Hermitization

To work with non-hermitian matrices we build the $2N \times 2N$ hermitian matrix

$$H(z) = \begin{pmatrix} 0 & z - J \\ \bar{z} - J^\dagger & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & A_z \\ A_z^\dagger & 0 \end{pmatrix} \tag{1}$$

and deal with the resolvent matrix of $H(z)$, which depends on a new complex variable w:

$$G(w, z; J) \equiv [w - H(z)]^{-1} = -\begin{pmatrix} \dfrac{w}{A_z A_z^\dagger - w^2} & \dfrac{A_z}{A_z^\dagger A_z - w^2} \\ \dfrac{A_z^\dagger}{A_z A_z^\dagger - w^2} & \dfrac{w}{A_z^\dagger A_z - w^2} \end{pmatrix} \equiv \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix}. \tag{2}$$

In the second equality we used the inverse of a partitioned matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & A^{-1}B(CA^{-1}B - D)^{-1} \\ D^{-1}C(BD^{-1}C - A)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Note that the blocks $G^{\alpha\beta}$ are functions of z and w and depend on J, i.e., $G^{\alpha\beta} = G^{\alpha\beta}(w, z; J)$. The resolvent we are interested in is $(z - J)^{-1} = A_z^{-1}$, and can be recovered by taking the limit $w \to \mathrm{i}0$ of Eq. (2) (the imaginary component of the limit helps regularize the denominators):

$$\lim_{w \to \mathrm{i}0} G(w, z; J) = -\begin{pmatrix} 0 & A_z^{-\dagger} \\ A_z^{-1} & 0 \end{pmatrix}.$$

The resolvent is just the component 21 of the resulting limit,

$$\frac{1}{z - J} = -\lim_{w \to \mathrm{i}0} G^{21}(w, z; J). \tag{3}$$

Perturbation in power series

The idea is to treat the random matrix J as a perturbation in Eq. (1),

$$H(z) = H_0(z) - \mathcal{J}, \quad \text{where } H_0(z) \equiv \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \text{ and } \mathcal{J} \equiv \begin{pmatrix} 0 & J \\ J^\dagger & 0 \end{pmatrix},$$

and then compute the resolvent $G(w, z; J)$ by expanding in series of powers of J:

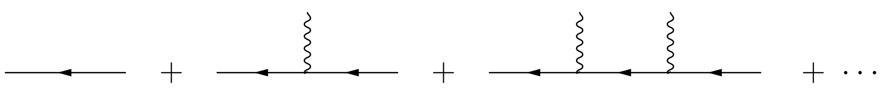
$$G(w, z; J) = \Big[w - H_0(z) + \mathcal{J}\Big]^{-1} = G_0(w, z) \sum_{n=0}^{\infty} [-\mathcal{J} G_0(w, z)]^n, \tag{4}$$

where we defined the unperturbed resolvent,

$$G_0(w, z) \equiv G(w, z; 0) = [w - H_0(z)]^{-1}.$$

We will refer to $G_0(w, z)$ as the *free propagator*. We will realize later that it is useful to represent the geometric series in Eq. (4) with Feynman diagrams:

$$G(w, z; J) = G_0(w, z) + G_0(w, z)(-\mathcal{J})G_0(w, z) + G_0(w, z)(-\mathcal{J})G_0(w, z)(-\mathcal{J})G_0(w, z) + \cdots$$

$=$  (5)

All the terms in the expansion are $2N \times 2N$ matrices, and the products appearing in each term are matrix multiplications. At this point, it is convenient to introduce some notation. The $2N \times 2N$ matrices we are working with can be thought of as a direct product of 2×2 matrices with $N \times N$ matrices. From this point of view, the components of a generic $2N \times 2N$ matrix A can be written as $A_{ab}^{\alpha\beta}$, where Greek indices are either 1 or 2 and Latin indices run from 1 to N.

Average over the disorder

To evaluate the average density of the eigenspectrum $\rho(z) = \pi^{-1}\bar{\partial}\,\mathrm{tr}\,\langle R(z)\rangle_J$, we need to compute the average over the disorder of the resolvent, Eq. (2). We do so by averaging term by term the series in Eq. (5), applying Wick's theorem, and using the covariance for the components of $\mathcal{J}$—which depend on the particular ensemble of matrices we are considering. With the notation introduced in the previous section, the first and second order moments of our ensemble

$$\langle J_{ij}\rangle_J = 0, \quad \langle |J_{ij}|^2\rangle_J = \frac{1}{N}, \quad \langle J_{ij}J_{ji}\rangle_J = \frac{\eta}{N},$$

can be expressed as the covariance for the components of $\mathcal{J}$:

$$\langle \mathcal{J}^{\alpha\beta}_{ab}\mathcal{J}^{\gamma\delta}_{cd}\rangle = \frac{1}{N}\,\delta_{ad}\delta_{bc}\Big[\sigma^{+}_{\alpha\beta}\sigma^{-}_{\gamma\delta} + \sigma^{-}_{\alpha\beta}\sigma^{+}_{\gamma\delta} + \eta\Big(\sigma^{+}_{\alpha\beta}\sigma^{+}_{\gamma\delta} + \sigma^{-}_{\alpha\beta}\sigma^{-}_{\gamma\delta}\Big)\Big]. \tag{6}$$

Here we introduced for convenience the 2×2 matrices

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

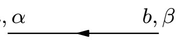
It will be convenient to recast Eq. (6) into

$$\begin{aligned}\langle \mathcal{J}^{\alpha\beta}_{ab}\mathcal{J}^{\gamma\delta}_{cd}\rangle &= \frac{1}{N}\left[\pi^1_{\alpha\delta}\delta_{ad}\,\pi^2_{\beta\gamma}\delta_{bc} + \pi^2_{\alpha\delta}\delta_{ad}\,\pi^1_{\beta\gamma}\delta_{bc} + \eta\Big(\pi^1_{\alpha\gamma}\delta_{ad}\,\pi^2_{\beta\delta}\delta_{bc} + \pi^2_{\alpha\gamma}\delta_{ad}\,\pi^1_{\beta\delta}\delta_{bc}\Big)\right] \\ &= \frac{1}{N}\delta_{ad}\delta_{bc}\left[\sum_{r,s}\pi^r_{\alpha\delta}\sigma^1_{rs}\pi^s_{\beta\gamma} + \eta\sum_{t,u}\pi^t_{\alpha\gamma}\sigma^1_{tu}\pi^u_{\beta\delta}\right],\end{aligned} \tag{7}$$

where we introduced three new 2×2 matrices:

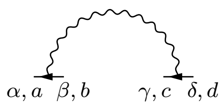
$$\pi^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Averaging the series (5) over the disorder gives rise to covariance factors like Eq. (6). These factors dictate how indices should be contracted. To understand the structure of the resuling series, it is useful to represent each contraction with diagrams. First, the free propagator $G_0(w,z)^{\alpha\beta}_{ab}$ has row and column indices (α, a) and (β, b) which can be represented at each end of a *propagator* line:

$G_0(w,z)^{\alpha\beta}_{ab} =$ 

.

The propagator line has a direction to keep track of the index order. Similarly, when we invoke Wick's theorem and pick a particular pair of $\mathcal{J}'s$ in the series (5) to compute the covariance, we are connecting their associated loose wiggled lines in (5). In diagrams,

$\langle \mathcal{J}^{\alpha\beta}_{ab}\mathcal{J}^{\gamma\delta}_{cd}\rangle =$ 

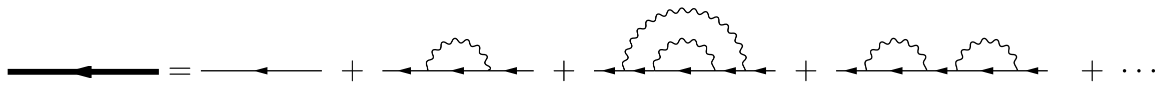
.

The non-crossing approximation consists in neglecting all the crossing diagrams—those that cannot be drawn on a plane, with the wavy lines drawn on the upper half plane above the straight arrow line, withouth the wavy lines crossing (Ahmadian et al, 2015).

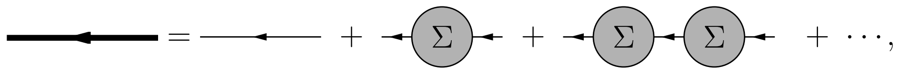
We will denote the averaged $G(w,z;J)$ by $\mathcal{G}(w,z)$ and we will represent its propagator with a thick directed line:

$\mathcal{G}(w,z) \equiv \langle G(w,z;J)\rangle_J =$.

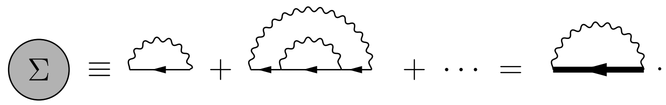
The full propagator results from averaging term by term of the series (5) and, under the non-crossing approximation, can be represented as an infinite series of the type

 (8)

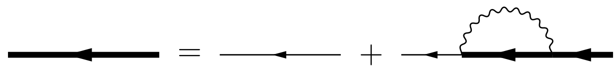
Note that this sum has the following structure

 (9)

where blobs represent the sum of all *irreducible* diagrams, defined as the set of diagrams that cannot be split in two by just removing a single line:

 (10)

In the last equality we used Eq. (8) to relate the sum of all irreducible diagrams inside the outer wavy line with $\mathcal{G}(w,z)$. We will refer to the sum of all irreducible diagrams as the *self-energy*, denoted by Σ. We can go further and apply recursively the decomposition (9), which combined with the definition (10) leads to the so-called Dyson-Schwinger relation

$\mathcal{G}(w,z) =$ 

$$= G_0(w,z) + G_0(w,z)\Sigma(w,z)\mathcal{G}(w,z). \tag{11}$$

A direct consequence of Schwinger-Dyson relation is that finding the averaged Green function $\mathcal{G}(w,z)$ is analogous to finding the self-energy $\Sigma(w,z)$. In fact, we can just express $\mathcal{G}(w,z)$ in terms of $\Sigma(w,z)$ by multiplying the last line of Eq. (11) by $G_0(w,z)^{-1}$ on the left and by $\mathcal{G}(w,z)^{-1}$ on the right, leading to

$$\mathcal{G}(w,z) = \frac{1}{G_0(w,z)^{-1} - \Sigma(w,z)} = \frac{1}{w - H_0(z) - \Sigma(w,z)}. \tag{12}$$

This expression can be calculated self-consistently. Diagram (10) represents the relation

$$\Sigma(w,z)^{\alpha\delta}_{ad} = \sum_{\beta,\gamma}\sum_{b,c}\langle \mathcal{J}^{\alpha\beta}_{ab}\mathcal{G}(w,z)^{\beta\gamma}_{bc}\mathcal{J}^{\gamma\delta}_{cd}\rangle_J,$$

which, after inserting the covariance (7) for our ensemble, becomes

$$\Sigma(w,z)^{\alpha\delta}_{ad} = \delta_{ad}\left(\pi^1_{\alpha\delta}\operatorname{tr}\mathcal{G}^{22}(w,z) + \pi^2_{\alpha\delta}\operatorname{tr}\mathcal{G}^{11}(w,z) + \eta\sigma^+_{\alpha\delta}\operatorname{tr}\mathcal{G}^{21}(w,z) + \eta\sigma^-_{\alpha\delta}\operatorname{tr}\mathcal{G}^{12}(w,z)\right).$$

In matrix form,

$$\Sigma(w,z) = \begin{pmatrix} -\mathrm{i}g_2\mathbf{1} & \eta g_{21}\mathbf{1} \\ \eta g_{12}\mathbf{1} & -\mathrm{i}g_1\mathbf{1}\end{pmatrix}, \tag{13}$$

where $\mathbf{1}$ is the $N\times N$ identity matrix and where we have introduced the scalar functions $g_1 \equiv \mathrm{i}\operatorname{tr}\mathcal{G}^{11}$, $g_2 \equiv \mathrm{i}\operatorname{tr}\mathcal{G}^{22}$, $g_{12} \equiv \operatorname{tr}\mathcal{G}^{12}$, and $g_{21} \equiv \operatorname{tr}\mathcal{G}^{21}$, which have to be determined self-consistently. To determine these functions, we rewrite Eq. (12) using the explicit form of Σ in Eq. (13), so that

$$\mathcal{G}(w,z) = \begin{pmatrix} w+\mathrm{i}g_2 & -z-\eta g_{21} \\ -\bar{z}-\eta g_{12} & w+\mathrm{i}g_1\end{pmatrix}^{-1} = \begin{pmatrix} (w+\mathrm{i}g_1)K^{-1} & (z+\eta g_{21})K^{-1} \\ (\bar{z}+\eta g_{12})K^{-1} & (w+\mathrm{i}g_2)K^{-1}\end{pmatrix} \tag{14}$$

with

$$K \equiv (z+\eta g_{21})(\bar{z}+\eta g_{12}) + (g_1 - \mathrm{i}w)(g_2 - \mathrm{i}w),$$

and use the definition of the scalar functions. We get

$$g_1(w,z) = (g_1 - \mathrm{i}w)\,\mathrm{tr}(K^{-1}), \qquad g_{12}(w,z) = -(z + \eta g_{21})\,\mathrm{tr}(K^{-1}), \tag{15}$$

$$g_2(w,z) = (g_2 - \mathrm{i}w)\,\mathrm{tr}(K^{-1}), \qquad g_{21}(w,z) = -(\bar{z} + \eta g_{12})\,\mathrm{tr}(K^{-1}). \tag{16}$$

Some leaps here. For $w = \mathrm{i}\epsilon$, with $\epsilon > 0$, relations (15)–(16) can be solved if

$$g_1(w,z) = g_2(w,z) \equiv g(w,z), \quad \text{and} \quad g_{21}(w,z) = \bar{g}_{12}(w,z) \equiv h(w,z),$$

with $g(w,z)$ and $h(w,z)$ obeying

$$g = \frac{g+\epsilon}{|z+\eta h|^2 + (g+\epsilon)^2}, \tag{17}$$

$$h = -\frac{\bar{z} + \eta\bar{h}}{|z+\eta h|^2 + (g+\epsilon)^2}. \tag{18}$$

We are looking for positive solutions of g. This is because g is the ensemble average of the normalized trace of $\mathrm{i}G_{11}(w,z;J)$, and the limit $w \to \mathrm{i}\epsilon$ of the matrix $\mathrm{i}G_{11}(\mathrm{i}\epsilon,z;J)$ is $\epsilon[A_z A_z^\dagger + \epsilon^2]^{-1}$, which for $\epsilon > 0$ is a positive-definite matrix and, therefore, has a positive trace.

Eq. (17) can be rearranged into a more convenient form defining $\gamma \equiv g + \epsilon$:

$$\epsilon = \gamma\left(1 - K(z,\gamma)\right), \tag{19}$$

where

$$K(z,\gamma) \equiv \frac{1}{|z+\eta h|^2 + \gamma^2}.$$

There are two possible ways for Eq. (19) to hold:

- One possibility is that $g = 0$ in the limit $\epsilon \to 0$ (i.e., the limit $\gamma \to 0^+$). In that case Eq. (19) can be satisfied provided that $\lim_{\gamma\to 0^+} K(z,\gamma) < 1$, that is, as long as

$$|z+\eta h|^2 > 1. \tag{20}$$

  Eq. (18) becomes in this case $h(z+\eta h) = -1$, which leads to $h = (-z + \sqrt{z^2 - 4\eta})/2\eta$ (we assume the square root includes the principal value and its negative), and the condition to satisfy, Eq. (20), reads

$$|z + \sqrt{z^2 - 4\eta}|^2 > 4.$$

  This inequality defines the region of the complex plane outside of an ellipse centered at the origin and major and minor axes $1+\eta$ and $1-\eta$, respectively (SHOW THAT). [the ellipse has focus at $c = \pm\sqrt{a^2 - b^2} = \pm 2\sqrt{\eta} < 1 + \eta$ and eccentricity $e = 2\sqrt{\eta}/(1+\eta)$]

- The other possibility is that g stays finite in the limit $\epsilon \to 0$. This requires $\lim_{\gamma\to g^+} K(z,\gamma) = 1^+$, which leads to the condition

$$1 = |z+\eta h|^2 + g^2.$$

  This condition has to be satisfied together with Eq. (18), which for this case takes the form

$$-\bar{z} = h + \eta\bar{h}. \tag{21}$$

  This relation can be solved for h

$$h = \frac{\eta z - \bar{z}}{1 - \eta^2},$$

which we can plug into Eq. (21) to isolate g:

$$g(z) = \frac{1}{1-\eta^2}\sqrt{(1-\eta^2)^2 - (1+\eta^2)z\bar{z} + \eta(z^2+\bar{z}^2)}.$$

Note, on the other hand, that $K(z,\gamma)$ is a decreasing function of γ and therefore, in this case, we should have $\lim_{\gamma\to 0^+} K(z,\gamma) \geq K(z,g) = 1$ or, equivalently,

$$|z+\eta h|^2 \leq 1. \tag{22}$$

This inequality defines the complementary of region given by (20). This solution is therefore expected to hold for all the z inside the ellipse defined above.

To summarize, the full Green function is

$$\mathcal{G} = \begin{pmatrix} -\mathrm{i}g(z) & \bar{h}(z) \\ h(z) & -\mathrm{i}g(z) \end{pmatrix} \otimes \mathbf{1}$$

where $g(z)$ and $h(z)$ are, if we denote by D the ellipse defined by Eq. (22),

$$g(z) = \begin{cases} \dfrac{1}{1-\eta^2}\sqrt{(1-\eta^2)^2 - (1+\eta^2)z\bar{z} + \eta(z^2+\bar{z}^2)} & \text{for } z \in D, \\ 0 & \text{for } z \notin D \end{cases} \tag{23}$$

$$h(z) = \begin{cases} (\eta z - \bar{z})/(1-\eta^2) & \text{for } z \in D, \\ (-z+\sqrt{z^2-4\eta})/2\eta & \text{for } z \notin D. \end{cases} \tag{24}$$

Once we have $h(z)$ we can evaluate the spectrum density using the limit (3),

$$\rho(z) = \frac{1}{\pi}\frac{\partial}{\partial\bar{z}}\left\langle \mathrm{tr}\,\frac{1}{z-J}\right\rangle_J = -\frac{1}{\pi}\frac{\partial}{\partial\bar{z}}h(z) = \begin{cases} [\pi(1-\eta^2)]^{-1} & \text{if } z \in D, \\ 0 & \text{if } z \notin D. \end{cases}$$

Two-point Green function

We need to compute

$$G(z_1,z_2) = \left\langle \mathrm{tr}\,\frac{1}{z_1-\mathbf{J}}\frac{1}{\bar{z}_2-\mathbf{J}^\dagger}\right\rangle. \tag{25}$$

To this end, we will proceed analogously as in the previous section: we will work with the extended Hermitian matrix H (Eq. (1)), define the associated Green function (Eq. (2)), compute the quantity

$$\left\langle G^{\alpha_1\beta_1}_{a_1b_1}(w_1,z_1;J)G^{\alpha_2\beta_2}_{a_2b_2}(w_2,z_2;J)\right\rangle_J,$$

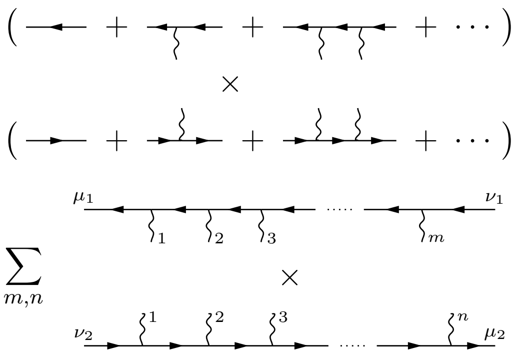
and then take an appropriate limit to extract the component containing $G(z_1,z_2)$, in Eq. (25). This object can be written in a slightly leaner way as

$$F_{\mu_1,\nu_2\,;\,\mu_2,\nu_1}(1;2) \equiv \left\langle G_{\mu_1\nu_1}(1;J)G_{\mu_2\nu_2}(2;J)\right\rangle_J, \tag{26}$$

where we introduced the shorthand notation for the arguments $(1) \equiv (w_1,z_1)$, $(2) \equiv (w_2,z_2)$, as well as for the generalized indices of the $2N$-dimensional space $\mu_i = (\alpha_i,a_i)$ and $\nu_i = (\beta_i,b_i)$. The apparently funny arrangement of the indices of F will prove useful later.

The product of Green functions inside the angle brackets can be represented with an extension of diagram (5):

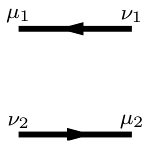
$$G_{\mu_1\nu_1}(w_1, z_1; J)\, G_{\mu_2\nu_2}(w_2, z_2; J) =$$



(27)

[...] If we average over the ensemble the series resulting from Eq. (27) and we apply the non-crossing approximation, we'll get a series of 'ladder diagrams' (Ahmadian et al, 2015). These digrams can be written as a sum of a *disconnected* and a *connected* terms, denoted by F^0 and F^D respectively. The disconnected sum is just

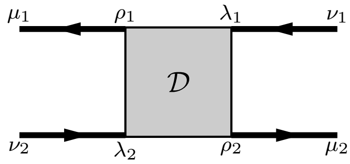
$$F^0_{\mu_1\nu_2\,;\,\mu_2\nu_1}(1;2) = \mathcal{G}_{\mu_1\nu_1}(1)\, \mathcal{G}_{\mu_2\nu_2}(2) =$$



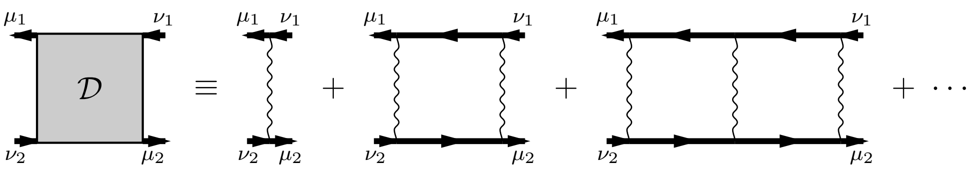
The connected sum F^D is the sum of the ladder diagrams where the two Green functions are connected by at least one wavy line:

$$F^D_{\mu_1,\nu_2\,;\,\mu_2,\nu_1}(1;2) = \sum_{\rho_1,\rho_2,\lambda_1,\lambda_2} \mathcal{G}_{\lambda_2\nu_2}(2)\, \mathcal{G}_{\mu_1\rho_1}(1)\, \mathcal{D}_{\rho_1\lambda_2\,;\,\rho_2\lambda_1}(1;2)\, \mathcal{G}_{\lambda_1\nu_1}(1)\, \mathcal{G}_{\mu_2\rho_2}(2)$$

$$=$$

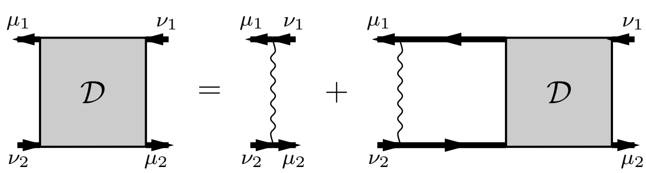


where we introduced a symbol for the sum of all amputated (i.e., without external legs) ladder diagrams:



(28)

This symbol represents the core contribution of the interaction and is usually called the *dressed vertex*, because it contains all higher-order corrections. The *bare* vertex is given by the first term in Eq. (28). As we did for the self-energy, it is possible to relate the dressed and the bare vertices through a recurrent equation. More specifically, the geometric series in (28) can be rexpressed as



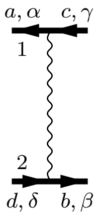
(29)

Adding the external legs gives rise to the relation obeyed by the averaged two-point Green function, Eq. (26). We discuss this relation in the next section.

Bethe-Salpeter equation

We use the following index convention for the bare vertex

$$\Gamma_{0\,ad;bc}^{\,\alpha\delta;\beta\gamma}(1;2)\Big| \equiv$$



$$= \left\langle \mathcal{J}_{ac}^{\alpha\gamma}\mathcal{J}_{bd}^{\beta\delta}\right\rangle = \frac{1}{N}\,\delta_{ad}\delta_{bc}\left[\sum_{r,t}\pi_{\alpha\delta}^{r}\,\sigma_{rt}^{1}\,\pi_{\gamma\beta}^{t} + \eta\sum_{s,u}\pi_{\alpha\beta}^{s}\sigma_{su}^{1}\pi_{\gamma\delta}^{u}\right]. \tag{30}$$

The Bethe-Salpeter equation, with Einstein summation convention and simplified indices, reads

$$F_{\mu_1\nu_2\,;\,\mu_2\nu_1}(1;2) = F^{0}_{\mu_1\nu_2\,;\,\mu_2\nu_1}(1;2) + F^{0}_{\mu_1\nu_2\,;\,\lambda_2\rho_1}(1;2)\Gamma_0^{\rho_1\lambda_2\,;\,\rho_2\lambda_1}F_{\lambda_1\rho_2\,;\,\mu_2\nu_1}(1;2) \tag{31}$$

This sum can be represented in matrix form if we use the following prescription.

$$F_{\mu_1,\nu_2\,;\,\mu_2,\nu_1}(1;2) \equiv \left\langle G(1;J)|_{\mu_1\nu_1} \otimes G^T(2;J)|_{\nu_2\mu_2}\right\rangle_J$$

Row indices are μ_1 and ν_2, and column indices are μ_2, ν_1. The subindex indicates the subspace of the tensor product.

$\mu_1\nu_2$ \ $\mu_2\nu_1$	11	21	12	22
11	$(11;11)$	$(11;21)$	$(12;11)$	$(12;21)$
12	$(11;12)$	$(11;22)$	$(12;12)$	$(12;22)$
21	$(21;11)$	$(21;21)$	$(22;11)$	$(22;21)$
22	$(21;12)$	$(21;22)$	$(22;12)$	$(22;22)$

where

$$(\alpha\beta;\gamma\delta) \equiv \mathrm{tr}\langle G^{\alpha\beta}(1)G^{\gamma\delta}(2)\rangle.$$

With this notation

$$\mathbf{\Gamma}_0 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & \eta & 0\\ 0 & \eta & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \qquad \mathbf{F}_0 = \begin{pmatrix} -g(z_1)g(z_2) & -\mathrm{i}g(z_1)h(z_2) & -\mathrm{i}\bar{h}(z_1)g(z_2) & \bar{h}(z_1)h(z_2)\\ -\mathrm{i}g(z_1)\bar{h}(z_2) & -g(z_1)g(z_2) & \bar{h}(z_1)\bar{h}(z_2) & -\mathrm{i}\bar{h}(z_1)g(z_2)\\ -\mathrm{i}h(z_1)g(z_2) & h(z_1)h(z_2) & -g(z_1)g(z_2) & -\mathrm{i}g(z_1)h(z_2)\\ h(z_1)\bar{h}(z_2) & -\mathrm{i}h(z_1)g(z_2) & -\mathrm{i}g(z_1)\bar{h}(z_2) & -g(z_1)g(z_2)\end{pmatrix}$$

and the relation (31) can be expressed as a matrix equation:

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0\mathbf{\Gamma}_0\mathbf{F} \quad \rightarrow \quad \mathbf{F} = (\mathbf{1} - \mathbf{F}_0\mathbf{\Gamma}_0)^{-1}\mathbf{F}_0.$$

Carrying out this matrix product and extracting the appropriate component we get our final Green function in Eq (25). For $z_1, z_2 \in D$, it is

$$F_{22\,;\,11}(1;2) = \frac{1-\eta^2 + (1+\eta^2)\bar{w}_1 w_2 - |w_1|^2 - |w_2|^2 + \eta\,(w_1w_2 + \bar{w}_1\bar{w}_2 - \bar{w}_1^2 - w_2^2)}{|\eta\,(w_1 - w_2) + \bar{w}_1 - \bar{w}_2|^2}, \tag{32}$$

where $w_i \equiv (z_i - \eta\bar{z}_i)/(1-\eta^2) = -\bar{h}_i$, see Eq. (24). For $z_1, z_2 \notin D$ we have instead

$$F_{22\,;\,11}(1;2) = \frac{h_1\bar{h}_2}{1 - h_1\bar{h}_2}, \quad \text{with} \quad h_i = \frac{-z_i + \sqrt{z_i^2 - 4\eta}}{2\eta}. \tag{33}$$

References

Ahmadian Y, Fumarola F, Miller KD (2015) Properties of networks with partially structured and partially random connectivity. Physical Review E 91:012,820, 10.1103/physreve.91.012820