

Equidistributed sequences and the analytic properties of a strange class of L -functions

Daniel Miller

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1 Motivation

Let E/\mathbf{Q} be an elliptic curve without complex multiplication. By an old theorem of Faltings, the quantities

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \mathrm{tr} \rho_{E,l}(\mathrm{fr}_p)$$

determine E up to isogeny. The starting point of this investigation is the corollary of a theorem of Harris, that the collection $\{\mathrm{sgn} a_p(E)\}_p$ in fact determines E up to isogeny. Ramakrishna had the insight that this fact means the “strange L -function”

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \mathrm{sgn} a_p(E)p^{-s}}$$

determines E up to isogeny. In this note, I define a more general class of strange L -functions, and show that their analytic properties are closely tied to the equidistribution of the $a_p(E)$.

Here is a brief discussion of this generalization in the case of a non-CM curve E/\mathbf{Q} . It is convenient to repack these traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the $\theta_p(E)$ are well-defined angles laying in the interval $[0, \pi]$. Write $\mu_{\mathrm{ST}} = \frac{2}{\pi} \sin^2 \theta \, d\theta$. Then the Sato–Tate conjecture (now a theorem) tells us that for any continuous function $f: [0, \pi] \rightarrow \mathbf{C}$, we have:

$$\left| \frac{1}{\pi(C)} \sum_{p \leq C} f(\theta_p) - \int_0^\pi f \, d\mu_{\mathrm{ST}} \right| = o(1)$$

as $C \rightarrow \infty$. It is well-known that this is equivalent to the analytic continuation of all the L -functions $L(\mathrm{sym}^k E, s)$. We take as our starting point the stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(C)} \sum_{p \leq C} f(\theta_p) - \int_0^\pi f \, d\mu_{\mathrm{ST}} \right| = O_f(C^{-\frac{1}{2}+\epsilon}).$$

They prove that this conjecture implies the Riemann Hypothesis for E . I prove that not only does their conjecture imply the Riemann Hypothesis for all $L(\text{sym}^k E, s)$, it also does for all the strange L -functions

$$L_f(E, s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

These results make perfect sense in a much more general context, and I will prove them there. In [section 2](#) I set up this context and carefully define strange L -functions there. In [section 3](#), I prove basic analytic properties of the strange L -functions and connect their analytic properties with the equidistribution of a sequence. Finally, in [section 4](#), I apply the general results to the following cases: a non-CM elliptic curve E/\mathbf{Q} , the product $E_1 \times E_2$ of a pair of non-isogenous non-CM elliptic curves over \mathbf{Q} , and the Jacobian of a generic genus-2 curve C/\mathbf{Q} .

2 Definitions

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| \leq 1\}$. Write \mathbf{D}^∞ for the set of sequences in \mathbf{D} indexed by the primes, i.e. $\boldsymbol{\lambda} \in \mathbf{D}^\infty$ is $(\lambda_2, \lambda_3, \dots)$.

Definition 2.1. Let $\boldsymbol{\lambda} \in \mathbf{D}^\infty$. The associated *strange L -function* is given by

$$L(\boldsymbol{\lambda}, s) = \prod_p \frac{1}{1 - \lambda_p p^{-s}},$$

wherever this product converges.

We will see that the analytic properties of $L(\boldsymbol{\lambda}, s)$ are closely tied to estimates for the sums $A_{\boldsymbol{\lambda}}(x) = \sum_{p \leq x} \lambda_p$. One often gets such estimates in the context of equidistribution, which we consider next.

For the remainder of this section, let X be a compact separable metric space with no isolated points. We write X^∞ for the space of sequences in X indexed by rational primes, i.e. points $\boldsymbol{x} \in X^\infty$ are of the form $\boldsymbol{x} = (x_2, x_3, \dots)$. By [\[Eng89, Cor. 2.3.16 & Th. 4.2.2\]](#), the compact space X^∞ is metrizable and separable, also with no isolated points.

Definition 2.2. For $\boldsymbol{x} \in X^\infty$ and $C > 0$, write \boldsymbol{x}^C for the probability measure given by

$$\int_X f \, d\boldsymbol{x}^C = \boldsymbol{x}^C(f) = \frac{1}{\pi(C)} \sum_{p \leq C} f(x_p).$$

Let μ be a Borel measure on X . Recall that \boldsymbol{x} is *μ -equidistributed* if $\boldsymbol{x}^C \rightarrow \mu$ weakly, i.e. $\boldsymbol{x}^C(f) \rightarrow \mu(f)$ for all $f \in C(X)$. In fact, we can extend this to not-necessarily-continuous functions as follows:

Theorem 2.3 (Mazzone). *Let μ be a Borel measure on X and let $f : X \rightarrow \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if $\boldsymbol{x}^C(f) \rightarrow \mu(f)$ for all μ -equidistributed \boldsymbol{x} .*

Proof. This follows directly from the proof of [Maz95, Th. 1]. \square

Fix a Borel measure μ on X , and write $C^{\text{ae}}(X, \mu)$ for the space of bounded, almost-everywhere continuous functions $f: X \rightarrow \mathbf{C}$.

Theorem 2.4. *Endowed with the supremum norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $C^{\text{ae}}(X, \mu)$ is a Banach space.*

Proof. This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero. \square

Definition 2.5. Let $f \in C^{\text{ae}}(X, \mu)$, $\mathbf{x} \in X^\infty$. The associated *strange L -function* is defined as

$$L_f(\mathbf{x}, s) = L(f(\mathbf{x}), s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}$$

for all $s \in \mathbf{C}$ for which the product converges.

Our typical source of a strange L -function is as follows. $X = G^\natural$, the space of conjugacy classes in a compact Lie group, and $f: G^\natural \rightarrow \mathbf{C}$ one of the “angles” of [KS99]. More precisely, let G be a compact Lie group and $\rho: G \rightarrow \text{U}(d)$ an irreducible representation. Following [KS99, Le. 1.0.9], write $\varphi_1^\rho, \dots, \varphi_d^\rho$ for the sequence of functions $G^\natural \rightarrow [0, 2\pi)$ such that for each $x \in G^\natural$, the unitary conjugacy class $\rho(x)$ has eigenvalues $e^{i\varphi_1^\rho(x)}, \dots, e^{i\varphi_d^\rho(x)}$, and $\varphi_1^\rho(x) \leq \dots \leq \varphi_d^\rho(x)$. We have, using Serre’s notation $L(\rho, s)$, the identity:

$$L(\rho, s) = \prod_{j=1}^{\deg \rho} L_{\exp(i\varphi_j^\rho)}(\mathbf{x}, s).$$

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let $X = [0, a_1) \times \dots \times [0, a_r)$. Given $x = (x_1, \dots, x_r) \in X$, we write $[0, x) = [0, x_1) \times \dots \times [0, x_r)$.

Definition 2.6. Given X as above, and $\mathbf{x} \in X^\infty$, the *star-discrepancy* of \mathbf{x} with respect to a Borel measure μ on X is:

$$\text{disc}(\mathbf{x}^C, \mu) = \sup_{x \in X} |\mathbf{x}^C(1_{[0, x)}) - \mu(1_{[0, x)})|.$$

The following result is essential:

Theorem 2.7 (Koksma–Hlawka). *Let X be as above. Let $f: X \rightarrow \mathbf{C}$ be such that $f \, dx$ is a measure with bounded variation. Let μ be a probability measure on X . Then*

$$|\mathbf{x}^C(f) - \mu(f)| \leq \text{Var}(f) \text{disc}(\mathbf{x}^C, \mu).$$

Proof. This is [Ökt99, Th. 3.2]. \square

Recall that if f on $[0, a]$ is C^2 , then $\text{Var}(f) = \int |f'|^2$.

3 Preliminary results

Theorem 3.1. *Let $\lambda \in \mathbf{D}^\infty$. Then $L(\lambda, s)$ defines a holomorphic function on the region $\{\Re s > 1\}$. Moreover, on that region,*

$$\log L(\lambda, s) = \sum_{p^n} \frac{\lambda_p^n}{p^{ns}}.$$

Proof. Expanding the product for $L(\lambda, s)$ formally, we have

$$L(\lambda, s) = \sum_{n \geq 1} \frac{\prod_{p|n} \lambda_p^{v_p(n)}}{n^s}.$$

An easy comparison with Riemann's zeta function tells us that the series expansion is holomorphic on $\{\Re s > 1\}$. By [Apo76, Th. 11.7], the product formula holds on the same region. The formula for $\log L(\lambda, s)$ comes from [Apo76, 11.9 Ex. 2]. \square

Theorem 3.2. *Assume $A_\lambda(x) = O(x^{\frac{1}{2}+\epsilon})$. Then $L(\lambda, s)$ converges on $\{\Re > \frac{1}{2}\}$, and $\log L(\lambda, s)$ has no poles on that region.*

Proof. Standard reductions reduce this to showing that

$$\sum_p \frac{\lambda_p}{p^s} \quad \text{and} \quad \sum_p \frac{\log(p)\lambda_p}{p^s}$$

converge on that region. We deal with $\sum \log(p)\lambda_p p^{-s}$; the other is similar. Use Abel summation:

$$\sum_{p \leq x} \frac{\lambda_p}{p^s} = \frac{\log x}{x^s} A_\lambda(x) - \int_2^x \frac{1-s \log t}{t^{s+1}} A_\lambda(t) dt.$$

We show that the first term approaches zero and that the integral converges absolutely. We have:

$$\left| \frac{\log x}{x^s} A_\lambda(x) \right| \ll \frac{\log x}{x^{\Re s}} x^{\frac{1}{2}+\epsilon}.$$

Since ϵ is arbitrary, the exponent of x is negative. Moreover,

$$\begin{aligned} \int_2^x \frac{1}{t^{s+1}} |A_\lambda(t)| dt &\ll \int_2^x \frac{1}{t^{\Re s+1}} t^{\frac{1}{2}+\epsilon} dt \\ \int_2^x \frac{\log t}{t^{s+1}} |A_\lambda(t)| dt &\ll \int_2^x \frac{\log t}{t^{\Re s+1}} t^{\frac{1}{2}+\epsilon} dt. \end{aligned}$$

Both these integrals converge because ϵ is arbitrary. \square

Corollary 3.3. *Let $X = [0, a_1) \times \cdots \times [0, a_r)$, with probability measure μ . If $\mathbf{x} \in X^\infty$ satisfies $\text{disc}(\mathbf{x}^C, \mu) \ll C^{-\frac{1}{2}+\epsilon}$, then $L_f(\mathbf{x}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$, and satisfies the Riemann Hypothesis, for all f bounded and almost-everywhere continuous with $\mu(f) = 0$.*

Proof. Koksma–Hlawka tells that if $\mu(f) = 0$, then $|\mathbf{x}^C(f)| \leq \text{Var}(f) \text{disc}(\mathbf{x}^C, \mu)$, i.e. $\mathbf{x}^C(f) \ll C^{-\frac{1}{2}+\epsilon}$. Thus the sequence $f(\mathbf{x})$ satisfies $A_{f(\mathbf{x})}(x) \ll x^{\frac{1}{2}+\epsilon}$, and the result follows from [Theorem 3.2](#). \square

4 Applications

Theorem 4.1. *Let E/\mathbf{Q} be a non-CM elliptic curve. Assume the Akiyama–Tanigawa conjecture for E . Then all the L -functions $L_f(\boldsymbol{\theta}, s)$ have analytic continuation to $\{\Re s > \frac{1}{2}\}$ and satisfy the Riemann Hypothesis. In particular, this holds for all $L(\text{sym}^k E, s)$.*

Proof. Akiyama–Tanigawa conjecture that $\text{disc}(\boldsymbol{\theta}^C, \text{ST}) \ll C^{-\frac{1}{2}+\epsilon}$ implies the first by [Corollary 3.3](#). The second part follows from the fact that any $L(\text{sym}^k E, s)$ can be written as a product of L_f ’s. \square

Theorem 4.2. *Let E_1, E_2 be two non-isogenous, non-CM elliptic curves over \mathbf{Q} . Assume the Akiyama–Tanigawa conjecture for the product $E_1 \times E_2$. Then for any $f: [0, \pi] \rightarrow \mathbf{C}$ that is not almost everywhere*

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