Galois representations from nilpotent completions

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Introductory sources for some of the following material are [KKL98] and [Iha91].

1 Generalities on nilpotent completion

Let Γ be a profinite group. There is a natural action of Γ on itself by inner automorphisms: ${}^x y = xyx^{-1}$. The image of the induced homomorphism $\Gamma \to \operatorname{Aut}\Gamma$ is a normal subgroup because

$$\phi\, {}^x(-)\phi^{-1}(y) = \phi\left(x\phi(y)x^{-1}\right) = \phi(x)y\phi(x)^{-1} = {}^{\phi(x)}y.$$

In other words, the homomorphism $\Gamma \to \operatorname{Aut}\Gamma$ is $\operatorname{Aut}\Gamma$-equivariant. Write $\operatorname{Out}\Gamma$ for the quotient of $\operatorname{Aut}\Gamma$ by the subgroup of inner automorphisms.

Write $[-,-]$ for the standard bracket operation: $[x,y] = xyx^{-1}y^{-1}$. We will frequently use the following identities relating $[-,-]$ and inner automorphisms (see [Hal76, 10.2]):

$$\begin{aligned} [x,y]^{-1} &= [y,x] \\ [x,[y,z]] &= {}^x[y,z]\cdot[z,y] \\ {}^x[y,z] &= [x,[y,z]]\cdot[y,z]. \end{aligned} \tag{1}$$

We define the *lower central series* of Γ by the rule

$$\begin{aligned} \Gamma_1 &= \Gamma \\ \Gamma_{n+1} &= [\Gamma,\Gamma_n]. \end{aligned}$$

Here $[\Gamma,\Gamma_n]$ is the closed subgroup of Γ_n generated by all $[x,y]$ with $x \in \Gamma$, $y \in \Gamma_n$. The identity ${}^x[y,z] = [x,[y,z]]\cdot[y,z]$ implies that Γ_{n+1} is normal in Γ_n, and that inner automorphisms act trivially on the quotient Γ_n/Γ_{n+1}. Thus, each quotient Γ_n/Γ_{n+1} is abelian, so it we can define

$$\operatorname{Lie}\Gamma = \bigoplus_{n\geqslant 1}(\operatorname{Lie}\Gamma)_n = \bigoplus_{n\geqslant 1}\Gamma_n/\Gamma_{n+1}.$$

I claim that the bracket on Γ induces the structure of a graded Lie algebra on $\mathfrak{g} = \operatorname{Lie}\Gamma$. Tautologically, $[\mathfrak{g}_1,\mathfrak{g}_1] \subset \mathfrak{g}_2$.

$$[x_n,[y_m,z_m]] = {}^{x_n}.$$

References

[Hal76] Marshall Hall, Jr. *The theory of groups*. AMS Chelsea Publishing Co., New York, NY, 1976. Reprinting of the 1968 edition.

[Iha91] Yasutaka Ihara. Braids, Galois groups, and some arithmetic functions. In *Proceedings of the International Congress of Mathematicians, Vol. I, I (Kyoto, 1990)*, pages 99–120, Tokyo, 1991. Math. Soc. Japan.

[KKL98] Helmut Koch, Susanne Kukkuk, and John Labute. Nilpotent local class field theory. *Acta Arith.*, 83(1):45–64, 1998.