Local Langlands for GL(n) over p-adic fields [after Peter Scholze]

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This is an exposition of Scholze's paper [Sch13]. For the remainder of this note, let F be a finite extension of $\mathbf{Q}_p$. Let $\mathcal{O}$ be the ring of integers of F, and let $k = \mathcal{O}/\mathfrak{m}$ be the residue field of $\mathcal{O}$. Write ϖ for a uniformizer of F.

1 Statement of the conjectures

Here we follow [Wed08]. Let Γ be an arbitrary locally profinite group.

Definition 1.1. *An* admissible representation of Γ *is a (possibly infinite-dimensional) vector space π with Γ-action such that:*

1. *For all open subgroups $U \subset \Gamma$, the space $\mathrm{H}^0(U, \pi)$ is finite-dimensional.*
2. $\pi = \varinjlim_{U \subset \Gamma} \mathrm{H}^0(U, \pi)$.

Note that we did not mention what field π is a vector space over. This is intentional. Since no mention is made of the topology on the field, the notion of "admissible representation" only depends on its isomorphism type as an abstract field. We will only consider vector spaces over fields abstractly isomorphic to $\mathbf{C}$, e.g. $\overline{\mathbf{Q}}_l$.

For each $n \geqslant 1$, let $\mathcal{A}_n(F)$ be the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. This has a subset $\mathcal{A}_n^{\mathrm{cusp}}(F)$ consisting of those π for which $\hom(\pi, \mathrm{ind}_P^{\mathrm{GL}_n(F)} \rho) = 0$ for all parabolic subgroups $P \subset \mathrm{GL}_n(F)$. Such π are called *supercuspidal*; equivalently the matrix coefficients of π are compact support modulo the center of $\mathrm{GL}_n(F)$.

Let $W_F \subset \mathrm{Gal}(\overline{F}/F)$ be the set of σ such that for some $r \in \mathbf{Z}$, we have $\sigma(x) \equiv \mathrm{fr}^r(x) \mod \mathfrak{m}_{\overline{F}}$ for all $x \in \mathcal{O}_{\overline{F}}$. This is the *Weil group* of F.

Definition 1.2. *A* Weil-Deligne representation *of W_F is a pair $\rho = (r, N)$, where $r : W_F \to \mathrm{GL}(V)$ is a continuous representation (V is a finite-dimensional vector space with the discrete topology) and $N : V \to V$ is a nilpotent linear map such that*

$$\mathrm{Ad}\, r(\gamma)(N) = |\gamma| \cdot N$$

for all $\gamma \in W_F$.

Here $|\cdot| : W_F \to \mathbf{R}^\times$ is defined by $|\sigma| = |\varpi^r|$, where $r \in \mathbf{Z}$ is such that $\sigma \equiv \mathrm{fr}^r \pmod p$. One says $\rho = (r, N)$ is *Frobenius semisimple* if $r : W_F \to \mathrm{GL}(V)$ is semisimple. Note that as with admissible representations, no mention is made of the field over which V is a vector space. As above, we will work with either $\mathbf{C}$ or $\overline{\mathbf{Q}}_l$. It turns out that a Weil-Deligne representation is the same thing as an "honest" representation of the pro-algebraic *Weil-Deligne group* $\mathrm{WD}_F = W_F \ltimes \mathbf{G}_{\mathrm{a}}$, via the action $\gamma \cdot x = |\gamma|^{-1} x$. Essentially, this is because representations of $\mathbf{G}_{\mathrm{a}}$ are exactly "choices of nilpotent endomorphisms."

For each $n \geqslant 1$, let $\mathcal{G}_n(F)$ be the set of equivalence classes of Frobenius-semisimple n-dimensional Weil-Deligne representations of W_F. It has a distinguished subset $\mathcal{G}_n^{\mathrm{irr}}(F)$ consisting of irreducible representations.

Conjecture 1.3 (local Langlands)**.** *There is a unique set of bijections* $\{\mathrm{rec}_n : \mathcal{A}_n(F) \to \mathcal{G}_n(F)\}$ *such that:*

1. rec_1 *is induced by local Class Field Theory.*
2. *The maps* rec_n *preserve L- and ε- factors:*

$$\begin{aligned} L(\pi_1 \oplus \pi_2) &= L(\mathrm{rec}(\pi_1) \otimes \mathrm{rec}(\pi_2)) \\ \varepsilon(\pi_1 \oplus \pi_2) &= \varepsilon(\mathrm{rec}(\pi_1) \otimes \mathrm{rec}(\pi_2)). \end{aligned}$$

3. *If* $\chi \in \mathcal{A}_1$*, then* $\mathrm{rec}(\pi \otimes \chi) = \mathrm{rec}(\pi) \otimes \mathrm{rec}_1(\chi)$.
4. *If* π *has central character* χ*, then* $\det \circ \mathrm{rec}(\pi) = \mathrm{rec}_1(\chi)$.

We will not define the L- and ε- factors for general representations. The one example is:

$$L(\rho, s) = \det\left(1 - q^{-s} r(\mathrm{fr}^{-1}), (\ker N)^I\right)^{-1},$$

where $q = \#k$ and $I \subset W_F$ is the *inertia group* of F. Henniart proved in [Hen93] that these requirements characterize the reciprocity map uniquely. Moreover, for such a correspondence, π is supercuspidal if and only if $\mathrm{rec}(\pi)$ is irreducible. Moreover, π is a subquotient of the parabolic induction of some $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ if and only if $\mathrm{rec}(\pi) = \mathrm{rec}(\pi_1) \oplus \cdots \oplus \mathrm{rec}(\pi_r)$. The local Langlands conjecture for $\mathrm{GL}(n)$ was originally proved by Harris and Taylor [HT01].

2 Moduli spaces of p-divisible groups

The presentation of p-divisible group here follows that of Messing in [Mes72]. Recall that p is the residue characteristic of F. Let $\mathbf{Z}(p^\infty) = \mathbf{Q}_p/\mathbf{Z}_p$; this is an ind-cyclic p-torsion group sometimes called the Prüfer group.

Definition 2.1. *Let S be a scheme. A* p-divisible group *over S is an fppf sheaf $G_{/S}$ such that:*

1. $\hom(\mathbf{Z}(p^\infty), G) \xrightarrow{\sim} G$ *"p-torsion"*
2. $G \xrightarrow{p} G$ *is an epimorphism "p-divisible"*
3. *Each* $G[p^n] = \hom(\mathbf{Z}/p^n, G)$ *is a finite flat group scheme on S.*

It follows that $G = \varinjlim G[p^n]$, where each $G[p^n]$ is a finite flat group scheme for which multiplication by p^i induces an isomorphism $G[p^{n+i}] \xrightarrow{p^i} G[p^n]$. Thus this definition agrees with the more traditional one. The main examples are:

1. The constant p-divisible group $\mathbf{Z}(p^\infty)$.
2. $\mathbf{G}_\mathrm{m}[p^\infty] = \boldsymbol{\mu}_{p^\infty} = \varinjlim \boldsymbol{\mu}_{p^n}$.
3. If $A_{/S}$ is an abelian scheme, $A[p^\infty] = \varinjlim A[p^n]$.

If $G_{/S}$ is a p-divisible group, we put $\mathrm{Lie}(G) = \mathrm{Lie}(G[p^n])$ for any $n \gg 0$. This is a locally free $\mathscr{O}_S$-module. We put $\dim(G) = \dim(\mathrm{Lie}\, G)$. Each $G[p^n]$ will be locally free of rank p^{nh} for some fixed $h = \mathrm{ht}(G)$, called the *height* of G. Recall that ϖ is a uniformizer in $\mathcal{O} = \mathcal{O}_F$.

Definition 2.2. *Let $S_{/\mathcal{O}}$ be a scheme. A* ϖ-divisible group *on S is a p-divisible group $G_{/S}$ together with a homomorphism $\mathcal{O} \to \mathrm{End}(G)$ such that the induced action of $\mathcal{O}$ on $\mathrm{Lie}(G)$ agrees with the usual one.*

Let $G_{0/k}$ be a ϖ-divisible group. We define a functor $\mathfrak{X}_{G_0}$ on connected artinian schemes over $\mathcal{O}$ by letting $\mathfrak{X}_{G_0}(S)$ be the set of isomorphism classes of ϖ-divisible groups $G_{/S}$ with an isomorphism $G \otimes k \xrightarrow{\sim} G_0$. Suppose now that $\dim(G_0) = 1$ and $\mathrm{ht}(G_0) = n$. Then for each $m \geqslant 1$, let $\mathfrak{X}_{G_0,m}(S)$ be the set of isomorphism classes of $G \in \mathfrak{X}_{G_0}(S)$ together with $x_1, \ldots, x_n \in G[p^m]$ such that $\langle x_1, \ldots, x_n \rangle = G[p^m]$ as relative Cartier divisors. Clearly $\mathrm{GL}_n(\mathcal{O}/\mathfrak{m}^{m+1})$ acts on $\mathfrak{X}_{G_0,m}$ for each m. So if we put

$$H_{G_0} = \varinjlim_m \mathrm{H}^0\left(\mathrm{R}\psi_{\mathfrak{X}_{G_0}} \overline{\mathbf{Q}_l}\right),$$

then H_{G_0} is a complex of $\overline{\mathbf{Q}_l}$-vector spaces with $\mathrm{GL}_n(\mathcal{O}) \times W_F$-action.

Here we use formal nearby cycle sheaves in the sense of Berkovich [Ber96], though he calls them vanishing cycle sheaves. If $\mathfrak{X}_{/\mathcal{O}}$ is a formal scheme, there is a functor $\psi : \mathrm{Sh}_{\text{ét}}(\mathfrak{X}_{\overline{F}}) \to \mathrm{Sh}_{\text{ét}}(\mathfrak{X}_{\overline{k}})$, where $(\psi\mathscr{F})(U) = \mathscr{F}(V)$, where $V = \mathfrak{V}_{\overline{F}}$ and $\mathfrak{V}_{/\mathcal{O}_{\overline{F}}}$ is the unique étale cover whose special fiber is U.

For each $r \geqslant 1$, let F_r/F be the unique degree-r unramified extension of F. Using a relative version of Dieudonné theory, Scholze notices that to each $\beta \in \Gamma_0(\varpi^m) \backslash \mathrm{GL}_n(\mathcal{O}_r)/\Gamma_0(\varpi^m)$, there is associated a unique one-dimensional ϖ-divisible group $\overline{G}_\beta$ of height n over $\mathcal{O}_r$, hence a formal scheme $\mathfrak{X}_{\beta,m}$ parameterizing ϖ-divisible lifts of $\overline{G}_\beta$. The scheme $\mathfrak{X}_{\beta,m}$ admits a $\mathrm{GL}_n(\mathcal{O}_r/\mathfrak{m}^m)$-action. Put

$$H_\beta = \varinjlim_m \mathrm{H}^0\left(\mathrm{R}\psi_{\mathfrak{X}_{\beta,m}} \overline{\mathbf{Q}_l}\right);$$

this is a virtual $W_{F_r} \times \mathrm{GL}_n(\mathcal{O}_r)$-representation.

3 Orbital integrals and transfer

The basic idea is follows. Let $\tau \in I \cdot \mathrm{fr}^r \subset W_F$. Recall that F_r is the degree-r unramified extension of F. For $h \in C_c^\infty(\mathrm{GL}_n(F_r))$, define $h^\vee(x) = h({}^{\mathrm{t}}x^{-1})$. Define an element $\phi_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F_r))$ by

$$\phi_{\tau,h}(\beta) = \begin{cases} \mathrm{tr}(\tau \times h^\vee, H_\beta) & \beta \in \mathrm{GL}_n(\mathcal{O}_r)\,\mathrm{diag}(\varpi, 1, \ldots, 1)\,\mathrm{GL}_n(\mathcal{O}_r) \\ 0 & \text{otherwise} \end{cases} \tag{$*$}$$

Then $\phi_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F_r))$ has values in $\mathbf{Q}$, independent of l. Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. We would like to characterize $\mathrm{rec}(\pi)$ by:

$$\mathrm{tr}(\phi_{\tau,h}, \pi) = \mathrm{tr}(\tau, \mathrm{rec}(\pi))\,\mathrm{tr}(h, \pi),$$

but this does not work because $\phi_{\tau,h}$ does not act on π. We must "push down" $\phi_{\tau,h}$ to a function $f_{\tau,h} \in \mathrm{GL}_n(F)$. This is done via requiring that orbital integrals match as follows.

Here, we loosely follow [AC89, p. 1.3]. For $\gamma \in \mathrm{GL}_n(F)$ and $\delta \in \mathrm{GL}_n(F_r)$, put

$$\begin{aligned} G_\gamma &= \{x \in \mathrm{GL}_n(F) : x^{-1}\gamma x = \gamma\} \\ G_{\delta,\mathrm{fr}} &= \{x \in \mathrm{GL}_n(F_r) : x^{-1}\gamma \mathrm{fr}(x) = \gamma\}, \end{aligned}$$

where $\mathrm{fr} : F_r \to F_r$ is the Frobenius map. Define

$$\begin{aligned} \mathrm{O}_\gamma(f) &= \int_{G_\gamma \backslash \mathrm{GL}_n(F)} f(x^{-1}\gamma x)\,\mathrm{d}\dot{x} && f \in C_c^\infty(\mathrm{GL}_n(F)) \\ \mathrm{TO}_\delta(\phi) &= \int_{G_{\delta,\mathrm{fr}} \backslash \mathrm{GL}_n(E)} \phi(x^{-1}\delta \mathrm{fr}(x))\,\mathrm{d}\dot{x} && \phi \in C_c^\infty(\mathrm{GL}_n(F_r)). \end{aligned}$$

Define the *norm map* $\mathrm{N} : \mathrm{GL}_n(F_r) \to \mathrm{GL}_n(F)$ by

$$\mathrm{N}(g) = g \cdot \mathrm{fr}(g) \cdots \mathrm{fr}^{r-1}(g).$$

Given $\phi \in C_c^\infty(\mathrm{GL}_n(F_r))$, there exists $f \in C_c^\infty(\mathrm{GL}_n(F))$, called the function *associated to* ϕ, such that for all regular $\delta \in \mathrm{GL}_n(F_r)$, we have

$$\mathrm{O}_\gamma(f) = \begin{cases} 0 & \gamma \text{ is not a norm} \\ \mathrm{TO}_\delta(\phi) & \gamma = \mathrm{N}(\delta) \end{cases}.$$

Even though the function f associated to ϕ is not well-defined, its traces are, so we have the following result.

Theorem 3.1. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Then there exists a unique $\rho \in \mathcal{G}_n(F)$ such that*

$$\mathrm{tr}(f_{\tau,h}, \pi) = \mathrm{tr}(\tau, \rho)\,\mathrm{tr}(h, \pi)$$

for all $\tau \in \mathrm{fr}^r \cdot I$ and $h \in C_c^\infty(\mathrm{GL}_n(F))$, where $f_{\tau,h}$ is associated to $\phi_{\tau,h}$ defined as in (∗). If we put

$$\mathrm{rec}(\pi) = \rho\left(\frac{n-1}{2}\right) \qquad \text{(Tate twist)},$$

then $\pi \mapsto \mathrm{rec}(\pi)$ realizes the local Langlands correspondence of Conjecture 1.3.

4 Global theory

Just as local class field theory was first proved via global class field theory, we need to embed the "local problem" of defining $\mathrm{rec}(\pi)$ into the "global problem" of associating Galois representations to automorphic representations.

First, Scholze uses induction on n to show that Theorem 3.1 follows from the result restricted to the class of either essentially square-integrable (square-integrable up to a twist by a power of the determinant) or "generalized Speh representations."

One can realize F as $\mathbf{F}_v$ for $v \mid p$ a place of a CM field $\mathbf{F}$ satisfying certain technical hypotheses [Sch13, §8]. One then constructs a central division algebra $\mathbf{D}$ over $\mathbf{F}$ together with an involution $\dagger$ of the second kind such that the $\mathbf{F}^\dagger$-group G_0 defined by

$$G_0(A) = \{g \in (A \otimes_{\mathbf{F}^\dagger} \mathbf{D})^\times : gg^\dagger = 1\}$$

is unitary of signature $(1, n-1)$ at one infinite place, and $(0, n)$ at all other infinite places. Define an $\mathbf{F}$-group by

$$G(A) = \{g \in (A \otimes_{\mathbf{F}} \mathbf{D})^\times : gg^\dagger = 1\}.$$

For each irreducible representation ξ of G, one has a $\Gamma_{\mathbf{F}} \times G(\mathbf{A}_\mathrm{f})$ representation

$$H_\xi = \varinjlim \mathrm{H}^\bullet_{\text{ét}}\left(\mathrm{Sh}_K(G), \mathscr{V}_\xi(\overline{\mathbf{Q}_l})\right),$$

in which $\mathscr{V}_\xi$ is the standard automorphic vector bundle on the Shimura variety $\mathrm{Sh}_K(G)$ associated to ξ.

If π is an automorphic representation of G, put $W_\xi(\pi) = \hom_{G(\mathbf{A}_\mathrm{f})}(\pi_\mathrm{f}, H_\xi)$. This is a $\Gamma_{\mathbf{F}}$-module. For a representation π_p of $\mathrm{GL}_n(F)$ of the special type specified above, the correspondence $\pi_p \leftrightarrow \mathrm{rec}(\pi_p)$ roughly occurs in $W_\xi(\pi)$ for some ξ. Since $G(\mathbf{Q}_p) = \mathrm{GL}_n(F) \times D^\times \times \mathbf{Q}_p^\times$ for the division algebra $D_{/\mathbf{Q}_p}$ of invariant $\frac{1}{n}$, this is at least plausible. In fact, $\pi_p \leftrightarrow \mathrm{rec}(\pi_p)$ only occurs at the level of "Grothendieck groups tensored with $\mathbf{Q}$." One needs Scholze's Lemma 3.2 to deduce Theorem 3.1.

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