CS682A Project: Quantum Bayesian Coherence

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Abstract

In this project report, I would describe this paper [3] on Quantum Bayesian Coherence. The main aim of this report is to show how we can interpret Born rule as an addition to the normal principles of Dutch Book Coherence and its implications.

1 Introduction

Quantum Bayesianism (QBism) [5] provides an interpretation to quantum mechanics where it rejects the notion of the quantum state corresponding to some physical reality. It is an attempt to interpret quantum mechanics like probability theory, with the quantum states represented as bayesian probabilities and the quantum mechanics provides rules to relate these probabilities for different events. In QBism, the quantum state corresponds to an agents degree of belief. So there can be as many quantum states as many agents, hence quantum states are not an objective property of nature but rather they are a subjective property of the agent.

To understand why one needs such an alternative interpretation, consider the standard interpretation of quantum mechanics where the quantum corresponds to some physical reality. Note that principle of locality is an important part of the classical physics, which states that an object is affected by only things close in its vicinity or an object cannot be influenced by far away surroundings. However, with phenomenons like quantum entanglement, we have state changes between two far away separated objects. Hence, if quantum states represent some physical reality, then our nature would have to be non local or not obey the principle of locality. This motivates perspectives like QBism where we do not view quantum states as describing some real properties of the nature. To understand rigorously how QBism can help solve important problems, refer to [2], [4]. Also, please find a recent interesting paper [6] that provides evidence for how the quantum measurements are subjective. This can again provide some motivation to consider the perspective of QBism, where even the measurements (along with quantum states) are supposed to be subjective and different for different agents.

Please note that I have not given rigorous arguments for explaining the importance behind QBism. Also, I have not gone through the references mention above [2], [4], [6] in detail. I have provided some loose arguments to help motivate the idea of QBism. The major concern of my report would not be understanding why QBism but it would be about understanding the role and the importance of Born Rule in it. The aim of my project report is to describe how we can view Born rule as an addition to the normal rules of probability.

The outline of the project report is as follows:

- The section 2 of the report would show what it means for a quantum state to be a degree of belief. We would show how a quantum state can be mapped to a probability distribution and highlight the importance of SIC measurements in enabling us arrive at a simpler representation.
- The section 3 of the report contains the major argument. We first arrive at a representation of Born rule in terms of SIC and then show it can be seen as an addition to Dutch Book Coherence. The subsection 3.1 provides some background about personalist bayesian probability and Dutch Book Coherence. The subsection 3.2 contains the argument to interpret Born rule as addition to rules of probability.
- The section 4 of the report provides some justification for assuming the Born rule in terms of SIC as the basic axiom of quantum mechanics. It attempts to derive properties of the quantum state space we created using SIC in section 2 by starting only with our Born rule interpretation of section 3.

2 Expressing Quantum State Space in terms of SIC

Let H_d denote a Hilbert space of dimension d and we represent a quantum state of the system using the density matrix representation.

Hence, ρ is represented by a density matrix or equivalently $\rho \in L(H_d)$ is P.S.D linear operator and $Tr(\rho) = 1$. I would not explain the density matrix representation of a quantum state and properties associated with it, a reference to learn about them [1]

Next, we define a minimal informationally complete POVM (IC-POVM) over the Hilbert space H_d as follows:

IC-POVM: A collection of P.S.D. linear operators over $L(H_d)$ as $E = \{E_i | \forall i = 1 : d^2\}$ such that:

1. $\sum_{i=1}^{d^2} E_i = I$

2. E_i's are linearly independent.

Note: We have d^2 linearly independent linear operators E_i's over vector space $L(H_d)$, hence the set $\{E_i\}$ also forms a **basis** for the vector space $L(H_d)$. This is because the dimension of space $L(H_d)$ is d^2, hence any d^2 linearly independent vectors of this space would form a basis for it.

Now, using Born Rule to compute the probabilities for the outcomes of above defined IC-POVM measurement:

$p(i) = Tr(\rho E_i)$ [Proof in [1]]

Since, E_i's span the space $L(H_d)$, we can write the quantum state ρ as follows: $\rho = \sum_{j=1}^{d^2} \alpha_j E_j$

Hence, $p(i) = \sum_j \alpha_j Tr(E_j E_i)$

Consider a matrix M s.t $M_{i,j} = Tr(E_j E_i)$, then the equation simplifies to $|p\rangle = M|\alpha\rangle$

If M is invertible, we would have

$$|p\rangle = M|\alpha\rangle \implies |\alpha\rangle = M^{-1}|p\rangle \tag{1}$$

Hence, we are able to obtain an injective mapping from a quantum state to a probability distribution. However, note that this mapping is not surjective, we cannot have every probability distribution giving us a valid quantum state ($Tr(\rho) = 1$, ρ is P.S.D.). To state more clearly, the probability distributions $|p\rangle$ belong to the probability simplex Δ_{d^2} [stating without justification, I am not defining the probability simplex formally] and there is an injective mapping from the convex set of quantum states to the probability distribution in the simplex.

2.1 Obtaining a simple expression for M using SIC

We wish to obtain a simple expression for M, preferrably make it close to identity or a diagonal matrix. This would help us in computing the inverse and obtain a simpler mapping between quantum states and probability distributions. For this task, we minimise the Forbenius Distance between I and M: $F = \sum_{i,j}(\delta_{i,j} - M_{i,j})^2$. Frobenius distance provides us with an estimate on how much M deviates from identity matrix, hence minimising it would help us to obtain a simple expression for M.

$$\begin{aligned} F &= \sum_{i,j}(\delta_{i,j} - M_{i,j})^2 \\ &= \sum_{i=1}^{d^2}(\delta_{i,i} - M_{i,i})^2) + \sum_{i \neq j}(\delta_{i,j} - M_{i,j})^2) \\ &= \sum_{i=1}^{d^2}(1 - Tr(E_i^2))^2) + \sum_{i \neq j}(Tr(E_i E_j))^2 \end{aligned} \tag{2}$$

Now, using Cauchy Schwarz Inequality: For a set of n numbers λ_r, we have $\sum_r \lambda_r^2 \geq \frac{1}{n}(\sum_r \lambda_r)^2$, with equality if all λ_r are equal.

Hence, from the application for Cauchy Schwarz Inequality on $\sum_{i=1}^{d^2}(1 - Tr(E_i^2))^2)$ and $\sum_{i \neq j}(Tr(E_i E_j))^2$ in (1):

$$F \geq \frac{1}{d^2}(\sum_{i=1}^{d^2}(1 - Tr(E_i^2)))^2 + \frac{1}{d^4 - d^2}(\sum_{i \neq j} Tr(E_i E_j))^2$$

Now, consider the case of equality in Cauchy Schwartz, we obtain that $1 - Tr(E_i^2)$ should be constant for all i and $Tr(E_i E_j)$ should be constant for all $i \neq j$. This implies

$$Tr(E_i^2) = m \quad \forall i \tag{3}$$

$$Tr(E_i E_j) = n \quad \forall i \neq j \tag{4}$$

Now, since E_i form POVM, hence $I = \sum_{i=1}^{d^2} E_i$ This implies $I^2 = \sum_{i,j} E_i E_j$
Taking trace on both sides and using the equality case results from (3), (4) from Cauchy Schwartz,

$$\begin{aligned} Tr(I^2) = d = \sum_{i,j} Tr(E_i E_j) &= \sum_i Tr(E_i^2) + \sum_i Tr(E_i E_j) \\ &= m * d^2 + n(d^4 - d^2) \end{aligned}$$

This implies:

$$m + (d^2 - 1)n = \frac{1}{d} \tag{5}$$

Consider the trace of E_k

$$\begin{aligned} Tr(E_k) &= Tr(IE_k) \\ &= Tr(\sum_i E_i E_k) \\ &= \sum_{i \neq k} Tr(E_i E_k) + Tr(E_k E_k) \\ &= m + (d^2 - 1) \end{aligned}$$

Using (5),

$$Tr(E_k) = \frac{1}{d} \tag{6}$$

With the relations (3),(4), the Forbenius distance equals its lower bound, hence the task of minimising the Frobenius distance is now same as minimising the lower bound on F obtained using Cauchy Schwartz. Hence, to minimise the Frobenius Distance F, we need to maximise m as then the lower bound given in the equation would be minimised. We would use the following claim to find obtain m for which the lower bound is minimised. The proof for the claim can be found in Appendix 6.1.

Claim: If A is positive definite matrix with Tr(A)=1, then $Tr(A^2) \leq Tr(A)$ and the equality holds iff largest eigenvalue of A is 1

Now, observe using eq. (6) $Tr(dE_k) = d * \frac{1}{d} = 1$ and $Tr((dE_k)^2) = md^2$ Note that maximising m is the same as maximising md^2. Hence, by the above claim $Tr((dE_k)^2) = md^2$ is maximised when largest eigen value of dE_k is 1 and the max value is given as $Tr(dE_k^2) = Tr(dE_k) = 1$

Since, $Tr(dE_k)$= Sum of Eigenvalues = 1 and all eigenvalues are positive with the largest eigenvalue 1, this implies dE_k has one eigenvalue 1 and the rest of the eigenvalues are zero. These conditions can certainly be fulfilled if dE_k is rank 1 projection matrix.

Hence, with Π_k denoting a rank-1 projection matrix

$$E_k = \frac{1}{d}\Pi_k = \frac{1}{d} |\phi_k\rangle \langle\phi_k| \tag{7}$$

This would lead to minimising the lower bound of Frobenius distance F. Using the equation 3, 4 we can derive further information on these projectors:

$$Tr(\Pi_i \Pi_j) = \frac{d\delta_{ij} + 1}{d + 1} \tag{8}$$

To formally state the result:

Lower bound on the Forbenius Distance between I and M would be minimised iff $E_i = \frac{1}{d}\Pi_i$ and $Tr(\Pi_i \Pi_j) = \frac{d\delta_{ij}+1}{d+1}$

Notice the importance of the claim we used above, that leads us to say that out result is both necessary and sufficient, as the minimising the lower bound is equivalent to maximising Tr($(dE_i)^2$).

Also, note that under these measurement operators, the trace of the product of the operators becomes symmetric i.e. $Tr(E_i E_j) = Tr(E_j E_i)$, hence they are also known as SIC or symmetric informationally complete POVM.

Now, we come to the main part where we show why SIC measurements help. Observe that with SIC the matrix M becomes a circulant matrix and invertible [Refer Appendix 6.2 to get idea on inverse of a circulant matrix]. Now, simplifying the expression in (1), we obtain:

$$\rho = \sum_i ((d + 1)p(i) - \frac{1}{d})\Pi_i \tag{9}$$

Hence, for SIC measurements we get the above mapping from quantum states to probability distributions.

2.2 Quantum State Space

In the equation (9), we cannot have every probability distribution giving rise to a valid quantum state. We wish to characterize which probability distributions would be valid or find equations that show the quantum state space in terms of probability distributions. Since the set of quantum states is convex with extreme points of the convex set as pure quantum states. [Proof in [1]] Hence, if we can characterize the valid probability distributions for the pure quantum states, we would know the distributions for every other quantum states. This is because every other quantum state would be a convex combination of the pure quantum states, hence knowing the probability distributions for pure quantum states allows us to understand the complete quantum state space.

To derive the valid probability distributions for the pure quantum states, we first describe the structure coefficients as they would be useful in derivation.

Since, $\{E_i\}$ form a basis for $L(H_d)$, this implies that $\{\Pi_i\}$ also form a basis for the same space. Now, $\Pi_i\Pi_i \in L(H_d)$, hence we would have the following:

$$\Pi_i\Pi_j = \sum_{k=0}^{d^2} \alpha_{ijk}\Pi_k \tag{10}$$

Here α_{ijk} is called the **structure coefficient**. Some useful properties for structure coefficients:

1. Take trace on both sides of equation (10) and using equation (8):

$$\sum_{k=0}^{d^2} \alpha_{ijk} = \frac{d\delta_{ij}+1}{d+1} \tag{11}$$

2. Proof can be found in Appendix 6.5

$$\alpha_{ijk} = \frac{1}{d}((d+1)Tr(\Pi_i\Pi_j\Pi_k) - \frac{d\delta_{ij}+1}{d+1}) \tag{12}$$

3.Proof can be found in Appendix 6.6

$$\sum_{i=1}^{d^2} \alpha_{ijk} = d\delta_{jk} = \sum_{j=1}^{d^2} \alpha_{ijk} = d\delta_{ik} \tag{13}$$

Now, we would derive the probability distributions for pure quantum states.

Now, using equation 9:

$$\begin{aligned}
\rho^2 &= \sum_{i,j}((d+1)^2 p(i)p(j) - \frac{d+1}{d}p(i) - \frac{d+1}{d}p(j) + \frac{1}{d^2})\Pi_i\Pi_j \\
&= \sum_{i,j}((d+1)^2 p(i)p(j) - \frac{d+1}{d}p(i) - \frac{d+1}{d}p(j) + \frac{1}{d^2})\sum_k \alpha_{ijk}\Pi_k \quad \text{(Using 10)} \\
&= \sum_{i,j,k}((d+1)^2 \alpha_{ijk}p(i)p(j) - \frac{d+1}{d}\alpha_{ijk}p(i) - \frac{d+1}{d}\alpha_{ijk}p(j) + \frac{1}{d^2}\alpha_{ijk})\Pi_k
\end{aligned} \tag{14}$$

Observe, using equation 13

$$\frac{1}{d^2}\sum_{i,j}\alpha_{ijk} = \frac{1}{d^2}\sum_i d * \delta_{ik} = \frac{1}{d}$$

$$\frac{d+1}{d}\sum_{i,j}p(i)\alpha_{ijk} = \frac{d+1}{d}\sum_i p(i)d * \delta_{ik} = (d+1)p(k)$$

Similarly,

$$\frac{d+1}{d}\sum_{i,j}p(j)\alpha_{ijk} = (d+1)p(k)$$

Substituting these values in the equation 14,

$$\rho^2 = \sum_k((\sum_{i,j}(d+1)^2\alpha_{ijk}p(i)p(j)) - 2(d+1)p(k) + \frac{1}{d})\Pi k$$

For a pure quantum state, $\rho^2 = \rho$, comparing the above equation with $\rho = \sum_k((d+1)p(k) - \frac{1}{d})\Pi_k$ gives:

$$\begin{aligned}
p(k) &= (\sum_{i,j}(d+1)\alpha_{ijk}p(i)p(j)) - 2p(k) + \frac{2}{d(d+1)} \\
&= \frac{d+1}{3}\sum_{i,j}\alpha_{ijk}p(i)p(j) + \frac{2}{3d(d+1)}
\end{aligned}$$

Hence, for a pure quantum state the valid probability distribution are those that satisfy the following equation:

$$p(k) = \frac{d+1}{3} \sum_{i,j} \alpha_{ijk} p(i)p(j) + \frac{2}{3d(d+1)} \tag{15}$$

The above d^2 quadratic equations can be reduced into two simpler equations. The proof can be found in Appendix 6.7

$$\sum_i p(i)^2 = \frac{2}{d(d+1)} \tag{16}$$

$$\sum_{i,j,k} \alpha_{ijk} p(i)p(j)p(k) = \frac{4}{d(d+1)^2} \tag{17}$$

The equations 16, 17 define the quantum state space representation in terms of probability distribution. Hence, using SIC measurements we can restrict the probability distributions for pure quantum states to satisfy 16, 17. Hence, now we know which probability distribution would provide us with valid quantum states via equation 9.

3 Born Rule as addition to Coherence

Consider the quantum state ρ and POVM $\{Gj\}$. We express the Born rule now in terms of SIC i.e. we use the equation (9) along with the Born Rule to compute the probabilities of the outcome of the POVM $\{Gj\}$ measurement:

$$\begin{aligned}
q(j) &= Tr(\rho G_j) \\
&= Tr(\sum_{i=1}^{d^2} ((d+1)p(i) - \frac{1}{d})\Pi_i G_j) \\
&= (d+1)\sum_{i=1}^{d^2} p(i) Tr(\Pi_i G_j) - \frac{1}{d}\sum_{i=1}^{d^2} Tr(\Pi_i G_j) \\
&= (d+1)\sum_{i=1}^{d^2} p(i) r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i)
\end{aligned}$$

where

$$r(j|i) = Tr(\Pi_i G_j) \tag{18}$$

Hence, the Born rule expressed in the terms of SIC becomes

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \tag{19}$$

We would now briefly discuss about the personalist bayesian probability and the Dutch Book Coherence. This is important as it forms the major argument of the paper, how we can interpret the Born rule as as addition to the Dutch Book Coherence by expressing it in terms of SIC. The section 3.1 talks about Dutch Book Coherence and then the section 3.2 describes how to view the equation 19 as an addition to Coherence.

3.1 Personalist Bayesian Probability & Dutch Book Coherence

The probability p(A) for an event A represents the degree of belief of the agent. Consider a lottery ticket as **Worth $1 if A**, if the agent is willing to either buy this bet or sell this bet for an amount p(A), this assigns meaning to p(A) or the probability of an event A as per the personalist Bayesian perspective. Buying the bet means that the agent pays amount $p(A) and then if the event A occurs, it gets backs the amount $1, else it receives nothing. Similarly, selling the bet means that the agent has convinced some other agent to buy this bet. Hence, it receives the amount $p(A) at the start before the bet and then if A occurs it has to pay the amount $1 to the other agent, else it does not pay anything.

Hence, the definition of p(A) as per the personalist Bayesian perspective:

If an agent is ready to buy or sell the bet: [Worth $1 if A] for a total amount of $p(A)

This assigns subjectivity to p(A), different agents can play the same bet for different amounts depending on their beliefs about the occurance of event A. However, the agent is assumed to be rational and it is expected to assign probabilities(amount at which the bet is bought or sold) to different events in such a manner that does not lead to sure loss situations. Hence, we can derive certain principles that relate probability assignment between various events and help the agent avoid sure loss situations. These principles form the **Dutch Book Coherence** and we can also show that the basic axioms of probability are among these principles.

Justification for $0 \leq p(A) \leq 1$: Consider the case of $p(A) > 1$ and the agent buys the bet [Worth \$1 if A] for a amount of \$p(A). In this case, even if the event A occurs, it is at a loss since it gets \$1 while it paid more than that to play the bet. Hence, it is a sure loss situation it. Similarly, for the case of $p(A < 0)$ and the agent sells the same bet. It is again a sure loss situation, as the total finance of the agent is \$p(A)-1 or \$p(A) (depending on whether A occurs or not), which is negative in both the cases.

Justification for $p(A \cup B) = p(A) + p(B)$ *if A,B are mutually exclusive*:

Consider the case of $p(A \cup B) > p(A) + p(B)$, the agent plays the following bets:

1. It buys the bet of [Worth \$1 if A] for the amount $p(A \cup B)$

2. It sells the bet of [Worth \$1 if A] for \$p(A)

3. It sells the bet of [Worth \$1 if B] for \$p(B)

The initial balance of the agent is: \$p(A) + \$p(B) - \$$p(A \cup B) < 0$ We can calculate the final balance in each of the three cases and it would turn out to be negative. Hence, the agent is at a sure loss with this configuration. A similar argument can be used to rule out the possibility of $p(A \cup B) < p(A) + p(B)$, consider the same bets as used in the previous case but the agent now sells the first bet on A and it buys the other two bets. Hence, to avoid sure loss situations for the case of two mutually exclusive events, the agent must assign $p(A \cup B) = p(A) + p(B)$

3.2 Core Argument

Notice the equation 19 representing Born rule in terms of SIC. Here, $p(i)$ denotes the probability of outcome i of the SIC measurement and $q(j)$ denotes the probability of outcome j of the POVM measurement. The term $\sum_i p(i)r(j|i)$ is interesting as if you think of $r(j|i)$ as the conditional probability of obtaining j in POVM measurement given we obtained i in SIC measurement, then $\sum_i p(i)r(j|i)$ is the total probability of obtaining j in POVM measurement if we had performed a SIC measurement before it.

Hence, if before POVM measurement we perform an SIC measurement, then the total probability of the outcome j for POVM measurement is:

$$Prob(j) = P(j) = \sum_i p(i)r(j|i)$$

Now, the equation 19 can essentially be seen as:

$$q(j) = Func(P(j), \sum_i r(j|i)) \tag{20}$$

Let us now think about a classical scenario where an event B is conditioned on an event A. Hence, we would have

$$p(B) = \sum_A p(A)p(B|A)$$

Now, imagine it is revealed that B is no longer conditioned on A. Let us denote the probability of B after this revelation by q(B). Now, there is no rule in classical probability theory that suggests that q(B) and p(B) must be related. After its revealed that B is no longer conditioned on A, classically it makes sense that q(B) is completely independent of p(B). The same argument can be constructed in the Dutch Book Coherence sense that an agent first plays a conditional lottery/bet on the event B and it is revealed to the agent that B is no longer conditioned on A. After the revelation the agent would play the bet at price \$$q(B)$ and there is no rule in Dutch Book Coherence that would suggest the agent to incorporate some information from \$$p(B)$ in \$$q(B)$.

However, if you look at equation 20, there actually exists a rule that relates $q(j)$ to $p(j)$! Lets interprets performing POVM $\{Gj\}$ measurement on quantum state ρ as follows:

1. First we perform an SIC measurement $\{\Pi_i\}$ and the POVM measurement is conditioned on the SIC measurement.

2. Then it is revealed that POVM measurement is no longer conditioned on the SIC measurement.

Hence, equation 20 actually gives a relation to relate Prob(j) in step 2 to the Prob(j) in step 1. Such a relation does not exist in the classical case but Born rule expressed in terms of SIC (Eq 19) provides us with a relation to relate them. Hence, Born rule can be interpreted as an addition to the normal principles in Dutch Book Coherence, so in a way quantum mechanics provides us with an additional rule to normal rules of probability.

This suggests that we should not view Born rule as something that assigns probability to the outcomes of the measurement of an objective quantum state. Rather there is no objective quantum state, quantum state is a degree of belief of the agent given by equation (9) and the Born Rule (19) provides the agent with a rule that guides it to form beliefs on the related events. Just like the Bayes rule, two agents can have different priors or subjective beliefs, but we do not want the agents to update their beliefs randomly after observing the data. Hence, the Bayes rule provides a relation so that the two priors don't diverge to very different posteriors. Similarly, the Born Rule (19) provides a relation so that two agents with different beliefs(different quantum states ρ) do not update their beliefs very differently after the POVM $\{G_j\}$ measurement is performed.

In the following content of the report, I would denote the SIC measurement as the counterfactual measurement and the POVM measurement as the real measurement. So, its like the quantum agent first hypothesis a counterfactual (SIC) measurement before observing the real (POVM) measurement and the Born rule (19) provides the relation between these two measurements.

4 Deriving Quantum State Space

In this section, we would explore if we assume the born role equation 19 as the basic postulate of the quantum mechanics and forget about the SIC measurements, can we recover some parts of the quantum state space which we derived in the section 2 (16), (17). Essentially, can we construct a probability simplex starting with the born rule which is similar to the probability simplex specified by 16, 17

Hence, now we do not have any SIC measurements and only equation 19 is true. So, before performing a real measurement, we perform a counterfactual measurement and the outcome probability are given by equation. Since, outcome probabilities for real measurements must be valid measurements, we must have:

$$0 \leq (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \leq 1 \tag{21}$$

Represent the the probabilities for counter factual measurements $\{p(i)|i = 1 : d^2\}$ by vector $|p\rangle$ and represent the conditional probabilities of real measurements on counterfactuals by a matrix R i.e. $R_{i,j} = r(j|i)$

Now, we define two sets P and Q where P is collection of the vectors $|p\rangle$ and Q is a collection of the matrices R. The set P can be interpreted as the State Space with the elements $|p\rangle \in P$ as the states. Hence, our goal now is to show if P can be shown as isomorphic to the quantum state space derived in section 2, specified by 16, 17.

We further impose restrictions of Consistency and Maximality on these sets as follows:

- Consistency: For all $|p\rangle \in P$ and $R \in Q$, the inequality 21 is satisfied
- Maximality: P and Q is maximal among all the consistent sets i.e. for all $P^{'}$ and $Q^{'}$ satisfying consistency, then $P^{'} \supseteq P$ and $Q^{'} \supseteq Q$ implies that $P^{'} = P$ and $Q^{'} = Q$

The importance of making P and Q as maximally consistent sets is that it implies both the sets P and Q must be convex [Proof in Appendix 6.3] . This is useful because the quantum state space representation in terms of probabilities in section 2 was also convex, hence starting with a convex set P captures some similarity with state space.

Hence, for the rest of the section 5 we would assume that the sets P and Q are maximally consistent. Now, we try find some relations between the sets P and Q.

Going in the reverse direction, estimating the probability of the counterfactual measurement after observing the outcome of the real measurement: $Prob(i|j) = Prob(j|i) * Prob(i)/Prob(j)$

Now, $Prob(j|i) = r(j|i)$ and $Prob(j) = q(j)$

Also, we can have a counterfactual measurement for which every outcome is equally possible, hence substitute Prob(i)= $\frac{1}{d^2} \forall i = 1 : d^2$ in the equation 19 we obtain, $Prob(j) = q(j) = \frac{1}{d^2}\sum_{i=1}^{d^2} r(j|i)$

Hence,

$$\begin{aligned} Prob(i|j) &= Prob(j|i) * Prob(i)/Prob(j) \\ &= \frac{r(j|i)}{\sum_k r(j|k)} \end{aligned} \tag{22}$$

We will now make an assumption which will be called: **Principle of Reciprocity**:

Assumption 1: For all $R \in Q$, the posterior probabilities for the counterfactual measurements $Prob(i|j)$ are also valid prior for such measurements i.e. $|k_j\rangle \in P$ where $k_j(i) = Prob(i|j)$. Also, all valid priors $|p\rangle \in P$ can be obtained from the equation 22

Hence, this provides us with a relation between the sets P and Q, as every element in the set P can be obtained via elements of Q.

The assumption can be justified for the case of POVM measurements in quantum systems. [See the complete proof in Appendix 6.4]

4.1 Basis Distributions

Since there is no restriction on the real measurements, we can have a case where the outcome probabilities for the real and counterfactual measurements are equal for all the outcomes. Note that this also imposes the restriction that the total number of outcomes for the real measurements are d^2. Hence, we have special type of real measurement s.t. $q(j) = p(j) \forall j = 1 : d^2$. Also, for these special measurements equation 19 implies:

$$p(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \tag{23}$$

Among these special measurements, consider the case when the probability for every outcome of the counterfactual measurement is same i.e. $p(i) = \frac{1}{d^2} \forall i$

Substituting this in equation 23 implies

$$\sum_i r(j|i) = 1 \tag{24}$$

Thus, we obtain a constraint on the conditional probabilities for these special measurements. Incorporating this constraint in equation in 23

$$p(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d} \tag{25}$$

Define a d^2 square matrix M s.t. $M_{i,j} = d + 1 - \frac{1}{d}$ for i=j and and $M_{i,j} = -\frac{1}{d}$ for $i \neq j$

Now, the above equation 25 be written as follows with R and $|p\rangle$ as defined previously in the start of section 5:

$$MR\,|p\rangle = |p\rangle \tag{26}$$

We would now make an assumption in our state space that would help us obtain a simple expression for R.

Assumption 2: For the state space P, the elements of state space $|p\rangle \in P$ span the full simplex Δ_{d^2}

With this assumption, the equation 26 can be simplified to $MR = I$. A short argument for its proof: Say the solution set of the equation 26 is S and observe that if $|p\rangle \in S$ span the full simplex Δ_{d^2}, then there must be at least d^2 independent vectors $|p\rangle \in S$ (because the full simplex Δ_{d^2} is a subspace of dimension d^2). Hence, we have d^2 independent solutions to the equation 26 which can be possible only if $MR = I$

Observe that M is a circulant matrix, refer Appendix 6.2 to get idea on inverse of a circulant matrix.

After computing the inverse of M, the equation 26 under the Assumption 2 implies $R = M^{-1}$, the elements of R given as:

$$r(j|i) = \frac{1}{d+1}(\delta_{i,j} + \frac{1}{d}) \tag{27}$$

Now, using the conditional probabilities given by the above equation 27 and uniform distribution for counterfactual measurement i.e. $p(i) = \frac{1}{d^2}$, we compute the posterior probability for counterfactual measurement using equation 22:

$$\begin{aligned} Prob(i|j) &= \frac{r(j|i)}{\sum_k r(j|k)} \\ &= \frac{1}{d+1}(\delta_{i,j} + \frac{1}{d}) \end{aligned}$$

By the principle of the reciprocity, the state formed by these posterior probabilities is a valid prior or $|p\rangle \in P$ where $p(i) = Prob(i|j)$.

Let us denote these states formed by the above posterior probabilities as the basis distributions $|e_k\rangle$. Hence, we have a set of d^2 basis distribution $\{e_k | k = 1 : d^2\}$ where $e_k(i) = \frac{1}{d+1}(\delta_{i,k} + \frac{1}{d})$.

Observe that all the basis distributions states satisfy

$$\sum_i e_k(i)^2 = \frac{2}{d(d+1)} \tag{28}$$

This equation is similar to the equation 16, the result we obtained in the case of pure quantum state representations using SIC. Hence, we have been able to arrive at certain states (basis distributions) that preserve some properties of the quantum state space derived in section 2, specified by 16, 17

To summarise, we started with certain special type of real measurements and under the Assumption 2, we were able to arrive at the basis states (posterior states under these special measurements). The basis states are special in the sense that they preserve some properties of the quantum state space derived in section 2.

Note: The basis distributions derived above can also be seen as the extreme points of the set P (P or the state space is a simplex, essentially a convex set). I did not completely understand this argument, so I cannot justify this. The argument for this can be found in the main reference [3] page 32. This interpretation of basis distribution as extreme points of the set P would be used in the next section.

4.2 Basis State Preparation

Let us define a special type of real measurements as ISU Measurements, such that whenever the outcome probabilities of the counterfactual measurements are uniform, it implies we have uniform distribution for the outcomes of real experiment too.

Hence, for an ISU measurement, if $p(i) = \frac{1}{d^2}$ $\forall$ i then we would have $q(j) = \frac{1}{m}$ $\forall = 1 : m$.

Substituting the uniform distribution for p and q in the equation 19 (Born Rule) and simplifying, we obtain

$$\sum_{i=1}^{d^2} r_{ISU}(j|i) = \frac{d^2}{m} \tag{29}$$

Incorporating the constraint given by 29, the basic axiom of Born Rule for an ISU measurement becomes:

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r_{ISU}(j|i) - \frac{d}{m} \tag{30}$$

Also, denote the posterior state for counterfactual measurement under the ISU measurement using the mechanism described in 5.1 by $|s\rangle$ s.t. $s(i) = Prob(i|j)$.

By principle of reciprocity, the state $|s\rangle$ is valid i.e. $|s\rangle \in P$ Now, using equation (22) and (29):

$$s(i) = \frac{r_{ISU}(j|i)}{\sum_k r_{ISU}(j|k)} = \frac{m}{d^2} r_{ISU}(j|i) \tag{31}$$

Substituting the expression for r_{ISU} from (31) in (30):

$$q(j) = \frac{d^2(d+1)}{m}\sum_{i=1}^{d^2} p(i)s(i) - \frac{d}{m} \tag{32}$$

Note that the above equation is valid for any $|p\rangle \in P$. This follows from the consistency of P and Q as defined in section 5.1, the equation 32 is just a simplified version of the basic axiom 19 for the case of ISU measurements. Since, $q(j)$ is valid probability, we must have the following true for all $|p\rangle \in P$ and for all $|q\rangle$ s.t. $|q\rangle$ is the posterior state of some ISU measurement

$$\frac{d}{d+1} \leq \sum_{i=1}^{d^2} p(i)s(i) \leq \frac{d+m}{d^2(d+1)} \tag{33}$$

We now make an another assumption regarding our state space:

Assumption 3: All the extreme points $|p\rangle \in P$ are generated as the posterior state of ISU measurement with d outcomes

The assumption states that if $|s\rangle$ is a basis state (extreme point) then it can be generated with ISU measurement with d outcomes i.e. we have m=d in equation (33) Also, the equation (33) is valid for all $|p\rangle \in P$, hence its also valid for all the extreme points of P. Hence, any two extreme points $|p\rangle$ and $|s\rangle$ must satisfy

$$\frac{d}{d+1} \leq \sum_{i=1}^{d^2} p(i)s(i) \leq \frac{2}{d^2(d+1)} \tag{34}$$

Also, by (28) the equality on right hand side must hold when $|p\rangle = |s\rangle$

Thus we have introduced a lower bound on the inner product of two extreme points which is interestingly also satisfied by any two quantum states in SIC Representation [Proof can be found in Appendix 6.8]. Hence, with the ISU measurements and the Assumption 3, we have been able to preserve some more properties of the quantum state space specified by 16, 17

The inequality (34) leads more interesting properties about the extreme points of state space P, these can be found in the section 5.3 and 5.4 of the main reference paper [3].

5 Conclusion

The major contribution of the paper [3] is to view Born Rule as an addition to Dutch Book coherence. They achieved this by introducing the idea of counterfactual SIC measurements, which lead to a specific interpretation of Born Rule as a relation between the outcome probabilities for the real measurements and the outcome probabilities for the real measurements if they were conditioned on the SIC measurements. There is no rule in classical probability theory to relate the probabilities for these two cases, but in quantum mechanics they are related via the equation (19) Hence, Born rule is an addition to the normal rules of Dutch-book coherence.

The following questions still remain unanswered:

1. Born rule is just interpreted like Dutch Book Coherence but we do not have a precise explanation using lottery tickets to explain it, we only interpret it similar to that. So we cannot precisely explain its origin or answer the why behind it. Neither can we explain what might go wrong if its not followed, unlike what we did with the normal Dutch book coherence principles that an agent would head for a sure loss if they are not followed.

2. In section 3 of the report, by assuming the Born Rule equation 19 and some other assumptions, we were able to obtain some properties of the quantum state space specified by (16, 17). But we could not not show that the state space P is isomorphic to that specified by (16, 17). Since we were able to derive (16) using the basis distributions in state space P, the next step would be to find whether we can derive equation (17) too for the state space P.

6 Appendix

6.1

Claim: If A is positive definite matrix with Tr(A)=1, then $Tr(A^2) \leq Tr(A)$ and the equality holds iff largest eigenvalue of A is 1

Proof:

A is P.S.D, hence eigenvalues $\lambda_i \geq 0$ and $\sum_i \lambda_i$ = Tr(A) = 1 $\implies 0 \leq \lambda_i \leq 1$

Now, observe that by Spectral Value Decomposition of A, we obtain that eigenvalues of A^2 are $\{\lambda_i^2\}$ where $\{\lambda_i\}$ are the eigenvalues of A

Hence, $Tr(A^2) = \sum_i \lambda_i^2 \leq \sum_i \lambda_i = 1$ ($\lambda_i^2 \leq \lambda_i$ because $0 \leq \lambda_i \leq 1$)

Now, for the case of equality: $\sum_i \lambda_i = 1$ Taking square on both sides and noting that $\sum_i \lambda_i^2 = 1$, we obtain: $\sum_{i \neq j} \lambda_i \lambda_j = 0$

Since, each λ_i is non negative, this implies that $\lambda_i \lambda_j = 0$ for $i \neq j$ and if we also want to satisfy the equation $\sum_i \lambda_i = 1$, then the only solution is that exactly one eigenvalue must be 1 and all the rest should equal 0.

The converse in the case of equality is trivial to prove as if largest eigenvalue is 1, then $\sum_i \lambda_i = 1$ would imply that the remaining eigenvalues have to be zero since A is P.S.D.

6.2

We discuss special circulant matrices with all the non diagonal elements as equal. (This serves our case since both circulant matrices in section 2.1 and section 4.1 were satisfy this) Hence, we have a circulant matrix of dimension n*n, with diagonal element as α and the non diagonal elements as β. Now, this matrix is diagonalisable with the $M = UDU$ where U is the Discrete Fourier Transform matrix. Hence, the eigenvalues of M are $\alpha + (n-1)\beta$ and $\alpha - \beta$ with multiplicity 1 and n-1.

Now, M is invertible iff D is invertible. Since D is a diagonal matrix with values as the eigenvalues of M, hence if all eigenvalues of M are nonzero then M would be invertible.

Also, note that $M^{-1} = UD^{-1}U$ and hence can be computed easily if you know the eigenvalues of M as the diagonal elements of D^{-1} would simply be the inverse of the diagonal elements of D.

6.3

Consider $|p_1\rangle, |p_2\rangle \in P$ that satisfy the inequality 21 for all $R \in Q$. Also, consider $|q_1\rangle, |q_2\rangle$ as the outcome probabilities for the real measurements obtained from Eq. 19 corresponding to $|p_1\rangle, |p_2\rangle$. Then it can be seen easily that $\alpha * |p_1\rangle + (1-\alpha)|p_2\rangle$ for $\alpha \in [0,1]$ also belongs to P:

$$
\begin{aligned}
q(j) &= (d+1)\sum_{i=1}^{d^2}(\alpha * p_1(i) + (1-\alpha)p_2(i))r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \\
&= \alpha\left((d+1)\sum_{i=1}^{d^2} p_1(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i)\right) + (1-\alpha)\left((d+1)\sum_{i=1}^{d^2} p_2(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i)\right) \\
&= \alpha * q_1(j) + (1-\alpha) * q_2(j) \quad \text{(we know } q_1 \text{ and } q_2 \text{ are valid probabilities)}
\end{aligned}
$$

6.4

Consider a POVM $\{G_j\}$ on a quantum system. Then, by eq. (19): $r(j|i) = Tr(\Pi_i G_j)$

Also, $\sum_{i=1}^{d^2} r(j|i) = Tr((\sum_{i=1}^{d^2} \Pi_i)G_j) = d^2 * Tr(G_j)$ (because $\sum_i E_i = I$ and $E_i = \frac{1}{d}\Pi_i$)

Using Eq (22) and define $\rho_j = \frac{G_j}{Tr(G_j)}$, we can see that $Prob(i|j)$ is again a valid quantum state in SIC representation.

So, the posterior probability of outcomes of SIC measurement is again a valid probability distribution which can be mapped to a quantum state.

$$
\begin{aligned}
Prob(i|j) &= \frac{Tr(\Pi_i G_j)}{d * Tr(G_j)} \\
&= \frac{1}{d} Tr(\rho_j \Pi_i)
\end{aligned}
$$

6.5

Using the equation (10):

$$\begin{aligned}
Tr(\Pi_i\Pi_j\Pi_k) &= Tr(\sum_m \alpha_{ijm}\Pi_m\Pi_k) \\
&= Tr(\alpha_{ijk}\Pi_k^2 + \sum_{m\neq k}\alpha_{ijm}\Pi_m\Pi_k) \\
&= \alpha_{ijk}Tr(\Pi_k^2) + \sum_{m\neq k}\alpha_{ijm}Tr(\Pi_m\Pi_k) \\
&= \alpha_{ijk} + \sum_{m\neq k}\alpha_{ijm}\frac{1}{d+1} \quad \text{(Using eq (8))} \\
&= \alpha_{ijk} + \frac{1}{d+1}(\frac{d\delta_{ij}+1}{d+1} - \alpha_{ijk}) \quad \text{(Using eq (11))} \\
&= \frac{d\delta_{ij}+1}{d+1} + \frac{d}{d+1}\alpha_{ijk}
\end{aligned}$$

6.6

Using the equation (10):

$$\begin{aligned}
\sum_i Tr(\Pi_i\Pi_j\Pi_k) &= \sum_i Tr(\Pi_i\sum_m \alpha_{jkm}\Pi_m) \\
&= \sum_m\sum_i \alpha_{jkm}Tr(\Pi_i\Pi_m) \\
&= \sum_m \alpha_{jkm}\sum_i \frac{d\delta_{mi}+1}{d+1}) \quad \text{(Using eq (8)))} \\
&= \sum_m \alpha_{jkm}(1 + \sum_{i\neq m}\frac{1}{d+1}) \\
&= \sum_m d*\alpha_{jkm} = d*\frac{d\delta_{jk}+1}{d+1} \quad \text{(Using eq (11)))}
\end{aligned}$$

Using the equation (12),

$$\begin{aligned}
\sum_{i=1}^{d^2}\alpha_{ijk} &= \sum_{i=1}^{d^2}\frac{1}{d}((d+1)Tr(\Pi_i\Pi_j\Pi_k) - \frac{d\delta_{ij}+1}{d+1}) \\
&= \frac{1}{d}((d+1)\sum_{i=1}^{d^2}Tr(\Pi_i\Pi_j\Pi_k) - \sum_{i=1}^{d^2}\frac{d\delta_{ij}+1}{d+1}) \\
&= \frac{1}{d}((d+1)d*\frac{d\delta_{jk}+1}{d+1} - d) = \delta_{jk}
\end{aligned}$$

6.7

Using the equation (9):

$$\begin{aligned}
Tr(\rho^2) &= Tr(\sum_{i,j}((d+1)^2p(i)p(j) - \frac{d+1}{d}p(i) - \frac{d+1}{d}p(j) + \frac{1}{d^2})\Pi_i\Pi_j) \\
&= \sum_{i,j}((d+1)^2p(i)p(j) - \frac{d+1}{d}p(i) - \frac{d+1}{d}p(j) + \frac{1}{d^2})Tr(\Pi_i\Pi_j) \\
&= \sum_{i,j}((d+1)^2p(i)p(j) - \frac{d+1}{d}p(i) - \frac{d+1}{d}p(j) + \frac{1}{d^2})\frac{d\delta_{i,j}+1}{d+1}
\end{aligned}$$

Observe,

$$\sum_{i,j}\frac{1}{d^2}\frac{d\delta_{i,j}+1}{d+1} = \frac{1}{d^2}*\sum_i\frac{d^2-1}{d+1} + 1 = d$$

$$\sum_{i,j}\frac{d+1}{d}p(i)\frac{d\delta_{i,j}+1}{d+1}=\frac{d+1}{d}\sum_i p(i)*(\frac{d^2-1}{d+1}+1)=(d+1)\sum_i p(i)=d+1$$

$$\begin{aligned}\sum_{i,j}(d+1)^2p(i)p(j)\frac{d\delta_{i,j}+1}{d+1}&=(d+1)^2(\sum_i p(i)^2+\sum_{i\neq j}p(i)p(j)\frac{1}{d+1})\\&=(d+1)^2(\sum_i p(i)^2+\frac{1}{d+1}\sum_i p(i)((\sum_j p(j))-p(i))\\&=(d+1)^2(\sum_i p(i)^2+\frac{1}{d+1}\sum_i p(i)-p(i)^2)\\&=d(d+1)(\sum_i p(i)^2)+d+1\end{aligned}$$

Substituting these values in the expression for $Tr(\rho^2)$ gives:

$$Tr(\rho^2)=d(d+1)(\sum_i p(i)^2)+d+1-2*(d+1)+d=d(d+1)(\sum_i p(i)^2)-1$$

Since, $Tr(\rho^2)=1$

$$\sum_i p(i)^2=\frac{2}{d(d+1)}$$

Now, the equation 17 is easy to derive by multiplying the equation 15 by p_k and taking sum over the index k on both sides of the equation. Using the result derived above for $\sum_i p(i)^2$, this can be reduced to equation 17

6.8

Consider any two quantum states in SIC representation i.e. by equation 9 we have two quantum states as follows:

$$\rho_1=\sum_i((d+1)p_1(i)-\frac{1}{d})\Pi_i$$
$$\rho_2=\sum_i((d+1)p_2(i)-\frac{1}{d})\Pi_i$$

Now, **without proof** I make the following claim:

Claim: Product of two PSD matrices is PSD

So, by the above claim we would have $\rho_1\rho_2$ as PSD and since Trace of a PSD matrix is positive, so $Tr(\rho_1\rho_2)\geq 0$. Using the SIC representation for ρ_1,ρ_2 we have the following:

$$\sum_{i,j}\left((d+1)^2p_1(i)p_2(j)-\frac{(d+1)}{d}p_1(i)-\frac{(d+1)}{d}p_2(j)+\frac{1}{d^2}\right)Tr(\Pi_i\Pi_j)\geq 0$$

$$\sum_{i,j}\left((d+1)^2p_1(i)p_2(j)-\frac{(d+1)}{d}p_1(i)-\frac{(d+1)}{d}p_2(j)+\frac{1}{d^2}\right)\frac{d*\delta_{i,j}+1}{d+1}\geq 0\quad\text{(Using 8)}$$

$$\sum_{i,j}\left((d+1)^2p_1(i)p_2(j)-\frac{(d+1)}{d}p_1(i)-\frac{(d+1)}{d}p_2(j)+\frac{1}{d^2}\right)\frac{d*\delta_{i,j}+1}{d+1}\geq 0$$

Now, we can resolve each of these terms as we have done previosuly in the section 6.7. Simplifying this way we obtain:

$$d(d+1)\sum_{i,j}p_1(i)p_2(j)-1\geq 0$$

This essentially implies:

$$\sum_{i,j}p_1(i)p_2(j)=\langle p_1||p_2\rangle\geq\frac{1}{d+1}$$

Hence, for any two quantum states in SIC representation we have the same lower bound as in the inequality (34)

Acknowledgement

The ideas and proofs presented in this report are not my own, my report is a summary of paper [3]. However, every idea, proof and argument presented in this report is in my own words. Also, I understand every concept and argument presented in this report, except a few places where I have mentioned clearly that I could not understand the complete argument and have tried to explain whatever I understood.

To clarify again, I do not understand how QBism can help solve important problems, but I have mentioned some references (which I have not read thoroughly) in the introduction section for the sake of motivating the discussion in this report.

Also, at the section 4.1 I have mentioned clearly in a Note that I do not understand completely why the basis distributions can be considered as the extreme points of the state space P

References

[1] Mixed states and measurement. `https://www.cs.cmu.edu/~odonnell/quantum15/lecture16.pdf`.

[2] Carlton M Caves, Christopher A Fuchs, and Rüdiger Schack. Subjective probability and quantum certainty. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 38(2):255–274, 2007.

[3] Christopher A. Fuchs and Ruediger Schack. Quantum-Bayesian Coherence. *arXiv e-prints*, page arXiv:0906.2187, Jun 2009.

[4] Christopher A Fuchs, Maximilian Schlosshauer, and Blake C Stacey. My struggles with the block universe. *arXiv preprint arXiv:1405.2390*, 2014.

[5] Richard Healey. Quantum-bayesian and pragmatist views of quantum theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2017 edition, 2017.

[6] Massimiliano Proietti, Alexander Pickston, Francesco Graffitti, Peter Barrow, Dmytro Kundys, Cyril Branciard, Martin Ringbauer, and Alessandro Fedrizzi. Experimental rejection of observer-independence in the quantum world. *arXiv preprint arXiv:1902.05080*, 2019.