The geometric Satake correspondence

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First, recall that if $X_{/S}$ and $Y_{/S}$ are schemes, there is a "hom-scheme" defined by

$$\mathbf{hom}(X,Y)(T) = \hom_T(X_{/T}, Y_{/T}).$$

In general, $\mathbf{hom}(X,Y)$ will only be an fpqc sheaf.

Let $\mathbf{D} = \operatorname{Spec}(\mathbf{Z}[\![t]\!])$ and $\mathbf{D}^* = \operatorname{Spec}(\mathbf{Z}(\!(t)\!))$. Then we have a pullback diagram:

$$\begin{tikzcd} \mathbf{D}^* \arrow[r, hook] \arrow[d] & \mathbf{D} \arrow[d] \\ \mathbf{G}_{\mathrm{m}} \arrow[r, hook] & \mathbf{A}^1 \end{tikzcd}$$

Thus, if $X_{/S}$ is an arbitrary scheme, we get a commutative diagram:

$$\begin{tikzcd} \mathbf{hom}(\mathbf{D}^*, X) & \mathbf{hom}(\mathbf{D}, X) \arrow[l, hook'] \\ \mathbf{hom}(\mathbf{G}_{\mathrm{m}}, X) \arrow[u] & \mathbf{hom}(\mathbf{A}^1, X) \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

In particular, if T is a torus, there is a natural map

$$\mathrm{X}_*(T) = \mathbf{hom}(\mathbf{G}_{\mathrm{m}}, T) \to \mathbf{hom}(\mathbf{D}^*, T) = G(\!(t)\!).$$

For $\lambda \in \mathrm{X}_*(T)$, write t^λ for the image in $G(\!(t)\!)$. Let G be a reductive group, and put $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$. It turns out that perverse sheaves on Gr_G are best understood in light of orbits $\mathrm{Gr}_G^\lambda = \overline{G[\![t]\!] \cdot t^\lambda}$ for $\lambda \in \mathrm{X}_*(T)$, where $T \subset G$ is a maximal torus.

If $G = T$ is already a torus, then $\mathrm{Gr}_T = \mathrm{X}_*(T)$, and each Gr_T^λ is a single point.

Note that $\mathrm{X}_*(T)(A) = T(A[t^{\pm 1}])$.