Torsion in the cohomology of Bianchi groups

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For $d \in \mathbf{Z}$ a square-free integer, write $\mathcal{O}_d$ for the ring of integers of the quadratic field $\mathbf{Q}(\sqrt{d})$. The plan is to compute explicitly, for any ideals $\mathfrak{a}, \mathfrak{n} \subset \mathcal{O}_d$, the cohomology $H(\mathfrak{n}, \mathfrak{a}) = \mathrm{H}^1(\Gamma(\mathfrak{n}), \mathcal{O}_d/\mathfrak{a})$. Note that whenever $\mathfrak{a} \mid \mathfrak{a}', \mathfrak{n} \mid \mathfrak{n}'$, we have a commutative diagram

$$\begin{tikzcd} H(\mathfrak{n}, \mathfrak{a}') \arrow[r, "\mathrm{res}_{\mathfrak{n}}^{\mathfrak{n}'}"] \arrow[d, "\mathrm{red}_{\mathfrak{a}}^{\mathfrak{a}'}"] & H(\mathfrak{n}', \mathfrak{a}') \arrow[d, "\mathrm{red}_{\mathfrak{a}}^{\mathfrak{a}'}"] \\ H(\mathfrak{n}, \mathfrak{a}) \arrow[r, "\mathrm{res}_{\mathfrak{n}}^{\mathfrak{n}'}"] & H(\mathfrak{n}', \mathfrak{a}). \end{tikzcd}$$

Put $H(\mathfrak{a}) = \varinjlim_{\mathfrak{n}} H(\mathfrak{n}, \mathfrak{a})$. Commutativity of the above diagram yields maps $\mathrm{red}_{\mathfrak{a}}^{\mathfrak{a}'} : H(\mathfrak{a}') \to H(\mathfrak{a})$. I conjecture that $\mathrm{red}_{\mathfrak{a}}^{\mathfrak{a}'}$ is surjective. In other words, for any $c \in H(\mathfrak{n}, \mathfrak{a})$ and $\mathfrak{a} \mid \mathfrak{a}'$, there exists $\mathfrak{n} \mid \mathfrak{n}'$ such that $\mathrm{res}_{\mathfrak{n}}^{\mathfrak{n}'}(c)$ lies in $\mathrm{red}_{\mathfrak{a}}^{\mathfrak{a}'} H(\mathfrak{n}', \mathfrak{a}')$. My goal is to computationally verify this conjecture in some special cases.

Recall

$$\mathrm{coind}_{\Gamma(\mathfrak{n})}^{\mathrm{SL}_2(\mathcal{O}_d)}(\mathcal{O}_d/\mathfrak{a}) = C(\mathrm{SL}_2(\mathcal{O}_d/\mathfrak{n}), \mathcal{O}_d/\mathfrak{a}),$$

the space of $\mathcal{O}_d/\mathfrak{a}$-valued functions on $\mathrm{SL}_2(\mathcal{O}_d/\mathfrak{n}) = \Gamma(\mathfrak{n})\backslash \mathrm{SL}_2(\mathcal{O}_d)$, with the usual action $(\gamma \cdot \xi)(x) = \xi(x\gamma)$. If we denote by $I_{\mathfrak{n},\mathfrak{a}}$ this coinduced module, then Shapiro's lemma tells us that

$$H(\mathfrak{n}, \mathfrak{a}) = \mathrm{H}^1\left(\mathrm{SL}_2(\mathcal{O}_d), I_{\mathfrak{n},\mathfrak{a}}\right).$$

Actually here, I can just compute $H_{\mathfrak{n},\nu} = \mathrm{H}^1(\Gamma(\mathfrak{n}), \mathbf{Z}/p^\nu)$. Fix a presentation $\mathrm{SL}_2(\mathcal{O}_d) = \langle G \mid R\rangle$. Then $H_{\mathfrak{n},\nu}$ is the cohomology of

$$I_{\mathfrak{n},\nu} \xrightarrow{\mu} C(G, I_{\mathfrak{n},\nu}) \xrightarrow{\Lambda} C(R, I_{\mathfrak{n},\nu}).$$