Equidistributed sequences and the analytic properties of a strange class of L-functions

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1 Motivation

Let $E_{/\mathbf{Q}}$ be an elliptic curve without complex multiplication. By an old theorem of Faltings, the quantities

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \operatorname{tr}\rho_{E,l}(\mathrm{fr}_p)$$

determine E up to isogeny. The starting point of this investigation is the corollary of a theorem of Harris, that the collection $\{\operatorname{sgn} a_p(E)\}_p$ in fact determines E up to isogeny. Ramakrishna had the insight that this fact means the "strange L-function"

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \operatorname{sgn} a_p(E) p^{-s}}$$

determines E up to isogeny. In this note, I define a more general class of strange L-functions, and show that their analytic properties are closely tied to the equidistribution of the $a_p(E)$.

Here is a brief discussion of this generalization in the case of a non-CM curve $E_{/\mathbf{Q}}$. It is convenient to repackage these traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the $\theta_p(E)$ are well-defined angles laying in the interval $[0, \pi]$. Write $\mu_{\mathrm{ST}} = \frac{2}{\pi}\sin^2\theta \,\mathrm{d}\theta$. Then the Sato–Tate conjecture (now a theorem) tells us that for any continuous function $f\colon [0,\pi] \to \mathbf{C}$, we have:

$$\left| \frac{1}{\pi(C)} \sum_{p \leqslant C} f(\theta_p) - \int_0^\pi f \,\mathrm{d}\mu_{\mathrm{ST}} \right| = o(1)$$

as $C \to \infty$. It is well-known that this is equivalent to the analytic continuation of all the L-functions $L(\operatorname{sym}^k E, s)$. We take as our starting point the stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(C)} \sum_{p \leqslant C} f(\theta_p) - \int_0^\pi f \,\mathrm{d}\mu_{\mathrm{ST}} \right| = O_f(C^{-\frac{1}{2}+\epsilon}).$$

They prove that this conjecture implies the Riemann Hypothesis for E. I prove that not only does their conjecture imply the Riemann Hypothesis for all $L(\operatorname{sym}^k E, s)$, it also does for all the strange L-functions

$$L_f(E,s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

These results make perfect sense in a much more general context, and I will prove them there. In section 2 I set up this context and carefully define strange L-functions there. In section 3, I prove basic analytic properties of the strange L-functions, and in section 4, I prove the main results connecting the analytic properties of strange L-functions with the equidistribution of a sequence. Finally, in section 6, I apply the general results to the following cases: a non-CM elliptic curve $E_{/\mathbf{Q}}$, the product $E_1 \times E_2$ of a pair of non-isogenous non-CM elliptic curves over $\mathbf{Q}$, and the Jacobian of a generic genus-2 curve $C_{/\mathbf{Q}}$.

2 Definitions

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| \leqslant 1\}$. Write $\mathbf{D}^\infty$ for the set of sequences in $\mathbf{D}$ indexed by the primes, i.e. $\boldsymbol{\lambda} \in \mathbf{D}^\infty$ is $(\lambda_2, \lambda_3, \dots)$.

Definition 2.1. Let $\boldsymbol{\lambda} \in \mathbf{D}^\infty$. The associated *strange L-function* is given by

$$L(\boldsymbol{\lambda}, s) = \prod_p \frac{1}{1 - \lambda_p p^{-s}},$$

wherever this product converges.

We will see that the analytic properties of $L(\boldsymbol{\lambda}, s)$ are closely tied to estimates for the sums $A_{\boldsymbol{\lambda}}(x) = \sum_{p \leqslant x} \lambda_p$. One often gets such estimates in the context of equidistribution, which we consider next.

For the remainder of this section, let X be a compact separable metric space with no isolated points. We write X^∞ for the space of sequences in X indexed by rational primes, i.e. points $\boldsymbol{x} \in X^\infty$ are of the form $\boldsymbol{x} = (x_2, x_3, \dots)$. By [Eng89, Cor. 2.3.16 & Th. 4.2.2], the compact space X^∞ is metrizable and separable, also with no isolated points.

Definition 2.2. For $\boldsymbol{x} \in X^\infty$ and $C > 0$, write $\boldsymbol{x}^C$ for the probability measure given by

$$\int_X f \, \mathrm{d}\boldsymbol{x}^C = \boldsymbol{x}^C(f) = \frac{1}{\pi(C)} \sum_{p \leqslant C} f(x_p).$$

Let μ be a Borel measure on X. Recall that $\boldsymbol{x}$ is *μ-equidistributed* if $\boldsymbol{x}^C \to \mu$ weakly, i.e. $\boldsymbol{x}^C(f) \to \mu(f)$ for all $f \in C(X)$. In fact, we can extend this to not-necessarily-continuous functions as follows:

Theorem 2.3 (Mazzone)**.** *Let μ be a Borel measure on X and let $f \colon X \to \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if $\boldsymbol{x}^C(f) \to \mu(f)$ for all μ-equidistributed $\boldsymbol{x}$.*

Proof. This follows directly from the proof of [Maz95, Th. 1]. □

Fix a Borel measure μ on X, and write $C^{\mathrm{ae}}(X,\mu)$ for the space of bounded, almost-everywhere continuous functions $f\colon X \to \mathbf{C}$.

Theorem 2.4. *Endowed with the supremum norm* $\|f\|_\infty = \sup_{x\in X} |f(x)|$, $C^{\mathrm{ae}}(X,\mu)$ *is a Banach space.*

Proof. This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero. □

Definition 2.5. Let $f \in C^{\mathrm{ae}}(X,\mu)$, $\boldsymbol{x} \in X^\infty$. The associated *strange L-function* is defined as

$$L_f(\boldsymbol{x}, s) = L(f(\boldsymbol{x}), s) = \prod_p \frac{1}{1 - f(x_p)p^{-s}}$$

for all $s \in \mathbf{C}$ for which the product converges.

Our typical source of a strange L-function is as follows. $X = G^\natural$, the space of conjugacy classes in a compact Lie group, and $f\colon G^\natural \to \mathbf{C}$ one of the “angles” of [KS99]. More precisely, let G be a compact Lie group and $\rho\colon G \to \mathrm{U}(d)$ an irreducible representation. Following [KS99, Le. 1.0.9], write $\varphi_1^\rho, \dots, \varphi_d^\rho$ for the sequence of functions $G^\natural \to [0, 2\pi)$ such that for each $x \in G^\natural$, the unitary conjugacy class $\rho(x)$ has eigenvalues $e^{i\varphi_1^\rho(x)}, \dots, e^{i\varphi_d^\rho(x)}$, and $\varphi_1^\rho(x) \leqslant \cdots \leqslant \varphi_d^\rho(x)$. We have, using Serre’s notation $L(\rho, s)$, the identity:

$$L(\rho, s) = \prod_{j=1}^{\deg \rho} L_{\exp(i\varphi_j^\rho)}(\boldsymbol{x}, s).$$

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let $X = [0, a_1) \times \cdots \times [0, a_r)$. Given $x = (x_1, \dots, x_r) \in X$, we write $[0, x) = [0, x_1) \times \cdots \times [0, x_r)$.

Definition 2.6. Given X as above, and $\boldsymbol{x} \in X^\infty$, the *star-discrepancy* of $\boldsymbol{x}$ with respect to a Borel measure μ on X is:

$$\mathrm{disc}(\boldsymbol{x}^C, \mu) = \sup_{x\in X} \left| \boldsymbol{x}^C(1_{[0,x)}) - \mu(1_{[0,x)}) \right|.$$

The following result is essential:

Theorem 2.7 (Koksma–Hlawka)**.** *Let X be as above. Let $f\colon X \to \mathbf{C}$ be such that $f\,\mathrm{d}x$ is a measure with bounded variation. Let μ be a probability measure on X. Then*

$$|\boldsymbol{x}^C(f) - \mu(f)| \leqslant \mathrm{Var}(f)\,\mathrm{disc}(\boldsymbol{x}^C, \mu).$$

Proof. This is [Ökt99, Th. 3.2]. □

3 Preliminary results

Theorem 3.1. *Let $\boldsymbol{\lambda} \in \mathbf{D}^\infty$. Then $L(\boldsymbol{\lambda}, s)$ defines a holomorphic function on the region $\{\Re s > 1\}$. Moreover, on that region,*

$$\log L(\boldsymbol{\lambda}, s) = \sum_{p^n} \frac{\lambda_p^n}{p^{ns}}.$$

Proof. Expanding the product for $L(\boldsymbol{\lambda}, s)$ formally, we have

$$L(\boldsymbol{\lambda}, s) = \sum_{n \geqslant 1} \frac{\prod_{p|n} \lambda_p^{v_p(n)}}{n^s}.$$

An easy comparison with Riemann's zeta function tells us that the series expansion is holomorphic on $\{\Re s > 1\}$. By [Apo76, Th. 11.7], the product formula holds on the same region. The formula for $\log L(\boldsymbol{\lambda}, s)$ comes from [Apo76, 11.9 Ex. 2]. □

Theorem 3.2. *Assume $A_{\boldsymbol{\lambda}}(x) = O(x^{\frac{1}{2}+\epsilon})$. Then $L(\boldsymbol{\lambda}, s)$ converges on $\{\Re > \frac{1}{2}\}$, and $\log L(\boldsymbol{\lambda}, s)$ has no poles on that region.*

Proof. Standard reductions reduce this to showing that

$$\sum_p \frac{\lambda_p}{p^s} \qquad \text{and} \qquad \sum_p \frac{\log(p)\lambda_p}{p^s}$$

converge on that region. We deal with $\sum \log(p)\lambda_p p^{-s}$; the other is similar. Use Abel summation:

$$\sum_{p \leqslant x} \frac{\lambda_p}{p^s} = \frac{\log x}{x^s} A_{\boldsymbol{\lambda}}(x) - \int_2^x \frac{1 - s \log t}{t^{s+1}} A_{\boldsymbol{\lambda}}(t)\, \mathrm{d}t.$$

We show that the first term approaches zero and that the integral converges absolutely. We have:

$$\left| \frac{\log x}{x^s} A_{\boldsymbol{\lambda}}(x) \right| \ll \frac{\log x}{x^{\Re s}} x^{\frac{1}{2}+\epsilon}.$$

Since ϵ is arbitrary, the exponent of x is negative. Moreover,

$$\int_2^x \frac{1}{t^{s+1}} |A_\lambda(t)|\, \mathrm{d}t \ll \int_2^x \frac{1}{t^{\Re s+1}} t^{\frac{1}{2}+\epsilon}\, \mathrm{d}t$$
$$\int_2^x \frac{\log t}{t^{s+1}} |A_{\boldsymbol{\lambda}}(t)|\, \mathrm{d}t \ll \int_2^x \frac{\log t}{t^{\Re s+1}} t^{\frac{1}{2}+\epsilon}\, \mathrm{d}t.$$

Both these integrals converge because ϵ is arbitrary. □

4 Main results

Let $E_{/\mathbf{Q}}$ be an elliptic curve, or more generally, let M be a motive. The associated analytic L-function $L(M,s)$ is of the form

$$L(M,s) = \prod_p P_p(M, p^{-s})^{-1},$$

where the $P_p(M,t) \in \mathbf{Z}[t]$ have absolute value 1. In the case of $E_{/\mathbf{Q}}$, we have $pt^2 - a_p t + 1$, which are normalized to

$$(t - e^{i\theta_p})(t - e^{-i\theta_p}) = t^2 - 2\cos(\theta_p)t + 1 = t^2 - \frac{a_p}{\sqrt{p}}t + 1.$$

Let $d = \deg P_p(M,t)$. Then we can write

$$P_p(M,t) = (t - e^{i\theta_p^{(1)}}) \dots (t - e^{-i\theta_p^{(d)}}),$$

where $\theta^{(1)} < \cdots < \theta^{(d)}$ in $[0, 2\pi]$. Then

$$L(M,s) = L(\boldsymbol{\theta}^{(1)}, s) \cdots L(\boldsymbol{\theta}^{(d)}, s)$$

More general example:

$$L(\operatorname{sym}^k E, s) = L(\boldsymbol{\theta}^k, s) L(\boldsymbol{\theta}^{k-1}$$

5 Connection to Serre's perspective

Let G be a compact connected Lie group, $G^\natural$ the space of conjugacy classes in G, and $\boldsymbol{x}$ a sequence in $G^\natural$. Given $\rho \in \widehat{G}$, Serre defines an L-function

$$L(\rho, s) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}.$$

Given $x \in G^\natural$, the matrix $\rho(x)$ has eigenvalues $\lambda_p^{(1),\rho}, \dots, \lambda_p^{(\deg\rho),\rho}$ whose angles form a nondecreasing sequence in $[0, 2\pi]$. The functions $\lambda_p^{(j),\rho} \colon G^\natural \to \mathbf{C}$ are almost-everywhere continuous, and

$$L(\rho, s) = \prod_{j=0}^{\deg\rho} L(\lambda_p^{(j),\rho}, s) = \prod_{j=0}^{\deg\rho} L_{\lambda^{(j),\rho}}(\boldsymbol{x}, s).$$

6 Applications

Recall, following [Bug08] that the *irrationality exponent* $\mu(\alpha)$ a real irrational number α is the supremum of all real numbers μ such that

$$\left|\alpha - \frac{p}{q}\right| < q^{-\mu}$$

for infinitely many $p/q \in \mathbf{Q}$. Bugeaud proves that for any $\mu \geqslant 2$, there is an element ξ_μ of the Cantor set with $\mu(\xi_\mu) = \mu$. Moreover, by [KN74, ?], for every $\epsilon > 0$, the sequence $x_n = n\alpha \mod 1$ has discrepancy $\operatorname{disc}(\boldsymbol{x}^C) = \Omega(C^{-\frac{1}{\mu(\alpha)-1}-\epsilon})$.

Theorem 6.1. *Let $X = S^1$ with the natural Haar measure. For every $\eta \in (0, \frac{1}{2})$, there is a sequence $\boldsymbol{x} = (x_2, x_3, \dots) \in (S^1)^\infty$ such that for all $f \in C^\infty(S^1)^{\|\cdot\|_\infty \leqslant 1}$, the function $\log L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re > \frac{1}{2}\}$, but for all $\epsilon > 0$, $|\operatorname{disc}(\boldsymbol{x}^C)| = \Omega(C^{-\eta-\epsilon})$.*

Proof. Let $\mu > 3$, and let $\boldsymbol{x} = \{x_2, x_3, \dots\}$ be the sequence $x_{p_n} = e^{2\pi i n \xi_\mu}$. To prove that $\log L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re > \frac{1}{2}\}$, we need only to prove that $|A_{\exp(2\pi i m \boldsymbol{x})}(t)| \ll t^{1/2}$, uniformly for each $m \in \mathbf{Z}$. This follows easily from:

$$\left|\sum_{n=1}^{N} e^{2\pi i m n \alpha}\right| \leqslant \frac{|-1 + e^{2\pi i M n \alpha}|}{|-1 + e^{2\pi i a m}|} \leqslant ? \leqslant \frac{1}{2} m(\eta - 1) \ll_\eta m$$

□

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