A brief summary of thesis results

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Some quick notation. If X is some space and $f\colon X \to \mathbf{C}$ a function, and $\boldsymbol{x} = (x_2, x_3, x_5, \dots)$ a sequence in X, write

$$L_f(\boldsymbol{x}, s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}.$$

Theorem 1. *If, for $\alpha \in [1/2, 1]$, we have*

$$\left| \sum_{p \leqslant N} f(x_p) \right| \ll N^{\alpha+\epsilon},$$

then $\log L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re s > \alpha\}$. Conversely, if $L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re s > \alpha\}$ and moreover,

$$|\log L_f(\boldsymbol{x}, \sigma + it)| \ll |t|^{1-\epsilon},$$

for all $\sigma > \alpha$, then

$$\left| \sum_{p \leqslant N} f(x_p) \right| \ll \pi(N)^{\alpha+\epsilon}.$$

Roughly, this theorem says that analytic continuation of $\log L_f(\boldsymbol{x}, s)$ to $\{\Re s > \alpha\}$ is equivalent to the bound $|\sum_{p \leqslant N} f(x_p)| \ll N^{\alpha+\epsilon}$.

Theorem 2. *Let $d \geqslant 1$. For any $\alpha \in [0, 1/2]$, there exists a sequence $\boldsymbol{x}$ in $(\mathbf{R}/\mathbf{Z})^d$ that is uniformly distributed, such that*

1. *$D_N = \Omega(N^{-\alpha+\epsilon})$ (aka, big-O, but not big-O of anything smaller).*
2. *For any $f \in C^\infty(\mathbf{R}/\mathbf{Z})^d$ with $\int f = 0$, the function $\log L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re s > 1/2\}$.*

This says there are d-dimensional sequences whose discrepancy decays arbitrarily slowly, but whose L-functions are well behaved.

Theorem 3. *There exists a sequence $\boldsymbol{\theta}$ in $[0, \pi]$ such that for each p, $2\sqrt{p}\cos(\theta_p) \in \mathbf{Z}$ and satisfies the Hasse bound, and such that*

1. *The discrepancy D_N is not $\ll N^{-\epsilon}$ for any ϵ.*
2. *The functions $L(\operatorname{sym}^k \boldsymbol{\theta}, s)$ satisfy the Riemann Hypothesis (for k odd).*