

# Equidistribution and the analytic properties of a strange class of $L$ -functions

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## 1 Motivation

Let  $E/\mathbf{Q}$  be an elliptic curve without complex multiplication. By an old theorem of Faltings [Fal83], the quantities

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \mathrm{tr} \rho_{E,l}(\mathrm{fr}_p)$$

determine  $E$  up to isogeny. That is, if  $E_1$  and  $E_2$  satisfy  $a_p(E_1) = a_p(E_2)$  for all  $p$ , then  $E_1$  and  $E_2$  are isogenous. The starting point of this investigation is the corollary of a theorem of Harris, that the collection  $\{\mathrm{sgn} a_p(E)\}_p$  in fact determines  $E$  up to isogeny. Ramakrishna had the insight that this fact means the “strange  $L$ -function”

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \mathrm{sgn} a_p(E) p^{-s}}$$

determines  $E$  up to isogeny. In this note, I define a more general class of strange  $L$ -functions, and show that their analytic properties are closely tied to the equidistribution of the  $a_p(E)$ .

Here is a brief discussion of this generalization in the case of a non-CM curve  $E/\mathbf{Q}$ . It is convenient to repackage the traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the  $\theta_p(E)$  are well-defined angles laying in the interval  $[0, \pi]$ . Write  $\mathrm{dST} = \frac{2}{\pi} \sin^2 \theta \, \mathrm{d}\theta$ . Then the Sato–Tate conjecture (now a theorem [BL+11]) tells us that for any continuous function  $f: [0, \pi] \rightarrow \mathbf{C}$ , we have

$$\left| \frac{1}{\pi(C)} \sum_{p \leq C} f(\theta_p) - \int_0^\pi f \, \mathrm{dST} \right| = o(1)$$

as  $C \rightarrow \infty$ . It is well-known that this follows from the analytic continuation (past  $\Re s = 1$ ) and non-vanishing except at  $s = 1$  of all the  $L$ -functions

$L(\text{sym}^k E, s)$  [Ser68, A.1, Th.1]. We take as our starting point the much stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(C)} \sum_{p \leq C} f(\theta_p) - \int_0^\pi f \, d\mu_{\text{ST}} \right| = O_f(C^{-\frac{1}{2}+\epsilon})$$

for all continuous  $f$ . (Their conjecture is actually more general; we will discuss the precise statement later.) They prove that this conjecture implies the Riemann Hypothesis for  $E$ . I prove that not only does their conjecture imply the Riemann Hypothesis for all  $L(\text{sym}^k E, s)$ , it also does for all the strange  $L$ -functions

$$L_f(E, s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

These results make perfect sense in a much more general context, and I will prove them there. In section 2 I set up this context and carefully define strange  $L$ -functions. In section 3, I prove basic analytic properties of the strange  $L$ -functions and connect their analytic properties with the equidistribution of a sequence. In section 4, I apply these results where “everything is known,” i.e. varieties over function fields. Finally, in section 5, I apply the general results to the following cases: a non-CM elliptic curve  $E/\mathbf{Q}$ , the product  $E_1 \times E_2$  of a pair of non-isogenous non-CM elliptic curves over  $\mathbf{Q}$ , and the Jacobian of a generic genus-2 curve  $C/\mathbf{Q}$ .

## 2 Definitions

Let  $\mathbf{D} = \{z \in \mathbf{C} : |z| \leq 1\}$ . Write  $\mathbf{D}^\infty$  for the set of sequences in  $\mathbf{D}$  indexed by the primes, i.e.  $\mathbf{z} \in \mathbf{D}^\infty$  is  $(z_2, z_3, \dots)$ . The space  $\mathbf{D}^\infty$  is compact, and comes naturally equipped with the (product) Lebesgue measure, normalized to have mass 1.

**Definition 2.1.** Let  $\mathbf{z} \in \mathbf{D}^\infty$ . The associated *strange  $L$ -function* is given by

$$L(\mathbf{z}, s) = \prod_p \frac{1}{1 - z_p p^{-s}},$$

wherever this product converges.

Elementary topology tells us that  $L: \mathbf{D}^\infty \times \mathbf{C}^{\Re > 1} \rightarrow \mathbf{C}$  is continuous. We will see that for fixed  $\mathbf{z} \in \mathbf{D}^\infty$ , the analytic properties of  $L(\mathbf{z}, s)$  are closely tied to estimates for the sums  $A_{\mathbf{z}}(x) = \sum_{p \leq x} z_p$ . One often gets such estimates in the context of equidistribution, which we consider next.

For the remainder of this section, let  $X$  be a compact separable metric space with no isolated points. We write  $X^\infty$  for the space of sequences in  $X$  indexed by rational primes, i.e. points  $\mathbf{x} \in X^\infty$  are of the form  $\mathbf{x} = (x_2, x_3, \dots)$ . By [Eng89, Cor.2.3.16, Th.4.2.2], the compact space  $X^\infty$  is metrizable and separable, also with no isolated points.

**Definition 2.2.** For  $\mathbf{x} \in X^\infty$  and  $C > 0$ , write  $\mathbf{x}^C$  for the probability measure given by

$$\int_X f d\mathbf{x}^C = \mathbf{x}^C(f) = \frac{1}{\pi(C)} \sum_{p \leq C} f(x_p).$$

Let  $\mu$  be a Borel measure on  $X$ . Recall that  $\mathbf{x}$  is  $\mu$ -*equidistributed* if  $\mathbf{x}^C \rightarrow \mu$  weakly, i.e.  $\mathbf{x}^C(f) \rightarrow \mu(f)$  for all  $f \in C(X)$ . In fact, we can extend this to not-necessarily-continuous functions as follows:

**Theorem 2.3** (Mazzone). *Let  $\mu$  be a Borel measure on  $X$  and let  $f: X \rightarrow \mathbf{C}$  be bounded and measurable. Then  $f$  is continuous almost everywhere if and only if  $\mathbf{x}^C(f) \rightarrow \mu(f)$  for all  $\mu$ -equidistributed  $\mathbf{x}$ .*

*Proof.* This follows directly from the proof of [Maz95, Th.1].  $\square$

Fix a Borel measure  $\mu$  on  $X$ , and write  $C^{\text{ae}}(X, \mu)$  for the space of bounded, almost-everywhere continuous functions  $f: X \rightarrow \mathbf{C}$ .

**Theorem 2.4.** *Endowed with the supremum norm  $\|f\|_\infty = \sup_{x \in X} |f(x)|$ ,  $C^{\text{ae}}(X, \mu)$  is a Banach space.*

*Proof.* This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero.  $\square$

**Definition 2.5.** Let  $f \in C^{\text{ae}}(X, \mu)^{\|\cdot\|_\infty \leq 1}$ ,  $\mathbf{x} \in X^\infty$ . The associated *strange  $L$ -function* is defined as

$$L_f(\mathbf{x}, s) = L(f(\mathbf{x}), s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}$$

for all  $s \in \mathbf{C}$  for which the product converges.

Our typical source of a strange  $L$ -function is as follows. Let  $G$  be a compact connected Lie group and  $X = G^\natural$ , the space of conjugacy classes of  $G$ . Then  $G^\natural$  inherits the Haar measure from  $G$ . Given any sequence  $\mathbf{x} \in (G^\natural)^\infty = G^{\natural, \infty}$  and function  $f \in C^{\text{ae}}(G^\natural)^{\|\cdot\|_\infty \leq 1}$ , we can define  $L_f(\mathbf{x}, s)$ . This is related to Serre's  $L$ -functions from [Ser68, A.2] as follows.

**Theorem 2.6.** *Let  $G$  be a compact connected Lie group,  $\rho \in \widehat{G}$  an irreducible unitary representation of  $G$ . Then there exist functions  $\lambda_\rho^1, \dots, \lambda_\rho^{\deg \rho}: G^\natural \rightarrow S^1$ , continuous away from the set  $\{\det(1 - \rho) = 0\}$ , such that for every  $x \in G^\natural$ , there are angles  $\theta_1, \dots, \theta_{\deg \rho} \in [0, 2\pi)$ , satisfying  $\theta_1 \leq \dots \leq \theta_{\deg \rho}$ , such that  $\lambda_\rho^j(x) = e^{i\theta_j}$  and moreover*

$$\det(1 - \rho(x)t) = \prod_{j=0}^{\deg \rho} (1 - \lambda_\rho^j(x)t).$$

*Proof.* This follows easily from [KS99, Lem.1.0.9].  $\square$

Recall that for  $\rho \in \widehat{G}$ , Serre defines  $L(\rho, s) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}$ . Using his notation, there is the identity

$$L(\rho, s) = \prod_{j=1}^{\deg \rho} L_{\lambda_\rho^j}(\mathbf{x}, s).$$

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let  $G$  be a compact connected semisimple Lie group. We will define discrepancy for sequences in  $G^\natural$ .

Let  $G^{\text{sc}}$  be the simply-connected cover of  $G$ . Choose a maximal torus  $T \subset G^{\text{sc}}$ ; let  $W = N(T)/T$  be the Weyl group. Let  $\mathfrak{t} = \text{Lie}(T)$  and recall that the kernel of  $\exp: \mathfrak{t} \rightarrow T$  is generated by the nodal vectors associated to the root system  $R(G^{\text{sc}}, T)$  [Lie7-9, 9.6 Pr.11]. Write  $\{t_1, \dots, t_r\} \subset \mathfrak{t}$  for these vectors. The exponential map  $\exp: \mathfrak{t} \rightarrow T$  induces an isomorphism  $\mathfrak{t}/(\langle t_i \rangle \rtimes W) \rightarrow G^\natural$ . Given  $x = (x_1, \dots, x_r) \in [0, 1]^r$ , write

$$I_x = \left\{ \sum_{i=1}^r a_i t_i : a_i \in [0, x_i] \right\} \subset \mathfrak{t}.$$

**Definition 2.7.** With the setup as above, let  $\mu, \nu$  be probability measures on  $G^\natural$ . The *discrepancy* between  $\mu$  and  $\nu$  is

$$\text{disc}(\mu, \nu) = \sup_{x \in [0, 1]^r} |\mu(\exp I_x) - \nu(\exp I_x)|.$$

If  $\nu = dx$ , the Haar measure on  $G^\natural$ , we simply write  $\text{disc}(\mu)$  for  $\text{disc}(\mu, dx)$ .

The Koksma–Hlawka inequality bounds the difference between the Haar integral and weighted average of a function on  $G^\natural$  in terms of the discrepancy of the sequence and the variation of the function.

The following result is essential:

**Theorem 2.8** (Koksma, Hlawka). *Let  $G$  be as above. Let  $f: G^\natural \rightarrow \mathbf{C}$  be such that  $f dx$  is a measure with bounded variation. Then*

$$\left| \mathbf{x}^C(f) - \int f dx \right| \leq \text{Var}(f) \text{disc}(\mathbf{x}^C).$$

*Proof.* This is [Ökt99, Th. 3.2]. □

We will often use the soft version of this inequality. Namely, assume  $\int f dx = 0$ . Then  $|\mathbf{x}^C(f)| \ll_f \text{disc}(\mathbf{x}^C)$  as  $C \rightarrow \infty$ . Here is another way of putting it. The sequence  $f(\mathbf{x})$  has  $|A_{f(\mathbf{x})}(C)| \ll_f \pi(C) \text{disc}(\mathbf{x}^C)$ .

### 3 Main results

**Theorem 3.1.** *Let  $\mathbf{z} \in \mathbf{D}^\infty$ . Then  $L(\mathbf{z}, s)$  defines a holomorphic function on the region  $\{\Re s > 1\}$ . Moreover, on that region,*

$$\log L(\mathbf{z}, s) = \sum_{p^n} \frac{z_p^n}{np^{ns}}.$$

*Proof.* Expanding the product for  $L(\mathbf{z}, s)$  formally, we have

$$L(\mathbf{z}, s) = \sum_{n \geq 1} \frac{\prod_{p|n} z_p^{v_p(n)}}{n^s}.$$

An easy comparison with Riemann's zeta function tells us that the series expansion is holomorphic on  $\{\Re s > 1\}$ . By [Apo76, Th. 11.7], the product formula holds on the same region. The formula for  $\log L(\mathbf{z}, s)$  comes from [Apo76, 11.9 Ex.2].  $\square$

**Theorem 3.2.** *Assume  $A_{\mathbf{z}}(x) \ll x^{\alpha+\epsilon}$ ,  $\alpha \in [\frac{1}{2}, 1]$ . Then  $\log L(\mathbf{z}, s)$  is holomorphic on  $\{\Re > \alpha\}$ .*

*Proof.* Split the sum for  $\log L$  into two pieces:

$$\log L(\mathbf{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geq 2} \frac{z_p^n}{np^{ns}}.$$

For each  $p$ , we have

$$\left| \sum_{n \geq 2} \frac{z_p^n}{np^{ns}} \right| \leq \sum_{n \geq 2} p^{-n\Re s} = p^{-2\Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leq \frac{1}{1 - p^{-\Re s}} \leq 2 + 2\sqrt{2},$$

so the second piece of  $\log L(\mathbf{z}, s)$  converges absolutely when  $\Re(s) > \frac{1}{2}$ . By [Ten95, II.1 Th.10], our bound on  $A_{\mathbf{z}}(x)$  yields the holomorphy of  $\sum z_p p^{-s}$  on  $\{\Re > \alpha\}$ .  $\square$

**Corollary 3.3.** *Let  $G$  be a compact connected semisimple Lie group,  $\mathbf{x} \in G^{\natural, \infty}$  satisfy  $\text{disc}(\mathbf{x}^C, dx) \ll C^{-\frac{1}{2}+\epsilon}$ . Then for every  $f \in C^{\text{ae}}(G^{\natural})^{\|\cdot\| \leq 1}$ ,  $L_f(\mathbf{x}, s)$  has analytic continuation to  $\{\Re s > \frac{1}{2}\}$ , and satisfies the Riemann Hypothesis, for all  $f$  bounded and almost-everywhere continuous with  $\mu(f) = 0$ .*

*Proof.* Koksma–Hlawka tells that if  $\mu(f) = 0$ , then  $\mathbf{x}^C(f) \ll C^{-\frac{1}{2}+\epsilon}$ . Thus the sequence  $f(\mathbf{x})$  satisfies  $A_{f(\mathbf{x})}(x) \ll x^{\frac{1}{2}+\epsilon}$ , and the result follows from Theorem 3.2.  $\square$

## 4 Strange $L$ -functions over function fields

Let  $k$  be a finite field of characteristic  $p$  and cardinality  $q$ . Let  $C/k$  be a nice curve in the sense of Poonen (i.e.,  $C$  is smooth, projective, and geometrically integral). Write  $K = k(C)$  for the function field of  $C$ . Fix a non-empty open subset  $U \subset C$  and a geometric point  $\infty \in U(\bar{k})$ . Fix a prime  $l \neq p$  and an embedding  $\overline{\mathbf{Q}}_l \hookrightarrow \mathbf{C}$ .

**Definition 4.1.** An  $l$ -adic sheaf  $\mathcal{F}$  on  $U$  is *good* if the following conditions hold.

1.  $\mathcal{F}$  is pure of weight zero.
2. Let  $G = \overline{\rho_{\mathcal{F}}(\pi_1(U_{\bar{k}}, \infty))}^{\text{Zar}}$ . Assume  $\rho_{\mathcal{F}}(\pi_1(U, \infty)) \subset G(\overline{\mathbf{Q}}_l)$ .

For any good sheaf  $\mathcal{F}$ , let  $\text{ST}(\mathcal{F})$  be a maximal compact subgroup of  $G(\mathbf{C})$ . For each  $u \in U$ , there is a well-defined conjugacy class  $\theta(u) = \rho(\text{fr}_u)^{\text{ss}} \in \text{ST}(\mathcal{F})^{\natural}$ . For any  $C > 0$ , write

$$\theta_{\mathcal{F}}^C = \frac{1}{\#\{u \in U : q_u \leq C\}} \sum_{q_u \leq C} \delta_{\theta(u)}.$$

Katz proves an equidistribution estimate for the  $\theta(u)$ 's.

**Theorem 4.2.** *Let  $\sigma$  be a non-trivial irreducible representation of  $\text{ST}(\mathcal{F})$ . Then*

$$|\theta_{\mathcal{F}}^C(\text{tr } \sigma)| \ll_{\mathcal{F}} \dim(\sigma) C^{-\frac{1}{2}}.$$

*Proof.* This is [Kat88, p.39]. □

Now let  $C^{\natural}(\text{ST}(\mathcal{F}))$  be the space of functions  $f: \text{ST}(\mathcal{F})^{\natural} \rightarrow \mathbf{C}$  satisfying:

$$\|f\|^{\natural} = \sum_{\sigma} \dim(\sigma) |\hat{f}(\sigma)| < \infty.$$

For such functions, we have:

$$|\theta_{\mathcal{F}}^C(f) - \mu(f)| \ll_{\mathcal{F}} \|f\|^{\natural} C^{-\frac{1}{2}}.$$

Thus for any  $f \in C^{\natural}(\text{ST}(\mathcal{F}))$ , the strange  $L$ -function  $L_f(\theta_{\mathcal{F}}, s)$  has analytic continuation to  $\{\Re s > \frac{1}{2}\}$  and satisfies the Riemann Hypothesis.

**Theorem 4.3.** *Let  $\mathbf{z} \in \mathbf{D}^{\infty}$ , and assume  $\log L(\mathbf{z}, s)$  has analytic continuation to  $\{\Re s > \alpha\}$ ,  $\alpha \in [\frac{1}{2}, 1]$ , and that for  $\sigma > \alpha$ , we have  $|\log L(\mathbf{z}, \sigma + it)| \ll |t|^{1-\epsilon}$ . Then  $|A_{\mathbf{z}}(x)| \ll x^{\alpha+\epsilon}$ .*

*Proof.* Recall that we can write

$$\log L(\mathbf{z}, p) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geq 2} \frac{z_p^n}{np^{ns}} = \sum_p \frac{z_p}{p^s} + O(\zeta(2\Re s)).$$

Thus, for any  $\epsilon > 0$ , our bound on  $|\log L(\mathbf{z}, \sigma + it)|$  implies the same bound for  $\sum \frac{z_p}{p^s}$  on  $\{\Re s > \alpha + \epsilon\}$ .

Let  $\gamma_T = \gamma_{1,T} + \gamma_{2,T} - \gamma_{3,T} - \gamma_{4,T}$  be the following contour:

$$\begin{aligned} \gamma_{1,T}(t) &= (\alpha + \epsilon) + it & t \in [-T, T] \\ \gamma_{2,T}(t) &= t + iT & t \in [\alpha + \epsilon, 1 + \epsilon] \\ \gamma_{3,T}(t) &= (1 + \epsilon) + it & t \in [-T, T] \\ \gamma_{4,T}(t) &= t - iT & t \in [\alpha + \epsilon, 1 + \epsilon]. \end{aligned}$$

By [Apo76, Th.11.18],

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{-\gamma_{3,T}} \sum_p \frac{z_p}{p^s} x^z \frac{dz}{z} =^* \sum_{p \leq x} z_p.$$

Let  $h(z)$  be the analytic continuation of  $\sum z_p p^{-s}$  to  $\{\Re > \alpha\}$ . Since  $\int_{\gamma} h(z) \frac{dz}{z} = 0$ , we obtain

$$\left| \sum_{p \leq z} z_p \right| \ll \left| \int_{\gamma_{T,1}} h(z) x^z \frac{dz}{z} \right| + \left| \int_{\gamma_{T,2}} h(z) x^z \frac{dz}{z} \right| + \left| \int_{\gamma_{T,4}} h(z) x^z \frac{dz}{z} \right|.$$

We know that  $|h(\sigma + it)| \ll |t|$ , so we can bound:

$$\left| \int_{\gamma_{T,2}} h(z) \frac{dz}{z} \right| = \left| \int_{\alpha+\epsilon}^{1+\epsilon} \frac{h(t+iT)x^{t+iT}}{t+iT} dt \right| \ll (1+\alpha)x^{1+\alpha}T^{-1},$$

and similarly for  $\int_{\gamma_{4,T}}$ . Finally, we note that

$$\left| \int_{\gamma_{T,1}} h(z) x^z \frac{dz}{z} \right| \ll \int_{-T}^T |t|^{1-\epsilon} \frac{x^{\alpha+\epsilon}}{(\alpha+\epsilon)^2 + t^2} dt \ll x^{\alpha+\epsilon}.$$

Letting  $T \rightarrow \infty$  we obtain the desired result.  $\square$

## 5 Applications

Recall, following [Bug08] that the *irrationality exponent*  $\mu(\alpha)$  a real irrational number  $\alpha$  is the supremum of all real numbers  $\mu$  such that

$$\left| \alpha - \frac{p}{q} \right| < q^{-\mu}$$

for infinitely many  $p/q \in \mathbf{Q}$ . Bugeaud proves that for any  $\mu \geq 2$ , there is an element  $\xi_\mu$  of the Cantor set with  $\mu(\xi_\mu) = \mu$ . Moreover, by [KN74, ?], for every  $\epsilon > 0$ , the sequence  $x_n = n\alpha \bmod 1$  has discrepancy  $\text{disc}(\mathbf{x}^C) = \Omega(C^{-\frac{1}{\mu(\alpha)-1}-\epsilon})$ .

**Theorem 5.1.** *Let  $X = S^1$  with the natural Haar measure. For every  $\eta \in (0, \frac{1}{2})$ , there is a sequence  $\mathbf{x} = (x_2, x_3, \dots) \in (S^1)^\infty$  such that for all  $f \in C^\infty(S^1)^{\|\cdot\|_\infty \leq 1}$ , the function  $\log L_f(\mathbf{x}, s)$  has analytic continuation to  $\{\Re > \frac{1}{2}\}$ , but for all  $\epsilon > 0$ ,  $|\text{disc}(\mathbf{x}^C)| = \Omega(C^{-\eta-\epsilon})$ .*

*Proof.* Let  $\mu > 3$ , and let  $\mathbf{x} = \{x_2, x_3, \dots\}$  be the sequence  $x_{p_n} = e^{2\pi i n \xi_\mu}$ . To prove that  $\log L_f(\mathbf{x}, s)$  has analytic continuation to  $\{\Re > \frac{1}{2}\}$ , we need only to prove that  $|A_{\exp(2\pi i m \mathbf{x})}(t)| \ll t^{1/2}$ , uniformly for each  $m \in \mathbf{Z}$ . This follows easily from:

$$\left| \sum_{n=1}^N e^{2\pi i m n \alpha} \right| \leq \frac{|-1 + e^{2\pi i M n \alpha}|}{|-1 + e^{2\pi i a m}|} \leq ? \leq \frac{1}{2} m(\eta - 1) \ll_\eta m$$

□

**Theorem 5.2.** *Let  $E/\mathbf{Q}$  be a non-CM elliptic curve, and put  $\theta = \theta(E)$ . Assume that  $\text{disc}(\theta^C) \ll C^{-\frac{1}{2}+\epsilon}$ . Then if  $f \in C^{\text{ae}}([0, \pi], \text{ST})^{\|\cdot\|_\infty \leq 1}$ , the strange  $L$ -function  $L_f(\theta, s)$  has analytic continuation to  $\{\Re > \frac{1}{2}\}$  and satisfy the Riemann Hypothesis. In particular, this holds for all  $L(\text{sym}^k E, s)$ .*

*Proof.* The first conclusion follows from [Corollary 3.3](#). The second part follows from the fact that any  $L(\text{sym}^k E, s)$  can be written as a product of  $L_f$ 's, namely the  $L_{\lambda_{\text{sym}^k}^j}$ 's in [section 2](#). □

**Theorem 5.3.** *Fix  $f \in C^{\text{ae}}([0, \pi], \text{ST})^{\|\cdot\|_\infty \leq 1}$  that is not almost everywhere constant.*

*Let  $E_1, E_2$  be two non-isogenous, non-CM elliptic curves over  $\mathbf{Q}$ . Assume the Akiyama–Tanigawa conjecture for the product  $E_1 \times E_2$ . Then for any  $f: [0, \pi] \rightarrow \mathbf{C}$  that is not almost everywhere*

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**Definition 5.4.** Let  $x \in \mathbf{R}^r$ . Following [\[Bug08\]](#), we say that the *irrationality exponent* of  $x$  is the supremum of all real numbers  $\mu$  such that

**Theorem 5.5.** *For  $x \in \mathbf{R}^d$ , write  $d(x, \mathbf{Z}) = \min\{|x - n| : n \in \mathbf{Z}\}$ . For  $x \in \mathbf{R}$ , we have the error bound:*

$$\left| \sum_{n \leq N} e^{2\pi i \langle x, nm \rangle} \right| \leq \frac{1}{d(\langle x, m \rangle, \mathbf{Z})}.$$

*Proof.* Note that  $d(x, \mathbf{Z}) \leq |\sin(\pi x)|$ . This, combined with the duplication formula  $\cos(2x) = 1 - 2\sin^2(x)$  gives us

$$\left| \sum_{n \leq N} e^{2\pi i \langle x, nm \rangle} \right| \leq \frac{2}{|e^{2\pi i \langle x, m \rangle} - 1|} = \frac{1}{|\sin(\pi \langle x, m \rangle)|} \leq \frac{1}{d(\langle x, m \rangle, \mathbf{Z})}.$$

□

## References

- [AT99] Shigeki Akiyama and Yoshio Tanigawa. “Calculation of values of  $L$ -functions associated to elliptic curves”. In: *Math. Comp.* 68.227 (1999), pp. 1201–1231.
- [Apo76] Tom M. Apostol. *Introduction to analytic number theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.



- [BL+11] Tom Barnet-Lamb et al. “A family of Calabi-Yau varieties and potential automorphy II”. In: *Publ. Res. Inst. Math. Sci.* 47.1 (2011), pp. 29–98.
- [Lie7–9] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics. Translated from the 1975 and 1982 French originals by Andrew Pressley. Berlin: Springer-Verlag, 2005.
- [Bug08] Yann Bugeaud. “Diophantine approximation and Cantor sets”. In: *Math. Ann.* 341.3 (2008), pp. 677–684.
- [Eng89] Ryszard Engelking. *General topology*. Second. Vol. 6. Sigma Series in Pure Mathematics. Translated from the Polish by the author. Heldermann Verlag, Berlin, 1989.
- [Fal83] G. Faltings. “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern”. In: *Invent. Math.* 73.3 (1983), pp. 349–366.
- [Kat88] Nicholas M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups*. Vol. 116. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1988.
- [KS99] Nicholas M. Katz and Peter Sarnak. *Random matrices, Frobenius eigenvalues, and monodromy*. Vol. 45. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1999.
- [KN74] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], 1974.
- [Maz95] Fernando Mazzone. “A characterization of almost everywhere continuous functions”. In: *Real Anal. Exchange* 21.1 (1995/96).
- [Ökt99] G. Ökten. “Error reduction techniques in quasi-Monte Carlo integration”. In: *Math. Comput. Modelling* 30.7-8 (1999), pp. 61–69.
- [Ser68] Jean-Pierre Serre. *Abelian  $l$ -adic representations and elliptic curves*. McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute. W. A. Benjamin, Inc., New York-Amsterdam, 1968.
- [Ten95] Gérald Tenenbaum. *Introduction to analytic and probabilistic number theory*. Vol. 46. Cambridge Studies in Advanced Mathematics. Translated from the second French edition (1995) by C. B. Thomas. Cambridge University Press, Cambridge, 1995.