > 0 \wedge (n = 0 \vee n = -1 \vee n > 0)) \vee x \leq 0) \qquad \text{(Distributivity)}$$

$$\equiv y = \sum_{k=0}^{x} nk \wedge ((x > 0 \wedge n \geq -1) \vee x \leq 0)$$

$$\Longleftarrow\; y = \sum_{k=0}^{x} nk \wedge ((x > 0 \wedge n \geq -1) \vee (x \leq 0 \wedge n \geq -1))$$

$$\equiv y = \sum_{k=0}^{x} nk \wedge (n \geq -1 \wedge (x > 0 \vee x \leq 0)) \qquad \text{(Distributivity)}$$

$$\equiv y = \sum_{k=0}^{x} nk \wedge n \geq -1 \equiv A \qquad \text{(Tautology)}$$

$\boxed{\textbf{1.8}}$ Proving quantified formulas

Reminder: in order to prove a statement $\forall x.P(x)$, one can proceed as follows:

- Fix an arbitrary a.
- For this fixed a, show that $P(a)$ is satisfied.

In order to prove $\exists x.P(x)$, it suffices to show $P(b)$ for some b.
Prove the following statements.

(a) $\forall x \in \mathbb{Z}.\exists y \in \mathbb{Z}.x < y$
(b) $\exists x \in \mathbb{Z}.\forall y \in \mathbb{Z}.y \neq x \vee y \neq 0$

Hint: Try to prove the statements as formal as possible.

Answer of exercise $\boxed{\textbf{1.8}}$

(a) Fix an arbitrary $x \in \mathbb{Z}$. To show: $\exists y \in \mathbb{Z}.x < y$. Proof: We have $x < x + 1$.
(b) We show: $\forall y \in \mathbb{Z}.y \neq 13 \vee y \neq 0$. Proof: Fix an arbitrary $y \in \mathbb{Z}$.
Case 1: $y = 13$**.** In this case, $y \neq 0$, and hence $y \neq 13 \vee y \neq 0$ is satisfied.
Case 2: $y \neq 13$**.** In this case, $y \neq 13 \vee y \neq 0$ is satisfied.
In any case, we have shown that $y \neq 13 \vee y \neq 0$ is satisfied.