Counterexamples related to the Sato–Tate conjecture for CM abelian varieties*

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31 March 2017

1 Introduction and motivation

Let $K/\mathbf{Q}$ be a finite Galois extension, $A_{/K}$ a g-dimensional abelian variety. Assume that A has CM defined over K; then there is a CM field F together with an isomorphism $F \simeq \operatorname{End}_K(A)_{\mathbf{Q}}$. Let $\mathfrak{a} = \operatorname{Lie}(A)$. Then the determinant of the action of K on $\mathfrak{a}$ (viewed as an F-vector space) gives a map $\det_{\mathfrak{a}}\colon \mathrm{R}_{K/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}} \to \mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}$. Put $G_A = \operatorname{im}(\det_{\mathfrak{a}})$ and $G_A^1 = \operatorname{im}(\det_{\mathfrak{a}})^{\mathrm{N}_{F/\mathbf{Q}}=1}$. The group G_A is the motivic Galois group of A, and $\mathrm{ST}(A)$, the Sato–Tate group of A, is a maximal compact subgroup of $G_A^1(\mathbf{C})$. It is obvious that the l-adic Galois representation associated with A, $\rho_l\colon G_{\mathbf{Q}} \to \mathrm{GL}_{2g}(\mathbf{Q}_l)$, has image in $(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}})(\mathbf{Q}_l)$; it actually is a map $\rho_l\colon G_{\mathbf{Q}} \to G_A(\mathbf{Q}_l)$.

Unitary representations of $\mathrm{ST}(A)$ are just characters, and basic representation theory tells us that all such representations are induced by an (algebraic) character of G_A defined over $\overline{\mathbf{Q}}$. For $r \in \mathrm{X}^*(G_A)$, there is an L-function $L(r_*\rho_l, s)$ coming from the composite Galois representation $r \circ \rho_l\colon G_{\mathbf{Q}} \to G_A(\mathbf{Q}_l) \to \overline{\mathbf{Q}_l}^{\times}$. The Sato–Tate conjecture for A says that all $L(r_*\rho_l, s)$ have non-vanishing analytic continuation past $\Re = 1$, and the Generalized Riemann hypothesis for A says that all $L(r_*\rho_l, s)$ satisfy the Riemann hypothesis.

Choose an isomorphism $(\mathbf{R}/\mathbf{Z})^d \simeq \mathrm{ST}(A)$, and put

$$\mathrm{D}_x(A) = \sup_{t\in[0,1]^d} \left| \frac{1}{\pi_K(x)} \sum_{\mathrm{N}(\mathfrak{p})\leqslant x} 1_{[0,t)}(\theta_{\mathfrak{p}}) - \int 1_{[0,t)} \right|.$$

Akiyama and Tanigawa conjectured that for non-CM elliptic curves, $\mathrm{D}_x(E) \ll x^{-\frac{1}{2}+\epsilon}$. We call the "Akiyama–Tanigawa conjecture" for A the discrepancy de-

*Notes for a talk given in Cornell's Number Theory Seminar.

cays like $\mathrm{D}_x(A) \ll x^{-\frac{1}{2}+\epsilon}$. Via the Koksma–Hlawka inequality, the Akiyama–Tanigawa conjecture implies that for all bounded-variation functions f on $\mathrm{ST}(A)$, the estimate

$$\left| \sum_{\mathrm{N}(\mathfrak{p}) \leqslant x} f(\theta_{\mathfrak{p}}) \right| \ll \mathrm{Var}(f) x^{\frac{1}{2}+\epsilon}.$$

For $r \in \mathrm{X}^*(G_A)$, this estimate implies the Riemann hypothesis for the L-function $L(r_* \rho_l, s)$.

Analogy with Artin L-functions (go into detail here!) seems to suggest that if all $L(r_* \rho_l, s)$ satisfy the Riemann Hypothesis, then the Akiyama–Tanigawa conjecture for A holds. We'll show: this converse is false, in a limited sense.

2 Diophantine approximation

Let $x \in [0,1]$ be irrational. It is well known that the sequence $(x \mod 1, 2x \mod 1, 3x \mod 1, \dots)$ is equidistributed in $[0,1]$. What is less well known is that the rate of convergence of empirical measures from this sequence to the uniform measure is governed by the irrationality measure of x. The irrationality measure $\mu(x)$ is the supremum of the set of $w \geqslant 1$ such that there are infinitely many p/q with $\left|\mu - \frac{p}{q}\right| \leqslant q^{-w}$. Let's generalize this to higher-dimensional space.

3 Fake Satake parameters

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