Vanishing cycles, cyclotomic polynomials, and monodromy

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1 Motivation

This is inspired by the discussion at the beginning of [Mil68, §9]. Let f be a regular function on $\mathbf{C}^{n+1}$. Let $V = V(f)$; this is an n-dimensional complex variety. Fix an isolated critical point $x \in V$. For $\varepsilon > 0$, let S_ε be a (real) sphere of radius ε centered at x, and put $K_\varepsilon = V \cap S_\varepsilon$. Define $\phi_\varepsilon : S_\varepsilon \smallsetminus K_\varepsilon \to S^1$ by

$$\phi_\varepsilon(x) = \frac{f(x)}{|f(x)|}.$$

The "fibration theorem" tells us that ϕ_ε is a fibration; we put $F_s = \phi_\varepsilon^{-1}(s) \subset S_\varepsilon \smallsetminus K_\varepsilon$ for $s \in S^1$. It is known that F_s is a smooth $2n$-dimensional real manifold. Let h be the canonical generator of $\pi_1(S^1)$. There is an endomorphism h_* of $\mathrm{H}_n(F_s, \mathbf{Z})$. Let Δ be its characteristic polynomial. Milnor claims that Grothendieck has proved Δ is a product of cyclotomic polynomials.

2 Reinterpretation

Let f, V, ... be as above. Instead of considering the homology of individual fibers F_s of the map ϕ_ε, we consider the pushforward $\mathrm{R}^n\phi_{\varepsilon,*}\mathbf{Z}$ as a sheaf on S^1. This is a local system, so we can consider it as a representation of the group $\pi_1(S^1)$. Hopefully, there is a theorem that $\mathrm{R}^n\phi_{\varepsilon,*}\mathbf{Z}$ is quasi-unipotent.

Consider [GRR72] and [DK73]

References

[DK73] Pierre Deligne and Nicholas Katz, editors. *Groupes de monodromie en géométrie algébrique. II*, volume 340 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébriqe du Bois-Marie 1967–1969 (SGA 7 II).

[GRR72] Alexandre Grothendieck, M. Raynaud, and D.S. Rim, editors. *Groupes de monodromie en géométrie algébrique. I*, volume 288 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébriqe du Bois-Marie 1967–1969 (SGA 7 I).

[Mil68] John Milnor. *Singular points of complex hypersurfaces*, volume 61 of *Annals of Mathematics Studies*. Princeton University Press, 1968.