A counterexample relating exponential sums and discrepancy

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For a prime p, let

$$T_p = \left\{ \frac{a}{2\sqrt{p}} : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p} \right\}$$
$$\Theta_p = \cos^{-1}(T_p).$$

Since applying continuous increasing functions preserves discrepancy, we have:

$$\operatorname{disc}(T_p, \mathrm{Leb}) \ll p^{-1/2}$$
$$\operatorname{disc}\left(\Theta_p, \frac{1}{2}\sin(t)\,\mathrm{d}t\right) \ll p^{-1/2}.$$

We claim that starting with $\theta_2 \in \Theta_2$, we can choose θ_p such that we preserve the inequalities:

$$\frac{1}{4\log x} \leqslant \operatorname{disc}(\{\theta_p\}_{p \leqslant x}) \leqslant \frac{4}{\log x}$$
$$\left| \sum_{p \leqslant x} U_1(\theta_p) \right| \leqslant 2\sqrt{x}$$

Recall that

$$U_1(\theta) = \frac{\sin(2\theta)}{\sin\theta}.$$

We can run this for all $p \leqslant 10^5$. Recall that $\pi(10^5) \approx 10000$.

Here is what we get:

Conjecture 1. *There exists a sequence of $\theta_p \in \Theta_p$ such that the following identities always hold:*

$$\frac{1}{4\log x} \leqslant \operatorname{disc}(\{\theta_p\}_{p \leqslant x}) \leqslant \frac{4}{\log x}$$
$$\left| \sum_{p \leqslant x} U_1(\theta_p) \right| \leqslant 2\sqrt{x}.$$

Figure 1: Plot of $\sum_{p \leqslant x} U_1(\theta_p)$

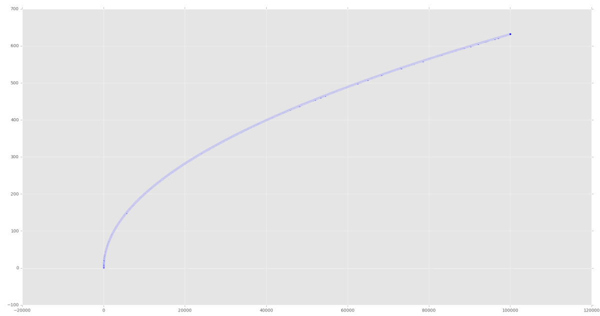
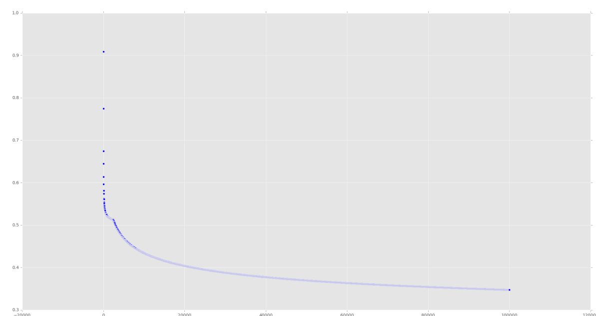


Figure 2: Plot of $\operatorname{disc}(\{\theta_p\}_{p \leqslant x})$



Next, choose $\bar{\rho}_l \colon G_{\mathbf{Q}} \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_l)$ to which we can apply Ramakrishna et. al.'s machinery. Define

$$\Theta_p(\bar{\rho}_l) = \left\{ \cos\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \operatorname{tr} \bar{\rho}_l(\mathrm{fr}_p) \pmod{l} \right\}.$$

Conjecture 2. *There exists a sequence of $\theta_p \in \Theta_p(\bar{\rho}_l)$ such that*

$$\operatorname{disc}(\{\theta_p\}_{p \leqslant x}) = \Omega\left(\frac{1}{\log x}\right)$$

$$\left| \sum_{p \leqslant x} U_1(\theta_p) \right| \ll \sqrt{x}.$$

Corollary 1. *There exists an (infinitely ramified) Galois representation $\rho_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ such that if we set $a_p = \operatorname{tr} \rho_l(\mathrm{fr}_p)$, then*

1. $a_p \in \mathbf{Z}$

2. $|a_p| \leqslant 2\sqrt{p}$ *.*

3. The $\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ *satisfy*

$$\operatorname{disc}(\{\theta_p\}_{p\leqslant x}) = \Omega\left(\frac{1}{\log x}\right)$$

$$\left|\sum_{p\leqslant x} U_1(\theta_p)\right| \ll \sqrt{x}.$$

and hence $L(\rho_l, s)$ *satisfies the Riemann Hypothesis.*

1 Towards a proof

Let $\bar{\rho}_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$ be a Galois representation. For each prime p, define

$$\Theta_p(l) = \left\{\cos\left(\frac{a}{2\sqrt{p}}\right) : a \in \mathbf{Z}, |a| \leqslant 2\sqrt{p}, a \equiv \operatorname{tr}\bar{\rho}_l(\mathrm{fr}_p) \pmod{l}\right\}.$$

It is easy to check that

$$\operatorname{disc}\left(\Theta_p(l), \frac{1}{2}\sin(t)\,\mathrm{d}t\right) \ll lp^{-1/2}.$$

We are looking for a way to choose $\theta_p \in \Theta_p(l)$ such that

1. $\operatorname{disc}(\{\theta_p\}_{p\leqslant x})$ decays like $1/\log x$
2. $\left|\sum_{p\leqslant x} U_1(\theta_p)\right|$ grows like $\sqrt{x}$.

To do this, suppose we have chosen $\{\theta_q\}_{q<p}$. In choosing θ_p, we want to simultaneously move the discrepancy towards $1/\log p$, while making sure that the U_1-sum doesn't get too big.

(Fact: if $\{x_1, \ldots, x_N\}$ and $\{y_1, \ldots, y_N\}$ are two sequences, then

$$|\operatorname{disc}(\{x_1, \ldots, x_N\}) - \operatorname{disc}(\{y_1, \ldots, y_N\})| \leqslant 2\|x - y\|_0.$$

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