Distributions and the Amice transform

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All topological spaces are tacitly assumed to be Hausdorff. Let X be a topological space, $\mathcal{O}$ a topological ring.

1 Topological preliminaries

Definition 1.1. We write $\mathscr{C}^0(X,\mathcal{O})$ for the set of continuous functions $f\colon X \to \mathcal{O}$. We give $\mathscr{C}^0(X,\mathcal{O})$ the compact-open topology, i.e. the topology generated by sets of of the form

$$B_{C,I} = \{f \in \mathscr{C}(X,\mathcal{O})\colon f(U) \subset I\},$$

for any compact $C \subset X$, $I \subset \mathcal{O}$.

Note that a *basis* of open sets for $\mathscr{C}^0(X,\mathcal{O})$ is given by finite intersections of the $B_{C,I}$. By definition of "topology generated by," to show a map $\phi\colon Y \to \mathscr{C}^0(X,\mathcal{O})$ is continuous, it suffices to show that $\phi^{-1}B_{C,I}$ is open for all C, I.

Lemma 1.2. *The natural map $\mathcal{O} \times \mathscr{C}^0(X,\mathcal{O}) \to \mathscr{C}^0(X,\mathcal{O})$ is continuous.*

Proof. Let $a \in \mathcal{O}$, $f \in \mathscr{C}^0(X,\mathcal{O})$ such that $af \in B_{C,I}$. Since multiplication $\mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is continuous, for each $c \in C$, there exists open neighborhoods $J_c \ni a$, $J'_c \ni f(c)$ such that $J_c \cdot J'_c \subset I$. By compactness of C, we get open $J \ni a$, $J' \supset f(C)$ such that $J \cdot J' \subset I$. It follows that

$$J \cdot B_{C,J'} \subset I,$$

and since $f \in B_{C,J'}$, we are done. $\square$

Thus for any space X, the module $\mathscr{C}^0(X,\mathcal{O})$ is a topological $\mathcal{O}$-module.

Lemma 1.3. *Let $\phi\colon X \to Y$ be a continuous map of topological spaces. Then $\phi^*\colon \mathscr{C}^0(Y,\mathcal{O}) \to \mathscr{C}^0(X,\mathcal{O})$, $f \mapsto f \circ \phi$, is continuous.*

Proof. Note that for compact $C \subset X$ and open $I \subset \mathcal{O}$, we have

$$(\phi^*)^{-1}B_{C,I} = \{f \in \mathscr{C}(Y,\mathcal{O})\colon f(\phi(C)) \subset I\} = B_{\phi(C),I}.$$

Since $\phi(C)$ is compact, we are done. $\square$

To sum things up: $\mathscr{C}^0(-,\mathcal{O})$ is a contravariant functor from (Hausdorff) topological spaces to topological $\mathcal{O}$-modules. If $\mathcal{O}$ is linearly topologized, then $\mathscr{C}^0(-,\mathcal{O})$ takes values in linearly topologized $\mathcal{O}$-modules. Clearly the same proofs work for $\mathscr{C}^0(-,M)$ and any topological $\mathcal{O}$-module M.

Definition 1.4. Write $\mathscr{D}_0(X,\mathcal{O})$ for the continuous dual of $\mathscr{C}^0(X,\mathcal{O})$. That is, an element $\mu \in \mathscr{D}_0(X,\mathcal{O})$ is a continuous linear functional $\mathscr{C}^0(X,\mathcal{O}) \to \mathcal{O}$. One often writes

$$\int_X f(x)\,\mathrm{d}\mu(x) = \mu(f),$$

for $f \in \mathscr{C}^0(X,\mathcal{O})$.

Lemma 1.5. *Let $\phi\colon X \to Y$ be a proper map. Then $\phi_*\colon \mathscr{D}_0(X,\mathcal{O}) \to \mathscr{D}_0(Y,\mathcal{O})$, given by $(\phi_*\mu)f = \mu(\phi^* f)$, is well-defined.*

Proof. Clearly the expression $(\phi_*\mu)f = \mu(\phi^* f)$ is well-defined. What we need to check is that $\phi_*\mu$ is also a distribution. Let $I \subset \mathcal{O}$ be open. Since μ is a continuous, there exists compact $C_i \subset X$ and open $J_i \subset \mathcal{O}$ such that $\mu(\bigcap B_{C_i,J_i}) \subset I$. Note that

$$\begin{aligned}(\phi_*\mu)\left(\bigcap B_{\phi(C_i),J_i}\right) &= \mu\left(\phi^*\left(\bigcap B_{\phi(C_i),J_i}\right)\right)\\ &\subset \mu\left(\bigcap B_{\phi^{-1}(\phi(C_i)),J_i}\right)\\ &\subset \mu\left(\bigcap B_{C_i,J_i}\right).\end{aligned}$$

We used the properness of ϕ in that $\phi^{-1}(\phi(C_i))$ is continuous. $\square$

We give $\mathscr{D}_0(X,\mathcal{O})$ the *open-open topology*, namely that generated by sets of the form

$$B_{C,I,J} = \{\mu\colon \mu(B_{C,I}) \subset J\}.$$

Theorem 1.6. *The rule $\mathscr{D}_0(-,\mathcal{O})$ is a (covariant) functor from the category of topological spaces with proper maps to topological $\mathcal{O}$-modules.*

Proof. All we need to do is check that if $\phi\colon X \to Y$ is proper, then $\phi_*\colon \mathscr{D}_0(X,\mathcal{O}) \to \mathscr{D}_0(Y,\mathcal{O})$ is continuous. Fix an open $B_{C,I,J} \subset \mathscr{D}_0(Y,\mathcal{O})$. It is easy to check that $\phi_*(B_{\phi^{-1}(C),I,J}) \subset B_{C,I,J}$, so we are done. $\square$

Often we will have interesting dense subspaces of $\mathscr{C}^0(X,\mathcal{O})$. For example, if X is totally disconnected, write $\mathscr{C}^\infty(X,\mathcal{O})$ for the subspace of locally constant functions. If X has some kind of analytic structure, we write $\mathscr{C}^\dagger(X,\mathcal{O})$ for the space of locally analytic functions. In general, if $\mathscr{C}^*(X,\mathcal{O}) \supset \mathscr{C}^\infty(X,\mathcal{O})$, then write $\mathscr{D}_*(X,\mathcal{O})$ for the topological dual of $\mathscr{C}^*(X,\mathcal{O})$. The inclusions $\mathscr{C}^\infty \hookrightarrow \mathscr{C}^* \hookrightarrow \mathscr{C}^0$ induce embeddings $\mathscr{D}_0 \hookrightarrow \mathscr{D}_* \hookrightarrow \mathscr{D}_\infty$. So an, e.g. locally analytic distribution is just a functional on $\mathscr{C}^\infty$ that admits a continuous extension to $\mathscr{C}^\dagger$.

2 Convolution

Henceforth, all (abstract) topological spaces are assumed compact. Moreover, we assume $\mathcal{O}$ has a *linear topology*—that is, it has a basis of neighborhoods of zero given by additive subgroups. Let X, Y be two (compact) topological spaces. Let $\mathrm{pr}_X, \mathrm{pr}_Y$ be the obvious projection maps. We have an induced map

$$\mathrm{pr}_X^* \otimes \mathrm{pr}_Y^* \colon \mathscr{C}^0(X, \mathcal{O}) \otimes \mathscr{C}^0(Y, \mathcal{O}) \to \mathscr{C}^0(X \times Y, \mathcal{O}).$$

Namely, it sends $f \otimes g$ to the map $(x, y) \mapsto f(x)g(y)$. We make the following assumption:

$$\text{The map } \mathrm{pr}_X^* \otimes \mathrm{pr}_Y^* \text{ has dense image.} \tag{dense}$$

This is satisfied for example if X and Y are profinite, or if X and Y are smooth manifolds.

Theorem 2.1. *Let X, Y satisfy* (dense). *Then there is a unique map $\times \colon \mathscr{D}_0(X, \mathcal{O}) \otimes \mathscr{D}_0(Y, \mathcal{O}) \to \mathscr{D}_0(X \times Y, \mathcal{O})$ such that for all $\lambda \in \mathscr{D}_0(X, \mathcal{O})$, $\mu \in \mathscr{D}_0(Y, \mathcal{O})$, $f \in \mathscr{C}^0(X, \mathcal{O})$ and $g \in \mathscr{C}^0(Y, \mathcal{O})$, we have*

$$\int_{X \times Y} f(x)g(y)\, \mathrm{d}(\lambda \times \mu)(x, y) = \left(\int_X f(x)\, \mathrm{d}\lambda(x) \right) \left(\int_Y g(y)\, \mathrm{d}\mu(y) \right).$$

Moreover, we have the Fubini-Tonelli theorem*:*

$$\begin{aligned} \int_{X \times Y} h(x, y)\, \mathrm{d}(\lambda \times \mu)(x, y) &= \int_X \int_Y h(x, y)\, \mathrm{d}\mu(y)\, \mathrm{d}\lambda(x) \\ &= \int_Y \int_X h(x, y)\, \mathrm{d}\lambda(x)\, \mathrm{d}\mu(y), \end{aligned}$$

for any $h \in \mathscr{C}^0(X \times Y, \mathcal{O})$.

Proof. Uniqueness of convolution follows trivially from (dense). By continuity, it suffices to show that $\lambda \times \mu$ is continuous on $\mathscr{C}^0(X, \mathcal{O}) \otimes \mathscr{C}^0(Y, \mathcal{O})$ with respect to the subspace topology. Given a neighborhood of zero $J \subset \mathcal{O}$, we know there exists C_X, J_X such that $\mu(B_{C_X, J_X})\ldots$

[finish later... technicalities.] $\square$

We are especially interested in the case where G is a profinite group. We take $m : G \times G \to G$ to be the multiplication map. We often write $*$ for the composite

$$\mathscr{D}_0(G, \mathcal{O}) \otimes \mathscr{D}_0(G, \mathcal{O}) \xrightarrow{\times} \mathscr{D}_0(G \times G, \mathcal{O}) \xrightarrow{m_*} \mathscr{D}_0(G, \mathcal{O}).$$

That is,

$$\int_G f\, \mathrm{d}(\lambda * \mu) = \int_G \int_G f(xy)\, \mathrm{d}\lambda(x)\, \mathrm{d}\mu(y).$$

Theorem 2.2. *Let G be profinite. Then convolution makes $\mathscr{D}_0(G,\mathcal{O})$ into an associative algebra (possibly without unit).*

Proof. This is purely formal. $\square$

3 The Amice transform

Let X be a profinite space. For the remainder of this section, we assume that $\mathcal{O}$ is a profinite ring. Let M be a profinite $\mathcal{O}$-module.

Definition 3.1. Fix a continuous map $\psi\colon X \to M$. Symbolically, the *Amice transform* induced by ψ is the map ${}^{\psi}\mathrm{A}\colon \mathscr{D}_0(X,\mathcal{O}) \to M$ given by

$$ {}^{\psi}\mathrm{A}_{\mu} = \int_X \psi(x)\,\mathrm{d}\mu(x). \tag{1} $$

Theorem 3.2. *The equation* (1) *induces a well-defined continuous $\mathcal{O}$-linear map.*

Proof. Since $M = \varprojlim M/I$ for $I \subset \mathcal{O}$ open, we may assume M itself is finite, hence discrete. Thus $\mathscr{C}^0(X,M) = \mathscr{C}^0(X,\mathcal{O}) \otimes M$; the theorem essentially follows. Explicitly, ψ is locally constant, so put

$$ {}^{\psi}\mathrm{A}_{\mu} = \sum_{m\in M} \mu(\chi_{\psi^{-1}(m)}) \cdot m. $$

Note that

$$ \begin{aligned} {}^{\psi}\mathrm{A}^{-1}(m) &= \{\mu \in \mathscr{D}_0(X,\mathcal{O})\colon\ {}^{\psi}\mathrm{A}_{\mu} = m\} \\ &\supset \bigcap_{n\in M} B_{\psi^{-1}(n),M,\delta_{m,n}m}. \end{aligned} $$

It follows that ${}^{\psi}\mathrm{A}$ is continuous. Linearity is trivial. $\square$

Given the isomorphism $\mathscr{C}^0(X,M) = \mathscr{C}^0(X,\mathcal{O}) \otimes M$ (one has to be careful when M is not finite) we see that the Amice transform is essentially $\mu \mapsto \mu(\psi)$. The following is a "non-commutative Fubini-Tonelli."

Lemma 3.3. *Let $\langle\cdot,\cdot\rangle\colon M \times M \to M$ be a bilinear pairing. Then*

$$ {}^{\langle\varphi,\psi\rangle}\mathrm{A}_{\lambda\times\mu} = \langle {}^{\varphi}\mathrm{A}_{\lambda}, {}^{\psi}\mathrm{A}_{\mu}\rangle. $$

Proof. To be precise, we are showing that

$$ \int_X\int_X \langle\varphi(x),\psi(y)\rangle\,\mathrm{d}\lambda(x)\,\mathrm{d}\mu(y) = \left\langle \int_X \varphi(x)\,\mathrm{d}\lambda(x), \int_X \psi(x)\,\mathrm{d}\mu(y) \right\rangle. $$

It suffices to prove the result when M is finite and $\varphi = m\chi_E$, $\psi = n\chi_F$. This is a computation:

$$\begin{aligned}
{}^{\langle\varphi,\psi\rangle}\mathrm{A}_{\lambda*\mu} &= \iint \langle m,n\rangle \chi_{E\times F}\,\mathrm{d}(\lambda\times\mu)(x,y) \\
&= \langle m,n\rangle(\lambda\times\mu)(\chi_E\otimes\chi_F) \\
&= \langle \lambda(m\chi_E), \mu(n\chi_F)\rangle,
\end{aligned}$$

which is exactly $\langle {}^{\varphi}\mathrm{A}_\lambda, {}^{\psi}\mathrm{A}_\mu\rangle$. □

We are particularly interested in the case where $X = G$ is a profinite group, $M = A$ is an associative (but possibly non-commutative) $\mathcal{O}$-algebra, and $\langle a,b\rangle = ab$.

Theorem 3.4. *Let $\psi\colon G \to A^\times$ be a continuous homomorphism. Then ${}^{\psi}\mathrm{A}$ respects multiplication.*

Proof. This is purely formal:

$$\begin{aligned}
{}^{\psi}\mathrm{A}_{\lambda*\mu} &= \int_G \psi(x)\,\mathrm{d}(\lambda*\mu)(x) \\
&= \int_G\int_G \psi(xy)\,\mathrm{d}\lambda(x)\,\mathrm{d}\mu(y) \\
&= \int_G\int_G \psi(x)\psi(y)\,\mathrm{d}\lambda(x)\,\mathrm{d}\mu(y) \\
&= \int_G \psi(x)\,\mathrm{d}\lambda(x)\int_G \psi(y)\,\mathrm{d}\mu(y) \\
&= {}^{\psi}\mathrm{A}_\lambda\,{}^{\psi}\mathrm{A}_\mu.
\end{aligned}$$

□

For any profinite group G, we have the profinite group algebra $\mathcal{O}[\![G]\!]$. There is an obvious (continuous) injection $G \hookrightarrow \mathcal{O}[\![G]\!]$. It is for this map that the Amice transform becomes really interesting.

Theorem 3.5. *Let $\iota\colon G \hookrightarrow \mathcal{O}[\![G]\!]$ the natural map. The Amice transform induces an isomorphism ${}^{\iota}\mathrm{A}\colon \mathscr{D}_\infty(G,\mathcal{O}) \xrightarrow{\sim} \mathcal{O}[\![G]\!]$.*

Proof. This is well-known. □

As a corollary, we see that $\mathscr{D}_0(G,\mathcal{O})$ and $\mathscr{D}_\dagger(G,\mathcal{O})$ are naturally subalgebras of $\mathcal{O}[\![G]\!]$. One generally applies this machinery to the simplest case—namely $G = \mathbf{Z}_p$. For that group there is a well-known isomorphism $\mathcal{O}[\![G]\!] \simeq \mathcal{O}[\![t]\!]$, given by $x \mapsto (1+t)^x = \sum_{n \geqslant 0} \binom{x}{n} t^n$. In light of this, the Amice transform is generally written

$$\mathrm{A}_\mu = \int_{\mathbf{Z}_p} (1+t)^x \, \mathrm{d}\mu(x) = \sum_{n \geqslant 0} \left(\int_{\mathbf{Z}_p} \binom{x}{n} \, \mathrm{d}\mu(x) \right) t^n.$$

It realizes various (now commutative) algebras of distributions on $\mathbf{Z}_p$ as more-or-less explicit subalgebras of $\mathcal{O}[\![t]\!]$, generally defined by conditions on the growth rate of coefficients.