A counterexample relating exponential sums and discrepancy

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December 19, 2016

For a prime p, let

$$T_p = \left\{ \frac{a}{2\sqrt{p}} : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p} \right\}$$
$$\Theta_p = \cos^{-1}\left(T_p\right).$$

Since applying continuous increasing functions preserves discrepancy, we have:

$$\operatorname{disc}(T_p, \mathrm{Leb}) \ll p^{-1/2}$$
$$\operatorname{disc}\left(\Theta_p, \frac{1}{2}\sin(t)\,\mathrm{d}t\right) \ll p^{-1/2}.$$

We claim that starting with $\theta_2 \in \Theta_2$, we can choose θ_p such that we preserve the inequalities:

$$\frac{1}{4\log x} \leqslant \operatorname{disc}(\{\theta_p\}_{p\leqslant x}) \leqslant \frac{4}{\log x}$$
$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \leqslant 2\sqrt{x}$$

Recall that

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin\theta}.$$

We can run this for all $p \leqslant 10^5$. Recall that $\pi(10^5) \approx 10000$.

Here is what we get:

Conjecture 1. *There exists a sequence of $\theta_p \in \Theta_p$ such that the following identities always hold:*

$$\frac{1}{4\log x} \leqslant \operatorname{disc}(\{\theta_p\}_{p\leqslant x}) \leqslant \frac{4}{\log x}$$
$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \leqslant 2\sqrt{x}.$$

Figure 1: Plot of $\sum_{p \leqslant x} U_1(\theta_p)$

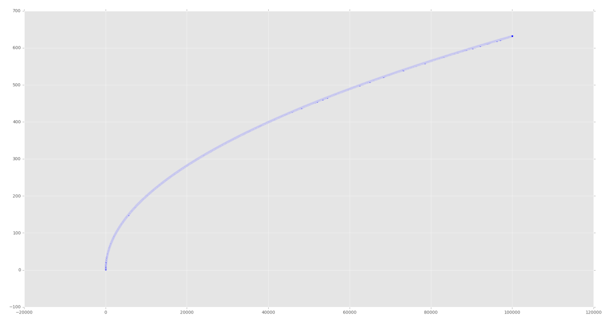
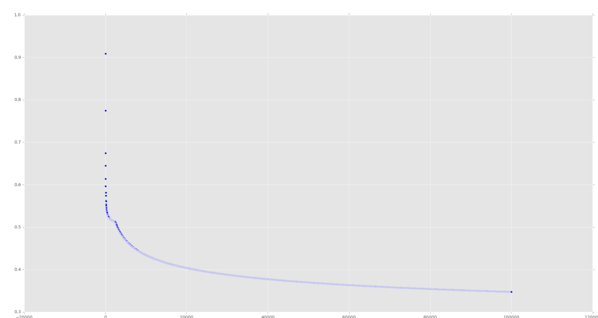


Figure 2: Plot of $\operatorname{disc}(\{\theta_p\}_{p \leqslant x})$



Next, choose $\bar{\rho}_l \colon G_{\mathbf{Q}} \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_l)$ to which we can apply Ramakrishna et. al.'s machinery. Define

$$\Theta_p(\bar{\rho}_l) = \left\{ \cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \operatorname{tr} \bar{\rho}_l(\mathrm{fr}_p) \pmod{l} \right\}.$$

Conjecture 2. *There exists a sequence of $\theta_p \in \Theta_p(\bar{\rho}_l)$ such that*

$$\operatorname{disc}(\{\theta_p\}_{p \leqslant x}) = \Omega\left(\frac{1}{\log x}\right)$$

$$\left| \sum_{p \leqslant x} U_1(\theta_p) \right| \ll \sqrt{x}.$$

Corollary 1. *There exists an (infinitely ramified) Galois representation $\rho_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that if we set $a_p = \operatorname{tr} \rho_l(\mathrm{fr}_p)$, then*

1. $a_p \in \mathbf{Z}$

2. $|a_p| \leqslant 2\sqrt{p}$ *.*

3. The $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ *satisfy*

$$\operatorname{disc}(\{\theta_p\}_{p\leqslant x}) = \Omega\left(\frac{1}{\log x}\right)$$
$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \ll \sqrt{x}.$$

and hence $L(\rho_l, s)$ *satisfies the Riemann Hypothesis.*

1 Towards a proof

Let $\bar{\rho}_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$ be a Galois representation. For each prime p, define

$$\Theta_p(l) = \left\{\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \operatorname{tr}\bar{\rho}_l(\mathrm{fr}_p) \pmod{l}\right\}.$$

It is easy to check that

$$\operatorname{disc}\left(\Theta_p(l), \frac{1}{2}\sin(t)\,\mathrm{d}t\right) \ll lp^{-1/2}.$$

We are looking for a way to choose $\theta_p \in \Theta_p(l)$ such that

1. $\operatorname{disc}(\{\theta_p\}_{p\leqslant x})$ decays like $1/\log x$
2. $\left|\sum_{p\leqslant x} U_1(\theta_p)\right|$ grows like $\sqrt{x}$.

To do this, suppose we have chosen $\{\theta_q\}_{q<p}$. In choosing θ_p, we want to simultaneously move the discrepancy towards $1/\log p$, while making sure that the U_1-sum doesn't get too big.

(Fact: if $\{x_1, \ldots, x_N\}$ and $\{y_1, \ldots, y_N\}$ are two sequences, then

$$|\operatorname{disc}(\{x_1, \ldots, x_N\}) - \operatorname{disc}(\{y_1, \ldots, y_N\})| \leqslant 2\|x - y\|_\infty.$$

)

It's actually quite simple. Note that:

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin\theta} = -U_1(\pi - \theta).$$

The basic idea is: set $\theta_3 \approx \pi - \theta_2$, $\theta_7 \approx \pi - \theta_5$, etc. and we can choose θ_2, θ_5 etc. arbitrarily, meaning good discrepancy, while the sum should approximately cancel out. First, since U_1 has bounded derivative, we know that

$$|U_1(\theta) - U_1(\varphi)| \ll |\theta - \varphi|$$

So, if $p_1 < p_2$ are sequential primes, we have

$$|\theta_{p_2} - (\pi - \theta_{p_1})| \ll p_1^{-1/2},$$

so

$$\begin{aligned}|U_1(\theta_{p_1}) + U_1(\theta_{p_2})| &\leqslant |U_1(\theta_{p_1}) - U_1(\pi - \theta_{p_1})| + |U_1(\pi - \theta_{p_1}) - U_1(\theta_{p_2})| \\ &\ll |\theta_{p_2} - (\pi - \theta_{p_1})| \\ &\ll p_1^{-1/2}.\end{aligned}$$

So,

$$\left|\sum_{p \leqslant x} U_1(\theta_p)\right| \ll \sum_{p \leqslant x} p^{-1/2} \ll \int_1^x t^{-1/2}\,\mathrm{d}t \ll \sqrt{x}.$$

(Same argument works for all U_{odd} because they all satisfy $U_{\mathrm{odd}}(\pi - \theta) = -U_{\mathrm{odd}}(\theta)$. In contrast, $U_{\mathrm{even}}(\pi - \theta) = U_{\mathrm{even}}(\theta)$.)

2 A legit proof!

Theorem 1. *Fix a prime l. Suppose we have chosen, for all primes p, some arbitrary residue class $\bar{a}_p \in \mathbf{F}_l$, and set*

$$\Theta_p(l) = \left\{\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \bar{a}_p \pmod{l}\right\}.$$

Then there exists a choice of $\theta_p \in \Theta_p(l)$ such that

1. *The sequence $\{\theta_p\}$ is equidistributed with respect to the Sato–Tate measure $\frac{2}{\pi}\sin^2\theta\,\mathrm{d}\theta$.*
2. *The discrepancy* $\mathrm{disc}(\{\theta_p\}_{p \leqslant x}, \mathrm{ST}) \gg \frac{1}{\log x}$.
3. $\left|\sum_{p \leqslant x} U_{\mathrm{odd}}(\theta_p)\right| \ll \sqrt{x}$.

Proof. Enumerate the primes $p_1 < p_2 < \cdots$. We will choose $\theta_{p_{\mathrm{odd}}} \in [0, \pi/2)$ so that the discrepancy of the sequence $\{\theta_{p_{\mathrm{odd}}}\}$ behaves as required in that interval. We'll then set $\theta_{p_{2i}} \approx \pi - \theta_{p_{2i-1}}$.

Everything comes down to: if $p < q$ are sequential primes and we have already chosen θ_p, we need to be able to choose θ_q so that $|U_1(\theta_p) + U_1(\theta_q)| \ll p^{-1/2}$. Since $\frac{\mathrm{d}U_1}{\mathrm{d}\theta} = -2\sin(\theta)$, we have (roughly)

$$|U_1(\theta) - U_1(\varphi)| \ll \max(\theta, \varphi) \cdot |\theta - \varphi|$$

for $\theta, \varphi \in [0, \pi/2)$.

Start with $t_p = \frac{a_p}{2\sqrt{p}}$ and $t_q = \frac{a_q}{2\sqrt{q}}$. We can guarantee that $|t_p - (\pi - t_q)| \ll p^{-1/2}$.

Fact:
$$|\cos^{-1}(1-x)-\cos^{-1}(1-(x+\sqrt{x}))| \ll x^{1/5}.$$
So roughly,
$$|\theta_p-\theta_q| \ll p^{-1/5},$$
After taking $\cos^{-1}$, all we can guarantee is that
$$|\theta_p-\theta_q| \ll$$
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Let's think systematically. We're picking t_1 and t_2 close to 1, which is where $(\cos^{-1})'$ blows up. But there shouldn't be very many of them close to 1. Aka,
$$\left|\frac{\#\{p \leqslant x : \theta_p \in [0,t)\}}{\pi(x)} - \int_0^t \mathrm{dST}\right| \ll \frac{1}{\log x}$$
$$\frac{\#\{p \leqslant x : \theta_p \in [0,t)\}}{\pi(x)} \ll t^2 + \frac{1}{\log x}.$$

We want to know, given x, how small the smallest $\theta_p, p \leqslant x$ is. Roughly, for what t is
$$\#\{p \leqslant x : \theta_p \in [0,t)\} < 1?$$
We already know that
$$\#\{p \leqslant x : \theta_p \in [0,t)\} \ll \frac{x}{\log x}\left(t^2+\frac{1}{\log x}\right).$$
This is frustrating, because it means, essentially, that our convergence to the Sato–Tate measure is so slow (by design) that we can't *ever* guarantee that no θ_p lies in some small interval. But there's something easier. For each $p \leqslant x$, we start by choosing $a_p \in \mathbf{Z}$. How close can a_p be to $2\sqrt{p}$? Numerical experiments **(prove this!)** show that for $t_p = \frac{a_p}{2\sqrt{p}}$, we have
$$|1-t_p| \gg p^{-1/2}.$$
This is key! That means θ_p won't be too small. In particular, we can control how close θ_p and θ_q will be.

We already have chosen θ_p. We want to choose a_q so that $\cos^{-1}(\frac{a_q}{2\sqrt{q}}) \approx \pi-\theta_p$, i.e.
$$\frac{a_q}{2\sqrt{q}} \approx \sin(\theta_p).$$
We can ensure
$$\left|\frac{a_q}{2\sqrt{q}} - \cos(\pi-\theta_p)\right| \ll p^{-1/2}.$$
Moreover, we know that $|\pm 1 - \frac{a_q}{2\sqrt{q}}| \gg q^{-1/2}$, and likewise for a_p. Thus,
$$|\theta_p-\theta_q| = \left|\cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right) - \pi + \cos^{-1}\left(\frac{a_q}{2\sqrt{q}}\right)\right| \ll p^{-1/2}\cdot ?$$
$\square$