Absolute continuity and Fourier coefficients

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Consider the compact Lie group $\mathrm{SU}(2)$. It has an obvious maximal torus, namely

$$T = \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

The Weyl group is

$$W = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right\},$$

whose non-trivial element acts on T by $\theta \mapsto -\theta$. It is well-known that the map $T/W \to \mathrm{SU}(2)^\natural$ is a bijection. We use it to make a couple definitions. First, note that for any function on T, we will write

$$f(\theta) = f \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}.$$

Moreover, for $f \in L^1(T)$, we have the Fourier coefficients:

$$\widehat{f}(m) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{im\theta} \, \mathrm{d}\theta.$$

Definition 1. A function $f \in L^1(\mathrm{SU}(2)^\natural)$ is *absolutely continuous* if it is the descent of a W-invariant absolutely continuous function on T. In other words, $\mathrm{AC}(T/W) = \mathrm{AC}(T)^W$.

Recall that $f \in C(T)$ is absolutely continuous if there exists $g \in L^1(T)$ for which

$$f(\theta) = f(0) + \int_0^\theta g(t) \, \mathrm{d}t, \qquad \theta \in [0, 2\pi).$$

Note that if $f \in \mathrm{AC}(T/W)$, the corresponding g may not descend to T/W.

Theorem 1. *If $f \in \mathrm{AC}(T/W)$, then $\widehat{f}(m) =? \widehat{g}(m)$.*

Proof. We compute directly:

$$\begin{aligned}
\widehat{f}(m) &= \frac{1}{2\pi}\int_0^{2\pi}\left(f(0)+\int_0^{\theta} g(t)\,\mathrm{d}t\right)e^{im\theta}\,\mathrm{d}\theta \\
&= \frac{f(0)}{2\pi}\int_0^{2\pi} e^{im\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} g(t)\int_t^{2\pi} e^{im\theta}\,\mathrm{d}\theta\mathrm{d}t \\
&= f(0)\delta_{m=0} - \frac{i}{2m\pi}\int_0^{2\pi} g(t)(e^{2\pi i m} - e^{imt})\,\mathrm{d}t \\
&= \begin{cases} f(0) & m=0 \\ \frac{i}{m}\widehat{g}(m) & \text{else} \end{cases}.
\end{aligned}$$

□

Theorem 2. *Let* $f \in \mathrm{AC}(T/W)$*. Then*

$$\|S_k f - f\|_\infty \ll k^{-1/2} \cdot \|f'\|_2.$$

Proof. Recall that $S_k f(x) = \sum_{|m|\leqslant k} \widehat{f}(m)e^{imx}$. Note that

$$\begin{aligned}
|S_k f(x) - f(x)| &\leqslant \sum_{|m|>k} |\widehat{f}(m)| \\
&\ll \sum_{|m|>k} \frac{1}{m}|\widehat{g}(m)| \\
&\leqslant \sqrt{\sum_{|m|>k} m^{-2}}\sqrt{\sum_{|m|>k} |\widehat{g}(m)|^2} \\
&\ll k^{-1/2}\cdot\|f'\|_2.
\end{aligned}$$

□