

Counterexamples related to the Sato–Tate conjecture

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Motivation and background

Discrepancy and Dirichlet series

Main theorem

Idea of the proof

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Use discrepancy (Kolmogorov–Smirnov statistic).

Akiyama–Tanigawa Conjecture

$$D_N = \sup_{x \in [0, \pi]} \left| \frac{1}{\pi(N)} \sum_{p \leq N} 1_{[0, x)}(\theta_p) - \int 1_{[0, x)}(\theta) \, dST(\theta) \right|.$$

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Theorem (Mazur)

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Theorem (Bucar–Kedlaya). Assume analytic continuation of $L(\mathrm{sym}^k E, s)$, GRH, and functional equation for all $k \geq 1$. Then $D_N \ll N^{-\frac{1}{4}+\epsilon}$.

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Common ingredient. Erdős–Turán–Koksma inequality: from a bound on $\left| \sum_{p \leq N} \text{tr } \rho(x_p) \right|$ to a bound on D_N .

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Let $\epsilon > 0$. Then there exists an infinitely ramified representation $\rho: G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{Z}_l)$ such that $\theta_p \in B_\epsilon(\pi/2)$ for a density one set of primes.

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Answer (Khare–Larsen–Ramakrishna). No!

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Answer. Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Definition

Let $\{\theta_p\}$ be a sequence in $[0, \pi]$, μ a measure on $[0, \pi]$. The *discrepancy* is

$$D_N(\{\theta_p\}, \mu) = \sup_{x \in [0, \pi]} \left| \frac{1}{\pi(N)} \sum_{p \leq N} 1_{[0, x)}(\theta_p) - \int 1_{[0, x)}(\theta) d\mu(\theta) \right|.$$

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Fact. $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

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For $k \geq 1$,

$$L(\mathrm{sym}^k \rho, s) = \prod_p \det \left(1 - \mathrm{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix} p^{-s} \right)^{-1}$$

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Example (Ramakrishna). $L_{\text{sgn}}(s) = \prod_p (1 - \text{sgn}(a_p) p^{-s})^{-1}$.

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If $\left| \sum_{p \leq N} f(\theta_p) \right| \ll N^{\alpha+\epsilon}$, then $L_f(s)$ admits a nonvanishing analytic continuation to $\Re > \alpha$.

Dirichlet series—basic facts

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Main theorem

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5. Fix $\alpha \in (0, \frac{1}{3})$. The discrepancy will decay like $\pi(N)^{-\alpha}$.

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Idea of the proof

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Fix a finite set U of primes. Then there exists a finite set N of primes such that

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Corollary. Given $\rho_n: G_{\mathbf{Q}, R_n} \rightarrow \mathrm{GL}_2(\mathbf{Z}/l^n)$, can choose $\mathrm{tr} \rho_{n+1}(\mathrm{fr}_p)$ for all p in a finite set.

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Controlling ramified primes

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Fact: constant in $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$ only depends on $\bar{\rho}$.

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Make U_1 so large that for $p > \max U_1$, $l^2 < \log p$.

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If $f \in C([0, \pi])$, $f \circ \cos^{-1}: [-1, 1] \rightarrow \mathbf{C}$ is Lipschitz, and $f(\pi - \theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

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Can get equidistribution with respect to μ with non-continuous probability distribution functions.

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