RESEARCH STATEMENT

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My primary interests are in algebraic geometry. Some of my current and recent projects and interests include generating series for tautological sheaves on Hilbert Schemes, investigating a version of mirror symmetry for K3 surfaces, constructing a mirror map for non-abelian FJRW theories, and investigating derived categories using matrix factorizations. These are described briefly below.

Generating Series for Hilbert Schemes. Let S be a smooth projective surface over $\mathbb{C}$ (or in analytical terms, a compact complex manifold of complex dimension 2). The Hilbert scheme of n points $S^{[n]}$ is a smooth variety of dimension $2n$ that parameterizes unordered sets of n points on S. When the points collide the Hilbert scheme also keeps track of some non-reduced structure. Given a vector bundle F of rank r on S, one obtains the so called tautological vector bundle $F^{[n]}$ of rank rn on $S^{[n]}$ via $F^{[n]} := q_*(p^*F)$, where p and q are the projections from the universal subscheme $Z \subset S \times S^{[n]}$. More colloquially, one can say that the fiber of $F^{[n]}$ over a point p of $S^{[n]}$ is the direct sum of the fibers of F over the points parameterized by p.

One might like to study numerical invariants of these tautological bundles. One such invariant is the top Chern class $c_{2n}(F^{[n]})$. Geometrically, this counts the (finite) number of points where an appropriate number of general sections of $F^{[n]}$ drop rank. Another such invariant is the holomorphic Euler characteristic of the determinant, defined as $\chi(\det F^{[n]}) := \sum_{i=0}^{2n} \dim H^i(S^{[n]}, \det F^{[n]})$, where the H^i are the sheaf cohomology groups. Via the Hirzebruch-Riemann-Roch formula, this can be computed using Chern numbers of $F^{[n]}$ and $S^{[n]}$.

A result of Ellingsrud, Göttche, and Lehn [EGL01] says that these invariants are polynomials in the Chern numbers of F and S. Further, when they are organized into generating series, even more structure is revealed. For a vector bundle V of rank r

$$\sum_{n=1}^{\infty} \chi\left(\det V^{[n]}\right) z^n = g_r(z)^{\chi(\det V)} \cdot f_r(z)^{\frac{1}{2}\chi(\mathcal{O}_S)} \cdot A_r(z)^{c_1(V).K_S - \frac{1}{2}K_S^2} \cdot B_r(z)^{K_S^2}.$$

Here K_S is the canonical divisor of S, and g_r, f_r, A_r, and B_r are power series in z whose coefficients are polynomials in r (the rank of V). We also have, for F a vector bundle of rank s,

$$\sum_{n=0}^{\infty} c_{2n}(F^{[n]}) w^n = V_s(w)^{c_2(F)} \cdot W_s(w)^{\chi(c_1(F))} \cdot X_s(w)^{\frac{1}{2}\chi(S)} \cdot Y_s(w)^{K_S \cdot c_1(F) - \frac{1}{2}K^2} \cdot Z_s(w)^{K_S^2}$$

The series g_r and f_r were given explicitly in [EGL01], but the remaining series were more mysterious. Lehn had conjectured formulas for them in rank 1 case [Leh99].

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In [Joh20], I proved that certain numerical predictions related to a "strange duality" conjecture were equivalent to the following relationship between the two sets of series.

Conjecture 1. *Define a change of variables between z and w via $w = zV_s(w)^{2-s}$, and also let $s = r + 1$. Then we have the following:*

$$g_r(z) = V_s(w) \cdot W_s(w) \tag{C1}$$

$$f_r(z) = \frac{X_s(w)}{V_s(w)^{4(2-s)} \left(\frac{dz}{dw}\right)^2} \tag{C2}$$

$$A_r(z) = Y_s(w) \tag{C3}$$

$$B_r(z) = Z_s(w) \tag{C4}$$

I also supported this conjecture with some special cases and computational data. Inspired by this, as well as by recent work of Voisin [Voi19], the authors Marian, Oprea, and Pandharipande gave formulas for V, W, and X [MOP19] [MOP21], from which one can verify C1 and C2, as well as Lehn's conjecture.

Although the relationships C3 and C4 have a simpler form, the series involved are the more mysterious ones. I am very interested in continuing to understand these. One can, for example, restrict to the case of trivial bundles. Proving the conjecture in this case is still open and would resolve "half" of the remaining question. In this case, one can write out the localization formulas involved somewhat explicitly. One obtains many interesting relationships, including some that can be related to the qt-Catalan numbers. Although some of the relationships suggested by this approach can be proven independently of the conjecture, proof of the conjecture itself remains elusive.

0.0.1. *Related Work.* My collaborations with Oprea, Pandharipande, Lim, and Arbesfeld in [JOP21] and [AJL$^+$21] study questions about rationality of generating series for torsion quotients of trivial bundles. Although these results do not seem to directly apply to Conjecture 1, there is certainly some relationship. For example, [AJL$^+$21] includes a Segre/Verlinde correspondence that seems to mirror Conjecture 1. I was able to prove a more general version of this correspondence, and I hope to be able to understand how that could lead to a more general version of Conjecture 1.

Arkadij Bojko has recently done work that involves a Segre/Verlinde correspondence, this time for Calabi Yau 4 folds [Boj21]. In communicating with him, it appears that a similar generalization applies to his work as well. It would be interesting to investigate this further.

Landau-Ginzburg Mirror Symmetry. My masters thesis at BYU involved proving part of a mirror symmetry theorem for FJRW theory. Starting with a quasihomogeneous polynomial W and a group of diagonal symmetries G (satisfying certain properties), one can construct an *A-model* theory, given by the FJWR construction [FJR13]. At its most basic, the A-model is a graded vector space, but in fact it can be extended to the much richer structure of a cohomological field theory. There is also a B-model theory that also takes as input a polynomial and group, satisfying slightly different conditions. A certain construction produces a "transposed" polynomial and group W^T and G^T. The assertion of mirror symmetry here is that the A-model of (W, G) should be isomorphic to the B-model of (W^T, G^T). My thesis proved this isomorphism at the level of Frobenius algebras for a large class of polynomials [Joh11], see also [FJJS12].

The adjective "diagonal" describing the group G above means that it acts on the underlying vector space as a diagonal matrix, or, equivalently, it acts on the polynomial by scaling each variable independently: $g.x_i = \lambda_{g,i} \cdot x_i$. More recent work suggests that versions of these constructions and mirror symmetric theorems should exist for a wider class of groups. I am currently working with a group at BYU, led by Nathan Priddis, which is constructing the mirror isomorphism for the A and B model vector spaces when the group G includes elements that act by permuting the variables. It appears that this will be an interesting and productive direction.

BHK Mirror Symmetry for K3 Surfaces. As a special case of the setup in the previous section, suppose that one has a quasihomogeneous polynomial W in 4 variables and a group of diagonal symmetries. The vanishing locus of W defines a (usually singular) surface in weighted projective space. One can further quotient this by the group G. Under appropriate numerical conditions, and after resolving singularities, the resulting surface will be a K3 surface S.

A K3 surface S ("named in honor of Kummer, Kähler, Kodaira and of the beautiful mountain K2 in Kashmir"[1]) is a smooth projective surface with trivial canonical bundle that is not an abelian surface. The cohomology group $H^2(S, \mathbb{Z})$ has the structure of an even unimodular lattice. This lattice is always isomorphic to a fixed "K3 lattice" L_{K3} of rank 22. The Picard group $\mathrm{Pic}(S)$ of S is the sublattice of $H^2(S, \mathbb{Z})$ generated by classes that can be represented by algebraic curves in S. The Picard group plays an important role in the study of K3 surfaces.

One can do the same thing for the transposed polynomial and group W^T, G^T (the transpose here is as in the previous section), again obtaining another K3 surface S^T. These are called in [CP18] a BHK (Bergland-Hübsch-Krawitz) mirror pair.

Another version of mirror symmetry for K3 surfaces, known as *lattice polarized* K3 mirror symmetry, is due to Dolgachev [Dol96]. An M-polarizable K3 surface is a pair (S, M) where S is a K3 surface and M is a lattice that can be embedded primitively in $\mathrm{Pic}(S)$. The pairs (S, M) and (S', M') are said to be a lattice polarized mirror pair if $M^\perp = M' \oplus U$, where the orthogonal complement is taken in L_{K3}, and U is the rank 2 hyperbolic lattice.

One can ask whether a BHK mirror pair can yield a lattice polarized mirror pair. In works such as [CP18], [CLPS14], and [ABS14] it is proposed (and proved in several cases), that if one takes the M to be the invariant lattice of a certain automorphism σ, then this is indeed true. In other words, $(S, \mathrm{Fix}_\sigma(\mathrm{Pic}\, S))$ and $(S^T, \mathrm{Fix}_{\sigma^T}(\mathrm{Pic}\, S^T))$ form a lattice polarized mirror pair. This leads to efforts to understand and compute the invariant lattices of $K3$ surfaces for these types of automorphisms. Currently, I am working on such a project with Nathan Priddis and Joshua Fullwood.

Matrix Factorizations and the Godeaux surface. The derived category of a variety is an important invariant that has received much attention since its introduction by Grothendieck and Verdier. In some cases, the derived category is

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well understood, while other cases remain more mysterious. The nicest situation is when the derived category has a *full exceptional collection*, that is, it can be built up out of a finite list of relatively simple objects. This is the case, for example, with the projective space $\mathbb{P}^n$.

In [BvBS13], the author investigates the derived category of the Godeaux surface X: the surface given by the quotient by an action of a group G of order 5 of the surface defined by $x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$ in $\mathbb{P}^3$. There, they find an exceptional collection of length 11 that is maximal (it can not be extended to a longer one). However, it is not full—there are some objects not generated by this collection. Such objects are sometimes known as phantoms.

A theorem of Orlov [Orl09] relates the derived category of a variety to the *singularity category*. The singularity category is in turn related to the category of *matrix factorizations* (see [PV14]). In a joint project with Nathan Priddis and Andreas Hochenegger, we hope to study the derived category of X by studying G-equivariant matrix factorizations of the polynomial $x_1^5+x_2^5+x_3^5+x_4^5$. These matrix factorizations have some computational advantages compared to the derived category, which will perhaps allow us to shine a light on some of these phantoms. However, the relationships involved are very subtle. If successful, this project will not only give us information about this specific example, but clarify some of the issues involved in moving between these various categories.

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