Analytic and arithmetic properties of a new class of L-functions

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1 Introduction

The work in this paper is inspired by the following example of Ramakrishna. Let $E_{/\mathbf{Q}}$ be a non-CM elliptic curve. Let l be an odd prime and $\rho_{E,l}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ the associated representation. Recall that $a_p = \mathrm{tr}(\rho_{E,l}(\mathrm{fr}_p))$, and these satisfy the Hasse bound $|a_p| < 2\sqrt{p}$. Then we have the following curious L-function with only one Euler factor at each prime:

$$L_{\mathrm{sgn}}(E,s) = \prod_p \frac{1}{1-\mathrm{sgn}(a_p)p^{-s}}.$$

We are interested in the analytic and arithmetic properties of a class of L-functions generalized from this one.

Definition 1.1. Let $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{Z}_l)$ be geometric in the sense of [FM95]. Assume the Sato–Tate group of ρ is well-defined; denote it by $\mathrm{ST}(\rho)$. Let $\eta\colon \mathrm{ST}(\rho)^\natural \to \mathbf{R}$ be a function of bounded variation.

Some conventions. Let X be a compact topological space and write $\boldsymbol{x} = (x_2, x_3, \dots)$, $\boldsymbol{y}$, etc. for sequences in X indexed by the prime numbers. Given such a sequence, we write

$$\boldsymbol{x}^C(f) = \frac{1}{\pi(C)} \sum_{p\leqslant C} f(x_p).$$

So the $\boldsymbol{x}^C$ are probability measures on X.

Lemma 1.2 (Abel summation)**.** *Let $\{x_p\}$ be a sequence of real numbers, $\phi \in C^1(\mathbf{R})$. Then*

$$\sum_{p\leqslant C} \phi(p)x_p = \phi(C)\sum_{p\leqslant C} x_p - \int_2^C \phi'(x)\sum_{p\leqslant x} x_p \,\mathrm{d}x.$$

Proof. Simply note that if $p_1, \dots, p_n$ is an enumeration of the primes $\leqslant X$, we have

$$\begin{aligned}\int_2^C \phi'(x) \sum_{p \leqslant x} x_p \,\mathrm{d}x &= \sum_{p \leqslant C} x_p \int_{p_n}^C \phi' + \sum_{i=1}^{n-1} \sum_{p \leqslant p_{i+1}} x_p \int_{p_i}^{p_{i+1}} \phi' \\ &= (\phi(C) - \phi(p_n)) \sum_{p \leqslant C} x_p + \sum_{i=1}^{n-1} (\phi(p_{i+1}) - \phi(p_i)) \sum_{p \leqslant p_{i+1}} x_p \\ &= \phi(C) \sum_{p \leqslant C} x_p - \sum_{p \leqslant X} \phi(p) x_p,\end{aligned}$$

as desired. □

Lemma 1.3. *Let (X, μ) be a separable metric space with Radon measure whose support is X. Let f be a bounded function on X. Then the following condition holds:*

$\lim_{C \to \infty} \boldsymbol{x}^C(f) = \mu(f)$ *for all μ-equidistributed sequences $\boldsymbol{x}$*

if and only if f is continuous almost everywhere.

Proof. This follows from [CV92], Corollary 1 and Remark 3.

[Give my own proof.] □

2 General setting

Let G be a compact, connected Lie group, $\boldsymbol{x} = \{x_p\}$ a sequence in $G^\natural$. We write $\widehat{G}$ for the collection of irreducible unitary representations of G.

Definition 2.1. Let $\eta \colon G^\natural \to \mathbf{C}$ be bounded and continuous almost everywhere. Then the associated *curious L-function* is

$$L_\eta(s) = \prod_p \frac{1}{1 - \eta(x_p) p^{-s}},$$

wherever the product converges.

Definition 2.2. For $\rho \in \widehat{G}$, the associated L-function is defined following [Ser68]:

$$L(s, \rho) = \prod_p \frac{1}{\det(1 - \rho(x_p) p^{-s})}.$$

Lemma 2.3. *Assume $\|\eta\|_\infty \leqslant 1$. Then the product formula for $L_\eta(s)$ converges absolutely on $\{\Re s > 1\}$. The function L_η is holomorphic on that region.*

Proof. By [Kno56, §3.7, Th. 5], the product for $L_\eta(s)$ converges whenever $\Re s > 1$. The rest is well-known "general nonsense" about Dirichlet series. $\square$

We are interested in the analytic continuation of $L_\eta(s)$ past $\Re s = 1$, in particular to line $\Re s = \frac{1}{2}$.

Lemma 2.4. *Assume $\sum \frac{\eta(x_p)}{p^s}$ converges to a holomorphic function on $\{\Re s > s_0\}$, $s_0 \in \left[\frac{1}{2}, 1\right]$. Then $L_\eta(s)$ can be analytically continued to a holomorphic function on $\{\Re s > s_0\}$.*

Proof. By [Apo76, 11.9, Ex. 2], on the domain of absolute convergence for L_η, we have

$$L_\eta(s) = \exp\left(\sum_p \sum_{\nu \geqslant 1} \frac{\eta(x_p)^\nu}{\nu p^{\nu s}}\right).$$

So, it suffices to prove that the argument of exp converges on $\{\Re s > s_0\}$. Now note that

$$\left|\sum_{n \geqslant 2} \frac{\eta(x_p)^\nu}{\nu p^{\nu s}}\right| \leqslant \sum_{\nu \geqslant 2} (p^{-\Re s})^\nu = p^{-2s} \frac{1}{1 - p^{-s}}.$$

Since $p \geqslant 2$ and $\Re s > 1/2$, we have $1 - 2^{-1/2} < 1 - p^{-s} < 1$, so the argument of exp converges if and only if $\sum_p \left(\frac{\eta(x_p)}{p^s} + p^{-2\Re s}\right)$ does. But $\sum p^{-2\Re s}$ converges absolutely, so we the desired result. $\square$

...definition of star-discrepancy on $G^\natural$...

Theorem 2.5. *If $\operatorname{disc}^*(\boldsymbol{x}^C) = O(C^{-\frac{1}{2}+\epsilon})$, then $\left|\int f - \boldsymbol{x}^C(f)\right| = O_f(C^{-\frac{1}{2}+\epsilon})$*

Theorem 2.6. *Assume that $\left|\int_{G^\natural} f - \boldsymbol{x}^C(f)\right| = O_f(C^{-\frac{1}{2}+\epsilon})$. If $\int \eta = 0$, then L_η has analytic continuation to $\{\Re s = 1/2\}$, and $\log L_\eta$ has no poles in that region.*

Proof. By Lemma 1.2 with $\phi(x) = x^{-s}$, we have

$$\begin{aligned}
\sum_{p \leqslant C} \frac{\eta(x_p)}{p^s} &= C^{-s} \sum_{p \leqslant C} \eta(x_p) + s \int_2^C \sum_{p \leqslant x} \eta(x_p) \frac{\mathrm{d}x}{x^{s+1}} \\
&= C^{-s} \operatorname{Li}(C) O(C^{-\frac{1}{2}+\epsilon}) + s \int_2^C \operatorname{Li}(x) O(x^{-\frac{1}{2}+\epsilon}) \frac{\mathrm{d}x}{x^{s+1}}.
\end{aligned}$$

Since $\operatorname{Li}(x) = O(x/\log x)$, the first term is $O(C^{\frac{1}{2}-s+\epsilon}/\log C) = o(1)$. We prove that the integral is absolutely convergent. Since $\Re s + \frac{1}{2} > 1$ and ϵ is arbitrary,

$$\int_2^C \frac{x^{\epsilon - \frac{1}{2} - \Re s}}{\log x} \mathrm{d}x$$

converges, and the proof is complete. $\square$

Theorem 2.7. *Let $\eta\colon G^\natural \to \mathbf{C}$ be bounded and have bounded variation, and moreover $\int \eta = 0$. Then*

$$\sum_p \frac{\eta(x_p)}{p^s} \qquad \text{and} \qquad \sum_p \frac{\log p}{p^s}\eta(x_p)$$

are holomorphic on the region $\{\Re s > \frac{1}{2}\}$.

Proof. We use Abel summation:

$$\sum_{p\leqslant C} \frac{\log p}{p^s}\eta(x_p) = \frac{\log C}{C^s}\sum_{p\leqslant C}\eta(x_p) - \int_2^C \frac{1-s\log x}{x^{s+1}}\sum_{p\leqslant x}\eta(x_p)\,\mathrm{d}x.$$

We show the the first term approaches zero and that the integral converges absolutely. We have:

$$\left|\frac{\log C}{C^s}\sum_{p\leqslant C}\eta(x_p)\right| \ll \frac{\log C}{C^s}\frac{C}{\log C}C^{-\frac{1}{2}+\epsilon} = C^{1-s-\frac{1}{2}+\epsilon}.$$

Since ϵ is arbitrary, the exponent of C is negative. Moreover:

$$\int_2^C \frac{1}{x^{s+1}}\left|\sum_{p\leqslant x}\eta(x_p)\right|\mathrm{d}x \ll \int_2^C \frac{1}{x^{s+1}}\frac{x}{\log x}x^{-\frac{1}{2}+\epsilon}\,\mathrm{d}x$$

$$\int_2^C \frac{\log x}{x^{s+1}}\left|\sum_{p\leqslant x}\eta(x_p)\right|\mathrm{d}x \ll \int_2^C \frac{\log x}{x^{s+1}}\frac{x}{\log x}x^{-\frac{1}{2}+\epsilon}\,\mathrm{d}x.$$

Both of these integrals converge because ϵ is arbitrary. $\square$

Corollary 2.8. *If $\rho \in \widehat{G}$, then RH + analytic continuation to $\{\Re s > \frac{1}{2}\}$ holds for $L(s,\rho)$.*

Proof. Take logarithmic derivative, reduce to the previous theorem. $\square$

3 Discrepancy on compact Lie groups

Let G be a compact, connected Lie group. Let $G^\natural$ be the space of conjugacy classes of G. We will define *star-discrepancy* for sequences on $G^\natural$. Let T be a maximal torus in G. Then the exponential map $\exp\colon \mathfrak{t} \to T$ is surjective. Choose a basis $\{t_1, \dots, t_r\}$ for $\mathfrak{t}$. Then we can identify $G^\natural$ with

$$\int_{G^\natural} f(x)\,\mathrm{d}x = \frac{1}{\#W}\int_T \det(1-\mathrm{Ad}(t^{-1})|_{\mathfrak{g}/\mathfrak{t}})f(t)\,\mathrm{d}t.$$

4 Some examples

Let G be a compact connected abelian lie group, and $g \in G$ such that $g^{\mathbf{Z}} \subset G$ is equidistributed. Then for any function η on G with $\|\eta\|_\infty \leqslant 1$, we have a curious L-function:

$$L_\eta(g, s) = \prod_p \frac{1}{1 - \eta(g^p)p^{-s}}.$$

[Bla bla general nonsense.]

For example, let $G = \mathbf{R}/\mathbf{Z}$ and $\alpha \in \mathbf{R}$ be an algebraic irrational, for example $\alpha = \sqrt{2}$. Then the corresponding L-function is:

$$L_{\exp(2\pi i t)}(\alpha, s) = \prod_p \frac{1}{1 - \exp(2\pi i \alpha p)p^{-s}}$$

5 Special Unitary group

For $G = \mathrm{SU}(2)$, the space of conjugacy classes is $[0, \pi]$, with θ corresponding to the matrix $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$. Note that the trace of the n-th symmetric power is $\sin((n+1)\theta)/\sin(\theta)$.

$$f(\pi - x) = f(\pi) + \mu([0, x]) \Rightarrow \mu([0, x]) = f(\pi - x) - f(\pi)$$

So

$$\int_0^x \frac{\mathrm{d}}{\mathrm{d}t}(f(\pi - t)) = f(\pi - x) - f(\pi)$$

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\pi - t) = -f'(\pi - t).$$

So the variation is:

$$\int_0^\pi \left| \frac{\mathrm{d}}{\mathrm{d}\theta} \frac{\sin((n+1)\theta)}{\sin\theta} \right| \mathrm{d}\theta$$

$n = 1$,4
$n = 2$, 8
$n = 3$, $(8(9 + 2\sqrt{6}))/9$
$n = 4$, 17

Let G be a compact connected Lie group, $T \subset G$ a maximal torus. Then we have the exponential map $\exp\colon \mathfrak{t} \twoheadrightarrow T$ with kernel Γ.

E.g. $\mathbf{R}^2/\mathbf{Z}^2$. Here $\{(1, 3), (0, 1)\}$ is a basis, so we could get, as basic sets:

$$\{(\lambda + 3\mu, \mu) \mod 1\}$$

Suppose we have a lattice $\Gamma \subset V$ with basis $\gamma_1, \ldots, \gamma_r$. Call a "rectangle" a set of the form:

$$I_{\boldsymbol{\alpha}} = \{t_1\gamma_1 + \cdots + t_r\gamma_r : 0 \leqslant t_1 < \alpha_1, \ldots, 0 \leqslant t_r < \alpha_r\}$$

Suppose $\Gamma = \mathbf{Z}^r \subset \mathbf{R}^r$. If $r = 2$ and we choose a basis as above, we are looking at:

$$I_{\alpha_1,\alpha_2} = \{(t_1, 3t_1 + t_2) : t_1 \in [0, \alpha_1), t_2 \in [0, \alpha_2)\}$$

6 Koksma–Hlawka

Let $[0, \pi]^r$ be our space. The μ-star discrepancy is:

$$D^*(\boldsymbol{x}^N) = \sup_{x \in [0,\pi]^r} \left| \frac{1}{N} \sum_{n \leqslant N} 1_{[0,x]}(x_n) - \int 1_{[0,x]} \, \mathrm{d}\mu \right|$$

Let f be a function on $[0, \pi]^r$. We say f has *bounded variation* if there is a measure μ such that

$$\int_{[0,x]} f' = \mu[0, x] = f(x)$$

Thus, the variation of f is $\int |f'|$.

For $[0, \pi]^r$, we have

$$\mu([0, \alpha_1] \times [0, \alpha_2]) = \int_0^{\alpha_2} \int_0^{\alpha_1} \frac{\mathrm{d}}{\mathrm{d}x_1} \frac{\mathrm{d}}{\mathrm{d}x_2} f = f(\alpha_1, \alpha_2) - f(0, 0)$$

Note:

$$f(x_1, \ldots, x_n) = f(0) + \int_0^{x_n} \cdots \int_0^{x_1} g(t_1, \ldots, t_n) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

$$\begin{aligned}\int_0^{x_2} \int_0^{x_1} \partial_{1,2} g(t_1, t_2) \, \mathrm{d}t_1 \mathrm{d}t_2 &= \int_0^{x_2} \partial_2 g(x_1, t_2) - \partial_2 g(0, t_2) \, \mathrm{d}t_2 \\ &= g(x_1, x_2) - g(x_1, 0) - g(0, x_2) + g(0, 0)\end{aligned}$$

7 Analytic continuation

Consider our usual $L_\eta(s)$. Recall that $(-L'/L)_\eta(s)$ is governed by

$$\log L_\eta(s) = \sum_{p^\nu} \frac{\eta(x_p)}{\nu p^{\nu s}}.$$

This we split up as follows:

$$\log L_\eta(s) = \sum_p \frac{\eta(x_p)}{p^s} + \sum_{\nu \geqslant 2} \frac{1}{\nu} \sum_p \frac{\eta(x_p)}{p^{\nu s}}$$

Let $H(s) = \sum_p \eta(x_p) p^{-s}$; then

$$\log L_\eta(s) = H(s) + \sum_{\nu \geqslant 2} \frac{1}{\nu} H(\nu s)$$

Theorem 7.1. *Suppose $|\boldsymbol{x}^C(f) - \int f| = O(C^{-\frac{1}{2}+\epsilon})$. If $\int \eta \neq 0$, then L_η has a pole of order $\int \eta$ at $s = 1$.*

Proof. It suffices to prove that $\log L_\eta(1+\epsilon) = -(\int \eta) \log \epsilon + O(1)$. This is a simple computation:

$$\begin{aligned}
\log L_\eta(1+\epsilon) &= \sum_p \frac{\eta(x_p)}{p^{1+\epsilon}} + O(1) \\
&= \int_2^\infty \frac{x}{\log x} (\mu(\eta) + x^{-\frac{1}{2}+}) \frac{\mathrm{d}x}{x^{2+\epsilon}} + O(1) \\
&= \mu(\eta) \int_2^\infty \frac{\mathrm{d}x}{x^{1+\epsilon} \log x} + O(1) \\
&= -\mu(\eta) \operatorname{Ei}(-\epsilon \log 2) + O(1) \\
&= -\mu(\eta) \log \epsilon + O(1).
\end{aligned}$$

□

Try analytically continuing

$$(-L'/L)_\eta(s) = \sum_{p^\nu} \frac{\log(p)\eta(x_p)^\nu}{p^{\nu s}}$$

We follow the easy proof. Start with:

$$\pi^{-s}\Gamma(s)\frac{1}{n^{2s}} = \int_0^\infty e^{-\pi n^2 y} y^s \frac{\mathrm{d}y}{y}$$

Make a sum:

$$\begin{aligned}
\sum_{p^\nu} \log(p)\eta(x_p)^\nu \pi^{-s}\Gamma(s)\frac{1}{(p^\nu)^{2s}} &= \sum_{p^\nu} \log(p)\eta(x_p)^\nu \int_0^\infty e^{-\pi n^2 y} y^s \frac{\mathrm{d}y}{y} \\
\pi^{-s}\Gamma(s)(-L'/L)_\eta(2s) &= \int_0^\infty \sum_{p^\nu} \log(p)\eta(x_p)^\nu e^{-\pi (p^\nu)^2 y} y^s \frac{\mathrm{d}y}{y}
\end{aligned}$$

So we are interested in the “series under the integral”:

$$\varphi(y) = \sum_{p^\nu} \log(p)\eta(x_p)^\nu e^{-\pi p^{2\nu} y}$$

The series for φ converges absolutely on $\{\Re > 0\}$. Better,

$$\sum_p \log(p) \sum_{\nu \geqslant 1} M^\nu (e^{-\pi})^{(p^2)^\nu y}$$

Recall the *Mellin transform*:

$$(\mathcal{M}f)(s) = \int_0^\infty (f(t) - f(\infty))t^s \,\frac{\mathrm{d}t}{t}$$

Thus we have the identity:

$$\pi^{-s}\Gamma(s)(-L'/L)_\eta(2s) = (\mathcal{M}\varphi)(s)$$

First, we need good bounds for $\varphi(y)$, i.e. it needs to be a constant plus $O(e^{-cy^\alpha})$, i.e. very fast decay. Better, write

$$\varphi(y) = \sum_{\nu \geqslant 1} \sum_p \log(p)\eta(x_p)^\nu \exp(-\pi p^{2\nu} y)$$

Try Abel summation for some fixed ν:

$$\begin{aligned}
&\sum_{p \leqslant C} \log(p)\eta(x_p)^\nu \exp(-\pi p^{2\nu} y) \\
&= \log(C)\exp(-\pi C^{2\nu} y) \sum_{p \leqslant C} \eta^\nu(x_p) + \int_2^C (1 - 2\pi\nu y x^{2\nu}\log x)\exp(-\pi y x^{2\nu})\,(*)\,\frac{\mathrm{d}x}{x} \\
&= C\exp(-\pi C^{2\nu} y)(\mu(\eta^\nu) + O(C^{-\frac{1}{2}+\epsilon})) + \int_2^\infty \cdots \\
&\approx \int_2^\infty (1 - 2\pi\nu y x^{2\nu}\log x)\exp(-\pi y x^{2\nu}) \sum_{p \leqslant x} \eta^\nu(x_p)\,\frac{\mathrm{d}x}{x} \\
&\ll \int_2^\infty \nu y x^{2\nu}\log(x)\exp(-\pi y x^{2\nu})\frac{x}{\log x}(\mu(\eta^\nu) + x^{-\frac{1}{2}+\epsilon})\,\frac{\mathrm{d}x}{x} \\
&\ll \int_2^\infty \nu y x^{2\nu}\exp(-\pi y x^{2\nu})(\mu(\eta^\nu) + x^{-\frac{1}{2}+\epsilon})\,\mathrm{d}x \\
&= \nu y \mu(\eta^\nu) \int_2^\infty x^{2\nu}(e^{-\pi y})^{x^{2\nu}}\,\mathrm{d}x +
\end{aligned}$$

8 Different perspective

Recall that our curious L-function can be written as a Dirichlet series:

$$L_\eta(s) = \sum_{n \geqslant 1} \frac{\prod_{p^\nu \| n} \eta(x_p)^\nu}{n^s}$$

We can write $\eta(n) = \prod_{p^\nu \| n} \eta(x_p)^\nu$; then η is a completely multiplicative function $\mathbf{N} \to \mathbf{C}$.

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