Counterexamples related to the Sato–Tate conjecture for CM abelian varieties*

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1 Introduction and motivation

Let $K/\mathbf{Q}$ be a finite Galois extension, $A_{/K}$ a g-dimensional abelian variety. Assume that A has CM defined over K; then there is a CM field F together with an isomorphism $F \simeq \operatorname{End}_K(A)_{\mathbf{Q}}$. Let $\mathfrak{a} = \operatorname{Lie}(A)$. Then the determinant of the action of K on $\mathfrak{a}$ (viewed as an F-vector space) gives a map $\det_{\mathfrak{a}} \colon \mathrm{R}_{K/\mathbf{Q}} \mathbf{G}_{\mathrm{m}} \to \mathrm{R}_{F/\mathbf{Q}} \mathbf{G}_{\mathrm{m}}$. Put $G_A = \operatorname{im}(\det_{\mathfrak{a}})$ and $G_A^1 = \operatorname{im}(\det_{\mathfrak{a}})^{\mathrm{N}_{F/\mathbf{Q}}=1}$. The group G_A is the motivic Galois group of A, and $\mathrm{ST}(A)$, the Sato–Tate group of A, is a maximal compact subgroup of $G_A^1(\mathbf{C})$. It is obvious that the l-adic Galois representation associated with A, $\rho_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_{2g}(\mathbf{Q}_l)$, has image in $(\mathrm{R}_{F/\mathbf{Q}} \mathbf{G}_{\mathrm{m}})(\mathbf{Q}_l)$; it actually is a map $\rho_l \colon G_{\mathbf{Q}} \to G_A(\mathbf{Q}_l)$.

Unitary representations of $\mathrm{ST}(A)$ are just characters, and basic representation theory tells us that all such representations are induced by an (algebraic) character of G_A defined over $\overline{\mathbf{Q}}$. For $r \in \mathrm{X}^*(G_A)$, there is an L-function $L(r_*\rho_l, s)$ coming from the composite Galois representation $r \circ \rho_l \colon G_{\mathbf{Q}} \to G_A(\mathbf{Q}_l) \to \overline{\mathbf{Q}_l}^\times$. The Sato–Tate conjecture for A says that all $L(r_*\rho_l, s)$ have non-vanishing analytic continuation past $\Re = 1$, and the Generalized Riemann hypothesis for A says that all $L(r_*\rho_l, s)$ satisfy the Riemann hypothesis.

Choose an isomorphism $(\mathbf{R}/\mathbf{Z})^d \simeq \mathrm{ST}(A)$, and put

$$\mathrm{D}_x(A) = \sup_{t \in [0,1]^d} \left| \frac{1}{\pi_K(x)} \sum_{\mathrm{N}(\mathfrak{p}) \leqslant x} 1_{[0,t)}(\theta_{\mathfrak{p}}) - \int 1_{[0,t)} \right| .$$

Akiyama and Tanigawa conjectured that for non-CM elliptic curves, $\mathrm{D}_x(E) \ll x^{-\frac{1}{2}+\epsilon}$. We call the "Akiyama–Tanigawa conjecture" for A the discrepancy de-

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cays like $\mathrm{D}_x(A) \ll x^{-\frac{1}{2}+\epsilon}$. Via the Koksma–Hlawka inequality, the Akiyama–Tanigawa conjecture implies that for all bounded-variation functions f on $\mathrm{ST}(A)$, the estimate

$$\left|\sum_{\mathrm{N}(\mathfrak{p})\leqslant x} f(\theta_{\mathfrak{p}})\right| \ll \mathrm{Var}(f)x^{\frac{1}{2}+\epsilon}.$$

For $r \in \mathrm{X}^*(G_A)$, this estimate implies the Riemann hypothesis for the L-function $L(r_*\rho_l, s)$.

Analogy with Artin L-functions (go into detail here!) seems to suggest that if all $L(r_*\rho_l, s)$ satisfy the Riemann Hypothesis, then the Akiyama–Tanigawa conjecture for A holds. We'll show: this converse is false, in a limited sense.

2 Diophantine approximation

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3 Fake Satake parameters

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