Counterexamples related to the Sato–Tate conjecture

Daniel Miller

25 April 2017

Cornell University

Outline

Motivation and background

Discrepancy and Dirichlet series

Main theorem

Sketch of proof

Motivation and background

$E_{/\mathbf{Q}}$ non-CM elliptic curve, l prime, $\rho_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$.

Fact: $a_p = \operatorname{tr} \rho_l(\mathrm{fr}_p) = p + 1 - \#E(\mathbf{F}_p)$, $|a_p| \leqslant 2\sqrt{p}$. (Hasse)

Satake parameter: $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$.

Sato–Tate measure: $\mathrm{ST} = \frac{2}{\pi}\sin^2\theta$ (Haar measure on $\mathrm{SU}(2)^\natural$).

Theorem (Taylor et. al.)

$\{\theta_p\}$ *is* ST-*equidistributed.*

Quantify rate of convergence of $\frac{1}{\pi(N)}\sum_{p\leqslant N}\delta_{\theta_p}$ to ST.

Use discrepancy (Kolmogorov–Smirnov statistic).

Akiyama–Tanigawa conjecture

$$D_N = \sup_{x \in [0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p \leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta) \, \mathrm{d}\, \mathsf{ST}(\theta) \right|.$$

Conjecture (Akiyama–Tanigawa)

$D_N \ll N^{-\frac{1}{2}+\epsilon}$.

There is a variant of this conjecture for CM elliptic curves. Also for CM abelian varieties, even motives.

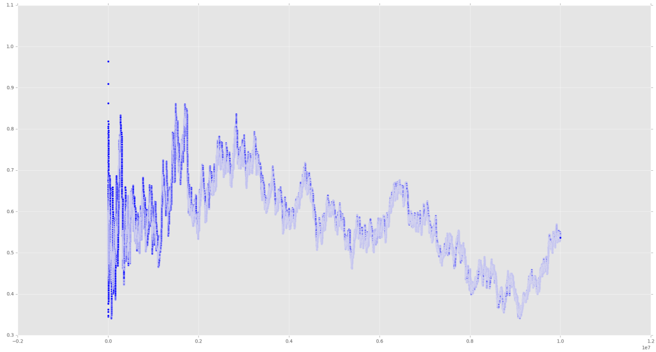
Theorem (Akiyama–Tanigawa)

Akiyama–Tanigawa conjecture $\Rightarrow$ *Riemann hypothesis for* E.

Theorem (Mazur)

Akiyama–Tanigawa conjecture $\Rightarrow$ *Riemann hypothesis for* $\operatorname{sym}^k E$

Computational evidence



$$\sqrt{\pi(N)} \cdot \mathrm{D}_N \text{ for } y^2 + y = x^3 - x,\ N \leqslant 10^7.$$

Related results

Theorem (Bucar–Kedlaya). Assume analytic continuation of $L(\mathrm{sym}^k E, s)$, GRH, and functional equation for all $k \geqslant 1$. Then $\mathrm{D}_N \ll N^{-\frac{1}{4}+\epsilon}$.

Theorem (Rouse–Thorner). Same result (with explicit constants) for a modular form of arbitrary weight .

Theorem (Niederreiter). Let $E_{/F}$, where F is a function field. Then $\mathrm{D}_N \ll N^{-\frac{1}{4}+\epsilon}$.

Theorem (Rosengarten). For $G = \mathrm{ST}(M)$ semisimple, GRH $\Rightarrow \mathrm{D}_N \ll N^{-\alpha_G+\epsilon}$, where $\alpha_G \to 0$ as $\mathrm{rk}\, G \to \infty$.

Common ingredient. Erdös–Turán inequality: from a bound on $\left|\sum_{p\leqslant N} \mathrm{tr}\, \rho(x_p)\right|$ to a bound on D_N.

Context

Question. Is the Sato–Tate conjecture a Galois-theoretic result?

Theorem (Pande)
Let $\epsilon > 0$. Then there exists an infinitely ramified representation $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that $\theta_p \in B_\epsilon(\pi/2)$ for a density one set of primes.

Theorem (Khare–Rajan)
Any $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ is ramified at a density zero set of primes.

Question (Serre). Can you control the growth of $\pi_{\mathrm{ram}(\rho)}(x)$?

Answer (Khare–Larsen–Ramakrishna). No!

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the growth of $\pi_{\mathrm{ram}(\rho)}(x)$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Answer. Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Discrepancy

Definition

Let $\{\theta_p\}$ be a sequence in $[0,\pi]$, μ a measure on $[0,\pi]$. The *discrepancy* is

$$\mathrm{D}_N(\{\theta_p\},\mu) = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta)\,\mathrm{d}\mu(\theta) \right|.$$

Fact. $\{\theta_p\}$ are μ-equidistributed if and only if $D_N \to 0$.

Fact. $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

Definition

For $k \geqslant 1$,

$$L(\operatorname{sym}^k \rho, s) = \prod_p \det\left(1 - \operatorname{sym}^k \left(\begin{smallmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{smallmatrix}\right) p^{-s}\right)^{-1}$$

Definition

For $f \colon [0, \pi] \to \mathbf{C}$ of bounded variation with $\mu(f) = 0$,

$$L_f(s) = \prod_p \left(1 - f(\theta_p) p^{-s}\right)^{-1}$$

Example (Ramakrishna). $L_{\mathrm{sgn}}(s) = \prod_p \left(1 - \operatorname{sgn}(a_p) p^{-s}\right)^{-1}$.

Dirichlet series—basic facts

Theorem (M.)

If $\left|\sum_{p\leqslant N} f(\theta_p)\right| \ll N^{\alpha+\epsilon}$*, then* $L_f(s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$*.*

Corollary

If $D_N \ll N^{\alpha-1+\epsilon}$*, then* $L_f(s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$*.*

Definition

$U_k(\theta) = \frac{\sin((k+1)\theta)}{\sin\theta} = \operatorname{tr}\operatorname{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix}.$

Theorem

If $\left|\sum_{p\leqslant N} U_k(\theta_p)\right| \ll N^{\alpha+\epsilon}$*, then* $L(\operatorname{sym}^k \rho, s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$*.*

Main theorem

Ingredients

1. Fix a rational prime $l \geqslant 5$.
2. Fix an odd, absolutely irreducible, weight 2 representation $\bar{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$. ρ will be a lift of $\bar{\rho}$.
3. Fix a function $h\colon \mathbf{R}^+ \to \mathbf{R}_{\geqslant 1}$ which increases slowly to infinity. We will have $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$.
4. Fix an absolutely continuous probability measure μ on $[0, \pi]$, with probability density function $f(\theta) \ll \sin\theta$. The angles $\{\theta_p\}$ will be μ-equidistributed.
5. Fix $\alpha \in \left(0, \frac{1}{3}\right)$. The discrepancy will decay like $\pi(N)^{-\alpha}$.

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the growth of $\pi_{\mathrm{ram}(\rho)}(x)$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Main theorem

Theorem (M.)

Let l, $\bar{\rho}$, h, μ, and α be as above. Then there exists $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that

1. $\rho \equiv \bar{\rho} \pmod{l}$.
2. $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$. *(Yes to Q4.* $\log x$, $\log^{10^{10}} x$, $A^{-1}(x)$*)*
3. *For each unramified p, $a_p = \mathrm{tr}\,\rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.*
4. $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(N)^{-\alpha})$. *(Yes to Q1–Q3.)*
5. *If $(\theta \mapsto \pi - \theta)_* \mu = \mu$, then for each odd k, $L(\mathrm{sym}^k \rho, s)$ satisfies the Riemann hypothesis. (Yes to Q5.)*

Sketch of proof

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
2. Construct ρ with $\operatorname{tr} \rho(\mathrm{fr}_p)$ close to the choices from 1.
3. Control the growth $\pi_{\mathsf{ram}}(x)$.
4. Inductive process.
5. Riemann hypothesis for $L(\operatorname{sym}^k \rho, s)$, k odd.

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
2. Construct ρ with $\operatorname{tr} \rho(\mathrm{fr}_p)$ close to the choices from 1.
3. Control the growth $\pi_{\mathsf{ram}}(x)$.
4. Inductive process.
5. Riemann hypothesis for $L(\operatorname{sym}^k \rho, s)$, k odd.

Prescribing discrepancy decay

Theorem (M.)

If $\alpha \in \left(0, \frac{1}{3}\right)$, there exists a sequence $(x_2, x_3, x_5, \dots)$ in $[-1,1]$ such that $|D_N - \pi(N)^{-\alpha}| \ll \pi(N)^{-1}$.

(Can have $x_p = \frac{a_p}{2\sqrt{p}}$ for $a_p \in \mathbf{Z}$ satisfying the Hasse bound.)

Fact: Discrepancy is invariant under pushforward by cos and $\cos^{-1}$.

Idea: Construct ρ so that $\frac{a_p}{2\sqrt{p}} \approx x_p$.

Fact: If $(x_p^{(1)})$ is a sequence with $|x_p - x_p^{(1)}| \ll p^{-\frac{1}{2}+\epsilon}$, then $\mathrm{D}_N^{(1)} = \Theta(\pi(N)^{-\alpha})$.

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
2. Construct ρ with $\operatorname{tr}\rho(\mathrm{fr}_p)$ close to the choices from 1.
3. Control the growth $\pi_{\mathrm{ram}}(x)$.
4. Inductive process.
5. Riemann hypothesis for $L(\operatorname{sym}^k \rho, s)$, k odd.

Lifting Galois representations

Construct ρ as $\varprojlim \rho_n$, where $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/l^n)$.

For all n, require $\det \rho_n \equiv \kappa \pmod{l^n}$ (l-adic cyclotomic character)

Passage from ρ_n to ρ_{n+1} is governed by $\mathrm{H}^i(G_{\mathbf{Q},R_n}, \mathrm{Ad}^0\,\bar{\rho})$, $i = 1, 2$.

Theorem (Khare–Larsen–Ramakrishna)
Fix a finite set U of primes. Then there exists a finite set N of primes such that

$$\mathrm{H}^1(G_{\mathbf{Q},R_n \cup N}, \mathrm{Ad}^0\,\bar{\rho}) \xrightarrow{\sim} \prod_{p \in R_n} \mathrm{H}^1(G_{\mathbf{Q}_p}, \mathrm{Ad}^0\,\bar{\rho}) \times \prod_{p \in U} \mathrm{H}^1_{\mathrm{nr}}(G_{\mathbf{Q}_p}, \mathrm{Ad}^0\,\bar{\rho})$$

Corollary. Given $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/l^n)$, can choose $\operatorname{tr} \rho_{n+1}(\mathrm{fr}_p)$ for all p in a finite set. (Finitely many more ramified primes.)

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
2. Construct ρ with $\operatorname{tr}\rho(\mathrm{fr}_p)$ close to the choices from 1.
3. Control the growth $\pi_{\mathsf{ram}}(x)$.
4. Inductive process.
5. Riemann hypothesis for $L(\operatorname{sym}^k \rho, s)$, k odd.

Controlling ramified primes

Given $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/l^n)$ and choices of $\operatorname{tr}\rho_{n+1}(\mathrm{fr}_p)$ (mod l^{n+1}), need to add finite set N to R_n.

Each $p \in N$ is chosen from a positive-density set of primes.

So p can be arbitrarily large!

If $\pi_R(x) \leqslant h(x)\,(\forall x)$, can force this for $\pi_{R\cup N}(x)$.

$\pi_{\mathrm{ram}(\bar{\rho})}(x) \leqslant h(x)$ may not hold—scale h to make this true!

Fact: constant in $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$ only depends on $\bar{\rho}$.

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
2. Construct ρ with $\operatorname{tr}\rho(\mathrm{fr}_p)$ close to the choices from 1.
3. Control the growth $\pi_{\mathrm{ram}}(x)$.
4. Inductive process.
5. Riemann hypothesis for $L(\operatorname{sym}^k \rho, s)$, k odd.

Lifting Galois representations—first stage

Lift from $\mathbf{Z}/l$ to $\mathbf{Z}/l^2$.

Fix a large finite set U_1 of primes.

For $p \in U_1$, can choose $a_p \in \mathbf{Z}$ subject only to $|a_p| \leqslant 2\sqrt{p}$ and $a_p \equiv \operatorname{tr} \rho_1(\mathrm{fr}_p) \pmod{l}$.

We can ensure $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant \frac{l}{2\sqrt{p}}$.

For $N \leqslant \max U_1$, $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(N)^{-\alpha})$.

Make U_1 so large that for $p > \max U_1$, $l^2 < \log p$.

Main theorem

Theorem (M.)
Let l, $\bar{\rho}$, h, μ, and α be as above. Then there exists $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that

1. $\rho \equiv \bar{\rho} \pmod{l}$.
2. $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$.
3. *For each unramified p, $a_p = \operatorname{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.*
4. $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(N)^{-\alpha})$.
5. *If $(\theta \mapsto \pi - \theta)_* \mu = \mu$, then for each odd k, $L(\operatorname{sym}^k \rho, s)$ satisfies the Riemann hypothesis.*

Lifting Galois representations—inductive step

Lift from $\mathbf{Z}/l^n$ to $\mathbf{Z}/l^{n+1}$.

Have already chosen a_p for $p \in U_n$. (1–5 hold)

Fix a really huge $U_{n+1} \supset U_n$.

For $p \in U_{n+1} \smallsetminus U_n$, can choose $a_p \in \mathbf{Z}$ subject only to $|a_p| \leqslant 2\sqrt{p}$ and $a_p \equiv \operatorname{tr} \rho_n(\mathrm{fr}_p) \pmod{l^n}$.

We can ensure $\left|\frac{a_p}{2\sqrt{p}} - x_p\right| \leqslant \frac{l^n}{2\sqrt{p}}$. ($l^n \ll \log p$).

For $N \leqslant \max U_{n+1}$, $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(N)^{-\alpha})$.

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
2. Construct ρ with $\operatorname{tr} \rho(\mathrm{fr}_p)$ close to the choices from 1.
3. Control the growth $\pi_{\mathrm{ram}}(x)$.
4. Inductive process.
5. Riemann hypothesis for $L(\operatorname{sym}^k \rho, s)$, k odd.

Riemann hypothesis

How can we make $L(\operatorname{sym}^k \rho, s)$, k odd, satisfy the Riemann hypothesis?

$\left|\sum_{p \leqslant N} U_k(\theta_p)\right| \ll N^{\frac{1}{2}+\epsilon}$ implies RH for $L(\operatorname{sym}^k \rho, s)$.

When k is odd, $U_k(\pi - \theta) = -U_k(\theta)$.

Enumerate the primes $p_1 = 2, q_1 = 3, p_2 = 5, q_2 = 7, \ldots$

If $\theta_q \approx \pi - \theta_p$, then $U_k(\theta_q) \approx -U_k(\theta_p)$ (within $p^{-\frac{1}{2}}$).

$$\begin{aligned}\left|\sum_{p \leqslant N} U_k(\theta_p)\right| &= \left|\sum_{p_i, q_i \leqslant N} \left(U_k(\theta_{p_i}) + U_k(\theta_{q_i})\right)\right| \\ &\ll \sum_{n \leqslant N} n^{-\frac{1}{2}} \\ &\ll N^{\frac{1}{2}}.\end{aligned}$$

Consequences

If $f \in C([0,\pi])$, $f \circ \cos^{-1}\colon [-1,1] \to \mathbf{C}$ is Lipschitz, and $f(\pi - \theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

For μ any "bump measure," there exits ρ with $\{\theta_p\}$ μ-equidistributed.

Can get equidistribution with respect to μ with non-continuous probability distribution functions.

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the growth of $\pi_{\mathrm{ram}(\rho)}(x)$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Conclusion & further questions

Conclusions.

1. The Akiyama–Tanigawa conjecture is stronger that it seems.
2. (Nearly) arbitrary "Sato–Tate distributions" are possible.
3. Pathological Galois representations can satisfy the Riemann hypothesis.

Questions.

1. Can we construct representations with $\mathrm{D}_N \ll N^{-\alpha+\epsilon}$, $\alpha > \frac{1}{2}$?
2. Is there a counterexample to GRH $\Rightarrow$ A–T for elliptic curves?
3. Can we prove anything about D_N for CM elliptic curves?

Thank you!