Counterexamples related to the Sato–Tate conjecture

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25 April 2017

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Outline

Motivation and background

Sato–Tate conjecture

$E_{/\mathbf{Q}}$ non-CM elliptic curve, l prime, $\rho_l \colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$.

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Use discrepancy (Kolmogorov–Smirnov statistic).

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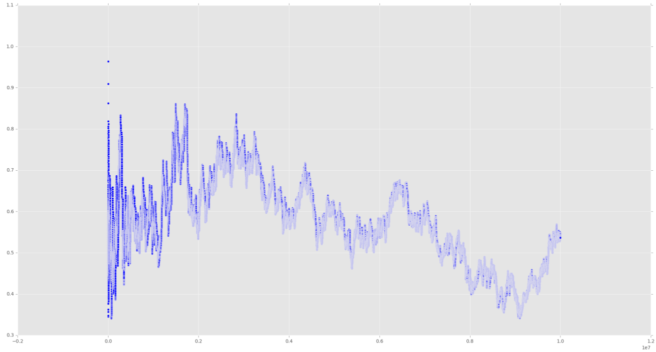
Computational evidence



$$\sqrt{\pi(N)} \cdot \mathrm{D}_N \text{ for } y^2 + y = x^3 - x,\ N \leqslant 10^7.$$

Related results

Theorem (Bucar–Kedlaya). Assume analytic continuation of $L(\mathrm{sym}^k E, s)$, GRH, and functional equation for all $k \geqslant 1$. Then $\mathrm{D}_N \ll N^{-\frac{1}{4}+\epsilon}$.

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Common ingredient. Erdös–Turán inequality: from a bound on $\left|\sum_{p \leqslant N} \mathrm{tr}\, \rho(x_p)\right|$ to a bound on D_N.

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Answer (Khare–Larsen–Ramakrishna). No!

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Answer. Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Discrepancy

Definition

Let $\{\theta_p\}$ be a sequence in $[0,\pi]$, μ a measure on $[0,\pi]$. The *discrepancy* is

$$\mathrm{D}_N(\{\theta_p\},\mu) = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta)\,\mathrm{d}\mu(\theta) \right|.$$

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Fact. $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

Dirichlet series

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For $k \geqslant 1$,

$$L(\operatorname{sym}^k \rho, s) = \prod_p \det\left(1 - \operatorname{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix} p^{-s}\right)^{-1}$$

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Example (Ramakrishna). $L_{\mathrm{sgn}}(s) = \prod_p \left(1 - \operatorname{sgn}(a_p) p^{-s}\right)^{-1}$.

Dirichlet series—basic facts

Theorem (M.)

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5. Fix $\alpha \in \left(0, \frac{1}{3}\right)$. The discrepancy will decay like $\pi(N)^{-\alpha}$.

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Sketch of proof

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If $\alpha \in \left(0, \frac{1}{3}\right)$, *there exists a sequence* $(x_2, x_3, x_5, \dots)$ *in* $[-1, 1]$ *such that* $|D_N - \pi(N)^{-\alpha}| \ll \pi(N)^{-1}$.

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Fix a finite set U of primes. Then there exists a finite set N of primes such that

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Given $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/I^n)$ and choices of $\operatorname{tr}\rho_{n+1}(\mathrm{fr}_p) \pmod{I^{n+1}}$, need to add finite set N to R_n.

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Fact: constant in $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$ only depends on $\bar{\rho}$.

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Enumerate the primes $p_1 = 2, q_1 = 3, p_2 = 5, q_2 = 7, \ldots$

If $\theta_q \approx \pi - \theta_p$, then $U_k(\theta_q) \approx -U_k(\theta_p)$ (within $p^{-\frac{1}{2}}$).

$$\begin{aligned}\left|\sum_{p\leqslant N} U_k(\theta_p)\right| &= \left|\sum_{p_i,q_i\leqslant N} \left(U_k(\theta_{p_i}) + U_k(\theta_{q_i})\right)\right| \\ &\ll \sum_{n\leqslant N} n^{-\frac{1}{2}} \\ &\ll N^{\frac{1}{2}}.\end{aligned}$$

Consequences

If $f \in C([0,\pi])$, $f \circ \cos^{-1}\colon [-1,1] \to \mathbf{C}$ is Lipschitz, and $f(\pi - \theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

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Can get equidistribution with respect to μ with non-continuous probability distribution functions.

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the growth of $\pi_{\mathrm{ram}(\rho)}(x)$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Conclusion & further questions

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1. Can we construct representations with $\mathrm{D}_N \ll N^{-\alpha+\epsilon}$, $\alpha > \frac{1}{2}$?
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3. Can we prove anything about D_N for CM elliptic curves?

Thank you!