KEY

Unit 6 Group Work
PCHA 2021-22 / Dr. Kessner

No Calculator

1. Evaluate:

a. $\binom{7}{1} = 7$

b. $\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$

c. $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3!} = 35$

d. $\binom{7}{4} = \binom{7}{3} = 35$

e. $\binom{12}{2} = \frac{12 \cdot 11}{2} = 66$

f. $\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3!} = 220$

g. $\binom{12}{9} = \binom{12}{3} = 220$

h. $\binom{12}{10} = \binom{12}{2} = 66$

i. $\binom{100}{99} = \binom{100}{1} = 100$

j. $\binom{2000}{2} = \frac{2000 \cdot 1999}{2} = 1999000$

2. Let $\{a_k\}_{k=1}^{\infty} = \{\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \cdots\}$.

a. What type of sequence is this? Write recursive and explicit formulas for a_k.

geometric

$a_{n+1} = -\frac{1}{2}a_n$ recursive

$$a_n = a_1 \cdot r^{n-1}$$
$$= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{n-1}$$

b. Let S_n be the n^{th} partial sum of the sequence $\{a_k\}$. Express S_n (for this particular sequence) in summation notation.

$$S_n = \sum_{k=1}^{n} a_k$$
$$= \sum_{k=1}^{n} \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1}$$

c. Write a formula for the actual sum S_n (for this particular sequence).

$$S_n = \frac{a_1(1-r^n)}{1-r}$$
$$= \frac{\frac{1}{2}\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)}$$
$$= \frac{1}{3}\left(1-\left(-\frac{1}{2}\right)^n\right)$$

$\Leftarrow$ note $\lim_{n\to\infty} S_n = \frac{1}{3}$ because $\left(-\frac{1}{2}\right)^n \to 0$ as $n \to \infty$

d. What is the sum of the infinite series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$? (Surprising?)

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{1}{2}}{1-\left(-\frac{1}{2}\right)}$$
$$= \frac{1/2}{3/2}$$
$$= \frac{1}{3}$$

3. Expand $(2 - x^2)^4$.

$$(2-x^2)^4 = 2^4 + 4 \cdot 2^3(-x^2) + 6 \cdot 2^2(-x^2)^2 + 4(2^1)(-x^2)^3 + (-x^2)^4$$
$$= 16 - 32x^2 + 24x^4 - 8x^6 + x^8$$

Find the x^6 term in $(2 - x^2)^5$.

$$\binom{5}{2}(2)^2(-x^2)^3 = -10 \cdot 4x^6$$
$$= -40x^6$$

Find the x^8 term in $(2 - x^2)^5$.

$$\binom{5}{1}(2)^1(-x^2)^4 = 10x^8$$

Hypergeometric

4. Suppose you have 7 red and 3 white marbles in a bag. You pick 6 of the marbles from the bag (without replacement).

a. What is the probability that you pick 6 red marbles?

$$P(6\text{red}) = \frac{\binom{7}{6}\binom{3}{0}}{\binom{10}{6}} = \frac{7}{210} = \frac{1}{30}$$

$$\binom{10}{6} = \binom{10}{4} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot} = 210$$

b. What is the probability that you pick 4 red (and 2 white marbles)?

$$P(4\text{red}) = \frac{\binom{7}{4}\binom{3}{2}}{\binom{10}{6}} = \frac{35\cdot 3}{210} = \frac{105}{210} = \frac{1}{2}$$

$$\binom{7}{4} = \binom{7}{3} = \frac{7\cdot 6\cdot 5}{3\cdot 1} = 35$$

c. What is the probability that you pick 2 red marbles?

$$P(2\text{red}) = 0$$

5. Suppose you have 50 black and 50 white marbles in a bag. You sample 5 marbles with replacement (in other words, you pick a marble, look at it, and put it back, 5 times). Let B be the number of times you pick a black marble. Calculate all of the 6 probabilities $P(B=0), P(B=1), ..., P(B=5)$. *Hint:* You only have to calculate half of these. Verify that $1 = \sum_{k=0}^{5} P(B=k)$.

binomial

$P(B=0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32} = P(B=5)$

$P(B=1) = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = \frac{5}{32} = P(B=4)$

$P(B=2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32} = P(B=3$

check: $\sum_{k=0}^{5} P(B=k) = \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32} = 1$ ✓

Now suppose you have 75 black and 25 white marbles in the bag. You again sample 5 marbles with replacement. Calculate the probabilities $P(B=0), P(B=1), ..., P(B=5)$ and again verify that $1 = \sum_{k=0}^{5} P(B=k)$.

$P(B=0) = \binom{5}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

$P(B=1) = \binom{5}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^4 = \frac{15}{1024}$

$P(B=2) = \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 = \frac{90}{1024}$

$P(B=3) = \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = \frac{270}{1024}$

$P(B=4) = \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 = \frac{405}{1024}$

$P(B=5) = \binom{5}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^0 = \frac{243}{1024}$

$\left(\frac{1}{4}\right)^5 = \frac{1}{2^{10}} = \frac{1}{1024}$

$\binom{5}{2} = \frac{5 \cdot 4}{2} = 10 = \binom{5}{3}$

check: $1 + 15 + 90 + 270 + 405 + 243 = 1024$ ✓