Compactly supported cohomology of discrete groups

Daniel Miller

August 7, 2016

1 Motivation

Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup. We are interested in the cohomology $\mathrm{H}^\bullet(\Gamma, \operatorname{sym}^k \mathbf{C})$. These groups are isomorphic to $\mathrm{H}^\bullet(\Gamma\backslash\mathfrak{H}, \widetilde{\operatorname{sym}^k \mathbf{C}})$, where $\mathfrak{H}$ is the upper half plane and $\widetilde{\operatorname{sym}^k \mathbf{C}}$ is the local system on $\Gamma\backslash\mathfrak{H}$ with monodromy $\operatorname{sym}^k \mathbf{C}$. Once we've introduced the symmetric spaces $S_\Gamma = \Gamma\backslash\mathfrak{H}$, it seems natural to also consider their cohomology with compact supports: $\mathrm{H}^\bullet_{\mathrm{c}}(S_\Gamma, \widetilde{V})$, where $V = \operatorname{sym}^k \mathbf{C}$.

More generally, let $G_{/\mathbf{Q}}$ be a split semisimple group, $K \subset G(\mathbf{R})$ a maximal compact subgroup, and $X = G(\mathbf{R})/Z(\mathbf{R})K$ the associated symmetric space. For $\Gamma \subset G(\mathbf{Q})$ a congruence subgroup, we have the quotient $S_\Gamma = \Gamma\backslash X$. If V is a representation of G, there is an induced local system $\widetilde{V}$ on S_Γ, and once again $\mathrm{H}^\bullet(\Gamma, V) = \mathrm{H}^\bullet(S_\Gamma, \widetilde{V})$. Once again, it is natural to consider $\mathrm{H}^\bullet_{\mathrm{c}}(S_\Gamma, \widetilde{V})$.

Of course, we can work in the greatest possible generality. Suppose Γ is an arbitrary (discrete) group. Let X be a contractible space on which Γ acts properly discontinuously. There is a natural (exact) functor $\widetilde{\cdot}\colon \mathrm{Mod}_{\mathbf{C}}(\Gamma) \to \mathrm{Sh}(\Gamma\backslash X)$, $V \mapsto \mathbf{C}_{\Gamma\backslash X} \otimes V$. And we have the functor "sections with compact support" $\Gamma_{\mathrm{c}}\colon \mathrm{Sh}(\Gamma\backslash X) \to \mathrm{Ab}$. This gives us two functors at the level of derived categories: $\mathrm{D}(\mathrm{Mod}(\Gamma)) \to \mathrm{D}(\mathrm{Ab})$, namely

$$V \mapsto \mathrm{R}\Gamma(\widetilde{V})$$
$$V \mapsto \mathrm{R}\Gamma_{\mathrm{c}}(\widetilde{V}).$$

First, it is not at all clear whether $\mathrm{H}^\bullet_{\mathrm{c}}(\Gamma\backslash X, \widetilde{V})$ is independent of X.

2 An example

Let F be a number field, $G = \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}$. Then $G(\mathbf{R}) = \prod_{v|\infty} F_v^\times$. If $\mathrm{N}\colon F_\infty \to \mathbf{R}$ is the norm map, then (up to finite index), a maximal compact subgroup $K \subset G(\mathbf{R})$ is given by the $\mathbf{R}$-points of the anisotropic group $G^{\mathrm{N}=1}$. The quotient $G_\infty/\mathbf{R}$ is topologically a finite disjoint union of Euclidean spaces. Let $\Gamma \subset G(F) = F^\times$ be a congruence subgroup—that is Γ is commensurable with $O_F^\times$. What is the quotient $\Gamma\backslash G_\infty/K$?

For example, if $F = \mathbf{Q}(\sqrt{d})$ is a real quadratic field, we want a unit $\varepsilon \in O_F^\times$ of infinite order. We are then interested in $\varepsilon^{\mathbf{Z}} \backslash F_\infty^\times$.

...

Let F be a number field, $\Gamma \subset F^\times$ a torsion-free group commensurable with $O_F^\times$. We know that $\Gamma \simeq \mathbf{Z}^{r+s-1}$. Does $\Gamma \backslash F_\infty^\times / K$ have a natural volume?

More generally, let $K_{\mathrm{f}} \subset \mathbf{A}_{F,\mathrm{f}}^\times$ be open compact. Put

$$Y_{K_{\mathrm{f}}} = F^\times \backslash \mathbf{A}_F^\times / K_\infty^\circ K_{\mathrm{f}}.$$

If K_{f} is sufficiently small (i.e., torsion-free) then $Y_{K_{\mathrm{f}}}$ is naturally a Riemannian manifold.

Start with the stupidest example, $F = \mathbf{Q}$. Then $K_\infty^\circ = 1$, so we are interested in $\mathbf{Q}^\times \backslash \mathbf{A}^\times / K_{\mathrm{f}}$. Suppose $K_{\mathrm{f}} = \Gamma(n) = \ker(\widehat{\mathbf{Z}}^\times \to (\mathbf{Z}/n)^\times)$... this won't have finite volume.

3 Tamagawa numbers of tori

Let $T_{/F}$ be a torus. Takashi Ono has found a formula for the Tamagawa number of T, i.e. the volume $T(F) \backslash T(\mathbf{A}_F)$. Put

$$\begin{aligned} h(T) &= \#\,\mathrm{H}^1(F, T^\vee) \\ i(T) &= \#Ш^1(T). \end{aligned}$$

Then $\mathrm{vol}(T(F) \backslash T(\mathbf{A}_F)) = \tau(T) = h(T)/i(T)$. [See Milne, ADT. Here $T^\vee$ is the dual torus in the sense of Langlands.]

[This isn't right, because $T(F) \backslash T(\mathbf{A}_F)$ shouldn't have finite volume! Never mind actually, it should have finite volume if and only if $T(F) \backslash T(\mathbf{A}_F)/K$ does.]

First, let's review some stuff about tori, their character groups, and Galois representations. Let L/K be a finite (possibly non-Galois) extension, $G = \mathrm{R}_{L/K} T$. Then $T^\vee$ is a Gal_L-module, and $(\mathrm{R}_{L/K} T)^\vee = \mathrm{ind}_L^K T^\vee$. Thus

$$h_L(T) = \#\,\mathrm{H}^1(L, T^\vee) = \#\,\mathrm{H}^1(K, \mathrm{ind}_L^K T^\vee) = h_K(\mathrm{R}_{L/K} T).$$

So $h(T)$ does not really depend on K, as long as we restrict appropriately. For example, $h_L(\mathbf{G}_{\mathrm{m}L}) = h_K(\mathrm{R}_{L/K} \mathbf{G}_{\mathrm{m}})$. I'm pretty sure the same holds for $Ш(T)$.

So, if $T = \mathbf{G}_{\mathrm{m}}$, we should have $h(T) = i(T) = 1$, whence $\tau(\mathbf{G}_{\mathrm{m}}) = 1$. Let's try this directly. Let $\omega = \frac{\mathrm{d}t}{t}$. Let

$$\rho(\mathbf{G}_{\mathrm{m}/F}) = \lim_{s \to 1} \frac{1}{s-1} L(F, s) = \frac{2^r (2\pi)^s \#\,\mathrm{Pic}(O_F) \Omega_F}{\#\boldsymbol{\mu}(F) \sqrt{|D_F|}},$$

where $L(F, s)$ is the Artin L-function of F.

What is the maximal $\mathbf{Q}$-split torus of $\mathrm{R}_{F/\mathbf{Q}} \mathbf{G}_{\mathrm{m}}$? Clearly the diagonal embedding $\mathbf{G}_{\mathrm{m}} \hookrightarrow \mathrm{R}_{F/\mathbf{Q}} \mathbf{G}_{\mathrm{m}}$ is split. Suppose $F/\mathbf{Q}$ is non-Galois, e.g. $\mathbf{Q}(\sqrt[4]{2})/\mathbf{Q}$.

3.1 Locally symmetric spaces for $\mathbf{G}_\mathrm{m}$

Let F be a number field, $G = \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}$. We claim that the diagonal $\mathbf{G}_\mathrm{m} \subset G$ is a maximal split torus. Note that

$$\hom(\mathbf{G}_\mathrm{m}, G) = \hom_{\mathrm{Gal}_\mathbf{Q}}(G^\vee, \mathbf{Z}) = \hom(\mathrm{ind}_F^\mathbf{Q}\,\mathbf{Z}, \mathbf{Z}) = \hom(\mathbf{Z}, \mathrm{res}_F^\mathbf{Q}\,\mathbf{Z}) = \mathbf{Z}.$$

The result follows.

Thus, the symmetric spaces we are interested in are of the form

$$S_{K_\mathrm{f}} = F^\times \backslash \mathbf{A}_F^\times / \mathbf{R}^+ K_\mathrm{f},$$

for $K_\mathrm{f} \subset \mathbf{A}_{F,\mathrm{f}}^\times$ open compact. Here $\mathbf{R}_{>0} \hookrightarrow F_\infty^\times$ via the diagonal embedding. At infinity, the space we're interested in is $F_\infty^\times/\mathbf{R}^+$, which is is isomorphic (as a Lie group) to $(F_\infty^\times)^{\mathrm{N}=1}$. It is well-known that $(F_\infty^\times)^{\mathrm{N}=1}/O_F^\times$ is compact, which tells us that for $\Gamma \subset F^\times$ a torsion-free arithmetic subgroup, the double quotient $\Gamma\backslash F_\infty^\times/\mathbf{R}^+$ is a torus. Its volume (with respect to the natural Haar measure induced from F_∞) should be computed in a similar manner to the regulator of F.