Lifting one-dimensional Galois representations

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1 Brief review of the setup

Throughout, $\Gamma = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and, for each finite set S of primes, $\Gamma_S = \mathrm{Gal}(\mathbf{Q}_S/S)$, where $\mathbf{Q}_S$ is the maximal extension of $\mathbf{Q}$ unramified outside S. Let k be a finite field of characteristic p. Fix a continuous irreducible representation $\bar\rho : \Gamma \to \mathrm{GL}_n(k)$. For each set S of primes such that $\bar\rho$ factors through Γ_S, we have a formal scheme $\mathcal{X}_S = \mathcal{X}_S(\bar\rho)$. It is given by its functor of points $\mathcal{X}_S : \mathsf{C}_{\mathrm{W}(k)} \to \mathsf{Set}$. Here, $\mathrm{W}(k)$ is the ring of Witt vectors of k and $\mathsf{C}_{\mathrm{W}(k)}$ is the category of artinian local $\mathrm{W}(k)$-algebras with residue field k. For A such an algebra, the set $\mathcal{X}_S(A)$ consists of lifts $\rho : \Gamma_S \to \mathrm{GL}_n(A)$ of $\bar\rho$, up to strict equivalence. Write $\mathfrak{t}_{\mathcal{X}_S} = \mathcal{X}_S(k[\varepsilon])$ for the tangent space of $\mathcal{X}_S$ at $\bar\rho$. It is well-known that there is a natural isomorphism

$$\mathfrak{t}_{\mathcal{X}_S} = \mathrm{H}^1(\Gamma_S, \mathrm{Ad}\,\bar\rho),$$

where $\mathrm{Ad}\,\bar\rho$ is Γ_S-module $\mathfrak{gl}_n(k)$, with action $\sigma \cdot x = \mathrm{Ad}(\bar\rho(\sigma))(x)$. Moreover, there is a good "obstruction theory" for lifting deformations of $\bar\rho$. Given a surjection $A \twoheadrightarrow A_0$ in $\mathsf{C}_{\mathrm{W}(k)}$ for which the kernel I is principle and annihilated by $\mathfrak{m}_A$, there is associated to each $\rho_0 \in \mathcal{X}_S(A_0)$ an *obstruction class* $o(\rho_0) \in \mathrm{H}^2(\Gamma_S, \mathrm{Ad}\,\bar\rho)$, the vanishing of which is necessary and sufficient for the existence of a lift of ρ_0 to A. If such a lift ρ exists, the set of lifts of ρ_0 admits a natural action of $\mathfrak{t}_{\mathcal{X}_S}$, which we denote $(c, \rho) \mapsto c \cdot \rho$, which makes the set of lifts a $\mathfrak{t}_{\mathcal{X}_S}$-torsor.

For any rational prime l, write $\Gamma_l = \mathrm{Gal}(\overline{\mathbf{Q}_l}/\mathbf{Q})$. The representation $\bar\rho : \Gamma_S \to \mathrm{GL}_n(k)$ restricts to a representation $\bar\rho_l = \bar\rho|_{\Gamma_l}$, and we write $\mathcal{X}_l = \mathcal{X}_l(\bar\rho)$ for the formal scheme classifying strict equivalence classes of lifts of $\bar\rho_l$ to representations $\rho_l : \Gamma_l \to \mathrm{GL}_n(A)$. The operation $\rho \mapsto \rho|_{\Gamma_l}$ induces a morphism $\mathcal{X}_S \to \mathcal{X}_l$ for each l. Just as above, there is a natural isomorphism $\mathfrak{t}_{\mathcal{X}_l} = \mathrm{H}^1(\Gamma_v, \mathrm{Ad}\,\bar\rho)$, and obstructions to lifts live in $\mathrm{H}^2(\Gamma_v, \mathrm{Ad}\,\bar\rho)$.

Put $\mathcal{X}_{\partial S} = \prod_{l \in S} \mathcal{X}_l$. Clearly $\mathfrak{t}_{\mathcal{X}_{\partial S}} = \bigoplus_{l \in S} \mathfrak{t}_{\mathcal{X}_l}$. For us, a *set of local conditions* is a formal subscheme $\mathcal{M} \subset \mathcal{X}_{\partial S}$. Given a set of local conditions $\mathcal{M}$, define $\mathcal{X}_{\mathcal{M}} = \mathcal{X} \times_{\mathcal{X}_{\partial S}} \mathcal{M}$. That is, for $A \in \mathsf{C}_{\mathrm{W}(k)}$, the set $\mathcal{X}_{\mathcal{M}}(A)$ consists of those $\rho \in \mathcal{X}_S(A)$ such that $(\rho|_{\Gamma_l})_{l \in S}$ lies in $\mathcal{M}(A)$. If, as will always be the case,

$\mathcal{M} = \prod_{l\in S} \mathcal{M}_l$ with each $\mathcal{M}_l \subset \mathcal{X}_l$, it is clear that

$$\mathfrak{t}_{\mathcal{X}_\mathcal{M}} = \ker\left(\mathfrak{t}_{\mathcal{X}_S} \to \bigoplus_{l\in S} \mathfrak{t}_{\mathcal{X}_l}/\mathfrak{t}_{\mathcal{M}_l}\right) = \ker\left(\mathrm{H}^1(\Gamma_S, \operatorname{Ad}\bar\rho) \to \bigoplus_{l\in S} \frac{\mathrm{H}^1(\Gamma_l, \operatorname{Ad}\bar\rho)}{\mathfrak{t}_{\mathcal{M}_l}}\right).$$

There is also an obstruction theory for $\mathcal{X}_\mathcal{M}$. [work this out!]

It is natural to ask whether the morphism $\mathcal{X}_S \to \mathcal{X}_\mathcal{M}$ is (formally) smooth. This is true if and only if, for each square-zero extension $A \twoheadrightarrow A_0$, an element $\rho_0 \in \mathcal{X}_S(A_0)$ lifts to A if and only if $(\rho_0|_{\Gamma_l})_{l\in S} \in \mathcal{M}(A_0)$ lifts to a $(\rho_l)_{l\in S} \in \mathcal{X}_\mathcal{M}(A)$. It is easy to check that this holds if and only if

$$\mathrm{H}^1_{\mathcal{M}^\perp}(\Gamma_S, \operatorname{Ad}\bar\rho^*) = \ker\left(\mathrm{H}^1(\Gamma_S, \operatorname{Ad}\bar\rho^*) \to \left(\bigoplus_{l\in S} \mathrm{H}^1(\Gamma_l, \operatorname{Ad}\bar\rho^*)\right)/\mathfrak{t}_\mathcal{M}^\perp\right).$$

Here, $\mathfrak{t}_\mathcal{M}^\perp \subset \bigoplus_{l\in S} \mathrm{H}^1(\Gamma_l, \operatorname{Ad}\bar\rho^*)$ is the orthogonal complement of $\mathfrak{t}_\mathcal{M}$ under the pairing $\mathfrak{t}_\mathcal{M} \times \bigoplus_{l\in S} \mathrm{H}^1(\Gamma_l, \operatorname{Ad}\bar\rho^*) \to \mathbf{Q}/\mathbf{Z}$ induced by the cup-products

$$\smile : \mathrm{H}^1(\Gamma_l, \operatorname{Ad}\bar\rho) \times \mathrm{H}^1(\Gamma_l, \operatorname{Ad}\bar\rho^*) \to \mathrm{H}^2(\Gamma_l, \boldsymbol{\mu}_p) \hookrightarrow \mathbf{Q}/\mathbf{Z}.$$

2 The one-dimensional case

Let Γ, k be as above. Fix a continuous character $\bar\chi : \Gamma \to k^\times$. For each appropriate S, write $\mathcal{X}_S = \mathcal{X}_S(\bar\chi)$. Note that $\operatorname{Ad}\bar\chi = k$ (the trivial representation). Because of this, a number of things can be computed using various duality theorems. First, note that for any profinite group G, we have $\mathrm{H}^\bullet(G, k) = \mathrm{H}^\bullet(G, \mathbf{Z}/p) \otimes k$. Thus by [NSW08, 8.6.9], there is a natural isomorphism

$$\mathrm{H}^2(\Gamma_l, k) = \mathrm{H}^0(\Gamma_l, \boldsymbol{\mu}_p)^\vee \otimes k = \begin{cases} k & \text{if } l \equiv 1 \pmod p \\ 0 & \text{otherwise} \end{cases},$$

that is, $\mathcal{X}_l$ is singular if and only if $l \equiv 1 \pmod p$. We can compute tangent spaces:

$$\begin{aligned} \mathfrak{t}_{\mathcal{X}_l} &= \mathrm{H}^1(\Gamma_l, k) \\ &= \mathrm{H}^1(\Gamma_l, \boldsymbol{\mu}_p)^\vee \otimes k \\ &= \hom(\widehat{\mathbf{Z}} \times \mathbf{Z}/(l-1) \times \mathbf{Z}_l, k) \\ &= \begin{cases} k^2 & \text{if } l = p \text{ or } l \equiv 1 \pmod p \\ k & \text{otherwise} \end{cases} \end{aligned}$$

In other words, $\mathcal{X}_p$ is smooth and two-dimensional, and the $\mathcal{X}_l$ ($l \neq p$) are singular with $\dim(\mathfrak{t}_{\mathcal{X}_l}) = 2$ ($l \equiv 1$) or smooth and one-dimensional ($l \not\equiv 1$).

Let $K/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field k. Write $\mathcal{O}_n = \mathcal{O}/\mathfrak{m}^{n+1}$. Given $\bar\chi : \Gamma \to \mathcal{O}_0^\times$ and a lift $\chi_n : \Gamma \to \mathcal{O}_n^\times$ ($n \geqslant 2$), we are interested in the existence (possibly after enlarging S) of a lift of χ_n to $\mathcal{O}$. This will be accomplished by making

$$Ш^1_S(\boldsymbol{\mu}_p) = \ker\left(\mathrm{H}^1(\Gamma_S, \boldsymbol{\mu}_p) \to \bigoplus_{l \in S} \mathrm{H}^1(\Gamma_l, \boldsymbol{\mu}_p)\right)$$

vanish, which should be possible via adding nice primes.

References

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*. Second. Vol. 323. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, 2008.