Synthetic differential geometry via examples

Daniel Miller

August 7, 2016

1 The main examples

Synthetic differential geometry (SDG) lives inside a topos $\mathcal{E}$, where we have fixed a commutative ring object R. We will construct two examples, denoted $\mathcal{E}_{\mathrm{sm}}$ and $\mathcal{E}_{\mathrm{alg}}$, which represent "smooth" and "algebraic" theories. Their construction works in parallel, starting with a Lawvere theory $\mathbb{T}$. The theory $\mathbb{T}_{\mathrm{sm}}$ has as objects all finite-dimensional $\mathbf{R}$-vector spaces, with morphisms all smooth maps. The theory $\mathbb{T}_{\mathrm{alg}}$ has as objects all finite free k-modules (with polynomial maps) for some fixed base ring k. That is, for E, F finite free k-modules, we put $\hom_{\mathbb{T}_{\mathrm{alg}}}(E, F) = \mathrm{S}(E^\vee) \otimes F$.

The category $\mathsf{Alg}_{\mathbb{T}}$ of $\mathbb{T}$-algebras is, by definition, the category of product-preserving functors $\mathbb{T} \to \mathsf{Set}$. In particular, if $W \in \mathbb{T}$ is a commutative ring object (we shall call such objects "Weil algebras") then $\hom(W, -)\colon \mathbb{T} \to \mathsf{Set}$ is a commutative ring object in $\mathsf{Alg}_{\mathbb{T}}$. Write $\mathsf{Wei}_{\mathbb{T}}$ for the category of commutative ring objects in $\mathsf{Alg}_{\mathbb{T}}$.

When $\mathbb{T} = \mathbb{T}_{\mathrm{sm}}$, write $C^\infty\text{-}\mathsf{Alg} = \mathsf{Alg}_{\mathbb{T}_{\mathrm{sm}}}$; we call objects of $C^\infty\text{-}\mathsf{Alg}$ "smooth algebras." Write $C^\infty\colon \mathsf{Man}^\circ \to \mathsf{Alg}_{\mathbb{T}_{\mathrm{sm}}}$ for the functor $C^\infty(M)(V) = \hom_{\mathrm{sm}}(M, V)$. It is known that C^∞ is fully faithful. In particular, if $V \in \mathbb{T}_{\mathrm{sm}}$, then $C^\infty(V) \in C^\infty\text{-}\mathsf{Alg}$, and $C^\infty(V)(U) = \hom_{\mathrm{sm}}(V, U)$.

For $\mathbb{T} = \mathbb{T}_{\mathrm{alg}}$, the category $\mathsf{Alg}_{\mathbb{T}_{\mathrm{alg}}} = \mathsf{Alg}_k$, the category of "honest" k-algebras. Namely, if A is a k-algebra, consider the functor $E \mapsto A \otimes_k E$ (this is a functor because polynomial maps of finite free modules still make sense after base change). It is easy to check that all objects of $\mathsf{Alg}_{\mathbb{T}_{\mathrm{alg}}}$ are of this form.

If $\mathbb{T}$ is a Lawvere theory, put $\mathsf{Aff}_{\mathbb{T}} = \mathsf{Alg}_{\mathbb{T}}^\circ$, and write $\mathrm{Spec}\colon \mathsf{Alg}_{\mathbb{T}}^\circ \to \mathsf{Aff}_{\mathbb{T}}$ for the obvious equivalence. Put $\mathcal{E}_{\mathbb{T}} = \widehat{\mathsf{Aff}_{\mathbb{T}}}$. For $A \in \mathsf{Alg}_{\mathbb{T}}$, we also write $\mathrm{Spec}(A)$ for the presheaf $\mathrm{Spec}(B) \mapsto \hom_{\mathsf{Alg}_{\mathbb{T}}}(A, B)$.