Math 100, Writing Assignment #2

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1 The Problem

A pirate ship captures a treasure of 1000 golden coins. The treasure has to be split among the 5 pirates: 1, 2, 3, 4, and 5 in order of rank. The pirates have the following important characteristics: infinitely smart, bloodthirsty, greedy. Starting with pirate 5 they can make a proposal how to split up the treasure. This proposal can either be accepted or the pirate is thrown overboard. A proposal is accepted if a majority of the pirates agree on it (a tie means the proposal is rejected). What proposal should pirate 5 make?

What happens if there are 6, 7, or more pirates? Generalize! (This problem has a lot to offer.) Your task here is to solve the problem and then come up with a way to explain the solution and write up a good solution. You will need to introduce/define some terms from probability theory.

2 The Solution

First, let us assume that each pirate's top priority is to stay alive, followed by greed, then their desire to shed blood. Since they are infinitely smart, they will each be aware of every possible situation and understand that they all share the same priorities. We can also assume that if they are offered the same number of coins as the previous case, they will vote against the proposal, preferring to throw someone overboard if they receive the same profit.

2.1 The 5th Pirate

In order to see how Pirate 5 makes his decision, we will look at each case building up to 5 pirates, starting with only two pirates, then three pirates, and so on. This will allow us to see how each pirate makes his decision on whether to vote for or against the proposal and how much the proposing pirate should offer them.

Case 1: Two Pirates Pirate 2, sadly, has no possible way to give a successful proposal.

Since he is need of two votes to have a succesful proposal, he must 1) vote in favor for himself, and 2) have Pirate 1 vote in favor of the proposal. What would seem most logical is that he simply offer Pirate 1 all of the coins; however, Pirate 1 would receive the same number of coins if Pirate 2 was thrown overboard. Since the pirates are all bloodthirsty, Pirate 1 would vote against the proposal, thereby securing himself all of the money and killing as many people as possible.

Case 2: Three Pirates Pirate 3 will propose to take all the coins.

From Case 1 we know that Pirate 2 will be thrown overboard if his proposal is not accepted, so he will vote in favor of Pirate 3's proposal no matter what it is. Needing only two votes to have a majority (two of three), Pirate 3 needs only his own additional vote to be successful. He can then award himself all of the coins, having not needed to "buy" any votes. The pirates will receive the following number of coins, respectively:

$$(P_1, P_2, P_3) = (0, 0, 1000).$$

Case 3: Four Pirates Pirate 4 will give out 2 coins, and take 998 for himself.

With four pirates, three votes in favor of the proposal are needed for it to pass. Pirate 4 cannot possibly give Pirate 3 more than 1000 coins, and so will lose his vote no matter what the proposal. Therefore, he must win over Pirate 1 and Pirate 2 to secure their votes.

If Pirate 4's proposal is not accepted, Pirate 1 and 2 will not receive any coins. Knowing this, Pirate 4 will offer each 1 coin, giving them a better result and gaining their votes. Since Pirate 3 will make no contribution, he will get nothing. Pirate 4 will then keep the 998 remaining coins for himself.

$$(P_1, P_2, P_3, P_4) = (1, 1, 0, 998).$$

Case 4: Five Pirates Pirate 5 will take 997 coins and give out 3 coins.

Pirate 5 will need a total of three votes to pass his proposal. He can guarantee his own, and can be sure that Pirate 4 will not vote for him; Pirate 5 cannot be as profitable as possible if he tries to secure Pirate 4's vote.

If Pirate 5's proposal is not accepted, Pirate 3 would receive nothing from Case 3 and so can be bought off with 1 coin. Both Pirate 1 and 2 would receive 1 coin each from Case 3, making them each purchasable for 2 coins; however, he needs only one of their votes. This leaves us with two possible proposals:

$$(P_1, P_2, P_3, P_4, P_5) = \begin{cases} (0, 2, 1, 0, 997) \\ (2, 0, 1, 0, 997) \end{cases}.$$

In either situation, Pirate 5 will have a successful proposal and will have maximized his profit.

2.2 The 6th & 7th Pirates

Case 4 introduced the possibility of two different outcomes, changing how pirates may cast their vote when we expand to six, seven, or more pirates. We must now take into consideration the probability of which outcome will be chosen in the preceding case, and look at each pirates expected value for voting yes or no.

Case 5: Six Pirates

Pirate 6 will need four votes to win the majority. His own is guaranteed. Pirate 5 is not the most profitable choice to try to win over, but Pirate 4, who in Case 4 receives nothing, can easily be bought off with 1 coin. This leaves us with the need for two more votes and three pirates to gain them from: Pirate 1, 2, or 3.

Pirate 3 knows that he can expect to get 1 coin from Case 4, so if he is to be selected, he must receive 2 coins.

Now, since Case 4 has not occurred yet, Pirates 1 and 2 are subject to a probability of being selected in Case 4 if they vote against Case 5. We must now introduce their ***expected value*** from Case 4 to know how much they need to receive in Case 5 to be won over.

Definition 2.1. *The expected value, or the average value to be gained from all possible outcomes, can be defined as*

$$E(X) = \sum_{i=0}^{\infty} x_i\, p_i$$

where $E(X)$ is the expected value, x_i is the value a given situation, and p_i is the probability of that situation occurring, assuming absolute convergence of the sum.

For our purposes, we can generalize

$$\begin{aligned} E(X) &= x_1\, p_1 + x_2\, p_2 \\ &= x_1\, p_1 + (0)\cdot(1 - p_1) \\ &= x_1\, p_1 + 0 \\ &= x_1\, p_1 \end{aligned}$$

as there are only two possible situations; the chosen pirate will receive some number x_1 of coins or none at all.

For Pirate 1 and 2, we see

$$E(X) = x_1\, p_1 = (2)\cdot(\frac{1}{2}) = 1.$$

Since both Pirates 1 and 2 have an expected value of 1 coin in Case 4, they can be won over in Case 5 with 2 coins. However, Pirate 6 only needs two votes from Pirates 1, 2, and 3, who all can be bought with 2; this gives us the following possible proposals:

$$(P_1, P_2, P_3, P_4, P_5, P_6) = \begin{cases} (2, 2, 0, 1, 0, 995) \\ (2, 0, 2, 1, 0, 995) \\ (0, 2, 2, 1, 0, 995) \end{cases}.$$

Pirate 6 need only choose one of these proposals to ensure that his proposition passes and that he has maximized his profit.

Case 6: Seven Pirates

Pirate 7 once again needs four votes to pass his proposal. His own is guaranteed and we know Pirate 6's vote will be lost. Pirate 5 can be bought with 1 coin since he expects to get nothing from the previous case, and Pirate 4 can be won by giving him 2 coins versus the 1 coin he would receive otherwise. To determine Pirate 1, 2, and 3's buy-off amount, we must look at their expected value from Case 5:

$$E(X) = x_1\, p_1 = (2)\cdot(\frac{2}{3}) = \frac{4}{3}.$$

This expected value is still less than two but greater than one, so each can be bought off with 2 coins. Then resulting possible proposals are as follows:

$$(P_1, P_2, P_3, P_4, P_5, P_6, P_7) = \begin{cases} (2, 2, 0, 0, 1, 0, 995) \\ (2, 0, 2, 0, 1, 0, 995) \\ (2, 0, 0, 2, 1, 0, 995) \\ (0, 2, 2, 0, 1, 0, 995) \\ (0, 2, 0, 2, 1, 0, 995) \\ (0, 0, 2, 2, 1, 0, 995) \end{cases}.$$

Pirate 7 is able to keep 995 coins, while giving away only 5 to secure votes for his proposal.

2.3 The General Case, n Pirates

We will now begin to form general expressions for the probability, expected value, and profit for any given pirate proposition, with the total number of pirates, $n \in \mathbb{N}, n \geq 5$; this will allow us to calculate the information with a variety of combinations.

Case n: The General Case, $n \geq 5$ Pirates

First, we want to create a genral expression for the expected value. To generalize this, we must think about what probablity is.

Definition 2.2. *In our case,* ***probability*** *is the number of possibilities that a given pirate has to be selected to receive 2 coins per the number of combinations we can choose from. We can write this as:*

$$p_1(n) = \frac{\binom{k-1}{j-1}}{\binom{k}{j}}$$

where k is the number of pirates that Pirate n can choose to give coins and j is the number of pirates he actually chooses ($j, k \in \mathbb{N}$). Here, k choose j represesnts the number of chosen pirates out of the number of available pirates. Once a specific pirate is chosen, there are $k-1$ pirates left, and $j-1$ will be chosen. So the different combinations that a specific pirate can have is $k-1$ choose $j-1$.

Because we can ignore the $n-2$, $n-1$, and n^{th} pirates as possible choices,

$$k = n - 3$$

and

$$j = \begin{cases} \frac{n-3}{2}, & n \text{ is odd} \\ \frac{n-2}{2}, & n \text{ is even} \end{cases} .$$

We can calculate

$$\binom{k}{j} = \frac{k!}{j!(k-j)!}.$$

By subsituting this formula into our equation for $p_1(n)$:

$$
\begin{aligned}
p_1(n) &= \frac{\binom{k-1}{j-1}}{\binom{k}{j}} \\
&= \frac{\frac{(k-1)!}{(j-1)!(k-j)!}}{\frac{k!}{j!(k-j)!}} \\
&= \frac{(k-1)!}{(j-1)!(k-j)!} \cdot \frac{j!(k-j)!}{k!} \\
&= \frac{(k-1)!}{(j-1)!} \cdot \frac{j \cdot (j-1)!}{k \cdot (k-1)!} \\
p_1(n) &= \frac{j}{k}
\end{aligned}
$$

Inputting our values for k and j in, and assuming x_1 is 2, we see our expected value as:

$$
\begin{aligned}
E(X) &= x_1\, p_1 \\
&= 2 \cdot \frac{j}{k} \\
&= 2 \cdot \frac{\frac{n-3}{2}}{n-3} \qquad n \textit{ is odd} \\
&= \frac{n-3}{n-3} \\
E(X) &= 1 \\
&= 2 \cdot \frac{\frac{n-2}{2}}{n-3} \qquad n \textit{ is even} \\
E(X) &= \frac{n-2}{n-3}
\end{aligned}
$$

$$
E(X) = \begin{cases} 1, & n \text{ is odd} \\ \frac{n-2}{n-3}, & n \text{ is even} \end{cases}
$$

We see that when n is odd, the expected value will continually be 1 and for even values we get a number greater than 1 but less than 2. Therefore, we can say that Pirate n will give 2 coins to j pirates. This gives the pirate a profit as follows:

$$
P_n(n) = \begin{cases} 1002 - n, & n \text{ is odd} \\ 1001 - n, & n \text{ is even} \end{cases}
$$

and a combination of proposals

$$(P_1, P_2, ..., P_{n-3}, P_{n-2}, P_{n-1}, P_n) = \begin{cases} (2, 2, ..., 0, 1, 0,\ P_n(n)) \\ ... \\ ... \\ (0, 0, ..., 2, 1, 0,\ P_n(n)) \end{cases}.$$

However, when we reach the case of the 1001^{st} pirate, we find a glitch; the 1000^{th} pirate only receives 1 coin in his proposition, so when Pirate 1001 makes his, Pirate 1000 is added to the group of pirates that can be selected to receive 2 coins. This changes the pool of pirates, which we labeled k, to

$$k = n - 2,$$

which in turn changes our expected value for that case to

$$E_{P_{1001}}(X) = \frac{n-3}{n-2} = \frac{998}{999}.$$

Our expected value has finally dropped below 1, which enables the 1002^{nd} pirate to lower the price of votes from 2 coins to 1 coin. This gives him the following combination of proposals:

$$(P_1, P_2, ..., P_{998}, P_{999}, P_{1000}, P_{1001}, P_{1002}) = \begin{cases} (1, 1, ..., 0, 0, 0, 0, 499) \\ ... \\ ... \\ (0, 0, ..., 1, 0, 1, 0, 499) \end{cases}$$

where Pirate 999 and 1001 receive 0 coins, as they received the maximum in the last proppposition, and Pirate 1000 remains one of the pirates available for selection. The expected value for this situation is $E_{P_{1002}}(X) = \frac{501}{999}$, thus keeping us below the 1 coin expected value.

Beginning with Pirate 1003's proposal, we can see that all but the $n-1$ pirate now enter the pool of pirates who may be selected to receive 1 coin. This changes everything we thought we knew about the patterns of pirate selection and profit; our pool of pirates becomes $k = n - 2$, the number selected becomes

$$j = \begin{cases} \frac{n}{2}, & n \text{ is even} \\ \frac{n-1}{2}, & n \text{ is odd} \end{cases}$$

and the expected value becomes

$$E(X) = \begin{cases} \frac{n}{2n-4}, & n \text{ is even} \\ \frac{n-1}{2n-4}, & n \text{ is odd} \end{cases},$$

for $n \geq 1003$. This changes Pirate n's payoff to

$$P_n(n) = \begin{cases} 1000 - \frac{n}{2}, & n \text{ is even} \\ 1000 - \frac{n-1}{2}, & n \text{ is odd} \end{cases}.$$

But what happens for Pirate 2002? He will be short 1 coin to pay everyone off for his proposal, thereby losing a necessary vote to keep from being thrown overboard. He has no chance of success.

However, Pirate 2003 has a guaranteed vote from Pirate 2002 (who would prefer to live), and so can pass his proposal, changing his votes needed to $\frac{n-3}{2}$.

Proposition 2.3. *For pirates $n \geq 2000$, we can find the next surviving pirate as*

$$P_{n(m)} = P_{1999+2^m}, \quad m \in \mathbb{N}.$$

Proof. (By induction)

Base Case: m=1

$$P_{1999+2^m} = P_{1999+2^1} = P_{1999+2} = P_{2001}$$

which we know to be true from earlier.

Inductive Step:

Suppose that $n = 1999 + 2^m$ for $n > 2000$. Then $1999 + 2^m + x$ gives the number of the next pirate to survive, where x represents the number of pirates from $P_{n(m)}$ to $P_{n(m+1)}$. Then the number of votes needed, v, is

$$v = \begin{cases} \frac{1999+x+2^m}{2} + 1, & x \text{ is odd (making the number of pirates even)} \\ \frac{1999+x+2^m+1}{2}, & x \text{ is even (making the number of pirates odd)} \end{cases}.$$

We can also express v as the number of votes bought combined with the number of additional votes from the x pirates:

$$v = 1000 + x$$

This gives us the following two equations, which we can solve for x:

$$(1) \qquad \begin{aligned} \frac{1999+x+2^m}{2}+1 &= 1000+x \\ 2001+x+2^m &= 2000+2x \\ x &= 2^m+1 \end{aligned}$$

$$(2) \qquad \begin{aligned} \frac{1999+x+2^m+1}{2} &= 1000+x \\ 2000+x+2^m &= 2000+2x \\ x &= 2^m \end{aligned}$$

However we are loooking for the next case, so x must be at its smallest value; therefore $x = 2^m$ and so must be even, resulting in an odd number of pirates. Substituting this in, we see

$$\begin{aligned} P_{n(m)+x} &= P_{1999+2^m+x} \\ &= P_{1999+2^m+2^m} \\ &= P_{1999+2^{m+1}} \\ &= P_{n(m+1)} \end{aligned}$$

And so the statement holds true for all m.

$\square$

When $P_n = P_{1999+2^m}$, the pirate will distribute 1 coin to 1000 pirates from $[P_1, P_{n-x-1}]$, will give 0 coins to pirates from $[P_{n-x}, P_{n-1}]$, and will take 0 coins for himself. When $P_n \neq P_{1999+2^m}$, that pirate will be thrown overboard, and the case will collapse until it reaches the last successful proposal.

2.4 Conclusion

In short, the following is how the pirates will give their propositions:

- P_1
  P_1 will take 1000 coins for himself.
- P_2
  P_2 will be thrown overboard, collapsing to P_1.
- P_3
  P_3 will give P_1 and P_2 0 coins, and take 1000 for himself.

- P_4
  P_4 will give P_1 and P_2 1 coin each, P_3 0 coins, and take 998 for himself.

- P_5
  P_5 will give either P_1 or P_2 2 coins, P_3 1 coin, P_4 0 coins, and take 997 for himself.

- $P_6 \leq P_n \leq P_{100}$
  P_n will give P_1 through P_{n-3} 0 or 2 coins, P_{n-2} 1 coin, P_{n-1} 0 coins, and take $1002 - n$ coins for $n = 2a + 1$ pirates or $1001 - n$ for $n = 2a$ pirates, $a \in \mathbb{N}$.

- P_{1001}
  P_{1001} will give P_1 through P_{998} 0 or 2 coins, P_{1000} 0 or 2 coins, P_{999} 1 coin, and will take 1 coin for himself.

- P_{1002}
  P_{1002} will give P_1 through P_{998} 0 or 1 coins, P_{1000} 0 or 1 coins, P_{999} and P_{1001} 0 coins, and will take 499 coins for himself.

- $P_{1003} \leq P_n \leq P_{2001}$
  P_n will give P_1 through P_{n-2} 0 or 1 coins, P_{n-1} 0 coins, and will take $1000 - \frac{n}{2}$ coins when $n = 2a$ or $1000 - \frac{n-1}{2}$ coins when $n = 2a + 1$ for himself.

- P_{2002}
  P_{2002} cannot win enough votes, and so will be thrown overboard, collapsing to P_{2001}.

- $P_{2003} \leq P_n$
  When $P_n = P_{1999+2^m}$, he will give P_1 through $P_{1999+2^{m-1}}$ 0 or 1 coins, $P_{2000+2^{m-1}}$ through P_{1998+2^m} 0 coins, and take 0 coins for himself.
  When $P_n \neq P_{1999+2^m}$, that pirate will be thrown overboard and the case will collapse until $P_n = P_{1999+2^m}$.