Towards an adelic completed cohomology

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Let F be a number field, G a split reductive group over F. For an open compact subgroup $K_\mathrm{f} \subset G(\mathbf{A}_\mathrm{f})$, we have the associated locally symmetric space

$$Y(K_\mathrm{f}) = G(F)\backslash G(\mathbf{A}_F)/K_\mathrm{f} K_\infty Z_\infty.$$

Let $E \supset F$. If ρ is a representation of G defined over E, then there is a compatible system of sheaves $\mathscr{V}_\rho$ on the $Y(K_\mathrm{f})$. One often studies the cohomology

$$\mathrm{H}^\bullet(\mathscr{V}_\rho) = \varinjlim_{K_\mathrm{f}} \mathrm{H}^\bullet(Y(K_\mathrm{f}), \mathscr{V}_\rho).$$

We can do better. Since G is split, it and ρ descend to objects over $\mathcal{O}_E$. Thus we can define the *adelic completed cohomology*

$$\mathbf{H}^\bullet(\mathscr{V}_\rho) = \mathbf{Q} \otimes \varprojlim_{\mathfrak{a}\subset \mathcal{O}_E} \varinjlim_{K_\mathrm{f}} \mathrm{H}^\bullet(Y(K_\mathrm{f}), \mathscr{V}_\rho/\mathfrak{a}).$$

Since $\mathbf{A}_{E,\mathrm{f}} = \mathbf{Q} \otimes \varprojlim_\mathfrak{a} \mathcal{O}_E/\mathfrak{a}$, this is naturally a $\mathbf{A}_{E,\mathrm{f}}$-module. Moreover, it has a continuous action of $G(\mathbf{A}_{F,\mathrm{f}})$. Unfortunately, this will probably not be very interesting. Note that for each $\mathfrak{a} \subset O_E$,

$$\mathscr{V}_\rho/\mathfrak{a} = \prod_{\mathfrak{p}|\mathfrak{a}} \mathscr{V}_\rho/\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})}.$$

It follows that $\mathbf{H}^\bullet(\mathscr{V}_\rho)$ is a restricted direct product of the $\varprojlim_e \varinjlim_{K_\mathrm{f}} \mathrm{H}^\bullet(Y(K_\mathrm{f}), \mathscr{V}_\rho/\mathfrak{p}^e)$. This is very close to Emerton's completed cohomology.

The question is what topology to put on $\mathbf{H}^\bullet(\mathscr{V}_\rho)$, and what sort of inductive and projective limits to take. It would need to live inside a "good" category of representations of $G(\mathbf{A}_{F,\mathrm{f}})$. These would have to be $\mathbf{A}_{E,\mathrm{f}}$-modules of infinite rank.

1 The case of tori

Let $G = T$ be a torus over F, and put $E = F$. For $K_\mathrm{f} \subset \mathbf{A}_\mathrm{f}^\times$, the quotient $Y(K_\mathrm{f}) = T(F)\backslash T(\mathbf{A}_F)/K_\mathrm{f} T(F_\infty)$ is a finite set. In particular, for $\rho = 1$, our cohomology is concentrated in degree zero. We have

$$\mathbf{H}^0 = \mathbf{Q} \otimes \varprojlim_{\mathfrak{a}\subset \mathcal{O}_F} \varinjlim_{K_\mathrm{f}} (\mathcal{O}_E/\mathfrak{a})^{\pi_0(Y(K_\mathrm{f}))}$$

Suppose T is split, so that T descends to a torus over O_F. Then the $K(\mathfrak{a}) = \ker(T(\widehat{\mathcal{O}_F}) \to T(\mathcal{O}_F/\mathfrak{a}))$ form a system of neighborhoods of the identity in $T(\mathbf{A}_\mathrm{f})$.

Let's try $T = \mathrm{GL}(1)$, $F = \mathbf{Q}$. Then

$$\begin{aligned} Y(K(n)) &= \mathbf{Q}^\times \backslash \mathbf{A}^\times / K(n) R^\times \\ &= \widehat{\mathbf{Z}}^\times / K(n) \\ &= (\mathbf{Z}/n)^\times. \end{aligned}$$

For given $a \in \mathbf{Z}$, the inductive limit $\varinjlim \mathrm{H}^0(Y_0(K(n)), \mathbf{Z}/a)$ is just $C^\infty(\widehat{\mathbf{Z}}^\times, \mathbf{Z}/a)$, that is continuous maps $\widehat{\mathbf{Z}}^\times \to \mathbf{Z}/a$. From the definition of projective limits, we get that $\mathbf{H}^0$ is the space of continuous maps $\widehat{\mathbf{Z}}^\times \to \mathbf{A}$. This is a huge monstrosity! But how is this a representation of $\mathbf{A}_\mathrm{f}^\times$?