Completed cohomology, deformation theory, and a torsion local Langlands correspondence

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1 Definitions

Consider the group $\mathrm{GL}(2)_{/\mathbf{Q}}$. For each open compact $K \subset \mathrm{GL}_2(\mathbf{A}_\mathrm{f})$, let X_K be the variety over $\mathbf{Q}$ underlying the (compactification of the quotient

$$Y_K(\mathbf{C}) = \mathrm{GL}_2(\mathbf{Q})\backslash \left(\mathbf{H}^{\pm} \times \mathrm{GL}_2(\mathbf{A}_\mathrm{f})\right)/K.$$

The projective system $\{X_K\}$ admits an action of $\mathrm{GL}_2(\mathbf{A}_\mathrm{f})$. Moreover, if ρ is a representation of $\mathrm{GL}(2)_{/\mathbf{Q}}$, there is a canonical sheaf, also denoted ρ, on the projective system $\{X_K\}$.

Let k be a finite field, $\mathrm{W}(k)$ its ring of Witt vectors. For any Artinian $\mathrm{W}(k)$-algebra A, put

$$\mathrm{H}^\bullet(\rho)_A = \varinjlim_{K \subset \mathrm{GL}_2(\mathbf{A}_\mathrm{f})} \mathrm{H}^1_{\text{ét}}\left((X_K)_{\overline{\mathbf{Q}}}, \rho_A\right).$$

This is an $A[\Gamma_\mathbf{Q} \times \mathrm{GL}_2(\mathbf{A}_\mathrm{f})]$-module.

If $w \geqslant 0$ is an integer, we put $\mathrm{H}^\bullet(w)_A = \mathrm{H}^\bullet(\mathrm{sym}^{w-2})_A$. If $A = \varprojlim A_i$ is a pro-artinian $\mathrm{W}(k)$-module, put $\mathrm{H}^\bullet(\rho)_A = \varprojlim \mathrm{H}^\bullet(\rho)_{A_i}$.

2 Some deformation theory

For a residual representation $\bar\rho : \Gamma_\mathbf{Q} \to \mathrm{GL}_2(k)$, we write $\mathfrak{X} = \mathfrak{X}(\bar\rho)$ for the deformation functor classifying lifts $\Gamma_{\mathbf{Q},S} \to \mathrm{GL}_2(k)$, for some unspecified S. To be precise, we are considering $\mathfrak{X}(\bar\rho)$ as an ind-(formal scheme). Assume $\bar\rho$ is odd and absolutely irreducible; then $\bar\rho$ is modular. By [Eme11, 1.2.6], there is a natural isomorphism

$$\bar\pi(\bar\rho) \simeq \hom_{\Gamma_\mathbf{Q}}(\bar\rho, \mathrm{H}^1_k)$$

of $\mathrm{GL}_2(\mathbf{A}_\mathrm{f})$-modules, assuming some technical conditions on $\bar\rho$. In particular, the hom-set is non-zero.

We define a functor $\mathfrak{H}(\bar\rho)$ on local artinian $\mathrm{W}(k)$-algebras with residue field k. For such an algebra A, we let $\mathfrak{H}(\bar\rho)(A)$ be the set of pairs (ρ, f), where $\rho \in \mathfrak{X}(\bar\rho)(A)$ and $f : \rho \to \mathrm{H}^1_A$ is $A[\Gamma_\mathbf{Q}]$-linear and reduces to some specified $\bar f : \bar\rho \hookrightarrow \mathrm{H}^1_k$.

3 Ordinary parts

We work out [Eme10a; Eme10b] for the group $\mathrm{GL}_2(\mathbf{Q}_p)$. Let k be a finite field, $\mathrm{W}(k)$ its ring of Witt vectors, and A an artinian local $\mathrm{W}(k)$-algebra. Let M be a locally profinite abelian group, $M^+ \subset M$ an open sub-semigroup. Let π be a finitely generated A-module with smooth M^+-action. Put

$$\pi^{\mathrm{ord}} = \hom_{M^+}(A[M], \pi).$$

Theorem 3.1. *The natural map $\pi^{\mathrm{ord}} \to \pi$ given by evaluation at 1 induces an isomorphism between π^{ord} and the maximal $A[M^+]$-submodule of π on which M^+ acts invertibly.*

Proof. This is essentially the proof of [Eme10a, 3.1.5]. Let B be the image of $A[M^+]$ in $\mathrm{End}_A(\pi)$. Then there is a $B = \prod B_{\mathfrak{p}}$, where each $B_{\mathfrak{p}}$ is local Artinian. This induces a decomposition $\pi = \prod \pi_{\mathfrak{p}}$. Call $\mathfrak{p}$ *ordinary* if M^+ acts invertibly on $\pi_{\mathfrak{p}}$, and *non-ordinary* otherwise. We claim that if $\mathfrak{p}$ is ordinary, then $(\pi_{\mathfrak{p}})^{\mathrm{ord}} = \pi_{\mathfrak{p}}$, and that $(\pi_{\mathfrak{p}})^{\mathrm{ord}} = 0$ otherwise. The first claim is obvious: if M^+ acts invertibly on $\pi_{\mathfrak{p}}$, then its action extends uniquely to one of M. If some $m^+ \in M^+$ does not act invertibly on $\pi_{\mathfrak{p}}$, it acts nilpotently, and we may as well assume that $m^+\pi_{\mathfrak{p}} = 0$. But then for $\phi : A[M] \to \pi$, we have

$$\phi(m) = m^+ \cdot \phi\left((m^+)^{-1} m\right) = 0,$$

so $\phi = 0$. $\square$

Now let $M \subset \mathrm{GL}_2(\mathbf{Q}_p)$ be the subgroup $\begin{pmatrix} * & \\ & * \end{pmatrix}$ of diagonal matrices, and M^+ be the sub-semigroup consisting of those $\begin{pmatrix} a & \\ & b \end{pmatrix}$ with $|a| \geqslant |b|$. Note that if we put $K = \mathrm{GL}_2(\mathbf{Z}_p) \subset \mathrm{GL}_2(\mathbf{Q}_p)$, then by [Bum97, 4.6.2], the natural map $M^+ \to K\backslash G/K$ is surjective. In particular, if π is a spherical representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, it should be determined by its restriction to M^+. In fact, it should be determined by the action of $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$. If π is a smooth G-representation, we define

$$\pi^{\mathrm{ord}} = \hom_{M^+}\left(A[M], \pi^{N_0}\right)_{M-\text{finite}},$$

where $N_0 = \begin{pmatrix} 1 & \\ \mathbf{Z}_p & 1 \end{pmatrix}$. In the notation of [Eme10a], the functor $(-)^{\mathrm{ord}}$ is $\mathrm{Ord}_{\overline{B}}$. By [Eme10a, 4.4.6], if π is a smooth G-representation and ρ is a smooth B-representation, then there is a natural isomorphism

$$\hom_G\left(\mathrm{ind}_B^G \rho, \pi\right) = \hom_M\left(\rho, \pi^{\mathrm{ord}}\right).$$

Consider $\mathrm{R}^1\pi^{\mathrm{ord}}$

4 Representations of $\mathrm{GL}(2)_{/\mathbf{Q}}$

Consider the split reductive group $\mathrm{GL}(2)_{/\mathbf{Q}}$. It has maximal torus

$$T = \begin{pmatrix} * & \\ & * \end{pmatrix} \subset \mathrm{GL}(2).$$

We identify $\mathrm{X}^*(T)$ with $\mathbf{Z}^2 = \langle \chi_1, \chi_2 \rangle$ via $\chi_i(g) = g_{ii}$. We have $\mathfrak{gl}(2) = \mathfrak{t} \oplus \mathfrak{gl}(2)_{\chi_1-\chi_2} \oplus \mathfrak{gl}(2)_{\chi_2-\chi_1}$. In particular, if we put $\alpha = \chi_1 - \chi_2$, we have $R = \{\pm\alpha\}$. We identify $\mathrm{X}_*(T)$ with $\mathrm{X}^*(T)$ in the obvious way, e.g. $\chi_1(g) = \begin{pmatrix} g & \\ & 1 \end{pmatrix}$. Under this identification, $(\pm\alpha)^\vee = \pm\alpha$, and the group $W \simeq S_2$ is generated by $(\chi_1, \chi_2) \mapsto (\chi_2, \chi_1)$.

The root lattice $Q = \mathbf{Z} \cdot R = \mathbf{Z}\alpha$ consists of all $n\chi_1 - n\chi_2$. Similarly, $X_0 = \{n\chi_1 + n\chi_2\}$. The weight lattice is $P = \mathbf{Z}\lambda = \{\frac{n}{2}(\chi_1 - \chi_2)\}$, where $\lambda = \frac{1}{2}\alpha$. Thus $P^+ = \mathbf{Z}_{\geqslant 0}\lambda$. The space of dominant weights is $X^+ = 2\mathbf{N} \cdot \lambda + X_0 = \{m\chi_1 + n\chi_2 : m \geqslant n\}$.

The standard representation sym^1 of $\mathrm{GL}(2)$ has highest weight χ_1. Similarly, sym^k has highest weight $k\chi_1$. So $\mathrm{sym}^k \otimes \det^d$ has highest weight $k\chi_1 + d(\chi_1 + \chi_2) = (d+k)\chi_1 + \chi_2$. To sum it up, we have the following:

Theorem 4.1. *Up to isomorphism, every irreducible representation of $\mathrm{GL}(2)$ is of the form $\mathrm{sym}^k \otimes \det^d$ for $k \geqslant 0$, $d \in \mathbf{Z}$. Such a representation has highest weight $(d+k, d)$.*

5 Locally symmetric spaces

We will continue to work with the group $\mathrm{GL}(2)_{/\mathbf{Q}}$. Let $A = Z(G)$ be the maximal split central torus in G. Let $M = \bigcap_{\chi \in \mathrm{X}^*(G)} \ker(\chi) = \mathrm{SL}(2)$, $\mathfrak{m} = \mathrm{Lie}(M)$. Then $\mathrm{Lie}(A) = \mathfrak{t}$ and $\mathfrak{gl}(2) = \mathfrak{m} \oplus \mathfrak{t}$. For $K \subset \mathrm{GL}_2(\mathbf{A}_\mathrm{f})$, put

$$Y_K = \mathrm{GL}_2(\mathbf{Q}) A(\mathbf{R})^\circ \backslash \mathrm{GL}_2(\mathbf{A}) / K_\infty K.$$

Since $A(\mathbf{R})^\circ \backslash \mathrm{GL}_2(\mathbf{R}) / K_\infty = \mathbf{H}^\pm$, this can be rewritten as

$$Y_K = \mathrm{GL}_2(\mathbf{Q}) \backslash \left(\mathbf{H}^\pm \times \mathrm{GL}_2(\mathbf{A}_\mathrm{f}) \right) / K.$$

The space of connected components of Y_K is naturally isomorphic to $\widehat{\mathbf{Z}}^\times / \det(K)$. It is known that Y_K is a moduli space of elliptic curves with level-K structure. As such, it has the canonical structure of a curve over $\mathbf{Q}$.

References

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