Discrepancy bounds over function fields

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Let $C_{/\mathbf{F}_p}$ be smooth proper curve, $\mathcal{F}$ a smooth l-adic sheaf, pure of weight zero on C. Let G^{geom} be the geometric monodromy group of $\mathcal{F}$, and assume what is necessary (see §9.0 of Katz–Sarnak) for the Sato–Tate conjecture to hold. That is, let K be a maximal compact subgroup of $G^{\mathrm{geom}}(\mathbf{C})$ and $K^\natural$ the set of conjugacy classes in K. For each $c \in C$, we have the Frobenius conjugacy class $\vartheta(c) \in K^\natural$. Katz–Sarnak prove that, for each irreducible $\rho\colon K \to \mathrm{GL}(n, \mathbf{C})$, we have the bound

$$\left| \frac{1}{\#\{c \in |C| : \deg c \leqslant N\}} \sum_{c \in |C| : \deg c \leqslant N} \operatorname{tr} \rho(\vartheta(c)) \right| \ll \#\{c \in |C| : \deg c \leqslant N\}^{-\frac{1}{2}}.$$

In simpler terms, if we enumerate the $c \in |C|$ as $c_1, c_2, \dots$ with ascending degree, then

$$\left| \frac{1}{N} \sum_{n \leqslant N} \operatorname{tr} \rho(\vartheta(c_n)) \right| \ll N^{-\frac{1}{2}}. \tag{1}$$

I am wondering whether a similar bound is known on the discrepancy of the $\vartheta(c_n)$. For example, assume $K = \mathrm{SU}(2)$, so that $K^\natural = [0, \pi]$. Put

$$D_N(\{\vartheta(c_n)\}) = \sup_{0 \leqslant \alpha \leqslant \pi} \left| \frac{\#\{n \leqslant N : \vartheta(c_n) \in [0, \alpha)\}}{N} - \int_0^\alpha \frac{2}{\pi} \sin^2(\theta)\, \mathrm{d}\theta \right|.$$

It is natural to conjecture (Akiyama and Tanigawa have for elliptic curves over $\mathbf{Q}$) that

$$D_N(\{\vartheta(c_n)\}) \ll N^{-\frac{1}{2}+\epsilon}. \tag{2}$$

Is this known in the above case? Via the Koksma–Hlawka inequality, a discrepancy bound like (2) certainly implies the estimate (1), but I am wondering if (2) is known even in the case where $\mathcal{F}$ comes from a family of elliptic curves $E_{/C}$.

I have constructed a sequence of angles $\vartheta_n \in [0, \pi]$ such that $|\sum_{n \leqslant N} \operatorname{tr} \rho(\vartheta_n)| \ll_\rho 1$, but for which the discrepancy decays like $N^{-1/k}$ for arbitrary k. So analytically, there is no a priori reason that the truth of (1) should imply that of (2).