Operads for number theory

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Our main source is [HS87]. Let $\mathcal{C}$ be a strict monoidal category. Our main example is the category $\mathsf{Com}(k)$ of chain complexes over a (commutative, unital) ring k. The main objects of interest will be collections $O = \{O_n : n \geqslant 0\}$ in $\mathcal{C}$. To handle these, we introduce some formalism. Note that $\mathbf{N} = \{1, 2, \ldots\}$.

Let $\mathsf{tup} = \coprod_{d \geqslant 0} \mathsf{tup}_d$ be the set of *tuples*. For $d \geqslant 0$, tup_d is the set of pairs (m, S), where S is a finite subset of $\mathbf{N}^d$ and $m : S \to \mathbf{Z}_{\geqslant 0}$. We write $\dim(m) = d$ if $m \in \mathsf{tup}_d$. Define

$$|\cdot| : \mathsf{tup} \to \mathbf{Z}_{\geqslant 0} \qquad |(m,S)| = \sum_{s \in S} m_s$$

$$l : \mathsf{tup}_{d+1} \to \mathsf{tup}_d \qquad l(m,S) = \left((s_1, \ldots, s_d) \mapsto \sup_{(s_1,\ldots,s_{d+1}) \in S} s_{d+1}, \{(s_d, \ldots, s \right)$$

We will write $m = (m_1, \ldots, m_n)$ for an n-tuple of integers; we put $l(m) = n$ and $|m| = \sum m_i$. In general, we say a tuple $m = (m_{i_1,\ldots,i_d})$ has *dimension* d, write $\dim(m) = d$, $|m| = \sum m_{i_1,\ldots,i_d}$ and

Definition 1. *An* operad O *in* $\mathcal{C}$ *consists of the following data:*

- *A family of objects* $\{O_n : n \geqslant 0\}$ *in* $\mathcal{C}$*. For* $m = (m_1, \ldots, m_n)$*, we put* $O_m = O_{m_1} \otimes \cdots \otimes O_{m_n}$.
- *Multiplication maps* $\gamma_m : O_{\ell(m)} \otimes O_m \to O_{|m|}$.
- *A unit* $e : 1 \to O_0$.

If m *is a two-dimensional tuple, put* $O_m = \bigotimes_{i_1,i_2} O_{m_{i_1},m_{i_2}}$.

This data is required to satisfy the following properties:

1 Application

Here our inspiration is from [Bei12]. The fundamental idea is that a ring like $\mathbf{Z}_p$, a limit of non-free $\mathbf{Z}$-modules, should be "resolved." More precisely, consider

$$\mathbf{Z}_p^\flat = \operatorname{holim}_n \operatorname{cone}(\mathbf{Z} \xrightarrow{p^n} \mathbf{Z}).$$

The functor $F \widehat{\otimes} \mathbf{Z}_p := \operatorname{holim} \operatorname{cone}$

References

[Bei12] A. Beilinson. p-adic periods and derived de Rham cohomology. *J. Amer. Math. Soc.*, 25(3):715–738, 2012.

[HS87] V. A. Hinich and V. V. Schechtman. On homotopy limit of homotopy algebras. In *K-theory, arithmetic and geomtry (Moscos, 1984–1986)*, volume 1289 of *Lecture Notes in Math.*, pages 240–264. Springer, 1987.