L-functions of elliptic curves with complex multiplication

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Let $A_{/\mathbf{Q}}$ be an abelian variety with complex multiplication (over $\mathbf{Q}$!) by F. That is, if we write $\mathrm{End}^\circ(A) = \mathrm{End}_{\mathbf{Q}}(A) \otimes \mathbf{Q}$, then $F \simeq \mathrm{End}^\circ(A)$. Let p be a prime at which A is unramified. Then

$$\mathrm{fr}_p \in \mathrm{End}^\circ(A_{\mathbf{F}_p}) = \mathrm{End}^\circ(A) = F.$$

Moreover, for each prime l, the Galois representation $\rho_{A,l}$ takes values in $F_l^\times = (F \otimes \mathbf{Q}_l)^\times$. In fact, if we write O for the ring of integers of F, then $\rho_{A,l}\colon G_{\mathbf{Q}} \to O_l^\times$. Moreover, for each p, $\mathrm{fr}_p \in F^\times$ as a p-Weil number of weight 1, i.e. $|\mathrm{fr}_p| = \sqrt{p}$ under each embedding $F \hookrightarrow \mathbf{C}$.

We will define an algebraic Hecke character associated with E. For every unramified prime p, the Frobenius fr_p lives in $F^\times$. There is, for each $\sigma\colon F \hookrightarrow \mathbf{C}$, a weight-1 Hecke character $\chi_\sigma\colon \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$, such that

$$L^{\mathrm{alg}}(A, s) = \prod_{\sigma\colon F \hookrightarrow \mathbf{C}} L(s, \chi_\sigma).$$

Thus

$$L^{\mathrm{an}}(A, s) = L^{\mathrm{alg}}\left(A, s + \frac{1}{2}\right) = \prod_\sigma L\left(s + \frac{1}{2}, \chi_\sigma\right),$$

where $L(s+1/2, \chi_\sigma) = L(s, \chi_\sigma \|\cdot\|^{-1/2})$, for $\|\cdot\|\colon \mathbf{A}^\times \to \mathbf{R}^+$ the adele norm.

The Sato–Tate group for A is the compact torus $\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}^{\mathrm{N}=1}(\mathbf{R}) = F_\infty^{\times,\mathrm{N}=1}$, which is isomorphic to $\prod_{\sigma\in\Phi} S^1$. The group of characters of $\mathrm{ST}(A)$ is generated by $\{\chi_\sigma\}$, so we know that if the Akiyama–Tanigawa conjecture for A is true, then each $L(s, \prod \chi_\sigma^{m_\sigma})$ satisfies the Riemann Hypothesis. My counterexample shows: the converse does not hold! Even if all the $L(s, \prod \chi_\sigma^{m_\sigma})$ satisfy the Riemann Hypothesis, it does not follow that Akiyama–Tanigawa holds.

1 Tate's thesis

Consider $\mathbf{A} = \mathbf{A}_\mathbf{Q}$. A *Hecke character* is a continuous homomorphism $\chi\colon \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$. First, note that the obvious map $\mathbf{R}^\times \times \widehat{\mathbf{Z}}^\times \to \mathbf{A}^\times/\mathbf{Q}^\times$ is an isomorphism. The character χ is *algebraic of weight w* if $\chi|_{\mathbf{R}^+} = (-)^{-m}$; so $\|\cdot\|$, the adele norm, is algebraic of weight -1. Since $G_\mathbf{Q}^{\mathrm{ab}} = \widehat{\mathbf{Z}}^\times$, a "Hecke character" is just a Dirichlet character + a quasicharacter of $\mathbf{R}^\times$, which is determined by its weight (algebraic or not), and sign.

Let χ be an algebraic Hecke character of weight w, l a rational prime. Then there is a corresponding Galois representation which we'll write $\chi_l\colon \widehat{\mathbf{Z}}^\times \to S^1$, given by

$$\chi_l(x) = x_l^{-w}\chi_{\mathrm{f}}(x)$$

2 The whole story

Let $A_{/\mathbf{Q}}$ be an absolutely simple abelian variety with CM type (F,Φ) defined over $\mathbf{Q}$, where $F = \mathrm{End}(A)_\mathbf{Q}$ and $\mathrm{hom}(F,\mathbf{C}) = \Phi \sqcup \overline{\Phi}$. There is a Galois representation $\rho_{A,l}\colon G_\mathbf{Q} \to F_l^\times \subset \mathrm{GL}_{2g}(\mathbf{Q}_l)$. We wish first to compute the motivic Galois group G_A. This will contain the Sato–Tate group $\mathrm{ST}(A)$, which is equal to the Mumford–Tate group of A. The main thing we need to do is compute $\mathrm{X}^*(G_A)$, show that it is equal to $\widehat{\mathrm{ST}(A)}$, and demonstrate a "reciprocity law" relating motivic and Hecke L-functions between the two.

Let L be the Galois closure of F; there is a norm map $\mathrm{N}_{\Phi^{-1}}: L^\times \to L^\times$, which sends $x \mapsto \prod_{\sigma\in\Phi} \sigma^{-1}(x)$, keeping in mind that $\Phi \subset \mathrm{Gal}(L/\mathbf{Q})$. Claim: $\mathrm{N}_{\Phi^{-1}}: L^\times \to F^\times$; call this map $\psi\colon \mathrm{R}_{L/\mathbf{Q}}\,\mathbf{G}_\mathrm{m} \to \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}$. On the level of characters, we have

$$\mathrm{N}^*_{\Phi^{-1}}\colon \mathrm{X}^*(\mathrm{R}_{L/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}) \to \mathrm{X}^*(\mathrm{R}_{L/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}),$$

and $\ker(\mathrm{N}^*_{\Phi^{-1}}) = \mathrm{X}^*(G_A)$. Another definition is: $\psi = \mathrm{N}_{\Phi_E} \circ \mathrm{N}_{F/E}$, where E is the reflex field of (F, Φ).

Let $G_A = \mathrm{im}(\psi) \subset \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}$.

3 Correct picture

The key fact is: *no* abelian variety has CM defined over $\mathbf{Q}$. Let K be a number field (which we may take to contain F and be Galois over $\mathbf{Q}$) and $A_{/K}$ an absolutely simple abelian variety with CM defined over K, and $F = \mathrm{End}^\circ(A)$. Let $\mathfrak{a} = \mathrm{Lie}(A)$; this is a K-vector space with F-action. The determinant gives us a natural map $\det_\mathfrak{a}\colon \mathrm{R}_{K/\mathbf{Q}}\,\mathbf{G}_\mathrm{m} \to \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}$; Serre–Tate refer to this as ψ. The motivic Galois group of A is $G_A = \mathrm{im}(\det_\mathfrak{a})$, and the canonical subgroup is $G_A^1 = \mathrm{im}(\det_\mathfrak{a})^{\mathrm{N}_{F/\mathbf{Q}}=1} \subset \mathrm{SL}(2g)$. The Sato–Tate group of A is the maximal compact subgroup of $G_A^1(\mathbf{C})$; this is a compact torus whose representations coincide with the complex representations of G_A^1. Now, $\mathrm{X}^*(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}) \twoheadrightarrow \mathrm{X}^*(G_A^1)$, so any representation of G_A^1 is induced by one of $\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}$. Here, we're on familiar ground. For $r \in \mathrm{X}^*(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m})$, the function $L(r_*\rho_{A,l}, s)$ is writable in terms of Hecke characters. That is, there is an explicit Hecke character ω_r such that $L(r_*\rho_{A,l}, s) = L(s, \omega_r)$, possibly up to twist. Let's do the details!

First, for a prime l, write

$$\rho_l = \rho_{A,l}\colon G_\mathbf{Q} \to G_A(\mathbf{Q}_l) \subset (\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m})(\mathbf{Q}_l) = F_l^\times$$

for the associated l-adic Galois representation. For $\sigma\colon F \hookrightarrow \mathbf{C}$, there is a Hecke character χ_σ such that

$$\chi_\sigma(\mathfrak{p}) = \sigma(\rho_l(\mathrm{fr}_\mathfrak{p})),$$

from which it follows that $L^{\mathrm{alg}}(\sigma \circ \rho_l, s) = L(s, \chi_\sigma)$. Now $L(A, s) = L^{\mathrm{alg}}(A, s + 1/2)$, so we set $\omega_\sigma = \chi_\sigma \| \cdot \|^{-1/2}$. Then

$$L(A, s) = \prod_{\sigma\colon F \hookrightarrow \mathbf{C}} L(s, \omega_\sigma),$$

and for any $r = \sum a_\sigma \sigma \in \mathrm{X}^*(\mathrm{R}_{F/\mathbf{Q}}\, \mathbf{G}_{\mathrm{m}})$, $L(r_* \rho_l, s) = L(s, \omega_r)$, where we put

$$\omega_r = \prod_{\sigma\colon F \hookrightarrow \mathbf{C}} \omega_\sigma^{a_\sigma}.$$

Since each ω_r is a Hecke character, it has analytic continuation past $\Re = 1$, so the Sato–Tate conjecture holds for A. However, the "Diophantine Approximation counterexample" shows that even if each $L(r_* \rho_l, s)$ satisfies the Riemann Hypothesis, it does not immediately follow that the Akiyama–Tanigawa conjecture holds for A.