Proving representability of deformation functors

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Let $\mathbf{F}$ be a finite field, $\mathrm{W}(\mathbf{F})$ its ring of Witt vectors, and $\mathfrak{Ar}_{\mathrm{W}(\mathbf{F})}$ the category of finite local $\mathrm{W}(\mathbf{F})$-algebras with residue field $\mathbf{F}$. Let $\widehat{\mathfrak{Ar}}^{\circ}_{\mathrm{W}(\mathbf{F})}$ be the opposite of the category of complete local noetherian $\mathrm{W}(\mathbf{F})$-algebras with residue field $\mathbf{F}$. For $R \in \widehat{\mathfrak{Ar}}_{\mathrm{W}(\mathbf{F})}$, write $\mathrm{Spf}(R)$ for the corresponding object of $\widehat{\mathfrak{Ar}}^{\circ}_{\mathrm{W}(\mathbf{F})}$.

Let G be a profinite action, $V_{\mathbf{F}}$ a finite-dimensional $\mathbf{F}$-vector space with continuous G-action. In proving the representability of the deformation functor $D_{V_{\mathbf{F}}}$, you use the following theorem of SGA:

Theorem 1 (SGA 3, VIIb, Thm. 1.4)**.** *Let $R \rightrightarrows X$ be an equivalence relation in $\widehat{\mathfrak{Ar}}^{\circ}_{\mathrm{W}(\mathbf{F})}$ such that the first projection is flat. Then the quotient of X by R exists. It represents the functor on points defined by the equivalence relation.*

The problem is, *this theorem is false.*

Example 1. Consider the following group object in $\widehat{\mathfrak{Ar}}^{\circ}_{\mathrm{W}(\mathbf{F})}$:

$$\widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} = \mathrm{Spf}(\mathbf{F}[\![t]\!]), \qquad A \mapsto (\mathfrak{m}_A, +).$$

It admits a Frobenius endomorphism $\varphi \colon \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \to \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}}$, which on A-points is $\varphi(a) = a^p$. Let $\boldsymbol{\alpha}_p = \ker(\varphi)$. Then

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \xrightarrow{\varphi} \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \longrightarrow 0$$

is exact in the flat topology. Apply Theorem 1 to the equivalence relation

$$\boldsymbol{\alpha}_p \times \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \rightrightarrows \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \times \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \qquad (a,b) \mapsto (b, a+b).$$

The quotient has coordinate ring

$$R = \{f \in \mathbf{F}[\![t]\!] \colon f(t\otimes 1) = f(t \otimes 1 + 1 \otimes t) \text{ in } \mathbf{F}[\![t]\!] \otimes \mathbf{F}[t]/(t^p)\} = \mathbf{F}[\![t^p]\!].$$

In other words, the quotient $\widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}}/\boldsymbol{\alpha}_p \simeq \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}}$ via $\varphi\colon \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}} \to \widehat{\mathbf{G}}_{\mathrm{a}/\mathbf{F}}$. If Theorem 1 were true, the sequence

$$0 \longrightarrow \boldsymbol{\alpha}_p(A) \longrightarrow \mathfrak{m}_A \xrightarrow{\varphi} \mathfrak{m}_A \longrightarrow 0$$

would be exact for all $A \in \mathfrak{Ar}_{\mathrm{W}(\mathbf{F})}$. But this is clearly false.

So Theorem 1 as stated is false. The correct theorem (which does apply in your setting) is the following:

Theorem 2. *Let $R \rightrightarrows X$ be an equivalence relation in $\widehat{\mathfrak{Ar}}^{\circ}_{\mathrm{W}(\mathbf{F})}$ such that one projection is flat and one is smooth. Then the quotient of X by R exists. It represents the functor on points defined by the equivalence relation.*

Proof. Let X/R be the quotient. It suffices to prove that for all $A \in \mathfrak{Ar}_{\mathrm{W}(\mathbf{F})}$, the map $X(A) \to (X/R)(A)$ is surjective. It suffices to prove the existence of a section $X/R \to X$ of the projection map $X \to X/R$. The theorem in SGA tells us that $X \to X/R$ is a flat cover, so after base-change, the projection $X \to X/R$ is $R \to X \times_{X/R} X$, as in the following commutative diagram:

$$\begin{tikzcd}
R \arrow[drr, bend left] \arrow[dr, equal] \arrow[ddr, bend right] & & \\
& X \times_{X/R} X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & X/R
\end{tikzcd}$$

So $X \to X/R$ becomes smooth after an fppf base-change. By descent, $X \to X/R$ is itself smooth. In $\widehat{\mathfrak{Ar}}_{\mathrm{W}(\mathbf{F})}$, smooth maps admit sections. $\square$

To summarize. You define a framed deformation functor $D^{\square}_{V_{\mathbf{F}}}$, and want the categorical quotient $D^{\square}_{V_{\mathbf{F}}}/\widehat{\mathrm{PGL}}_d$ to be the same as the "presheaf quotient" $D_{V_{\mathbf{F}}}$. The theorem in SGA does not give you this—you really need the fact that $\widehat{\mathrm{PGL}}_d$ is smooth.