Fourier analysis on abelian Lie groups

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Let G be an abelian Lie group, $\widehat{G} = \hom(G, S^1)$ its unitary dual. Given $x \in G, y \in \widehat{G}$, write $\langle x, y\rangle = y(x)$. For $f \in L^1(G)$, we have its *Fourier transform* $\widehat{f}\colon \widehat{G} \to \mathbf{C}$, defined by

$$\widehat{f}(x) = \int f(y)\langle y, x\rangle \,\mathrm{d}y.$$

Recall the *convolution* of two functions is given by

$$(f * g)(x) = \int f(xy^{-1})g(y)\,\mathrm{d}y.$$

We now compute:

$$\begin{aligned}
\widehat{f * g}(x) &= \iint f(yz^{-1})g(z)\,\mathrm{d}z\langle y, x\rangle\,\mathrm{d}y \\
&= \iint f(yz^{-1})g(z)\langle y, x\rangle\,\mathrm{d}y\mathrm{d}z \\
&= \iint f(y)g(z)\langle yz, x\rangle\,\mathrm{d}y\mathrm{d}z \\
&= \int f(y)\langle y, x\rangle\,\mathrm{d}y \int g(z)\langle z, x\rangle\,\mathrm{d}z \\
&= \widehat{f}(x)\widehat{g}(x).
\end{aligned}$$

Now let $\mathfrak{g} = \mathrm{Lie}(G)$. This Lie algebra acts on $C^\infty(G)$ by

$$(Xf)(x) = \frac{\mathrm{d}f}{\mathrm{d}t}(\exp(tX)x)\bigg|_{t=0}$$

hence the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ acts on $C^\infty(G)$ via the obvious extension. Since $\mathfrak{g}$ is abelian, $\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$.) There is a natural function $\mathrm{tr}\colon \mathfrak{g} \to C^\infty(\widehat{G})$, given by

$$(\mathrm{tr}\, X)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\langle \exp(tX), x\rangle\bigg|_{t=0}.$$

This extends to a ring homomorphism $\operatorname{tr}\colon \mathcal{S}(\mathfrak{g}) \to C^\infty(\widehat{G})$. Now, we have an elementary computation:

$$\begin{aligned}
\widehat{Xf}(x) &= \int \frac{\mathrm{d}f}{\mathrm{d}t}(\exp(tX)y)\Big|_{t=0} \langle y, x\rangle \,\mathrm{d}y \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int f(\exp(tX)y)\langle y, x\rangle \,\mathrm{d}y\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int f(y)\langle \exp(-tX)y, x\rangle \,\mathrm{d}y\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle \exp(-tX), x\rangle \int f(y)\langle y, x\rangle \,\mathrm{d}y\Big|_{t=0} \\
&= -\operatorname{tr} X \widehat{f}(x).
\end{aligned}$$

In other words, we have the following identities:

$$\begin{aligned}
\widehat{f * g} &= \widehat{f}\widehat{g} \\
\widehat{Xf} &= -\operatorname{tr} X \widehat{f}
\end{aligned}$$