Noncommutative algebras and algebraic groups

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Let k be a fixed commutative ring, not necessarily a field. Let A be a unital k-algebra. The functor $A^\times : \mathsf{Sch}_k \to \mathsf{Set}$ given by

$$A^\times(X) = \Gamma(X, \mathscr{O}_X \otimes_k A)^\times$$

is, when A is "reasonable," represented by a group scheme which we denote $A^\times_{/k}$, or just $A^\times$. The purpose of this note is to relate algebraic properties of A with the group $A^\times$. Note for example that $\mathrm{M}_n(k)^\times = \mathrm{GL}_{n/k}$.

1 Foundations

For the moment, we work in maximal possible generality. Let S be a fixed base scheme. If $\mathscr{F}$ is a sheaf on S and $f\colon X \to S$ is an object in Sch_S, we write $\mathscr{F}_X = f^*\mathscr{F}$ for the pullback of $\mathscr{F}$ to X.

Theorem 1.1. *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_S$-module. Then the functor $\mathbf{V}(\mathscr{F})\colon \mathsf{Sch}_S \to \mathsf{Set}$ given by*

$$\mathbf{V}(\mathscr{F})(X) = \hom_{\mathscr{O}_X}(\mathscr{F}_X, \mathscr{O}_X)$$

is represented by an S-scheme, also denoted $\mathbf{V}(\mathscr{F})$. The functor $\mathbf{V}\colon S_{\mathrm{qc}}^\circ \to \mathsf{Sch}_S$ is left-exact.

Proof. That $\mathbf{V}(\mathscr{F})$ is representable is standard. To check exactness of $\mathbf{V}$, just note that

$$\begin{aligned}\mathbf{V}\left(\varinjlim \mathscr{F}_\alpha\right)(X) &= \hom_{\mathscr{O}_X}\left(\varinjlim \mathscr{F}_\alpha, \mathscr{O}_X\right) \\ &= \varprojlim \hom_{\mathscr{O}_X}(\mathscr{F}_\alpha, \mathscr{O}_X) \\ &= \left(\varprojlim \mathbf{V}(\mathscr{F}_\alpha)\right)(X).\end{aligned}$$

$\square$

Thus, to give a morphism of schemes $\mathbf{V}(\mathscr{F}) \times \mathbf{V}(\mathscr{G}) \to \mathbf{V}(\mathscr{H})$, it suffices to give a morphism of sheaves $\mathscr{H} \to \mathscr{F} \oplus \mathscr{G}$. Note that the functor $\mathbf{V}$, while trivially faithful, is definitely not full. Since $\mathbf{V}(\mathscr{F})(X) = \hom_{\mathscr{O}_X}(\mathscr{F}_X, \mathscr{O}_X)$ is clearly a (commutative) group, we see that $\mathbf{V}(\mathscr{F})$ admits a group structure.

Lemma 1.2. *Let $\mathscr{C}$ be a quasi-coherent $\mathscr{O}_S$-coalgebra. Then $\mathbf{V}(\mathscr{C})$ is naturally an S-algebra.*

Proof. In other words, the functor $\mathbf{V}(\mathscr{C})\colon \mathsf{Sch}_S \to \mathsf{Set}$ factors through the category Rin of associative unital rings. Let $\Delta\colon \mathscr{C} \to \mathscr{C} \otimes_{\mathscr{O}_S} \mathscr{C}$ be the comultiplication map, and $\eta\colon \mathscr{C} \to \mathscr{O}_S$ the conit. Then $\mathbf{V}(\mathscr{C})(X)$ is given an algebra structure via convolution:

$$
\begin{aligned}
1 &= \eta_X \\
f \cdot g &= (f \otimes g) \circ \Delta_X .
\end{aligned}
$$

The verification that with this structure, $\mathbf{V}(\mathscr{C})$ is an S-algebra is routine. $\square$

So $\mathbf{V}$ gives us a functor $\mathsf{cAlg}(S_{\mathrm{qc}}) \to \mathsf{Rin}_{/S}$. If $\mathscr{C}$ is an $\mathscr{O}_S$-coalgebra, then its dual sheaf $\mathscr{C}^\vee$ is naturally an $\mathscr{O}_S$-algebra.

Theorem 1.3. *Let $\mathscr{C}$ be a locally free $\mathscr{O}_S$-coalgebra. Then $\mathscr{M} \mapsto \mathscr{M}^\vee$ gives an equivalence between the category of locally free (of finite type) $\mathscr{C}$-comodules and the category of locally free (of finite type) $\mathscr{C}^\vee$-modules.*

Let S_{lf} be the category of locally free $\mathscr{O}_S$-modules of finite type.

Theorem 1.4. *Let $\mathscr{M} \in S_{\mathrm{lf}}$. Then the functor $\mathbf{W}(\mathscr{M})\colon \mathsf{Sch}_S \to \mathsf{Set}$ given by*

$$\mathbf{W}(\mathscr{M})(X) = \Gamma(X, \mathscr{M}_X)$$

is representable. Moreover, $\mathscr{M} \mapsto \mathbf{W}(\mathscr{M})$ gives a left-exact tensor-functor from S_{lf} to $\mathsf{Mod}(\mathbf{W}(\mathscr{O}_S))$.

Proof. We content ourselves with showing that $\mathbf{W}(\mathscr{M}) = \mathbf{V}(\mathscr{M}^\vee)$. Since $\mathscr{M}$ is locally free of finite type, it is self-dual. Thus

$$\mathbf{V}(\mathscr{M}^\vee)(X) = \hom_{\mathscr{O}_X}(\mathscr{M}_X^\vee, \mathscr{O}_X) = \Gamma(X, \mathscr{M}_X^{\vee\vee}) = \Gamma(X, \mathscr{M}_X)$$

as desired. $\square$

So we have a functor $\mathbf{W}\colon \mathsf{Rin}(S_{\mathrm{lf}}) \to \mathsf{Rin}_{/S}$.

2 Representations of groups and algebras

If $\mathscr{A}$ is a locally free $\mathscr{O}_S$-algebra, we write $\mathrm{GL}(\mathscr{A}) = \mathscr{A}^\times$ for the functor $X \mapsto \Gamma(X, \mathscr{A}_X)^\times$. This is representable, as it can easily be written as a fiber product of schemes. Many well-known algebraic groups arise via this construction, or one of its generalizations.

Theorem 2.1. *Let* $\mathscr{A} \in \mathsf{Rin}(S_{\mathrm{lf}})$. *Then there is a natural isomorphism* $\mathrm{Lie}(\mathscr{A}^\times) = (\mathbf{W}(\mathscr{A}), [\cdot, \cdot])$.

Proof. Recall that for $X \in \mathsf{Sch}_S$, we write $X[\epsilon]$ for the scheme whose underlying space is the same as X, but whose structure sheaf is $\mathscr{O}_X[\epsilon]/\epsilon^2$. Then

$$\begin{aligned} \mathrm{Lie}(\mathscr{A}^\times)(X) &= \ker(\Gamma(X, \mathscr{A}_X[\epsilon])^\times \to \Gamma(X, \mathscr{A}_X)^\times) \\ &= \Gamma(X, 1 + \epsilon \mathscr{A}_X) \\ &\simeq \mathbf{W}(\mathscr{A})(X). \end{aligned}$$

Checking that the bracket comes from the commutator on $\mathscr{A}$ is a simple computation. $\square$

3 The case of a field

Let k be a field of characteristic zero.