Thoughts on lie algebras

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August 7, 2016

Let k be a commutative ring. We will write $\mathcal{A}$ to denote the category of k-algebras, but much of this will work for $\mathcal{A}$ an arbitrary (sufficiently nice) category.

Let $\mathcal{A}^*$ be the category of augmented k-algebras (or more generally, the category of arrows $0 \to a$ in $\mathcal{A}$). Then the initial and terminal objects in $\mathcal{A}^*$ are the same. Let $\mathrm{Ab}(\mathcal{A}^*)$ denote the category of abelian group objects in $\mathcal{A}^*$. There is an obvious forgetful functor $\mathrm{Ab}(\mathcal{A}^*) \to \mathcal{A}^*$. It has a left adjoint which we denote by Ω. One has

$$\Omega(A, \varepsilon) = k \oplus \varepsilon_* \Omega^1_{A/k}$$

Now let $\mathrm{Cg}(\mathcal{A}^*)$ denote the category of cogroup objects in $\mathcal{A}^*$ – in other words group objects in $\mathcal{A}^{*\circ}$. So $\hom(G, -)$ is a group-valued functor for $G \in \mathrm{Cg}(\mathcal{A}^*)$.

If $G \in \mathrm{Cg}(\mathcal{A}^*)$ and $M \in \mathrm{Ab}(\mathcal{A}^*)$, then $\hom_{\mathcal{A}^*}(G, M)$ has two group operations, each of which distribute over the other. Thus $\hom_{\mathrm{Ab}(\mathcal{A}^*)}(\Omega G, -)$ is canonically an abelian-group functor, so ΩG, a priori only an abelian group object, is also an abelian co-group object in $\mathcal{A}^*$. Put $\mathfrak{g} = \Omega G$. I claim that $\mathfrak{g}$ has a natural action of G, the *adjoint action* $\mathrm{ad} : G \to \mathrm{Aut}\,\mathfrak{g}$. We first realize this action on

Given $g \in G(A)$, $X \in \mathfrak{g}(A)$

$$\begin{aligned} && g \in \hom_{\mathcal{A}^*}(G, A) \\ X \in \hom_{\mathcal{A}^*}(\mathfrak{g}, A) = \hom_{\mathrm{Ab}(\mathcal{A}^*)}(\mathfrak{g}, k \oplus A) &=& \hom_{\mathcal{A}^*}(G, k \oplus A) \end{aligned}$$

1 Some examples

Let A be (possibly non-associative) k-algebra. Let $\mathrm{Aut}(A)$ be the funtor $R \mapsto \mathrm{Aut}_R(A \otimes_k R)$. Put $G = \mathrm{Aut}(A)$; we want to compute $\mathfrak{g} = \mathrm{Lie}\, G$. We have

$$\mathfrak{g}(R) = \ker(G(R[\varepsilon]) \to G(R)) = \ker(\mathrm{Aut}_R(A_{R[\varepsilon]}) \to \mathrm{Aut}_R(A_R))$$

If $\phi : A \otimes R[\varepsilon] \to A \otimes R[\varepsilon]$ is an isomorphism of $R[\varepsilon]$-algebras in $\mathfrak{g}(R)$. Then ϕ is of the form $a \otimes r \mapsto (a + (\partial a)\varepsilon) \otimes r$, for $\partial : A_r \to A_R$ an R-linear map. One checks that ϕ is an automorphism if and only if ∂ is a derivation. We have:

$$\mathrm{Der}(A) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Aut}\, A)$$
$$\mathrm{End}(V) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{GL}\, V)$$

If we have $\rho : G \to \mathrm{Aut}\, A$, then we get $\rho : \mathrm{Lie}\, G \to \mathrm{Der}(A)$. In particular, the adjoint action of G on itself gives $\mathfrak{g} \to \mathrm{Der}(G)$.

Let's work things out for $G = \mathrm{GL}(V)$, where V is some k-module. We want to get a morphism $\mathfrak{g} \to \mathrm{Der}(G)$. We look at R-valued points. An element $X \in \mathfrak{g}(R)$ is identified with

$$\exp(X) = 1 + X\varepsilon \in \ker(\mathrm{GL}(V \otimes R[\varepsilon]) \to \mathrm{GL}(V \otimes R)).$$

The action $\mathrm{ad} : G \to \mathrm{Aut}\, G$ should give us $\mathrm{ad}(X) \in \ker(\mathrm{Aut}(G \otimes R[\varepsilon]) \to \mathrm{Aut}(G \otimes R))$. Indeed, $\mathrm{ad}(X)$ acts on S-valued points as $\mathrm{ad}(X) : G_{R[\varepsilon]}(S) \to G_{R[\varepsilon]}(S)$ as honest conjugation by $1 + \varepsilon X$.

$$g : R[\varepsilon][X_{ij}, \det^{-1}] \to S \leftrightarrow (g_{ij}) \in \mathrm{GL}(V \otimes S)$$
$$(1 + \varepsilon X) g (1 - \varepsilon X) = g + \varepsilon [X, g]$$

2 Some functors

We define a k-group functor $\mathrm{Der}(A)$ by $\mathrm{Der}(A)(R) = \mathrm{Der}_R(A \otimes_k R)$. In nice circumstances, this is a quasi-coherent $\mathscr{O}$-module. Define $\mathrm{Der}(A) \to \mathrm{Lie}(\mathrm{Aut}\, A)$ as follows. On R-valued points, we need

$$\mathrm{Der}_R(A \otimes R) \to \ker(\mathrm{Aut}(A)(R[\varepsilon]) \to \mathrm{Aut}(A)(R))$$
$$\mathrm{Der}_R(A \otimes R) \to \ker(\mathrm{Aut}_{R[\varepsilon]}(A \otimes R[\varepsilon]) \to \mathrm{Aut}_R(A \otimes R))$$

Let $\partial : A \otimes R \to A \otimes R$ be an R-derivation. Define $\phi = 1 + \varepsilon \cdot \partial$ by

$$\phi(a \otimes r) = a \otimes r + \partial(a) \otimes \varepsilon r.$$

In other words, $\phi = 1 + \partial \otimes \varepsilon$. We have

$$\begin{aligned} \phi(a \otimes r)\phi(b \otimes s) &= (a \otimes r + \partial(a) \otimes \varepsilon r)(b \otimes s + \partial(b) \otimes \varepsilon s) \\ &= ab \otimes rs + a\partial(b) \otimes \varepsilon rs + b\partial(a) \otimes \varepsilon rs \\ &= ab \otimes rs + (a\partial b + b\partial a) \otimes \varepsilon rs \\ &= ab \otimes rs + \partial(ab) \otimes \varepsilon rs. \end{aligned}$$

So ϕ is a ring homomorphism. Note that $1 + \varepsilon \cdot \partial$ and $1 - \varepsilon \cdot \partial$ are inverses, so ϕ is an automorphism. Conversely, one checks that all elements of $\mathrm{Lie}(\mathrm{Aut}\, A)(R)$ are of the form $1 + \varepsilon \cdot \partial$. In other words, $\mathrm{Der}(A) \to \mathrm{Lie}(\mathrm{Aut}\, A)$ is an isomorphism of group functors.

3 Invariant derivations

Let G be a k-group functor. Then we have a homomorphism $l : G \to \mathrm{Aut}\, G$, the "left regular representation." The induced infinitesimal representation $l : \mathfrak{g} \to \mathrm{Lie}(\mathrm{Aut}\, G) = \mathrm{Der}(G)$ should induce an isomorphism between $\mathfrak{g} = \mathrm{Lie}\, G$ and the algebra of invariant derivations of G. Let's see how this works. On points, we have

$$l(R) : \ker(G(R[\varepsilon]) \to G(R)) \to \ker(\mathrm{Aut}_{R[\varepsilon]}(G \otimes R[\varepsilon]) \to \mathrm{Aut}_R(G \otimes R))$$

Given $X \in \mathfrak{g}(R)$, we need an element $l(R)(X) \in \mathrm{Aut}_{R[\varepsilon]}(G \otimes R[\varepsilon])$. We define this on the functor of points: $l(R)(X)(S) : (G \otimes R[\varepsilon])(S) \to (G \otimes R[\varepsilon])(S)$ is $x \mapsto X \cdot x$.