Abstract class field theory

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This is a reworking of the abstract class field theory in Neukirch's book *Algebraic Number Theory*.

1 The setup

Let G be a profinite group. We will denote closed subgroups of G by lowercase letters g, h, ... If g, h are two subgroups of G, then (g, h) will denote the closed subgroup generated by g and h. Let A be a continuous G-module, written additively. If $g \subset G$, write A^g for the submodule of A consisting of elements fixed by g. If $g \subset h$ and the index $[h : g]$ is finite, we have the *norm map*

$$N_{h/g} : A^g \to A^h \qquad N_{h/g}x = \sum_{\sigma \in h/g} \sigma x$$

Suppose we have a continuous surjective homomorphism $d : G \to \hat{\mathbb{Z}}$. We will write $I = \ker d$ and $\tilde{g} = g \cap I$. If $g \subset G$ is of finite index, then $dg \subset \hat{\mathbb{Z}}$ is also of finite index. We define

$$e_g = [I : \tilde{g}]$$
$$f_g = [\hat{\mathbb{Z}} : dg]$$

We define $d_g = \frac{1}{f_g}$; the map $d_g : g \to \hat{\mathbb{Z}}$ is also a continuous surjection. There are relative "ramification" and "inertia" degrees:

$$e_{h/g} = [\tilde{h} : \tilde{g}]$$
$$f_{h/g} = [dh : dg]$$

Theorem 1. *If $g \subset h$ are subgroups of G, we have $[h : g] = e_{h/g} f_{h/g}$.*

Proof. If g is normal in h, the following exact sequence

$$1 \to \tilde{h}/\tilde{g} \to h/g \to dh/dg \to 1$$

yields the desired identity. For the general case, use the transitivity of e and f. □

We also define the *Weil group* $W = d^{-1}(\mathbb{Z})$. As usual, there is a relative version:

$$W_{h/\tilde{g}} = d_h^{-1}(\mathbb{Z}) \subset h/\tilde{g}$$

I should prove that the obvious map $W_{h/\tilde{g}} \to h/g$ is surjective. In fact, we have the following general theorem:

Theorem 2. *Let $d : G \to H$ be a continuous surjective homomorphism between profinite groups. If $X \subset H$ is dense and $W = d^{-1}(X)$, then X is dense in G.*

Proof. Let g be an open normal subgroup of G; we have to show that $W \to G/g$ is surjective. Note that for $I = \ker d$ we have $(g, I) = gI$ because g is open and the two subgroups are normal. Since G/gI is a finite quotient of $G/I \simeq H$, the map $W \to G/gI$ is surjective. Thus for $\sigma \in G/g$, we have $\sigma = w$ in G/gI for some $w \in W$. In other words, $\sigma w^{-1} = \tau i$, where $\tau \in g$ and $i \in I$. But then in G/g, we have $\sigma = iw$ which is in the image of W. □

2 Abstract valuation theory

If $g \subset G$, then d_g induces an isomorphism $g/\tilde{g} \to \hat{\mathbb{Z}}$. The element of g corresponding to $1 \in \hat{\mathbb{Z}}$ will be written ϕ_g and called the *Frobenius* of g. We let $F_{h/\tilde{g}} = d_h^{-1}(\mathbb{N})$. Since $\mathbb{N}$ surjects onto $\mathbb{Z}/n$ for all n, $\mathbb{N}$ is dense in $\hat{\mathbb{Z}}$, whence $F_{h/\tilde{g}}$ is dense.

Theorem 3. *Let $\sigma \in F_{h/\tilde{g}}$, and let $l = (\sigma, \tilde{g})$. Then*

1. $f_{h/l} = d_h(\sigma)$
2. $[h : l] < \infty$
3. $\tilde{l} = \tilde{g}$
4. $\sigma = \phi_l$.

Proof. 1. We have $f_{h/l} = [dh : dl] = [\hat{\mathbb{Z}} : d_h\sigma] = d_h\sigma$.

2. From the fundamental identity $[h : l] = e_{h/l} f_{h/l}$, it is sufficient to prove $e_{h/l} < \infty$. But $e_{h/l} = [\tilde{h} : \tilde{l}] \leqslant [\tilde{h} : \tilde{g}] = e_{h/g} < \infty$.

3. There is an obvious surjection $l/\tilde{g} \twoheadrightarrow l/\tilde{l} \simeq \hat{\mathbb{Z}}$. But $l/\tilde{g}$ is procyclic, being generated by σ, and it is a general theorem that if a surjection between procyclic groups must be an isomorphism.

4. The group dl is generated by $d_l(\sigma)$, so when we normalize, $d_l(\sigma) = 1$, i.e. $\sigma = \phi_l$. □

We now consider valuations on the G-module A. A *Henselian valuation* on A is a homomorphism $v : A^G \to \hat{\mathbb{Z}}$ with image containing $\mathbb{Z}$, such that for all finite index $g \subset G$, we have

$$v(N_{G/g} A^g) = f_g v(A^G)$$

This lets us define the valuations $v_g = \frac{1}{f_g} v \circ N_{G/g}$; these satisfy the obvious compatibility properties.

We call $\pi \in A^g$ a *prime* if $v_g(\pi) = 1$, and write $U_g = \ker v_g$. We will wrote π_g for the prime of g.

3 The reciprocity map

Recall that if G is a finite group, the *Tate cohomology* of a G-module A is a $\mathbb{Z}$-graded abelian group $\hat{H}^\bullet(G, A)$ defined as follows:

$$\hat{H}^n(G, A) = \begin{cases} H^n(G, A) & \text{if } n > 0 \\ A^G/NA & \text{if } n = 0 \\ {}_N A / \mathfrak{j}_G A & \text{if } n = -1 \\ H_{-n-1}(G, A) & \text{if } n < -1 \end{cases}$$

Here $N = N_G = \sum_{\sigma \in G} \sigma$, $\mathfrak{j}_G$ is the *augmentation ideal* generated by $\{\sigma - 1 : \sigma \in G\}$, and ${}_N A = \{x \in A : Nx = 0\}$. It turns out (though we will not use this fact) that $\hat{H}^\bullet$ forms a cohomology theory. We will only use $\hat{H}^0$ and $\hat{H}^{-1}$.

Returning to the case where G is profinite: from here on out, we will assume the following axiom:

$$\hat{H}^i(h/g, U_g) = 0 \qquad \text{for } i \in \{-1, 0\}$$

If $h \supset g \supset k$ are subgroups of G with $[h : k] < \infty$, then $N_{h/k} = N_{h/g} N_{g/k}$ implies $N_{h/k} A^k \subset N_{h/g} A^g$. Set, for arbitrary inclusions $h \supset g$:

$$N_{h/g} A^g = \bigcap_k N_{h/k} A^k$$

where k ranges over $h \supset k \supset g$ with $[h : k] < \infty$. The previous remark shows that this agrees with the usual definition if we already have $[h : g] < \infty$.

We define the *reciprocity map* first in a special case. It will be, for $G \supset h \supset g$ with $[G:h]$ finite, a map

$$r_{h/\tilde{g}} : F_{h/\tilde{g}} \to A^h / N_{h/\tilde{g}} A^{\tilde{g}}$$

For $\sigma \in F_{h/\tilde{g}}$, let $l = (\sigma, \tilde{g})$. A previous theorem shows that $[h:l] < \infty$, so it makes sense to define v_l and say that

$$r_{h/\tilde{g}}(\sigma) = N_{h/l}(\pi_l) \mod N_{h/\tilde{g}} A^{\tilde{g}}$$

First, we need to show that this is independent of the choice of π_l. Two different choices will differ by an element $u \in U_l$; it suffices to prove that $N_{h/l} u \in N_{h/\tilde{g}} A^g$. So we need $N_{h/l} u \in N_{h/k} A^k$ for all $h \supset k \supset \tilde{g}$ with $[h:k] < \infty$. Replacing k by $k \cap l$ if necessary, we may assume $k \subset l$. We then apply the axiom $\hat{H}^0(l/k, U_k) = 0$ to find $x \in U_k$ with $N_{l/k} x = u$. It follows that $N_{h/l} u = N_{h/l} N_{l/k} x = N_{h/k} x \in N_{h/k} A^k$, so $r_{h/\tilde{g}}$ is well-defined.

For the remainder of this section, fix $g \lhd h \subset G$ with $[G:g] < \infty$. We set $N = N_{\tilde{h}/\tilde{g}}$, and for general σ we set $\sigma_n = 1 + \cdots + \sigma^{n-1}$. This yields the formal identity

$$(\sigma - 1)\sigma_n = \sigma_n(\sigma - 1) = \sigma^n - 1$$

Theorem 4. *Fix $\phi, \sigma \in F_{h/\tilde{g}}$ with $d_h \phi = 1$ and $d_h \sigma = n$. If $l = (\sigma, \tilde{g})$, then*

$$N_{h/l} = \phi_n N = N \phi_n$$

Proof. Let $l = (\sigma, \tilde{h})$; we clearly have $N_{h/l} = N_{h/l_0} N_{l_0/l}$. Moreover, it is easy to see that $N_{h/l_0} = \phi_n$. To see that $N_{\tilde{h}/\tilde{g}} = N_{l_0/l}$, one simply needs to check that $l_0 = (l, \tilde{h})$ and $\tilde{g} = l \cap \tilde{h}$, which is not hard. Finally, ϕ_n and N commute because $\tilde{h}$ is normal in h. □

For arbitrary groups G and G-modules A, let $H_0(G, A) = A/\mathfrak{j}_G A$. In our case, we will consider $H_0(\tilde{h}/\tilde{g}, U_{\tilde{g}})$ and set $\mathfrak{j} = \mathfrak{j}_{\tilde{h}/\tilde{g}}$. It is easy to check that N descends to a map $H_0(\tilde{h}/\tilde{g}, U_{\tilde{g}}) \to U_{\tilde{h}}$. In fact, we have the following

Theorem 5. *Suppose $\phi \in h$ has $d_h \phi = 1$. Then N restricts to a map*

$$N : H_0(\tilde{h}/\tilde{g}, U_{\tilde{g}})^{\phi} \to N_{h/\tilde{g}} U_{\tilde{g}}$$

Proof. First, note that the action of h on $U_{\tilde{g}}$ is well-defined because $\tilde{g}$ is normal in h. Suppose $x = \bar{u} \in U_{\tilde{g}}$ with $\phi x = x$. Then we have

$$(\phi - 1)u = \sum (\tau_i - 1) u_i$$

for some $\tau_i \in \tilde{h}$ and $u_i \in U_{\tilde{g}}$. We wish to show that $Nu \in N_{h/m} U_m$ for all $h \supset m \supset \tilde{g}$ with $[h:m] < \infty$. It is clearly sufficient to assume $m \subset g$. □