On Sarnak's letter to Mazur

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Suppose we have an Euler product of the form

$$L(\rho, s) = \prod_{\mathfrak{p}} \frac{1}{\det(1 - \rho(\mathrm{fr}_{\mathfrak{p}}) \,\mathrm{N}(\mathfrak{p})^{-s})}$$

Write the characteristic polyomials

$$\det(1 - \rho(\mathrm{fr}_{\mathfrak{p}}) t) = \prod (1 - \lambda_{\mathfrak{p},i} t).$$

What follows is a well-known computation. First, note that

$$\frac{\mathrm{d}}{\mathrm{d}s} \log f(s) = \frac{f'}{f}(s).$$

Thus, we know that:

$$\begin{aligned}
-\frac{L'}{L}(\rho, s) &= -\frac{\mathrm{d}}{\mathrm{d}s} \log L(\rho, s) \\
&= -\frac{\mathrm{d}}{\mathrm{d}s} \sum_{\mathfrak{p}} \log \frac{1}{\prod_i (1 - \lambda_{\mathfrak{p},i} \,\mathrm{N}(\mathfrak{p})^{-s})} \\
&= \sum_{\mathfrak{p}} \sum_i \frac{\mathrm{d}}{\mathrm{d}s} \log(1 - \lambda_{\mathfrak{p},i} \,\mathrm{N}(\mathfrak{p})^{-s}) \\
&= -\sum_{\mathfrak{p}} \sum_i \frac{\mathrm{d}}{\mathrm{d}s} \sum_{j \geqslant 1} \frac{(\lambda_{\mathfrak{p},i} \,\mathrm{N}(\mathfrak{p})^{-s})^j}{j} \\
&= -\sum_{\mathfrak{p}} \sum_i \sum_{j \geqslant 1} \frac{\mathrm{d}}{\mathrm{d}s} \frac{\lambda_{\mathfrak{p},i}^j \,\mathrm{N}(\mathfrak{p})^{-js}}{j} \\
&= \sum_{\mathfrak{p}} \sum_i \sum_{j \geqslant 1} (\lambda_{\mathfrak{p},i}^j \log \mathrm{N}(\mathfrak{p})) \,\mathrm{N}(\mathfrak{p})^{-js} \\
&= \sum_{j \geqslant 1} \sum_{\mathfrak{p}} \frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{js}} \sum_i \lambda_{\mathfrak{p},i}^j
\end{aligned}$$

This, as a computation, is some general nonsense. What if the characteristic polynomials are $(1 - e^{i\theta_\mathfrak{p}} t)(1 - e^{-i\theta_\mathfrak{p}} t)$, and we are taking $\operatorname{sym}^n \rho$? Then the characteristic polynomials of $\operatorname{sym}^n \rho$ are

$$\prod_{a+b=n} (1 - e^{i\theta_\mathfrak{p} a} e^{-i\theta_\mathfrak{p} b} t) = \prod_{a+b=n} (1 - e^{i\theta_\mathfrak{p} (a-b)} t).$$

1 Deriving (6)

Here we prove that

$$\sum_{j=0}^{n} e^{i(n-2j)\theta} = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

This is a basic computation:

$$\begin{aligned}
\sum_{j=0}^{n} e^{i(n-2j)\theta} &= e^{in\theta} \sum_{j=0}^{n} (e^{-2i\theta})^j \\
&= e^{in\theta} \frac{(e^{-2i\theta})^{n+1} - 1}{e^{-2i\theta} - 1} \\
&= \frac{e^{i(n-2(n+1))\theta} - e^{in\theta}}{e^{-2i\theta} - 1} \\
&= \frac{e^{i(-n-2)\theta} - e^{in\theta}}{e^{-2i\theta} - 1} \\
&= \frac{e^{-i(n+1)\theta} - e^{i(n+1)\theta}}{e^{-i\theta} - e^{i\theta}} \\
&= \frac{\sin((n+1)\theta)}{\sin\theta},
\end{aligned}$$

the last step following from the well-known identify $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$. Define $U_n(\theta)$ to be that last function.

We know that

$$-\frac{L'}{L}(s, \operatorname{sym}^n \pi) = \sum_{r \geqslant 1} \sum_{p} \frac{\log p}{p^{rs}} U_n(r\theta_p)$$

2 General theory

For the moment, we look at the local theory. Start with an arbitrary invertible matrix $A(t)$ depending smoothly on t. Then Jacobi's formula tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det A(t) = \det A(t) \operatorname{tr}\left(A(t)^{-1} \frac{\mathrm{d}A}{\mathrm{d}t} t \right).$$

In other words, $\frac{\mathrm{d}}{\mathrm{d}t}\log\det A(t) = \mathrm{tr}\left(A(t)^{-1}\frac{\mathrm{d}A}{\mathrm{d}t}(t)\right)$.

So, for the function $L_{\mathfrak{p}}(\theta, s) = \det(1-\theta\,\mathrm{N}(\mathfrak{p})^{-s})^{-1}$, we can compute

$$\begin{aligned}
-\frac{L'_{\mathfrak{p}}}{L_{\mathfrak{p}}}(\theta, s) &= \mathrm{tr}\left((1-\theta\,\mathrm{N}(\mathfrak{p})^{-s})^{-1}\frac{\mathrm{d}}{\mathrm{d}s}(1-\theta\,\mathrm{N}(\mathfrak{p})^{-s})\right) \\
&= \mathrm{tr}\left(\sum_{r\geqslant 0}(\theta\,\mathrm{N}(\mathfrak{p})^{-s})^r\theta\,\mathrm{N}(\mathfrak{p})^{-s}\log\mathrm{N}(\mathfrak{p})\right) \\
&= \sum_{r\geqslant 0}\mathrm{tr}(\theta^r\,\mathrm{N}(\mathfrak{p})^{-rs}\theta\,\mathrm{N}(\mathfrak{p})^{-s}\log\mathrm{N}(\mathfrak{p})) \\
&= \log\mathrm{N}(\mathfrak{p})\sum_{r\geqslant 1}\frac{\mathrm{tr}(\theta^r)}{\mathrm{N}(\mathfrak{p})^{rs}}
\end{aligned}$$

So let's look at a global L-function

$$L(s) = \prod_{\mathfrak{p}}\det(1-\theta_{\mathfrak{p}}\,\mathrm{N}(\mathfrak{p})^{-s})^{-1}.$$

From the above computation, we have that

$$-\frac{L'}{L}(s) = \sum_{r\geqslant 1}\sum_{\mathfrak{p}}\frac{\log\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{rs}}\,\mathrm{tr}(\theta_{\mathfrak{p}}^r).$$

3 A misconception

Let $f = \sum a_n q^n$ be a modular cusp eigenform of weight k. There are two cadidates for the local L-factors of the L-function associated to f, namely

$$\begin{aligned}
L_p^{\mathrm{alg}}(f, s) &= (1 - a_p p^{-s} + p^{k-1}p^{-2s})^{-1} \\
L_p^{\mathrm{an}}(f, s) &= (1 - a_p p^{-(k-1)/2}p^{-s} + p^{-2s})^{-1} \\
&= L_p^{\mathrm{alg}}\left(s + \frac{k-1}{2}\right).
\end{aligned}$$

Essentially, the analytic L-function uses the normalized eigenvalues of Frobenius. Since we'll be doing analysis, we will always use the analytic L-function exclusively, and simply denote it by L. In particular, note if f is a weight-2 modular form correspoding to an elliptic curve $E_{/\mathbf{Q}}$, we have $L^{\mathrm{an}}(f, s) = L^{\mathrm{alg}}(E, s + 1/2)$, so that $L^{\mathrm{an}}(f, 1/2) = L^{\mathrm{alg}}(E, 1)$ is predicted by BSD.

4 Fancy approach

Let $G = \mathrm{SU}(2)$; this is a compact group. Let f be a weight-k modular cusp eigenform. We'll start without messing with symmetric powers. For each unramified prime p, we put

$$x_p = p^{-(k-1)/2} \begin{pmatrix} \alpha_1(p) & \\ & \alpha_2(p) \end{pmatrix} \in G^\natural,$$

where $\alpha_i(p)$ are the eigenvalues of $\rho_{E,l}(\mathrm{fr}_p)$. In other words, x_p is the (normalized ?) Satake parameter of π_p. The Sato-Tate conjecture tells us that $\{x_p\} \subset G^\natural$ is equidistributed.

For any representation ρ of G, we put, following Serre:

$$L(s,\rho) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}.$$

From stuff we already know, we have the computation

$$-\frac{L'}{L}(s, \mathrm{sym}^n) = \sum_{r \geqslant 1} \sum_p \frac{\log p}{p^{rs}} \operatorname{tr} \mathrm{sym}^n(x_p^r).$$

By Peter-Weil, the functions $\{\operatorname{tr} \mathrm{sym}^n\}$ form an orthonormal basis for $L^2(G^\natural)$.

More generally,

$$-\frac{L'}{L}(s,\rho) = \sum_{\nu \geqslant 1} \sum_{\mathfrak{p}} \frac{\log \mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{\nu s}} \operatorname{tr} \rho(x_{\mathfrak{p}}^\nu)$$

In other words, if we put the "Von-Mangoldt function"

$$\Lambda_\rho(\mathfrak{a}) = \begin{cases} \log \mathrm{N}(\mathfrak{p}) \operatorname{tr} \rho(x_{\mathfrak{p}}^\nu) & \text{if } \mathfrak{a} = \mathfrak{p}^\nu \\ 0 & \text{otherwise} \end{cases},$$

then

$$-\frac{L'}{L}(s,\rho) = \sum_{\mathfrak{a}} \frac{\Lambda_\rho(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s}.$$

5 Functional equation for algebraic L-functions

Here we follow [FPR94] in computing the conjectured functional equation for symmetric powers of an elliptic curve.

Let $E_{/\mathbf{Q}}$ be a non-CM elliptic curve. Consider the motive $\mathrm{H}^1(E)$. As Galois representations, we have the isomorphism $\mathrm{H}^1(E, \mathbf{Z}_l) \simeq T_l(E)^\vee$. But since [FPR94] defines local L-functions using geometric Frobenius, we have

$$P_p(\mathrm{H}^1(E), u) = \det(1 - \rho_{\mathrm{H}^1(E),l}(\text{geom. frob.}_p)u) = 1 - a_p u + p u^2.$$

Now consider $\operatorname{sym}^n \mathrm{H}^1(E)$. In general, suppose we have a 2×2 diagonal matrix $\theta = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}$. Here are a few of its symmetric powers:

$$\operatorname{sym}^2 \theta = \begin{pmatrix} \lambda^2 & & \\ & \lambda\mu & \\ & & \mu^2 \end{pmatrix}$$
$$\operatorname{sym}^3 \theta = \begin{pmatrix} \lambda^3 & & & \\ & \lambda^2\mu & & \\ & & \lambda\mu^2 & \\ & & & \mu^3 \end{pmatrix}$$
$$\cdots$$

This tells us that

$$\det(1 - (\operatorname{sym}^n \theta)u) = \prod_{a+b=n} (1 - \lambda^a \mu^b u) = \sum_{i=0}^{n} ()u^i$$

For sym^3, we have

$$\begin{aligned} \det(1 - (\operatorname{sym}^3 \theta)u) &= (1 - \lambda^3 u)(1 - \lambda^2 \mu u)(1 - \lambda\mu^2)(1 - \mu^3 u) \\ &= 1 - (\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)u + \cdots \end{aligned}$$

Notation: $\omega_s(x) = |x|^s$, as $\omega \colon K^\times \to \mathbf{C}^\times$. Here K is a local field.

References

[FPR94] Jean-Marc Fontaine and Bernadette Perrin-Riou. "Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L". In: *Motives*. Vol. 55. Proc. Sympos. Pure Math. 1994, pp. 599–706.