Smoothness and some deformation rings

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1 Notation

As is typical, if G is a profinite group and M a topological G-module, we write $\mathrm{H}^\bullet(G,M)$ for the continuous cohomology of G with coefficients in M. If k is a field and $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts continuously on M, we write $\mathrm{H}^\bullet(k,M)$ instead of $\mathrm{H}^\bullet(\mathrm{Gal}(k^{\mathrm{sep}}/k),M)$.

Suppose $\mathrm{Gal}_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on M. Then for each place v, the we write $\mathrm{H}^\bullet(v,M) = \mathrm{H}^\bullet(\mathbf{Q}_v,M)$. If M is unramified outside a set S of places, we write $\mathrm{H}^\bullet(S,M) = \mathrm{H}^\bullet(\mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q}),M)$, where $\mathbf{Q}_S$ is the maximal extension of $\mathbf{Q}$ unramified outside S.

Fix a finite field k of characteristic p. Let $\mathrm{W}(k)$ be the ring of Witt vectors of k, and let $\mathsf{C}_{\mathrm{W}(k)}$ be the category of local artinian $\mathrm{W}(k)$-algebras with residue field k. Fix once and for all a continuous representation $\bar\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(k)$. If S is a finite set of places outside which $\bar\rho$ is unramified, define a functor $\mathcal{X}_S(\bar\rho)\colon \mathsf{C}_{\mathrm{W}(k)} \to \mathsf{set}$ by letting

$$\mathcal{X}_S(\bar\rho)(A) = \{\text{deformations of } \bar\rho \text{ to } \mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q}) \to \mathrm{GL}_2(A)\}.$$

Since $\bar\rho$ is fixed, we will generally drop it from the notation. It is well known that there is a canonical isomorphism $\mathfrak{t}_{\mathcal{X}_S} \simeq \mathrm{H}^1(S,\mathrm{ad}\,\bar\rho)$. The functor $\mathcal{X}_S$ is smooth if and only if $\mathrm{H}^2(S,\mathrm{ad}\,\bar\rho) = 0$. More generally, if $A_1 \twoheadrightarrow A_0$ in $\mathsf{C}_{\mathrm{W}(k)}$ has kernel $\mathfrak{a}$ annihilated by $\mathfrak{m}_{A_1}$, then for each $\rho_0 \in \mathcal{X}_S(A_0)$, there is an "obstruction class" $o(\rho_0) \in \mathrm{H}^2(S,\mathrm{ad}\,\bar\rho)\otimes\mathfrak{a}$ whose vanishing is necessary and sufficient for the existence of a lift of ρ_0 to $\rho_1 \in \mathcal{X}_S(A_1)$.

For a place v of $\mathbf{Q}$, let $\mathcal{X}_v = \mathcal{X}_v(\bar\rho)$ classify deformations of $\bar\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}_v}/\mathbf{Q})}$. If S is a finite set of places of $\mathbf{Q}$, write $\mathcal{X}_{\partial S} = \prod_{v\in S}\mathcal{X}_v$. Clearly $\mathfrak{t}_{\mathcal{X}_{\partial S}} = \bigoplus_{v\in S}\mathrm{H}^1(v,\mathrm{ad}\,\bar\rho)$.

We will tacitly fix all determinants, which means that we deal with the cohomology of $\mathrm{ad}^\circ\,\bar\rho$, the space of trace-zero matrices.

2 Smoothness

Let $\mathcal{X}, \mathcal{Y} \colon \mathsf{C}_{\mathrm{W}(k)} \to \mathsf{set}$, and $f \colon \mathcal{X} \to \mathcal{Y}$ a morphism. One says f is *formally smooth* if whenever $A_1 \twoheadrightarrow A_0$ in $\mathsf{C}_{\mathrm{W}(k)}$, the natural map

$$\mathcal{X}(A_1) \to \mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A_1)$$

is surjective. In other words, if $x_0 \in \mathcal{X}(A_0)$ is such that $f(x_0)$ lifts to $\mathcal{Y}(A_1)$, then x_0 lifts to $\mathcal{X}(A_1)$. Clearly the composite of smooth morphisms is smooth.

3 Poitou-Tate duality

Let $V \in \mathsf{Rep}_k(\mathbf{Q})$ be unramified outside a finite set S of places. Suppose we have a set $\{L_v : v \in S\}$, where $L_v \subset \mathrm{H}^1(v, V)$. Let $L^\perp = \{L_v^\perp : v \in S\}$, where $L_v^\perp \subset \mathrm{H}^1(v, V^*)$ is the annihilator of L_v under the cup product. Define

$$\mathrm{H}^1_L(S, V) = \ker\left(\mathrm{H}^1(S, V) \to \bigoplus_{v \in S} \frac{\mathrm{H}^1(v, V)}{L_v} \right),$$

and similarly for $\mathrm{H}^1_{L^\perp}(S, V^*)$. Poitou-Tate duality gives us an exact sequence

$$\mathrm{H}^1(S, V) \to \bigoplus_{v \in S} \mathrm{H}^1(v, V) \to \mathrm{H}^1(S, V^*)^\vee \to \mathrm{H}^2(S, V) \to \bigoplus_{v \in V} \mathrm{H}^2(v, V).$$

Quotient out by L to obtain

$$\mathrm{H}^1(S, V) \to \bigoplus_{v \in S} \frac{\mathrm{H}^1(v, V)}{L_v} \to \mathrm{H}^1_{L^\perp}(S, V^*)^\vee \to \mathrm{H}^2(S, V) \to \bigoplus_{v \in V} \mathrm{H}^2(v, V). \quad (*)$$

When $L = 0$, we write $\text{Ш}^1_S(V) = \mathrm{H}^1_0(S, V)$.

4 Formal smoothness of deformation spaces

Let $\bar\rho$, $\mathcal{X}_S$, $\mathcal{X}_{\partial S}$ be as above.

Theorem 4.1. *$\partial \colon \mathcal{X}_S \to \mathcal{X}_{\partial S}$ is smooth if and only if $\text{Ш}^1_S(\mathrm{ad}^\circ \bar\rho^*) = 0$.*

Proof. Suppose we have $A_1 \twoheadrightarrow A_0$ in $\mathsf{C}_{\mathrm{W}(k)}$. Given $\rho_0 \in \mathcal{X}_S(A_0)$, the image $\partial(\rho_0) = (\rho_0|_V)_{v \in S}$ lifts to $\mathcal{X}_{\partial S}(A_1)$ if and only if $o(\partial \rho_0) \in \bigoplus_{v \in S} \mathrm{H}^2(v, \mathrm{ad}^\circ \bar\rho)$ vanishes. The original ρ_0 lifts to $\mathcal{X}_S(A_1)$ if and only if $o(\rho_0) \in \mathrm{H}^2(S, \mathrm{ad}^\circ \bar\rho)$ vanishes. In other words, smoothness is equivalent to the vanishing of

$$\ker\left(\mathrm{H}^2(S, \mathrm{ad}^\circ \bar\rho) \to \bigoplus_{v \in S} \mathrm{H}^2(v, \mathrm{ad}^\circ \bar\rho) \right) \simeq \text{Ш}^1_S(\mathrm{ad}^\circ \bar\rho^*)^\vee,$$

the isomorphism being a part of Poitou-Tate duality. □

Suppose we have a subfunctor $\mathcal{C} = \prod_{v \in S} \mathcal{C}_v \subset X_{\partial S}$; put $\mathfrak{c} = \mathfrak{t}_{\mathcal{C}}$. Define $\mathcal{X}_{\mathcal{C}}$ to be the pullback

$$\begin{tikzcd} \mathcal{X}_{\mathcal{C}} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, hook] \\ \mathcal{X}_S \arrow[r, "\partial"] & \mathcal{X}_{\partial S} \end{tikzcd}$$

Theorem 4.2. *If* $\mathrm{H}^1_{\mathfrak{c}^\perp}(S, \mathrm{ad}^\circ \bar{\rho}^*) = 0$*, then* $\mathcal{X}_{\mathcal{C}} \to \mathcal{C}$ *is smooth.*

Proof. From $(*)$, we already know that $\text{Ш}^1_S(\mathrm{ad}^\circ \bar{\rho}^*) = 0$, so $\mathcal{X} \to \mathcal{X}_{\partial S}$ is smooth. Suppose $A_1 \twoheadrightarrow A_0$ in $\mathsf{C}_{\mathrm{W}(k)}$, and let $\rho_0 \in \mathcal{X}_{\mathcal{C}}(A_0)$. If $\partial(\rho_0)$ lifts to $\widetilde{\partial \rho_0} \in \mathcal{C}(A_1)$, then because $\mathcal{X}_S \to \mathcal{X}_{\partial S}$ is smooth, ρ_0 lifts to $\rho_1 \in \mathcal{X}_S(A_1)$. The exact sequence $(*)$ tells us that

$$\mathfrak{t}_{\mathcal{X}_{\partial S}} = \partial_* \mathfrak{t}_{\mathcal{X}_S} + \mathfrak{c}.$$

It is well-known that lifts of $\partial(\rho_0)$ to $\mathcal{X}_{\partial S}(A_1)$ form a $\mathfrak{t}_{X_S} \otimes \mathfrak{a}$-torsor. In particular, there exists $t = \partial_* c + d \in \mathfrak{t}_{\mathcal{X}_{\partial S}}$ such that $c \in \mathfrak{t}_{\mathcal{X}_S}$, $d \in \mathfrak{c}$, and $t \cdot \partial(\rho_1) = \widetilde{\partial(\rho_0)} \in \mathcal{C}(A_1)$. Then

$$\partial(c \cdot \rho_1) = (\partial_* c) \cdot \partial(\rho_1) = (-l) \cdot \widetilde{\partial(\rho_0)} \in \mathcal{C}(A_1).$$

So $c \cdot \rho_1$ is a lift of ρ_0 to $\mathcal{X}_{\mathcal{C}}(A_1)$. □

Corollary 4.3. *If there is a smooth* $\mathcal{C} \subset \mathcal{X}_{\partial S}$ *such that* $\mathrm{H}^1_{\mathfrak{c}^\perp}(S, \mathrm{ad}^\circ \bar{\rho}^*) = 0$*, then* $\mathcal{X}_{\mathcal{C}}$ *is smooth.*

Proof. Note that $\mathcal{X}_{\mathcal{C}} \to *$ is the composite of $\mathcal{X}_{\mathcal{C}} \to \mathcal{C}$ and $\mathcal{C} \to *$, both of which are smooth. □