Counterexamples related to the Sato–Tate conjecture for CM abelian varieties*

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1 Introduction and motivation

Let $K/\mathbf{Q}$ be a finite Galois extension, $A_{/K}$ a g-dimensional abelian variety. Fix a rational prime l; then the l-adic Tate module of A gives a representation $\rho_l\colon G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_{2g}(\mathbf{Q}_l)$. The image actually lies in the subgroup $\mathrm{GSp}_{2g}(\mathbf{Q}_l)$ preserving the Weil pairing, but we won't worry about that. If we write $F = \mathrm{End}_K(A)_{\mathbf{Q}}$, then since ρ_l commutes with F, the representation actually takes values in $\mathrm{GL}_{2g/[F:\mathbf{Q}]}(F\otimes\mathbf{Q}_l)$. We are interested in the extreme case, when $[F:\mathbf{Q}] = 2g$. When this occurs, we say that A has complex multiplication defined over K. Write $\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}$ for the algebraic group whose functor of points is, for any $\mathbf{Q}$-algebra R, given by $(R\otimes\mathbf{Q}_l)^\times$. The representation ρ_l is a map $\rho_l\colon G_{\mathbf{Q}} \to (\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}})(\mathbf{Q}_l)$. The *motivic Galois group* of A is a $\mathbf{Q}$-subgroup $G_A \subset \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}$ such that for all l, $\overline{\mathrm{im}(\rho_l)}^{\mathrm{Zar}} = G_A(\mathbf{Q}_l)$. It has a canonical subgroup $G_A^1 = G_A^{\mathrm{N}_{F/\mathbf{Q}}=1}$, which we will not motivate here. There is a direct description of G_A. Let $\det_{\mathfrak{a}}\colon \mathrm{R}_{K/\mathbf{Q}} \to \mathrm{R}_{F/\mathbf{Q}}$ be induced by the determinant of the K-action on $\mathfrak{a} = \mathrm{Lie}(A)$, viewed as an F-vector space. Then $G_A = \mathrm{im}(\det_{\mathfrak{a}})$. The *Sato–Tate group* of A is the maximal compact subgroup of the torus $G_A^1(\mathbf{C})$. So $\mathrm{ST}(A) = (\mathbf{R}/\mathbf{Z})^d$ for some $1 \leqslant d \leqslant g$. If A has good reduction at $\mathfrak{p} \nmid l$, then $\rho_l(\mathrm{fr}_{\mathfrak{p}})$ actually lives in $F^\times$ and is independent of l. Write $\pi_{\mathfrak{p}} \in F^\times$ for this quantity; it is a $\mathfrak{p}$-Weil number of weight 1, i.e. $|\sigma(\pi_{\mathfrak{p}})| = \mathrm{N}(\mathfrak{p})^{1/2}$ for all $\sigma\colon F \hookrightarrow \mathbf{C}$. Even better, Shimura–Taniyama–Weil have constructed a continuous homomorphism $\varepsilon\colon \mathbf{A}_K^\times \to F^\times$ which agrees with $\det_{\mathfrak{a}}$ on $K^\times \subset \mathbf{A}_K^\times$, and for almost all $\mathfrak{p}$, sends a uniformizer $\varpi_{\mathfrak{p}}$ for $\mathfrak{p}$ to the element $\pi_{\mathfrak{p}}$.

For any $\sigma\colon F \hookrightarrow \mathbf{C}$, let $\chi_\sigma\colon \mathbf{A}_K^\times/K^\times \to \mathbf{C}^\times$ be the quasicharacter $\chi_\sigma(x) = \sigma(\varepsilon(x)\psi(x_\infty)^{-1})$. Here, $\varepsilon(x)\psi(x_\infty)^{-1} \in (F\otimes\mathbf{R})^\times$, and we write σ for the

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map $(F\otimes \mathbf{R})^\times \to \mathbf{C}^\times$ induced by σ. If we write also σ for the corresponding character of $\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m}$, then there is equality

$$L(\sigma_*\rho_l, s) = L(s, \chi_\sigma).$$

Given any $r = \sum m_\sigma \sigma \in \mathrm{X}^*(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m})$, put

, and $\theta_\mathfrak{p} = \frac{\rho_l(\mathrm{fr}_\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{1/2}}$ lies in $\mathrm{ST}(A)$. The *Sato–Tate conjecture* for A tells us that the $\theta_\mathfrak{p}$ are equidistributed in $\mathrm{ST}(A)$, i.e. for all $f\in C(\mathrm{ST}(A))$, we have

$$\int f(x)\,\mathrm{d}x = \lim_{x\to\infty} \frac{1}{\pi_K(x)} \sum_{\mathrm{N}(\mathfrak{p})\leqslant x} f(\theta_\mathfrak{p})$$

Serre outlined a way to prove the Sato–Tate conjecture. Note first that any character of $\mathrm{ST}(A)$ is induced by an algebraic character of G_A. Given $r\in \mathrm{X}^*(G_A)$, we associate $L(r_*\rho_l, s)$ with the composite Galois representation $r\circ\rho_l\colon G_\mathbf{Q}\to \overline{\mathbf{Q}_l}^\times$. To prove the Sato–Tate conjecture for A, it suffices to show that for all f, the function $L(r_*\rho_l, s)$ admits non-vanishing meromorphic continuation past $\Re = 1$, with at most a simple pole at $s=1$. We will see later how this is done. For our purposes, note that it also suffices to show that for all $r\in\mathrm{X}^*(G_A)$ which induce a nontrivial character of $\mathrm{ST}(A)$, a bound of the form

$$\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} r(\theta_\mathfrak{p})\right| = o\left(\pi_K(x)\right).$$

If we can replace $o(\pi_K(x))$ with $x^{-\frac{1}{2}+\epsilon}$, we will have established the Riemann hypothesis for $L(r_*\rho_l, s)$.

Unitary representations of $\mathrm{ST}(A)$ are just characters, and basic representation theory tells us that all such representations are induced by an (algebraic) character of G_A defined over $\overline{\mathbf{Q}}$. For $r\in\mathrm{X}^*(G_A)$, there is an L-function $L(r_*\rho_l, s)$ coming from the composite Galois representation $r\circ\rho_l\colon G_\mathbf{Q}\to G_A(\mathbf{Q}_l)\to \overline{\mathbf{Q}_l}^\times$. The Sato–Tate conjecture for A says that all $L(r_*\rho_l, s)$ have non-vanishing analytic continuation past $\Re = 1$, and the Generalized Riemann hypothesis for A says that all $L(r_*\rho_l, s)$ satisfy the Riemann hypothesis.

Choose an isomorphism $(\mathbf{R}/\mathbf{Z})^d\simeq \mathrm{ST}(A)$, and put

$$\mathrm{D}_x(A) = \sup_{t\in[0,1]^d}\left|\frac{1}{\pi_K(x)}\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} 1_{[0,t)}(\theta_\mathfrak{p}) - \int 1_{[0,t)}\right|.$$

Akiyama and Tanigawa conjectured that for non-CM elliptic curves, $\mathrm{D}_x(E)\ll x^{-\frac{1}{2}+\epsilon}$. We call the "Akiyama–Tanigawa conjecture" for A the discrepancy decays like $\mathrm{D}_x(A)\ll x^{-\frac{1}{2}+\epsilon}$. Via the Koksma–Hlawka inequality, the Akiyama–Tanigawa conjecture implies that for all bounded-variation functions f on

ST(A), the estimate

$$\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} f(\theta_{\mathfrak{p}})\right| \ll \mathrm{Var}(f) x^{\frac{1}{2}+\epsilon}.$$

For $r \in \mathrm{X}^*(G_A)$, this estimate implies the Riemann hypothesis for the L-function $L(r_*\rho_l, s)$.

Analogy with Artin L-functions (go into detail here!) seems to suggest that if all $L(r_*\rho_l, s)$ satisfy the Riemann Hypothesis, then the Akiyama–Tanigawa conjecture for A holds. We'll show: this converse is false, in a limited sense.

2 Diophantine approximation

Let $x \in [0,1]$ be irrational. It is well known that the sequence ($x \mod 1, 2x \mod 1, 3x \mod 1, \dots$) is equidistributed in $[0,1]$. What is less well known is that the rate of convergence of empirical measures from this sequence to the uniform measure is governed by the irrationality measure of x. The irrationality measure $\mu(x)$ is the supremum of the set of $w \geqslant 1$ such that there are infinitely many p/q with $\left|\mu - \frac{p}{q}\right| \leqslant q^{-w}$. Let's generalize this to higher-dimensional space. If $\vec{x} \in \mathbf{R}^d$, let $\omega_0(\vec{x})$ (resp. $\omega_{d-1}(\vec{x})$) be the supremum of the set of w such that there exist infinitely many $(n, \vec{m}) \in \mathbf{Z} \times \mathbf{Z}^d$ such that

$$\begin{aligned} |n\vec{x} - \vec{m}|_\infty &\leqslant |(n,\vec{m})|_\infty^{-w} \qquad \text{(resp.} \\ |n + \langle \vec{m}, \vec{x}\rangle| &\leqslant |(n,\vec{m})|_\infty^{-w}). \end{aligned}$$

There are quantities $\omega_i(\vec{x})$, $i = 0, \dots, d-1$, defined in terms of approximation of $\vec{x}$ in terms of rational linear projective varieties of dimension i, but we do not need them.

A theorem of Jarník says that if $w \geqslant 1/d$, then there exists $\vec{x} \in \mathbf{R}^d$ such that $\omega_0(\vec{x}) = w$ and $\omega_{d-1}(\vec{x}) = dw + d - 1$. It is clear that if $d = 1$, then $\omega_0(x) + 1$ is the traditional irrationality measure of x. The key fact we need is that

$$\begin{aligned} \frac{1}{d(n\vec{x}, \mathbf{Z}^d)} &\ll |n|^{\omega_0(\vec{x})+\epsilon} \\ \frac{1}{d(\langle \vec{m}, \vec{x}\rangle, \mathbf{Z})} &\ll |\vec{m}|_\infty^{\omega_{d-1}(\vec{x})+\epsilon}. \end{aligned}$$

Moreover, if we let $\vec{x} = (\vec{x} \mod \mathbf{Z}^d, 2\vec{x} \mod \mathbf{Z}^d, \dots)$ in $(\mathbf{R}/\mathbf{Z})^d$, then

$$\begin{aligned} \mathrm{D}_N(\boldsymbol{x}) &\ll N^{-\frac{1}{\omega_{d-1}(\vec{x})}+\epsilon} \\ \mathrm{D}_N(\boldsymbol{x}) &= \Omega\left(N^{-\frac{d}{\omega_0(\vec{x})}-\epsilon}\right). \end{aligned}$$

In particular, this means that if $\omega_0(\vec{x}) = w$ and $\omega_{d-1}(\vec{x}) = dw + d - 1$, then $\mathrm{D}_N(\boldsymbol{x}) \ll N^{-\frac{1}{dw+d-1}+\epsilon}$ and $\mathrm{D}_N(\boldsymbol{x}) = \Omega(N^{-\frac{d}{w}-\epsilon})$. By letting w get very large, we can ensure that $\mathrm{D}_N(\boldsymbol{x})$ decays very slowly.

3 Fake Satake parameters

Given a sequence (z_p) of complex numbers, the corresponding Dirichlet series is

$$L(s) = \prod (1 - z_p p^{-s})^{-1},$$

with terms omitted when z_p is undefined. The Riemann Hypothesis for $L(s)$ follows from a bound of the form $|\sum_{p \leqslant x} z_p| \ll x^{\frac{1}{2}+\epsilon}$. Now suppose $z_p = f(x_p)$, for $f \in C^\infty((\mathbf{R}/\mathbf{Z})^d)$, and $x_{p_n} = n\vec{x} \mod \mathbf{Z}^d$. Then we need to bound sums of the form $\sum_{n \leqslant N} f(n\vec{x})$ for f smooth. Suppose $f(\vec{y}) = e^{2\pi i \langle \vec{m}, \vec{y} \rangle}$. Then

$$\left| \sum_{n \leqslant N} f(n\vec{x}) \right| = \left| \frac{\left(e^{2\pi i \langle \vec{m}, \vec{x} \rangle}\right)^{N+1} - e^{2\pi i \langle \vec{m}, \vec{x} \rangle}}{e^{2\pi i \langle \vec{m}, \vec{x} \rangle} - 1} \right| \leqslant \frac{2}{\left| e^{2\pi i \langle \vec{m}, \vec{x} \rangle} - 1 \right|}.$$

It's easy to check that

$$\frac{1}{\left| e^{2\pi i \langle \vec{m}, \vec{x} \rangle} - 1 \right|} \leqslant \frac{1}{d(\langle \vec{m}, \vec{x} \rangle, \mathbf{Z})},$$

so we get the bound $\left| \sum_{n \leqslant N} f(n\vec{x}) \right| \ll |\vec{m}|_\infty^{\omega_{d-1}(\vec{x})+\epsilon}$.

This tells us that the L-function $\prod(1 - f(x_p)p^{-s})^{-1}$ satisfies the Riemann Hypothesis, but the x_p have very slowly decaying discrepancy.