Equidistributed sequences and the analytic properties of a strange class of L-functions

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1 Motivation

Let $E_{/\mathbf{Q}}$ be an elliptic curve without complex multiplication. By an old theorem of Faltings, the quantities

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \operatorname{tr} \rho_{E,l}(\mathrm{fr}_p)$$

determine E up to isogeny. The starting point of this investigation is the corollary of a theorem of Harris, that the collection $\{\operatorname{sgn} a_p(E)\}_p$ in fact determines E up to isogeny. Ramakrishna had the insight that this fact means the "strange L-function"

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \operatorname{sgn} a_p(E) p^{-s}}$$

determines E up to isogeny. In this note, I define a more general class of strange L-functions, and show that their analytic properties are closely tied to the equidistribution of the $a_p(E)$.

Here is a brief discussion of this generalization in the case of a non-CM curve $E_{/\mathbf{Q}}$. It is convenient to repackage these traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the $\theta_p(E)$ are well-defined angles laying in the interval $[0, \pi]$. Write $\mu_{\mathrm{ST}} = \frac{2}{\pi} \sin^2\theta \, \mathrm{d}\theta$. Then the Sato–Tate conjecture (now a theorem) tells us that for any continuous function $f\colon [0, \pi] \to \mathbf{C}$, we have:

$$\left| \frac{1}{\pi(C)} \sum_{p \leqslant C} f(\theta_p) - \int_0^\pi f \, \mathrm{d}\mu_{\mathrm{ST}} \right| = o(1)$$

as $C \to \infty$. It is well-known that this is equivalent to the analytic continuation of all the L-functions $L(\operatorname{sym}^k E, s)$. We take as our starting point the stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(C)} \sum_{p \leqslant C} f(\theta_p) - \int_0^\pi f \, \mathrm{d}\mu_{\mathrm{ST}} \right| = O_f(C^{-\frac{1}{2}+\epsilon}).$$

They prove that this conjecture implies the Riemann Hypothesis for E. I prove that not only does their conjecture imply the Riemann Hypothesis for all $L(\operatorname{sym}^k E, s)$, it also does for all the strange L-functions

$$L_f(E,s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

These results make perfect sense in a much more general context, and I will prove them there. In section 2 I set up this context and carefully define strange L-functions there. In section 3, I prove basic analytic properties of the strange L-functions, and in section 4, I prove the main results connecting the analytic properties of strange L-functions with the equidistribution of a sequence. Finally, in section 6, I apply the general results to the following cases: a non-CM elliptic curve $E_{/\mathbf{Q}}$, the product $E_1 \times E_2$ of a pair of non-isogenous non-CM elliptic curves over $\mathbf{Q}$, and the Jacobian of a generic genus-2 curve $C_{/\mathbf{Q}}$.

2 Definitions

Throughout this section, let X be a compact separable metric space with no isolated points. We write X^∞ for the space of sequences in X indexed by rational primes, i.e. points $\boldsymbol{x} \in X^\infty$ are of the form $\boldsymbol{x} = (x_2, x_3, \dots)$. By [Eng89, Cor. 2.3.16 & Th. 4.2.2], the compact space X^∞ is metrizable and separable, also with no isolated points.

Definition 2.1. For $\boldsymbol{x} \in X^\infty$ and $C > 0$, write $\boldsymbol{x}^C$ for the probability measure given by

$$\int_X f \, \mathrm{d}\boldsymbol{x}^C = \boldsymbol{x}^C(f) = \frac{1}{\pi(C)} \sum_{p \leqslant C} f(x_p).$$

Let μ be a Borel measure on X. Recall that $\boldsymbol{x}$ is *μ-equidistributed* if $\boldsymbol{x}^C \to \mu$ weakly, i.e. $\boldsymbol{x}^C(f) \to \mu(f)$ for all $f \in C(X)$. In fact, we can extend this to not-necessarily-continuous functions as follows:

Theorem 2.2 (Mazzone)**.** *Let μ be a Borel measure on X and let $f\colon X \to \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if $\boldsymbol{x}^C(f) \to \mu(f)$ for all μ-equidistributed $\boldsymbol{x}$.*

Proof. This follows directly from the proof of [Maz95, Th. 1]. □

Fix a Borel measure μ on X, and write $C^{\mathrm{ae}}(X,\mu)$ for the space of bounded, almost-everywhere continuous functions $f\colon X \to \mathbf{C}$.

Theorem 2.3. *Endowed with the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$, $C^{\mathrm{ae}}(X,\mu)$ is a Banach space.*

Proof. This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero. □

Definition 2.4. Let $f \in C^{\mathrm{ae}}(X, \mu)$, $\boldsymbol{x} \in X^\infty$. The associated *strange L-function* is defined as

$$L_f(\boldsymbol{x}, s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}$$

for all $s \in \mathbf{C}$ for which the product converges.

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let G be a compact connected Lie group.

3 Preliminary results

Here we make a yet more general definition. Given $\boldsymbol{\lambda} = (\lambda_2, \lambda_3, \dots)$, with $\|\boldsymbol{\lambda}\| = \sup_p |\lambda_p| \leqslant 1$, define

$$L(\boldsymbol{\lambda}, s) = \prod_p \frac{1}{1 - \lambda_p p^{-s}}.$$

Write $A_{\boldsymbol{\lambda}}(x) = \sum_{p \leqslant x} \lambda_p$. We make the following assumption: $A_{\boldsymbol{\lambda}}(x) = O(x^{\frac{1}{2}+\epsilon})$.

Theorem 3.1. *Assume $A_{\boldsymbol{\lambda}}(x) = O(x^{\frac{1}{2}+\epsilon})$. Then $L(\boldsymbol{\lambda}, s)$ converges on $\{\Re > \frac{1}{2}\}$, and $\log L(\boldsymbol{\lambda}, s)$ has no poles on that region.*

Proof. Standard reductions reduce this to showing that

$$\sum_p \frac{\lambda_p}{p^s} \qquad \text{and} \qquad \sum_p \frac{\log(p)\lambda_p}{p^s}$$

converge on that region. We deal with $\sum \log(p)\lambda_p p^{-s}$; the other is similar. Use Abel summation:

$$\sum_{p \leqslant x} \frac{\lambda_p}{p^s} = \frac{\log x}{x^s} A_{\boldsymbol{\lambda}}(x) - \int_2^x \frac{1 - s\log t}{t^{s+1}} A_{\boldsymbol{\lambda}}(t)\,\mathrm{d}t.$$

We show that the first term approaches zero and that the integral converges absolutely. We have:

$$\left|\frac{\log x}{x^s} A_{\boldsymbol{\lambda}}(x)\right| \ll \frac{\log x}{x^{\Re s}} x^{\frac{1}{2}+\epsilon}.$$

Since ϵ is arbitrary, the exponent of x is negative. Moreover,

$$\int_2^x \frac{1}{t^{s+1}} |A_\lambda(t)|\,\mathrm{d}t \ll \int_2^x \frac{1}{t^{\Re s+1}} t^{\frac{1}{2}+\epsilon}\,\mathrm{d}t$$
$$\int_2^x \frac{\log t}{t^{s+1}} |A_{\boldsymbol{\lambda}}(t)|\,\mathrm{d}t \ll \int_2^x \frac{\log t}{t^{\Re s+1}} t^{\frac{1}{2}+\epsilon}\,\mathrm{d}t.$$

Both these integrals converge because ϵ is arbitrary. $\square$

4 Main results

Let $E_{/\mathbf{Q}}$ be an elliptic curve, or more generally, let M be a motive. The associated analytic L-function $L(M,s)$ is of the form

$$L(M,s) = \prod_p P_p(M, p^{-s})^{-1},$$

where the $P_p(M,t) \in \mathbf{Z}[t]$ have absolute value 1. In the case of $E_{/\mathbf{Q}}$, we have $pt^2 - a_p t + 1$, which are normalized to

$$(t - e^{i\theta_p})(t - e^{-i\theta_p}) = t^2 - 2\cos(\theta_p)t + 1 = t^2 - \frac{a_p}{\sqrt{p}}t + 1.$$

Let $d = \deg P_p(M,t)$. Then we can write

$$P_p(M,t) = (t - e^{i\theta_p^{(1)}}) \dots (t - e^{-i\theta_p^{(d)}}),$$

where $\theta^{(1)} < \cdots < \theta^{(d)}$ in $[0, 2\pi]$. Then

$$L(M,s) = L(\boldsymbol{\theta}^{(1)}, s) \cdots L(\boldsymbol{\theta}^{(d)}, s)$$

More general example:

$$L(\operatorname{sym}^k E, s) = L(\boldsymbol{\theta}^k, s) L(\boldsymbol{\theta}^{k-1}$$

5 Connection to Serre's perspective

Let G be a compact connected Lie group, $G^\natural$ the space of conjugacy classes in G, and $\boldsymbol{x}$ a sequence in $G^\natural$. Given $\rho \in \widehat{G}$, Serre defines an L-function

$$L(\rho, s) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}.$$

Given $x \in G^\natural$, the matrix $\rho(x)$ has eigenvalues $\lambda_p^{(1),\rho}, \dots, \lambda_p^{(\deg\rho),\rho}$ whose angles form a nondecreasing sequence in $[0, 2\pi]$. The functions $\lambda_p^{(j),\rho} \colon G^\natural \to \mathbf{C}$ are almost-everywhere continuous, and

$$L(\rho, s) = \prod_{j=0}^{\deg\rho} L(\lambda_p^{(j),\rho}, s) = \prod_{j=0}^{\deg\rho} L_{\lambda^{(j),\rho}}(\boldsymbol{x}, s).$$

6 Applications

References

[AT99] Shigeki Akiyama and Yoshio Tanigawa. "Calculation of values of L-functions associated to elliptic curves". In: *Math. Comp.* 68.227 (1999), pp. 1201–1231.

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