Compatible families of Galois representations

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Let K/k be an extension of global fields. We say that a continuous homomorphism $\rho : G_K \to \mathrm{GL}(k_v)$ is *rational at* w if $\rho(I_w) = 1$ and the polynomial

$$\Phi_{\rho,w} = \det(T \cdot 1 - \rho(\varphi_w)) \in k_v[T]$$

is actually in $K[T]$.

Now let $\{\rho_v : G_K \to \mathrm{GL}_n(k_v)\}$ be a collection of continuous homomorphisms, where v ranges over the places of k. We say that $\{\rho_v\}$ is a *(strictly) compatible system* if there is a finite set S of places of k such that

1. if $v \notin S$, then ρ_v is rational at w for all $w \nmid v$
2. if $u, v \notin S$, then for all $w \nmid u, v$, we have $\Phi_{\rho_u,w} = \Phi_{\rho_v,w}$

We will write $\rho = \rho_\bullet = \{\rho_v\}$ for such a family. We let C be the set of all strictly compatible systems. Note that if $\rho = \{\rho_v\} \in \mathsf{C}$, there is a well-defined positive integer $d = \dim \rho$ defined by $d = \dim \rho_v$ for every v.

We would like C to be a neutral Tannakian category, as this would allow us to define a group "$\pi_1(\mathsf{C})$" which would classify strictly compatible families of representations.

The problem is: *what is the correct notion of a morphism $f : \alpha \to \beta$ for $\alpha, \beta \in \mathsf{C}$?* Surely it involves a collection $\{f_v : \alpha_v \to \beta_v\}$ of G_K-linear maps. But we would also want some kind of "compatibility condition." If we allow the f_v to be arbitrary, then C does not have quotients (just let $f_v = 0$ for half of the v, and $f_v = 1$ for the other half). We will attempt to imitate Deligne's construction of mixed motives in [1, 1.4]. Let $\mathbb{A}^f = \mathbb{A}^f_k$ be the ring of finite adeles over k. Recall that elements of $\mathbb{A}^f$ are collections $(a_v) \in \prod k_v$ with $a_v \in \mathfrak{o}_v$ for all but finitely many v. Instead of a family $\{\rho_v : G_K \to \mathrm{GL}_n(k_v)\}$ we will look at a single (continuous) representation $\rho : G_K \to \mathrm{GL}_n(\mathbb{A}^f)$. Of course, this means we have to say what the topology on $\mathbb{A}^f$ is. Essentially, note that (as a set) $\mathbb{A}^f = \varinjlim \mathbb{A}^f(S)$, where S ranges over all finite sets of places of k and $\mathbb{A}^f(S) = \prod_{v \in S} k_v \times \prod_{v \notin S} \mathfrak{o}_v$. We simply require that each $\mathbb{A}^f(S)$ be an open subring of $\mathbb{A}^f$. We now need to decide what it means for $\rho : G_K \to \mathrm{GL}_n(\mathbb{A}^f)$ to be "unramified" or "compatible."

First, we introduce a new ring $\mathbb{B}$. As a set, $\mathbb{B}$ consists of all sequences $(a_v) \in \mathbb{A}^f_k$ such that there is some finite set S of places of k, and $a \in K$ such that $a_v = a$ for all $v \notin S$. We assume K/k is Galois – that makes the field "$K \cap k_v$" well-defined. We give $\mathbb{B} \subset \mathbb{A}^f$ the subspace topology. It is a corollary of the strong approximation theorem [2, III.1, ex.1] that $\mathbb{B}$ is dense in $\mathbb{A}^f$. I would like to say that a "compatible family of representations" is a continuous representation $\rho : G_K \to \mathrm{GL}_n(\mathbb{A}^f)$ with the characteristic polynomial of $\rho(\varphi_w)$ an element of $\mathbb{B}[T]$ for all but finitely many w. This, however, includes no information about ramification. So, we will keep that information separate, i.e.

A *strictly compatible family* is a continuous representation $\rho : G_K \to \mathrm{GL}_n(\mathbb{A}^f_k)$ such that there exists a finite set S with

1. for $v \notin S$, ρ_v is unramified at w for all $w \nmid v$
2. for $u, v \notin S$ and $w \nmid u, v$, $\Phi_{\rho_u,w} = \Phi_{\rho_v,w}$.

We may have to stipulate that if $v \in S$ and u has the same residue characteristic as v, then $u \in S$.

The ring $\mathbb{B}$ may still be useful – this time for defining morphisms in C. A preliminary definition is follows: a morphism $f : \rho \to \eta$ is a G_k-linear map such that if $n = \dim \rho$, $m = \dim \eta$, we have

$$f \in M_{m \times n}(\mathbb{B})$$

It is easy to check that if $\rho, \eta \in \mathsf{C}$, then $\rho \oplus \eta \in \mathsf{C}$. Naively, we will define ρ^* in the usual way – the only question is whether the characteristic polynomials of Frobenii behave well. Suppose A is an invertible matrix. Then the characteristic polynomial of A is $\prod(T - \lambda)$ where λ runs over the eigenvalues of A (with multiplicity). If $Ax = \lambda x$, then $A^{-1}x = \lambda^{-1}x$, so the characteristic polynomial of A^{-1} is $\prod(X - \lambda^{-1})$. Essentially by the fundamental theorem of symmetric polynomials, this will be expressible in terms of the coefficients of the characteristic polynomial of A. Thus if $\Phi_{\rho_v, w}$ is K-rational, so will be $\Phi_{\rho_v^*, w}$. Moreover, $\Phi_{\rho_v^*, w}$ only depends on the coefficients of $\Phi_{\rho_v, w}$, so $\rho^* \in \mathsf{C}$ whenever $\rho \in \mathsf{C}$. Finally, similar considerations using symmetric polynomials show that if $\rho, \eta \in \mathsf{C}$, so is $\rho \otimes \eta$. We can define $\hom(\rho, \eta) = \rho^* \otimes \eta$, but I think that this forces us to not have quotients in C.

References

[1] Deligne, P. *Le groupe fondamental de la droite projective moins trois points.*

[2] Neukirch, J. *Algebraic number theory.*