RESEARCH STATEMENT

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I am interested in algebraic geometry. Some of my recent interests are moduli spaces of sheaves, strange duality for algebraic surfaces, Bridgeland stability, tropical geometry, and mirror symmetry.

1. Some Background: Moduli Spaces of Sheaves

An important theme in algebraic geometry is the construction and study of moduli spaces. One attempts to construct a space parameterizing some geometric objects (e.g. algebraic curves, vector bundles on a variety). Studying the geometry of this space often gives insight into the objects being parametrized.

Important examples of moduli spaces are moduli spaces of vector bundles on a variety X. First, one restricts to a smaller problem by considering only vector bundles with fixed discrete invariants. These invariants (including, e.g., the rank of the vector bundle) are contained in the chern character. The chern character of a vector bundle F is denoted by $\mathrm{ch}(F)$. The set of vector bundles with fixed chern character is still too large to be parametrized by a scheme of finite type, but by fixing an embedding of X into projective space, one can define a subset of the vector bundles called semistable vector bundles on X. One should also include some torsion free sheaves that are limits of semistable vector bundles. There is a well behaved moduli space for semistable sheaves, and furthermore, every vector bundle has a unique decomposition into (more precisely: has a filtration by) semistable sheaves. Let $\mathcal{M}(f)$ be the moduli space of semistable sheaves on X with chern character f.

One important property of a vector bundle F is its space of global sections, $H^0(X, F)$. One would like to have a formula for the dimension of this space in terms of the discrete invariants of the sheaf. Unfortunately this is not possible: two vector bundles with the same chern character can have spaces of global sections with different sizes. However, using sheaf cohomology, one can define the Euler characteristic as $\chi(F) := \sum_i (-1)^i \dim H^i(X, F)$. The Euler characteristic $\chi(F)$ is much better behaved than the number $\dim H^0(X, F)$. Not only is it additive in short exact sequences, but it can be computed from the chern character of a sheaf by the well known Hirzebruch-Riemann-Roch theorem. In some situations, one can expect that the higher cohomology of a sheaf vanishes so that the Euler characteristic gives you $\dim H^0(X, F)$.

2. Le Potier's Strange Duality for Surfaces

Suppose one is given two chern characters, e and f which are orthogonal in the sense that $\chi(E \otimes F) = 0$ for some (and hence any) sheaves with $\mathrm{ch}(E) = e$ and $\mathrm{ch}(F) = f$. One then expects that for general choices of E and F, the tensor product will have no cohomology at all (in particular, no sections). However, in nice situations, there will be a divisor (codimension 1 subvariety) $\Theta \subset \mathcal{M}(e) \times \mathcal{M}(f)$ consisting of points parameterizing sheaves (E, F) with $H^0(E \otimes F) > 0$. Fixing a point $F \in \mathcal{M}(f)$, one can also restrict this divisor to $\mathcal{M}(e) \times \{[F]\}$, obtaining a divisor Θ_F on $\mathcal{M}(e)$ consisting of points parameterizing sheaves E such that $H^0(E \otimes F) > 0$. Reversing the roles of e and f one obtains a divisor Θ_E on $\mathcal{M}(f)$. Recall that in algebraic geometry, a divisor determines a line bundle. We focus on the case of del Pezzo surfaces, where the divisors Θ_E are linearly equivalent as E varies (that is, they all determine the same line bundle, which, by a slight abuse, we also call Θ_E). The divisor Θ induces a pairing between the spaces of sections of the two line bundles Θ_F and Θ_E. The strange duality conjecture asserts that this is a perfect pairing, that is:

$$H^0(\mathcal{M}(f), \Theta_E) \cong H^0(\mathcal{M}(e), \Theta_F)^\vee \tag{1}$$

This conjecture is motivated by an analogous statements on curves (for a proof see [25] or [30], note that in the curve case the statement is complicated by that fact that the Θ_E are not linearly equivalent) and also by some simple cases. For example, when both chern characters have rank 1, the statement boils down to a duality $\bigwedge^n H^0(L) \cong \bigwedge^m H^0(L)^\vee$ where $n+m = \dim H^0(L)$ for some line bundle L on the surface (see [27]).

Marian and Oprea, in [25], used the following strategy for the proof of strange duality for curves. One sets $v = e^\vee + f$ and constructs a vector bundle V with $\mathrm{ch}(V) = v$. The condition $\chi(E \otimes F) = 0$ implies that we expect only finitely many short exact sequences

$$0 \to E^\vee \to V \to F \to 0 \tag{2}$$

with $\mathrm{ch}(E^\vee) = e^\vee$ and $\mathrm{ch}(F) = f$. Each such short exact sequence produces sections $s_F \in H^0(\mathcal{M}(e), \Theta_F)$ and $s_E \in H^0(\mathcal{M}(f), \Theta_E)$. One can check that the sections produced this way are independent and form candidate dual bases. The Verlinde formula (in the case of curves) allows one to compute the dimension of both sides of (1), and one can use a technique of Bertram [4] studying intersection theory on an appropriate Quot scheme to count the number of quotients (2) and conclude the proof.

For surfaces, a general analogue of the Verlinde formula does not seem to be available. However, inspired by the method of finitely many short exact sequences above, Aaron Bertram, Thomas Goller, and myself developed a novel technique to make computations that provide evidence for this conjecture. We prove (paper in preparation):

Theorem 1. *Let S be a del Pezzo surface. Let f be the chern character of an ideal sheaf of n points with $n \leq 7$. Let e satisfy $\chi(e \otimes f) = 0$. Suppose there exists a V with $\mathrm{ch}(V) = v := e^\vee + f$ so that $V^\vee$ is globally generated and has vanishing higher cohomology. Then the Euler characteristic $\chi(\Theta_E)$ is equal to the expected number of exact sequences* (2).

From this, under some additional assumptions, one can deduce that the strange duality gives a perfect pairing of $H^0(\mathcal{M}(f), \Theta_E)$ with a subspace of $H^0(\mathcal{M}(e), \Theta_F)$. In a few cases, we can compute the dimension of the latter to obtain the full isomorphism (1).

Our result is made tractable because f is assumed to be the chern character of an ideal sheaf of n points, so that $\mathcal{M}(f)$ is actually the Hilbert scheme of n points on S. For a fixed n, it is possible to obtain a formula in terms of e and the invariants of S for $\chi(\Theta_E)$ by using the factorization of the generating function given in [9]. One also needs to use localization techniques from [11] and [10] to compute some of the factors.

To obtain a count of the expected number of short exact sequences, we used multiple point formulas. We reinterpret the problem of counting maps from V to an ideal sheaf of n points as constructing a certain projective bundle over S and an immersion of it into a projective space and then counting its n-fold points (the points in the target that have n preimages).

There is a theory of computing the number of multiple points of an admissible map using the invariants of the spaces involved. Computing the number (in our case, as a function of e and the invariants of S for some fixed n) requires techniques from [24] and [7] and a non-trivial amount of computer computation. When the dust clears, the answer matches the (also non-trivial) computation of $\chi(\Theta_E)$ exactly for $n \leq 7$. At $n = 8$, the computation becomes unfeasible, and at the same time the proof of the multiple point formula in [24] breaks down and becomes conjectural.

One challenge is to show that the expected number of exact sequences (2) is correct. This should follow if one can show that there are finitely many. Although we expect that a general V has this property, there is some difficultly in even finding one such V! For a general del Pezzo this still remains elusive, but we were able to find some infinite families of choices of e on $\mathbb{P}^2$ where this can be checked rigorously.

Other progress in strange duality for surfaces includes results for K3 surfaces [28], abelian surfaces [5, 26] and for some 1 dimension sheaves [8, 33]. Ours result seems to be the first non-trival result with positive rank on both sides for del Pezzo surfaces.

2.1. **Bridgeland Stability.** Bridgeland's notion of stability conditions for the derived category of coherent sheaves (see [6]) and the associated moduli spaces seem to shed some light on this problem. To give some motivation for Bridgeland stability, recall from the first section that the notion of stability depended on a choice of an embedding of X in projective space. Different embeddings may give rise to different moduli spaces. In good situations, these different moduli spaces are actually birational, and variation of the embedding gives a way to study the birational geometry of the moduli space.

Bridgeland stability conditions take this idea further by not only changing which objects are stable, but allowing one to vary the abelian subcategory of the derived category. Thus, some of the semistable objects parametrized by a Bridgeland moduli space may not be sheaves at all, but rather complexes of sheaves. The space of Bridgeland stability conditions forms a complex manifold, and again one can study the birational geometry of the moduli space by observing how the moduli space changes as the stability condition is varied.

Bayer and Macri showed in [1] that e will define a stability condition on $\mathcal{M}(f)$ for which the the bundle Θ_E is ample, that is, it gives a preferred model of the moduli space on which to consider strange duality. Certain Bridgeland moduli spaces have a more tractable structure, e.g. they may be built from Grassmannians or Kronecker spaces (quotients of Grassmannians). This gives us some interesting examples where both sides of (1) can be computed.

Thinking in the derived category is also useful in our interpretation using finitely many exact sequences. In the case where e also has rank one, the maps $V \to F$ are not surjective. However, they are surjective when considered in an appropriate tilted category (an abelian subcategory of the derived category obtained from $\mathrm{Coh}(S)$ by a simple operation).

2.2. **Strange Duality Moving Forward.** One interesting project would be to generalize Theorem 1 by extending it to all n. The Euler characteristics are already packaged into a power series of a particular form in [9], so is natural to try to analyze the form of the power series formed by the Quot scheme counts. In fact, an application of Theorem 4.2 from [9] suggests that this could be possible—each of these numbers can be obtained as a certain coefficient in power series of form similar to that of the Euler characteristics. It is still not clear how this form will be preserved when the numbers are assembled into a single power series, but this seems like a promising approach.

It would be satisfying to complete the strange duality isomorphism by providing a computation of $H^0(\mathcal{M}(e), \Theta_F)$. As we noted above, using the appropriate Bridgeland moduli space makes this more tractable. We could extend our range of examples by understanding the more complicated Bridgeland moduli spaces formed by Kronecker spaces fibered over Kronecker spaces.

Looking further, our multiple point method seems to not depend much on the dimension of S. It should be straightforward to modify it to work for Fano three-folds. Unfortunately, we are not aware of any method to compute the number of sections of Θ_E on the Hilbert scheme of points of a three-fold. The Hilbert scheme of points on a three-fold is much less well behaved (e.g. is it reducible) than that on a surface. We may hope that the appropriate Bridgeland stability conditions will help us pick out a well behaved model to consider for strange duality.

2.3. **Problems in Bridgeland Stability.** Bridgeland stability conditions are an active area of current research. A central problem now is to prove the existence of stability conditions on three-folds. There were few methods to approach this until Bayer, Macri, and Toda proposed a conjectural inequality on the invariants of certain tilt stable objects in [3]. They showed that this inequality was equivalent to the existence of stability conditions. Since then, this inequality has been proven for several classes of threefolds [2, 21–23, 32], but the general case is still open. This would be a possible area of future investigation for me.

3. Tropical Geometry

Tropical geometry is a version of algebraic geometry that studies certain piecewise linear objects. In tropical geometry, we work over the min-plus semiring. As a set, this is $\mathbb{R}$, but with addition and multiplication replaced by new operations

$$a \oplus b = \min(a, b) \qquad a \odot b = a + b.$$

One can then define tropical polynomials in n variables with these new operations. The hypersurface corresponding a polynomial is defined not as the vanishing locus, but rather as the non-linear locus of the function corresponding to the polynomial. This produces a codimension 1 polyhedral subcomplex of $\mathbb{R}^n$.

There are some possibly surprising analogies to classical algebraic geometry. For example, Bezout's theorem for the intersection of curves in the plane holds, and the genus formula for (complex) smooth plane curves also gives the first Betti number of smooth tropical plane curves.

In my work I was interested in further investigating another analogy. Recall that in classical algebraic geometry $\binom{d+2}{2} - 1$ general points in the projective plane uniquely determine a curve of degree d. This is true in tropical geometry as well. I investigated the problem of determining when points in tropical space are general. I also generalized the problem by allowing points to impose certain higher codimension conditions. When these conditions are imposed on a given hypersurface, I ask when the conditions uniquely determine it, i.e. can the hypersurface be deformed while still satisfying the conditions? The answer uses the dual complex, a polyhedral decomposition of the Newton polygon of the polynomial in which the combinatorics of the hypersurface are encoded. A nice special case of my result in [18] is

Theorem 2. *Conditions of total codimension equal to the dimension of the space of hypersurfaces will uniquely determine the hypersurface if and only if the codimension of each condition matches the dimension of the corresponding cell in the dual complex and the subcomplex of the dual complex corresponding to the conditions is connected.*

In the same paper, I showed how one could generalize the results in [31] to produce a theory of *tropical weighted* matrices that takes into account my higher codimension conditions. I prove a Cramer's rule type theorem that allows one to determine whether the conditions uniquely determine a hypersurface by doing computations with the coordinates.

Theorem 3. *A square tropical weighted matrix is non-singular if and only if its tropical weighted determinant vanishes. A $(N-1)\times N$ tropical weighted matrix has a unique solution to the associated homogeneous system if and only if each maximal minor is non-singular, and this solution is given by computing the maximal minors.*

Tropical geometry caught the attention of the algebraic geometry world when Mikhalkin used tropical methods to prove a curve counting formula [29]. Is it possible that the higher codimension conditions introduced in my paper can be used to solve interesting enumerative problems? In any case, tropical geometry seems like a potentially fruitful and unifying subject. For example, Mark Gross has a book entitled *Mirror Symmetry and Tropical Geometry* [13] that explores the connections between tropical, holomorphic, and log geometry.

4. FJRW theory and Mirror Symmetry

For my master's thesis I did work in FJRW theory. FJRW theory produces an "A-model" cohomological field theory corresponding to a singularity defined by a quasi-homogeneous polynomial W

and a subgroup G of the symmetries of the polynomial [12]. On can form a "transpose" polynomial W^T and group G^T and then produce an orbifold Milnor ring that is the "B-model" [15,19,20]. The orbifold Milnor ring has the structure of a bi-graded Frobenius algebra, i.e. it is a finite dimensional bi-graded $\mathbb{C}$-algebra equipped with a pairing that is compatible with the multiplication. Restricting the FJRW theory to genus 0 three point invariants also gives a Frobenius algebra. A mirror symmetry conjecture predicts that these are isomorphic. Mark Krawitz proved that these are isomorphic as bi-graded vector spaces [20]. I developed some new techniques to prove the algebra isomorphism for a large class of polynomials W in [16, 17]:

Theorem 4. *Suppose W is a sum of loop or Fermat type polynomials and G is an admissible group of symmetries. Then the FJRW Frobenius algebra is isomorphic to the orbifold Milnor ring of W^T, G^T.*

Related to the project, I also wrote open source code to do computations in FJRW theory and on the moduli space of curves that has been used by other people.

Recently, there have been exciting developments in the FJRW/Milnor ring mirror symmetry picture. This would be a possible area for future work. He, Shen, and Webb in [14] show that in the un-orbifolded case, the two cohomological field theories match for a large class of polynomials. In order to extend this to the orbifold case, one would have to first give the construction of the full orbifold B-model, which as of yet is lacking.

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