A POSSIBLE NEW APPROACH TO DEFORMATION THEORY

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1. Motivation

Let F be a number field, k a local field with ring of integers $\mathcal{O}$ and field of fractions κ. Typically one starts with a continuous representation $\bar{\rho} : G_{F,S} \to \mathrm{GL}_n(\kappa)$ and studies lifts of $\bar{\rho}$ to representations $G_{F,S} \to \mathrm{GL}_n(A)$, where A ranges over local pseudocompact $\mathcal{O}$-algebras with residue field κ. There is generally a pseudocompact $\mathcal{O}$-algebra $R_{\bar{\rho}}^{\square}$ classifying such lifts. One proceeds to take quotients of the associated formal scheme, and hope that these quotients (and certain subfunctors) are representable.

2. Key ideas

There are two motivating ideas. First, instead of representations $G_{F,S} \to \mathrm{GL}_n(\kappa)$, work more categorically. Let $U = \mathrm{Spec}(\mathcal{O}_F) \smallsetminus S$ and consider κ-local systems on the étale site of U. For each artinian $\mathcal{O}$-algebra A, we have the category $\mathsf{LS}(U, A)$ of A-local systems on $U_{\text{ét}}$. Hopefully the functor $A \mapsto \mathsf{LS}(U, A)$ is a formal stack (though not algebraic) in some sense. In any case, one defines $\mathsf{LS}(U, \mathcal{O}) = \varprojlim \mathsf{LS}(U, A)$. Write π for the functor $L \mapsto L \otimes_{\mathcal{O}} \kappa$ from $\mathsf{LS}(U, \mathcal{O})$ to $\mathsf{LS}(U, \kappa)$. Given $L \in \mathsf{LS}(U, \kappa)$, we could write D_L for the "fiber" $\pi^{-1}(L)$, which we again hope to be a formal stack in some reasonable sense.

Everything up to now has merely been a change of language. The next main idea is that instead of working with $\mathcal{O}$-local systems or κ-local systems, we should work with the "derived category" $\mathsf{D}_c^b(U, k)$ of bounded constructible k-sheaves on $U_{\text{ét}}$. That is, let $\mathrm{PS}(\mathcal{O})$ be the category of pseudocompact $\mathcal{O}$-algebras. For any $A \in \mathrm{PS}(\mathcal{O})$, we have a category $\mathsf{D}_c^b(U, A)$ of bounded constructible A-sheaves on $U_{\text{ét}}$. One might hope that $\mathsf{D}_c^b(U, -)$ is again a formal stack in some sense.

The analogy of a residual representation $\bar{\rho} : G_{F,S} \to \mathrm{GL}_n(\kappa)$ is the choice of $L \in \mathsf{D}_c^b(\kappa)$. Hopefully the "fiber above L" gives some kind of formal stack $\mathcal{D}_L$. But it is not at all obvious that $\mathcal{D}_L$ will be representable, or even a stack.

If E is an elliptic curve over F, we can "spread out" E to an elliptic scheme over some $U \subset \mathrm{Spec}(O_F)$. Let $f : E \to U$ be the structure map; then $\mathsf{R}f_*\kappa \in \mathsf{D}_b^c(U, \kappa)$ is our analogue of the residual representation $\bar{\rho}_{E,\ell}$. The problem is, studying the "derived deformation theory" of $\mathsf{R}f_*\kappa$ is probably too complicated. A better place to start would be with the "derived deformation theory" of the cyclotomic character. Hopefully this would give us some kind of "derived version" of the Iwasawa algebra.

The mod-p cyclotomic character should be $\mathbf{F}_p(1)$ in $\mathsf{D}_b^c(U, \mathbf{F}_p)$. I need to understand what constructible (as opposed to smooth) sheaves on U look like. A major problem is that constructible sheaves might have ramification, in which case classifying constructible lifts would be the wrong thing.

So the question is: what (if any) is the concrete interpretation of $\mathsf{D}_c^b(U, A)$?