Equidistributed subgroups in compact Lie groups

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Let G be a compact connected Lie group, $\Gamma \subset G$ a dense free subgroup with two generators γ_1, γ_2. For $n \to \infty$, write Γ_n for the set of all products of n letters taken from $\{\gamma_1, \gamma_2\}$. Put

$$\mu_n(f) = 2^{-n} \sum_{\gamma \in \Gamma_n} f(\gamma).$$

Claim: μ_n converge to the Haar measure of G. It is sufficient to prove that $\mu_n(\operatorname{tr} \rho) \to 0$ for all non-trivial irreducible ρ.

Recall the left-translation operators

$$L_\gamma f(x) = f(\gamma^{-1} x).$$

Fact:

$$\mu_n = \left(\frac{L_{\gamma_1} + L_{\gamma_2}}{2} \right)^n.$$

What do we have to prove:

$$\left\| \left(\frac{L_{\gamma_1} + L_{\gamma_2}}{2} \right)^n \right\| < 1$$

for all n, otherwise. . . .

1 General perspective

Let G be a compact connected semisimple group. Then there is a dense subgroup $\Gamma \subset G$ generated by two elements. Claim: for "almost all" pairs γ_1, γ_2, the group $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is free and dense in G.

For any n, let Γ_n be the "ball" in Γ consisting of all products of n elements from the set $\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}\}$. Consider

$$\mu_n = \frac{1}{\# \Gamma_n} \sum_{\gamma \in \Gamma_n} \delta_\gamma.$$

Claim: if $\rho \in \widehat{G}$ (so ρ is an irreducible unitary representation of G) then $\mu_n(\operatorname{tr} \rho) \to 0$.

Note that:

$$\mu_1 = \frac{1}{4}\left(L_{\gamma_1} + L_{\gamma_1^{-1}} + L_{\gamma_2} + L_{\gamma_2^{-1}}\right)\Big|_{x=0}$$

What is $\delta_\gamma * f$?

$$\begin{aligned}(\delta_\gamma * f)(S) &= \iint 1_S(xy)\,\mathrm{d}\delta_\gamma(x) f(y)\mathrm{d}y \\ &= \int 1_S(\gamma y) f(y)\,\mathrm{d}y \\ &= \int 1_S(y) f(\gamma^{-1}y)\,\mathrm{d}y \\ &= \int_S L_\gamma f.\end{aligned}$$

In other words, $\delta_\gamma * f = L_\gamma f$. Also, let's see what is

$$\begin{aligned}(\delta_\gamma * \delta_\eta)(S) &= \iint 1_S(xy)\,\mathrm{d}\delta_\gamma(x)\mathrm{d}\delta_\eta(y) \\ &= \int 1_S(\gamma y)\,\mathrm{d}\delta_\eta(y) \\ &= 1_S(\gamma\eta) \\ &= \delta_{\gamma\eta}(S).\end{aligned}$$

In other words, $\delta_{\gamma_1} * \delta_{\gamma_2} = \delta_{\gamma_1\gamma_2}$.

So, if $\Gamma = \langle \gamma_1, \gamma_2 \rangle$ is free on two generators, then for

$$\mu = \frac{1}{4}\left(\delta_{\gamma_1} + \delta_{\gamma_2} + \delta_{\gamma_1^{-1}} + \delta_{\gamma_2^{-1}}\right)$$

the measure μ^{*n} is the n-th "empirical measure" μ_n above.

2 Simpler perspective

As before, let G be a compact semisimple Lie group, and let $\Gamma \subset G$ be a dense free subgroup on two generators γ_1, γ_2. Note that

$$\frac{1}{2^n} \sum_{\sigma\colon \{1,\ldots,n\}\to\{1,2\}} f(\gamma_{\sigma(1)}\gamma_{\sigma(2)}\cdots\gamma_{\sigma(n)}) = \left(\frac{L_{\gamma_1} + L_{\gamma_2}}{2}\right)^n f(1).$$

In particular, if $f = \operatorname{tr}\rho$ for some $\rho \in \widehat{G}$, we want to bound (in $\|\cdot\|_\infty$)

$$\sum_{n\leqslant N} (L_{\gamma_1} + L_{\gamma_2})^n$$

If $\rho \in \widehat{G}$, then fundamentally we want to bound:

$$\left\|\sum_{n\leqslant N} (L_{\gamma_1} + L_{\gamma_2})^n \operatorname{tr}\rho\right\|_\infty .$$

It will help if we first note that

$$\sum_{n\leqslant N}(L_{\gamma_1}+L_{\gamma_2})^n = ((L_{\gamma_1}+L_{\gamma_2})^{N+1}-(L_{\gamma_1}+L_{\gamma_2}))(L_{\gamma_1}+L_{\gamma_2}-1)^{-1}.$$

So the operator norm of the sum is bounded above by $\|L_{\gamma_1}+L_{\gamma_2}-1\|^{-1}$. Note that if $(L_{\gamma_1}+L_{\gamma_2}-1)f=\lambda f$, then $(L_{\gamma_1}+L_{\gamma_2})f=(\lambda+1)f$.

3 Towards a legitimate proof

As is very familiar by now, let G be a compact semisimple group, $\Gamma=\langle\gamma_1,\gamma_2\rangle\subset G$ a dense free subgroup on two generators. We claim that for $f\in C^\infty(G^\natural)$,

$$\sum_{n\leqslant N}\sum_{\sigma\colon\{1,\dots,n\}\to\{1,2\}} f(\gamma_{\sigma(1)}\dots\gamma_{\sigma(n)}) = \sum_{n\leqslant N}(L_{\gamma_1^{-1}}+L_{\gamma_2^{-1}})^n f(1) \ll_f 1.$$

First, it definitely holds that

$$(L_{\gamma_1^{-1}}+L_{\gamma_2^{-1}}-1)\sum_{n\leqslant N}(L_{\gamma_1^{-1}}+L_{\gamma_2^{-1}})^n = ((L_{\gamma_1^{-1}}+L_{\gamma_2^{-1}})^{N+1}-(L_{\gamma_1^{-1}}+L_{\gamma_2^{-1}}))$$

Valid question: does $L_{\gamma_1^{-1}}+L_{\gamma_2^{-1}}-1$ have any zeros?

Suppose $f(\gamma_1^{-1}x)+f(\gamma_2^{-1}x)=f(x)$ for some smooth function f.