Coleman integration via Tannakian formalism

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August 7, 2016

This is inspired by [Bes02]. Let k and K be fields and $\mathcal{A} \to \mathsf{Var}_k$ a category fibered in abelian K-enriched tensor categories. That is, for any variety $X_{/k}$, we have an abelian tensor category $\mathcal{A}(X)$, enriched over K. We suppose further that there is a natural isomorphism $\mathcal{A}(\operatorname{Spec} k) = \mathsf{Vec}_K$. Finally, we suppose that for $f\colon X \to Y$, the pullback $f^*\colon \mathcal{A}(Y) \to \mathcal{A}(X)$ is exact.

For a variety $X_{/k}$, there is an "$\mathcal{A}$-fundamental groupoid," denoted $\Pi_1^{\mathcal{A}}(X)$. We have $\operatorname{Ob}\Pi_1^{\mathcal{A}}(X) = X(k)$. For $x, y \in \operatorname{Ob}\Pi_1^{\mathcal{A}}(X)$, the "hom-space" $\hom_{\Pi_1^{\mathcal{A}}(X)}(x, y)$ will be a pro-variety over K, namely

$$\hom_{\Pi_1^{\mathcal{A}}(X)}(x, y)(A) = \operatorname{Isom}^{\otimes}(x_A^*, y_A^*).$$

Here $x^*\colon \mathcal{A}(X) \to \mathcal{A}(\operatorname{Spec} k) = \mathsf{Vec}_K$ is the pullback induced by $x\colon \mathcal{A}(\operatorname{Spec} k) \to \mathcal{A}(X)$, and x_A^* is the base-change of x^* to the category of finite free A-modules. Similarly for y.

Given a morphism $f\colon X \to Y$, there is an induced morphism $f_*\colon \Pi_1^{\mathcal{A}}(X) \to \Pi_1^{\mathcal{A}}(Y)$ of fundamental groupoids. On objects, it is just $f\colon X(k) \to Y(k)$, and on morphisms,

$$f^*(\lambda)_A = \lambda_{f^*A}\colon (fx)^*A = x^*(f^*A) \to y^*(f^*A) = (fy)^*A.$$

References

[Bes02] Amnon Besser. "Coleman integration using the Tannakian formalism". In: *Math. Ann.* 322.1 (2002), pp. 19–48.