p-adic L-functions and the Iwasawa main conjecture

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An impossible-to-find, but conceptual source seems to be [FK06]. A more accessible source may be [PR00].

Things more or less got started in [Rib76], and really got going in [MW84]. A more recent result is [SU14].

1 The start of the field

A lot of mathematical activity was kicked off by Ribet's influential paper [Rib76]. Let p be an odd prime, and let $C = \mathrm{Cl}\left(\mathbf{Q}(\mu_p)\right) \otimes \mathbf{F}_p$. Then $G_{\mathbf{Q}}$ acts on this through its quotient $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$, and if $\chi : G_{\mathbf{Q}} \to \mathbf{F}_p^\times$ is the corresponding cyclotomic character, we have a decomposition $C = \bigoplus_{i \in \mathbf{Z}/(p-1)} C(\chi^i)$, where

$$C(\chi^i) = \{c \in C : \sigma \cdot c = \chi^i(\sigma) c \text{ for all } \sigma \in G_{\mathbf{Q}}\}.$$

Ribet proved that if for $2 \leqslant k \leqslant p-3$ even, $C(\chi^k) \neq 0$ if and only if $v_p(B_k) > 0$. One direction had already been proved by Herbrand, so all Ribet has to do is prove that $v_p(B_k) > 0$ implies $C(\chi^k) \neq 0$. He does this by constructing the appropriate unramified extension of $\mathbf{Q}(\mu_p)$ using modular forms.

2 The Main Conjecture for modular forms

We follow [Kat04, III]. Let $\mathbf{Q}_\infty$ be the $\mathbf{Z}_p$-extension of $\mathbf{Q}$, with Galois group $\Gamma \xrightarrow{\sim} \mathbf{Z}_p^\times$ via the cyclotomic character κ. We are interested in modules over the Iwasawa algebra $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$, which is (non-canonically) isomorphic to the power-series ring $\mathbf{Z}_p[\![t]\!]$. If M is a Γ-module over $\mathbf{Z}_p$, we can interpret it as a representation of $G_{\mathbf{Q}}$ unramified outside p, hence a sheaf on the étale site of $\mathbf{Z}[\frac{1}{p}]$. Let $\mathfrak{o}_n = \mathbf{Z}[\zeta_{p^{n+1}}]$ and $\mathfrak{o}_{n,\{p\}} = \mathfrak{o}_n[\frac{1}{p}]$. We put

$$\mathbf{H}^i(M) = \varprojlim_n \mathrm{H}^i(\mathfrak{o}_{n,\{p\}}, M)$$

It turns out that $\mathbf{H}^i(M) = 0$ unless $i \in \{1, 2\}$. If M is a $\mathbf{Q}_p$-representation of Γ, we choose a Γ-stable lattice $M_0 \subset M$, and put

$$\mathbf{H}^i(M) = \mathbf{H}^i(M_0) \otimes \mathbf{Q}.$$

Let p be an odd prime, $k \geqslant 2$ and $N \geqslant 1$. We start with a normalized newform $f \in S_k(X_1(N))$. Choose a prime $\ell \nmid Np$, let $F_f = \mathbf{Q}(f)$, and let V_f be the associated two-dimensional $F_{f,\lambda}$-vector space with continuous $G_{\mathbf{Q}}$-action. Kato defines a canonical map $z : V_f \to \mathbf{H}^1(V_f)$.

References

[FK06] Takako Fukaya and Kazuya Kato. A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory. In *Proceedings of the St. Petersburg Mathematical Society. Vol. XII*, volume 219 of *Amer. Math. Soc. Transl. Ser. 2*, pages 1–85, Providence, RI, 2006. Amer. Math. Soc.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):117–290, 2004. Cohomologies p-adiques et applications arithmétiques.

[MW84] Barry Mazur and Andrew Wiles. Class fields of abelian extensions of $\mathbf{Q}$. *Invent. Math.*, 76(2):179–330, 1984.

[PR00] Bernadette Perrin-Riou. *p-adic L-functions and p-adic representations*, volume 3 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providnce, RI, 2000. Translated from the 1995 French original by Leila Schneps and revised by the author.

[Rib76] Kenneth Ribet. A modular construction of unramified p-extensions of $\mathbf{Q}(\mu_p)$. *Invent. Math.*, 34(3):151–162, 1976.

[SU14] Christopher Skinner and Eric Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 195(1):1–277, 2014.