The method of Hadamard–de la Vallée-Poussin

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August 7, 2016

We follow Deligne's *Weil II.* Let Γ be a group isomorphic to either $\mathbf{R}$ or $\mathbf{Z}$, ω_1 a non-trivial quasi-character $\omega_1 \colon \Gamma \to \mathbf{R}^+$. Let G a locally compact group which is an extension of Γ by a compact group G°. Let Σ be a countably infinite set, $(x_v)_{v\in\Sigma} \subset G^\natural$ a family of conjugacy classes in G. We assume:

A' If $\Gamma \simeq \mathbf{R}$, the extension G is trivial.

The following hypothesis is automatically true:

A" If $\Gamma \simeq \mathbf{Z}$, the center of G maps to a subgroup of finite index in Γ.

For $s \in \mathbf{C}$, put $\omega_s = \omega_1^s$. Write also ω_s for the composite map $G \to \Gamma \to \mathbf{C}^\times$, and put $\mathrm{N}v = \omega_{-1}(x_v)$. If $\Gamma \simeq \mathbf{Z}$, write q and deg the number > 1 such that the isomorphism between Γ and $\mathbf{Z}$ such that $\omega_1(\gamma) = q^{-\deg(\gamma)}$. One has $\omega_s = \omega_{s+2\pi i \log q}$. Writing deg for the composite map $G \to \Gamma^{\deg} \to \mathbf{Z}$, we have $\deg(v) = \deg(x_v)$.

Let g be an element of the center of G with non-trivial image in Γ. It exists by hypothesis. A complex representation $\tau \colon G \to \mathrm{GL}(V)$ is unitarisable if and only if $\tau(g)$ is: a g-invariant Hermitian structure yields a G-invariant structure by integration on the compact group $G/g^{\mathbf{Z}}$. If τ is irreducible, then $\tau(g)$ is a scalar, so there exists a unique real number σ such that $|\tau(g)| = \omega_\sigma(g)$, and $\tau \cdot \omega_{-\sigma}$ is unitarizable. We call σ the *real part* $\Re\tau$ of τ. We have $\Re(\tau\omega_s) = \Re(\tau) + \Re(s)$.

Let $\widetilde{G}$ the set of isomorphism classes of irreducible representations of G, and $\widehat{G}$ the set of unitary such representations. The sets $\{\tau\omega_s : s \in \mathbf{C}\}$ form a partition of $\widetilde{G}$, and the application $s \mapsto \tau\omega_s$ identifies $\{\tau\omega_s\}$ with the quotient of $\mathbf{C}$ by a discrete subgroup of $i\mathbf{R}$ if $\Gamma \simeq \mathbf{Z}$, and with $\mathbf{C}$ if $\Gamma \simeq \mathbf{R}$. We give $\widetilde{G}$ the structure of a Riemann surface via the disjoint union of these quotients.

B' For each $v \in \Sigma$, $\mathrm{N}v > 1$.

B" The infinite product $\prod_{v\in\Sigma}(1 - \mathrm{N}v^{-s})^{-1}$ converges absolutely for $\Re s > 1$.

For $\Gamma \simeq \mathbf{Z}$, these conditions tells us that $\deg(v) > 0$, and, for each $\epsilon > 0$:

$$\#\{v : \deg(v) = n\} = O(q^{(1+\epsilon)n}).$$

The hypothesis (B") tells us that for $\tau \in \widetilde{G}$, the infinite product

$$L(\tau) = \prod_{v\in\Sigma} \det(1 - \tau(x_v))^{-1}$$

converges absolutely for $\Re(\tau) > 1$. Each factor is holomorphic in τ for $\Re(\tau) > 0$, and $L(\tau)$ is holomorphic for $\Re(\tau) > 1$. We put $L(\tau, s) = L(\tau\omega_s)$.

For a representation τ not necessarily irreducible, we define $L(\tau)$ and $L(\tau, s)$ as above. Put $L'(\tau) = \frac{\mathrm{d}}{\mathrm{d}s} L(\tau, s)\big|_{s=0}$. On the domain of convergence:

$$-\frac{L'}{L}(\tau) = \sum_{v\in\Sigma, n>0} \log(\mathrm{N}v) \operatorname{tr}\tau(x_v^n).$$

We can generalize this to τ a virtual representation, i.e. an element of the Grothendieck group of representations of G. For τ unitary and σ real > 1, $\omega_\sigma\tau$ in the domain of convergence, we get

$$-\frac{L'}{L}(\omega_\sigma\tau) = \sum_{v\in\Sigma, n>0} (\log(\mathrm{N}v)(\mathrm{N}v)^{-n\sigma}) \operatorname{tr}\tau(x_v^n).$$

Let mu be a measure on $G^\natural$. For each virtual unitary representation τ of G, we put

$$\widehat{\mu}(\tau) = \int \operatorname{tr}\tau \,\mathrm{d}\mu.$$

The integral converges if the total mass $|\mu| < \infty$. We call the function $\tau \mapsto \widehat{\mu}(\tau)$ the *Fourier transform* of μ. If we don't require that τ be unitary, we call it the *Fourier–Laplace transform.* If μ is positive with finite total mass, we have for each unitary virtual representation ρ:

$$\widehat{\mu}(\rho\otimes\bar{\rho}) \geqslant 0 \qquad (\text{for } \mu \geqslant 0).$$

We see that, for each $\sigma > 1$, $\Lambda_\sigma(\tau) = -\frac{L'}{L}(\omega_\sigma\tau)$ is the Fourier transform of the positive measure with finite total mass:

$$\mu_\sigma = \sum_{v\in\Sigma, n>0} \log(\mathrm{N}v)\mathrm{N}v^{-n\sigma}\delta_{x_v^n}.$$

on G. For each virtual unitary representation ρ, with real character $\geqslant 0$, one has $\Lambda_\sigma(\rho) \geqslant 0$ for $\sigma > 1$; in particular, for each unitary virtual representation ρ, one has:

$$\Lambda_\sigma(\rho\otimes\bar{\rho}) \geqslant 0 \qquad (\text{for } \sigma > 1).$$

For $\tau \in \widehat{G}$, let $\nu(\tau)$ be the order of the pole (or opposite of the order of zero) of L at $\tau\omega_1$. We extend ν to an additive function on the Grothendieck group of G. For τ in this group, the function $-\frac{L'}{L}(\tau\omega_s)$ has at most simple poles, and the residue at $\tau\omega_1$ is $\nu(\tau)$. We get that for each unitary virtual representation of G:

$$\nu(\rho\otimes\bar{\rho}) \geqslant 0.$$

It is classic that the non-vanishing of $L(\tau)$ on $\Re(\tau) = 1$ implies the equidistribution of the x_v. The case where $\Gamma \simeq \mathbf{R}$ is treated in detail in Serre's *Abelian l-adic representations and elliptic curves.* We treat here the case where $\Gamma \simeq \mathbf{Z}$. Assume the following:

C The function $L(\tau)$ admits a meromorphic continuation to $\Re\tau \geqslant 1$; on this domain at has a simple pole at ω_1, and at the other pole, it does not vanish on $\Re(\tau) = 1$.

D $\Gamma \simeq \mathbf{Z}$.

Let z be a central element of G, with degree $d > 0$. The space $G^\natural$ is the quotient of G by the compact group $G/g^{\mathbf{Z}}$ acting by interior automorphisms. It is the disjoint union of $G^\natural_n$, where $G^\natural_n$ is the set of conjugacy classes of degree n, and multiplication by z^k induces an isomorphism from $G^\natural_n$ to $G^\natural_{n+kd}$.

Consider the following measure on $G^\natural$:

$$\mu^\natural = \sum_{v \in \Sigma, n > 0} \deg(v) q^{-n \deg(v)} \delta_{x_v^n},$$

where

- $\mathrm{d}g =$ the Haar measure, where G° has volume 1,
- $\mu_0 =$ the product of $\mathrm{d}g$ with the characteristic function of $\{\omega_{-1} > 1\}$.
- $\mu_0^\natural$ the projection of μ_0 onto $G^\natural$.

Prop. If (C) and (D) are true, the Fourier-Laplace transform of $\mu^\natural - \mu_0^\natural$ (which converges for $\tau \in \widehat{G}$ with $\Re(\tau) > 0$) extends to a holomorphic function on $\{\Re\tau > 0\}$.

The Fourier–Laplace transform of μ is $-\frac{1}{\log q}\frac{L'}{L}(\omega_1\tau)$. That of $\mu_0^\natural$ vanishes away from ω_s, where it has value $\frac{q^{-s}}{1-q^{-s}}$. On the domain $\{\Re\tau \geqslant 0\}$, both $\widehat{\mu}^\natural$ and $\widehat{\mu}_0^\natural$ are meromorphic, with a simple pole with residue $1/\log q$ at ω_0. The proposition follows.

Theorem: assuming (C) and (D), for each i, the measure $z^{-n}\mu^\natural|_{G^\natural_{i+nd}}$ on $G^\natural_i$ converges weakly to $z^{-n}\mu_0^\natural|_{G^\natural_{i+nd}}$ as $n \to \infty$.