Basic facts about Hodge structures

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Let K/k be a finite Galois extension with Galois group Γ. We write $\mathbf{S} = \mathrm{R}_{K/k}\,\mathbf{G}_\mathrm{m}$ and let $\mathrm{Hdg}(K/k) = \mathrm{Rep}_k(\mathbf{S})$. By definition, for any k-algebra A, we have

$$\mathbf{S}(A) = (A \otimes_k K)^\times.$$

Recall that there is a canonical k-algebra isomorphism $K \otimes_k K \to \prod_\Gamma K$, given by $x \otimes y \mapsto (x\gamma(y))_\gamma$. It induces a canonical isomorphism $\mathbf{S}_K \xrightarrow{\sim} \prod_\Gamma \mathbf{G}_\mathrm{m}$, given on A-points by

$$a \otimes x \mapsto (a\gamma(x))_\gamma.$$

Write $z_\gamma : \mathbf{S}_K \to \mathbf{G}_\mathrm{m}$ for the component $a \otimes x \mapsto a\gamma(x)$. Note that $\mathrm{X}^*(\prod_\Gamma \mathbf{G}_\mathrm{m}) = \mathbf{Z}^\Gamma$. So $z : \mathbf{S}_K \xrightarrow{\sim} \prod_\Gamma \mathbf{G}_\mathrm{m}$ induces an isomorphism $z^* : \mathbf{Z}^\Gamma \xrightarrow{\sim} \mathrm{X}^*(\mathbf{S}_K)$.

There is a canonical homomorphism $w : \mathbf{G}_\mathrm{m} \to \mathbf{S}$ given by $w(a) = a \otimes 1$ on A-points. The induced map $w^* : \mathrm{X}(\mathbf{S}_K) \to \mathbf{Z}$ is just $(z_\gamma) \mapsto \sum z_\gamma$. If $V \in \mathrm{Hdg}(K/k)$, we write $w^* V$ for the induced representation of $\mathbf{G}_\mathrm{m}$. Note that there is a canonical decomposition

$$w^* V = \bigoplus_{\chi \in \mathrm{X}^*(\mathbf{G}_\mathrm{m})} (w^* V)_\chi.$$

The *weight* of V, denoted $\mathrm{wt}(V)$, is by definition the set $\{\chi \in \mathrm{X}^*(\mathbf{G}_\mathrm{m}) : (w^* V)_\chi \neq 0\}$. We say that V is *pure of weight* n if $\mathrm{wt}(V) = \{n\}$.

If $V \in \mathrm{Hdg}(K/k)$, there is a decomposition

$$V_K = \bigoplus_{\chi \in \mathrm{X}^*(\mathbf{S}_K)} V_{K,\chi}.$$

The *type* of V, denoted $\mathrm{tp}(V)$, is by definition the set $\{\chi \in \mathbf{Z}^\Gamma : (z^* V)_\chi \neq 0\}$. Alternatively, could think of $\mathrm{tp}(V)$ as being a subset of $\mathrm{X}^*(\mathbf{S}_K)$.

It is a general theorem that $\mathrm{X}_*(\mathbf{S}_K) = \mathrm{X}^*(\mathbf{S}_K)^\vee$. Since $\mathbf{Z}^\Gamma$ is self-dual as a representation of Γ, we have that $\mathrm{X}_*(\mathbf{S}_K) = \mathbf{Z}^\Gamma$. For $\gamma \in \Gamma$, write $z_\gamma^\vee : \mathbf{G}_\mathrm{m} \to \mathbf{S}_K$ for the cocharacter dual to $z_\gamma : \mathbf{S}_K \to \mathbf{G}_\mathrm{m}$. Write $\mu_{\gamma,h} : \mathbf{G}_\mathrm{m} \to \mathrm{GL}(V_K)$ for the composite $\mu_{\gamma,h} = h \circ z_\gamma^\vee$. More generally, for $\chi \in \mathbf{Z}^\Gamma$, we have $\mu_{\chi,h} = h \circ \chi^\vee : \mathbf{G}_\mathrm{m} \to \mathrm{GL}(V_K)$.