Absolute continuity and Fourier coefficients

Daniel Miller

November 18, 2016

Consider the compact Lie group SU(2). It has an obvious maximal torus, namely

$$T = \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$$

The Weyl group is

$$W = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right\},$$

whose non-trivial element acts on T by $\theta \mapsto -\theta$. It is well-known that the map $T/W \to \mathrm{SU}(2)^\natural$ is a bijection. We use it to make a couple definitions. First, note that for any function on T, we will write

$$f(\theta) = f \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}.$$

Moreover, for $f \in L^1(T)$, we have the Fourier coefficients:

$$\widehat{f}(m) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-im\theta} \, \mathrm{d}\theta.$$

Definition 1. A function $f \in L^1(\mathrm{SU}(2)^\natural)$ is *absolutely continuous* if it is the descent of a W-invariant absolutely continuous function on T. In other words, $\mathrm{AC}(T/W) = \mathrm{AC}(T)^W$.

Recall that $f \in C(T)$ is absolutely continuous if there exists $g \in L^1(T)$ for which

$$f(\theta) = f(0) + \int_0^\theta g(t) \, \mathrm{d}t, \qquad \theta \in [0, 2\pi).$$

Note that if $f \in \mathrm{AC}(T/W)$, the corresponding g may not descend to T/W.

Theorem 1. *If $f \in \mathrm{AC}(T/W)$ and $m \neq 0$, then $\widehat{f}(m) = -\frac{i}{m}\widehat{g}(m)$.*

Proof. We compute directly:

$$
\begin{aligned}
\widehat{f}(m) &= \frac{1}{2\pi}\int_0^{2\pi}\left(f(0)+\int_0^{\theta} g(t)\,\mathrm{d}t\right)e^{-im\theta}\,\mathrm{d}\theta \\
&= \frac{f(0)}{2\pi}\int_0^{2\pi} e^{-im\theta}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} g(t)\int_t^{2\pi} e^{-im\theta}\,\mathrm{d}\theta\mathrm{d}t \\
&= \frac{i}{2m\pi}\int_0^{2\pi} g(t)(e^{-2\pi im}-e^{-imt})\,\mathrm{d}t \\
&= \frac{ie^{-2\pi im}}{2\pi m}\int_0^{2\pi} g(t)\,\mathrm{d}t - \frac{i}{2\pi m} g(t)e^{-imt}\,\mathrm{d}t \\
&= -\frac{i}{m}\widehat{g}(m).
\end{aligned}
$$

□

Recall that for $k \geqslant 0$, we write $S_k f$ for the k-th partial Fourier series for f, namely

$$S_k f(x) = \sum_{|m|\leqslant k} \widehat{f}(m)e^{imx}.$$

It is known that if f is absolutely continuous, then $S_k f \to f$ uniformly. We give a quantitative bound on the rate of convergence.

Theorem 2. *Let $f \in \mathrm{AC}(T/W)$. Then*

$$\|S_k f - f\|_\infty \ll k^{-1/2}\cdot\|f'\|_2.$$

Proof. Let g be such that $f(x) = f(0) + \int_0^x g(t)\,\mathrm{d}t$. We then compute

$$
\begin{aligned}
|S_k f(x) - f(x)| &\leqslant \sum_{|m|>k} |\widehat{f}(m)| \\
&\ll \sum_{|m|>k} \frac{1}{m}|\widehat{g}(m)| \\
&\leqslant \sqrt{\sum_{|m|>k} m^{-2}}\sqrt{\sum_{|m|>k} |\widehat{g}(m)|^2} \\
&\ll k^{-1/2}\cdot\|f'\|_2,
\end{aligned}
$$

using Cauchy–Schwartz for the third inequality and the Plancherel theorem for the fourth. □

Theorem 3. *Fix $x \in T$ with $\omega_0(x)$ finite, and let $x_n = nx \mod \pi \in T/W$. Then if $f \in \mathrm{AC}(T/W)$ with $\int_0^\pi f(t)\,\mathrm{d}t = 0$, we have*

$$\left|\sum_{n\leqslant N} f(x_n)\right| \ll ?$$

Proof. We begin by splitting the sum in the theorem into two parts. Let $k \geqslant 0$ be arbitrary. Then

$$\left|\sum_{n \leqslant N} f(x_n)\right| \leqslant \sum_{|m| \leqslant k} |\widehat{f}(m)| \left|\sum_{n \leqslant N} e^{imx_n}\right| + \sum_{n \leqslant N} |S_k f(x_n) - f(x_n)|.$$

Recall that $|\widehat{f}(m)| \leqslant \frac{1}{|m|}|\widehat{g}(m)|$ and the Fourier coefficients of g are bounded. Moreover, we already know that

$$\left|\sum_{n \leqslant N} e^{imx_n}\right| \ll_{\epsilon,x} |m|^{\omega_0(x)+\epsilon},$$

which tells us that

$$\sum_{|m| \leqslant k} |\widehat{f}(m)| \left|\sum_{n \leqslant N} e^{imx_n}\right| \ll_f \sum_{|m| \leqslant k} |m|^{-1+\omega_0(x)+\epsilon} \ll_f \frac{1}{\omega_0(x)+\epsilon}(k^{\omega_0(x)+\epsilon} - 1)$$

Combining everything with the previous bound on $\|S_k f - f\|_\infty$, we get

$$\left|\sum_{n \leqslant N} f(x_n)\right| \ll_{f,x,\epsilon} \log(k)|m|^?$$

$\square$

...we only get $\ll N^{\alpha(\omega_0(x)+\epsilon)} + N^{1-\alpha/2}$. Best that can be done is
$\max(aw, 1 - a/2)$
$aw = 1 - a/2$
$(w + 1/2)a = 1$
$a = 1/(w + 1/2)$
So, for $w \in (1/2, 1)$, the best power of N we can get as a bound for the sums is

$$\frac{w}{w + 1/2}$$

As $w \to 1$, the power is $< 2/3$.