Conceptual approach to Fontaine's period rings

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1 Brief overview

The ideas here are inspired by [Fon94] and [Sch12]. Fix a ring Λ, an ideal $\mathfrak{p} \subset \Lambda$, and a Λ-algebra V. The category $\mathcal{T}^{\leqslant m}(\mathfrak{p}) = \mathcal{T}_\Lambda^{\leqslant m}(V, \mathfrak{p})$ of $\mathfrak{p}$-*adic pro-infinitesimal* Λ-*thickenings of* V *of order* $\leqslant m$ consists of pairs (D, θ), where D is a $\mathfrak{p}$-adically complete Λ-algebra, $\theta : D \to V$ is a surjection, and moreover $(\ker \theta)^{m+1} = 0$. The category $\mathcal{T}^\infty(\mathfrak{p})$ of $\mathfrak{p}$-*adic pro-infinitesimal* Λ-*thickenings* of V consists of similar pairs, except that now we require D to be separated and complete with respect to $I_D = \ker(\theta)$.

Theorem 1. *Let $\mathfrak{p} = (p)$, and suppose V is p-adically complete and $\mathrm{fr} : V/p \to V/p$ is surjective. Then $\mathcal{T}^\infty(p)$ has an initial object $\mathbf{A}_{\mathrm{inf}}(V/\Lambda)$.*

Proof. One constructs $\mathbf{A}_{\mathrm{inf}}$ directly. Start by putting $V^\flat = \varprojlim_{\mathrm{fr}} V/p$. Our assumptions on V make the map $(-)^\sharp : V^\flat \to V$ given by

$$x^\sharp = \lim_{n\to\infty} \widetilde{x_n}^{p^n}$$

a well-defined multiplicative map. It induces a ring map $\theta : W(V^\flat) \to V$ by $[x] \mapsto x^\sharp$. The ring $\mathbf{A}_{\mathrm{inf}}(V/\Lambda)$ is the completion of $W(V^\flat)_\Lambda$ with respect to $\ker(\theta : W(V^\flat)_\Lambda \to V)$. The basic idea is: suppose we have $\theta : D \twoheadrightarrow V$. Define $(-)^\sharp : V^\flat \to D$ just as above; this is multiplicative, and induces a unique $\theta : W(V^\flat)_\Lambda \to D$, which in turn factors uniquely through its completion $\mathbf{A}_{\mathrm{inf}}(V/\Lambda)$. $\square$

The ring $\mathbf{B}_{\mathrm{dR}}^+(V/\Lambda)$ is the completion of $\mathbf{A}_{\mathrm{inf}}(V/\Lambda)[\frac{1}{p}]$ with respect to $\ker(\theta)$.

$$\log[\cdot] : \mathrm{U}^\times \to \mathbf{B}_{\mathrm{dR}}$$

The category pf consists of $\mathbf{F}_p$-algebras A such that $\mathrm{fr} : R \to R$ is surjective. The category Pf consists of p-adically separated and complete rings A such that $A/p \in \mathsf{pf}$. There are functors $- \otimes \mathbf{F}_p : \mathsf{Pf} \to \mathsf{pf}$ and $W(-) : \mathsf{pf} \to \mathsf{Pf}$. Note that $(W, \otimes \mathbf{F}_p)$ is an adjoint pair.

$$\mathsf{Pf} \xrightarrow{W} \mathsf{pf}$$

If $\mathbf{B}_{\mathrm{cris}}(V/\Lambda)$ is a divided power envelope, then its universal property should give a map $\mathbf{B}_{\mathrm{cris}} \to \mathbf{B}_{\mathrm{dR}}$. The trickier ring is $\mathbf{B}_{\mathrm{st}}$. Also note that

$$\mathrm{gr}^\bullet \mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{HT}}.$$

2 Some categories and functors

Let A be a reduced $\mathbf{F}_p$-algebra. Then $\mathrm{fr} : A \to A$ is injective, so $\varprojlim_{\mathrm{fr}} A = \bigcap \mathrm{fr}^n(A)$. This motivates our general definition of $\mathrm{fr}^\infty(A) = \varprojlim_{\mathrm{fr}} A$. The ring $\mathrm{fr}^\infty(A)$ is an "$\mathbf{F}_p$-algebra with splitting." That is, it comes with a canonical section $(-)^{1/p} : (a_0, a_1, \dots) \mapsto (a_1, \dots)$ of Frobenius. Let pf denote the category of such algebras. That is, an object of pf is a pair $(A, (-)^{1/p})$, where A is an $\mathbf{F}_p$-algebra and $(-)^{1/p} : A \to A$ is a section of $\mathrm{fr} : A \to A$. So fr^∞ is a functor $\mathsf{Alg}(\mathbf{F}_p) \to \mathsf{pf}$. In fact, one can easily check that

$$\hom_{\mathsf{Alg}(\mathbf{F}_p)}(A, B) = \hom_{\mathsf{pf}}((A, (-)^{1/p}), \mathrm{fr}^\infty B),$$

i.e. fr^∞ is right-adjoint to the forgetful functor $\mathsf{pf} \to \mathsf{Alg}(\mathbf{F}_p)$.

Let $\mathsf{Alg}(p)$ be the category of p-adically complete algebras, and let Pf be the full subcategory mapping onto $\mathsf{pf} \subset \mathsf{Alg}(\mathbf{F}_p)$. There is the obvious functor $\otimes \mathbf{F}_p : \mathsf{Pf} \to \mathsf{pf}$. Moreover, there is a functor "take Witt vectors" $W : \mathsf{pf} \to \mathsf{Pf}$ that satisfies

$$\hom_{\mathsf{Pf}}(W(A), B) = \hom_{\mathsf{pf}}(A, B/p).$$

3 A bestiary of period rings

Call a *quasi-perfectoid ring* a commutative p-adic Banach algebra A such that $\mathrm{fr} : A^\circ/p \to A^\circ/p$ is surjective. For the remainder, let A be a quasi-perfectoid ring. We agree that $\mathbf{A}_*$ will take values in $\mathbf{Z}_p$-algebras, while $\mathbf{B}_*$ will take values in $\mathbf{Q}_p$-algebras. In fact, when both are defined, $\mathbf{B}_* = \mathbf{A}_*[\frac{1}{p}]$. For simplicity, we assume p is odd.

3.1 $\mathbf{A}_{\mathrm{inf}}$

The ring $\mathbf{A}_{\mathrm{inf}}(A)$ is the "universal p-adic pro-infinitesimal formal thickening of A°." That is, it has a surjective ring map $\theta : \mathbf{A}_{\mathrm{inf}}(A) \to A^\circ$ for which $\mathbf{A}_{\mathrm{inf}}(A)$ is complete with respect to $\ker(\theta)$. Explicitly,

$$\mathbf{A}_{\mathrm{inf}}(A) = W(\mathrm{fr}^\infty A^\circ/p),$$

and $\theta([a]) = a^\sharp$. Note that $\mathbf{A}_{\mathrm{inf}}(A)$ has a natural filtration in which $\mathrm{fil}^r \mathbf{A}_{\mathrm{inf}} = (\ker\theta)^r$.

3.2 $\mathbf{B}_{\mathrm{inf}}$

This is just $\mathbf{B}_{\mathrm{inf}}(A) = \mathbf{A}_{\mathrm{inf}}(A)[\frac{1}{p}]$. Note that $\theta : \mathbf{A}_{\mathrm{inf}}(A) \to A^\circ$ extends uniquely to $\theta : \mathbf{B}_{\mathrm{inf}}(A) \to A$, so $\mathbf{B}_{\mathrm{inf}}$ inherits the filtration from $\mathbf{A}_{\mathrm{inf}}$.

3.3 $\mathrm{U}^\times$

We define two subgroups of $\mathrm{fr}^\infty(A^\circ/p)$:

$$\mathrm{U}^1(A) = \{x \in \mathrm{fr}^\infty(A^\circ/p) : x^\sharp \equiv 1 \pmod p\}$$
$$\mathrm{U}^\times(A) = \{x \in \mathrm{fr}^\infty(A^\circ/p) : |x^\sharp - 1| < 1\}.$$

Clearly $\mathrm{U}^1 \subset \mathrm{U}^\times$. Note that a better-motivated definition would be $\mathrm{U}^1(A) = \{a \in A : |a-1| < \frac{1}{p}\}$. The logarithm function

$$\log(a) = \sum_{n \geqslant 1} (-1)^{n+1} \frac{(a-1)^n}{n}$$

is easily checked to converge, so it gives a continuous homomorphism (with this definition of U^1) $\mathrm{U}^1(A) \to A$.

3.4 $\mathbf{B}_{\mathrm{dR}}$

First we define $\mathbf{B}_{\mathrm{dR}}^+(A)$ to be the completion of $\mathbf{B}_{\mathrm{inf}}(A)$ with respect to $\ker(\theta)$. There is a continuous homomorphism $\log[\cdot] : \mathrm{U}^\times \to \mathbf{B}_{\mathrm{dR}}^+$. The ring $\mathbf{B}_{\mathrm{dR}}$ should be an appropriate localization of $\mathbf{B}_{\mathrm{dR}}^+$.

3.5 $\mathbf{B}_{\mathrm{HT}}$

We set $\mathbf{B}_{\mathrm{HT}}^{+} = \mathrm{gr}^{\geqslant 0}\,\mathbf{B}_{\mathrm{dR}}$, and $\mathbf{B}_{\mathrm{HT}} = \mathrm{gr}^{\bullet}\,\mathbf{B}_{\mathrm{dR}}$. Hopefully, $\mathbf{B}_{\mathrm{HT}}(A) = \bigoplus_{n\in\mathbf{Z}} \mathrm{U}^{\times}(A)^{\otimes n}$. But this only works if $\dim_{\mathbf{Q}_p} \mathrm{U}^{\times}(A) = 1$. Indeed, we needed that to define $\langle\xi\rangle = \mathrm{gr}^1\,\mathbf{B}_{\mathrm{dR}}$ anyways.

3.6 $\mathbf{B}_{\mathrm{cris}}$

Let $\mathbf{A}_{\mathrm{cris}}(A)$ be the universal p-adically complete formal divided-power thickening of A°, and $\mathbf{B}_{\mathrm{cris}}(A) = \mathbf{A}_{\mathrm{cris}}(A)[\frac{1}{p}]$. By definition, there is a map $\theta : \mathbf{A}_{\mathrm{cris}}(A) \to A^{\circ}$ inducing $\theta : \mathbf{B}_{\mathrm{cris}}(A) \to A$. Moreover, we get a natural map (injective if A is a field) $\mathbf{B}_{\mathrm{cris}}(A) \to \mathbf{B}_{\mathrm{dR}}(A)$. Moreover, the map $\log[\cdot] : \mathrm{U}^{\times} \to \mathbf{B}_{\mathrm{dR}}$ factors through $\log[\cdot] : \mathrm{U}^{\times} \to \mathbf{B}_{\mathrm{cris}}$.

3.7 $\mathbf{B}_{\mathrm{st}}$

Let $\mathbf{B}_{\mathrm{st}}^{+}(A)$ be the initial object among $\mathbf{B}_{\mathrm{cris}}^{+}(A)$-algebras S together with $\lambda : \mathrm{frac}(\mathrm{fr}^{\infty} A^{\circ}/p)^{\times} \to S$ extending $\log[\cdot] : \mathrm{U}^{\times}(A) \to S$. One has

$$\mathbf{B}_{\mathrm{st}}^{+} = \mathrm{Sym}^{\bullet}(\mathrm{frac}(\mathrm{fr}^{\infty} A^{\circ}/p)^{\times}) \otimes_{\mathrm{Sym}^{\bullet}((\mathrm{fr}^{\infty} A^{\circ}/p)^{\times})} \mathbf{B}_{\mathrm{cris}}^{+}(A).$$

Again, if A is a field, then $\mathrm{frac}(\mathrm{fr}^{\infty} A^{\circ}/p)^{\times}/\mathrm{fr}^{\infty}(A^{\circ}/p)^{\times} = A^{\flat\times}/A^{\flat\circ\times}$ is a one-dimensional $\mathbf{Q}_p$-vector space, so we have a (non-canonical) isomorphism $\mathbf{B}_{\mathrm{st}}^{+} = \mathbf{B}_{\mathrm{cris}}[X]$. Finally, $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{st}}^{+} \otimes_{\mathbf{B}_{\mathrm{cris}}^{+}} \mathbf{B}_{\mathrm{cris}}$.

References

[Fon94] Jean-Marc Fontaine. Le corps des périodes p-adiques. *Astérisque*, (223):59–111, 1994.

[Sch12] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.