Counterexamples related to the Sato–Tate conjecture

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Outline

Motivation and background

Discrepancy and Dirichlet series

Main theorem

Idea of the proof

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Sato–Tate Conjecture

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Use discrepancy (Kolmogorov–Smirnov statistic).

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$$D_N = \sup_{x \in [0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p \leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta) \, \mathrm{d}\, \mathsf{ST}(\theta) \right|.$$

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Related results

Theorem (Bucar–Kedlaya). Assume analytic continuation of $L(\mathrm{sym}^k E, s)$, GRH, and functional equation for all $k \geqslant 1$. Then $\mathrm{D}_N \ll N^{-\frac{1}{4}+\epsilon}$.

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Common ingredient. Erdös–Turán–Koksma inequality: from a bound on $\left|\sum_{p \leqslant N} \mathrm{tr}\, \rho(x_p)\right|$ to a bound on D_N.

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Theorem (Pande)

Let $\epsilon > 0$. *Then there exists an infinitely ramified representation* $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ *such that* $\theta_p \in B_\epsilon(\pi/2)$ *for a density one set of primes.*

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Answer (Khare–Larsen–Ramakrishna): no!

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Answer: Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Discrepancy

Definition

Let $\{\theta_p\}$ be a sequence in $[0,\pi]$, μ a measure on $[0,\pi]$. The *discrepancy* is

$$\mathrm{D}_N(\{\theta_p\},\mu) = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x)}(\theta_p) - \int_0^x \mathrm{d}\mu \right|.$$

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Fact: $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

Dirichlet series

Definition

For $k \geqslant 1$,

$$L(\operatorname{sym}^k \rho, s) = \prod_p \det\left(1 - \operatorname{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix} p^{-s}\right)^{-1}$$

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Example (Ramakrishna): $L_{\operatorname{sgn}}(s) = \prod_p \left(1 - \operatorname{sgn}(a_p) p^{-s}\right)^{-1}$.

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5. Fix $\alpha \in \left(0, \frac{1}{3}\right)$. The discrepancy D_N will decay like $\pi(N)^{-\alpha}$.

Theorem

Let l, $\bar{\rho}$, h, μ, and α be as above. Then there exists $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that

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4. $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(x)^{-\alpha})$. (Yes to Q1–Q3.)
5. If $(\theta \mapsto \pi - \theta)_* \mu = \mu$, then for each odd k, $L(\operatorname{sym}^k \rho, s)$ satisfies the Riemann hypothesis.

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2. $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$. (Yes to Q4. $\log x$, $\log^{10^{10}} x$, $A^{-1}(x)$)
3. For each unramified p, $a_p = \operatorname{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.
4. $\mathrm{D}_N(\{\theta_p\}, \mu) = \Theta(\pi(x)^{-\alpha})$. (Yes to Q1–Q3.)
5. If $(\theta \mapsto \pi - \theta)_* \mu = \mu$, then for each odd k, $L(\operatorname{sym}^k \rho, s)$ satisfies the Riemann hypothesis. (Yes to Q5.)

Idea of the proof

Prescribing discrepancy decay

Theorem

If $\alpha \in \left(0, \frac{1}{3}\right)$, *there exists a sequence* $(x_2, x_3, x_5, \dots)$ *in* $[-1, 1]$ *such that* $|D_N - \pi(N)^{-\alpha}| \ll \pi(N)^{-1}$.

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Fact: If $(x_p^{(1)})$ is a sequence with $|x_p - x_p^{(1)}| \ll p^{-1/2+\epsilon}$, then $\mathrm{D}_N^{(1)} = \Theta(\pi(N)^{-\alpha})$.

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Construct ρ as $\varprojlim \rho_n$, where $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/I^n)$.

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Fact: constant in $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$ only depends on $\bar{\rho}$.

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$\ll N^{\frac{1}{2}}$.

Consequences

If $f \in C([0,\pi])$, $f \circ \cos^{-1} \colon [-1,1] \to \mathbf{C}$ is Lipschitz, and $f(\pi - \theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

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Can get equidistribution with respect to μ with non-continuous probability distribution functions.

Questions?