Riemann Hypothesis for “fake L-functions” coming from elliptic curves

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1 Introduction

Let $E_{/\mathbf{Q}}$ be an elliptic curve, l an odd prime, and $\rho_{E,l} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ the associated Galois representation. For p not dividing the conductor of E, we define

$$\begin{aligned} a_p &= p + 1 - \#E(\mathbf{F}_p) \\ &= \operatorname{tr} \rho_{E,l}(\mathrm{fr}_p). \end{aligned}$$

Then the Hasse bound $|a_p| \leqslant 2\sqrt{p}$ allows us to define an angle $\theta_p = \theta_p(E) \in [0, \pi]$ by the equation

$$2 \cos \theta_p = \frac{a_p}{\sqrt{p}}.$$

Let $S([0,\pi])$ be the space of step functions on $[0,\pi]$. For $\eta \in S([0,\pi])$ with $\|\eta\|_\infty \leqslant 1$, we define a “fake L-function” $L_\eta(E,s)$ by an infinite product

$$L_\eta(E,s) = \prod_p \left(1 - \eta(\theta_p) p^{-s}\right)^{-1}$$

where here and in the remainder of the document we take the product over all unramified primes of E.

Our goal is to show that, assuming a kind of “strong Sato–Tate conjecture” for E, the $L_\eta(E,s)$ satisfy the appropriate analogue of the Riemann Hypothesis.

2 Heuristics

Just for this section, let $\{\theta_p\}$ be a collection of i.i.d. random variables with joint distribution μ_{ST} and f a function on $[0,\pi]$. Then $\{f(\theta_p)\}$ is a collection of i.i.d. random variables with expected value $\mathbf{E}[f(\theta_p)] = \mu_{\mathrm{ST}}(f)$. The strong law of large numbers tells us that

$$\frac{1}{\pi(X)} \sum_{p \leqslant X} f(\theta_p) \to \mu_{\mathrm{ST}}(f).$$

We want quantitative bound on the convergence. A basic but involved computation yields

$$\mathbf{E}\left[\left(\frac{1}{\pi(X)} \sum_{p \leqslant X} f(\theta_p) - \mu_{\mathrm{ST}}(f)\right)^2\right] = \frac{1}{\pi(X)} \operatorname{Var}(f)$$

So heuristically, if the θ_p come from an elliptic curve, we expect

$$\left|\frac{1}{\pi(X)} \sum_{p \leqslant X} f(\theta_p) - \int f \, \mathrm{d}\mu_{\mathrm{ST}}\right| = O\left(\operatorname{Var}(f) X^{-\frac{1}{2}+\epsilon}\right).$$

3 First properties

Throughout this section, $E_{/\mathbf{Q}}$ is an elliptic curve and all notation is as above.

Lemma 3.1. *The product for $L_\eta(E, s)$ converges absolutely on the region $\{\Re s > 1\}$.*

Proof. By [Kno56, §3.7, Th. 5], it suffices to prove that the sum $\sum_p \frac{|\eta(\theta_p)|}{p^s}$ converges when $\Re s > 1$. This is simple:

$$\sum_p \frac{|\eta(\theta_p)|}{p^s} \leqslant \sum_n n^{-s} = \zeta(s)$$

which is already known to converge. □

Let μ_{ST} be the Sato–Tate measure $\frac{2}{\pi} \sin^2 \theta \, \mathrm{d}\theta$ and, for each $X > 0$, define a measure $\mu^X = \frac{1}{\pi(X)} \sum_{p \leqslant X} \delta_{\theta_p}$. Then the Sato–Tate conjecture states that $\mu^X \to^* \mu_{\mathrm{ST}}$, in the sense that for each $f \in C([0, \pi])$,

$$\lim_{X \to \infty} \frac{1}{\pi(X)} \sum_{p \leqslant X} f(\theta_p) = \int_{[0,\pi]} f \, \mathrm{d}\mu_{\mathrm{ST}}.$$

To simplify notation, for any measure μ, write $\mu(f) = \int f \, \mathrm{d}\mu$.

Conjecture 3.2 (Akiyama–Tanigawa)**.**

$$\sup_{x\in[0,\pi]} \left|(\mu^X - \mu_{\mathrm{ST}})(\chi_{[x,\pi]})\right| = O(X^{-\frac{1}{2}+\epsilon}).$$

This is a trivial rewriting of the statement in [AT99, Conj. 1].

Theorem 3.3. *Assume the Akiyama–Tanigawa conjecture. Then for any step function* $\eta\colon [0,\pi] \to [-1,1]$*, we have*

$$\left|(\mu^X - \mu_{\mathrm{ST}})(\eta)\right| = O_\eta(X^{-\frac{1}{2}+\epsilon}).$$

Proof. We may write $\eta = \sum \lambda_i \chi_{[x_i,\pi]}$. Then

$$\begin{aligned}
\left|(\mu^X - \mu_{\mathrm{ST}})(\eta)\right| &= \left|\sum \lambda_i(\mu^X - \mu_{\mathrm{ST}})(\chi_{[x_i,\pi]})\right| \\
&\leqslant \sum |\lambda_i| \left|(\mu^X - \mu_*)(\chi_{[x_i,\pi]})\right| \\
&= \left(\sum |\lambda_i|\right) O(X^{-\frac{1}{2}+\epsilon}).
\end{aligned}$$

□

Theorem 3.4.

4 Functions with non-zero expected value

Any integrable function on $[0,\pi]$ can be written as $c+\eta$, where $c \in \mathbf{R}$ and $\mu_{\mathrm{ST}}(\eta) = 0$. Then

$$\begin{aligned}
L_{c+\eta}(E,s) &= \prod_p \left(1 - (c + \eta(\theta_p))p^{-s}\right)^{-1} \\
&= \prod_p \left(1 - \eta(\theta_p)p^{-s+\log_p c}\right)^{-1}.
\end{aligned}$$

References

[AT99] Shigeki Akiyama and Yoshio Tanigawa. “Calculation of values of L-functions associated to elliptic curves”. In: *Math. Comp.* 68.227 (1999), pp. 1201–1231.

[Kno56] Konrad Knopp. *Infinite sequences and series.* Translated by Frederick Bagemihl. Dover Publications, Inc., New York, 1956.