Foundations of deformation theory

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1 Generalities

If C is an arbitrary categorys, we will often identify objects in C with their functor of points. In other words, we write

$$X(S) = h_X(S) = \hom(S, X)$$

for $X, S \in \mathsf{C}$.

Theorem 1.1. *If C is a category with finite limits, G is a group object in C, and Γ is a finite group, then the functor $X \mapsto \hom_{\mathsf{Grp}}(\Gamma, GX)$ is representable.*

Proof. We will define the representing object as an equalizer. Consider the products $\prod_\Gamma G$ and $\prod_{\Gamma\times\Gamma} G$; these come with projection maps $\pi_\sigma : \prod_\Gamma G \to G$ and $\pi_{\sigma,\tau} : \prod_{\Gamma\times\Gamma} G \to G$ for $\sigma, \tau \in \Gamma$. We will write $m : G \times G \to G$ for the multiplication morphism. Define $f : \prod_\Gamma G \to \prod_{\Gamma\times\Gamma} G$ by $f_{\sigma,\tau} = \pi_{\sigma\tau}$. Similarly, define $g : \prod_\Gamma G \to \prod_{\Gamma\times\Gamma} G$ by $g_{\sigma,\tau} = m \circ (\pi_\sigma \times \pi_\tau)$. Note that in terms of functors of points, $h_{\prod_I G}(X) = \hom_{\mathsf{Set}}(I, GX)$ for any finite set I. As maps $\hom_{\mathsf{Set}}(\Gamma, GX) \to \hom_{\mathsf{Set}}(\Gamma\times\Gamma, GX)$, f and g send $s : \Gamma \to GX$ to $(\sigma,\tau) \mapsto s(\sigma\tau)$ and $(\sigma,\tau) \mapsto s(\sigma)s(\tau)$. One easily sees that our desired representing object is the equalizer of the diagram

$$\prod_\Gamma G \underset{g}{\overset{f}{\rightrightarrows}} \prod_{\Gamma\times\Gamma} G.$$

□

If G and Γ are as in the theorem, we will write G^Γ for the object representing $X \mapsto \hom_{\mathsf{Grp}}(\Gamma, GX)$.

Definition 1.2. *A category C is* cofiltered *if for any finite category I and any diagram $F : I \to \mathsf{C}$, there is an object $c \in \mathsf{C}$ that admits a natural transformation $\alpha : \Delta_c \to F$.*

Here, as is common, Δ_c denotes the constant functor $I \to \mathsf{C}$ given by $i \mapsto c$, with all morphisms going to 1_c. Now let C be an arbitrary category. We will write $\widehat{\mathsf{C}}$ for the *pro-category* of C. An object in C is a functor $I \to \mathsf{C}$ for some small cofiltered category I. We will formally write $\varprojlim_{i\in I} c_i$ for such an object. If $\varprojlim_{j\in J} d_j$ is another object in $\widehat{\mathsf{C}}$, then we define

$$\hom_{\widehat{\mathsf{C}}}\left(\varprojlim c_i, \varprojlim d_j\right) = \varprojlim_j \varinjlim_i \hom_{\mathsf{C}}(c_i, d_j)$$

Our main example of a pro-category is $\widehat{\mathsf{fGrp}}$, the category of profinite groups (here fGrp is the category of finite groups). It is well-known that $\widehat{\mathsf{fGrp}}$ is equivalent to the category of compact hausdorff totally disconnected groups with continuous homomorphisms.

Let C, C' be categories with finite limits. One says a functor $F : \mathsf{C} \to \mathsf{C}'$ is *left exact* if F commutes with all finite limits. Note that a functor $F : \mathsf{C} \to \mathsf{Set}$ commuting with finite limits extends uniquely to a functor $F : \widehat{\mathsf{C}} \to \mathsf{Set}$ via $F(\varprojlim c_i) = \varprojlim F(c_i)$. One says that $F : \mathsf{C} \to \mathsf{Set}$ is *pro-representable* if $F : \widehat{\mathsf{C}} \to \mathsf{Set}$ is representable.

Theorem 1.3 (prop 3.1 of [2]). *Let* C *be a category with finite limits. A functor* $F : \mathsf{C} \to \mathsf{Set}$ *is pro-representable if and only if* F *is left-exact.*

In fact, Grothendieck proves that if F is pro-representable, then $F = \varinjlim h_{X_i}$, where the X_i are indexed by a filtered poset I, and such that the maps $X_i \to X_j$ for $i \leqslant j$ are epimorphisms.

2 Categories of commutative rings

Unless explicitly said otherwise, all rings will be commutative and unital. We write Ring for the category of (commutative, unital) rings. Let lRing be the category of local rings and local ring homomorphisms.

Let $\mathcal{O}$ be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field κ. We let $\mathsf{Ar} = \mathsf{Ar}_{\mathcal{O}}$ be the category of local $\mathcal{O}$-algebras A that are artinian as $\mathcal{O}$-modules and such that the structure map $\mathcal{O} \to A$ induces an isomorphism $\kappa \xrightarrow{\sim} A/\mathfrak{m}_A$. By definition, the morphisms in Ar are homomorphisms of local $\mathcal{O}$-algebras. Let $\widehat{\mathsf{Ar}}$ be the pro-category of Ar. By definition, an object in $\widehat{\mathsf{Ar}}$ is a formal cofiltered inverse limit $\varprojlim A_\alpha$ where the A_α are in Ar. Just as with profinite groups, we can identify the formal inverse limit $\varprojlim A_\alpha$ with its actual inverse limit in the category of rings. As with all pro-categories, $\widehat{\mathsf{Ar}}$ admits cofiltered limits, and the inc. lusion functor $\mathsf{Ar} \hookrightarrow \widehat{\mathsf{Ar}}$ preserves finite limits.

3 The deformation functor

Let $\mathcal{O}$ be a complete local ring, and let G be a group scheme over $\mathcal{O}$. We are interested in topologizing the groups $G(A)$ for pro-artinian $\mathcal{O}$-algebras A. In fact, we simply note that G preserves limits, and write $A = \varprojlim A_i$, where each A_i is in Ar. We give $G(A) = \varprojlim G(A_i)$ the inverse limit topology, where each $G(A_i)$ is discrete. One can readily verify (**check this**) that if G is an affine algebraic group, this recovers the standard way of topologizing $G(A)$.

Let Γ be a profinite group, and suppose we have a continuous homomorphism $\eta : \Gamma \to G(\kappa)$. We are interested in lifts of η to homomorphisms $\rho : \Gamma \to G(A)$ for $A \in \widehat{\mathsf{Ar}}$.

4 Notes

The notation $\mathbb{V}(\mathcal{F})$ just means $Spec(S^\bullet(\mathcal{F}))$ for a quasi-coherent sheaf $\mathcal{F}$, where $S^\bullet$ denotes "take symmetric algebra."

Let C be pro-artinian $\mathcal{O}$-algebras, $\widehat{C}$ its pro-category. Let $C_{/\kappa}$ be artinian $\mathcal{O}$-algebras with $\mathcal{O}$-algebra maps $A \to \kappa$, and similarly for $\widehat{C_{/\kappa}}$. There is an obvious inclusion $C_{/\kappa} \to C$. Finally, let lC be artinian local $\mathcal{O}$-algebras with residue field κ. Note that lC is a full (!) subcategory of $C_{/\kappa}$. The functor $lC \to C_{/\kappa}$ has an adjoint, namely $(A \to \kappa) \mapsto A_{\ker(A \to \kappa)}$. In other words,

$$\hom_{C_{/\kappa}}(A, B) = \hom_{lC}(A_{\mathfrak{m}}, B)$$

for $A \in C_{/\kappa}$ and $B \in lC$. I am hoping that this extends to an adjunction between $\widehat{C_{/\kappa}}$ and $\widehat{lC}$. It might be worth reading SGA 3 or [2] to find out what kinds of limits and colimits C and the other categories have.

5 The relevant categories

Recall that a (commutative) ring A is *pseudocompact* if A has a basis $\{\mathfrak{a}_\alpha\}$ of neighborhoods of 0 such that each $\mathfrak{a}_\alpha$ is an ideal of finite colength – that is $A/\mathfrak{a}_\alpha$ has finite length as an A-module. A good source for pseudocompact rings is the first couple sections of [1, $\mathrm{VII_B}$]. The category $\mathsf{PC}(A)$ of pseudocompact A-algebras is just the pro-category of the category $\mathsf{Art}(A)$ of finite length A-algebras, and one defines a pseudocompact A-module in the obvious way. That is, a pseudocompact A-module is an filtered projective limit of topological A-modules of finite length.

Let A-alg be the category of A-algebras. The inclusion $\mathsf{PC}(A) \hookrightarrow A$-$\mathsf{alg}$ has a left adjoint, the "completion functor" which assigns to an A-algebra B the projective limit $\hat{B} = \varprojlim B/\mathfrak{b}$, where $\mathfrak{b}$ ranges over all ideals $\mathfrak{b} \subset B$ with $B/\mathfrak{b}$ of finite length over A.

Now let $\mathcal{O}$ be a pseudocompact local ring, and κ the residue field. The category $\mathsf{PC}(\mathcal{O})_\kappa$ consists of pseudocompact $\mathcal{O}$-algebras A together with a $\mathcal{O}$-algebra map $A \to \kappa$. Since

$$\begin{tikzcd} \mathcal{O} \arrow[r] \arrow[dr] & A \arrow[d] \\ & \kappa \end{tikzcd}$$

commutes, $A \to \kappa$ is surjective, so it picks out a maximal ideal $\mathfrak{m}$ of A. From [1, $\mathrm{VII_B}$ 0.1.1], we know that A is a direct product of local pseudocompact $\mathcal{O}$-algebras, and thus $\mathfrak{m}$ picks out one of those local rings with residue field κ.

The category $\mathsf{LPC}(\mathcal{O})_\kappa$ is the subcategory of $\mathsf{PC}(\mathcal{O})_\kappa$ consisting of *local* pseudocompact $\mathcal{O}$-algebras. The inclusion $\mathsf{LPC}(\mathcal{O})_\kappa \to \mathsf{PC}(\mathcal{O})_\kappa$ has a left adjoint. To $A \to \kappa$ in $\mathsf{PC}(\mathcal{O})_\kappa$, one assigns $A_\mathfrak{m} \to \kappa$, where $\mathfrak{m} = \ker(A \to \kappa)$.

Now we reverse arrows. Let $S = \mathrm{Spec}(\mathcal{O})$ and consider Aff_S, the category of affine schemes over S. The category Vaf_S is the opposite category to $\mathsf{PC}(A)$. We call objects of Vaf_S *formal schemes over* S. For a pseudocompact $\mathcal{O}$-algebra A, we denote by $\mathrm{Spf}(A)$ the corresponding formal S-scheme. The projection $\mathcal{O} \twoheadrightarrow \kappa$ corresponds to $s : \mathrm{Spf}(\kappa) \to \mathrm{Spf}(\mathcal{O})$, and we write Vaf_S^s for the category of "s-pointed formal schemes over S," that is commutative diagrams

$$\begin{tikzcd} s \arrow[r] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$

Finally, cVaf_S^s denotes the subcategory of Vaf_S^s consisting of connected formal schemes, i.e. Spf of local rings. To summarize, we have categories and functors

$$\mathsf{cVaf}_S^s \leftrightarrow \mathsf{Vaf}_S^s \to \mathsf{Vaf}_S \leftrightarrow \mathsf{Aff}_S$$

where $\leftrightarrow$ means the inclusion has a right adjoint.

6 Hom-functors

Let C be an arbitrary category enriched over topological spaces that admits finite products and arbitrary filtered inductive limits. If G is a group object in C and Γ is a profinite group, then one can prove (cf. my earlier notes) that the functor $X \mapsto \hom_{\mathsf{topGrp}}(\Gamma, G(X))$ is represented by an object we will denote by G^Γ. Note that G^Γ can be constructed directly.

7 Deformation functors

Suppose we start with a group object G in Aff_S, i.e. $G = \mathrm{GL}(n)$. One can check that completion $\mathsf{Aff}_S \to \mathsf{Vaf}_S$ commutes with finite products, so $\hat{G}$ is a group object in Vaf_S. Thus, from here on out, we will begin with a group object G in Vaf_S.

Given a group object G in Vaf_S and a profinite group Γ, by the previous section there is G^Γ in Vaf_S such that $G^\Gamma(X) = \hom_{\mathsf{tpGp}}(\Gamma, G(X))$. Let $s : \mathrm{Spf}(\kappa) \to \mathrm{Spf}(\mathcal{O})$, and suppose we have picked an s-valued point of G^Γ, i.e. $\bar{\rho} \in G^\Gamma(s) = \hom(\Gamma, G(\kappa))$. Write $D^\square_{\bar{\rho}} = (\bar{\rho} : s \to G^\Gamma)^\wedge$, i.e. $D^\square_{\bar{\rho}}$ is the connected component of $\bar{\rho}$ in G^Γ. I claim that $D^\square_{\bar{\rho}}$ is what one would expect from the notation, i.e. $D^\square_{\bar{\rho}}(A)$ is the set of continuous representations $\rho : \Gamma \to G(A)$ lifting $\bar{\rho}$. But this is easy, for by definition, for $X \in \mathsf{cVaf}^s_S$:

$$D^\square_{\bar{\rho}}(s \to X) = \hom(s \to X, (\bar{\rho} : s \to G^\Gamma)^\wedge) = \hom_{s,S}(X, G^\Gamma)$$

The following diagram commutes:

$$\begin{tikzcd} \hom(X, G^\Gamma) \arrow[r, "\sim"] \arrow[d] & \hom(\Gamma, G(X)) \arrow[d] \\ \hom(s, G^\Gamma) \arrow[r, "\sim"] & \hom(\Gamma, G(s)) \end{tikzcd}$$

Thus, if $f : X \to G^\Gamma$ corresponds with $\eta : \Gamma \to G(X)$, then its reduction $\bar{\eta} : \Gamma \to G(s)$ is equal to $f \circ s$, so $\bar{\eta} = \bar{\rho}$ iff $f \circ s = \bar{\rho}$, which occurs iff f respects the basepoint $\bar{\rho}$. The result follows.

Now let $\bar{e} : s \to S \to G$ be the special point of the identity section. Denote by $\hat{G}$ the completion $(\bar{e} \to G)^\wedge$. One checks that $\hat{G}(A) = \{g \in G(A) : \bar{g} = 1\}$, and so it makes sense to set $D_{\bar{\rho}} = D^\square_{\bar{\rho}}/\hat{G}$, where $\hat{G}$ acts on $D^\square_{\bar{\rho}}$ by conjugation (induced from the natural action of G on Γ^G by conjugation).

So, we have $\hat{G} \times D_{\bar{\rho}} \rightrightarrows D_{\bar{\rho}}$, and if the coequalizer exists, $D_{\bar{\rho}}$ is representable. The question now is under what generality we can mod out by group actions. Böckle cites theorem 1.4 of [1, $\mathrm{VII_B}$] to prove that under certain circumstances, $D^\square_{\bar{\rho}}/\hat{G}$ exists. Essentially, all he needs is for $\hat{G} \times D^\square_{\bar{\rho}} \to D^\square_{\bar{\rho}}$ to be an equivalence relation, with the projection "topologically flat" (I should find out what that means).

The first thing is that one can replace $\hat{G}$ with $\hat{G}/\hat{Z}$ in the quotient, e.g. $\mathrm{GL}(n)$ with $\mathrm{PGL}(n)$. I think that $\widehat{G/Z} = \hat{G}/\hat{Z}$, so we can restrict to the case when G and Z are varieties (because we should be able to go all the way back to Aff_S). If G/Z is smooth, things should work.

Perhaps if everything is affine, quotients always exist (?) In terms of rings, we have $\mathcal{O}_{\hat{G}} \rightrightarrows R^\square_{\bar{\rho}} \hat{\otimes} \mathcal{O}_{\hat{G}}$, and the equalizer (in commutative rings) certainly exists. So perhaps deformation functors are *always* representable in a big enough category. The question is whether $D_{\bar{\rho}}$ is at all nice.

References

[1] M. Demazure, P. Gabriel and A. Grothendieck, *Seminaire de Geometrie Algebrique 3*

[2] Grothendieck, A. *Technique de descente et théorèmes d'existence en géométrie algébrique II*, Séminaire Bourbaki exp. 195, 1958.