Equidistribution and the analytic properties of a strange class of L-functions

Daniel Miller

September 5, 2016

1 Motivation

Let $E_{/\mathbf{Q}}$ be an elliptic curve without complex multiplication. By an old theorem of Faltings [Fal83], the sequence

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \operatorname{tr} \rho_{E,l}(\mathrm{fr}_p)$$

determines E up to isogeny. That is, if E_1 and E_2 satisfy $a_p(E_1) = a_p(E_2)$ for all E, then E_1 and E_2 are isogenous. The starting point of this investigation is the corollary of a theorem of Harris, that the sequence $(\operatorname{sgn} a_p(E))_p$ in fact determines E up to isogeny. Ramakrishna had the insight that this fact means the "strange L-function"

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \operatorname{sgn} a_p(E) p^{-s}}$$

determines E up to isogeny. In this note, we define a more general class of strange L-functions, and show that their analytic properties are closely tied to the distribution of the $a_p(E)$.

Here is a brief discussion of this generalization in the case of a non-CM curve $E_{/\mathbf{Q}}$. It is convenient to repackage the traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the $\theta_p(E)$ are well-defined angles laying in the interval $[0, \pi]$. Write $\mathrm{dST} = \frac{2}{\pi} \sin^2 \theta \, \mathrm{d}\theta$. Then the Sato–Tate conjecture, now a theorem [BLGHT11], tells us that for any continuous function $f\colon [0, \pi] \to \mathbf{C}$, we have

$$\left| \frac{1}{\pi(N)} \sum_{p \leqslant N} f(\theta_p) - \int_0^\pi f \, \mathrm{dST} \right| = o(1)$$

as $N \to \infty$. It is well-known that this follows from the analytic continuation past $\Re s = 1$ and non-vanishing except at $s = 1$ of all the L-functions $L(\operatorname{sym}^k E, s)$

[Ser68, A.1 Th.1]. We take as our starting point the much stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(N)} \sum_{p \leqslant N} f(\theta_p) - \int_0^\pi f \, \mathrm{d}\mu_{\mathrm{ST}} \right| \ll_f N^{-\frac{1}{2}+\epsilon}$$

for all f of bounded variation. (Their conjecture is actually more precise; we will discuss their exact statement later.) They prove that this conjecture implies the Riemann Hypothesis for E. We prove that not only does their conjecture imply the Riemann Hypothesis for all $L(\operatorname{sym}^k E, s)$, it also does for all the strange L-functions

$$L_f(E,s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

where f varies over almost-everywhere continuous functions on $[0,\pi]$.

These results make perfect sense in a much more general context, and we prove them there. In Section ? I set up this context and carefully define strange L-functions. In Section ?, I prove basic analytic properties of the strange L-functions and connect their analytic properties with the equidistribution of a sequence. In Section ?, I apply these results where "everything is known," i.e. varieties over function fields. Finally, in Section ?, I apply the general results to the following cases: a non-CM elliptic curve $E_{/\mathbf{Q}}$, the product $E_1 \times E_2$ of a pair of non-isogenous non-CM elliptic curves over $\mathbf{Q}$, and the Jacobian of a generic genus-2 curve $C_{/\mathbf{Q}}$.

2 Strange L-functions

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| \leqslant 1\}$. Write $\mathbf{D}^\infty$ for the set of sequences in $\mathbf{D}$ indexed by the primes, i.e. $\boldsymbol{z} \in \mathbf{D}^\infty$ is $(z_2, z_3, \dots)$. The space $\mathbf{D}^\infty$ is compact, and comes naturally equipped with the (product) Lebesgue measure, normalized to have mass 1.

Definition 2.1. Let $\boldsymbol{z} \in \mathbf{D}^\infty$. The associated *strange L-function* is given by

$$L(\boldsymbol{z}, s) = \prod_p \frac{1}{1 - z_p p^{-s}},$$

wherever this product converges.

Elementary topology tells us that $L\colon \mathbf{D}^\infty \times \mathbf{C}^{\Re > 1} \to \mathbf{C}$ is continuous. We will see that for fixed $\boldsymbol{z} \in \mathbf{D}^\infty$, the analytic properties of $L(\boldsymbol{z}, s)$ are closely tied to estimates for the sums $A_{\boldsymbol{z}}(x) = \sum_{p \leqslant x} z_p$. One often gets such estimates in the context of equidistribution, which we consider next.

For the remainder of this section, let X be a compact separable metric space with no isolated points. We write X^∞ for the space of sequences in X indexed by rational primes, i.e. points $\boldsymbol{x} \in X^\infty$ are of the form $\boldsymbol{x} = (x_2, x_3, \dots)$. By [Eng89, Cor.2.3.16, Th.4.2.2], the compact space X^∞ is metrizable and separable, also with no isolated points.

Definition 2.2. For $\boldsymbol{x} \in \mathbf{D}^\infty$ and $N > 0$, write $\boldsymbol{x}^N$ for the probability measure on $\mathbf{D}$ given by

$$\int_X f \, \mathrm{d}\boldsymbol{x}^N = \boldsymbol{x}^N(f) = \frac{1}{\pi(N)} \sum_{p \leqslant N} f(x_p).$$

Let μ be a Borel measure on X. Recall that $\boldsymbol{x}$ is *μ-equidistributed* if $\boldsymbol{x}^N \to \mu$ weakly, i.e. $\mu(f) = \lim_{N\to\infty} \boldsymbol{x}^N(f)$ for all $f \in C(X)$. In fact, we can extend this to not-necessarily-continuous functions as follows:

Theorem 2.3 (Mazzone)**.** *Let μ be a Borel measure on X and let $f\colon X \to \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if $\mu(f) = \lim_{N\to\infty} \boldsymbol{x}^N(f)$ for all μ-equidistributed $\boldsymbol{x}$.*

Proof. This follows directly from the proof of [Maz96, Th.1]. □

Fix a Borel measure μ on X, and write $C^{\mathrm{ae}}(X,\mu)$ for the space of bounded, almost-everywhere continuous functions $f\colon X \to \mathbf{C}$.

Lemma 2.4. *Endowed with the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$, $C^{\mathrm{ae}}(X,\mu)$ is a Banach space.*

Proof. This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero. □

Definition 2.5. Let $f \in C^{\mathrm{ae}}(X,\mu)^{\|\cdot\|_\infty \leqslant 1}$, $\boldsymbol{x} \in X^\infty$. The associated *strange L-function* is defined as

$$L_f(\boldsymbol{x}, s) = L(f(\boldsymbol{x}), s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}$$

for all $s \in \mathbf{C}$ for which the product converges.

Our typical source of a strange L-function is as follows. Let G be a compact connected Lie group and $X = G^\natural$, the space of conjugacy classes of G. Then $G^\natural$ inherits the Haar measure from G. Given any sequence $\boldsymbol{x} \in (G^\natural)^\infty = G^{\natural,\infty}$ and function $f \in C^{\mathrm{ae}}(G^\natural)^{\|\cdot\|_\infty \leqslant 1}$, we can define $L_f(\boldsymbol{x}, s)$. This is related to Serre's L-functions from [Ser68, A.2] as follows.

Theorem 2.6. *Let G be a compact connected Lie group, $\rho \in \widehat{G}$ an irreducible unitary representation of G. Then there exist functions $\lambda_\rho^1, \ldots, \lambda_\rho^{\deg\rho}\colon G^\natural \to S^1$, continuous away from the set $\{\det(1-\rho) = 0\}$, such that for every $x \in G^\natural$, there are angles $\theta_1, \ldots, \theta_{\deg\rho} \in [0, 2\pi)$, satisfying $\theta_1 \leqslant \cdots \leqslant \theta_{\deg\rho}$, such that $\lambda_\rho^j(x) = e^{i\theta_j}$ and moreover*

$$\det(1 - \rho(x)t) = \prod_{j=0}^{\deg\rho} (1 - \lambda_\rho^j(x)t).$$

Proof. This follows easily from [KS99, Lem.1.0.9]. □

Recall that for $\rho \in \widehat{G}$, Serre defines $L(\rho, s) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}$. Using his notation, there is the identity

$$L(\rho, s) = \prod_{j=1}^{\deg \rho} L_{\lambda_\rho^j}(\boldsymbol{x}, s).$$

3 Discrepancy

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let $r \geqslant 1$ be an integer, and consider the space $[0,1]^r$. For $x, y \in [0,1]^r$, we write $x < y$ (resp. $x \leqslant y$) if $x = (x_1, \dots, x_r)$, $y = (y_1, \dots, y_r)$ and $x_i < y_i$ (resp. $x_i \leqslant y_i$) for all i. Given $x < y \in [0,1]^r$, write

$$\begin{aligned} I_x &= \{z \in [0,1]^r : z < x\} \\ I_{x,y} &= \{z \in [0,1]^r : x \leqslant z < y\}. \end{aligned}$$

Definition 3.1. Let μ, ν be probability measures on $[0,1]^r$. The *star-discrepancy* between μ and ν is

$$\operatorname{disc}^\star(\mu, \nu) = \sup_{x \in [0,1]^r} |\mu(I_x) - \nu(I_x)|,$$

the *discrepancy* between μ and ν is

$$\operatorname{disc}(\mu, \nu) = \sup_{x < y \in [0,1]^r} |\mu(I_{x,y}) - \nu(I_{x,y})|,$$

and the *isotropic discrepancy* between μ and ν is

$$\operatorname{disc}^{\mathrm{iso}}(\mu, \nu) = \sup_{C \subset [0,1]^r} |\mu(C) - \nu(C)|,$$

where C ranges over open and closed convex subsets of $[0,1]^r$.

Let λ be the Lebesgue measure on $[0,1]^r$, $\boldsymbol{x}$ a sequence in $[0,1]^r$. We write $\operatorname{disc}^*(\boldsymbol{x}^N) = \operatorname{disc}^*(\boldsymbol{x}^N, \lambda)$ for $* \in \{\varnothing, \star, \mathrm{iso}\}$.

Theorem 3.2. *Let $\boldsymbol{x}$ be a sequence in $[0,1]^r$. Then*

$$\begin{aligned} \operatorname{disc}(\boldsymbol{x}^N) &\leqslant \operatorname{disc}^{\mathrm{iso}}(\boldsymbol{x}^N) \leqslant (4r\sqrt{r} + 1) \operatorname{disc}(\boldsymbol{x}^N)^{1/r}, \\ \operatorname{disc}^\star(\boldsymbol{x}^N) &\leqslant \operatorname{disc}(\boldsymbol{x}^N) \leqslant 2^r \operatorname{disc}^\star(\boldsymbol{x}^N). \end{aligned}$$

Proof. The first inequality is Theorem 1.6, and the second is Example 1.2, both from [KN74, Ch.2]. □

We can use the above to define discrepancy for sequences in $G^\natural$, the space of conjugacy classes in a compact connected semisimple Lie group.

Let G^{sc} be the simply-connected cover of G. Choose a maximal torus $T \subset G^{\mathrm{sc}}$; let $W = \mathrm{N}(T)/T$ be the Weyl group. Let $\mathfrak{t} = \mathrm{Lie}(T)$ and recall that the kernel of $\exp\colon \mathfrak{t} \twoheadrightarrow T$ is generated by the nodal vectors associated to the root system $\mathrm{R}(G^{\mathrm{sc}}, T)$ [Bou05, 9.6 Pr.11]. Write $\{t_1, \dots, t_r\} \subset \mathfrak{t}$ for these vectors. The exponential map $\exp\colon \mathfrak{t} \to T$ induces an isomorphism $\mathfrak{t}/(\langle t_i\rangle \rtimes W) \to G^\natural$. In particular, we can use the basis $\{t_1, \dots, t_r\}$ to identify $\mathfrak{t}/\langle t_i\rangle$ with $[0,1]^r/W$. Let $p_G\colon [0,1]^r \twoheadrightarrow G^\natural$ be the surjection $(x_1, \dots, x_r) \mapsto \exp(\sum x_i t_i)$. This is a $\#W$-to-one map almost everywhere, so for a measure μ on $G^\natural$ the "pullback measure"

$$p_G^*\mu(f) = \mu\left(\frac{1}{\#W}\sum_{w\in W} w^* f\right)$$

makes sense.

Definition 3.3. With the setup as above, let μ, ν be probability measures on $G^\natural$. The $*$-*discrepancy* between μ and ν is

$$\operatorname{disc}^*(\mu,\nu) = \operatorname{disc}^*(p_G^*\mu, p_G^*\nu).$$

Example 3.4. Let $G = \mathrm{SU}(2)$ with maximal torus

$$T = \left\{\begin{pmatrix} e^{2\pi i t} & \\ & e^{-2\pi i t}\end{pmatrix} : -1 \leqslant t < 1\right\}.$$

Then $W = S_2$, whose nontrivial element acts via $t \mapsto -t$.

If ν is the Haar measure on $G^\natural$, we simply write $\operatorname{disc}(\mu)$ for $\operatorname{disc}(\mu, \mathrm{Haar})$.

The Koksma–Hlawka inequality bounds the difference between the Haar integral and weighted average of a function on $G^\natural$ in terms of the discrepancy of the sequence and the variation of the function.

The following result is essential:

Theorem 3.5 (Koksma, Hlawka)**.** *Let G be as above. Let $f\colon G^\natural \to \mathbf{C}$ be such that $f\,\mathrm{d}x$ is a measure with bounded variation. Then*

$$\left|\boldsymbol{x}^C(f) - \int f\,\mathrm{d}x\right| \leqslant \operatorname{Var}(f)\operatorname{disc}(\boldsymbol{x}^C).$$

Proof. This is [Ökt99, Th. 3.2]. □

We will often use the soft version of this inequality. Namely, assume $\int f\,\mathrm{d}x = 0$. Then $|\boldsymbol{x}^C(f)| \ll_f \operatorname{disc}(\boldsymbol{x}^C)$ as $C \to \infty$. Here is another way of putting it. The sequence $f(\boldsymbol{x})$ has $|A_{f(\boldsymbol{x})}(C)| \ll_f \pi(C)\operatorname{disc}(\boldsymbol{x}^C)$.

Theorem 3.6. *Let $\boldsymbol{x}$ be a sequence in $[0,1]^r$. Then*

$$\operatorname{disc}^{\mathrm{iso}}(\boldsymbol{x}^N, \mu) = \sup_{P\subset[0,1]^r} \left|\boldsymbol{x}^N(P) - \mu(P)\right|,$$

where P ranges over all open and closed convex polytopes contained in $[0,1]^r$.

Proof. We follow the proof of [KN74, Ch.2 Th.1.5]. Clearly the supremum in question is bounded above by isotropic discrepancy, so we only need to show the opposite bound. Let $C \subset [0,1]^r$ be a convex set. Suppose C contains $x_{i_1}, \dots, x_{i_a}$. Then C contains P, the convex hull of $\{x_{i_1}, \dots, x_{i_a}\}$.

Use the fact that given a convex set, and a point not in the interior of the set, the two can be separated by a hyperplane. Intersect half-planes, and get $P \subset C \subset Q$, with P and Q polytopes, and $\boldsymbol{x}^N(P) = \boldsymbol{x}^N(C) = \boldsymbol{x}^N(Q)$. This yields (via $a \leqslant b \leqslant c$ implies $|b| \leqslant \max\{|a|, |c|\}$)

$$|\boldsymbol{x}^N(C) - \mu(C)| \leqslant \max\{|\boldsymbol{x}^N(P) - \mu(P)|, |\boldsymbol{x}^N(Q) - \mu(Q)|\}.$$

□

4 Main results

Theorem 4.1. *Let $\boldsymbol{z} \in \mathbf{D}^\infty$. Then $L(\boldsymbol{z}, s)$ defines a holomorphic function on the region $\{\Re s > 1\}$. Moreover, on that region,*

$$\log L(\boldsymbol{z}, s) = \sum_{p^n} \frac{z_p^n}{np^{ns}}.$$

Proof. Expanding the product for $L(\boldsymbol{z}, s)$ formally, we have

$$L(\boldsymbol{z}, s) = \sum_{n \geqslant 1} \frac{\prod_{p|n} z_p^{v_p(n)}}{n^s}.$$

An easy comparison with Riemann's zeta function tells us that the series expansion is holomorphic on $\{\Re s > 1\}$. By [Apo76, Th. 11.7], the product formula holds on the same region. The formula for $\log L(\boldsymbol{z}, s)$ comes from [Apo76, 11.9 Ex.2]. □

Theorem 4.2. *Assume $A_{\boldsymbol{z}}(x) \ll x^{\alpha+\epsilon}$, $\alpha \in [\frac{1}{2}, 1]$. Then $\log L(\boldsymbol{z}, s)$ is holomorphic on $\{\Re > \alpha\}$.*

Proof. Split the sum for $\log L$ into two pieces:

$$\log L(\boldsymbol{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geqslant 2} \frac{z_p^n}{np^{ns}}.$$

For each p, we have

$$\left| \sum_{n \geqslant 2} \frac{z_p^n}{np^{ns}} \right| \leqslant \sum_{n \geqslant 2} p^{-n\Re s} = p^{-2\Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leqslant \frac{1}{1 - p^{-\Re s}} \leqslant 2 + 2\sqrt{2},$$

so the second piece of $\log L(\boldsymbol{z}, s)$ converges absolutely when $\Re(s) > \frac{1}{2}$. By [Ten95, II.1 Th.10], our bound on $A_{\boldsymbol{z}}(x)$ yields the holomorphy of $\sum z_p p^{-s}$ on $\{\Re > \alpha\}$. $\square$

Corollary 4.3. *Let G be a compact connected semisimple Lie group, $\boldsymbol{x} \in G^{\natural,\infty}$ satisfy* $\operatorname{disc}(\boldsymbol{x}^C, \mathrm{d}x) \ll C^{-\frac{1}{2}+\epsilon}$. *Then for every $f \in C^{\mathrm{ae}}(G^\natural)^{\|\cdot\|\leqslant 1}$, $L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$, and satisfies the Riemann Hypothesis, for all f bounded and almost-everywhere continuous with $\mu(f) = 0$.*

Proof. Koksma–Hlawka tells that if $\mu(f) = 0$, then $\boldsymbol{x}^C(f) \ll C^{-\frac{1}{2}+\epsilon}$. Thus the sequence $f(\boldsymbol{x})$ satisfies $A_{f(\boldsymbol{x})}(x) \ll x^{\frac{1}{2}+\epsilon}$, and the result follows from Theorem 4.2. $\square$

5 Strange L-functions over function fields

Let k be a finite field of characteristic p and cardinality q. Let $C_{/k}$ be a nice curve in the sense of Poonen (i.e., C is smooth, projective, and geometrically integral). Write $K = k(C)$ for the function field of C. Fix a non-empty open subset $U \subset C$ and a geometric point $\infty \in U(\bar{k})$. Fix a prime $l \neq p$ and an embedding $\overline{\mathbf{Q}_l} \hookrightarrow \mathbf{C}$.

Definition 5.1. An l-adic sheaf $\mathcal{F}$ on U is *good* if the following conditions hold.

1. $\mathcal{F}$ is pure of weight zero.
2. Let $G = \overline{\rho_{\mathcal{F}}(\pi_1(U_{\bar{k}}, \infty))}^{\mathrm{Zar}}$. Assume $\rho_{\mathcal{F}}(\pi_1(U, \infty)) \subset G(\overline{\mathbf{Q}}_l)$.

For any good sheaf $\mathcal{F}$, let $\mathrm{ST}(\mathcal{F})$ be a maximal compact subgroup of $G(\mathbf{C})$.For each $u \in U$, there is a well-defined conjugacy class $\theta(u) = \rho(\mathrm{fr}_u)^{\mathrm{ss}} \in \mathrm{ST}(\mathcal{F})^\natural$. For any $C > 0$, write

$$\boldsymbol{\theta}_{\mathcal{F}}^C = \frac{1}{\#\{u \in U : q_u \leqslant C\}} \sum_{q_u \leqslant C} \delta_{\theta(u)}.$$

Katz proves an equidistribution estimate for the $\theta(u)$'s.

Theorem 5.2. *Let σ be a non-trivial irreducible representation of* $\mathrm{ST}(\mathcal{F})$. *Then*

$$|\boldsymbol{\theta}_{\mathcal{F}}^C(\operatorname{tr}\sigma)| \ll_{\mathcal{F}} \dim(\sigma) C^{-\frac{1}{2}}.$$

Proof. This is [Kat88, p.39]. $\square$

Now let $C^\natural(\mathrm{ST}(\mathcal{F}))$ be the space of functions $f\colon \mathrm{ST}(\mathcal{F})^\natural \to \mathbf{C}$ satisfying:

$$\|f\|^\natural = \sum_\sigma \dim(\sigma)|\widehat{f}(\sigma)| < \infty.$$

For such functions, we have:

$$|\boldsymbol{\theta}_{\mathcal{F}}^C(f) - \mu(f)| \ll_{\mathcal{F}} \|f\|^\natural C^{-\frac{1}{2}}.$$

Thus for any $f \in C^\natural(\mathrm{ST}(\mathcal{F}))$, the strange L-function $L_f(\boldsymbol{\theta}_{\mathcal{F}}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$ and satisfies the Riemann Hypothesis.

Theorem 5.3. *Let $\boldsymbol{z} \in \mathbf{D}^\infty$, and assume $\log L(\boldsymbol{z}, s)$ has analytic continuation to $\{\Re > \alpha\}$, $\alpha \in [\frac{1}{2}, 1]$, and that for $\sigma > \alpha$, we have $|\log L(\boldsymbol{z}, \sigma + it)| \ll |t|^{1-\epsilon}$. Then $|A_{\boldsymbol{z}}(x)| \ll x^{\alpha+\epsilon}$.*

Proof. Recall that we can write

$$\log L(\boldsymbol{z}, p) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geqslant 2} \frac{z_p^n}{np^{ns}} = \sum_p \frac{z_p}{p^s} + O(\zeta(2\Re s)).$$

Thus, for any $\epsilon > 0$, our bound on $|\log L(\boldsymbol{z}, \sigma + it)|$ implies the same bound for $\sum \frac{z_p}{p^s}$ on $\{\Re > \alpha + \epsilon\}$.

Let $\gamma_T = \gamma_{1,T} + \gamma_{2,T} - \gamma_{3,T} - \gamma_{4,T}$ be the following contour:

$$\begin{aligned}
\gamma_{1,T}(t) &= (\alpha + \epsilon) + it \qquad t \in [-T, T] \\
\gamma_{2,T}(t) &= t + iT \qquad t \in [\alpha + \epsilon, 1 + \epsilon] \\
\gamma_{3,T}(t) &= (1 + \epsilon) + it \qquad t \in [-T, T] \\
\gamma_{4,T}(t) &= t - iT \qquad t \in [\alpha + \epsilon, 1 + \epsilon].
\end{aligned}$$

By [Apo76, Th.11.18],

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{-\gamma_{3,T}} \sum_p \frac{z_p}{p^s} x^z \frac{\mathrm{d}z}{z} =^* \sum_{p \leqslant x} z_p.$$

Let $h(z)$ be the analytic continuation of $\sum z_p p^{-s}$ to $\{\Re > \alpha\}$. Since $\int_\gamma h(z) \frac{\mathrm{d}z}{z} = 0$, we obtain

$$\left| \sum_{p \leqslant z} z_p \right| \ll \left| \int_{\gamma_{T,1}} h(z) x^z \frac{\mathrm{d}z}{z} \right| + \left| \int_{\gamma_{T,2}} h(z) x^z \frac{\mathrm{d}z}{z} \right| + \left| \int_{\gamma_{T,4}} h(z) x^z \frac{\mathrm{d}z}{z} \right|.$$

We know that $|h(\sigma + it)| \ll |t|$, so we can bound:

$$\left| \int_{\gamma_{T,2}} h(z) \frac{\mathrm{d}z}{z} \right| = \left| \int_{\alpha+\epsilon}^{1+\epsilon} \frac{h(t + iT) x^{t+iT}}{t + iT} \,\mathrm{d}t \right| \ll (1 + \alpha) x^{1+\alpha} T^{-1},$$

and similarly for $\int_{\gamma_{4,T}}$. Finally, we note that

$$\left| \int_{\gamma_{T,1}} h(z) x^z \frac{\mathrm{d}z}{z} \right| \ll \int_{-T}^{T} |t|^{1-\epsilon} \frac{x^{\alpha+\epsilon}}{(\alpha + \epsilon)^2 + t^2} \,\mathrm{d}t \ll x^{\alpha+\epsilon}.$$

Letting $T \to \infty$ we obtain the desired result. □

6 Applications

Recall, following [Bug08] that the *irrationality exponent* $\mu(\alpha)$ a real irrational number α is the supremum of all real numbers μ such that

$$\left| \alpha - \frac{p}{q} \right| < q^{-\mu}$$

for infinitely many $p/q \in \mathbf{Q}$. Bugeaud proves that for any $\mu \geqslant 2$, there is an element ξ_μ of the Cantor set with $\mu(\xi_\mu) = \mu$. Moreover, by [KN74, ?], for every $\epsilon > 0$, the sequence $x_n = n\alpha \mod 1$ has discrepancy $\operatorname{disc}(\boldsymbol{x}^C) = \Omega(C^{-\frac{1}{\mu(\alpha)-1}-\epsilon})$.

Theorem 6.1. *Let $X = S^1$ with the natural Haar measure. For every $\eta \in (0, \frac{1}{2})$, there is a sequence $\boldsymbol{x} = (x_2, x_3, \dots) \in (S^1)^\infty$ such that for all $f \in C^\infty(S^1)^{\|\cdot\|_\infty \leqslant 1}$, the function $\log L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re > \frac{1}{2}\}$, but for all $\epsilon > 0$, $|\operatorname{disc}(\boldsymbol{x}^C)| = \Omega(C^{-\eta-\epsilon})$.*

Proof. Let $\mu > 3$, and let $\boldsymbol{x} = \{x_2, x_3, \dots\}$ be the sequence $x_{p_n} = e^{2\pi i n \xi_\mu}$. To prove that $\log L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re > \frac{1}{2}\}$, we need only to prove that $|A_{\exp(2\pi i m \boldsymbol{x})}(t)| \ll t^{1/2}$, uniformly for each $m \in \mathbf{Z}$. This follows easily from:

$$\left|\sum_{n=1}^{N} e^{2\pi i m n \alpha}\right| \leqslant \frac{|-1 + e^{2\pi i M n \alpha}|}{|-1 + e^{2\pi i a m}|} \leqslant ? \leqslant \frac{1}{2} m(\eta - 1) \ll_\eta m$$

□

Theorem 6.2. *Let $E_{/\mathbf{Q}}$ be a non-CM elliptic curve, and put $\boldsymbol{\theta} = \boldsymbol{\theta}(E)$. Assume that $\operatorname{disc}(\boldsymbol{\theta}^C) \ll C^{-\frac{1}{2}+\epsilon}$. Then if $f \in C^{\mathrm{ae}}([0, \pi], \mathrm{ST})^{\|\cdot\|_\infty \leqslant 1}$, the strange L-function $L_f(\boldsymbol{\theta}, s)$ has analytic continuation to $\{\Re > \frac{1}{2}\}$ and satisfy the Riemann Hypothesis. In particular, this holds for all $L(\operatorname{sym}^k E, s)$.*

Proof. The first conclusion follows from Corollary 4.3. The second part follows from the fact that any $L(\operatorname{sym}^k E, s)$ can be written as a product of L_f's, namely the $L_{\lambda^j_{\mathrm{sym}^k}}$'s in Section **??**. □

Theorem 6.3. *Fix $f \in C^{\mathrm{ae}}([0, \pi], \mathrm{ST})^{\|\cdot\|_\infty \leqslant 1}$ that is not almost everywhere constant.*

Let E_1, E_2 be two non-isogenous, non-CM elliptic curves over $\mathbf{Q}$. Assume the Akiyama–Tanigawa conjecture for the product $E_1 \times E_2$. Then for any $f\colon [0, \pi] \to \mathbf{C}$ that is not almost everywhere

7 A collection of counterexamples

In [AT99, ?], Akiyama and Tanigawa claim that for $E_{/\mathbf{Q}}$, the "discrepancy conjecture" $\operatorname{disc}(\boldsymbol{\theta}^C) \ll C^{-\frac{1}{2}+\epsilon}$ is equivalent to the Riemann Hypothesis for $L(E, s)$. In this section, I construct a collection of examples which show that their conjecture is false for any motive with positive-dimensional Sato–Tate group.

Throughout this section, $|\cdot|_\infty$ is the sup-norm, and $|\cdot|$ can be any of the (commensurable) p-norms on a finite-dimensional real vector space.

Definition 7.1. Let $x \in \mathbf{R}^r$ be such that $x_1, \ldots, x_r$ are $\mathbf{Q}$-linearly independent. Following [Lau09], we define r-dimensional *irrationality exponents* as the suprema $\omega_0(x)$ and $\omega_{r-1}(x)$ of the sets of w for which there are infinitely many $m = (m_0, \ldots, m_r) \in \mathbf{Z}^{r+1}$ for which

$$\max\{|m_0 x_i - m_i|\} \leqslant |m|_\infty^{-w}$$
$$|m_0 + m_1 x_1 + \cdots + m_r x_r| \leqslant |m|_\infty^{-w}$$

respectively.

Given $x \in \mathbf{R}^r$, write $d(x, \mathbf{Z}^r) = \min_{m \in \mathbf{Z}^r} |x - m|$.

Lemma 7.2. *Let $x \in \mathbf{R}^r$ with $|x|_\infty \leqslant 1$ and $\omega_0(x)$ (resp. $\omega_{r-1}(x)$) is finite. Then*

$$\frac{1}{d(nx, \mathbf{Z}^r)} \ll_{\epsilon,x} n^{\omega_0(x)+\epsilon} \qquad \text{as } n \to \infty, \text{ (resp.)}$$
$$\frac{1}{d(\langle m, x\rangle, \mathbf{Z})} \ll_{\epsilon,x} |m|^{\omega_{r-1}(x)+\epsilon} \qquad \text{as } m \to \infty \text{ in } \mathbf{Z}^r\ .$$

Proof. Let $\epsilon > 0$. Then there are only finitely many $n \in \mathbf{N}$ (resp. $m \in \mathbf{Z}^r$) such that the inequalities in Definition 7.1 hold with $\omega_0(x) + \epsilon$ (resp. $\omega_{r-1}(x) + \epsilon$). In other words, there exist $C_0, C_{r-1} > 0$ such that

$$\max\{|m_0 x_i - m_i|\} \geqslant C_0 |m|_\infty^{-\omega_0(x)-\epsilon}$$
$$|m_0 + m_1 x_1 + \cdots + m_r x_r| \geqslant C_{r-1} |m|_\infty^{-\omega_{r-1}(x)-\epsilon}.$$

for all $m \neq 0$. We consider the first inequality, temporarily setting $|\cdot| = |\cdot|_\infty$. Then $d(nx, \mathbf{Z}^r) = \max\{|nx_i - m_i|\}$ for some m_i such that $|m_i - nx_i| < 1$. Thus $|(n, m_1, \ldots, m_r)| \leqslant \max\{|n|, |nx_i|\} \leqslant |n|$. In particular,

$$d(nx, \mathbf{Z}^r) \geqslant C_0 |n|^{-\omega_0(x)-\epsilon},$$

which implies $\frac{1}{d(nx,\mathbf{Z}^r)} \ll |n|^{\omega_0(x)+\epsilon}$, the implied constant depending on both x and ϵ.

For the second inequality, temporarily set $|\cdot| = |\cdot|_1$, and note that $d(m_1 x_1 + \cdots + m_r x_r, \mathbf{Z}) = |m_0 + m_1 x_1 + \cdots + m_r x_r|$ for $|m_0| \leqslant |(m_1, \ldots, m_r)| \cdot |x| + 1$. Thus $|(m_0, \ldots, m_r)|_\infty \leqslant 2|x||(m_1, \ldots, m_r)|$, giving us

$$d(m_1 x_1 + \cdots m_r x_r, \mathbf{Z}) \geqslant C'_{r-1} |(m_1, \ldots, m_r)|^{-\omega_{r-1}(x)-\epsilon},$$

which implies $\frac{1}{d(\langle m,x\rangle,\mathbf{Z})} \ll |m|^{\omega_{r-1}(x)+\epsilon}$, the implied constant again depending on both x and ϵ. □

Let $\mathbf{T}^r = (\mathbf{R}/\mathbf{Z})^r$, with Haar measure normalized to have total mass one. Given $x \in \mathbf{T}^r$, we define $\omega_0(x)$ and $\omega_{r-1}(x)$ as in Definition 7.1, choosing any coset representative of x. This definition is independent of the choice. Recall that for $f \in L^1(\mathbf{T}^r)$, the *Fourier coefficients* of f are, for $m \in \mathbf{Z}^r$

$$\widehat{f}(m) = \int_{\mathbf{T}^r} e^{2\pi i \langle m, x\rangle}\, \mathrm{d}x,$$

where $\langle m, x\rangle = m_1 x_1 + \cdots + m_r x_r$ is the usual inner product.

Theorem 7.3 (Jarník)**.** *Let $w \geqslant 1/r$. Then there exists $x \in \mathbf{R}^r$ such that $\omega_0(x) = w$ and $\omega_{r-1}(x) = rw + r - 1$.*

Theorem 7.4. *Fix $x \in \mathbf{T}^r$ with $\omega_{r-1}(x)$ finite. Then*

$$\left|\sum_{n\leqslant N} e^{2\pi i\langle m,nx\rangle}\right| \ll_{\epsilon,x} |m|^{\omega_{r-1}(x)+\epsilon}$$

as m ranges over $\mathbf{Z}^r \smallsetminus 0$.

Proof. First, note the easy bound:

$$\left|\sum_{n\leqslant N} e^{2\pi in\langle m,x\rangle}\right| = \left|\frac{e^{2\pi iN\langle m,x\rangle}-1}{e^{2\pi i\langle m,x\rangle}-1}\right| \leqslant \frac{2}{|e^{2\pi i\langle m,x\rangle}-1|}.$$

Since $|e^{2\pi i\langle m,x\rangle} - 1| = \sqrt{2-2\cos(2\pi\langle m,x\rangle)}$ and $\cos(2\theta) = 1 - 2\sin^2(\theta)$, we obtain $\left|\sum_{n\leqslant N} e^{2\pi in\langle m,x\rangle}\right| \leqslant \frac{1}{|\sin(\pi\langle m,x\rangle)|}$. It is easy to check that $|\sin(\pi t)| \geqslant d(t,\mathbf{Z})$, hence $\left|\sum_{n\leqslant N} e^{2\pi in\langle m,x\rangle}\right| \leqslant \frac{1}{d(\langle m,x\rangle,\mathbf{Z})}$. The final estimate comes from Lemma 7.2. □

Theorem 7.5. *Assume $\omega_{r-1}(x) < \infty$. Let $f \in L^1(\mathbf{T}^r)$ with $\widehat{f}(0) = 0$ and suppose the Fourier coefficients of f satisfy the bound $|\widehat{f}(m)| \ll |m|^{-\frac{1}{r-1}-\omega_{r-1}-\epsilon}$. Then*

$$\left|\sum_{n\leqslant N} f(nx)\right| \ll_{f,x} 1.$$

Proof. Write f as a Fourier series:

$$f(x) = \sum_{m\in\mathbf{Z}^r} \widehat{f}(m)e^{2\pi i(m\cdot x)}.$$

Since $\int f = 0$, we have $\widehat{f}(0) = 0$. Thus we can compute

$$\begin{aligned}
\left|\sum_{n\leqslant N} f(nx)\right| &= \left|\sum_{n\leqslant N}\sum_{m\in\mathbf{Z}^r\smallsetminus 0} \widehat{f}(m)e^{2\pi in(m\cdot x)}\right| \\
&\leqslant \sum_{m\in\mathbf{Z}^r\smallsetminus 0} |\widehat{f}(m)| \left|\sum_{n\leqslant N} e^{2\pi in(m\cdot x)}\right| \\
&\ll_{x,\epsilon} \sum_{m\in\mathbf{Z}^r\smallsetminus 0} |m|^{-\frac{1}{r-1}-\omega_{r-1}(x)-\epsilon}|m|^{\omega_{r-1}(x)+\epsilon/2} \\
&\ll_{x,\epsilon} \sum_{m\in\mathbf{Z}^r\smallsetminus 0} |m|^{-\frac{1}{r-1}-\epsilon/2}.
\end{aligned}$$

The sum converges since the exponent is less than $-\frac{1}{r-1}$, and it doesn't depend on N, whence the result. □

Corollary 7.6. *Assume $\omega_{r-1}(x) < \infty$, and let $f \in C^\infty(\mathbf{T}^r)$ with $\widehat{f}(0) = 0$. Then $\left|\sum_{n \leqslant N} f(nx)\right| \ll_{f,x} 1$.*

Proof. This follows from Theorem 7.5 and the fact that the Fourier coefficients of a smooth function decay faster than $|m|^k$, for any $k \in (-\infty, -1]$. □

Theorem 7.7. *If $\omega_0(x) < \infty$, then the sequence $\boldsymbol{x} = (nx)_{n \geqslant 1}$ in $\mathbf{T}^r$ has discrepancy*

$$\operatorname{disc}(\boldsymbol{x}^N) = \Omega\left(2^{-r\left(2+\frac{1}{\omega_0(x)}\right)-\epsilon} N^{-\frac{r}{\omega_0(x)}-\epsilon}\right).$$

Proof. We follow the proof of [KN74, Ch.2, Th.3.3]. First, given $\epsilon > 0$, there exists $\delta > 0$ such that $\frac{r}{\omega_0(x)-\delta} = \frac{r}{\omega_0(x)} + \epsilon$.

By the definition of $\omega_0(x)$, there exist infinitely many $(q, m_1, \ldots, m_r)$ with $q > 0$ such that

$$|qx - m|_\infty \leqslant (\max\{q, |m|_\infty|\})^{-\omega_0(x)+\delta/2}.$$

Since $\max\{q, |m|_\infty\} \geqslant q$, we derive the stronger statement that for infinitely many $q \to \infty$, there exists $m = (m_1, \ldots, m_r) \in \mathbf{Z}^r$ such that $|qx - m|_\infty \leqslant q^{-\omega_0(x)+\delta/2}$, or, equivalently, $|x - \frac{m}{q}| \leqslant q^{-1-\omega_0(x)+\delta/2}$. Pick such a q, and let $N = \lfloor q^{\omega_0(x)-\delta} \rfloor$. Then for $n \leqslant N$, we have $|nx - \frac{n}{q}m| \leqslant q^{-1-\delta/2}$. Thus, for $n \leqslant N$, each nx is within $q^{-1-\delta/2}$ of the grid $\frac{1}{q}\mathbf{Z}^r \subset \mathbf{T}^r$. Thus, they miss a box with side lengths $q^{-1} - 2q^{-1-\delta/2}$. For q sufficiently large, $q^{-1} - 2q^{-1-\delta/2} \geqslant 1/2q$, so the (non-star) discrepancy of $\boldsymbol{x}^N$ is bounded below by $2^{-r}q^{-r}$. Since $q^{\omega_0(x)-\delta} \leqslant 2N$, the (non-star) discrepancy at N is bounded below by

$$2^{-r}\left((2N)^{\frac{1}{\omega_0(x)+\delta}}\right)^{-r} = 2^{-r-\frac{r}{\omega_0(x)+\delta}} N^{-\frac{r}{\omega_0(x)+\delta}} = 2^{-r\left(1+\frac{1}{\omega_0(x)}\right)-\epsilon} N^{-\frac{r}{\omega_0(x)}-\epsilon}.$$

Since r-dimensional star-discrepancy is bounded below by 2^{-r} times non-star discrepancy, we obtain the final result. □

The key point in the above theorem is that

$$\operatorname{disc}(\boldsymbol{x}^N) = \Omega_{x,r,\epsilon}\left(N^{-\frac{r}{\omega_0(x)}-\epsilon}\right).$$

Theorem 7.8. *Let $\eta \in (0,1)$. Then there exists $x \in \mathbf{T}^r$ such that for all $f \in C^\infty(\mathbf{T}^r)$ with $\widehat{f}(0) = 0$, the estimate*

$$\left|\sum_{n \leqslant N} f(nx)\right| \ll_{f,x} 1$$

holds, but for which

$$\operatorname{disc}(\boldsymbol{x}^N) = \Omega_{\epsilon,r,x}\left(N^{-\eta-\epsilon}\right).$$

Proof. Let $w = \frac{r}{\eta} \geqslant \frac{1}{r}$. By Theorem 7.3, there exists $x \in \mathbf{T}^r$ with $\omega_0(x) = w$ and $\omega_{r-1}(x) = rw + r - 1$. The result follows easily from Corollary 7.6 and Theorem 7.7. □

Lemma 7.9 (Moser). *Let f be a smooth, nonnegative function on $[0,1]^r$ such that $\int f = 1$ and f vanishes only on the boundary of $[0,1]^r$. Then there is a unique factorization*

$$f(x_1, \ldots, x_r) = f_1(x_1) f_2(x_1, x_2) \cdots f_r(x_1, \ldots, x_r)$$

of f into smooth functions such that

$$\int_0^1 f_i(x_1, \ldots, x_{i-1}, t) \, \mathrm{d}t = 1$$

for all $1 \leqslant i \leqslant r$.

Proof. We prove this by induction on r. For $r = 1$, the claim is trivial. Otherwise, fix $(x_1, \ldots, x_{r-1})$. Then we are trying to solve the following problem. Find a factorization $g(t) = \lambda h(t)$, where $\int h = 1$. This has the obvious (unique) solution $h(t) = g(t)/(\int g)$. Thus, we have:

$$\begin{aligned} f_{r-1}(x_1, \ldots, x_{r-1}) &= \int_0^1 f(x_1, \ldots, x_{r-1}, t) \, \mathrm{d}t \\ f_r(x_1, \ldots, x_r) &= f(x_1, \ldots, x_r) / f_{r-1}(x_1, \ldots, x_{r-1}). \end{aligned}$$

□

Lemma 7.10. *Let λ be the Lebesgue measure on $[0,1]^r$, and $\mu = f\lambda$ where $f \geqslant 0$ is smooth, and $f \neq 0$ on the interior of $[0,1]^r$. Then there is a diffeomorphism $u \colon [0,1]^r \to [0,1]^r$, identity on the boundary, such that $u_* \lambda = \mu$.*

Proof. We follow [Mos65]. First, use Lemma 7.9 to factor f as a product

$$f(x_1, \ldots, x_r) = f_1(x_1) f_2(x_1, x_2) \cdots f_r(x_1, \ldots, x_r).$$

Let

$$u_i(x_1, \ldots, x_i) = \int_0^{x_i} f_i(x_1, \ldots, x_{i-1}, t) \, \mathrm{d}t.$$

Then each u_i is a strictly increasing function, and $u = (u_1, \ldots, u_r)$ is a diffeomorphism of the unit square, which is the identity on the boundary. Moreover,

$$\det(\operatorname{Jac} u) = \prod \frac{\mathrm{d}u_i}{\mathrm{d}x_i} = \prod f_i = f.$$

Now, by the change of variables formula,

$$\int \phi \, \mathrm{d}u_*^{-1} \lambda = \int \phi \circ u^{-1} \, \mathrm{d}\lambda = \int \phi \det(\operatorname{Jac} u) \, \mathrm{d}\lambda = \int \phi \, \mathrm{d}\mu,$$

i.e. $\mu = u_*^{-1} \lambda$. □

Theorem 7.11. *Let μ, f be as above. Then there exists a sequence $\boldsymbol{x}$ in $[0,1]^r$ such that* $\mathrm{disc}(\boldsymbol{x}^N,\mu) = \Omega(N^{-r\eta-\epsilon})$*, but for which* $|\sum g(x_n)| \ll_g 1$ *for all smooth g with $\mu(g) = 0$.*

Proof. By Lemma 7.10, there exists a boundary-preserving diffeomorphism $u\colon [0,1]^r \to [0,1]^r$, such that $u_*\lambda = \mu$, where λ is the Lebesgue measure as above.

Start with a sequence $\boldsymbol{y}_n = ny$, where y is as in Theorem 7.8. Let $\boldsymbol{x} = u_*\boldsymbol{y}$, i.e. $x_n = u(y_n)$. Then, if $\phi \in C^\infty([0,1]^r)$, the composite $\phi\circ u$ is also smooth, so

$$\left|\sum_{n\leqslant N}\phi(u_* y_n)\right| = \left|\sum_{n\leqslant N}(\phi\circ u)(y_n)\right| \ll_{\phi\circ u, y} 1.$$

Thus, all we need is a lower bound on the discrepancy. The proof of Theorem 7.7 tells us that for infinitely many $N\to\infty$, there is an r-ball B_N with volume $CN^{-\eta-\epsilon}$ (C not depending on N) that does not contain any of $y_1,\dots,y_N$. By [Pol01, Th.2.1], for N sufficiently large, the set $u(B_N)$ is convex, and moreover $\mu(u(B_N)) = \lambda(B_N)$. Thus, since $\boldsymbol{x}^N(u(B_N)) = \varnothing$, we have

$$\mathrm{disc}^{\mathrm{iso}}(\boldsymbol{x}^N,\mu) \geqslant CN^{-\eta-\epsilon},$$

and thus

$$\mathrm{disc}(\boldsymbol{x}^N,\mu) = \Omega(N^{-r\eta-\epsilon})$$

as desired.

[Need: bounds relating discrepancy and isotropic discrepancy to hold for non-Lebesgue measure.] □

Theorem 7.12. *Let $[0,1]^r,\mu$ be as above. Then there exists $\boldsymbol{x}$ such that for all smooth f, $L_f(\boldsymbol{x},s)$ satisfies the Riemann Hypothesis (analytic continuation and no zeros on $\{\Re > \frac{1}{2}\}$, but for which* $\mathrm{disc}(\boldsymbol{x}^N) = \Omega(N^?)$.

References

[Apo76] Tom M. Apostol. *Introduction to analytic number theory.* Springer-Verlag, New York-Heidelberg, 1976. Undergraduate Texts in Mathematics.

[AT99] Shigeki Akiyama and Yoshio Tanigawa. Calculation of values of L-functions associated to elliptic curves. *Math. Comp.*, 68(227):1201–1231, 1999.

[BLGHT11] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor. A family of Calabi-Yau varieties and potential automorphy II. *Publ. Res. Inst. Math. Sci.*, 47(1):29–98, 2011.

[Bou05] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9.* Elements of Mathematics. Springer-Verlag, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[Bug08] Yann Bugeaud. Diophantine approximation and Cantor sets. *Math. Ann.*, 341(3):677–684, 2008.

[Eng89] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.

[Kat88] Nicholas M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups*, volume 116 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1988.

[KN74] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences.* Wiley-Interscience [John Wiley & Sons], 1974. Pure and Applied Mathematics.

[KS99] Nicholas M. Katz and Peter Sarnak. *Random matrices, Frobenius eigenvalues, and monodromy*, volume 45 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.

[Lau09] Michel Laurent. On transfer inequalities in Diophantine approximation. In *Analytic number theory*, pages 306–314. Cambridge Univ. Press, Cambridge, 2009.

[Maz96] Fernando Mazzone. A characterization of almost everywhere continuous functions. *Real Anal. Exchange*, 21(1), 1995/96.

[Mos65] Jürgen Moser. On the volume elements on a manifold. *Trans. Amer. Math. Soc.*, 120:286–294, 1965.

[Ökt99] G. Ökten. Error reduction techniques in quasi-Monte Carlo integration. *Math. Comput. Modelling*, 30(7-8):61–69, 1999.

[Pol01] B. T. Polyak. Convexity of nonlinear image of a small ball with applications to optimization. *Set-Valued Anal.*, 9(1-2):159–168, 2001.

[Ser68] Jean-Pierre Serre. *Abelian l-adic representations and elliptic curves.* McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Ten95] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*, volume 46 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas.