Galois representations for CM counterexample

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Let $K/\mathbf{Q}$ be a finite Galois extension, $\{\theta_\mathfrak{p}\}$ a sequence in $(\mathbf{R}/\mathbf{Z})^d$ indexed by primes of K, such that $\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} r(\theta_\mathfrak{p})\right| = O(1)$ for all nontrivial $r \in \mathrm{X}^*((\mathbf{R}/\mathbf{Z})^d)$. Suppose $\{\vartheta_\mathfrak{p}\}$ is a sequence with $|\vartheta_\mathfrak{p} - \theta_\mathfrak{p}|_\infty \ll \mathrm{N}(\mathfrak{p})^{-\frac{1}{2}}$. We wish to establish a bound $\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} r(\vartheta_\mathfrak{p})\right| \ll x^{\frac{1}{2}+\epsilon}$.

Now, Taylor's theorem (the multivariate version) tells us that for any $f \in C^\infty((\mathbf{R}/\mathbf{Z})^d)$, we have near $(a_1, \dots, a_d)$:

$$f(x_1, \dots, x_d) = f(a_1, \dots, a_d) + \sum_{i=1}^d \frac{\mathrm{d}f}{\mathrm{d}x_i}(a_1, \dots, a_d)(x_i - a_i) + O(|x-a|_\infty^2).$$

In particular, I think we can say that $|f(x) - f(y)| \ll |x-y|$, the implied constant depending on the max of the $\left|\frac{\mathrm{d}f}{\mathrm{d}x_i}\right|_\infty$. In particular, we can compute:

$$\begin{aligned}
\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} r(\vartheta_\mathfrak{p})\right| &\leqslant \left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} r(\vartheta_\mathfrak{p})\right| + \sum_{\mathrm{N}(\mathfrak{p})\leqslant x} |r(\theta_\mathfrak{p}) - r(\vartheta_\mathfrak{p})| \\
&\ll 1 + \sum_{\mathrm{N}(\mathfrak{p})\leqslant x} |\theta_\mathfrak{p} - \vartheta_\mathfrak{p}|_\infty \\
&\ll \int_1^x \frac{\mathrm{d}t}{\sqrt{t}} = \sqrt{x}.
\end{aligned}$$

Hopefully we can say something similar about discrepancy. Now suppose we have a "fake sequence" $\{\vartheta_\mathfrak{p}\}$. We want to construct a Galois representation $\rho\colon G_K \to \left(\mathbf{Z}_l^\times\right)^d$, possibly infinitely ramified, such that $\rho(\mathrm{fr}_\mathfrak{p}) \in \mathbf{Q}^\times$??

$\rho(\mathrm{fr}_\mathfrak{p})/\mathrm{N}(\mathfrak{p})^{\frac{1}{2}}$

$(F \otimes \mathbf{Q}_l)^\times \simeq (\mathbf{Q}_l^\times)^d$

Let's consider the case when F is a quadratic CM field, and

$$\mathrm{ST} \subset (\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_\mathrm{m})^{\mathrm{N}_{F/\mathbf{Q}}=1}(\mathbf{C}).$$

Note that $\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}(\mathbf{C}) = (F\otimes\mathbf{C})^\times = (\mathbf{C}^\times)^2$, and $\mathrm{N}_{F/\mathbf{Q}}(z_1,z_2) = z_1\overline{z_2}$. So ST is a maximal compact subgroup of $\{(z_1, \overline{z_1}^{-1})\}$ consisting of those $|z_1| = 1$. We want a Galois representation $\rho\colon G_K \to (F\otimes\mathbf{Q}_l)^\times$ such that $\rho(\mathrm{fr}_{\mathfrak{p}}) \in F$ is 1-Weil. (Maybe? Maybe 0-Weil).

We can assume that K is a CM field. Then $\mathbf{A}_K^\times/K^\times$ should be a bit easier to understand. We can choose a prime l at which F splits, so that $F\otimes\mathbf{Q}_l = (\mathbf{Q}_l)^2$.

First of all, $\mathbf{A}_K = \mathbf{C}\times\prod' K_{\mathfrak{p}}$, so $\mathbf{A}_K^\times = \mathbf{C}^\times\times\prod' K_{\mathfrak{p}}^\times$. We are interested in maps $\mathbf{A}_K^\times/K^\times K_\infty^\times \to F_l^\times$.

$\rho\colon G_K \to F_l^\times$, each $\rho(\mathrm{fr}_{\mathfrak{p}}) \in F$ is $\mathfrak{p}$-Weil of weight 1. The renormalization $\rho(\mathrm{fr}_{\mathfrak{p}})\,\mathrm{N}(\mathfrak{p})^{-1/2}$, which lies in a compact torus in $(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}})^{\mathrm{N}_{F/\mathbf{Q}}=1}(\mathbf{C})$, needs to be close to $\vartheta_{\mathfrak{p}}$

Fix a finite prime $\mathfrak{p}$ of K. In the elliptic curve case, $(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}})^{\mathrm{N}_{F/\mathbf{Q}}=1}(\mathbf{C}) = \mathbf{C}^\times$, with embedding $F^\times \hookrightarrow \mathbf{C}^\times$. (There are two embeddings, but the subfield is the same.)

We are trying to solve the equation $\mathrm{N}_{F/\mathbf{Q}}(x) = \mathrm{N}(\mathfrak{p})^{1/2}$.

Make things very explicit. Let $F = \mathbf{Q}(\sqrt{-d})$, which is a well-defined subfield of $\mathbf{C}$. Fix $\mathfrak{p}$ a finite prime of F; then there is a set $T_{\mathfrak{p}} = \{x\in F : \mathrm{N}_{F/\mathbf{Q}}(x) = \mathrm{N}(\mathfrak{p})\}$.

In $\mathbf{Q}(i)$, we have $\mathfrak{p} = (1+4i)$, with $\mathrm{N}(\mathfrak{p}) = 5$. What elements have $|x| = \sqrt{5}$, i.e. $\mathrm{N}(x) = \mathrm{N}(\mathfrak{p})$? In that case, x and $1+4i$ will differ by a unit. In other words, there are only finitely many such x, and $\#T_{\mathfrak{p}}$ is bounded.

In general, in the quadratic CM case, $|x| = \mathrm{N}(x)^{1/2}$, so we are looking at $T_{\mathfrak{p}} = \{x\in F : \mathrm{N}_{F/\mathbf{Q}} = \mathrm{N}(\mathfrak{p})\}$. If $\mathrm{N}(x) = \mathrm{N}(\mathfrak{p})$, then the ideal $\mathfrak{p}\mid x$, which means that in fact $\mathfrak{p} = \langle x\rangle$. If $\mathfrak{p}$ is principal, than generators differ by units, and $O_F^\times$ is finite.

1 Possible concrete case

Let's address abelian 2-folds with CM. Let $A_{/K}$ be an abelian 2-fold; then $\mathrm{End}_K(A)_{\mathbf{Q}}$ has rank 4 over $\mathbf{Q}$. If A has CM, then $F = \mathrm{End}_K(A)_{\mathbf{Q}}$ has $[F:\mathbf{Q}] = 4$. The field F is totally imaginary, so $\mathrm{rk}\,O_F^\times = 1$. The motivic Galois group G_A^1 should be a two-dimensional torus. In particular, we want to "cut out" a two-dimensional subgroup of $(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}})^{\mathrm{N}_{F/\mathbf{Q}}=1}$, which is three-dimensional. Let F^+ be the totally real subfield of F. Then $(\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}})^{\mathrm{N}_{F/F^+}=1}$ is such a group.

Fix a prime $\mathfrak{p}$ of K. We are looking for prospective "Frobenius at $\mathfrak{p}$" in $F^\times$, such that for all $\sigma\colon F\hookrightarrow\mathbf{C}$, $|\sigma(x)| = \mathrm{N}(\mathfrak{p})^{1/2}$.

Fundamentally, this is the problem. Let F be a CM field with $[F:\mathbf{Q}] = 2g$. Then $\mathrm{rk}\,O_F^\times = g-1$.

Let $\ell\colon F_\infty = \mathbf{C}^g \to \mathbf{R}^g$ be the map $\ell(z_1,\dots,z_g) = (\log|z_1|,\dots,\log|z_g|)$. Then $\ell(O_F^\times)$ is a discrete lattice in $(\mathbf{R}^g)^{\mathrm{tr}=0}$.

2 Product formula

Suppose $\sum x_p p^{-s}$ converges conditionally when $\Re s > 1/2$. We wish to prove that for $x_n = \prod x_p^{v_p(n)}$, the series $\sum a_n n^{-s}$ also converges conditionally when $\Re s > 1/2$. Write $\operatorname{supp}(n) \leqslant x$ if all primes dividing n are $\leqslant x$. We know that the product $\prod(1 - x_p p^{-s})^{-1}$ converges conditionally. So, note that

$$\prod_{p \leqslant x} \frac{1}{1 - x_p p^{-s}} = \sum_{\operatorname{supp}(n) \leqslant x} x_n n^{-s} = \sum_{p \leqslant x} x_p p^{-s} + \sum_{\substack{S \subset \{p \leqslant x\} \\ \# S > 1}} \sum_{?}$$

$$\prod_{p \leqslant x} (1 - x_p p^{-s})^{-1} = \prod_{p \leqslant x} \sum_{r \geqslant 0} x_p^r p^{-rs} = \sum_{n \geqslant 1 : supp(n) \subset \{p \leqslant x\}} x_n n^{-s}.$$

Question: if $\{p \leqslant x\} = \{p_1, p_2, \ldots, p_n\}$, then

$$\begin{aligned}
\sum_{a,b \geqslant 1} x_{p_1}^a x_{p_2}^b p_1^{-as} p_2^{-bs} &= \left(\sum_{a \geqslant 1} (x_{p_1} p_1^{-s})^a \right) \left(\sum_{b \geqslant 1} (x_{p_2} p_2^{-s})^b \right) \\
&= \frac{x_{p_1} p_1^{-s}}{1 - x_{p_1} p_1^{-s}} \frac{x_{p_2} p_2^{-s}}{1 - x_{p_2} p_2^{-s}} \\
&\leqslant x_{p_1} x_{p_2} p_1^{-s} p_2^{-s}
\end{aligned}$$

Thus,

$$\sum_{\operatorname{supp}(n) \leqslant x} x_n n^{-s} = \sum_{p \leqslant x} x_p p^{-s} + O\left(\sum_{S \subset \{p \leqslant x\}} p_S^{-\Re s} \right)$$

The question is: does $\sum_{n \leqslant x} x_n n^{-s}$ converge? We already know that $\sum_{\operatorname{supp}(n) \leqslant x} x_n n^{-s}$ converges.

$$\sum_{n \leqslant x} x_n n^{-s} = \sum_{\operatorname{supp}(n) \leqslant x} x_n n^{-s} - \sum_{\operatorname{supp}(n) \leqslant x, n > x} x_n n^{-s}$$

Can we show that $\sum_{\operatorname{supp}(n) \leqslant x, n > x} x_n n^{-s}$ converges / approaches zero in some way?