

Equidistribution and the analytic properties of a strange class of L -functions

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1 Motivation

Let E/\mathbf{Q} be an elliptic curve without complex multiplication. By an old theorem of Faltings [Fal83], the quantities

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \mathrm{tr} \rho_{E,l}(\mathrm{fr}_p)$$

determine E up to isogeny. That is, if E_1 and E_2 satisfy $a_p(E_1) = a_p(E_2)$ for all p , then E_1 and E_2 are isogenous. The starting point of this investigation is the corollary of a theorem of Harris, that the collection $\{\mathrm{sgn} a_p(E)\}_p$ in fact determines E up to isogeny. Ramakrishna had the insight that this fact means the “strange L -function”

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \mathrm{sgn} a_p(E) p^{-s}}$$

determines E up to isogeny. In this note, I define a more general class of strange L -functions, and show that their analytic properties are closely tied to the equidistribution of the $a_p(E)$.

Here is a brief discussion of this generalization in the case of a non-CM curve E/\mathbf{Q} . It is convenient to repackage the traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the $\theta_p(E)$ are well-defined angles laying in the interval $[0, \pi]$. Write $\mathrm{dST} = \frac{2}{\pi} \sin^2 \theta \, \mathrm{d}\theta$. Then the Sato–Tate conjecture (now a theorem [BL+11]) tells us that for any continuous function $f: [0, \pi] \rightarrow \mathbf{C}$, we have

$$\left| \frac{1}{\pi(C)} \sum_{p \leq C} f(\theta_p) - \int_0^\pi f \, \mathrm{dST} \right| = o(1)$$

as $C \rightarrow \infty$. It is well-known that this follows from the analytic continuation (past $\Re s = 1$) and non-vanishing except at $s = 1$ of all the L -functions

$L(\text{sym}^k E, s)$ [Ser68, A.1, Th.1]. We take as our starting point the much stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(C)} \sum_{p \leq C} f(\theta_p) - \int_0^\pi f \, d\mu_{\text{ST}} \right| = O_f(C^{-\frac{1}{2}+\epsilon})$$

for all continuous f . (Their conjecture is actually more general; we will discuss the precise statement later.) They prove that this conjecture implies the Riemann Hypothesis for E . I prove that not only does their conjecture imply the Riemann Hypothesis for all $L(\text{sym}^k E, s)$, it also does for all the strange L -functions

$$L_f(E, s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

These results make perfect sense in a much more general context, and I will prove them there. In [section 2](#) I set up this context and carefully define strange L -functions. In [section 3](#), I prove basic analytic properties of the strange L -functions and connect their analytic properties with the equidistribution of a sequence. In [section 4](#), I apply these results where “everything is known,” i.e. varieties over function fields. Finally, in [section 5](#), I apply the general results to the following cases: a non-CM elliptic curve E/\mathbf{Q} , the product $E_1 \times E_2$ of a pair of non-isogenous non-CM elliptic curves over \mathbf{Q} , and the Jacobian of a generic genus-2 curve C/\mathbf{Q} .

2 Definitions

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| \leq 1\}$. Write \mathbf{D}^∞ for the set of sequences in \mathbf{D} indexed by the primes, i.e. $\mathbf{z} \in \mathbf{D}^\infty$ is (z_2, z_3, \dots) . The space \mathbf{D}^∞ is compact, and comes naturally equipped with the (product) Lebesgue measure, normalized to have mass 1.

Definition 2.1. Let $\mathbf{z} \in \mathbf{D}^\infty$. The associated *strange L -function* is given by

$$L(\mathbf{z}, s) = \prod_p \frac{1}{1 - z_p p^{-s}},$$

wherever this product converges.

Elementary topology tells us that $L: \mathbf{D}^\infty \times \mathbf{C}^{\Re > 1} \rightarrow \mathbf{C}$ is continuous. We will see that for fixed $\mathbf{z} \in \mathbf{D}^\infty$, the analytic properties of $L(\mathbf{z}, s)$ are closely tied to estimates for the sums $A_{\mathbf{z}}(x) = \sum_{p \leq x} z_p$. One often gets such estimates in the context of equidistribution, which we consider next.

For the remainder of this section, let X be a compact separable metric space with no isolated points. We write X^∞ for the space of sequences in X indexed by rational primes, i.e. points $\mathbf{x} \in X^\infty$ are of the form $\mathbf{x} = (x_2, x_3, \dots)$. By [Eng89, Cor.2.3.16, Th.4.2.2], the compact space X^∞ is metrizable and separable, also with no isolated points.

Definition 2.2. For $\mathbf{x} \in X^\infty$ and $C > 0$, write \mathbf{x}^C for the probability measure given by

$$\int_X f d\mathbf{x}^C = \mathbf{x}^C(f) = \frac{1}{\pi(C)} \sum_{p \leq C} f(x_p).$$

Let μ be a Borel measure on X . Recall that \mathbf{x} is μ -*equidistributed* if $\mathbf{x}^C \rightarrow \mu$ weakly, i.e. $\mathbf{x}^C(f) \rightarrow \mu(f)$ for all $f \in C(X)$. In fact, we can extend this to not-necessarily-continuous functions as follows:

Theorem 2.3 (Mazzone). *Let μ be a Borel measure on X and let $f: X \rightarrow \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if $\mathbf{x}^C(f) \rightarrow \mu(f)$ for all μ -equidistributed \mathbf{x} .*

Proof. This follows directly from the proof of [Maz95, Th.1]. \square

Fix a Borel measure μ on X , and write $C^{\text{ae}}(X, \mu)$ for the space of bounded, almost-everywhere continuous functions $f: X \rightarrow \mathbf{C}$.

Theorem 2.4. *Endowed with the supremum norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $C^{\text{ae}}(X, \mu)$ is a Banach space.*

Proof. This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero. \square

Definition 2.5. Let $f \in C^{\text{ae}}(X, \mu)^{\|\cdot\|_\infty \leq 1}$, $\mathbf{x} \in X^\infty$. The associated *strange L -function* is defined as

$$L_f(\mathbf{x}, s) = L(f(\mathbf{x}), s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}$$

for all $s \in \mathbf{C}$ for which the product converges.

Our typical source of a strange L -function is as follows. Let G be a compact connected Lie group and $X = G^\natural$, the space of conjugacy classes of G . Then G^\natural inherits the Haar measure from G . Given any sequence $\mathbf{x} \in (G^\natural)^\infty = G^{\natural, \infty}$ and function $f \in C^{\text{ae}}(G^\natural)^{\|\cdot\|_\infty \leq 1}$, we can define $L_f(\mathbf{x}, s)$. This is related to Serre's L -functions from [Ser68, A.2] as follows.

Theorem 2.6. *Let G be a compact connected Lie group, $\rho \in \widehat{G}$ an irreducible unitary representation of G . Then there exist functions $\lambda_\rho^1, \dots, \lambda_\rho^{\deg \rho}: G^\natural \rightarrow S^1$, continuous away from the set $\{\det(1 - \rho) = 0\}$, such that for every $x \in G^\natural$, there are angles $\theta_1, \dots, \theta_{\deg \rho} \in [0, 2\pi)$, satisfying $\theta_1 \leq \dots \leq \theta_{\deg \rho}$, such that $\lambda_\rho^j(x) = e^{i\theta_j}$ and moreover*

$$\det(1 - \rho(x)t) = \prod_{j=0}^{\deg \rho} (1 - \lambda_\rho^j(x)t).$$

Proof. This follows easily from [KS99, Lem.1.0.9]. \square

Recall that for $\rho \in \widehat{G}$, Serre defines $L(\rho, s) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}$. Using his notation, there is the identity

$$L(\rho, s) = \prod_{j=1}^{\deg \rho} L_{\lambda_\rho^j}(\mathbf{x}, s).$$

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let G be a compact connected semisimple Lie group. We will define discrepancy for sequences in G^\natural .

Let G^{sc} be the simply-connected cover of G . Choose a maximal torus $T \subset G^{\text{sc}}$; let $W = N(T)/T$ be the Weyl group. Let $\mathfrak{t} = \text{Lie}(T)$ and recall that the kernel of $\exp: \mathfrak{t} \rightarrow T$ is generated by the nodal vectors associated to the root system $R(G^{\text{sc}}, T)$ [Lie7-9, 9.6 Pr.11]. Write $\{t_1, \dots, t_r\} \subset \mathfrak{t}$ for these vectors. The exponential map $\exp: \mathfrak{t} \rightarrow T$ induces an isomorphism $\mathfrak{t}/(\langle t_i \rangle \rtimes W) \rightarrow G^\natural$. Given $x = (x_1, \dots, x_r) \in [0, 1]^r$, write

$$I_x = \left\{ \sum_{i=1}^r a_i t_i : a_i \in [0, x_i] \right\} \subset \mathfrak{t}.$$

Definition 2.7. With the setup as above, let μ, ν be probability measures on G^\natural . The *discrepancy* between μ and ν is

$$\text{disc}(\mu, \nu) = \sup_{x \in [0, 1]^r} |\mu(\exp I_x) - \nu(\exp I_x)|.$$

If $\nu = dx$, the Haar measure on G^\natural , we simply write $\text{disc}(\mu)$ for $\text{disc}(\mu, dx)$.

The Koksma–Hlawka inequality bounds the difference between the Haar integral and weighted average of a function on G^\natural in terms of the discrepancy of the sequence and the variation of the function.

The following result is essential:

Theorem 2.8 (Koksma, Hlawka). *Let G be as above. Let $f: G^\natural \rightarrow \mathbf{C}$ be such that $f dx$ is a measure with bounded variation. Then*

$$\left| \mathbf{x}^C(f) - \int f dx \right| \leq \text{Var}(f) \text{disc}(\mathbf{x}^C).$$

Proof. This is [Ökt99, Th. 3.2]. □

We will often use the soft version of this inequality. Namely, assume $\int f dx = 0$. Then $|\mathbf{x}^C(f)| \ll_f \text{disc}(\mathbf{x}^C)$ as $C \rightarrow \infty$. Here is another way of putting it. The sequence $f(\mathbf{x})$ has $|A_{f(\mathbf{x})}(C)| \ll_f \pi(C) \text{disc}(\mathbf{x}^C)$.

3 Main results

Theorem 3.1. *Let $\mathbf{z} \in \mathbf{D}^\infty$. Then $L(\mathbf{z}, s)$ defines a holomorphic function on the region $\{\Re s > 1\}$. Moreover, on that region,*

$$\log L(\mathbf{z}, s) = \sum_{p^n} \frac{z_p^n}{np^{ns}}.$$

Proof. Expanding the product for $L(\mathbf{z}, s)$ formally, we have

$$L(\mathbf{z}, s) = \sum_{n \geq 1} \frac{\prod_{p|n} z_p^{v_p(n)}}{n^s}.$$

An easy comparison with Riemann's zeta function tells us that the series expansion is holomorphic on $\{\Re s > 1\}$. By [Apo76, Th. 11.7], the product formula holds on the same region. The formula for $\log L(\mathbf{z}, s)$ comes from [Apo76, 11.9 Ex.2]. \square

Theorem 3.2. *Assume $A_{\mathbf{z}}(x) \ll x^{\alpha+\epsilon}$, $\alpha \in [\frac{1}{2}, 1]$. Then $\log L(\mathbf{z}, s)$ is holomorphic on $\{\Re > \alpha\}$.*

Proof. Split the sum for $\log L$ into two pieces:

$$\log L(\mathbf{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geq 2} \frac{z_p^n}{np^{ns}}.$$

For each p , we have

$$\left| \sum_{n \geq 2} \frac{z_p^n}{np^{ns}} \right| \leq \sum_{n \geq 2} p^{-n\Re s} = p^{-2\Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leq \frac{1}{1 - p^{-\Re s}} \leq 2 + 2\sqrt{2},$$

so the second piece of $\log L(\mathbf{z}, s)$ converges absolutely when $\Re(s) > \frac{1}{2}$. By [Ten95, II.1 Th.10], our bound on $A_{\mathbf{z}}(x)$ yields the holomorphy of $\sum z_p p^{-s}$ on $\{\Re > \alpha\}$. \square

Corollary 3.3. *Let G be a compact connected semisimple Lie group, $\mathbf{x} \in G^{\natural, \infty}$ satisfy $\text{disc}(\mathbf{x}^C, d\mathbf{x}) \ll C^{-\frac{1}{2}+\epsilon}$. Then for every $f \in C^{\text{ae}}(G^{\natural})^{\|\cdot\| \leq 1}$, $L_f(\mathbf{x}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$, and satisfies the Riemann Hypothesis, for all f bounded and almost-everywhere continuous with $\mu(f) = 0$.*

Proof. Koksma–Hlawka tells that if $\mu(f) = 0$, then $\mathbf{x}^C(f) \ll C^{-\frac{1}{2}+\epsilon}$. Thus the sequence $f(\mathbf{x})$ satisfies $A_{f(\mathbf{x})}(x) \ll x^{\frac{1}{2}+\epsilon}$, and the result follows from Theorem 3.2. \square

4 Strange L -functions over function fields

Let k be a finite field of characteristic p and cardinality q . Let C/k be a nice curve in the sense of Poonen (i.e., C is smooth, projective, and geometrically integral). Write $K = k(C)$ for the function field of C . Fix a non-empty open subset $U \subset C$ and a geometric point $\infty \in U(\bar{k})$. Fix a prime $l \neq p$ and an embedding $\overline{\mathbf{Q}}_l \hookrightarrow \mathbf{C}$.

Definition 4.1. An l -adic sheaf \mathcal{F} on U is *good* if the following conditions hold.

1. \mathcal{F} is pure of weight zero.
2. Let $G = \overline{\rho_{\mathcal{F}}(\pi_1(U_{\bar{k}}, \infty))}^{\text{Zar}}$. Assume $\rho_{\mathcal{F}}(\pi_1(U, \infty)) \subset G(\overline{\mathbf{Q}}_l)$.

For any good sheaf \mathcal{F} , let $\text{ST}(\mathcal{F})$ be a maximal compact subgroup of $G(\mathbf{C})$. For each $u \in U$, there is a well-defined conjugacy class $\theta(u) = \rho(\text{fr}_u)^{\text{ss}} \in \text{ST}(\mathcal{F})^{\natural}$. For any $C > 0$, write

$$\theta_{\mathcal{F}}^C = \frac{1}{\#\{u \in U : q_u \leq C\}} \sum_{q_u \leq C} \delta_{\theta(u)}.$$

Katz proves an equidistribution estimate for the $\theta(u)$'s.

Theorem 4.2. *Let σ be a non-trivial irreducible representation of $\text{ST}(\mathcal{F})$. Then*

$$|\theta_{\mathcal{F}}^C(\text{tr } \sigma)| \ll_{\mathcal{F}} \dim(\sigma) C^{-\frac{1}{2}}.$$

Proof. This is [Kat88, p.39]. □

Now let $C^{\natural}(\text{ST}(\mathcal{F}))$ be the space of functions $f: \text{ST}(\mathcal{F})^{\natural} \rightarrow \mathbf{C}$ satisfying:

$$\|f\|^{\natural} = \sum_{\sigma} \dim(\sigma) |\hat{f}(\sigma)| < \infty.$$

For such functions, we have:

$$|\theta_{\mathcal{F}}^C(f) - \mu(f)| \ll_{\mathcal{F}} \|f\|^{\natural} C^{-\frac{1}{2}}.$$

Thus for any $f \in C^{\natural}(\text{ST}(\mathcal{F}))$, the strange L -function $L_f(\theta_{\mathcal{F}}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$ and satisfies the Riemann Hypothesis.

Theorem 4.3. *Let $\mathbf{z} \in \mathbf{D}^{\infty}$, and assume $\log L(\mathbf{z}, s)$ has analytic continuation to $\{\Re s > \alpha\}$, $\alpha \in [\frac{1}{2}, 1]$, and that for $\sigma > \alpha$, we have $|\log L(\mathbf{z}, \sigma + it)| \ll |t|^{1-\epsilon}$. Then $|A_{\mathbf{z}}(x)| \ll x^{\alpha+\epsilon}$.*

Proof. Recall that we can write

$$\log L(\mathbf{z}, p) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geq 2} \frac{z_p^n}{np^{ns}} = \sum_p \frac{z_p}{p^s} + O(\zeta(2\Re s)).$$

Thus, for any $\epsilon > 0$, our bound on $|\log L(\mathbf{z}, \sigma + it)|$ implies the same bound for $\sum \frac{z_p}{p^s}$ on $\{\Re s > \alpha + \epsilon\}$.

Let $\gamma_T = \gamma_{1,T} + \gamma_{2,T} - \gamma_{3,T} - \gamma_{4,T}$ be the following contour:

$$\begin{aligned} \gamma_{1,T}(t) &= (\alpha + \epsilon) + it & t \in [-T, T] \\ \gamma_{2,T}(t) &= t + iT & t \in [\alpha + \epsilon, 1 + \epsilon] \\ \gamma_{3,T}(t) &= (1 + \epsilon) + it & t \in [-T, T] \\ \gamma_{4,T}(t) &= t - iT & t \in [\alpha + \epsilon, 1 + \epsilon]. \end{aligned}$$

By [?],

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-\gamma_{3,T}} \sum_p \frac{z_p}{p^s} x^z \frac{dz}{z} =^* \sum_{p \leq x} z_p.$$

Let $h(z)$ be the analytic continuation of $\sum z_p p^{-s}$ to $\{\Re > \alpha\}$. Since $\int_{\gamma} h(z) \frac{dz}{z} = 0$, we obtain

$$\left| \sum_{p \leq x} z_p \right| \ll \left| \int_{\gamma_{T,1}} h(z) x^z \frac{dz}{z} \right| + \left| \int_{\gamma_{T,2}} h(z) x^z \frac{dz}{z} \right| + \left| \int_{\gamma_{T,4}} h(z) x^z \frac{dz}{z} \right|.$$

We know that $|h(\sigma + it)| \ll |t|$, so we can bound:

$$\left| \int_{\gamma_{T,2}} h(z) \frac{dz}{z} \right| = \left| \int_{\alpha+\epsilon}^{1+\epsilon} \frac{h(t+iT) x^{t+iT}}{t+iT} dt \right| \ll (1+\alpha) x^{1+\alpha} T^{-1},$$

and similarly for $\int_{\gamma_{4,T}}$. Finally, we note that

$$\left| \int_{\gamma_{T,1}} h(z) x^z \frac{dz}{z} \right| \ll \int_{-T}^T |t|^{1-\epsilon} \frac{x^{\alpha+\epsilon}}{(\alpha+\epsilon)^2 + t^2} dt \ll x^{\alpha+\epsilon}.$$

Letting $T \rightarrow \infty$ we obtain the desired result. \square

5 Applications

Theorem 5.1. *Let E/\mathbf{Q} be a non-CM elliptic curve, and put $\theta = \theta(E)$. Assume that $\text{disc}(\theta^C) \ll C^{-\frac{1}{2}+\epsilon}$. Then if $f \in C^{\text{ae}}([0, \pi], \text{ST})^{\|\cdot\|_{\infty} \leq 1}$, the strange L -function $L_f(\theta, s)$ has analytic continuation to $\{\Re > \frac{1}{2}\}$ and satisfy the Riemann Hypothesis. In particular, this holds for all $L(\text{sym}^k E, s)$.*

Proof. The first conclusion follows from [Corollary 3.3](#). The second part follows from the fact that any $L(\text{sym}^k E, s)$ can be written as a product of L_f 's, namely the $L_{\lambda_{\text{sym}^k}^j}$'s in [section 2](#). \square

Theorem 5.2. *Fix $f \in C^{\text{ae}}([0, \pi], \text{ST})^{\|\cdot\|_{\infty} \leq 1}$ that is not almost everywhere constant.*

Let E_1, E_2 be two non-isogenous, non-CM elliptic curves over \mathbf{Q} . Assume the Akiyama–Tanigawa conjecture for the product $E_1 \times E_2$. Then for any $f: [0, \pi] \rightarrow \mathbf{C}$ that is not almost everywhere

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