Division algebras and spin groups

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August 7, 2016

1 Totally unrelated but possibly interesting question

Let k be a field, D a division algebra with center k. One can define a norm map $N : D \to k$ that is multiplicative. Clearly $N(k^\times) = (k^\times)^n$, where $n = \dim_k D$. So $N(D^\times) \supset (k^\times)^n$, and it is natural to ask whether they are equal. Put $\mathrm{H}(D) = N(D^\times)/N(k^\times)$. The question is: when is $\mathrm{H}(D) = 0$?

This is essentially the same question as the following. Put $D^{(1)} = \ker(N : D^\times \to k^\times)$. There is a natural isomorphism $D^\times/k^\times \xrightarrow{\sim} \mathrm{Aut}(D)$, sending $a \in D^\times$ to $b \mapsto aba^{-1}$. Hubbard asked whether $D^{(1)} \to \mathrm{Aut}(D)$ is surjective, i.e. whether $D^{(1)} \to D^\times/k^\times$ is surjective. Choosing $a \in D^\times$, we see that the only way that a can be in the image of $D^{(1)}$ is for $N(a) \in (k^\times)^n$. So we could say that the failure of $D^{(1)} \to D^\times/k^\times$ to be surjective is measured by $N(D^\times)/N(k^\times)$. Alternatively, we could think of $D^{(1)}$ and $D^\times$ as affine group schemes over k. The image of $D^{(1)} \to D^\times/\mathbf{G}_m$ is normal, so the quotient $D^{(1)}\backslash D^\times/\mathbf{G}_m$ exists as a variety over k, and we could ask about its structure.

Even better, the quotient $D^\times/\mathbf{G}_m D^{(1)}$ exists as a commutative group scheme over k. It should have dimension zero, so if it is étale, it will correspond to a G_k-module.

I think there is an easy solution. Write D^1 instead of $D^{(1)}$ and consider both D^1 and $D^\times$ as algebraic groups over k. The sequence

$$1 \longrightarrow D^1 \longrightarrow D^\times \xrightarrow{N} \mathbf{G}_m \longrightarrow 1$$

of algebraic groups over k is exact in the étale topology, so we get a long exact sequence in cohomology.

$$1 \longrightarrow D^1 \longrightarrow D^\times \xrightarrow{N} k^\times \longrightarrow \mathrm{H}^1(k, D^1) \longrightarrow \mathrm{H}^1(k, D^\times) \longrightarrow 0$$

It follows that $N(D^\times) = \ker(k^\times \to \mathrm{H}^1(k, D^1))$. That kernel is actually computable. Recall that $\mathrm{H}^1(k, D^1) = \mathrm{H}^1(G_k, D^1(k^s))$. An element $\lambda \in k^\times$ is send to the cocycle $\varphi_\lambda : G_k \to D^1(k^s)$ defined as follows. Choose a lift $\widetilde{\lambda} \in D^1(k^s)$ of λ, and put $\varphi_\lambda(\sigma) = \sigma(\widetilde{\lambda})/\widetilde{\lambda}$. It is easy to check that $\varphi_\lambda = 0$ in $\mathrm{H}^1(k, D^1)$ if and only if there exists $\widetilde{\lambda}$ with $\sigma(\widetilde{\lambda}) = \widetilde{\lambda}$, i.e. if and only if λ is in the image of $N(D^1)$, i.e. I'm not sure where this proof is going…it turns out to be a rather sophisticated proof that $N(D^\times) = N(D^\times)$.

2 A problem of Hubbard

John Hubbard suggested the following problem to me. Let k be a field (the main example I have in mind is a number fields, but the problem can be stated in much greater generality). Let (V, q) be a "quadratic space over k," i.e. V is a finite-dimensional k-vector space and q is a quadratic form on V. For example, we could have $k = \mathbf{Q}$, $V = \mathbf{Q}^{\oplus 4}$, and $q(x_1, \dots, x_4) = x_1^2 + x_2^2 + x_3^2 - 7x_4^2$. We can form the *orthogonal group* of q, $\mathrm{O}(q) \subset \mathrm{GL}(V)$; this is an algebraic group over k. The subgroup $\mathrm{O}^+(q) \subset \mathrm{O}(q)$ of "isometries" with determinant one is called the special orthogonal group of q. It turns out that there is a natural embedding of $\mathrm{O}^+(q)$ into the group of units of a particular associative algebra.

Let $C(V,q)$ be the *Clifford algebra* of (V,q), i.e. the quotient

$$C(V,q) = T(V)/\langle v \otimes v - q(v) : x \in V \rangle$$

where $T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$ is the Tensor algebra of V. The involution $-1 : V \to V$ induces an involution γ of $C(V,q)$, and we put

$$C^\circ(V,q) = C(V,q)^\gamma.$$

Key fact: $k^\times/(k^\times)^2 = \mathrm{H}^2(k, \mu_2) = \mathrm{H}^2(k, \mathbf{G}_m)[2]$. Moreover, all division algebras associated to elements of $\mathrm{H}^2(k,\mu_2)$ are quaternion algebras. We have the following exact sequences:

$$1 \longrightarrow \mathrm{O}^+(q) \longrightarrow \mathrm{O}(q) \xrightarrow{\det} \mu_2 \longrightarrow 1$$

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \Gamma(q) \xrightarrow{\alpha} \mathrm{O}(q) \longrightarrow 1$$

$$1 \longrightarrow \mathrm{Pin}(q) \longrightarrow \Gamma(q) \xrightarrow{N} \mathbf{G}_m \longrightarrow 1$$

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Pin}(q) \xrightarrow{\alpha} \mathrm{O}(q) \longrightarrow 1$$

From the last sequence, we get $\mathrm{H}^1(\mathrm{O}_n) \to \mathrm{H}^2(\mu_2)$, which will associate a division algebra to each n-dimensional quadratic form over k.

In general, if D is an n^2-dimensional division algebra over k, then the map $N : D^\times \to k^\times$ will factor through the n-th power map, i.e. $N = (-)^n \circ N_{rd}$, where N_{rd} is the so-called *reduced norm*.