Counterexamples related to the Sato–Tate conjecture

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Outline

Motivation

Sato–Tate Conjecture

$E_{/\mathbf{Q}}$ non-CM elliptic curve, $a_p = \operatorname{tr} \rho_l(\mathrm{fr}_p) \in \mathbf{Z}$, $|a_p| \leqslant 2\sqrt{p}$.

Satake parameter: $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$.

Sato–Tate measure: $\mathrm{ST} = \frac{2}{\pi}\sin^2\theta\,\mathrm{d}\theta$ (Haar measure on $\mathrm{SU}(2)^\natural$).

Theorem (Taylor et. al.)

$\{\theta_p\}$ *is equidistributed with respect to* ST.

Quantify rate of convergence of $\frac{1}{\pi(N)}\sum_{p\leqslant N}\delta_{\theta_p}$ to ST.

Use discrepancy (Kolmogorov–Smirnov statistic).

Akiyama–Tanigawa Conjecture

$$D_N = \sup_{x \in [0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p \leqslant N} 1_{[0,x)}(\theta_p) - \int_0^x \mathrm{d}\,\mathsf{ST} \right|.$$

Conjecture (Akiyama–Tanigawa)

$D_N \ll N^{-\frac{1}{2}+\epsilon}$.

There is a variant of this conjecture for CM elliptic curve.

Theorem (Akiyama–Tanigawa)

Akiyama–Tanigawa conjecture $\Rightarrow$ *Riemann Hypothesis for* E.

Theorem (Mazur)

Akiyama–Tanigawa conjecture $\Rightarrow$ *Riemann Hypothesis for* $\mathrm{sym}^k E$

Context

Pande: is the Sato–Tate conjecture a Galois-theoretic result?

Theorem (Pande)

Let $\epsilon > 0$. Then there exists an infinitely ramified representation $\rho \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that $\theta_p \in B_\epsilon(\pi/2)$ for a density one set of primes.

Theorem (Khare–Rajan)

Any $\rho \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ is ramified at a density zero set of primes.

Question (Serre): can you *a priori* control the ratio $\frac{\pi_{\mathrm{ram}(\rho)}(x)}{\pi(x)}$?

Answer (Khare–Larsen–Ramakrishna): no!

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the ratio $\frac{\pi_{\mathrm{ram}(\rho)}(x)}{\pi(x)}$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Answer: Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Discrepancy

Definition

Let $\{\theta_p\}$ be a sequence in $[0,\pi]$, μ a measure on $[0,\pi]$. The *discrepancy* is

$$\mathrm{D}_N(\{\theta_p\},\mu) = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x)}(\theta_p) - \int_0^x \, \mathrm{d}\mu \right|.$$

$\{\theta_p\}$ are μ-equidistributed if and only if $D_N \to 0$.

Fact: $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

Dirichlet series

Definition

For $k \geqslant 1$,

$$L(\operatorname{sym}^k \rho, s) = \prod_p \det\left(1 - \operatorname{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix} p^{-s}\right)^{-1}$$

Definition

For $f\colon [0,\pi] \to \mathbf{C}$ of bounded variation with $\mu(f) = 0$,

$$L_f(s) = \prod_p \left(1 - f(\theta_p)p^{-s}\right)^{-1}$$

Example: $L_{\mathrm{sgn}}(s) = \prod_p (1 - \operatorname{sgn}(a_p)p^{-s})^{-1}$.

Dirichlet series—results

Theorem

If $\left|\sum_{p\leqslant N} f(\theta_p)\right| \ll N^{\alpha+\epsilon}$*, then* $L_f(s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$*.*

Corollary

If $D_N \ll N^{-\alpha+\epsilon}$*, then* $L_f(s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$*.*

Definition

$$U_k(\theta) = \frac{\sin((k+1)\theta)}{\sin\theta} = \operatorname{tr}\operatorname{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix}.$$

Theorem

If $\left|\sum_{p\leqslant N} U_k(\theta_p)\right| \ll N^{\alpha+\epsilon}$*, then* $L(\operatorname{sym}^k \rho, s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$*.*

Main theorem

Ingredients

1. Fix a rational prime $l \geqslant 7$.

2. Fix an odd, absolutely, weight 2 representation $\bar{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$. ρ will be a lift of $\bar{\rho}$.

3. Fix a function $h\colon \mathbf{R}^+ \to \mathbf{R}_{\geqslant 1}$ which increases slowly to infinity. We will have $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$.

4. Fix an absolutely continuous measure μ on $[0, \pi]$, with bounded probability density function. The angles $\{\theta_p\}$ will be μ-equidistributed.

5. Fix $\alpha \in \left(0, \frac{1}{2}\right)$. The discrepancy D_N will decay like $\pi(N)^{-\alpha}$.

Theorem

Let l, $\bar{\rho}$, h, μ, and α be as above. Then there exists $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that

1. $\rho \equiv \bar{\rho} \pmod{l}$.
2. $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$. (Yes to Q4.)
3. For each unramified p, $a_p = \operatorname{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.
4. $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(x)^{-\alpha})$. (Yes to Q1–Q3.)
5. If $(\theta \mapsto \pi - \theta)_* \mu = \mu$, then for each odd k, $L(\operatorname{sym}^k \rho, s)$ satisfies the Riemann hypothesis. (Yes to Q5.)

Idea of the proof

Prescribing discrepancy decay

Theorem

If $\alpha \in \left(0, \frac{1}{2}\right)$*, there exists a sequence* $(x_2, x_3, x_5, \dots)$ *in* $[-1,1]$ *such that* $|D_N - \pi(N)^{-\alpha}| \ll \pi(N)^{-1}$*.*

Fact: Discrepancy is invariant under pushforward by cos and $\cos^{-1}$.

Idea: Construct ρ so that $\frac{a_p}{2\sqrt{p}} \approx x_p$.

Fact: If $(x_p^{(1)})$ is a sequence with $|x_p - x_p^{(1)}| \ll p^{-1/2+\epsilon}$, then $\mathrm{D}_N^{(1)} = \Theta(\pi(N)^{-\alpha})$.

Lifting Galois representations

What is a deformation?

Weight?

What does it take to lift from $\mathbf{Z}/l^n$ to $\mathbf{Z}/l^{n+1}$?

Lifting Galois representations—first stage

Lift from $\mathbf{Z}/l$ to $\mathbf{Z}/l^2$.

Lifting Galois representations—inductive step

Lift from $\mathbf{Z}/l^n$ to $\mathbf{Z}/l^{n+1}$.

Consequences

If $f \in C([0,\pi])$, $f \circ \cos^{-1}\colon [-1,1] \to \mathbf{C}$ is Lipschitz, and $f(\pi-\theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

Questions?