A counterexample relating exponential sums and discrepancy

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For a prime p, let

$$T_p = \left\{ \frac{a}{2\sqrt{p}} : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p} \right\}$$
$$\Theta_p = \cos^{-1}\left(T_p\right).$$

Since applying continuous increasing functions preserves discrepancy, we have:

$$\mathrm{D}(T_p, \mathrm{Leb}) \ll p^{-1/2}$$
$$\mathrm{D}\left(\Theta_p, \frac{1}{2}\sin(t)\,\mathrm{d}t\right) \ll p^{-1/2}.$$

We claim that starting with $\theta_2 \in \Theta_2$, we can choose θ_p such that we preserve the inequalities:

$$\frac{1}{4\log x} \leqslant \mathrm{D}(\{\theta_p\}_{p\leqslant x}) \leqslant \frac{4}{\log x}$$
$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \leqslant 2\sqrt{x}$$

Recall that

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin\theta}.$$

We can run this for all $p \leqslant 10^5$. Recall that $\pi(10^5) \approx 10000$.

Here is what we get:

Conjecture 0.1. *There exists a sequence of $\theta_p \in \Theta_p$ such that the following identities always hold:*

$$\frac{1}{4\log x} \leqslant \mathrm{D}(\{\theta_p\}_{p\leqslant x}) \leqslant \frac{4}{\log x}$$
$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \leqslant 2\sqrt{x}.$$

Figure 1: Plot of $\sum_{p \leqslant x} U_1(\theta_p)$

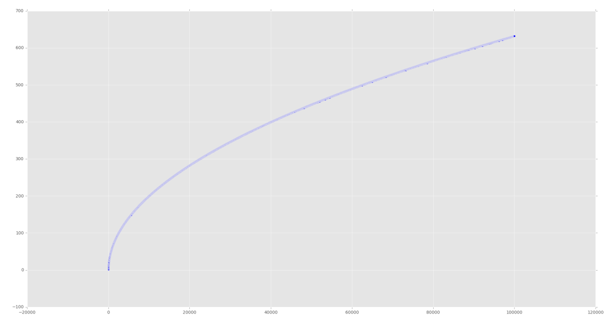
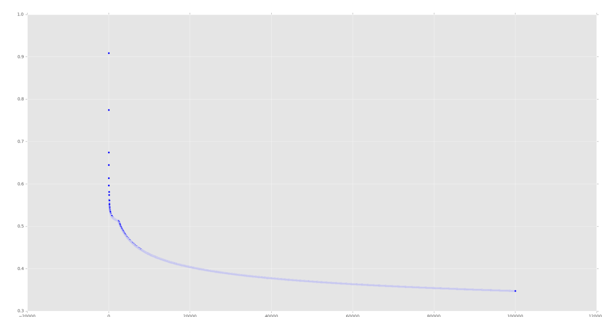


Figure 2: Plot of $\mathrm{D}(\{\theta_p\}_{p \leqslant x})$



Next, choose $\bar{\rho}_l \colon G_{\mathbf{Q}} \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_l)$ to which we can apply Ramakrishna et. al.'s machinery. Define

$$\Theta_p(\bar{\rho}_l) = \left\{ \cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \operatorname{tr} \bar{\rho}_l(\mathrm{fr}_p) \pmod{l} \right\}.$$

Conjecture 0.2. *There exists a sequence of $\theta_p \in \Theta_p(\bar{\rho}_l)$ such that*

$$\mathrm{D}(\{\theta_p\}_{p \leqslant x}) = \Omega\left(\frac{1}{\log x}\right)$$

$$\left| \sum_{p \leqslant x} U_1(\theta_p) \right| \ll \sqrt{x}.$$

Corollary 0.3. *There exists an (infinitely ramified) Galois representation $\rho_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that if we set $a_p = \operatorname{tr} \rho_l(\mathrm{fr}_p)$, then*

1. $a_p \in \mathbf{Z}$

2. $|a_p| \leqslant 2\sqrt{p}$ *.*

3. The $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ *satisfy*

$$\mathrm{D}(\{\theta_p\}_{p\leqslant x}) = \Omega\left(\frac{1}{\log x}\right)$$
$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \ll \sqrt{x}.$$

and hence $L(\rho_l, s)$ *satisfies the Riemann Hypothesis.*

1 Towards a proof

Let $\bar{\rho}_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$ be a Galois representation. For each prime p, define

$$\Theta_p(l) = \left\{\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \operatorname{tr}\bar{\rho}_l(\mathrm{fr}_p) \pmod{l}\right\}.$$

It is easy to check that

$$\mathrm{D}\left(\Theta_p(l), \frac{1}{2}\sin(t)\,\mathrm{d}t\right) \ll lp^{-1/2}.$$

We are looking for a way to choose $\theta_p \in \Theta_p(l)$ such that

1. $\mathrm{D}(\{\theta_p\}_{p\leqslant x})$ decays like $1/\log x$
2. $\left|\sum_{p\leqslant x} U_1(\theta_p)\right|$ grows like $\sqrt{x}$.

To do this, suppose we have chosen $\{\theta_q\}_{q<p}$. In choosing θ_p, we want to simultaneously move the discrepancy towards $1/\log p$, while making sure that the U_1-sum doesn't get too big.

(Fact: if $\{x_1, \dots, x_N\}$ and $\{y_1, \dots, y_N\}$ are two sequences, then

$$|\,\mathrm{D}(\{x_1, \dots, x_N\}) - \mathrm{D}(\{y_1, \dots, y_N\})| \leqslant 2\|x - y\|_\infty.$$

)

It's actually quite simple. Note that:

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin\theta} = -U_1(\pi - \theta).$$

The basic idea is: set $\theta_3 \approx \pi - \theta_2$, $\theta_7 \approx \pi - \theta_5$, etc. and we can choose θ_2, θ_5 etc. arbitrarily, meaning good discrepancy, while the sum should approximately cancel out. First, since U_1 has bounded derivative, we know that

$$|U_1(\theta) - U_1(\varphi)| \ll |\theta - \varphi|$$

So, if $p_1 < p_2$ are sequential primes, we have

$$|\theta_{p_2} - (\pi - \theta_{p_1})| \ll p_1^{-1/2},$$

so

$$\begin{aligned}|U_1(\theta_{p_1}) + U_1(\theta_{p_2})| &\leqslant |U_1(\theta_{p_1}) - U_1(\pi - \theta_{p_1})| + |U_1(\pi - \theta_{p_1}) - U_1(\theta_{p_2})| \\ &\ll |\theta_{p_2} - (\pi - \theta_{p_1})| \\ &\ll p_1^{-1/2}.\end{aligned}$$

So,

$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \ll \sum_{p\leqslant x} p^{-1/2} \ll \int_1^x t^{-1/2}\,\mathrm{d}t \ll \sqrt{x}.$$

(Same argument works for all U_{odd} because they all satisfy $U_{\text{odd}}(\pi - \theta) = -U_{\text{odd}}(\theta)$. In contrast, $U_{\text{even}}(\pi - \theta) = U_{\text{even}}(\theta)$.)

2 A legit proof!

Theorem 2.1. *Fix a prime l. Suppose we have chosen, for all primes p, some arbitrary residue class $\bar{a}_p \in \mathbf{F}_l$, and set*

$$\Theta_p(l) = \left\{\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \bar{a}_p \pmod{l}\right\}.$$

Then there exists a choice of $\theta_p \in \Theta_p(l)$ such that

1. *The sequence $\{\theta_p\}$ is equidistributed with respect to the Sato–Tate measure $\frac{2}{\pi}\sin^2\theta\,\mathrm{d}\theta$.*
2. *The discrepancy $\mathrm{D}(\{\theta_p\}_{p\leqslant x}, \mathrm{ST}) \gg \frac{1}{\log x}$.*
3. *$\left|\sum_{p\leqslant x} U_{\text{odd}}(\theta_p)\right| \ll \sqrt{x}$.*

Proof. Enumerate the primes $p_1 < p_2 < \cdots$. We will choose $\theta_{p_{\text{odd}}} \in [0, \pi/2)$ so that the discrepancy of the sequence $\{\theta_{p_{\text{odd}}}\}$ behaves as required in that interval. We'll then set $\theta_{p_{2i}} \approx \pi - \theta_{p_{2i-1}}$.

Everything comes down to: if $p < q$ are sequential primes and we have already chosen θ_p, we need to be able to choose θ_q so that $|U_1(\theta_p) + U_1(\theta_q)| \ll p^{-1/2}$. Since $\frac{\mathrm{d}U_1}{\mathrm{d}\theta} = -2\sin(\theta)$, we have (roughly)

$$|U_1(\theta) - U_1(\varphi)| \ll \max(\theta, \varphi)\cdot|\theta - \varphi|$$

for $\theta, \varphi \in [0, \pi/2)$.

Start with $t_p = \frac{a_p}{2\sqrt{p}}$ and $t_q = \frac{a_q}{2\sqrt{q}}$. We can guarantee that $|t_p - (\pi - t_q)| \ll p^{-1/2}$.

Fact:
$$|\cos^{-1}(1-x)-\cos^{-1}(1-(x+\sqrt{x}))| \ll x^{1/5}.$$
So roughly,
$$|\theta_p-\theta_q| \ll p^{-1/5},$$
After taking $\cos^{-1}$, all we can guarantee is that
$$|\theta_p-\theta_q| \ll$$
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Let's think systematically. We're picking t_1 and t_2 close to 1, which is where $(\cos^{-1})'$ blows up. But there shouldn't be very many of them close to 1. Aka,
$$\left|\frac{\#\{p \leqslant x : \theta_p \in [0,t)\}}{\pi(x)} - \int_0^t \mathrm{dST}\right| \ll \frac{1}{\log x}$$
$$\frac{\#\{p \leqslant x : \theta_p \in [0,t)\}}{\pi(x)} \ll t^2 + \frac{1}{\log x}.$$
We want to know, given x, how small the smallest $\theta_p, p \leqslant x$ is. Roughly, for what t is
$$\#\{p \leqslant x : \theta_p \in [0,t)\} < 1?$$
We already know that
$$\#\{p \leqslant x : \theta_p \in [0,t)\} \ll \frac{x}{\log x}\left(t^2 + \frac{1}{\log x}\right).$$
This is frustrating, because it means, essentially, that our convergence to the Sato–Tate measure is so slow (by design) that we can't *ever* guarantee that no θ_p lies in some small interval. But there's something easier. For each $p \leqslant x$, we start by choosing $a_p \in \mathbf{Z}$. How close can a_p be to $2\sqrt{p}$? Numerical experiments **(prove this!)** show that for $t_p = \frac{a_p}{2\sqrt{p}}$, we have
$$|1-t_p| \gg p^{-1/2}.$$
This is key! That means θ_p won't be too small. In particular, we can control how close θ_p and θ_q will be.

We already have chosen θ_p. We want to choose a_q so that $\cos^{-1}(\frac{a_q}{2\sqrt{q}}) \approx \pi-\theta_p$, i.e.
$$\frac{a_q}{2\sqrt{q}} \approx \sin(\theta_p).$$
We can ensure
$$\left|\frac{a_q}{2\sqrt{q}} - \cos(\pi-\theta_p)\right| \ll p^{-1/2}.$$
Moreover, we know that $|\pm 1 - \frac{a_q}{2\sqrt{q}}| \gg q^{-1/2}$, and likewise for a_p. Thus,
$$|\theta_p-\theta_q| = \left|\cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right) - \pi + \cos^{-1}\left(\frac{a_q}{2\sqrt{q}}\right)\right| \ll p^{-1/2}\cdot ?$$

Good news: numerical experiments show that we can get very good approximation to $U_1(\theta_q) \approx -U_1(\theta_p)$ for $p < q$ successive primes. This is fantastic!

Numerical experiments suggest that we can enforce

$$|U_1(\theta_p) + U_1(\theta_q)| \ll \frac{\log p}{p}.$$

$\square$

Let (X, μ) be a topological measure space. Suppose g is a non-trivial automorphism of X, such that $g_* \mu = \mu$. Suppose $g^2 = 1$. If we want to minimize

$$\left| \sum_{p \leqslant x} f(x_p) \right|,$$

while letting the discrepancy of $\{x_p\}$ vary arbitrarily. Suppose we can find a "good" subset $U \subset X$ such that $X = U \sqcup gU$. Choose $x_{p_{\text{odd}}} \in U$ to control the discrepancy, and then choose $x_{p_{\text{even}}} \approx g(x_{p_{\text{odd}}})$. For any $f \in C^\infty(X)$ such that $g^* f = -f$. Then

$$\sum_{p \leqslant x} f(x_p) = \sum (f(x_{p_{\text{even}}}) + f(x_{p_{\text{odd}}})) \approx \sum 0.$$

We know that near $\theta = 0$,

$$U_n(\theta) = n + C_n \theta^2 + O(\theta^3).$$

(I think this will hold for any f with $\int f = 0$ and $f(\pi - \theta) = f(\theta)$.)

3 Precise method

Let $\{p_1, p_2, \ldots, \}$ be an enumeration of the rational primes. Given $x \in \mathbf{R}$, write $\sum_{p_{\text{odd}} \leqslant x} a_p$ for the sum of all a_p for $p_i \leqslant x$ with i odd, and similarly for $\sum_{p_{\text{even}} \leqslant x}$. Suppose we have chosen $\theta_{p_{\text{odd}}} \in [0, \pi/2)$ so that $\mathrm{D}(\{\theta_{p_{\text{odd}}}\}_{p_{\text{odd}} \leqslant x})$ decays as desired. Suppose we choose $\theta_{p_{\text{even}}} \approx \pi - \theta_{p_{\text{odd}}}$. That is, for $p < q$ successive primes with $p = p_i$, i odd, we'll choose $\theta_q \approx \pi - \theta_p$.

We know that $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ for some $a_p \in \mathbf{Z}$ with $|a_p| \leqslant 2\sqrt{p}$. We want to choose $\theta_q \approx \pi - \theta_p$, i.e.

$$\cos^{-1}\left(\frac{a_q}{2\sqrt{q}}\right) \approx \pi - \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$$
$$\frac{a_q}{2\sqrt{q}} \approx -\frac{a_p}{2\sqrt{p}}.$$

since $\cos(\pi - \cos^{-1}(x)) = -x$. We can guarantee that

$$\left| \frac{a_q}{2\sqrt{q}} + \frac{a_p}{2\sqrt{p}} \right| \leqslant \frac{1}{\sqrt{q}}.$$

Claim: if x, y are "further than ϵ" from ± 1 and $|x-y| < \epsilon$, then $|\cos^{-1}(x) - \cos^{-1}(y)| \leqslant \sqrt{\epsilon}$. (Have checked with Wolfram Alpha, prove later.)

In conclusion, for each successive primes $p = p_{\text{odd}} < q = p_{\text{even}}$, if there is $\theta_p \in \Theta_p(l)$ chosen already, we can also choose $\theta_q \in \Theta_q(l)$ so that

$$|\theta_q - (\pi - \theta_p)| \ll l p^{-1/4}.$$

This is all that is needed, since we're looking at f that is of the form

$$f(\theta) = f(0) + C\theta^2 + O(\theta^3)$$

for θ close to zero. (In fact, this is true for *all* smooth, Weyl-invariant f, whether or not they satisfy $f(\theta) = -f(\pi - \theta)$.) The squaring "pushes the difference" back to $p^{-1/2}$. That is, for θ, φ close to zero, but at least ϵ away from zero, we have

$$|f(\theta) - f(\varphi)| \ll |\theta - \varphi|^2.$$

Now the question is, if $\theta_q \approx \pi - \theta_p$, how close is the discrepancy of $\{\theta_{p_{\text{odd}}}\}$ and $\{\theta_{p_{\text{even}}}\}$?

Better, how close are

$$\#\{p_{\text{odd}} \leqslant x : \theta_{p_{\text{odd}}} \leqslant t\} \qquad \text{and} \qquad \#\{p_{\text{odd}} \leqslant x : \theta_{p_{\text{odd}}} \leqslant t\}?$$

We know that $|\theta_p - \theta_q| \ll p^{-1/4}$. Actually, all we need is that if $\mathrm{D}(\{\theta_{p_{\text{odd}}}\}) \to 0$, then also $\mathrm{D}(\{\theta_{p_{\text{even}}}\}) \to 0$.

Suppose we have two sequences $\{x_n\}$ and $\{y_n\}$ such that $\mathrm{D}(\{x_n\}_{n \leqslant N}) \sim \frac{1}{\log N}$, and also $|x_n - y_n| \leqslant n^{-1/4}$. For some really big N, choose $M < N$, ideally $M \approx \log N$.

Look at

$$\limsup_{N \to \infty} \mathrm{D}(\{y_n\}_{M \leqslant n \leqslant N}) \leqslant M^{-1/4}.$$

With complete generality, we have:

$$|\mathrm{D}(\{x_n\}_{n \leqslant N}) - \mathrm{D}(\{x_n\}_{M \leqslant n \leqslant N})| \ll \frac{1}{M}$$

This is all we need.

Lemma 3.1. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in $\mathbf{R}_{\geqslant 0}$. Suppose $\nu = f\,\mathrm{d}x$ for a continuous function f. Then*

$$|\,\mathrm{D}^\star(\boldsymbol{x}^N, \nu) - \mathrm{D}^\star(\boldsymbol{y}^N, \nu)| \leqslant \epsilon \|f\|_\infty + \frac{\#\{n \leqslant N : |x_n - y_n| > \epsilon\}}{N}.$$

Proof. It is actually sufficient to just prove that

$$\mathrm{D}^\star(\boldsymbol{y}^N, \nu) \leqslant \mathrm{D}^\star(\boldsymbol{x}^N, \nu) + \epsilon \|f\|_\infty + \frac{\#\{n \leqslant N : |x_n - y_n| > \epsilon\}}{N}.$$

Start with an arbitrary interval $[0, t)$. Clearly

$$\#\{n \leqslant N : y_n < t\} \leqslant \#\{n \leqslant N : x_n < t + \epsilon\} + \#\{n \leqslant N : |x_n - y_n| > \epsilon\},$$

and also
$$\left|\frac{\#\{n \leqslant N : x_n < t+\epsilon\}}{N} - \mu[0,t+\epsilon)\right| \leqslant \mathrm{D}^\star(\boldsymbol{x}^N,\mu).$$
It follows that
$$\frac{\#\{n \leqslant N : y_n < t\}}{N} - \mu[0,t) \leqslant \mu[t,t+\epsilon) + \mathrm{D}^\star(\boldsymbol{x}^N,\mu) + \frac{\#\{n \leqslant N : |x_n - y_n| > \epsilon\}}{N}.$$
A similar argument with $[0,t-\epsilon)$ yields
$$\frac{\#\{n \leqslant N : y_n < t\}}{N} - \mu[0,t) \geqslant -\mu[t-\epsilon,t) - \mathrm{D}^\star(\boldsymbol{x}^N,\mu) - \frac{\#\{n \leqslant N : |x_n - y_n| > \epsilon\}}{N}.$$
Since the discrepancy is a supremum over t, we get
$$\mathrm{D}^\star(\boldsymbol{y}^N,\mu) \leqslant \mathrm{D}^\star(\boldsymbol{x}^N,\mu) + \|f\|_\infty \epsilon + \frac{\#\{n \leqslant N : |x_n - y_n| > \epsilon\}}{N}$$
as desired. □

This lemma has a powerful application.

Theorem 3.2. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be sequences in $\mathbf{R}$ and $\mu = f\,\mathrm{d}x$ be a measure induced by a continuous function f. Suppose that*

1. *$\boldsymbol{x}$ is μ-equidistributed.*
2. *$\|\boldsymbol{x}_{>N} - \boldsymbol{y}_{>N}\|_\infty \to 0$.*

Then $\boldsymbol{y}$ is also μ-equidistributed.

Proof. Recall that $\boldsymbol{x}_{>N} = (x_{N+1}, x_{N+2}, \dots)$, and that $\|\cdot\|_\infty$ is the supremum norm. Let $\varphi : \mathbf{N} \to \mathbf{N}$ be a function such that $\varphi(n) \to \infty$, but also $\varphi(n) = o(n)$. For example, we could have $\varphi(n) = \lfloor \log n \rfloor$. For any N, let $\epsilon = \|\boldsymbol{x}_{>\varphi(N)} - \boldsymbol{y}_{>\varphi(N)}\|_\infty$, and apply Lemma 3.1. Trivially, we know that
$$\#\{n \leqslant N : |x_n - y_n| > \epsilon\} \leqslant \varphi(N),$$
so we can write
$$\mathrm{D}(\boldsymbol{y}^N,\mu) \leqslant \mathrm{D}(\boldsymbol{x}^N,\mu) + 2\|\boldsymbol{x}_{>\varphi(N)} - \boldsymbol{y}_{>\varphi(N)}\|_\infty \cdot \|f\|_\infty + \frac{\varphi(N)}{N} \to 0.$$
Note that we do not control the rate of decay of $\mathrm{D}(\boldsymbol{y}^N,\mu)$. □

4 Summary of argument

Fix a prime l, and for each prime p, a choice of equivalence class $\bar{a}_p \in \mathbf{F}_l$. Define
$$\Theta_p(l) = \left\{\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \bar{a}_p \pmod{l}\right\}.$$
Claim: there is a choice of $\theta_p \in \Theta_p(l)$ such that

1. $\mathrm{D}^\star(\{\theta_p\}_{p\leqslant X})$ is not $O(X^{-\epsilon})$ for any $\epsilon > 0$.
2. For any $f \in C^\infty(\mathbf{R}/2\pi\mathbf{Z})^W$ with $f(\pi - \theta) = -f(\theta)$, we have

$$\left|\sum_{p\leqslant X} f(\theta_p)\right| \ll \sqrt{X}.$$

How do we construct this sequence $\{\theta_p\}_p$? Enumerate the primes as $\{p_n\}_{n\geqslant 1}$. Choose the sequence $\{\theta_{p_{2n-1}}\}_{n\geqslant 1}$ so that

1. $\theta_{p_{2n-1}} \in [0, 2\pi)$
2. $\mathrm{D}^\star\left(\{\theta_{p_{2n-1}}\}_{n\leqslant N}, 2\cdot \mathrm{ST}|_{[0,\pi/2)}\right) \to 0$, but slower than any $N^{-\epsilon}$. **(Prove this is possible!)**

We've proved that we can choose $\theta_{p_{2n}}$ so that $|\theta_{p_{2n}} - (\pi - \theta_{p_{2n-1}})| \ll lp^{-1/4}$, which implies (via Theorem 3.2) that

$$\mathrm{D}^\star\left(\{\theta_{p_{2n}}\}_{n\leqslant N}, 2\cdot \mathrm{ST}|_{[\pi/2,\pi)}\right) \to 0.$$

(We need to know that the discrepancy of $\pi/2 - x$ is the same as that of x.) It follows that the sequence $\{\theta_p\}$ formed by interleaving our "even" and "odd" indexed primes has discrepancy that goes to zero **(We need to prove that if x and y are sequences equidistributed with respect to measures supported on $[0,\pi/2)$ and $[\pi/2,\pi)$, then the "interleaved" sequence also has equidistribution and discrepancy which decays no faster than the slower of the two.)**

Note that our hypothesis on $\theta_{p_{2n}} \approx \pi - \theta_{p_{2n-1}}$, we have, for f as in the result,

$$|f(\theta_{p_{2n-1}}) + f(\theta_{p_{2n}})| \ll_f lp^{-1/2}.$$

(Problem here: this bound only works near "the edges." But also, the θs are better away from the edges.)