Obstruction theory via the cotangent complex

Daniel Miller

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Our main example is as follows. Let k be a finite field, $\mathrm{W}(k)$ its ring of Witt vectors. Consider the category $\mathsf{Vaf}_{\mathrm{W}(k)}$ of "formal varieties over $\mathrm{W}(k)$." It is the opposite category of the full category of topological $\mathrm{W}(k)$-algebras that are filtered projective limits of finite $\mathrm{W}(k)$-algebras. We will give $\mathsf{Vaf}_{\mathrm{W}(k)}$ a suitable subcanonical Grothendieck topology, and consider sheaves on it. Note that $\mathrm{Sh}(\mathsf{Vaf}_{\mathrm{W}(k)})$ comes with a commutative ring object – namely the forgetful functor $\mathrm{Spf}(A) \mapsto A$. We will denote this functor by $\mathscr{O}$. If $\mathcal{X}$ is a formal scheme, or just a sheaf on $\mathsf{Vaf}_{\mathrm{W}(k)}$, we will consider $\mathcal{X}$ as the topos $\mathrm{Sh}\left(\mathsf{Vaf}_{\mathrm{W}(k)}\right)_{/\mathcal{X}}$. This has an obvious commutative ring object $\mathscr{O}_{\mathcal{X}} = \mathscr{O} \times \mathcal{X}$. So for the rest of this note we will work with an arbitrary ringed topos $(\mathcal{X}, \mathscr{O})$, but the reader should keep in mind this specific example.

Our main reference is [Ill71]. Also be aware that we will sometimes work with sheaves on the category of connected, pointed $\mathrm{W}(k)$-formal varieties – that is the opposite category of the category of local profinite $\mathrm{W}(k)$-algebras with residue field k. We will do this in the context of specific deformation problems.

Brief justification that this generalization works. Let $\mathcal{X}$ be a topos, $\mathfrak{Top}$ the category of all topoi and geometric morphisms. Then the "slice functor" $x \mapsto \mathcal{X}_{/x}$ from $\mathcal{X} \to \mathfrak{Top}_{/\mathcal{X}}$ is a fully faithful embedding by [Joh77, 4.38]. So there is no loss replacing a formal scheme over $\mathrm{W}(k)$ with the topos of sheaves over this scheme, regarded as a topos over the topos of sheaves over $\mathrm{W}(k)$.

1 Cotangent complex for morphisms of topoi

If $\mathcal{X}, \mathcal{Y}$ are topoi, we call a *morphism* $f : \mathcal{X} \to \mathcal{Y}$ an adjoint pair (f^{-1}, f_*), where is a morphism $f_* : \mathcal{X} \to \mathcal{Y}$ and $f^{-1} : \mathcal{Y} \to \mathcal{X}$ is exact. If $\mathcal{X}$ and $\mathcal{Y}$ are ringed topoi, then a morphism $f : \mathcal{X} \to \mathcal{Y}$ must come with a morphism of ring objects $\mathscr{O}_{\mathcal{Y}} \to f_*\mathscr{O}_{\mathcal{X}}$, or equivalently $f^{-1}\mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$.

Definition 1.1 ([Ill71, II 1.2.7]). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of ringed topoi. The* cotangent complex *of $\mathcal{X}$ over $\mathcal{Y}$ is the simplicial $\mathscr{O}_{\mathcal{X}}$-module given by $L_{\mathcal{X}/\mathcal{Y}} = L_{\mathscr{O}_{\mathcal{X}}/f^{-1}\mathscr{O}_{\mathcal{Y}}}$.*

Here $L_{B/A}$ is defined as in [Ill71, II 1.2]. Our main example of interest is when $\mathcal{X}$ is a some deformation functor for a residual Galois representation $\bar{\rho}$. The representation $\bar{\rho}$ will correspond to $\bar{\rho} : \mathrm{Spf}(k) \to \mathcal{X}$, and we will be concerned with $L_{\bar{\rho}/\mathcal{X}} = L_{\mathrm{Spf}(k)/_{\bar{\rho}}\mathcal{X}}$. This is a simplicial k-vector space.

2 Obstruction theory

Our goal is as follows. Work over a base topos $\mathcal{S}$. Suppose $x_0 : \mathcal{X}_0 \to \mathcal{Y}$ is a morphism and I is an $\mathscr{O}_{\mathcal{X}_0}$-module. We are interested in extensions of x_0 to $x : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ has the same underlying topos as $\mathcal{X}_0$, but for which $\mathscr{O}_{\mathcal{X}}$ is a square-zero extension of $\mathscr{O}_{\mathcal{X}_0}$ by the ideal I.

Theorem 2.1. *Let $\mathcal{X}_0 \xrightarrow{x_0} \mathcal{Y}$ be a morphism over $\mathcal{Y}$, and I be an $\mathscr{O}_{\mathcal{X}_0}$-module. Then there is a canonical* obstruction class

$$o(x_0) \in \mathrm{Ext}^2_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{Y}}, I)$$

which is 0 *if and only if an extension of x_0 to $\mathcal{X} \to \mathcal{Y}$ exists. If such an extension exists, then the extensions are a $\mathrm{Ext}^1_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{Y}}, I)$-torsor, and each extension has automorphism group $\mathrm{Ext}^0_{\mathcal{X}_0}(L_{\mathcal{X}_0/\mathcal{Y}}, I)$.*

Proof. This is [Ill71, III 2.1.7], where $\mathcal{Y}_0 = \mathcal{Y}$ and the base topos is hidden from notation. □

3 One-dimensional representations

Let Γ be a finitely generated $\mathbf{Z}_p$-module. Write $\mathcal{X}_\Gamma$ for the deformation space parameterizing lifts of $1 \colon \Gamma \to \mathsf{k}^\times$. So $\mathcal{X}_\Gamma$ is a (formal) scheme over $\mathrm{W}(\mathsf{k})$. One way to understand the cotangent complex $\mathrm{L}_{\mathcal{X}_\Gamma/\mathrm{W}(\mathsf{k})}$ is by embedding $\mathcal{X}_\Gamma$ into a smooth scheme.

Let $\Gamma_\bullet \twoheadrightarrow \Gamma$ be a minimal free resolution of Γ as a $\mathbf{Z}_p$-module. So $\Gamma_\bullet = [\Gamma_1 \hookrightarrow \Gamma_0]$. Then we have a closed embedding $\mathcal{X}_\Gamma \hookrightarrow \mathcal{X}_{\Gamma_0} = \mathrm{Spf}(\mathrm{W}(\mathsf{k})[\![\Gamma_0]\!])$. Then [Ill71, III 3.3.6] tells us that

$$\mathrm{L}_{\mathcal{X}_\Gamma/\mathrm{W}(\mathsf{k})} = \left[\mathfrak{a}/\mathfrak{a}^2 \to \Omega^1_{\mathcal{X}_{\Gamma_0}/\mathrm{W}(\mathsf{k})} \otimes_{\mathrm{W}(\mathsf{k})[\![\Gamma_0]\!]} \mathrm{W}(\mathsf{k})[\![\Gamma]\!]\right],$$

where $\mathfrak{a} = \ker(\mathrm{W}(\mathsf{k})[\![\Gamma_0]\!] \twoheadrightarrow \mathrm{W}(\mathsf{k})[\Gamma])$.

4 Take two

Suppose $\Gamma = \mathbf{Z}_p^{\oplus r} \times \bigoplus_i \mathbf{Z}/p^{n_i}$. Then

$$R = \Lambda[\![\Gamma]\!] \simeq \Lambda[\![s_1, \dots, s_r, t_i]\!]/\langle 1 - (1 - t_i)^{p^{n_i}} \rangle.$$

This gives us an obvious surjection $\Lambda[\![\boldsymbol{s}, \boldsymbol{t}]\!] \twoheadrightarrow R$. Let $\mathfrak{a}$ be its kernel. Then

$$\mathrm{L}_{R/\Lambda} \simeq \left[\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\mathrm{d}} R\,\mathrm{d}(\boldsymbol{s}, \boldsymbol{t}) \right].$$

Now, more or less by definition, $\mathfrak{a} = \langle 1 - (1 - t_i)^{p^{n_i}} \rangle$.

References

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I*, volume 239 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[Joh77] P. T. Johnstone. *Topos theory*, volume 10 of *London Math. Soc. Monographs*. Academic Press, 1977.