Complexification

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So far we have only considered real Lie groups, even though some of our examples, like $\mathrm{GL}(n, \mathbf{C})$, seem like they should be "complex" objects in some way. So, without further ado, we make a definition.

Definition 1. A *complex Lie group* is a group which is also a complex manifold, such that all the group operations are analytic maps.

With this definition, it is easy to see that $\mathrm{GL}(n, \mathbf{C})$ is a complex Lie group, as is $\mathrm{Sp}(2n, \mathbf{C})$. Clearly, every complex Lie group can be made into a real Lie group in a trivial way, by forgetting the complex structure. That is, we have a kind of map (a functor, for those in the know)

$$(-)^{\mathrm{real}} \colon \{\text{complex Lie groups}\} \to \{\text{real Lie groups}\}.$$

It would be nice for there to be a kind of "inverse" map in the opposite direction (an adjoint functor, for those who care). That is, given a real Lie group G, we would like there to be a complex Lie group $G_{\mathbf{C}}$, together with a homomorphism $G \to G_{\mathbf{C}}$, such that for any complex Lie group H and a homomorphism $f \colon G \to H$, there exists a unique extension $\widetilde{f} \colon G_{\mathbf{C}} \to H$. We call $G_{\mathbf{C}}$ the *complexification* of G.

(Brief interlude on universal properties and uniqueness.)

Theorem 1. *Let $G \to H$ be a map from a real Lie group to a complex one, both connected, that induces an isomorphism $\mathfrak{g}_{\mathbf{C}} \to \mathfrak{h}$. If the induced map $\pi_1(G) \to \pi_1(H)$ is an isomorphism, then H is a complexification of G.*

Proof. We only need to check that H satisfies the universal property. Let H' be an arbitrary complex Lie group together with a homomorphism $f \colon G \to H'$. This gives us a real Lie algebra map $\mathrm{d}f \colon \mathfrak{g} \to \mathfrak{h}'$, which uniquely extends to a complex Lie algebra map $(\mathrm{d}f)_{\mathbf{C}} \colon \mathfrak{g}_{\mathbf{C}} \to \mathfrak{h}'$. Equivalently, $\mathrm{d}f$ extends uniquely to a complex Lie algebra map $\widetilde{\mathrm{d}f} \colon \mathfrak{h} \to \mathfrak{h}'$, and thus to a Lie group homomorphism $\widetilde{f} \colon \widetilde{H} \to H'$, where $\widetilde{H}$ is the universal cover of H. Since $\widetilde{f}$ comes from $f \colon G \to H$, it is $\pi_1(H)$-equivariant, and thus descends to a map $\widetilde{f} \colon H \to H'$. Since $\mathrm{d}\widetilde{f} = \widetilde{\mathrm{d}f}$, a $\mathbf{C}$-linear map, $\widetilde{f}$ is in fact complex analytic. $\square$

As an example, since $\pi_1(\mathrm{SL}(n,\mathbf{C}))=1$, the inclusion $\mathrm{SL}(n,\mathbf{R})\hookrightarrow\mathrm{SL}(n,\mathbf{C})$ makes $\mathrm{SL}(n,\mathbf{C})$ the complexification of $\mathrm{SL}(n,\mathbf{R})$.

Recall $\mathrm{U}(n)=\{g\in\mathrm{GL}(n,\mathbf{C}) : {}^{\mathrm{t}}g\cdot\overline{g}=1\}$, so its Lie algebra $\mathfrak{u}(n)=\{x\in\mathfrak{gl}(n,\mathbf{C}) : {}^{\mathrm{t}}x+\overline{x}=0\}$. Thus if $x\in i\mathfrak{u}(n)$, ${}^{\mathrm{t}}x=\overline{x}$, i.e. $i\mathfrak{u}(n)$ is exactly the space of Hermitian matrices.

Theorem 2. *Let $P=\exp(i\mathfrak{u}(n))$. Then $\exp\colon i\mathfrak{u}(n)\to P$ is a homeomorphism.*

Proof. (Well known. Otherwise, conjugate, ..., unitary matrix with blocks, ... commutes.) $\square$

Theorem 3. *Let K be a compact connected Lie group. Then the complexification of K exists.*

Proof. We know there is an embedding $K\subset\mathrm{U}(n)$ for some $n\geqslant 1$. Put $\mathfrak{k}=\mathrm{Lie}(K)$, $P=\exp(i\mathfrak{k})$, and $G=K\cdot P\subset\mathrm{GL}(n,\mathbf{C})$. Since G is the product of a closed subgroup and a compact subgroup of $\mathrm{GL}(n,\mathbf{C})$, general nonsense tells us that G is closed in $\mathrm{GL}(n,\mathbf{C})$. Moreover, $\mathrm{Lie}(G)=\mathrm{Lie}(K)+\mathrm{Lie}(P)=\mathfrak{k}+i\mathfrak{k}=\mathfrak{k}_{\mathbf{C}}$, a complex vector space, so we can use the exponential map $\mathfrak{k}_{\mathbf{C}}\to G$ to give G a complex structure. Since P is contractible, $K\to G$ induces an isomorphism $\pi_1(K)\to\pi_1(G)$. Thus $G=K_{\mathbf{C}}$. $\square$