

A counterexample relating exponential sums and discrepancy

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For a prime p , let

$$T_p = \left\{ \frac{a}{2\sqrt{p}} : a \in \mathbf{Z}, |a| \leq 2\sqrt{p} \right\}$$

$$\Theta_p = \cos^{-1}(T_p).$$

Since applying continuous increasing functions preserves discrepancy, we have:

$$\text{disc}(T_p, \text{Leb}) \ll p^{-1/2}$$

$$\text{disc}\left(\Theta_p, \frac{1}{2} \sin(t) dt\right) \ll p^{-1/2}.$$

We claim that starting with $\theta_2 \in \Theta_2$, we can choose θ_p such that we preserve the inequalities:

$$\frac{1}{4 \log x} \leq \text{disc}(\{\theta_p\}_{p \leq x}) \leq \frac{4}{\log x}$$

$$\left| \sum_{p \leq x} U_1(\theta_p) \right| \leq 2\sqrt{x}$$

Recall that

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin \theta}.$$

We can run this for all $p \leq 10^5$. Recall that $\pi(10^5) \approx 10000$.

Here is what we get:

Conjecture 1. *There exists a sequence of $\theta_p \in \Theta_p$ such that the following identities always hold:*

$$\frac{1}{4 \log x} \leq \text{disc}(\{\theta_p\}_{p \leq x}) \leq \frac{4}{\log x}$$

$$\left| \sum_{p \leq x} U_1(\theta_p) \right| \leq 2\sqrt{x}.$$

Figure 1: Plot of $\sum_{p \leq x} U_1(\theta_p)$

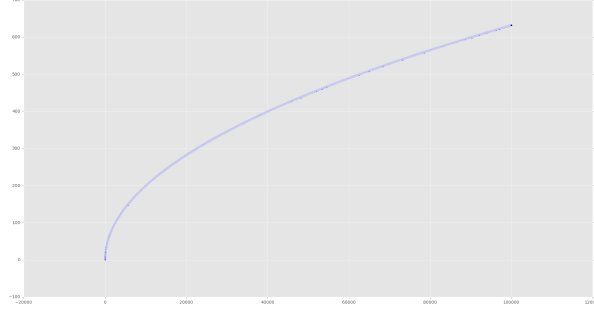
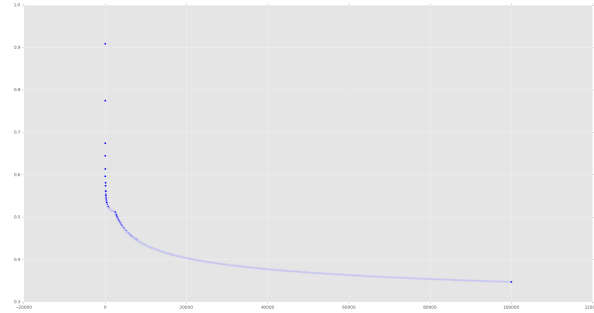


Figure 2: Plot of $\text{disc}(\{\theta_p\}_{p \leq x})$



Next, choose $\bar{\rho}_l: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_l)$ to which we can apply Ramakrishna et. al.'s machinery. Define

$$\Theta_p(\bar{\rho}_l) = \left\{ \cos^{-1} \left(\frac{a}{2\sqrt{p}} \right) : a \in \mathbf{Z}, |a| \leq 2\sqrt{p}, a \equiv \text{tr } \bar{\rho}_l(\text{fr}_p) \pmod{l} \right\}.$$

Conjecture 2. *There exists a sequence of $\theta_p \in \Theta_p(\bar{\rho}_l)$ such that*

$$\begin{aligned} \text{disc}(\{\theta_p\}_{p \leq x}) &= \Omega \left(\frac{1}{\log x} \right) \\ \left| \sum_{p \leq x} U_1(\theta_p) \right| &\ll \sqrt{x}. \end{aligned}$$

Corollary 1. *There exists an (infinitely ramified) Galois representation $\rho_l: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{Z}_l)$ such that if we set $a_p = \text{tr } \rho_l(\text{fr}_p)$, then*

1. $a_p \in \mathbf{Z}$

2. $|a_p| \leq 2\sqrt{p}$.

3. The $\theta_p = \cos^{-1} \left(\frac{a_p}{2\sqrt{p}} \right)$ satisfy

$$\text{disc}(\{\theta_p\}_{p \leq x}) = \Omega \left(\frac{1}{\log x} \right)$$

$$\left| \sum_{p \leq x} U_1(\theta_p) \right| \ll \sqrt{x}.$$

and hence $L(\rho_l, s)$ satisfies the Riemann Hypothesis.

1 Towards a proof

Let $\bar{\rho}_l: G_{\mathbf{Q}} \rightarrow \text{GL}_2(\mathbf{F}_l)$ be a Galois representation. For each prime p , define

$$\Theta_p(l) = \left\{ \cos^{-1} \left(\frac{a}{2\sqrt{p}} \right) : a \in \mathbf{Z}, |a| \leq 2\sqrt{p}, a \equiv \text{tr } \bar{\rho}_l(\text{fr}_p) \pmod{l} \right\}.$$

It is easy to check that

$$\text{disc} \left(\Theta_p(l), \frac{1}{2} \sin(t) dt \right) \ll lp^{-1/2}.$$

We are looking for a way to choose $\theta_p \in \Theta_p(l)$ such that

1. $\text{disc}(\{\theta_p\}_{p \leq x})$ decays like $1/\log x$
2. $\left| \sum_{p \leq x} U_1(\theta_p) \right|$ grows like \sqrt{x} .

To do this, suppose we have chosen $\{\theta_q\}_{q < p}$. In choosing θ_p , we want to simultaneously move the discrepancy towards $1/\log p$, while making sure that the U_1 -sum doesn't get too big.

(Fact: if $\{x_1, \dots, x_N\}$ and $\{y_1, \dots, y_N\}$ are two sequences, then

$$|\text{disc}(\{x_1, \dots, x_N\}) - \text{disc}(\{y_1, \dots, y_N\})| \leq 2\|x - y\|_{\infty}.$$

)

It's actually quite simple. Note that:

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin \theta} = -U_1(\pi - \theta).$$

The basic idea is: set $\theta_3 \approx \pi - \theta_2$, $\theta_7 \approx \pi - \theta_5$, etc. and we can choose θ_2, θ_5 etc. arbitrarily, meaning good discrepancy, while the sum should approximately cancel out. First, since U_1 has bounded derivative, we know that

$$|U_1(\theta) - U_1(\varphi)| \ll |\theta - \varphi|$$

So, if $p_1 < p_2$ are sequential primes, we have

$$|\theta_{p_2} - (\pi - \theta_{p_1})| \ll p_1^{-1/2},$$

so

$$\begin{aligned} |U_1(\theta_{p_1}) + U_1(\theta_{p_2})| &\leq |U_1(\theta_{p_1}) - U_1(\pi - \theta_{p_1})| + |U_1(\pi - \theta_{p_1}) - U_1(\theta_{p_2})| \\ &\ll |\theta_{p_2} - (\pi - \theta_{p_1})| \\ &\ll p_1^{-1/2}. \end{aligned}$$

So,

$$\left| \sum_{p \leq x} U_1(\theta_p) \right| \ll \sum_{p \leq x} p^{-1/2} \ll \int_1^x t^{-1/2} dt \ll \sqrt{x}.$$

(Same argument works for all U_{odd} because they all satisfy $U_{\text{odd}}(\pi - \theta) = -U_{\text{odd}}(\theta)$. In contrast, $U_{\text{even}}(\pi - \theta) = U_{\text{even}}(\theta)$.)

2 A legit proof!

Theorem 1. *Fix a prime l . Suppose we have chosen, for all primes p , some arbitrary residue class $\bar{a}_p \in \mathbf{F}_l$, and set*

$$\Theta_p(l) = \left\{ \cos^{-1} \left(\frac{a}{2\sqrt{p}} \right) : a \in \mathbf{Z}, |a| \leq 2\sqrt{p}, a \equiv \bar{a}_p \pmod{l} \right\}.$$

Then there exists a choice of $\theta_p \in \Theta_p(l)$ such that

1. *The sequence $\{\theta_p\}$ is equidistributed with respect to the Sato–Tate measure $\frac{2}{\pi} \sin^2 \theta d\theta$.*
2. *The discrepancy $\text{disc}(\{\theta_p\}_{p \leq x}, \text{ST}) \gg \frac{1}{\log x}$.*
3. $\left| \sum_{p \leq x} U_{\text{odd}}(\theta_p) \right| \ll \sqrt{x}.$

Proof. Enumerate the primes $p_1 < p_2 < \dots$. We will choose $\theta_{p_{\text{odd}}} \in [0, \pi/2)$ so that the discrepancy of the sequence $\{\theta_{p_{\text{odd}}}\}$ behaves as required in that interval. We'll then set $\theta_{p_{2i}} \approx \pi - \theta_{p_{2i-1}}$.

Everything comes down to: if $p < q$ are sequential primes and we have already chosen θ_p , we need to be able to choose θ_q so that $|U_1(\theta_p) + U_1(\theta_q)| \ll p^{-1/2}$. Since $\frac{dU_1}{d\theta} = -2 \sin(\theta)$, we have (roughly)

$$|U_1(\theta) - U_1(\varphi)| \ll \max(\theta, \varphi) \cdot |\theta - \varphi|$$

for $\theta, \varphi \in [0, \pi/2)$.

Start with $t_p = \frac{a_p}{2\sqrt{p}}$ and $t_q = \frac{a_q}{2\sqrt{q}}$. We can guarantee that $|t_p - (\pi - t_q)| \ll p^{-1/2}$.

Fact:

$$|\cos^{-1}(1-x) - \cos^{-1}(1-(x+\sqrt{x}))| \ll x^{1/5}.$$

So roughly,

$$|\theta_p - \theta_q| \ll p^{-1/5},$$

After taking \cos^{-1} , all we can guarantee is that

$$|\theta_p - \theta_q| \ll$$

Let's think systematically. We're picking t_1 and t_2 close to 1, which is where $(\cos^{-1})'$ blows up. But there shouldn't be very many of them close to 1. Aka,

$$\left| \frac{\#\{p \leq x : \theta_p \in [0, t]\}}{\pi(x)} - \int_0^t d\text{ST} \right| \ll \frac{1}{\log x}$$

$$\frac{\#\{p \leq x : \theta_p \in [0, t]\}}{\pi(x)} \ll t^2 + \frac{1}{\log x}.$$

We want to know, given x , how small the smallest $\theta_p, p \leq x$ is. Roughly, for what t is

$$\#\{p \leq x : \theta_p \in [0, t]\} < 1?$$

We already know that

$$\#\{p \leq x : \theta_p \in [0, t]\} \ll \frac{x}{\log x} \left(t^2 + \frac{1}{\log x} \right).$$

This is frustrating, because it means, essentially, that our convergence to the Sato–Tate measure is so slow (by design) that we can't *ever* guarantee that no θ_p lies in some small interval. But there's something easier. For each $p \leq x$, we start by choosing $a_p \in \mathbf{Z}$. How close can a_p be to $2\sqrt{p}$? Numerical experiments (**prove this!**) show that for $t_p = \frac{a_p}{2\sqrt{p}}$, we have

$$|1 - t_p| \gg p^{-1/2}.$$

This is key! That means θ_p won't be too small. In particular, we can control how close θ_p and θ_q will be.

We already have chosen θ_p . We want to choose a_q so that $\cos^{-1}(\frac{a_q}{2\sqrt{q}}) \approx \pi - \theta_p$, i.e.

$$\frac{a_q}{2\sqrt{q}} \approx \sin(\theta_p).$$

We can ensure

$$\left| \frac{a_q}{2\sqrt{q}} - \cos(\pi - \theta_p) \right| \ll p^{-1/2}.$$

Moreover, we know that $|\pm 1 - \frac{a_q}{2\sqrt{q}}| \gg q^{-1/2}$, and likewise for a_p . Thus,

$$|\theta_p - \theta_q| = \left| \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right) - \pi + \cos^{-1}\left(\frac{a_q}{2\sqrt{q}}\right) \right| \ll p^{-1/2}.$$

Good news: numerical experiments show that we can get very good approximation to $U_1(\theta_q) \approx -U_1(\theta_p)$ for $p < q$ successive primes. This is fantastic!

Numerical experiments suggest that we can enforce

$$|U_1(\theta_p) + U_1(\theta_q)| \ll \frac{\log p}{p}.$$

□

Let (X, μ) be a topological measure space. Suppose g is a non-trivial automorphism of X , such that $g_*\mu = \mu$. Suppose $g^2 = 1$. If we want to minimize

$$\left| \sum_{p \leq x} f(x_p) \right|,$$

while letting the discrepancy of $\{x_p\}$ vary arbitrarily. Suppose we can find a “good” subset $U \subset X$ such that $X = U \sqcup gU$. Choose $x_{p_{\text{odd}}} \in U$ to control the discrepancy, and then choose $x_{p_{\text{even}}} \approx g(x_{p_{\text{odd}}})$. For any $f \in C^\infty(X)$ such that $g^*f = -f$. Then

$$\sum_{p \leq x} f(x_p) = \sum (f(x_{p_{\text{even}}}) + f(x_{p_{\text{odd}}})) \approx \sum 0.$$

We know that near $\theta = 0$,

$$U_n(\theta) = n + C_n \theta^2 + O(\theta^3).$$

(I think this will hold for any f with $\int f = 0$ and $f(\pi - \theta) = f(\theta)$.)

3 Precise method

Let $\{p_1, p_2, \dots\}$ be an enumeration of the rational primes. Given $x \in \mathbf{R}$, write $\sum_{p_{\text{odd}} \leq x} a_p$ for the sum of all a_p for $p_i \leq x$ with i odd, and similarly for $\sum_{p_{\text{even}} \leq x}$. Suppose we have chosen $\theta_{p_{\text{odd}}} \in [0, \pi/2)$ so that $\text{disc}(\{\theta_{p_{\text{odd}}}\}_{p_{\text{odd}} \leq x})$ decays as desired. Suppose we choose $\theta_{p_{\text{even}}} \approx \pi - \theta_{p_{\text{odd}}}$. That is, for $p < q$ successive primes with $p = p_i$, i odd, we’ll choose $\theta_q \approx \pi - \theta_p$.

We know that $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ for some $a_p \in \mathbf{Z}$ with $|a_p| \leq 2\sqrt{p}$. We want to choose $\theta_q \approx \pi - \theta_p$, i.e.

$$\begin{aligned} \cos^{-1}\left(\frac{a_q}{2\sqrt{q}}\right) &\approx \pi - \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right) \\ \frac{a_q}{2\sqrt{q}} &\approx -\frac{a_p}{2\sqrt{p}}. \end{aligned}$$

since $\cos(\pi - \cos^{-1}(x)) = -x$. We can guarantee that

$$\left| \frac{a_q}{2\sqrt{q}} + \frac{a_p}{2\sqrt{p}} \right| \leq \frac{1}{\sqrt{q}}.$$

Claim: if x, y are “further than ϵ ” from ± 1 and $|x - y| < \epsilon$, then $|\cos^{-1}(x) - \cos^{-1}(y)| \leq \sqrt{\epsilon}$. (Have checked with Wolfram Alpha, prove later.)

In conclusion, for each successive primes $p = p_{\text{odd}} < q = p_{\text{even}}$, if there is $\theta_p \in \Theta_p(l)$ chosen already, we can also choose $\theta_q \in \Theta_q(l)$ so that

$$|\theta_q - (\pi - \theta_p)| \ll lp^{-1/4}.$$

This is all that is needed, since we’re looking at f that is of the form

$$f(\theta) = f(0) + C\theta^2 + O(\theta^3)$$

for θ close to zero. (In fact, this is true for *all* smooth, Weyl-invariant f , whether or not they satisfy $f(\theta) = -f(\pi - \theta)$.) The squaring “pushes the difference” back to $p^{-1/2}$. That is, for θ, φ close to zero, but at least ϵ away from zero, we have

$$|f(\theta) - f(\varphi)| \ll |\theta - \varphi|^2.$$

Now the question is, if $\theta_q \approx \pi - \theta_p$, how close is the discrepancy of $\{\theta_{p_{\text{odd}}}\}$ and $\{\theta_{p_{\text{even}}}\}$?

Better, how close are

$$\#\{p_{\text{odd}} \leq x : \theta_{p_{\text{odd}}} \leq t\} \quad \text{and} \quad \#\{p_{\text{odd}} \leq x : \theta_{p_{\text{odd}}} \leq t\}?$$

We know that $|\theta_p - \theta_q| \ll p^{-1/4}$. Actually, all we need is that if $\text{disc}(\{\theta_{p_{\text{odd}}}\}) \rightarrow 0$, then also $\text{disc}(\{\theta_{p_{\text{even}}}\}) \rightarrow 0$.

Suppose we have two sequences $\{x_n\}$ and $\{y_n\}$ such that $\text{disc}(\{x_n\}_{n \leq N}) \sim \frac{1}{\log N}$, and also $|x_n - y_n| \leq n^{-1/4}$. For some really big N , choose $M < N$, ideally $M \approx \log N$.

Look at

$$\limsup_{N \rightarrow \infty} \text{disc}(\{y_n\}_{M \leq n \leq N}) \leq M^{-1/4}.$$

With complete generality, we have:

$$|\text{disc}(\{x_n\}_{n \leq N}) - \text{disc}(\{x_n\}_{M \leq n \leq N})| \ll \frac{1}{M}$$

This is all we need.