

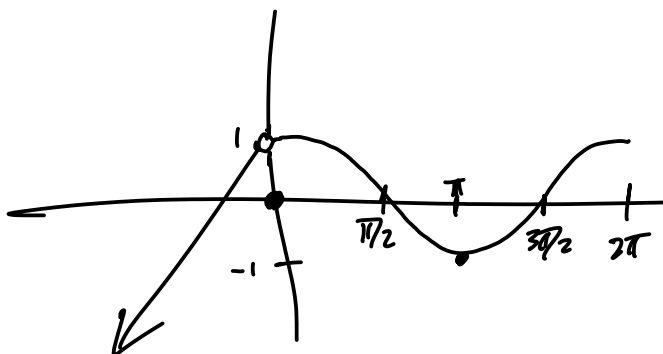
KEY

No calculator! Have fun!

1. Let

$$f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \cos x & \text{if } x > 0 \end{cases}$$

a) Sketch the graph of $f(x)$.



b) On what intervals is f increasing and/or decreasing? Is f bounded? Does it have any local or global maxima or minima?

increasing on $(-\infty, 0)$

increasing on $[\pi + 2\pi k, 2\pi + 2\pi k]$

decreasing on $[0 + \pi k, \pi + \pi k]$
 $k \in \mathbb{Z}^+$

global
local max at $2\pi + 2\pi k$

local min at $\pi + 2\pi k$

$k \in \mathbb{Z}^+$

$k = 0, 1, 2, \dots$

bounded above

c) Does f have any discontinuities? Where, and what type?

removable discontinuity at $x = 0$

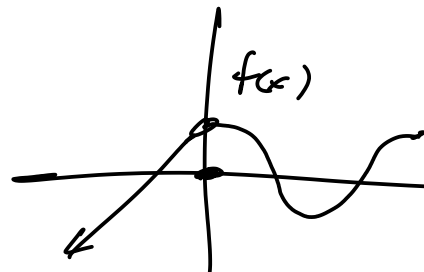
d) Describe the end behavior of f using limits.

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$\lim_{x \rightarrow \infty} f(x)$ does not exist

2. Consider the same function from the previous problem.

$$f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \cos x & \text{if } x > 0 \end{cases}$$



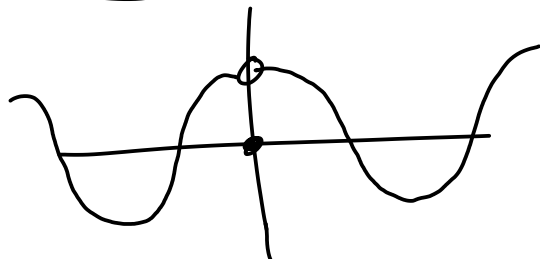
Sketch the graphs of the following transformed functions:

- $p(x) = -f(x)$
- $q(x) = f(|x|)$
- $r(x) = -f(|x|)$
- $s(x) = f(-|x|)$
- $t(x) = |f(-|x|)|$

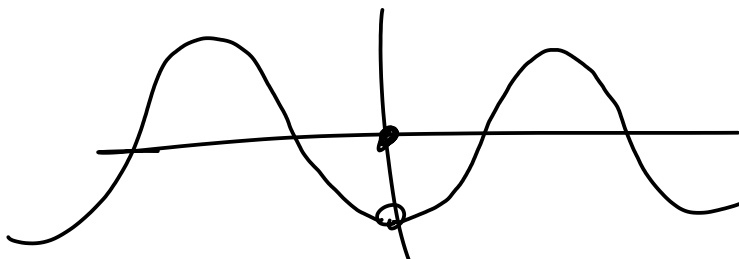
$$p(x) = -f(x)$$



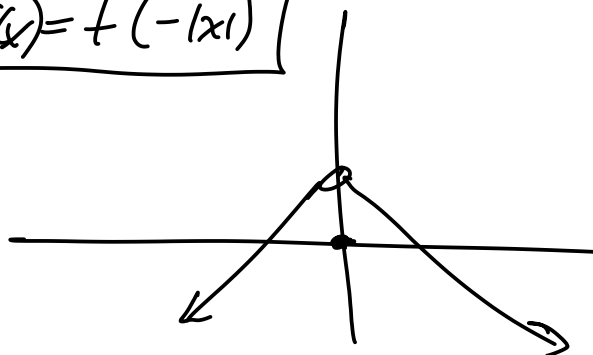
$$q(x) = f(|x|)$$



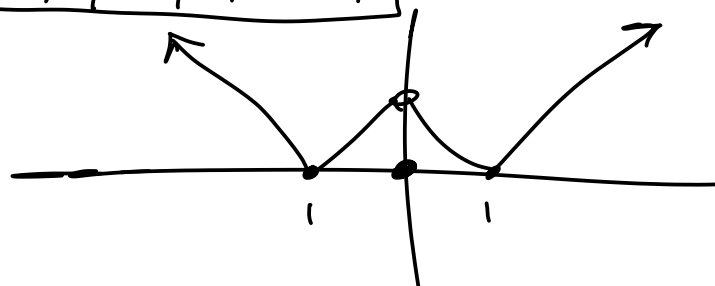
$$r(x) = -q(x) = -f(|x|)$$



$$s(x) = f(-|x|)$$



$$t(x) = |s(x)| = |f(-|x|)|$$



3. Factor the following polynomial completely, both over \mathbb{R} (as a product of real linear and irreducible quadratic factors) and over \mathbb{C} (as a product of complex linear factors). Sketch the graph of the function.

$$p(x) = x^4 + 6x^3 + 13x^2 + 12x + 4$$

potential rational roots: $\pm 1, 2, 4$

$$p(1) \neq 0$$

$$p(-1) = +1 - 6 + 13 - 12 + 4 = 0 \checkmark$$

$$\begin{array}{r|rrrrr} -1 & 1 & 6 & 13 & 12 & 4 \\ & & -1 & -5 & -8 & -4 \\ \hline & 1 & 5 & 8 & 4 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & 5 & 8 & 4 & 0 \\ & & -1 & -4 & -4 & \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$$x^2 + 4x + 4$$

$$(x+2)^2$$

$$\Rightarrow p(x) = (x+1)^2(x+2)^2$$

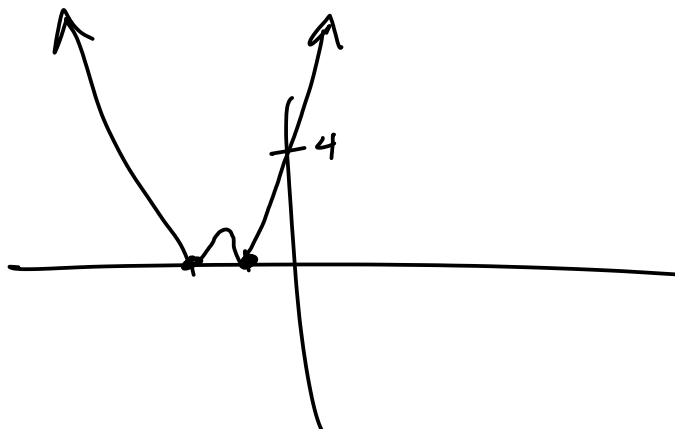
(over \mathbb{R} or \mathbb{C})

$$\Rightarrow p(x) = (x+1)(x^3 + 5x^2 + 8x + 4)$$

$$q(x)$$

$$q(-1) = -1 + 5 - 8 + 4 = 0 \checkmark$$

(-1 is a double root)



4. Factor the following polynomial completely, both over \mathbb{R} (as a product of real linear and irreducible quadratic factors) and over \mathbb{C} (as a product of complex linear factors). Sketch the graph of the function.

$$q(x) = x^5 + 2x^4 - 16x - 32$$

potential rational roots: $\pm 1, 2, 4, 8, 16, 32$

$$q(1) = 1 + 2 - 16 - 32 \neq 0$$

$$q(-1) = -1 + 2 + 16 - 32 \neq 0$$

$$q(2) = 32 + 32 - 32 - 32 = 0 \quad \checkmark$$

$$\underline{2} \overline{) 1 \ 2 \ 0 \ 0 \ -16 \ -32}$$

$$\underline{2 \ 8 \ 16 \ 32 \ 32}$$

$$\underline{-2) \ 1 \ 4 \ 8 \ 16 \ 16 \ 0}$$

$$\underline{-2 \ -4 \ -8 \ -16}$$

$$\underline{-2) \ 1 \ 2 \ 4 \ 8 \ 0}$$

$$\underline{-2 \ 0 \ -8}$$

$$\underline{1 \ 0 \ 4 \ 0}$$

$$x^2 + 4$$

$$(x+2i)(x-2i)$$

$$q(x) = (x-2)(x^4 + 4x^3 + 8x^2 + 16x + 16)$$

$$r(x)$$

$$r(2) \neq 0$$

$$r(-2) = 16 - 32 + 32 - 32 + 16 = 0 \quad \checkmark$$

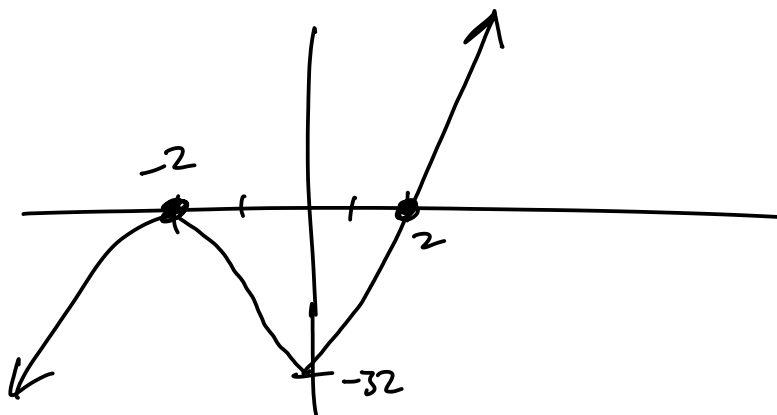
$$q(x) = (x-2)(x+2)(x^3 + 2x^2 + 4x + 8)$$

$$s(x)$$

$$s(-2) = -8 + 8 - 8 + 8 = 0 \quad \checkmark$$

$$\Rightarrow q(x) = (x-2)(x+2)^2(x^2+4) \text{ over } \mathbb{R}$$

$$= (x-2)(x+2)^2(x+2i)(x-2i) \text{ over } \mathbb{C}$$



5. Sketch the graph of the following rational function.

$$r(x) = \frac{x^3 + x^2 - x - 1}{x}$$

Write limits to describe its end behavior, and its behavior near asymptotes. Challenge: Describe its asymptotic end behavior.

factor $p(x) = x^3 + x^2 - x - 1$

$$p(1) = 0 \Rightarrow \begin{array}{r|rrrr} 1 & 1 & 1 & -1 & -1 \\ & & 1 & 2 & 1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$$x^2 + 2x + 1 = (x+1)^2$$

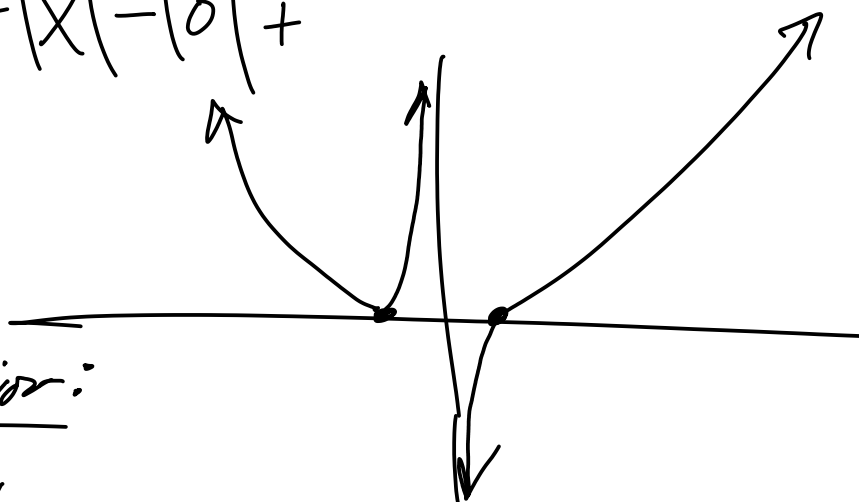
$$p(x) = (x-1)(x+1)^2$$

$$r(x) = \frac{(x-1)(x+1)^2}{x} \leftarrow \text{zeros at } \pm 1$$

$$x \leftarrow \text{asymptote at } x=0$$

		-1		0		1	
$(x+1)^2$	+	0	+	+	+	+	+
x	-	-	-	0	+	+	+
$x-1$	-	-	-	-	-	0	+
$r(x)$	+	0	+	X	-	0	+

end behavior
 $\lim_{x \rightarrow \pm\infty} r(x) = \infty$



asymptotic end behavior:

$$r(x) = \frac{x^3 + x^2 - x - 1}{x}$$

$$r(x) = x^2 + x - 1 - \frac{1}{x}$$

$$\begin{array}{r|rrrr} 0 & 1 & 1 & -1 & -1 \\ & 0 & 0 & 0 & \\ \hline & 1 & 1 & -1 & -1 \end{array}$$



$$r(x) \approx x^2 + x - 1 \text{ as } x \rightarrow \infty$$

asymptotically parabolic