AUTOMORPHIC REPRESENTATIONS AND DEFORMATION THEORY IN ARITHMETIC GEOMETRY

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ABSTRACT. This is a brief expository note, motivated by [Mor12], on the analogy between the character variety of the fundamental group of a hyperbolic knot, and the p-ordinary deformation space of a two-dimensional modular Galois representation. Hopefully this analogy should generalize to arbitrary reductive groups, or at least $\mathrm{GL}(n)$. In light of this, much of this note is an introduction to the relevant objects from a general perspective. So we introduce notion of a p-adic family of automorphic representations, and the (conjecturally) corresponding family of p-adic Galois representations in full generality.

CONTENTS

1. THE ANALOGY BETWEEN KNOTS AND PRIMES

Recall that a morphism $f : X \to Y$ of schemes is *étale* if it is flat and unramified. Equivalently, f is flat and for each $y \in Y$, the fiber X_y is the spectrum of a finite product of finite separable extensions of $k(y)$. If X is a connected noetherian scheme with chosen base point $x \in X$, the category of finite étale covers of X is canonically equivalent to the category of finite sets with continuous action of $\pi_1(X, x)$. Here $\pi_1(X, x)$ is the *étale fundamental group* of X at x, which we will denote $\pi_1(X)$ when x is clear. To save space, we will write $\pi_1(A)$ instead of $\pi_1(\operatorname{Spec} A)$. A good reference for all of this is [Mil13].

We start by recalling the analogy described in [Mor12, ch.3-4]. We should think of the circle S^1 as a $K(\mathbf{Z}, 1)$-space. The arithmetic analogue is $\operatorname{Spec}(\mathbf{F}_q)$ for any prime power q. Indeed, as is the case for any field, $\pi_1(\mathbf{F}_q) = \operatorname{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q) \simeq \widehat{\mathbf{Z}}$, generated by the Frobenius $\operatorname{fr}_q(x) = x^q$. The arithmetic analogue of a 3-manifold

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is $X = \operatorname{Spec}(O_F) \smallsetminus S$ for any number field F and finite set S of primes in O_F. Indeed, for constructible sheaves $\mathscr{F}$ on X, there is a "3-dimensional duality" $\operatorname{Ext}^\bullet(\mathscr{F}, \mathbf{G}_\mathrm{m}) = \mathrm{H}_c^{3-\bullet}(X, \mathscr{F})^\vee$ [Maz73, 2.4] similar to the classical Artin-Verdier duality $\operatorname{Ext}^\bullet(\mathscr{L}, \mathbf{Q}) = \mathrm{H}_c^{3-\bullet}(M, \mathscr{L})^\vee$ for local systems on a 3-manifold M.

In topology, a knot is an embedding $K : S^1 \hookrightarrow S^3$. More generally, we could consider any $K(\mathbf{Z}, 1) \hookrightarrow M$, where M is a 3-manifold. The arithmetic analogue is a map $\operatorname{Spec}(\mathbf{F}_q) \to \operatorname{Spec}(O_F)$ coming from a prime ideal $\mathfrak{p} \subset O_F$ with residue field $\mathbf{F}_q$. In topology, the standard way of studying a knot $K : S^1 \hookrightarrow S^3$ is to consider a small tubular neighborhood V_K of S^1 in S^3. For simplicity, we assume $F = \mathbf{Z}$; then the arithmetic analogue is an "infinitesimal étale neighborhood" of $\operatorname{Spec}(\mathbf{F}_p) \hookrightarrow \operatorname{Spec}(\mathbf{Z})$, namely $\operatorname{Spec}(\mathbf{Z}_p) \to \operatorname{Spec}(\mathbf{Z})$. The *peripheral group* of K is $\pi_1(V_K \smallsetminus K) \simeq \mathbf{Z}^2$, corresponding to $\pi_1(\operatorname{Spec}(\mathbf{Z}_p) \smallsetminus \operatorname{Spec}(\mathbf{F}_p)) = \pi_1(\mathbf{Q}_p)$. Finally, the *knot group* is $\Gamma_K = \pi_1(S^3 \smallsetminus K)$, corresponding to $\Gamma_{\mathbf{Q},p} = \pi_1(\mathbf{Z}[\frac{1}{p}])$. The peripheral map $\pi_1(V_K \smallsetminus K) \to \Gamma_K$ corresponds to the map $\pi_1(\mathbf{Q}_p) \to \Gamma_{\mathbf{Q},p}$.

Unfortunately, this analogy is not perfect. While the topological periphery group $\pi_1(V_K \smallsetminus K)$ is abelian, free on generators l, m (l for latitude, m for meridian) and $\pi_1(V_K \smallsetminus K) \to \pi_1(V_K)$ has kernel $\langle m \rangle$, the arithmetic case is a more complicated. The group $D_p = \pi_1(\mathbf{Q}_p)$ is a very complicated pro-solvable group. It can be written down in terms of generators and relations as in [NSW08, VII §5], but this is not very helpful. The group D_p does have a canonical quotient classifying *tame covers* of $\operatorname{Spec}(\mathbf{Q}_p)$ which has a presentation

$$\pi_1^{\mathrm{tame}}(\mathbf{Q}_p) = \langle \sigma, \tau : \sigma\tau\sigma^{-1} = \tau^p \rangle.$$

We can think of the degenerate case "$p = 1$" as being in exact analogy with topology. The kernel of $D_p \to \pi_1(\mathbf{Z}_p)$ is written I_p, and (in analogy with topology) called the *inertia group* at p.

2. Automorphic representations

2.1. **Adeles.** Let F be a number field. If v is a place of F, write F_v for the completion of F at v, and O_v for the ring of integers of F_v. Write $\mathbf{A} = \mathbf{A}_F$ for the ring of adeles of F; the most important fact about $\mathbf{A}$ is that it is a locally compact F-algebra. It can be defined in many ways:

- The topological direct limit $\varinjlim_S \mathbf{A}(S)$, where S ranges over all finite sets of valuations of F, and
$$\mathbf{A}(S) = \prod_{v \in S} F_v \times \prod_{v \notin S} O_v.$$
- The topological tensor product $(\mathbf{R} \times \widehat{\mathbf{Z}}) \otimes F$.
- A restricted direct product $\prod_v' (F_v, O_v)$, consisting of those tuples $(a_v) \in \prod_v F_v$ for which $a_v \in O_v$ for almost all v.
- Via [GS66], an initial object in the category of locally compact F-algebras with no proper open ideals, and with the intersection of all closed maximal ideals being 0.

Write $\mathbf{A}_\mathrm{f} = \widehat{\mathbf{Z}} \otimes F$ for the ring of *finite adeles*. It is totally disconnected.

In [Con12], it is shown that there is a unique fiber-product-preserving functor $(-)(\mathbf{A})$ from affine schemes of finite type over F to topological spaces, compatible with closed embeddings, for which $(\operatorname{Spec} F[t])(\mathbf{A}) = \mathbf{A}$ with its usual topology. In particular, if G is an algebraic group over F, the abstract group $G(\mathbf{A})$ carries the structure of a locally compact topological group. As such, it has a unique (up to scalar) Haar measure $\mathrm{d}g$.

2.2. **Hecke algebras.** A good reference for this section is [Fla79]. Let G be a connected reductive group over F, and let $\mathfrak{g} = \mathrm{Lie}(G)$. There exists an open subset of $U \subset \mathrm{Spec}(O_F)$ and a "spreading out" of G to a reductive group scheme on U. Up to finitely many places, this spreading out is well-defined. In particular, for almost all finite places v, the group $G(O_v)$ is well-defined for almost all v. It is a maximal (open) compact subgroup of $G(F_v)$. We normalize Haar measures on $G(F_v)$ so that $G(O_v)$ has volume 1, and choose the Haar measure on $G(\mathbf{A})$ to be the product of measure on each $G(F_v)$.

Let v be a finite place. Write $\mathcal{H}_v$ for the *Hecke algebra* consisting of continuous, locally constant, compactly supported functions $G(F_v) \to \mathbf{Q}$. Multiplication is convolution:

$$(f \star g)(x) = \int_{G(F_v)} f(g) g(y^{-1}x)\, \mathrm{d}y.$$

Even though this is written as an integral, it is a finite sum over double cosets of open compact subgroups of $G(F_v)$, so no analysis is involved. For almost all v, the algebra $\mathcal{H}_v$ comes with a canonical idempotent, $e_v = \chi_{G(O_v)}$. Write $\mathcal{H}_\mathrm{f}$ for the restricted tensor product (in the sense of [Fla79, §2]) of the $\mathcal{H}_v$ with respect to the e_v. It is the direct limit $\varinjlim_S \mathcal{H}(S)$, where $\mathcal{H}(S) = \bigotimes_{v\in S} \mathcal{H}_v$, and for $T \supset S$, the injection $\mathcal{H}(S) \to \mathcal{H}(T)$ is induced by the e_v for $v \in T \smallsetminus S$. We will also think of $\mathcal{H}_\mathrm{f}$ as the algebra of locally constant, compactly supported functions f on $G(\mathbf{A}_\mathrm{f})$.

The ring $F_\infty = F \otimes \mathbf{R}$ is a finite product of copies of $\mathbf{R}$ and $\mathbf{C}$, so $G(F_\infty)$ is naturally a Lie group. Fix a maximal compact subgroup $K_\infty \subset G(F_\infty)$. The Hecke algebra $\mathcal{H}_\infty = \mathcal{H}_\infty(G)$ is the convolution algebra of K_∞-finite distributions on $G(F_\infty)$ with support in K_∞. There is an isomorphism

$$U(\mathfrak{g}_\mathbf{C}) \otimes_{U(\mathfrak{k}_\mathbf{C})} M(K_\infty), \xrightarrow{\sim} \mathcal{H}_\infty \qquad D \otimes \mu \mapsto D \star \mu,$$

where $\mathfrak{k} = \mathrm{Lie}(K_\infty)$ and $M(K_\infty)$ is the algebra of measures on K_∞.

The *global Hecke algebra* of G is $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_\mathrm{f}$. We will be interested in special classes of representations of $\mathcal{H}$.

For a sufficiently large set S of finite places, it makes sense to define $e_S \in \mathcal{H}_\mathrm{f}$ to be the characteristic function of $\prod_{v\notin S} G(O_v)$.

An *admissible* representation of $\mathcal{H}_\mathrm{f}$ is an $\mathcal{H}_\mathrm{f}$-module V such that for each $v \in V$, there is a finite set S of places for which $e_S \cdot v = v$.

2.3. **Automorphic representations.** A good reference for this section is [BJ79].

Let F, G, … be as above. Let Z be the center of G, and choose a character $\omega : Z(F)\backslash Z(\mathbf{A}) \to \mathbf{C}^\times$. Write $L^2(G,\omega)$ for the space of measurable functions $f : G(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ such that

$$\begin{aligned} &f(zx) = \omega(z) f(x) && z \in Z(\mathbf{A}) \\ &\|f\|^2 = \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} |f(x)|^2\, \mathrm{d}x < \infty. \end{aligned}$$

The space $L^2(G,\omega)$ is a representation of $G(\mathbf{A})$ in the obvious way. Write $L^2_\mathrm{disc}(G,\omega)$ for the closed subspace generated by all irreducible closed subrepresentations. Let $\mathcal{A}(G,\omega) \subset L^2_\mathrm{disc}(G,\omega)$ be the space of K-finite vectors which are also $Z(\mathfrak{g}_\infty)$-finite. (It is a non-trivial fact that K-finite vectors are smooth.)

Then $\mathcal{A}(G,\omega)$ is naturally a $\mathcal{H}$-module, and as such, decomposes as a countable direct sum of irreducible representations with finite multiplicities:

$$\mathcal{A}(G,\omega) = \bigoplus_{\pi\in\widehat{G}_{\mathrm{c.a}}(\omega)} m(\pi)\pi. \tag{1}$$

We call the irreducible admissible representations of $\mathcal{H}$ appearing in (1) *automorphic representations* of G. By [Fla79, th.4], each automorphic representation π decomposes as a restricted tensor product $\bigotimes \pi_v$ of irreducible admissible representations of the $\mathcal{H}_v$.

In the remainder, we will often pass without comment between admissible representations of $G(\mathbf{A}_{\mathrm{f}})$ and admissible representations of $\mathcal{H}_{\mathrm{f}}$. This is not hard. Suppose $\pi : G(\mathbf{A}_{\mathrm{f}}) \to \mathrm{GL}(V)$ is an admissible representation. For $f \in \mathcal{H}_{\mathrm{f}}$ and $v \in V$, put

$$f \star v = \int_{G(\mathbf{A}_{\mathrm{f}})} f(x)\pi(x)\cdot v \,\mathrm{d}x.$$

This integral is actually a finite sum. Indeed, we can write f as a finite sum of scalars multiples of characteristic functions χ_{gK}, where $K \subset G(\mathbf{A}_{\mathrm{f}})$ is open, compact, and fixes v. For such a function, we see that

$$\chi_{gK} \star v = \int_{G(\mathbf{A}_{\mathrm{f}})} \chi_{gK}(x)\pi(x)v \,\mathrm{d}x = \int_K gv \,\mathrm{d}x = \mathrm{vol}(K)gv.$$

So the action of $\mathcal{H}_{\mathrm{f}}$ on V makes sense. Going the other way is also easy. If V is an admissible $\mathcal{H}_{\mathrm{f}}$-module and $g \in G(\mathbf{A}_{\mathrm{f}})$, choose open compact normal K such that $\chi_K \star v = v$. Inspired by the above, put $gv = \mathrm{vol}(K)^{-1}\chi_{gK} \star v$.

2.4. **Hecke eigensystems and L-functions.** Let π be an automorphic representation of G and choose a nonzero vector u in π. For almost all places v, the idempotent $e_v = \chi_{G(O_v)}$ in $\mathcal{H}_v$ fixes u (in this case, we say that π is *unramified* at v). In particular, the action of $\mathcal{H}_v$ on π factors through that of

$$\mathcal{H}_v(O_v) = e_v \mathcal{H}_v e_v = C_c^\infty(G(O_v)\backslash G(F_v)/G(O_v)).$$

Let S be a set of places outside which e_v fixes u. Let $\mathcal{H}(S) = \bigotimes_{v\notin S} \mathcal{H}_v(O_v)$. Then π is an irreducible admissible module over $\mathcal{H}(S)\otimes\bigotimes_{v\in S}\mathcal{H}_v$. Since $\mathcal{H}(S)$ is central in this algebra, it must act via a character $\chi : \mathcal{H}(S) \to \mathbf{C}$. The system of homomorphisms $\{\chi_v : \mathcal{H}_v(O_v) \to \mathbf{C} : v \notin S\}$ is called a *Hecke eigensystem.*

In the case $G = \mathrm{GL}(n)$, Hecke eigensystems have a particularly easy description. A character $\chi : \mathcal{H}_v(O_v) \to \mathbf{C}$ is uniquely determined by a semisimple conjugacy class $\sigma_v(\chi) \in \mathrm{GL}(n,\mathbf{C})$. If $\pi = \bigotimes_v \pi_v$ is an automorphic representation of $\mathrm{GL}(n)$, put $\sigma_v(\pi) = \sigma(\chi_{\pi_v})$ and (for finite v):

$$L_v(s,\pi) = \det\left(1 - \mathrm{N}(v)\cdot\sigma_v(\pi)^{-s}\right)^{-1}.$$

For S sufficiently large, we can define the *partial L-function* of π as

$$L_S(s,\pi) = \prod_{v\notin S} L_v(s,\pi).$$

This has the expected properties including analytic continuation, a functional equation.... In the case $G = \mathrm{GL}(n)$, an automorphic representation π is determined by $L(s,\pi)$.

3. Shimura varieties

For the rest of this note, the reader should keep in mind the example $F = \mathbf{Q}$, $G = \mathrm{GL}(2)$. Many of the definitions work in greater generality, but technicalities (which we wish to avoid) multiply endlessly.

3.1. **Locally symmetric spaces and their cohomology.**

Classically, one studies representations of a real semisimple group G by fixing a maximal compact K, setting $X = G/K$, and studying the regular representation of G on $C^\infty(\Gamma\backslash X)$ for $\Gamma \subset G$ a discrete group. Big examples are the (affine) modular curves $Y_0(n)$, coming from $\Gamma_0(n) \subset \mathrm{SL}(2, \mathbf{R})$. We will carry out this construction adelically.

Let G be a connected reductive group over $\mathbf{Q}$. Put $X = Z_\infty\backslash G(F_\infty)/K_\infty$. Let $K \subset G(\mathbf{A}_\mathrm{f})$ be an open compact subgroup. We define

$$\mathrm{Sh}_K(G) = G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_\mathrm{f}))/K.$$

A priori, this is only a topological space, but the quotient map $X \times G(\mathbf{A}_\mathrm{f})/K \to \mathrm{Sh}_K(G)$ gives $\mathrm{Sh}_K(G)$ the structure of a Riemannian manifold. Let (V, ρ) be a representation of G. There is an induced local system $\mathscr{V}_\rho$ of F-vector spaces on $\mathrm{Sh}_K(G)$, whose (global) sections are locally constant sections $s : X \times G(\mathbf{A}_\mathrm{f})/K \to V$ such that $s(\gamma x) = \rho(\gamma)s(x)$ for $\gamma \in G(\mathbf{Q})$.

The cohomology $\mathrm{H}^\bullet(\mathrm{Sh}_K(G), \mathscr{V}_\rho)$ is naturally an admissible $\mathcal{H}_\mathrm{f}$-module. For open compact $C \subset K$, the function χ_{gC} acts via the correspondence

$$\mathrm{Sh}_K(G) \twoheadleftarrow \mathrm{Sh}_{K\cap C}(G) \xrightarrow{\cdot g} \mathrm{Sh}_{K\cap g^{-1}Cg}(G) \twoheadrightarrow \mathrm{Sh}_K(G).$$

There is a standard compactification of $\mathrm{Sh}_K(G)$, namely its *Borel-Serre compactification* $\overline{\mathrm{Sh}}_K(G)$. Define the *cuspidal cohomology* to be

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_K(G), \mathscr{V}_\rho) = \ker\left(\mathrm{H}^\bullet(\overline{\mathrm{Sh}}_K(G), \mathscr{V}_\rho) \to \mathrm{H}^\bullet(\partial\,\mathrm{Sh}_K(G), \mathscr{V}_\rho)\right).$$

The cuspidal cohomology is also an admissible $\mathcal{H}_\mathrm{f}$-module. In fact, we have a *generalized Eichler-Shimura isomorphism* [Sch09, 4.1].

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_K(G), \mathscr{V}_\rho) = \bigoplus_{\substack{\pi \in \mathcal{A}_{\mathrm{cusp}}(G, \chi_\rho)\\ K\text{-spherical}}} \mathrm{H}^\bullet(\mathcal{H}_\infty, \pi_\infty \otimes \rho) \otimes \pi_\mathrm{f}^K.$$

The notation $\mathrm{H}^\bullet(\mathcal{H}_\infty, -)$ needs explanation. There is a good category of admissible $\mathcal{H}_\infty$-modules, and $\hom(\mathbf{C}, -)$ is left-exact. Its derived functor is the $(\mathfrak{g}_\infty, K_\infty)$-cohomology $\mathrm{H}^\bullet(\mathcal{H}_\infty -)$.

Better:

$$\mathrm{H}^\bullet_{\mathrm{cusp}}(\mathrm{Sh}(G), \mathscr{V}_\rho) = \bigoplus_{\pi \in \widehat{G}_\mathrm{c}(\omega)} \mathrm{H}^\bullet(\mathcal{H}_\infty, \pi_\infty \otimes \rho) \otimes \pi_\mathrm{f}.$$

3.2. **Canonical models.**

A good reference for this is [Moo98]. In subsection 3.1 we constructed $\mathrm{Sh}_K(G)$ as a Riemannian manifold. It turns out that there is a good definition of "canonical model" for Shimura varieties over number fields, and in that sense, all Shimura varieties $\mathrm{Sh}_K(G)$ have a canonical model over a number field called the *reflex field.* (We have been intentionally avoiding use of the Shimura datum necessary to define $\mathrm{Sh}_K(G)$ in full generality – the reflex field depends on this.)

Let E be the reflex field. Not only does the projective system $\mathrm{Sh}(G) = \varprojlim \mathrm{Sh}_K(G)$ descend to the reflex field, but the action of $G(\mathbf{A}_\mathrm{f})$ via correspondences descends in a canonical way. Moreover, in [Har85] it is shown that our local systems $\mathscr{V}_\rho$ descend in a functorial way to $G(\mathbf{A}_\mathrm{f})$-equivariant local systems on $\mathrm{Sh}(G)$.

The main reason we care about this is that if $\mathrm{Sh}_K(G)$ is smooth and $E = \mathbf{Q}$, then general theorems about étale cohomology tell us that

$$\mathrm{H}^\bullet_{\mathrm{sing},c}(\mathrm{Sh}_K(G), \mathscr{V}_\rho) = \mathrm{H}^\bullet_{\text{ét},c}(\mathrm{Sh}_K(G)_{\overline{\mathbf{Q}}}, \mathscr{V}_\rho(\overline{\mathbf{Q}_l}))$$

after choice of an isomorphism $\mathbf{C} \simeq \overline{\mathbf{Q}_l}$. The choice of an *arithmetic compactification* of $\mathrm{Sh}_K(G)$ lets us extend the action of $\mathcal{H}_\mathrm{f}$ to the étale cohomology of $\mathrm{Sh}_K(G)$.

4. Modular representations

4.1. **Modular curves.** In this section, algebraic groups, adeles, etc. will be taken over $\mathbf{Q}$. Let $n \geqslant 1$ be an integer. We define the following congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$:

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}) : c \equiv 0 \pmod n \right\}.$$

Let $K_0(n)$ be the induced subgroup of $\mathrm{GL}_2(\mathbf{A}_\mathrm{f})$. Write $Y_0(n)$ for the induced locally symmetric space:

$$Y_0(n) = \mathrm{Sh}_{K_0(n)}(\mathrm{GL}_2) = \mathrm{GL}_2(\mathbf{Q}) \backslash \mathrm{GL}_2(\mathbf{A}) / Z_\infty K_\infty K_0(n).$$

Here $Z_\infty = Z(\mathrm{GL}_d(\mathbf{R}))$ and $K_\infty = \mathrm{SO}(2) \subset \mathrm{GL}_2(\mathbf{R})$. Put $X = \mathrm{GL}_2(\mathbf{R})^+ / Z_\infty K_\infty$. Note that $\mathrm{GL}_2(\mathbf{R})^+/Z = \mathrm{SL}_2(\mathbf{R})$, so

$$\mathrm{GL}_2(\mathbf{R})^+ / Z_\infty K_\infty \xrightarrow{\sim} \mathfrak{H} = \{z \in \mathbf{C} : \Im z > 0\}$$

via $\gamma \mapsto \gamma \cdot i$. The strong approximation theorem tells us that

$$\mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(\mathbf{Q}) \, \mathrm{GL}_2(\mathbf{R}) K_0(n),$$

so the quotient $\mathrm{Sh}_{K_0(n)}(\mathrm{GL}_2)$ is just

$$(\mathrm{GL}_2(\mathbf{Q}) \cap K_0(n)) \backslash \mathfrak{H} = \Gamma_0(n) \backslash \mathfrak{H} = Y_0(n).$$

This will be a singular complex-analytic orbifold. There are two ways of realizing $Y_0(n)$ and its compactication $X_0(n)$ as curves over $\mathbf{Q}$:

(1) Interpret $Y_0(n)$ as a moduli space for elliptic curves with level structure. This moduli problem makes sense over $\mathbf{Q}$, so $Y_0(n)$ descends in a canonical way to $\mathbf{Q}$.
(2) Use the general theory of canonical models of Shimura varieties.

The former approach generalizes to a special class of Shimura varieties consisting of those of *PEL type* (standing for **P**olarization, **E**ndomorphism, and **L**evel structure). The theory of PEL-type Shimura varieties is interesting and useful, but we won't go into it here.

Instead, note that the space

$$\mathfrak{H}^\pm = Z_\infty \backslash \mathrm{GL}_2(\mathbf{R}) / \mathrm{SO}_2(\mathbf{R})$$

can be interpreted as the set of $\mathrm{GL}_2(\mathbf{R})$-conjugacy classes of homomorphisms $\mathbf{S} = \mathrm{R}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_\mathrm{m} \to \mathrm{GL}(2)_\mathbf{R}$ containing

$$h : (x, y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

This is all defined over $\mathbf{Q}$, so the theory of canonical models discussed in subsection 3.2 tells us that if $K \subset \mathrm{GL}_2(\mathbf{A}_\mathrm{f})$ is any open compact subgroup, the quotient

$$\mathrm{Sh}_K(\mathrm{GL}_2) = \mathrm{GL}_2(\mathbf{Q})\backslash(\mathfrak{H}^\pm \times \mathrm{GL}_2(\mathbf{A}_\mathrm{f}))/K$$

descends to a uniquely determined curve over $\mathbf{Q}$. Moreover, this curve has a well-defined smooth compactification also defined over $\mathbf{Q}$, so we don't need to worry about the difference between minimal and toroidal compactifications.

4.2. **The Eichler-Shimura construction.** Let $n \geqslant 3$. As above, write $Y_0(n)$ for the Shimura variety $\mathrm{Sh}_{K_0(n))}(\mathrm{GL}_2)$, and write $X_0(n)$ for its arithmetic compactification. We are interested in the cohomology $\mathrm{H}^1_\mathrm{cusp}(X_0(n), \mathscr{V}_{\mathrm{sym}^{k-2}})$. At the moment, this is just a $\mathbf{C}$-vector space with an action of $\mathcal{H}_\mathrm{f}$. However, in subsection 3.1, there is an automorphic decomposition

$$\mathrm{H}^1_\mathrm{cusp}(X_0(n), \mathscr{V}_{\mathrm{sym}^{k-2}}) = \bigoplus_{\substack{\pi \in \mathcal{A}(\mathrm{GL}_2) \\ \Gamma_0(n)\text{-spherical}}} \mathrm{H}^1(\mathfrak{gl}_2, \mathrm{SO}(2), \pi_\infty \otimes \mathrm{sym}^{k-2}) \otimes \pi_\mathrm{f}^{\Gamma_0(n)}.$$

The computation in [Har87, §3.4-3.6] tells us that $\mathrm{H}^1(\mathfrak{gl}_2, \mathrm{SO}(2), \pi_\infty \otimes \mathrm{sym}^{k-2}) = 0$ unless π is the automorphic representation coming from a weight-k cuspidal eigenform of level n, in which case $\mathrm{H}^1(\mathfrak{gl}_2, \mathrm{SO}(2), \pi_\infty \otimes \mathrm{sym}^{k-2}) = \mathbf{C} \oplus \overline{\mathbf{C}}$. In particular,

$$\mathrm{H}^1_\mathrm{cusp}(X_0(n), \mathscr{V}_{\mathrm{sym}^{k-2}}) = \bigoplus_{f \text{ eigen-cusp}} \mathbf{C} \oplus \overline{\mathbf{C}} = S_k(\Gamma_0(n)) \oplus \overline{S_k(\Gamma_0(n))}.$$

Recall that our modular curves are defined over $\mathbf{Q}$. So we can consider the cohomology spaces

$$\mathrm{H}^1_{\text{ét},\mathrm{cusp}}(X_0(n)_{\overline{\mathbf{Q}}}, \overline{\mathbf{Q}_l}) \simeq S_2(\Gamma_0(n), \mathbf{C}).$$

These have commuting actions of $\Gamma_{\mathbf{Q},ln} = \pi_1(\mathbf{Z}[\frac{1}{ln}])$ and $\mathcal{H}_\mathrm{f}$. So $\Gamma_{\mathbf{Q},ln}$ acts on each $\mathcal{H}_\mathrm{f}$-irreducible piece. These pieces are 2-dimensional, so we get, for each cuspidal eigenform f, a Galois representation $\rho_{f,l} : \Gamma_{\mathbf{Q},nl} \to \mathrm{GL}_2(\overline{\mathbf{Q}_l})$. The *Eichler-Shimura relation* basically tells us that the Hecke and Frobenius parameters for π_f and $\rho_{f,l}$ match up. That is, for all $p \nmid ln$, we have $\rho_{f,l}(\mathrm{fr}_p) = \sigma_p(\pi_f)$, or equivalently $L_p(\rho_{f,l}, s) = L_p(\pi_f, s)$.

It is known that $\rho_{f,l} : \Gamma_{\mathbf{Q},ln} \to \mathrm{GL}_2(\overline{\mathbf{Q}_l})$ factors through $\mathrm{GL}_2(K_{f,\lambda})$, where $K_f = \mathbf{Q}(a_p(f) : p \text{ prime})$ is a number field and λ is a place of K_f dividing l. An elementary argument shows that we can conjugate the image of $\rho_{f,l}$ to lie in $\mathrm{GL}_2(O_{f,\lambda})$, where $O_f = O_{K_f}$. In particular, we can reduce $\rho_{f,l}$ modulo λ to get a continuous representation

$$\bar{\rho}_{f,l} : \Gamma_{\mathbf{Q},ln} \to \mathrm{GL}_2(O_{f,\lambda}/\lambda) = \mathrm{GL}_2(\mathbf{F}_\lambda).$$

We say that a mod-l representation of $\Gamma_\mathbf{Q}$ is *modular* (better, *automorphic*) if it is of the form $\bar{\rho}_{f,l}$ for some f. Similarly, we say that an l-adic representation of $\Gamma_\mathbf{Q}$ is *modular* (better, *automorphic*) if it is of the form $\rho_{f,l}$ for some l.

In what follows, we will ignore the modular form f and just write π for the corresponding cuspidal automorphic representation of $\mathrm{GL}(2)$, keeping in mind that for some automorphic representations (those corresponding to Maass forms) we still don't know how to construct the associated Galois representations. For an automorphic representation π, write $\rho_{\pi,l}$ for the corresponding l-adic representation (assuming it exists).

4.3. **Interpolating modular representations.** The Hida families we will see later on are essentially p-adic families of cuspidal automorphic representations of GL(2). Even better, Hida constructed the corresponding p-adic family of Galois representations. Here we will give the (conjectural) bigger picture.

Let G be a connected reductive group over $\mathbf{Q}$. Recall we have a Hecke algebra $\mathcal{H} = \mathcal{H}_{\mathrm{f}} \otimes \mathcal{H}_\infty$, and automorphic representations of G are, by definition, a special class of irreducible representations of $\mathcal{H}$. Remember that $\mathcal{H}_{\mathrm{f}}$ is the convolution algebra of locally constant, compactly supported functions on $G(\mathbf{A}_{\mathrm{f}})$. If $K \subset G(\mathbf{A}_{\mathrm{f}})$ is open compact, then $e_K = \frac{1}{\mu(K)}\chi_K$ is an idempotent in $\mathcal{H}_{\mathrm{f}}$, and we write $\mathcal{H}(K) = e_K \mathcal{H}_{\mathrm{f}} e_K$ for algebra of locally constant, compactly supported, K-bi-invariant functions on $G(\mathbf{A}_{\mathrm{f}})$. If e_K acts trivially on an automorphic representation π, we say π is *K-spherical.*

The basic idea of a p-adic family of automorphic representations is that we should fix $K^p \subset G(\mathbf{A}_{\mathrm{f}}^p)$ (the *tame level*) and consider families of automorphic representations that are $K^p K_p$-spherical, where K_p ranges over a special class of subgroups of $G(\mathbf{Q}_p)$. One makes this rigorous via a modified Hecke algebra. Here we mainly follow [Urb11, 4.1]. Fix a prime p and assume G is split at p. Let (T, B) be a Borel pair, and let N be the unipotent radical of B. Define

$$\begin{aligned} I_r &= \{g \in G(\mathbf{Z}_p) : \bar{g} \in B(\mathbf{Z}/p^r)\} \\ T^- &= \{t \in T(\mathbf{Q}_p) : tN(\mathbf{Z}_p)t^{-1} \subset N(\mathbf{Z}_p)\} \\ \Delta_r^- &= I_r T^- I_r. \end{aligned}$$

There is an isomorphism $u : \mathbf{Z}_p[T^-/T(\mathbf{Z}_p)] \to C_c^\infty(I_r\backslash\Delta_r^-/I_r)$ by $t \mapsto u_t = \chi_{I_r t I_r}$. Here it is crucial that we normalize the Haar measure so that I_r has volume 1. So we call $\mathcal{U}_p = \mathbf{Z}_p[T^-/T(\mathbf{Z}_p)]$ and think of $\mathcal{U}_p$ as a single avatar for all of the $C_c^\infty(I_r\backslash\Delta_r^-/I_r, \mathbf{Z}_p)$. Our big Hecke algebra is

$$\mathbf{h} = C_c^\infty(K^p\backslash G(\mathbf{A}_{\mathrm{f}}^p)/K^p, \mathbf{Q}_p) \otimes \mathcal{U}_p.$$

Note that $\mathbf{h} \hookrightarrow \mathcal{H}_{\mathrm{f}} \otimes \mathbf{Q}_p$. So, if σ is an irreducible representation of $\mathbf{h}$, we will call σ *automorphic* if $\sigma = \pi_{\mathrm{f}}^{K^p I_r}|_{\mathbf{h}}$ for some honest automorphic representation π. Note that if $G \neq \mathrm{GL}(n)$, the representation π may not be unique.

A p-adic family of automorphic representations of G will contain $K^p I_r$-spherical representations for varying r. We will also require the weight to vary p-adically. So, let $\mathfrak{W}$ be the p-adic weight space determined by

$$\mathfrak{W}(A) = \hom_{\mathsf{cts}}(T(\mathbf{Z}_p), A^\times).$$

Let $\mathfrak{W}^{\mathrm{cl}} = \mathrm{X}^*(T)$ be the set of *classical weights.* We say an automorphic representation π is *cohomological of weight* λ if π appears in some $\mathrm{H}^\bullet_{\mathrm{cusp}}(\mathrm{Sh}_{K^p I_r}(G), \mathscr{V}_\lambda(\mathbf{C}))$. Given such a representation, we have the trace map $\mathrm{tr}_\pi : \mathbf{h} \to \overline{\mathbf{Q}_p}$, which characterizes π as an $\mathbf{h}$-module.

Definition. *A p-adic family $\boldsymbol{\sigma}$ of automorphic representations of G of tame level K^p consists of:*

- *An rigid subset $\mathfrak{U} \subset \mathfrak{W}$.*
- *A finite flat surjection $w : \mathfrak{T} \to \mathfrak{U}$.*
- *A $\mathbf{Q}_p$-linear map $J : \mathbf{h} \to \mathscr{O}(\mathfrak{T})$.*
- *A dense set $\boldsymbol{\sigma}^{\mathrm{cl}} \subset \mathfrak{U}^{\mathrm{cl}}$.*

We require that for each $\sigma \in \boldsymbol{\sigma}^{\mathrm{cl}}$, the weight $\lambda = w(\sigma)$ is dominant and the composite

$$J_\sigma : \mathbf{h} \to \mathscr{O}(\mathfrak{X}) \xrightarrow{\mathrm{ev}_\sigma} \overline{\mathbf{Q}_p}$$

is $m(\pi, \lambda)\,\mathrm{tr}_\pi$ for a cohomological representation π of weight λ.

Here $m(\pi, \lambda)$ is the *Euler-Poincaré characteristic*

$$m(\pi, \lambda) = \sum (-1)^i \dim \hom_{\mathbf{h}}(\pi^{K^p I_r}, \mathrm{H}^i_{\mathrm{cusp}}(\mathrm{Sh}_{K^p I_r}(G), \mathscr{V}_\lambda(\mathbf{C})).$$

In [Urb11], Urban constructed p-adic families containing the "finite slope" representations, for groups satisfying the Harish-Chandra condition (having discrete series at infinity).

Recall that for "nice" automorphic representations π, there should be a Galois representation $\rho_{\pi,p} : \Gamma_{\mathbf{Q}} \to {}^{\mathrm{L}}G(\overline{\mathbf{Q}_p})$ unramified almost everywhere, such that for unramified v, $\rho_{\pi,p}(\mathrm{fr}_v)$ is conjugate to the Satake parameter of π at v. For $G = \mathrm{GL}(n)$, this will just be a representation $\Gamma_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}_p})$. Note that even if multiplicity-one theorems fail for G, the trace tr_π determines $\rho_{\pi,p}$. So we will speak of "the Galois representation associated to tr_π," bearing in mind that we don't currently know how to construct $\rho_{\pi,p}$ for general G.

Conjecture. *Let $\boldsymbol{\sigma}$ be a p-adic family of automorphic representations of G with L-algebraic classical points. Then there is a continuous representation $\rho_{\boldsymbol{\sigma}} : \Gamma_{\mathbf{Q}} \to {}^{\mathrm{L}}G(\mathscr{O}(\mathfrak{X}))$ such for all $\sigma \in \boldsymbol{\sigma}^{\mathrm{cl}}$, the composite*

$$\rho_\sigma : \Gamma_{\mathbf{Q}} \to {}^{\mathrm{L}}G(\mathscr{O}(\mathfrak{X})) \xrightarrow{\mathrm{ev}_\sigma} {}^{\mathrm{L}}G(\overline{\mathbf{Q}_p})$$

is the Galois representation associated to J_σ.

This conjecture is *wide* open – we don't even know how to construct individual $\rho_{\pi,p}$ for most G. In [SU14], Skinner and Urban construct p-adic families of *pseudorepresentations* for "unitary similitude groups" associated to an imaginary quadratic field. For $G = \mathrm{GL}(2)$ and π corresponding to a p-ordinary form, Hida has constructed a big p-adic family containing π, and the associated p-adic family of Galois representations.

5. Deformation theory

5.1. **Motivation.** First let's consider the motivation for studying deformations of Galois representations. If X is a nice (that is smooth, projective and geometrically integral) variety over $\mathbf{Q}$, its étale cohomology $\mathrm{H}^\bullet_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_l)$ carries a continuous action of $\Gamma_{\mathbf{Q}}$. The "right" way to see this is as follows. Spread out X to a smooth proper scheme $\mathcal{X}$ over an open $U = \mathrm{Spec}(\mathbf{Z}) \smallsetminus S$. Write $\pi : \mathcal{X} \to U$ for the structure map. Then $\mathsf{R}^\bullet \pi_* \mathbf{Q}_l$ is a local system on $U_{\text{ét}}$. The étale version of covering space theory tells us that local systems correspond to representations of $\pi_1(U) = \Gamma_{\mathbf{Q},S}$.

The motivating example was the representation $\rho_{E,l}$ coming from an elliptic curve $E \xrightarrow{\pi} U$ via $\mathsf{R}^1 \pi_* \mathbf{Q}_l$. Langlands' conjectural framework tells us that there should exist an automorphic cuspidal representation π of $\mathrm{GL}(2)$ for which $\rho_{E,l} \simeq \rho_{\pi,l}$. We don't know how to construct π this directly. However, we *do* know (via Serre's conjecture) that $\bar\rho_{E,l}$ is automorphic. One of the main applications of deformation theory is to prove that the automorphy of $\bar\rho_{E,l}$ implies that of $\rho_{E,l}$.

More generally, given an l-adic Galois representation $\rho : \Gamma_{\mathbf{Q},S} \to \mathrm{GL}_n(\mathbf{Q}_l)$ that is suitably nice (*geometric*, in the sense of Fontaine-Mazur [FM95]), Langlands' program tells us that we should expect there to be a cuspidal automorphic representation π of $\mathrm{GL}(n)$ such that $\rho \simeq \rho_{\pi,l}$ (assuming we knew how to construct ρ_π in general). Proving that ρ is automorphic is very hard! However, we have a much better chance (in theory and in practice) of showing that $\bar{\rho} : \Gamma_{\mathbf{Q},S} \to \mathrm{GL}_n(\mathbf{F}_l)$ is automorphic. A theorem to the effect that "$\bar{\rho}$ automorphic $\Rightarrow \rho$ automorphic" is known as a *automorphy lifting theorem.* In practice, one has to impose many technical conditions on $\bar{\rho}$ and the automorphic representation with $\bar{\rho}_\pi \simeq \bar{\rho}$, and one uses groups like $\mathrm{GSp}(n)$ instead of $\mathrm{GL}(n)$.

5.2. **Representations of knot groups.** Our exposition here follows that of [Mor12, ch.13-14]. Let $K \subset S^3$ be a hyperbolic knot, $M = S^3 \smallsetminus K$ the knot complement, $\pi = \pi_1(M)$ the knot group. The uniformization $\mathbf{H}^3 \to M$ induces a representation $\pi \to \mathrm{Aut}(\mathbf{H}^3) = \mathrm{PSL}_2(\mathbf{C})$ which lifts to $\rho : \pi \to \mathrm{SL}_2(\mathbf{C})$. Introduce the *representation variety* $\mathrm{Rep}(\pi, \mathrm{SL}_2)$ of homomorphisms $\pi \to \mathrm{SL}_2(\mathbf{C})$. There are two ways of describing $\mathrm{Rep}(\pi, \mathrm{SL}_2)$. One elementary but useful approach is to write $\pi = \langle g_1, \dots, g_m | r_1, \dots, r_n \rangle$. The variety $\mathrm{Rep}(\pi, \mathrm{SL}_2)$ is just the subset of $\mathrm{SL}_2(\mathbf{C})^m$ cut out by the relations $r_1, \dots, r_n$. A more functorial definition is to require that for all $\mathbf{C}$-algebras A, a natural isomorphism

$$\hom_{\mathsf{grp}}(\pi, \mathrm{SL}_2(A)) \simeq \hom_{\mathsf{sch}/\mathbf{C}}(\operatorname{Spec} A, \mathrm{Rep}(\pi, \mathrm{SL}_2)).$$

The *character variety* of K is the geometric quotient

$$X_K = \mathrm{Rep}(\pi, \mathrm{SL}_2) /\!\!/ \mathrm{SL}_2 = \operatorname{Spec}\left(\mathbf{C}[\mathrm{Rep}(\pi, \mathrm{SL}_2)]^{\mathrm{SL}_2(\mathbf{C})}\right),$$

via the obvious action of $\mathrm{SL}(2)$ on $\mathrm{Rep}(\pi, \mathrm{SL}_2)$ via conjugation. The representation ρ is a point in X_K, and one is interested in the connected component $X_K(\rho)$.

5.3. **Deformation functors.** The analogous situation in number theory is much more complicated, partly because $\Gamma_S = \pi_1(\operatorname{Spec}(\mathbf{Z}) \smallsetminus S)$ is not a finitely presented group – it's a compact topological group which is (conjecturally) *topologically* finitely presented. So instead of looking for representations $\Gamma_S \to \mathrm{GL}_2(\mathbf{C})$, we should look for continuous representations $\Gamma_S \to \mathrm{GL}_2(A)$, where A is a topological $\mathbf{Z}_p$-algebra.

Briefly, a scheme X over k can be thought of in terms of its functor of points $X(-) : k\text{-}\mathsf{Alg} \to \mathsf{Set}$. In the arithmetic context, our deformation spaces will be *formal schemes* over $\mathbf{Z}_p$. For us, this just means that the test category consists of complete local pro-artinian $\mathbf{Z}_p$-algebras with residue field $\mathbf{F}_p$. If R is such an ring, we write $\mathrm{Spf}(R)$ to denote the functor $A \mapsto \hom(R, A)$. There is a way of making $\mathrm{Spf}(R)$ into a topological space with structure sheaf, but we will not need this.

The functorial approach to defining representation schemes works well. If π is an arbitrary profinite group, there is a formal scheme $\widehat{\mathrm{Rep}}(\pi, \mathrm{GL}_n)$, satisfying

$$\widehat{\mathrm{Rep}}(\pi, \mathrm{GL}_n)(A) = \hom_{\mathrm{cts}}(\pi, \mathrm{GL}_n\, A),$$

for any local pro-artinian $\mathbf{Z}_p$-algebra A with residue field $\mathbf{F}_p$. The problem is, $\widehat{\mathrm{Rep}}(\pi, \mathrm{GL}_n)$ is really horrible as a space – it generally has infinitely many different connected components. So before we do anything else, let's restrict to the connected component of $\bar{\rho}$, where $\bar{\rho} : \pi \to \mathrm{GL}_n(\mathbf{F}_p)$ has been fixed beforehand. The component $\mathfrak{X}^\square(\bar{\rho})$ represents continuous homomorphisms $\pi \to \mathrm{GL}_n(A)$ that reduce to $\bar{\rho}$ modulo p.

As before, we will quotient out by the natural action of $\mathrm{GL}(n)$, but here we should be careful because $\mathrm{GL}(n)$ does not preserve the component $\mathfrak{X}^{\square}(\bar{\rho})$. The correct thing to do is to first define

$$\widehat{\mathrm{GL}}_n(A) = \{g \in \mathrm{GL}_n(A) : g \equiv 1 \pmod{p})\} = \ker\left(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbf{F}_p)\right).$$

The action of $\widehat{\mathrm{GL}}(n)$ on $\widehat{\mathrm{Rep}}(\pi, \mathrm{GL}_n)$ preserves $\mathfrak{X}^{\square}(\bar{\rho})$. Now a miracle happens. Define

$$\mathfrak{X}(\bar{\rho})(A) = \mathfrak{X}^{\square}(A)/\widehat{\mathrm{GL}}_n(A).$$

Then in [Maz99, pr.1], it is proved that if $\bar{\rho}$ is absolutely irreducible and π satisfies a certain technical hypothesis (which will hold for all our examples), the functor $\mathfrak{X}(\bar{\rho})$ is represented by a complete local noetherian $\mathbf{Z}_p$-algebra $R_{\bar{\rho}}$ with residue field $\mathbf{F}_p$. That is, there is a representation $\rho : \pi \to \mathrm{GL}_n(R_{\bar{\rho}})$ lifting $\bar{\rho}$ such that any $\widehat{\mathrm{GL}}_n(A)$-equivalence class of lifts $\pi \to \mathrm{GL}_n(A)$ is induced by a unique continuous homomorphism $R_{\bar{\rho}} \to A$.

Suppose we have fixed a subgroup $I \subset \pi$. We call a representation $\rho : \pi \to \mathrm{GL}_2(A)$ *I-ordinary* if ρ^I is a free, rank-one, direct summand of ρ. Suppose $\bar{\rho}$ is absolutely irreducible and I-ordinary. Then we can define a subfunctor $\mathfrak{X}^{\circ}(\bar{\rho})$ of $\mathfrak{X}(\bar{\rho})$ by

$$\mathfrak{X}^{\circ}(\bar{\rho})(A) = \{\rho \in \mathfrak{X}(\bar{\rho})(A) : \rho \text{ is } I\text{-ordinary}\}.$$

By [Maz99, pr.3], $\mathfrak{X}^{\circ}(\bar{\rho})$ is represented by a complete local noetherian $\mathbf{Z}_p$-algebra $R^{\circ}_{\bar{\rho}}$ with residue field $\mathbf{F}_p$.

5.4. **The case $n = 1$.** The easiest example is when our representations take values in $\mathrm{GL}(1)$. Let $\pi = \pi_1(\mathbf{Z}[\frac{1}{p}])$ and $\bar{\rho} = \bar{\kappa} : \pi \to \mathrm{GL}_1(\mathbf{F}_p)$ be the *mod-p cyclotomic character*. This is defined, for $\sigma \in \Gamma_{\mathbf{Q}}$, by

$$\sigma(\zeta_p) = \zeta_p^{\bar{\kappa}(\sigma)}.$$

Let's start by computing $\widehat{\mathrm{Rep}}(\pi, \mathrm{GL}_1)$. Deformations $\rho : \pi \to A^{\times}$ factor through π^{ab}. Class field theory tells us that $\pi^{\mathrm{ab}} \simeq \mathbf{Z}_p^{\times}$. So

$$\widehat{\mathrm{Rep}}(\pi, \mathrm{GL}_1) = \mathrm{Spf}\left(\mathbf{Z}_p\left[\!\left[\mathbf{Z}_p^{\times}\right]\!\right]\right) \simeq \coprod_{\varepsilon : \pi \to \mathbf{F}_p^{\times}} \mathrm{Spf}\left(\mathbf{Z}_p[\![\mathbf{Z}_p]\!]\right),$$

where each $\mathrm{Spf}(\mathbf{Z}_p[\![\mathbf{Z}_p]\!])$ is the connected component of some ε. We see that $R_{\bar{\kappa}} \simeq \mathbf{Z}_p[\![\mathbf{Z}_p]\!] \simeq \mathbf{Z}_p[\![X]\!]$, via $[p] \leftrightarrow X + 1$.

5.5. **The main example.** Let $U \subset \mathrm{Spec}(\mathbf{Z})$ be open, and put $\pi = \pi_1(U)$. Let $E \xrightarrow{e} U$ be an elliptic curve. Choose a prime $p \notin U$. The p-torsion subscheme $E[p]$ is an étale cover of U, so we get an action of π on the underlying set $E[p] \simeq (\mathbf{Z}/p)^2$. This action preserves the group structure, so we have a representation

$$\bar{\rho} = \bar{\rho}_{E,p} : \pi_1(U) \to \mathrm{GL}_2(\mathbf{F}_p).$$

Another way of constructing $\bar{\rho}$ is to use the equivalence between étale-local $\mathbf{F}_p$-systems on U and $\mathbf{F}_p$-representations of π to realize $\mathsf{R}e_*\mathbf{F}_p$ as a mod-p representation of π.

We could consider the universal deformation ring $R_{\bar{\rho}}$ and its associated formal spectrum $\mathfrak{X}(\bar{\rho})$. Recall that if $K \hookrightarrow S^3$ is a hyperbolic knot, the "knot decomposition group" $D_K = \pi_1(\text{torus})$ is abelian, so

$$\rho_K|_{D_K} \sim \begin{pmatrix} \varepsilon & * \\ & \varepsilon^{-1} \end{pmatrix},$$

for some character $\varepsilon : \Gamma_K \to \mathbf{C}^\times$. In particular, $\varepsilon(I_K) = 1$. So to make the analogy between knots and primes more precise, we should require that our residual Galois representation $\bar\rho : \pi \to \mathrm{GL}_2(\mathbf{F}_p)$ be *p-ordinary* in the sense that

$$\bar\rho|_{D_p} \sim \begin{pmatrix} \varphi & * \\ & \psi \end{pmatrix},$$

where $\psi(I_p) = 1$. This is exactly "I_p-ordinary" as defined above. So there is a p-ordinary representation $\rho^\circ : \pi \to \mathrm{GL}_2(R^\circ_{\bar\rho})$ that is universal for p-ordinary deformations, i.e. $\mathrm{Spf}(R^\circ_{\bar\rho})$ represents the functor

$$\mathfrak{X}^\circ(\bar\rho)(A) = \{\rho \in \mathfrak{X}(\bar\rho)(A) : \rho \text{ is } p\text{-ordinary}\}.$$

6. Deformations of hyperbolic structures and Hida theory

We put together all the machinery we've developed to see an analogy between the space of hyperbolic structures on a 3-manifold and the formal spectrum of Hida's big Hecke algebras.

6.1. Deformation of hyperbolic structures.

Let K be a hyperbolic knot, $M = S^3 \smallsetminus K$ the knot complement. Put $\Gamma = \pi_1(M)$. Let $T_K = \mathrm{Teich}(\Gamma)$ be the space of injective homomorphisms $\Gamma \to \mathrm{Iso}(\mathbf{H}^3)$ with discrete image, taken up to conjugacy; this is the space of hyperbolic structures on M. Since $\mathrm{Iso}(\mathbf{H}^3) = \mathrm{PSL}_2(\mathbf{C})$, we get a map

$$\phi : T_K \to X_K.$$

It's not quite straightforward – you have to fudge for this to be well defined. On a small neighborhood, $T_K^\circ \to X_K^\circ$ is an isomorphism.

Also, $X_K^\circ \to \mathbf{C}$, $\rho \mapsto \mathrm{tr}\,\rho(m)$ is locally biholomorphic. Similarly, $\mathfrak{T}^\circ(\bar\rho) \to \mathfrak{X}^\circ(\bar\rho)$, choose a generator τ of the $\mathbf{Z}_p$-quotient of $\pi_1(\mathbf{Z}[\frac{1}{p}])$. Then $\rho \mapsto \mathrm{tr}\,\rho$ is p-adically bianalytic near $\bar\rho$.

6.2. Hida theory.

We showed explicitly how to construct the Galois representations associated to cuspidal eigenforms of weight 2. In fact, in [Del73], Deligne showed how to construct $\rho_{f,l}$ for *any* cusp-eigenform f of weight $k \geqslant 2$. Recall that such forms have a Fourier expansion

$$f(z) = \sum_{n \geqslant 1} a_n(f) e^{2\pi n z}.$$

Say that f is *p-ordinary* if $a_p(f)$ is a p-adic unit. This is equivalent to $\bar\rho_{f,p}$ being an extension of an unramified character by a character. In [Hid86a, Hid86b], Hida p-adically interpolated the Galois representations $\rho_{f,p}$ coming from p-ordinary f of varying weight and level.

Fix a prime $p \geqslant 5$ and an integer n prime to p. Let f be a p-ordinary modular form of level np^r. Then there is a p-adic family $\boldsymbol{f}$ of automorphic representations containing f. In fact, Hida *explicitly* constructs a completion $\mathbf{h}^\circ$ of $\mathbf{h}$ such that for $\mathfrak{T}^\circ = \mathrm{Spf}(\mathbf{h}^\circ)$, *all* p-ordinary forms of tame level n which are congruent to f lie in $\mathfrak{T}^\circ$. Moreover, there is the associated Galois representation $\rho_{\boldsymbol{f}} : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{h}^\circ)$, such that *all* p-ordinary modular Galois representations congruent to $\rho_{f,p}$ come from $\mathbf{h}^\circ$.

6.3. **The analogy.** Let f be a p-ordinary cuspidal eigenform. Put $\rho = \rho_{f,p}$. Let $\mathfrak{T}^\circ = \mathrm{Spf}(\mathbf{h}^\circ)$ be the corresponding p-adic family of modular forms, and $\rho_{\boldsymbol{f}} : \Gamma_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{h}^\circ)$ the Galois representation. Since $\rho_{\boldsymbol{f}} \equiv \bar{\rho}_{f,p} \pmod p$ and $\rho_{\boldsymbol{f}}$ is p-ordinary, we get a map $R^\circ_{\bar{\rho}_f} \to \mathbf{h}^\circ$, equivalently

$$\phi : \mathfrak{T}^\circ \to \mathfrak{X}^\circ(\bar{\rho}).$$

The *very* hard theorem of Wiles, etc. is that ϕ is an isomorphism if f satisfies some technical hypotheses. This is the analogy of the map $T_K^\circ \to X_K^\circ$ being an isomorphism in a small neighborhood.

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