Artin motives and the Tannakian formalism

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1 Finite groups and group schemes

Let k be a field, Γ a (finite) abstract group. Let Aff_k be the category of affine schemes over k. We can think of Γ as a presheaf on Aff_k via

$$\underline{\Gamma} : A \mapsto \begin{cases} \Gamma & \text{if } A \neq 0 \\ 1 & \text{if } A = 0. \end{cases}$$

It is easily seen that $\underline{\Gamma}$ is not a sheaf. However, it is a separated presheaf, and it is easy to check that its (Zariski) sheafification is represented by the affine group scheme

$$\Gamma = \operatorname{Spec}\left(k[\Gamma]^\vee\right).$$

Consider the general linear group $\mathrm{GL}(n)$ over k. As a group-valued functor, this is $A \mapsto \mathrm{GL}(n, A)$. We claim that to give a representation $\rho : \Gamma \to \mathrm{GL}(n)$ over k (i.e. a homomorphism $\Gamma \to \mathrm{GL}(n)$ of k-group schemes), it is equivalent to give a homomorphism of honest groups

$$\rho : \Gamma \to \mathrm{GL}(n, k).$$

This is not hard to check. Since sheafification is the left-adjoint to the inclusion functor from presheaves to sheaves, we have

$$\hom(\Gamma, \mathrm{GL}(n)) = \hom_{\widehat{\mathsf{Aff}_k}}(\underline{\Gamma}, \mathrm{GL}(n)) = \hom(\Gamma, \mathrm{GL}(n, k)),$$

whence the result. It follows that a k-representation of Γ in the scheme-theoretic sense is the same thing as a k-representation of Γ in the group-theoretic sense. We denote both categories by $\mathsf{Rep}_k(\Gamma)$.

Incidentally, much of this can be generalized. Let $\mathcal{C}$ be a site, and let $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ denote the category of presheaves (resp. sheaves) on $\mathcal{C}$. There is a diagram of functors.

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift left, "(-)_c"] & \widehat{\mathcal{C}} \arrow[l, shift left, "\hom(1,-)"] \arrow[r, shift left, "(-)^+"] & \widetilde{\mathcal{C}} \arrow[l, shift left, "i"] \end{tikzcd}$$

Here, X_c is the constant presheaf $C \mapsto X$, $(-)^+$ is sheafification, and i is the natural embedding. There are adjunctions $(-)_c \dashv \hom(1, -)$ and $(-)^+ \dashv i$. Our result above is a special case of the fact that if Γ is a group and $G \in \widetilde{\mathcal{C}}$ a group object, then

$$\hom_{\widetilde{\mathcal{C}}}(\Gamma_c^+, G) = \hom_{\widehat{\mathcal{C}}}(\Gamma_c, iG) = \hom(\Gamma, \hom(1, iG)) = \hom(\Gamma, G(1)).$$

It's worth noting that $X_c^+ = \coprod_X 1$, which can be seen via

$$\hom_{\widetilde{\mathcal{C}}}(X_c^+, F) = \hom_{\widehat{\mathcal{C}}}(X_c, iF) = \hom(X, \hom(1, iF)) = \prod_X \hom(1, F) = \hom_{\widetilde{\mathcal{C}}}\left(\coprod_X 1, F\right).$$

From this, we see that if Γ is an arbitrary (abstract) group, the associated (Zariski) sheaf Γ_c^+ is represented by the scheme $\coprod_\Gamma \operatorname{Spec}(k)$, which is only affine if Γ is finite.

2 Profinite groups and group schemes

Homomorphisms between topological groups and abstract groups will always be assumed continuous, and similarly for topological spaces. Let $X = \varprojlim X_\alpha$ be a profinite set, Y an affine scheme over k. We have

$$\hom(X, Y(1)) = \varinjlim \hom_{\mathsf{cts}}(X_\alpha, Y(1)) = \varinjlim \hom(X^+_{\alpha,c}, Y).$$

In other words, $Y \mapsto \hom(X, Y(1))$ is "represented" by the pro-scheme $\varprojlim X^+_{\alpha,c}$. We will write X for this pro-scheme, keeping in mind that $\varprojlim X^+_{\alpha,c}$ is not actually a scheme.

The main application is that if Γ is a profinite group and G is a group scheme over k, then

$$\hom_{\mathsf{cts}}(\Gamma, G(k)) = \hom(\Gamma, G),$$

where in the latter hom-set, Γ is interpreted as the formal projective limit

$$\Gamma = \varprojlim_U \operatorname{Spec}\left(k[\Gamma/U]^\vee\right).$$

So the categories of "scheme-theoretic k-representations of Γ" and "continuous k-representations of Γ" are equivalent, and we will write $\mathsf{Rep}_k(\Gamma)$ for both. Note that "continuous representations" means continuous when k is given the discrete topology.