Unit 9 Test
PCHA 2021-22 / Dr. Kessner

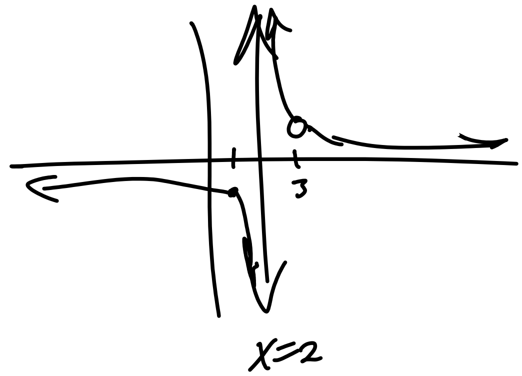
KEY

Name / Pledge:

No calculator! Have fun!

1. Evaluate the following limits, evaluating left and right side limits where applicable.

Let $f(x) = \dfrac{x-3}{(x-2)(x-3)}$ $= \dfrac{1}{x-2}$ if $x \neq 3$



a. $\lim\limits_{x\to\infty} f(x) = 0$

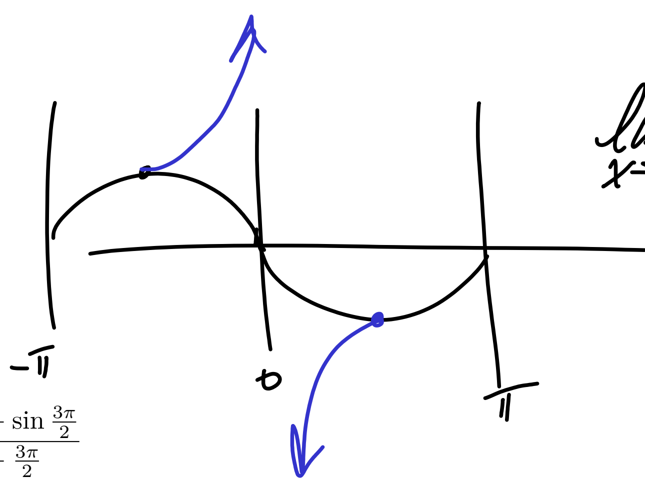
b. $\lim\limits_{x\to 2} f(x)$

$\lim\limits_{x\to 2^-} f(x) = -\infty$

$\lim\limits_{x\to 2^+} f(x) = +\infty$

c. $\lim\limits_{x\to 3} f(x) = 1$

d. $\lim\limits_{x\to 0}(-\csc x)$



$\lim\limits_{x\to 0^-} -\csc x = +\infty$

$\lim\limits_{x\to 0^+} -\csc x = -\infty$

e. $\lim\limits_{x\to\frac{3\pi}{2}} \dfrac{\sin x - \sin\frac{3\pi}{2}}{x - \frac{3\pi}{2}}$

$= \frac{d}{dx}(\sin x)\left(\frac{3\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$

2. a. Let $g(x) = 5x^3 + x$. Find $g'(x)$ using a limit definition.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \left[5(x+h)^3 + (x+h) - 5x^3 - x \right] \frac{1}{h}$$

$$= \lim_{h \to 0} \left[5(3x^2h + 3xh^2 + h^3) + h \right] \frac{1}{h}$$

$$= \lim_{h \to 0} 15x^2 + 15xh + 5h^2 + 1$$

$$= 15x^2 + 1$$

b. A little bird whispers to you: For any a,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

Let $f(x) = 2^x$. Find $f'(x)$ using a limit definition and the ancient knowledge you have gained from the bird.

$$f'(x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$

$$= \lim_{h \to 0} \frac{2^x 2^h - 2^x}{h}$$

$$= \lim_{h \to 0} 2^x \left(\frac{2^h - 1}{h} \right)$$

$$= 2^x \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$= 2^x \ln 2$$

3. Calculate the derivatives of the following functions.

a. $r(x) = x^3 + 3^x + x^{-3}$

$$r'(x) = 3x^2 + 3^x \ln 3 - 3x^{-4}$$

b. $s(x) = \log_2 \sin x$

$$s'(x) = \frac{1}{\sin x \ln 2} \cdot \cos x = \frac{1}{\ln 2} \cot x$$

c. $t(x) = \sec^2 x$

$$t'(x) = 2\sec x \cdot \sec x \tan x$$
$$= \frac{2\sin x}{\cos^3 x}$$

d. $t(x) = \sec^2 x = \dfrac{1}{\cos^2 x}$ using the quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$t'(x) = \frac{0(\ \ ) - 1(2\cos x)(-\sin x)}{\cos^4 x} = \frac{2\sin x}{\cos^3 x}$$

e. $t(x) = \sec^2 x = (\cos x)^{-2}$ using the product, power, and chain rules.

$$t'(x) = -2(\cos x)^{-3}(-\sin x)$$
$$= \frac{2\sin x}{\cos x}$$

4. a. Let $y = \sqrt[4]{x}$. Find $\dfrac{dy}{dx}$ in two ways: 1) the general power rule, 2) implicit differentiation (plus the integer power rule). Verify that your answers are the same.

$$\frac{dy}{dx} = \frac{1}{4}x^{-3/4}$$

$$y^4 = x$$

$$4y^3\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{4y^3} = \frac{1}{4x^{3/4}}$$

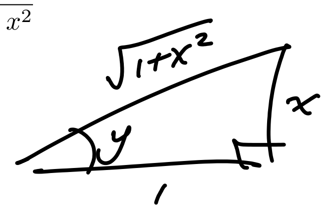
b. Let $y = \tan^{-1} x$ (inverse tan, not reciprocal). Use implicit differentiation to show that:

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\tan y = x$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \frac{1}{1+x^2}$$



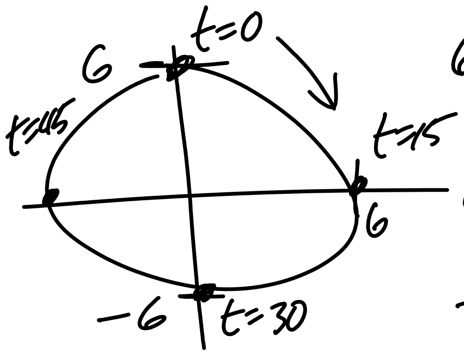
ii. Let $c(x) = \tan^{-1}(\tan x)$.

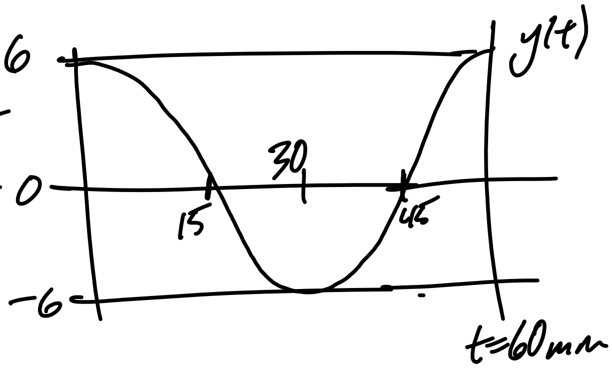
Use the formula above and the chain rule to find $c'(x)$.

$$c'(x) = \frac{1}{1+\tan^2 x} \cdot \sec^2 x = 1$$

5. You are quarantined at home and bored, contemplating a clock on your wall. Suppose the minute hand of the clock is 6 inches long. (What is the period in minutes?)

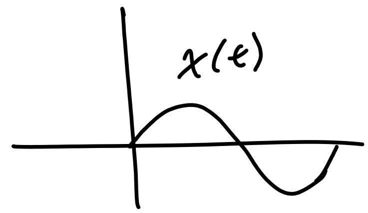
a. Let the origin be the center of the clock. Find parametric equations for the position of the tip of the minute hand. For example, at t=0, the position is $(0, 6)$, and at t=15 minutes, the position is $(6, 0)$.





$y(t) = 6\cos\left(\frac{2\pi}{60}t\right)$

$x(t) = 6\sin\left(\frac{2\pi}{60}t\right)$



b. Find equations for $x'(t)$ and $y'(t)$.

$x'(t) = 6\cos\left(\frac{2\pi}{60}t\right)\left(\frac{\pi}{30}\right) = \frac{\pi}{5}\cos\left(\frac{2\pi}{60}t\right)$

$y'(t) = -6\sin\left(\frac{2\pi}{60}t\right)\left(\frac{\pi}{30}\right) = -\frac{\pi}{5}\sin\left(\frac{2\pi}{60}t\right)$

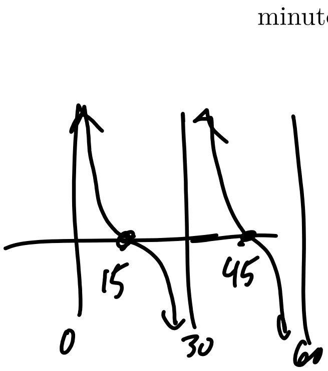
c. The *speed* is the magnitude of the velocity vector $\langle x'(t), y'(t)\rangle$. Show that the speed of the tip of the hour hand is $\frac{12\pi}{60}$. **Bonus:** Explain why this makes sense.

$|\langle x', y'\rangle| = \sqrt{\left(\frac{\pi}{5}\right)^2\cos^2(\ ) + \left(\frac{\pi}{5}\right)^2\sin^2(\ )} = \frac{\pi}{5} = \frac{12\pi}{60}$

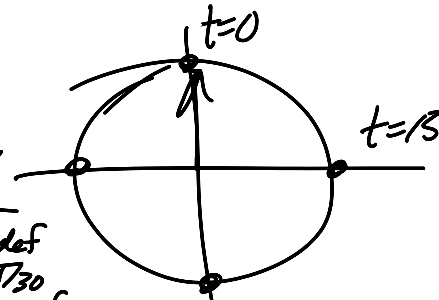
d. Let $s(t) = \frac{y(t)}{x(t)}$ be the slope of the minute hand (as a line). Find $s(t)$ and $s'(t)$ at $t = 0, 15, 30, 45$ minutes.

$s(t) = \frac{6\cos\left(\frac{\pi}{30}t\right)}{6\sin\left(\frac{\pi}{30}t\right)} = \cot\left(\frac{\pi}{30}t\right)$

$s'(t) = -\frac{\pi}{30}\csc^2\left(\frac{\pi}{30}t\right)$



t	s	s'
0	undef	undef
15	0	$-\pi/30$
30	undef	undef
45	0	$-\pi/30$



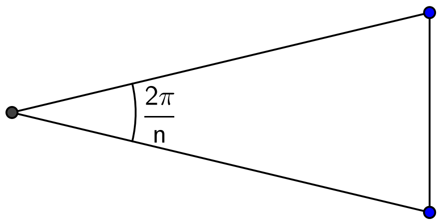
Bonus Assuming the differentiation rule for a^x, derive (!) the differentiation rule for $\log_a x$.

Bonus Derive the general power rule for derivatives: $\dfrac{d}{dx}x^r = rx^{r-1}$ for any $r \in \mathbb{R}$. *Hint:* Let $y = x^r$. Take the natural log of both sides, and use implicit differentiation.

Bonus Show that the vertex of the parabola $y = ax^2 + bx + c$ is located at $x = \frac{-b}{2a}$.

Bonus Parametrize a circle of radius 6 around the origin. Show that the position vector $\langle x(t), y(t) \rangle$ is always orthogonal to the velocity vector $\langle x'(t), y'(t) \rangle$, and that the velocity vector is orthogonal to the acceleration vector.

Bonus



Let $P(n)$ be the perimeter of the regular n-sided polygon inscribed in the unit circle. Show that

$$P(n) = n \cdot 2\sin(\frac{2\pi}{2n}) = 2n\sin(\frac{\pi}{n})$$

Show that

$$\lim_{n\to\infty} P(n) = 2\pi$$

Hint: Let $x = \frac{1}{n}$. Why does this make sense?