β_{k+1} by deleting the first m letters of β_k and appending word α_i to the result if the first letter of β_k is a_i. The computation stops in k steps if the length of β_k is less than m or the first letter of β_k is a_{n+1}.

Example. The 2-tag-system T_1 is defined by

$$a_1 \to a_2a_1a_3, \quad a_2 \to a_1, \quad a_3 \to STOP.$$

On the initial word $\beta = a_2a_1a_1$, the computation of T_1 is

$$a_2a_1a_1 \to a_1a_1 \to a_2a_1a_3 \to a_3a_1.$$

Minsky [8] proved the existence of a universal 2-tag-system, and, therefore, we will deal only with 2-tag-systems, which also have the following properties:

1. The computation of a tag-system stops only on a word beginning with the halting symbol a_{n+1}.
2. The productions α_i, $i \in \{1,\ldots,n\}$, are not empty.

Henceforth, tag-systems will be 2-tag-systems.

A universal Turing machine U simulates a tag-system as follows. Let T be a tag-system on $A = \{a_1,\ldots,a_{n+1}\}$ with productions $a_i \to \alpha_i$. To each letter $a_i \in A$ is associated a positive number N_i and codes A_i and $\tilde{A}_i$ (may be $A_i = \tilde{A}_i$), of the form u^{N_i} ($= uu\ldots u$, N_i times), where u is a string of symbols of the machine U.

The codes A_i (or $\tilde{A}_i$) are separated by marks on the tape of U.

For $i \in \{1,\ldots,n\}$, the production $a_i \to \alpha_i = a_{i1}a_{i2}\ldots a_{im_i}$ of the tag-system T is coded by

$$P_i = A_{im_i}A_{im_i-1}\ldots A_{i2}A_{i1}.$$

The initial word $\beta = a_ra_sa_t\ldots a_w$, to be transformed by the tag-system T, is coded by

$$S = A_rA_sA_t\ldots A_w \quad (S = \tilde{A}_r\tilde{A}_s\tilde{A}_t\ldots\tilde{A}_w).$$

The initial tape of the UTM is:

$$Q_L\,\underbrace{P_{n+1}P_n\ldots P_1P_0}_{P}\,\underbrace{A_rA_sA_t\ldots A_w}_{S}\,Q_R,$$

where Q_L and Q_R are respectively infinite to the left and to the right parts of the tape of the UTM and consist only of blank symbols, P_{n+1} is the code of the halting symbol a_{n+1}, P_0 is the additional code consisting of several marks, and the head of the UTM is located on the left side of the code S in the state q_1 (in the case of the machine in UTM(2,18) the head of the UTM is located on the right side of the code P_0 in the state q_1).

Let T be an arbitrary tag-system, S_1 and S_2 be the codes of the words β_1 and β_2, respectively, and $\beta_1 \xrightarrow{T} \beta_2$. Then the UTM U transforms:

$$Q_LP_{n+1}P_n\ldots P_1P_0S_1Q_R \overset{U}{\Rightarrow} Q_LP_{n+1}P_n\ldots P_1P_0RS_2Q_R$$

(R corresponds to the cells which were bearing the codes of the deleted first two symbols).

The work of the UTM can be divided into three stages:

(i) On the first stage, the UTM searches the code P_r corresponding to the code A_r and then the UTM deletes the codes A_r and A_s (i.e. it deletes the mark between them).

(ii) On the second stage, the UTM writes the code P_r in Q_R of the tape in the reversed order.

(iii) On the third stage the UTM restores its own tape for a new cycle of modelling.

The number N_i corresponding to the symbol a_i ($i \in \{1,\ldots,n+1\}$) of the tag-system has the property that there are exactly N_r marks at each cycle of modelling between the code P_r and the code A_r (in the case of the machine in UTM(2,18) there are $N_r + 1$ marks, but the additional mark in P_0 is deleted immediately at the beginning of the first stage). On the first stage of modelling, the head of the UTM goes through a number of marks in the part P equal to the number of symbols u in the code A_r.

After the first stage the tape of the UTM is

$$Q_L P_{n+1} P_n \ldots P_{r+1} P_r P'_{r-1} \ldots P'_1 P'_0 R' A'_r A_s A_t \ldots A_w Q_R$$

and the head of the UTM locates the mark between A'_r and A_s. Then the UTM deletes this mark and the second stage of modelling begins.

After the second stage, the tape of the UTM is

$$Q_L P_{n+1} P_n \ldots P_{r+1} P''_r P''_{r-1} \ldots P''_1 P''_0 R'' A_t \ldots A_w A_{r1} A_{r2} \ldots A_{rm_r} Q_R,$$

and the head of the UTM is located on the left side of P''_r and the third stage of modelling begins.

After the third stage, the tape of the UTM is

$$Q_L P_{n+1} P_n \ldots P_1 P_0 R A_t \ldots A_w A_{r1} A_{r2} \ldots A_{rm_r} Q_R$$

and the head of the UTM is located on the right side of R.

Let $a_1, a_2, \ldots a_k, b_1, b_2, \ldots b_k$ be the symbols of the Turing machine. $a_1 a_2 \ldots a_k R b_1 b_2 \ldots b_k$ means that, when the head of the UTM moves to the right the group of symbols $a_1 a_2 \ldots a_k$ is changed to the group $b_1 b_2 \ldots b_k$. It is analogous, when the head of the UTM moves to the left (R is changed to L).

$R a_1 a_2 \ldots a_k (b_1 b_2 \ldots b_k) L$ means that the group of symbols $a_1 a_2 \ldots a_k$ makes the direction of the motion of the head of the UTM change from the right to the left and changes itself to $b_1 b_2 \ldots b_k$. It is analogous, when the head of UTM moves to the left (R is changed to L).

4. The UTM with 24 states and 2 symbols

The symbols of the machine in UTM(24,2) (see [13]) are 0 (blank symbol) and 1; and the states are q_i ($i = 1,\ldots,24$).

$$N_1 = 1, \quad N_{k+1} = N_k + m_k + 2 \quad (k \in \{1,\ldots,n\}).$$

(iii) On the third stage of modelling:

$$\begin{array}{ll} dRb & (q_5 dbRq_5), \\ 1R1 & (q_5 11Rq_5). \end{array}$$

When the head of the UTM moves to the right and meets the mark c, both the third stage and the whole cycle of modelling are over. The UTM deletes the mark c and a new cycle of modelling begins:

$$cR1 \quad (q_5 c1Rq_1).$$

8. The UTM with 4 states and 6 symbols

The machine in UTM(4,6) (see [13]) simulates the following class of tag-systems:

$$(\mathcal{T}_2) \quad \begin{cases} a_i \to \alpha_i, \quad i \in \{1,\ldots,n\} \\ a_n \to a_n a_n \\ a_{n+1} \to STOP \end{cases}$$

where $\alpha_i = a_n a_n \beta_i$ and β_i are not empty.

We show the universality of tag-systems of type $\mathcal{T}_2$ in Lemma 8.1.

Lemma 8.1. For every tag-system T of type $\mathcal{T}$, there is the tag-system T' of type $\mathcal{T}_2$ which models T.

Proof. Can be provided in the same manner as that of Lemma 7.1. □

The symbols of the machine in UTM(4,6) are 0 (blank symbol), 1, b, $\overleftarrow{b}$, $\vec{b}$ and c; the states are q_i $(i = 1,\ldots,4)$.

$$N_1 = 1, \quad N_{k+1} = N_k + 2m_k \quad (k \in \{1,\ldots,n\}).$$

We note that $m_n = 2$. The code of the production $\alpha_i = a_n a_n a_{i3} \ldots a_{im_i}$ $(i \in \{1,\ldots,n\})$ of the tag-system is

$$P_i = b1b1^{N_{im_i}} bb1^{N_{im_i-1}} \ldots bb1^{N_{i3}} bb1^{N_n} bb1^{N_n - N_i}.$$

In particular,

$$P_n = b1b1^{N_n}bb, \quad P_0 = b, \quad P_{n+1} = \overleftarrow{b}b,$$

where $A_j = 1^{N_j}$, $j \in \{1,\ldots,n+1\}$ and the symbol b is a mark.

The code S of the initial word $\beta = a_r a_s a_t \ldots a_w$, to be transformed by the tag-system, is

$$S = 1^{N_r} c 1^{N_s} c 1^{N_t} \ldots c 1^{N_w}$$

and the symbol c is a mark.

The program of the machine in UTM(4,6):

$q_1\ 1\overleftarrow{b}\ Lq_1$	$q_2\ 10\ \ Rq_2$	$q_3\ 11\ \ Rq_3$	$q_4\ 10\ \ Rq_4$
$q_1\ b\vec{b}\ Rq_1$	$q_2\ b\vec{b}\ \ Lq_3$	$q_3\ b\overleftarrow{b}\ Rq_4$	$q_4\ bc\ \ Lq_2$
$q_1\ \vec{b}b\ Lq_1$	$q_2\ \vec{b}\overleftarrow{b}\ Rq_2$	$q_3\ \vec{b}b\ Rq_3$	$q_4\ \vec{b}\overleftarrow{b}\ Rq_4$
$q_1\ \overleftarrow{b}0\ Rq_1$	$q_2\ \overleftarrow{b}\vec{b}\ Lq_2$	$q_3\ \overleftarrow{b}\ \ —$	$q_4\ \overleftarrow{b}\ \ —$
$q_1\ 0\overleftarrow{b}\ Lq_1$	$q_2\ 01\ \ Lq_2$	$q_3\ 0c\ \ Rq_1$	$q_4\ 0c\ \ Lq_2$
$q_1\ c0\ \ Rq_4$	$q_2\ cb\ \ Rq_2$	$q_3\ c1\ \ Rq_1$	$q_4\ cb\ \ Rq_4$

(i) On the first stage of modelling:

$1\ L\ \overleftarrow{b}$ $\quad(q_1\ 1\overleftarrow{b}\ Lq_1)$,
$0\ L\ \overleftarrow{b}$ $\quad(q_1\ 0\overleftarrow{b}\ Lq_1)$,
$\vec{b}\ L\ b$ $\quad(q_1\ \vec{b}b\ Lq_1)$,
$L\ b\ (\vec{b})R$ $\quad(q_1\ b\vec{b}\ Rq_1)$,
$\overleftarrow{b}\ R\ 0$ $\quad(q_1\ \overleftarrow{b}0\ Rq_1)$,
$b\ R\ \vec{b}$ $\quad(q_1\ b\vec{b}\ Rq_1)$,
$R\ 1\ (\overleftarrow{b})L$ $\quad(q_1\ 1\overleftarrow{b}\ Lq_1)$.

If the head of the UTM moves to the right and meets the mark c, the first stage of modelling is finished. At that time the tape of the UTM is

$$Q_{\mathrm{L}}P_{n+1}P_n\dots P_{r+1}P_rP'_{r-1}\dots P'_1P'_0R'A'_rA_sA_t\dots A_wQ_{\mathrm{R}}$$

and the head of the UTM locates the symbol c between the codes of A'_r and A_s ($A'_j = 0^{N_j},\ j \in \{1,\dots,n+1\}$). $P'_k(k \in \{0,\dots,r-1\})$ differs from P_k by the fact that the marks b are changed to the marks $\vec{b}$ and A_j to A'_j ($j \in \{1,\dots,n+1\}$):

$$P'_k = \vec{b}0\vec{b}0^{N_{km_k}}\vec{b}\vec{b}0^{N_{km_k-1}}\dots\vec{b}\vec{b}0^{N_{k3}}\vec{b}\vec{b}0^{N_n}\vec{b}\vec{b}0^{N_n-N_k}.$$

After that the UTM deletes the mark c (q_1c0Rq_4) and the second stage of modelling begins (first of all, the UTM writes the mark c in the part Q_{R}).

(ii) On the second stage of modelling:

$0\ L1$ $\quad(q_2\ 01\ Lq_2)$,
$\overleftarrow{b}\ L\vec{b}$ $\quad(q_2\ \overleftarrow{b}\vec{b}\ Lq_2)$,
$0\ bL\ 1c$ $\quad(q_2\ b\vec{b}\ Lq_3,\ \ q_3\ 0c\ Rq_1,\ q_1\ \vec{b}b\ Lq_1,\ q_1\ c0\ Rq_4,\ q_4\ bc\ Lq_2,\ q_2\ 01\ Lq_2)$.

$L\ \vec{b}(\overleftarrow{b})R$ $\ (q_2\ \vec{b}\overleftarrow{b}\ Rq_2)$ and the UTM writes symbol 1 in Q_{R}.

As after the first stage of modelling there are exactly N_r marks $\vec{b}$ between the codes P_r and S, the UTM will write exactly N_r symbols 1 in Q_r. After that, when the head is located on the code P_r, the UTM will write $N_n - N_r$ more symbols 1 in Q_r. As a result there will be the code A_n.

$L1(0)R$ $(q_2 10 Rq_2)$ and the UTM writes the symbol 1 in Q_R. In this case

$$\begin{array}{ll}
1\ R\ 0 & (q_2\ 10\ Rq_2),\\
\vec{b}\ R\ \overleftarrow{b} & (q_2\ \vec{b}\overleftarrow{b}\ Rq_2),\\
c\ R\ b & (q_2\ cb\ Rq_2),\\
R\ 0\ (1)L & (q_2\ 01\ Lq_2).
\end{array}$$

$Lbb(\overleftarrow{b}\overleftarrow{b})R$ $(q_2\ b\ \vec{b}Lq_3,\ q_3\ b\ \overleftarrow{b}Rq_4,\ q_4\vec{b}\overleftarrow{b}Rq_4)$ and the UTM writes the mark c in Q_R. In this case

$$\begin{array}{ll}
1\ R\ 0 & (q_4\ 10\ Rq_4),\\
\vec{b}\ R\ \overleftarrow{b} & (q_4\ \vec{b}\overleftarrow{b}\ Rq_4),\\
c\ R\ b & (q_4\ cb\ Rq_4),\\
R\ 0\ (c)L & (q_4\ 0c\ Lq_2).
\end{array}$$

If the head of the UTM moves to the left and meets the pair $\overleftarrow{b}b$, then the UTM halts $(q_2\ b\ \vec{b}Lq_3,\ q_3\overleftarrow{b}-)$.

If the head of the UTM moves to the left and meets the pair $1b$, the second stage of modelling is over. At that time the tape of the UTM is

$$Q_L P_{n+1} P_n \ldots P_{r+1} P''_r P''_{r-1} \ldots P''_1 P''_0 R'' A_t \ldots A_w A_n A_n A_{r3} A_{r4} \ldots A_{rm_r} Q_R$$

and the head is located on the second left symbol of the code P''_r. P''_k $(k \in \{0, \ldots r\})$ differs from P_k by the fact that the marks b are changed to the marks $\vec{b}$.

Then $L1b(1b)R$ $(q_2\ b\vec{b}\ Lq_3,\ q_3\ 11\ Rq_3,\ q_3\ \vec{b}b\ Rq_3)$ and the UTM goes to the third stage of modelling.

(iii) On the third stage of modelling the UTM restores the tape in P:

$$\begin{array}{ll}
\vec{b}\ R\ b & (q_3\ \vec{b}b\ Rq_3),\\
1\ R\ 1 & (q_3\ 11\ Rq_3).
\end{array}$$

When the head of the UTM moves to the right and meets the mark c, the third stage and the whole cycle of modelling are over. The UTM deletes the mark c and a new cycle of modelling $(q_3 c 1 Rq_1)$ begins.

9. The UTM with 3 states and 10 symbols

The symbols of the machine in UTM(3,10) (see [15]) are 0 (blank symbol), $1, \overleftarrow{1}, \vec{1}, b$, $\overleftarrow{b}$, $\vec{b}, c$, $\overleftarrow{c}$, $\vec{c}$; and the states are q_1, q_2, q_3.

$$N_1 = 1, \quad N_{k+1} = N_k + 2m_k + 2 \quad (k \in \{1, \ldots, n\}).$$