SELMER GROUPS IN ARITHMETIC TOPOLOGY

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In the first couple sections I construct things in a relatively elementary way. The rest of this note looks at categorical foundations.

1. Arithmetic setup

Let F be a number field, S a finite set of places of F. Write $G_{F,S} = \pi_1(\mathrm{Spec}(O_F) \smallsetminus S)$ for the Galois group of the maximal extension of F unramified outside S. Let M be a $G_{F,S}$-module. The *S-Tate-Shafarevich group* of M is

$$Ш_S^\bullet(M) = \ker\left(\mathrm{H}^\bullet(G_{F,S}, M) \to \bigoplus_{v\in S} \mathrm{H}^\bullet(G_v, M)\right),$$

where $G_v = \pi_1(F_v)$ is the decomposition group at v. Let's start by giving a geometric definition of Ш.

Let $X = \mathrm{Spec}(O_F)$, and let $S \subset X$ be a closed subscheme. Write $i : S \hookrightarrow X$ and $j : U = X \smallsetminus S \hookrightarrow X$ for the inclusion maps. We should think of the $G_{F,S}$-module M as being a locally constant sheaf $\mathscr{F}$ on U. The question is: how should we think of $\bigoplus_{v\in S} \mathrm{H}^\bullet(G_v, M)$? Let S^+ be the infinitesimal étale neighborhood of S. Then $S^+ = \coprod_{v\in S} \mathrm{Spec}(O_{F,v})$. It follows that

$$\partial S = S^+ \smallsetminus S = \coprod_{v\in S} \mathrm{Spec}(F_v).$$

Locally constant sheaves on ∂S are the same thing as a collection of G_v-modules for $v \in S$. The analogue of "treating M as a G_v-module" is $j_*\mathscr{F}|_{\partial S}$. So our sheaf-theoretic Tate-Shafarevich group is

$$Ш_S^\bullet(\mathscr{F}) = \ker\left(\mathrm{H}^\bullet(U, \mathscr{F}) \to \mathrm{H}^\bullet(\partial S, j_*\mathscr{F}|_{\partial S})\right).$$

A common place for these groups to arise is in deformation theory. If $\bar\rho : G_{F,S} \to \mathrm{GL}_2(\mathbf{F}_q)$ is a Galois representation, one wants $Ш_S^1(\mathrm{ad}\,\bar\rho)$ to vanish. Often, by enlarging S cleverly, one can ensure this.

2. Topological analogue

Let M be a three manifold and let $L \subset M$ be a link (not just a knot – this is important). Put $U = M \smallsetminus L$, and let $\mathscr{L}$ be a local system on U. Let $j : U \hookrightarrow M$ be the inclusion. Let V_L be a tubular neighborhood of L, and put $\partial V_L = V_L \smallsetminus L$ (this deformation retracts onto a union of tori). The *topological Tate-Shafarevich group* is

$$Ш_L^\bullet(\mathscr{L}) = \ker\left(\mathrm{H}^\bullet(U, \mathscr{L}) \to \mathrm{H}^\bullet(\partial V_L, j_*\mathscr{L}|_{\partial V_L})\right).$$

General question: is $Ш_L^\bullet(\mathscr{L})$ an "already known object"? If so, what role does it play?

Let's look at a baby example. Let $K \subset S^3$ be a knot, $\mathscr{L}$ the constant sheaf $\mathbf{Z}$. Then

$$\begin{aligned}
Ш^1_K(\mathbf{Z}) &= \ker\left(\hom(\pi_1(U), \mathbf{Z}) \to \hom(\pi_1(\partial V_L), \mathbf{Z})\right) \\
&= \ker\left(\hom(G_K^{\mathrm{ab}}, \mathbf{Z}) \to \mathbf{Z}^2\right) \\
&= \left(G_K^{\mathrm{ab}}/\mathbf{Z}^2\right)^\vee,
\end{aligned}$$

where $\mathbf{Z}^2 \to G_K$ is the peripheral map. Since $G_K^{\mathrm{ab}} = \mathbf{Z}$, this "topological Tate-Shafarevich group" is cyclic. It's not clear to me whether we can say much about general $Ш^1_L(\mathscr{L})$.

3. Kan extensions

A source for some of this is [Rie14], available online at `http://www.math.harvard.edu/~eriehl/cathtpy.pdf`. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor. We get an induced natural transformation (of 2-functors?) $f^* : [\mathcal{D}, -] \to [\mathcal{C}, -]$. That is, for each category $\mathcal{E}$, there is a functor $f^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ that sends $g : \mathcal{D} \to \mathcal{E}$ to $gf : \mathcal{C} \to \mathcal{E}$. We say that f *admits left (resp. right) Kan extensions* (non-standard terminology) if $f^* : \mathcal{E}^{\mathcal{D}} \to \mathcal{D}^{\mathcal{C}}$ has left (resp. right) adjoints, which we denote L_f (resp. R_f). If f^* has both adjoints, we say that f *has Kan extensions*. In this case, there is an adjoint triple $(\mathrm{L}_f, f^*, \mathrm{R}_f)$ fitting into a diagram

$$\begin{tikzcd}
\mathcal{E}^{\mathcal{D}} \arrow[r, "f^*"] & \mathcal{E}^{\mathcal{C}}. \arrow[l, bend right=60, "\mathrm{L}_f"'] \arrow[l, bend left=60, "\mathrm{R}_f"]
\end{tikzcd}$$

To be more concrete, we have natural isomorphisms

$$\begin{aligned}
[\mathrm{L}_f g, h] &= [g, f^* h] \\
[f^* g, h] &= [g, \mathrm{R}_f h]
\end{aligned}$$

4. Derived functors

Now let $f : \mathcal{A} \to \mathcal{B}$ be an additive functor on abelian categories. We also write f for the induced functor $\mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ on categories of chain complexes modulo homotopy. Let $q : \mathsf{K}(-) \to \mathsf{D}(-)$ be the localization functor. We define (if they exist)

$$\begin{aligned}
\mathsf{L}f &= \mathrm{R}_q \bar{f} \\
\mathsf{R}f &= \mathrm{L}_q \bar{f}.
\end{aligned}$$

This deserves some explanation. We will concentrate on the right-derived functor $\mathsf{R}f : \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$. The functor $q_{\mathcal{A}} : \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ induces

$$q_{\mathcal{A}}^* : \mathsf{D}(\mathcal{B})^{\mathsf{D}(\mathcal{A})} \to \mathsf{D}(\mathcal{B})^{\mathsf{K}(\mathcal{A})}.$$

The image of $\bar{f}$ under its left adjoint is $\mathsf{R}f$. That is, there is a natural isomorphism

$$\hom_{[\mathsf{D}(\mathcal{A}), \mathsf{D}(\mathcal{B})]}(\mathsf{R}f, g) = \hom_{[\mathsf{K}(\mathcal{A}), \mathsf{D}(\mathcal{B})]}(\bar{f}, g \circ q_{\mathcal{A}}).$$

Putting $g = \mathsf{R}f$, the identity morphism $\mathsf{R}f \to \mathsf{R}f$ induces the unit $\eta_f : \bar{f} \to \mathsf{R}f \circ q_{\mathcal{A}}$. All morphisms $\bar{f} \to gq$ come from a unique $\mathsf{R}f \to g$ via η.

We say that $\mathsf{R}f = f$ if $\eta : \bar{f} \to \mathsf{R}f \circ q$ is an isomorphism. When this is the case, we will write f instead of $\mathsf{R}f$.

4.1. **Composition of derived functors.** Let $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ be additive functors. We will construct a canonical natural transformation $\mathsf{R}(g \circ f) \to \mathsf{R}g \circ \mathsf{R}f$. By the definition of $\mathsf{R}(-)$, we have

$$[\mathsf{R}(g \circ f), \mathsf{R}g \circ \mathsf{R}f] = [\overline{g \circ f}, \mathsf{R}g \circ \mathsf{R}f \circ q].$$

We construct a transformation $\overline{g \circ f} \to \mathsf{R}g \circ \mathsf{R}f \circ q$ as follows:

$$\overline{g \circ f} = \bar{g} \circ f \xrightarrow{\eta_g \circ f} \mathsf{R}g \circ q \circ f = \mathsf{R}g \circ \bar{f} \xrightarrow{\mathsf{R}g \circ \eta_f} \mathsf{R}g \circ \mathsf{R}f \circ q.$$

4.2. **Functoriality of derived functors.** Suppose we have $\alpha : f \to g$. There should be $\mathsf{R}\alpha : \mathsf{R}f \to \mathsf{R}g$. For this, it suffices to construct $[\mathsf{R}g, -] \to [\mathsf{R}f, -]$ via

$$[\mathsf{R}g, -] = [\bar{g}, q^*-] \xrightarrow{\bar{\alpha}^*} [\bar{f}, q^*-] = [\mathsf{R}f, -].$$

Suppose we have functors $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$, $h : \mathcal{A} \to \mathcal{C}$ together with $\alpha : h \to gf$. Suppose further that $\mathsf{R}f = f$. Then there is a canonical transformation

$$\mathsf{R}h \to \mathsf{R}(gf) \to \mathsf{R}g \circ f.$$

5. Derived Tate-Shafarevich groups

If $\mathcal{X}$ be a topos, let $\Gamma = \Gamma_{\mathcal{X}} = \hom(1_{\mathcal{X}}, -)$. Recall that a morphism of topoi (called a *geometric morphism* in [MLM94]) $f : \mathcal{X} \to \mathcal{Y}$ is an adjoint pair (f^*, f_*), where $f_* : \mathcal{X} \to \mathcal{Y}$ and f^* preserves limits. Note that f^* already preserves colimits. Write $\mathsf{D}(\mathcal{X})$ for the derived category of abelian group objects in $\mathcal{X}$. Since f^* is exact, we write $f^* : \mathsf{D}(\mathcal{Y}) \to \mathsf{D}(\mathcal{X})$ for the induced functor.

There is a canonical natural transformation $\Gamma_{\mathcal{Y}} \to \Gamma_{\mathcal{X}} \circ f^*$, constructed via

$$\Gamma_{\mathcal{Y}} \to \hom(1_{\mathcal{Y}}, f_* f^* -) = \hom(f^* 1_{\mathcal{Y}}, f^* -) = \Gamma_{\mathcal{X}} \circ f^*,$$

via the unit $1 \to f_* f^*$. We have seem that this gives $\mathsf{R}\Gamma_{\mathcal{Y}} \to \mathsf{R}\Gamma_{\mathcal{X}} \circ f^*$. We define the *$f$-Tate-Shafarevich group* to be

$$Ш_f = \ker\left(\mathsf{R}\Gamma_{\mathcal{Y}} \to \mathsf{R}\Gamma_{\mathcal{X}} \circ f^*\right).$$

The problem is, this doesn't exist (in general) as an object of the derived category. So we can either look at

$$Ш_f^\bullet(-) = \ker\left(\mathrm{H}^\bullet(Y, -) \to \mathrm{H}^\bullet(X, f^* -)\right)$$

or define

$$Ш_f = \operatorname{hok}\left(\mathsf{R}\Gamma_{\mathcal{Y}} \to \mathsf{R}\Gamma_{\mathcal{X}} \circ f^*\right)$$

the homotopy-kernel.

We could go even further and define the *Tate-Shafarevich category* to be

$$Ш(f) = \mathsf{D}(\mathcal{Y}) / \ker(Ш_f).$$

There is the obvious functor $\mathsf{D}(\mathcal{Y}) \to Ш(f)$.

References

[MLM94] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, 1994. Corrected reprint of the 1992 edition.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, 2014.