RESEARCH STATEMENT

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I am interested in algebraic geometry. Some of my interests are moduli spaces of curves and sheaves, strange duality for algebraic surfaces, tropical geometry, and FJRW theory and mirror symmetry.

1. Moduli Spaces of Non-Special Curves

An important theme in algebraic geometry is the construction and study of moduli spaces. One attempts to construct a space parameterizing some geometric objects (such as algebraic curves or vector bundles on a variety). Studying the geometry of this space often gives insight into the objects being parametrized.

Some of my recent work (joint with Alexander Polishchuk) involves moduli spaces parameterizing algebraic curves with non-special divisors. More precisely, a point of the space $\widetilde{\mathcal{U}}_{g,g}$ parameterizes a (possibly singular, but reduced) genus g algebraic curve C, together with g distinct points $p_1, \dots, p_g$ on the smooth locus of C, and a choice of tangent vector at each of the chosen points. The non-special condition says that the divisor $p_1 + \cdots + p_g$ moves in linear equivalence no more than expected, which in this case is not at all.

One nice feature of the scheme $\widetilde{\mathcal{U}}_{g,g}$ is that it is actually an affine variety. Recall that an affine variety is one that can be described as a subset of $\mathbb{C}^n$ by the vanishing of finitely many equations.

In detail, one proceeds as follows (see [Pol13]). The space $H^0(C, p_1 + \cdots + 2p_i + \cdots p_g)$ is two dimensional. Thus, there is a function f_i, unique up to an additive constant, whose expansion at p_i begins like $t^{-2} + \dots$, where t is a parameter determined up to first order by the choice of tangent vector at p_i. Next, the space $H^0(C, p_1 + \cdots + 3p_i + \cdots + p_g)$ is three dimensional. One can choose a function h_i in this space with expansion beginning like $t^{-3} + \cdots$, uniquely up to adding a constant and a multiple of f_i. One can make these choices unique by imposing some condition, for example, one can require that $f_i^3 - h_i^2$ has a pole of order no more that 2 at p_i.

Now, consider the product $f_i f_j$ for $i \neq j$. This lies in $H^0(C, p_1 + \cdots + 2p_i + \cdots + 2p_j + \cdots p_g)$, which has dimension $g+3$. A basis for this space is given by $1, h_i, h_j, f_1, \dots, f_g$. Hence we see that there must be an equation of the form

$$f_i f_j = \alpha_{ij} h_j + \alpha_{ji} h_i + \sum c_{ij}^k f_i + a_{ij}$$

(where the new letters introduced are constants). Similarly, one rewrites $f_i h_j$, $h_i h_j$, and h_i^2 using the basis $1, f_i^n, h_i f_i^n$ (for $i = 1, \dots, g; n \geq 0$). One now views the coefficients appearing in these equations as functions on $\widetilde{\mathcal{U}}_{g,g}$. In fact, these give affine coordinates for $\widetilde{\mathcal{U}}_{g,g}$ (see [Pol13]). One can find the relations among these coordinates (or in other words, the equations cutting out your variety) by using the Buchberger algorithm (the algorithm more commonly used to compute Grobner bases).

Once this affine scheme has been constructed, one wants to remove the choice of tangent vectors by quotienting out by the action of $(\mathbb{C}^\times)^n$. This requires a choice of a character of $(\mathbb{C}^\times)^n$. Different choices give different birational models of the moduli space of smooth marked curves $\mathcal{M}_{2,2}$.

In our recent paper [JP18], we studied the different chambers in the character lattice for the case of $g = 2$ and gave geometric characterizations of the stable curves in each space. We also used the explicit description in coordinates to describe a map which shows that one of these moduli spaces is obtained by blowing down the Weierstrass locus (that is, the locus where $h^1(C, p_1 + p_2) > 0$, or in other words, the locus of special divisors) in Smyth's $\mathcal{Z}$-stable compactification of $\mathcal{M}_{2,2}$.

1.1. **More about non-special curves.** The construction above can be generalized to the case where the number of marked points is greater that the genus. In this case, the resulting scheme $\tilde{\mathcal{U}}_{g,n}$, is not affine, but it is affine over a Grassmannian.

The map to a Grassmannian can be described nicely. Let $D = p_1 + \cdots + p_n$ be the divisor associated to the marked points, and $\mathcal{O}_D$ be its structure sheaf. Recall that there is a standard exact sequence of sheaves

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D \to 0$$

that gives rise to a exact sequence of vector spaces:

$$H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(D))$$

For a non-special curve the space $H^1(\mathcal{O}_C(D))$ is assumed to be 0. Furthermore, the choice of tangent vectors at each point is equivalent to the choice of an isomorphism between $H^0(\mathcal{O}_D)$ and a fixed $\mathbb{C}^n$. Hence, we obtain a map $\tilde{\mathcal{U}}_{g,n} \to \mathrm{Gr}(g, n)$ taking a curve with its points and tangent vectors to the quotient $\mathbb{C}^n \cong H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_C)$.

In [Pol16], Polishchuk considers the case of genus 1 and shows that $\tilde{\mathcal{U}}_{1,n}$ is not only affine over a projective scheme as above, but also projective over an affine scheme. It would be interesting to extend this result to genus 2 or higher. I have written some code to do the related computations. The situation seems more complicated than the genus one case, but I hope that the computations will give some insight into the structure at play and yield some some useful results.

Another interesting aspect of these moduli spaces is that they are also moduli spaces for minimal A^∞ structures on a certain graded associative algebra E_g (see [Pol13]). An A^∞ structure on an algebra R is a series of "multiplications" $m_1, m_2, m_3, \ldots$, where each m_i takes i inputs from R and gives one output. These multiplications are required to satisfy certain compatibility conditions. This is something I would be interested in studying more.

2. Le Potier's Strange Duality for Surfaces

2.1. **The Conjecture.** Another important example of a moduli space in the moduli space of coherent sheaves on a fixed variety. One usually fixes discrete invariants, such as the Chern character.

Suppose one is given two chern characters, e and f, on a surface S with corresponding moduli spaces $\mathcal{M}(e)$ and $\mathcal{M}(f)$. If we impose a certain orthogonality condition on e and f, one then expects that for general choices of $E \in \mathcal{M}(e)$ and $F \in \mathcal{M}(f)$, the tensor product $E \otimes F$ will have no sections. However, in nice situations, there will be a divisor (codimension 1 subvariety) $\Theta_F \subset \mathcal{M}(e)$ consisting of points parameterizing sheaves E with $H^0(E \otimes F) > 0$, and symmetrically we have $\Theta_E \subset \mathcal{M}(f)$. Recall that in algebraic geometry, a divisor determines a line bundle. We focus on the case of del Pezzo surfaces, where the divisors Θ_E are linearly equivalent as E varies (that is, they all determine the same line bundle, which, by a slight abuse, we also call Θ_E). The strange duality conjecture asserts that there is an induced a perfect pairing, that is:

$$H^0(\mathcal{M}(f), \Theta_E) \cong H^0(\mathcal{M}(e), \Theta_F)^\vee \tag{1}$$

This conjecture is motivated by an analogous statements on curves (for a proof see [MO07] or [Pau14], note that in the curve case the statement is complicated by that fact that the Θ_E are not linearly equivalent) and also by some simple cases. For example, when both chern characters have rank 1, the statement boils down to a duality $\bigwedge^n H^0(S, L) \cong \bigwedge^m H^0(S, L)^\vee$ where $n + m = \dim H^0(S, L)$ for some line bundle L on the surface (see [MO08]).

2.2. **The finite Quot scheme method.** Marian and Oprea, in [MO07], used the following strategy for the proof of strange duality for curves. One sets $v = e^\vee + f$ and constructs a vector bundle V with $\mathrm{ch}(V) = v$. The orthogonality condition implies that we expect only finitely many short exact sequences

$$0 \to E^\vee \to V \to F \to 0 \tag{2}$$

with $\mathrm{ch}(E^\vee) = e^\vee$ and $\mathrm{ch}(F) = f$. The sequences are parameterized by the Quot scheme $\mathrm{Quot}(V, f)$. Each such short exact sequence produces sections $s_F \in H^0(\mathcal{M}(e), \Theta_F)$ and $s_E \in H^0(\mathcal{M}(f), \Theta_E)$ (corresponding to divisors Θ_F and Θ_E, respectively). One can check that the sections produced this way are independent and that the strange duality pairing resticted to the subspaces generated by these sections is a perfect pairing. The Verlinde formula (in the case of curves) allows one to compute the dimension of both sides of (1), and one can use a technique of Bertram [Ber94] studying intersection theory on a larger Quot scheme to count the number of quotients (2). These numbers match, which completes the proof.

2.3. **Hilbert schemes, universal series, and multiple point formulas.** For surfaces, a general analogue of the Verlinde formula does not seem to be available. However, if we restrict ourselves to the case where f is the chern character of an ideal sheaf of n points, we obtain that $\mathcal{M}(f) \cong S^{[n]}$, the Hilbert scheme parameterizing sets of n unordered points on the surfaces. We still don't have a general closed formula for the Euler characteristics of line bundles on $S^{[n]}$, however, a theorem of Ellingsrud, Göttsche, and Lehn [EGL01] gives insight into the structure of these numbers, and provides an effective method for their computation. There is a map $\mathrm{Pic}(S) \to \mathrm{Pic}(S^{[n]}) : L \mapsto L_n$. We also have E, the line bundle associated to the exceptional divisor of the map from the Hilbert scheme to the symmetric product. We then have for any surface S,

$$\sum_{n=1}^{\infty} \chi\left(L_n \otimes E^{\otimes r}\right) z^n = g_r(z)^{\chi(L)} \cdot f_r(z)^{\frac{1}{2}\chi(\mathcal{O}_S)} \cdot A_r(z)^{L.K_S - \frac{1}{2}K_S^2} \cdot B_r(z)^{K_S^2}, \tag{3}$$

where $A_r(Z)$, $B_r(Z)$, $f_r(z)$, $g_r(z)$ are power series in z depending only on r (not on S or L). The series f_r and g_r are determined explicitly in [EGL01]. The series A_r and B_r can be determined for small n by localization computations on $\mathbb{P}^2$ (see [EGL01], [ES96], [ES87], and [Joh16]).

We wanted to compare these Euler characteristics to the number of sequences (2). To compute the latter, Aaron Bertram, Thomas Goller, and I in [BGJ16] developed a technique involving multiple points formulas. We reinterpret the problem of counting maps from V to an ideal sheaf of n points as constructing a certain projective bundle over S and a map of it into a projective space and then counting its n-fold points (the points in the target that have n preimages).

There is a theory of computing the number of multiple points of an admissible map using the invariants of the spaces involved. Computing the number (in our case, as a function of e and the invariants of S for some fixed n) requires techniques from [MR10] and [BS12] and a non-trivial amount of computer computation. When the dust clears, the answer matches the (also non-trivial) computation of $\chi(S^{[n]}, \Theta_E)$ exactly for $n \leq 7$. At $n = 8$, the computation becomes unfeasible, and at the same time the proof of the multiple point formula in [MR10] breaks down and becomes conjectural. The main result of [BGJ16] is:

Theorem 1. *Let S be a del Pezzo surface. Let f be the chern character of an ideal sheaf of n points with $n \leq 7$. Let e satisfy $\chi(e \otimes f) = 0$, have rank at least two, and have $c_1(e)$ sufficiently ample. Then the Euler characteristic $\chi(S^{[n]}, \Theta_E)$ is equal to the expected number of exact sequences (2).*

From this, under some additional assumptions, one can deduce that the strange duality gives a perfect pairing of $H^0(\mathcal{M}(f), \Theta_E)$ with a subspace of $H^0(\mathcal{M}(e), \Theta_F)$. In a few cases, we can compute the dimension of the latter to obtain the full isomorphism (1).

2.4. **More Universal Series.** The multiple point methods have some limitations. There is another way to interpret the expected length of the finite Quot scheme as the top codimensional Chern class of the so called *tautological bundle* $(V^*)^{[n]}$ on $S^{[n]}$ associated to V^* (the dual of the bundle in (2)). One can compute these using Nakajima's $\mathfrak{q}$-operators, as shown in Lehn's paper [Leh99]. I derived an algorithm from [Leh99] and wrote some code to compute these. Although the computation still becomes unfeasible for larger n, an advantage of this approach is that the Chern numbers

$c_{2n}((V^*)^{[n]})$ also satisfy universal properties. We have (see [EGL01], Theorem 4.2) for any surface S and any sheaf F (or more generally, any class $F \in K(S)$) of rank s:

$$\sum_{n=0}^{\infty} c_{2n}(F^{[n]})w^n = V_s(w)^{c_2(F)} \cdot W_s(w)^{\chi(c_1(F))} \cdot X_s(w)^{\frac{1}{2}\chi(S)} \cdot Y_s(w)^{K_S \cdot c_1(F) - \frac{1}{2}K^2} \cdot Z_s(w)^{K_S^2}$$

where $V_s(w), W_s(w), X_s(w), Y_s(w), Z_s(w)$ are power series in w depending only on s.

The finite Quot scheme method suggests then that there should be some interesting relationship between these series and the series for the Euler characteristics of line bundles (3). In [Joh18], I worked this expected relationship out:

Conjecture 2. *Define a change of variables between z and w via $w = zV_s(w)^{2-s}$, and also let $s = r + 1$. Then we have the following:*

$$g_r(z) = V_s(w) \cdot W_s(w) \tag{C1}$$

$$f_r(z) = \frac{X_s(w)}{V_s(w)^{4(2-s)} \left(\frac{dz}{dw}\right)^2} \tag{C2}$$

$$A_r(z) = Y_s(w) \tag{C3}$$

$$B_r(z) = Z_s(w) \tag{C4}$$

I verified this computationally up to order 6, and I also checked that it was consistent with other known facts such as the computation of the top Segre class of line bundles on K-trivial surfaces by Marian, Oprea, and Pandharipande [MOP15]. Inspired by my data and analysis in [Joh18], those three authors found and proved closed formulas for the series $V_s(w)$, $W_s(w)$, and $X_s(w)$ [MOP17a], [MOP17b]. With a little extra work, one can use these formulas to prove (C1) and (C2). Proofs of (C3) and (C4) are still elusive.

Additionally, my paper [Joh18] used some interesting combinatorial methods to prove a new result predicted by Conjecture 2.

Theorem 3. *For any smooth projective surface S*

$$c_{2n}(\mathcal{O}_p^{[n]}) = (-1)^n C_n$$

where $\mathcal{O}_p^{[n]}$ is the tautological sheaf on $S^{[n]}$ associated to $\mathcal{O}_p$, the structure sheaf of a point, and C_n is the nth Catalan number.

2.4.1. *Bridgeland stability and representations.* In the discussion above, we have not mentioned the choice of stability condition, which can give different birational models of the moduli space. Bayer and Macri showed in [BM14] that one can use the chern character e to define a Bridgeland stability condition so that the bundle Θ_E is ample on the resulting moduli space $\mathcal{M}(f)$. We view this as a preferred model on which to consider strange duality. Certain Bridgeland moduli spaces have a more tractable structure, e.g. they may be built from Grassmannians or Kronecker spaces (quotients of Grassmannians). This gives us some interesting examples where both sides of (1) can be computed.

A slightly stronger version of the strange duality conjecture asserts that the spaces of sections in question are dual not only as vector spaces, but as representations of $\mathrm{Aut}(S)$. (An autormorphism of S gives an automorphism of the moduli space (by pulling back sheaves), which the gives an automorphism of line bundles on the moduli space.) In some special cases, we can compute these representations using the Bridgeland moduli spaces. My thesis [Joh16] has some interesting examples. I would like to find more examples and see how generally I can make this work. A recent paper of Yuan [Yua18] has some examples that may be interesting to investigate further in this context.

3. Some Other Interests

3.1. **FJRW theory and Mirror Symmetry.** For my master's thesis I did work with Tyler Jarvis at BYU in FJRW theory. FJRW theory produces an "A-model" cohomological field theory corresponding to a singularity defined by a quasi-homogeneous polynomial W and a subgroup G of the symmetries of the polynomial [FJR13]. On can form a "transpose" polynomial W^T and group G^T and then produce an orbifold Milnor ring that is the "B-model" [Kra10, Kau06, IV90]. The orbifold Milnor ring has the structure of a bi-graded Frobenius algebra, i.e. it is a finite dimensional bi-graded $\mathbb{C}$-algebra equipped with a pairing that is compatible with the multiplication. Restricting the FJRW theory to genus 0 three point invariants also gives a Frobenius algebra. A mirror symmetry conjecture predicts that these are isomorphic. Mark Krawitz proved that these are isomorphic as bi-graded vector spaces [Kra10]. I developed some new techniques to prove the algebra isomorphism for a large class of polynomials W in [Joh11, JFJS12]:

Theorem 4. *Suppose W is a sum of loop or Fermat type polynomials and G is an admissible group of symmetries. Then the FJRW Frobenius algebra is isomorphic to the orbifold Milnor ring of W^T, G^T.*

This is an area I would like to revisit in the future,

3.2. **Code.** I enjoy the computational aspects of mathematics. I most often work with Sage, an open source mathematics project based on Python. I wrote code to do the computations in strange duality described above. I also wrote code to do computations in FJRW theory and on the moduli space of curves that has been used by other people. More recently, I wrote some code to compute any product of tautological classes on the moduli space of curves. I have packaged some of this code and uploaded it to the PyPI repository to allow for one command installation. See `https://pypi.python.org/pypi/mgn`.

3.3. **Tropical Geometry.** While at the University of Utah, I did some work in tropical geometry, producing the paper [Joh15] which gives conditions under which incidence conditions unique determine a tropical hypersurface. I would be interested in revisiting tropical geometry in the future. Because of its visual nature and low prerequisites, it may make a good topic for undergraduate research.

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