Tannakian categories

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1 Motivation

Throughout, k is an arbitrary field of characteristic zero. We will work over k, so all maps are tacitly assumed to be k-linear and all tensor product will be over k. Consider the following categories.

1.1 Representations of an algebraic group

For $G_{/k}$ an algebraic group, the category $\mathrm{Rep}(G)$ has as objects pairs (V, ρ), where V is a finite-dimensional k-vector space and $\rho : G \to \mathrm{GL}(V)$ is a homomorphism of k-groups. A morphism $(V_1, \rho_1) \to (V_2, \rho_2)$ in $\mathrm{Rep}(G)$ is a k-linear map $f : V_1 \to V_2$ such that for all k-algebras A and $g \in G(A)$, one has $f\rho_1(g) = \rho_2(g)f$, i.e. the following diagram commutes:

$$\begin{tikzcd} V_1 \otimes A \arrow[r, "f"] \arrow[d, "\rho_1(g)"] & V_2 \otimes A \arrow[d, "\rho_2(g)"] \\ V_1 \otimes A \arrow[r, "f"] & V_2 \otimes A. \end{tikzcd}$$

1.2 Representations of a Hopf algebra

Let H be a co-commutative Hopf algebra. The category $\mathrm{Rep}(H)$ has as objects H-modules that are finite-dimensional over k, and morphisms are k-linear maps. The algebra H acts on a tensor product $U \otimes V$ via its comultiplication $\Delta : H \to H \otimes H$.

1.3 Representations of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over k. The category $\mathrm{Rep}(\mathfrak{g})$ has as objects $\mathfrak{g}$-representations that are finite-dimensional as a k-vector space. There is a canonical isomorphism $\mathrm{Rep}(\mathfrak{g}) = \mathrm{Rep}(\mathcal{U}\mathfrak{g})$, where $\mathcal{U}\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

1.4 Continuous representations of a compact Lie group

Let K be a compact Lie group. The category $\mathrm{Rep}_{\mathbf{C}}(K)$ has as objects pairs (V, ρ), where V is a finite-dimensional complex vector space and $\rho : K \to \mathrm{GL}(V)$ is a continuous (hence smooth, by Cartan's theorem) homomorphism. Morphisms $(V_1, \rho_1) \to (V_2, \rho_2)$ are K-equivariant $\mathbf{C}$-linear maps $V_1 \to V_2$.

1.5 Graded vector spaces

Consider the category whose objects are finite-dimensional k-vector spaces V together with a direct sum decomposition $V = \bigoplus_{n \in \mathbf{Z}} V_n$. Morphisms $U \to V$ are k-linear maps $f : U \to V$ such that $f(U_n) \subset V_n$.

1.6 Hodge structures

Let V be a finite-dimensional $\mathbf{R}$-vector space. A *Hodge structure* on V is a direct sum decomposition $V_{\mathbf{C}} = \bigoplus V_{p,q}$ such that $\overline{V_{p,q}} = V_{q,p}$. If U, V are vector spaces with Hodge structures, a morphism $U \to V$ is a $\mathbf{R}$-linear map $f : U \to V$ such that $f(U_{p,q}) \subset V_{p,q}$. Write Hdg for the category of vector spaces with Hodge structure.

Let $\mathrm{Vec}(k)$ be the category of finite-dimensional k-vector spaces. For $\mathcal{C}$ any of the categories above, there is a faithful functor $\omega : \mathcal{C} \to \mathrm{Vec}(k)$. In our examples, it is just the forgetful functor. The main theorem will be that for $\pi = \mathrm{Aut}(\omega)$, the functor ω induces an equivalence of categories $\mathcal{C} \xrightarrow{\sim} \mathrm{Rep}(\pi)$. We proceed to make sense of the undefined terms in this theorem.

2 Main definitions

Our definitions follow [DM82]. As before, k is an arbitrary field of characteristic zero.

2.1 Tannakian category

A *k-linear category* is an abelian category $\mathcal{C}$ such that each V_1, V_2, the group $\hom(V_1, V_2)$ has the structure of a k-vector space in such a way that the composition map $\hom(V_2, V_3) \otimes \hom(V_1, V_2) \to \hom(V_1, V_3)$ is k-linear. For us, a *rigid k-linear tensor category* is a k-linear category $\mathcal{C}$ together with the following data:

1. An exact faithful functor $\omega : \mathcal{C} \to \mathrm{Vec}(k)$.
2. A bi-additive functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
3. Natural isomorphisms $\omega(V_1 \otimes V_2) \xrightarrow{\sim} \omega(V_1) \otimes \omega(V_2)$.
4. Isomorphisms $V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ for all $V_i \in \mathcal{C}$.
5. Isomorphisms $(V_1 \otimes V_2) \otimes V_3 \xrightarrow{\sim} V_1 \otimes (V_2 \otimes V_3)$

These data are required to satisfy the following conditions:

1. There exists an object $1 \in \mathcal{C}$ such that $\omega(1)$ is one-dimensional and such that the natural map $k \to \hom(1, 1)$ is an isomorphism.
2. If $\omega(V)$ is one-dimensional, there exists $V^{-1} \in \mathcal{C}$ such that $V \otimes V^{-1} \simeq 1$.
3. Under ω, the isomorphisms 3 and 4 are the obvious ones.

By [DM82, Pr. 1.20], this is equivalent to the standard (more abstract) definition. Note that all our examples in section 1 are rigid k-linear tensor categories. One calls the functor ω a *fiber functor*.

2.2 Automorphisms of a functor

Let $(\mathcal{C}, \otimes)$ be a rigid k-linear tensor category. In this setting, define a functor $\mathrm{Aut}(\omega)$ from k-algebras to groups by setting:

$$\begin{aligned}\mathrm{Aut}^{\otimes}(\omega)(A) &= \mathrm{Aut}^{\otimes}(\omega : \mathcal{C} \otimes A \to \mathrm{Rep}(A)) \\ &= \left\{ (g_V) \in \prod_{V \in \mathcal{C}} \mathrm{GL}(\omega(V) \otimes A) : g_1 = 1,\ g_{V_1 \otimes V_2} = g_{V_1} \otimes g_{V_2}, \text{ and } f g_{V_1} = g_{V_1} f \text{ for all } f, V_1, V_2 \right\}.\end{aligned}$$

In other words, an element of $\mathrm{Aut}(\omega)(A)$ consists of a collection (g_V) of A-linear automorphisms $g_V : \omega(V) \otimes A \xrightarrow{\sim} \omega(V) \otimes A$, where V ranges over objects in $\mathcal{C}$. This collection must satisfy:

1. $g_1 = 1_{\omega(1)}$
2. $g_{V_1 \otimes V_2} = g_{V_1} \otimes g_{V_2}$ for all $V_1, V_2 \in \mathcal{C}$, and
3. whenever $f : V_1 \to V_2$ is a morphism in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \omega(V_1)_A & \xrightarrow{f} & \omega(V_2)_A \\ \downarrow g_{V_1} & & \downarrow g_{V_2} \\ \omega(V_1)_A & \xrightarrow{f} & \omega(V_2)_A. \end{array}$$

2.3 Pro-algebraic group

Typically one only considers affine group schemes $G_{/k}$ that are *algebraic*, i.e. whose coordinate ring $\mathscr{O}(G)$ is a finitely generated k-algebra, or equivalently that admit a finite-dimensional faithful representation. Let $G_{/k}$ be an arbitrary affine group scheme, V an arbitrary representation of G over k. By [DM82, Cor. 2.4], one has $V = \varinjlim V_i$, where V_i ranges over the finite-dimensional subrepresentations of V. Applying this to the regular representation $G \to \mathrm{GL}(\mathscr{O}(G))$, we see that $\mathscr{O}(G) = \varinjlim \mathscr{O}(G_i)$, where G_i ranges over the algebraic quotients of G. That is, an arbitrary affine group scheme $G_{/k}$ can be written as a filtered projective limit $G = \varprojlim G_i$, where each G_i is an affine algebraic group over k. So we will speak of pro-algebraic groups instead of arbitrary affine group schemes.

If V is a finite-dimensional k-vector space and $G = \varprojlim G_i$ is a pro-algebraic k-group, representations $G \to \mathrm{GL}(V)$ factor through some algebraic quotient G_i. That is, $\hom(G, \mathrm{GL}(V)) = \varinjlim \hom(G_i, \mathrm{GL}(V))$. As a basic example of this, let Γ be a profinite group, i.e. a projective limit of finite groups. If we think of Γ as a pro-algebraic group, then algebraic representations $\Gamma \to \mathrm{GL}(V)$ are exactly those representations that are continuous when V is given the discrete topology.

3 Reconstruction theorem

First, suppose $\mathcal{C} = \mathrm{Rep}(G)$ for a pro-algebraic group G, and that $\omega : \mathrm{Rep}(G) \to \mathrm{Vec}(k)$ is the forgetful functor. Then the Tannakian fundamental group $\mathrm{Aut}^{\otimes}(\omega)$ carries no new information [DM82, Pr. 2.8]:

Theorem 3.1. *Let $G_{/k}$ be a pro-algebraic group, $\omega : \mathrm{Rep}(G) \to \mathrm{Vec}(k)$ the forgetful functor. Then $G \xrightarrow{\sim} \mathrm{Aut}^{\otimes}(G)$.*

The main theorem is the following, taken essentially verbatim from [DM82, Th. 2.11].

Theorem 3.2. *Let $(\mathcal{C}, \otimes, \omega)$ be a rigid k-linear tensor category. Then $\pi = \mathrm{Aut}^{\otimes}(\omega)$ is represented by a pro-algebraic group, and $\omega : \mathcal{C} \to \mathrm{Rep}(\pi)$ is an equivalence of categories.*

Often, the group $\pi_1(\mathcal{C})$ is "too large" to handle directly. For example, if $\mathcal{C}$ contains infinitely many simple objects, probably $\pi_1(\mathcal{C})$ will be infinite-dimensional. For $V \in \mathcal{C}$, let $\mathcal{C}(V)$ be the Tannakian subcategory of $\mathcal{C}$ generated by V. One puts $\pi_1(\mathcal{C}/V) = \pi_1(\mathcal{C}(V))$. It turns out that $\pi_1(\mathcal{C}/V) \subset \mathrm{GL}(\omega V)$, so $\pi_1(\mathcal{C}/V)$ is finite-dimensional. One has $\pi_1(\mathcal{C}) = \varprojlim \pi_1(\mathcal{C}/V)$.

4 Examples

4.1 Pro-algebraic groups

If $G_{/k}$ is a pro-algebraic group, then Theorem 3.1 tells us that if $\omega : \mathrm{Rep}(G) \to \mathrm{Vec}(k)$ is the forgetful functor, then $G = \mathrm{Aut}^{\otimes}(G)$. That is, $G = \pi_1(\mathrm{Rep}\, G)$.

4.2 Hopf algebras

Suppose H is a co-commutative Hopf algebra over k. Then $\pi_1(\operatorname{Rep} H) = \operatorname{Spec}(H^\circ)$, where H° is the *reduced dual* defined in [Car07]. Namely, for any k-algebra A, A° is the set of k-linear maps $\lambda : A \to k$ such that $\lambda(\mathfrak{a}) = 0$ for some two-sided ideal $\mathfrak{a} \subset A$ of finite codimension. The key fact here is that $(A \otimes B)^\circ = A^\circ \otimes B^\circ$, so that we can use multiplication $m : H \otimes H \to H$ to define comultiplication $m^* : H^\circ \to (H \otimes H)^\circ = H^\circ \otimes H^\circ$. From [DG80, II §6 1.1], if G is a linear algebraic group over an algebraically closed field k of characteristic zero, we get an isomorphism $\mathscr{O}(G)^\circ = k[G(k)] \otimes \mathcal{U}(\mathfrak{g})$. Here $k[G(k)]$ is the usual group algebra of the abstract group $G(k)$, and $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g} = \operatorname{Lie}(G)$, both with their standard Hopf structures.

[Note: one often calls $\mathscr{O}(G)^\circ$ the "space of distributions on G." If instead G is a real Lie group, then one often writes $\mathscr{H}(G)$ for the space of distributions on G. Let $K \subset G$ be a maximal compact subgroup, $\mathrm{M}(K)$ the space of finite measures on K. Then convolution $D \otimes \mu \mapsto D * \mu$ induces an isomorphism $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{M}(K) \xrightarrow{\sim} \mathscr{H}(G)$. In the algebraic setting, $k[G(k)]$ is the appropriate replacement for $\mathrm{M}(K)$.]

4.3 Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Then by [Mil07], $G = \pi_1(\operatorname{Rep} \mathfrak{g})$ is the unique connected, simply connected algebraic group with $\operatorname{Lie}(G) = \mathfrak{g}$. If $\mathfrak{g}$ is not semisimple, e.g. $\mathfrak{g} = k$, then things get a lot nastier. See the above example.

4.4 Compact Lie groups

By definition, the *complexification* of a real Lie group K is a complex Lie group $K_{\mathbf{C}}$ such that all morphisms $K \to \mathrm{GL}(V)$ factor uniquely through $K_{\mathbf{C}} \to \mathrm{GL}(V)$. It turns out that $K_{\mathbf{C}}$ is a complex algebraic group, and so $\pi_1(\operatorname{Rep} K) = K_{\mathbf{C}}$.

4.5 Graded vector spaces

To give a grading $V = \bigoplus_{n \in \mathbf{Z}} V_n$ on a vector space is equivalent to giving an action of the split rank-one torus $\mathbf{G}_{\mathrm{m}}$. On each V_n, $\mathbf{G}_{\mathrm{m}}$ acts via the character $g \mapsto g^n$. Thus $\pi_1(\text{graded vector spaces}) = \mathbf{G}_{\mathrm{m}}$.

4.6 Hodge structures

Let $\mathbf{S} = \mathrm{R}_{\mathbf{C}/\mathbf{R}}\, \mathbf{G}_{\mathrm{m}}$; this is defined by $\mathbf{S}(A) = (A \otimes \mathbf{C})^\times$ for $\mathbf{R}$-algebras A. One can check that the category Hdg of Hodge structures is equivalent to $\operatorname{Rep}_{\mathbf{R}}(\mathbf{S})$. Thus $\pi_1(\mathrm{Hdg}) = \mathbf{S}$.

References

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