Equidistribution and the analytic properties of a strange class of L-functions

Daniel Miller

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1 Motivation

Let $E_{/\mathbf{Q}}$ be an elliptic curve without complex multiplication. By an old theorem of Faltings [Fal83], the quantities

$$a_p(E) = p + 1 - \#E(\mathbf{F}_p) = \operatorname{tr} \rho_{E,l}(\mathrm{fr}_p)$$

determine E up to isogeny. That is, if E_1 and E_2 satisfy $a_p(E_1) = a_p(E_2)$ for all E, then E_1 and E_2 are isogenous. The starting point of this investigation is the corollary of a theorem of Harris, that the collection $\{\operatorname{sgn} a_p(E)\}_p$ in fact determines E up to isogeny. Ramakrishna had the insight that this fact means the "strange L-function"

$$L_{\mathrm{sgn}}(E, s) = \prod_p \frac{1}{1 - \operatorname{sgn} a_p(E) p^{-s}}$$

determines E up to isogeny. In this note, I define a more general class of strange L-functions, and show that their analytic properties are closely tied to the equidistribution of the $a_p(E)$.

Here is a brief discussion of this generalization in the case of a non-CM curve $E_{/\mathbf{Q}}$. It is convenient to repackage the traces of Frobenius as follows:

$$\theta_p(E) = \cos^{-1}(a_p(E)/2\sqrt{p}).$$

The Hasse Bound guarantees that the $\theta_p(E)$ are well-defined angles laying in the interval $[0, \pi]$. Write $\mathrm{dST} = \frac{2}{\pi} \sin^2\theta \, \mathrm{d}\theta$. Then the Sato–Tate conjecture (now a theorem [BL+11]) tells us that for any continuous function $f\colon [0,\pi] \to \mathbf{C}$, we have

$$\left| \frac{1}{\pi(C)} \sum_{p \leqslant C} f(\theta_p) - \int_0^\pi f \, \mathrm{dST} \right| = o(1)$$

as $C \to \infty$. It is well-known that this follows from the analytic continuation (past $\Re s = 1$) and non-vanishing except at $s = 1$ of all the L-functions

$L(\operatorname{sym}^k E, s)$ [Ser68, A.1, Th.1]. We take as our starting point the much stronger conjecture, due to Akiyama–Tanigawa [AT99], that

$$\left| \frac{1}{\pi(C)} \sum_{p \leqslant C} f(\theta_p) - \int_0^\pi f \, \mathrm{d}\mu_{\mathrm{ST}} \right| = O_f(C^{-\frac{1}{2}+\epsilon})$$

for all continuous f. (Their conjecture is actually more general; we will discuss the precise statement later.) They prove that this conjecture implies the Riemann Hypothesis for E. I prove that not only does their conjecture imply the Riemann Hypothesis for all $L(\operatorname{sym}^k E, s)$, it also does for all the strange L-functions

$$L_f(E,s) = \prod_p \frac{1}{1 - f(\theta_p(E))p^{-s}}$$

These results make perfect sense in a much more general context, and I will prove them there. In section 2 I set up this context and carefully define strange L-functions. In section 3, I prove basic analytic properties of the strange L-functions and connect their analytic properties with the equidistribution of a sequence. In section 4, I apply these results where "everything is known," i.e. varieties over function fields. Finally, in section 5, I apply the general results to the following cases: a non-CM elliptic curve $E_{/\mathbf{Q}}$, the product $E_1 \times E_2$ of a pair of non-isogenous non-CM elliptic curves over $\mathbf{Q}$, and the Jacobian of a generic genus-2 curve $C_{/\mathbf{Q}}$.

2 Definitions

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| \leqslant 1\}$. Write $\mathbf{D}^\infty$ for the set of sequences in $\mathbf{D}$ indexed by the primes, i.e. $\boldsymbol{z} \in \mathbf{D}^\infty$ is $(z_2, z_3, \dots)$. The space $\mathbf{D}^\infty$ is compact, and comes naturally equipped with the (product) Lebesgue measure, normalized to have mass 1.

Definition 2.1. Let $\boldsymbol{z} \in \mathbf{D}^\infty$. The associated *strange L-function* is given by

$$L(\boldsymbol{z}, s) = \prod_p \frac{1}{1 - z_p p^{-s}},$$

wherever this product converges.

Elementary topology tells us that $L\colon \mathbf{D}^\infty \times \mathbf{C}^{\Re > 1} \to \mathbf{C}$ is continuous. We will see that for fixed $\boldsymbol{z} \in \mathbf{D}^\infty$, the analytic properties of $L(\boldsymbol{z}, s)$ are closely tied to estimates for the sums $A_{\boldsymbol{z}}(x) = \sum_{p \leqslant x} z_p$. One often gets such estimates in the context of equidistribution, which we consider next.

For the remainder of this section, let X be a compact separable metric space with no isolated points. We write X^∞ for the space of sequences in X indexed by rational primes, i.e. points $\boldsymbol{x} \in X^\infty$ are of the form $\boldsymbol{x} = (x_2, x_3, \dots)$. By [Eng89, Cor.2.3.16, Th.4.2.2], the compact space X^∞ is metrizable and separable, also with no isolated points.

Definition 2.2. For $\boldsymbol{x} \in X^\infty$ and $C > 0$, write $\boldsymbol{x}^C$ for the probability measure given by

$$\int_X f \, \mathrm{d}\boldsymbol{x}^C = \boldsymbol{x}^C(f) = \frac{1}{\pi(C)} \sum_{p \leqslant C} f(x_p).$$

Let μ be a Borel measure on X. Recall that $\boldsymbol{x}$ is *μ-equidistributed* if $\boldsymbol{x}^C \to \mu$ weakly, i.e. $\boldsymbol{x}^C(f) \to \mu(f)$ for all $f \in C(X)$. In fact, we can extend this to not-necessarily-continuous functions as follows:

Theorem 2.3 (Mazzone)**.** *Let μ be a Borel measure on X and let $f \colon X \to \mathbf{C}$ be bounded and measurable. Then f is continuous almost everywhere if and only if $\boldsymbol{x}^C(f) \to \mu(f)$ for all μ-equidistributed $\boldsymbol{x}$.*

Proof. This follows directly from the proof of [Maz95, Th.1]. □

Fix a Borel measure μ on X, and write $C^{\mathrm{ae}}(X, \mu)$ for the space of bounded, almost-everywhere continuous functions $f \colon X \to \mathbf{C}$.

Theorem 2.4. *Endowed with the supremum norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $C^{\mathrm{ae}}(X, \mu)$ is a Banach space.*

Proof. This is an elementary corollary of the fact that a countable union of measure-zero sets has measure zero. □

Definition 2.5. Let $f \in C^{\mathrm{ae}}(X, \mu)^{\|\cdot\|_\infty \leqslant 1}$, $\boldsymbol{x} \in X^\infty$. The associated *strange L-function* is defined as

$$L_f(\boldsymbol{x}, s) = L(f(\boldsymbol{x}), s) = \prod_p \frac{1}{1 - f(x_p) p^{-s}}$$

for all $s \in \mathbf{C}$ for which the product converges.

Our typical source of a strange L-function is as follows. Let G be a compact connected Lie group and $X = G^\natural$, the space of conjugacy classes of G. Then $G^\natural$ inherits the Haar measure from G. Given any sequence $\boldsymbol{x} \in (G^\natural)^\infty = G^{\natural,\infty}$ and function $f \in C^{\mathrm{ae}}(G^\natural)^{\|\cdot\|_\infty \leqslant 1}$, we can define $L_f(\boldsymbol{x}, s)$. This is related to Serre's L-functions from [Ser68, A.2] as follows.

Theorem 2.6. *Let G be a compact connected Lie group, $\rho \in \widehat{G}$ an irreducible unitary representation of G. Then there exist functions $\lambda_\rho^1, \dots, \lambda_\rho^{\deg \rho} \colon G^\natural \to S^1$, continuous away from the set $\{\det(1 - \rho) = 0\}$, such that for every $x \in G^\natural$, there are angles $\theta_1, \dots, \theta_{\deg \rho} \in [0, 2\pi)$, satisfying $\theta_1 \leqslant \cdots \leqslant \theta_{\deg \rho}$, such that $\lambda_\rho^j(x) = e^{i\theta_j}$ and moreover*

$$\det(1 - \rho(x) t) = \prod_{j=0}^{\deg \rho} (1 - \lambda_\rho^j(x) t).$$

Proof. This follows easily from [KS99, Lem.1.0.9]. □

Recall that for $\rho \in \widehat{G}$, Serre defines $L(\rho, s) = \prod_p \det(1 - \rho(x_p)p^{-s})^{-1}$. Using his notation, there is the identity

$$L(\rho, s) = \prod_{j=1}^{\deg \rho} L_{\lambda_\rho^j}(\boldsymbol{x}, s).$$

The rest of our definitions concern discrepancy, which for now we define only in a special context. Let G be a compact connected semisimple Lie group. We will define discrepancy for sequences in $G^\natural$.

Let G^{sc} be the simply-connected cover of G. Choose a maximal torus $T \subset G^{\mathrm{sc}}$; let $W = \mathrm{N}(T)/T$ be the Weyl group. Let $\mathfrak{t} = \mathrm{Lie}(T)$ and recall that the kernel of $\exp\colon \mathfrak{t} \twoheadrightarrow T$ is generated by the nodal vectors associated to the root system $\mathrm{R}(G^{\mathrm{sc}}, T)$ [$\mathrm{Lie}_{7\text{–}9}$, 9.6 Pr.11]. Write $\{t_1, \dots, t_r\} \subset \mathfrak{t}$ for these vectors. The exponential map $\exp\colon \mathfrak{t} \to T$ induces an isomorphism $\mathfrak{t}/(\langle t_i \rangle \rtimes W) \to G^\natural$. Given $x = (x_1, \dots, x_r) \in [0,1]^r$, write

$$I_x = \left\{ \sum_{i=1}^r a_i t_i : a_i \in [0, x_i] \right\} \subset \mathfrak{t}.$$

Definition 2.7. With the setup as above, let μ, ν be probability measures on $G^\natural$. The *discrepancy* between μ and ν is

$$\mathrm{disc}(\mu, \nu) = \sup_{x \in [0,1]^r} \left| \mu(\exp I_x) - \nu(\exp I_x) \right|.$$

If $\nu = \mathrm{d}x$, the Haar measure on $G^\natural$, we simply write $\mathrm{disc}(\mu)$ for $\mathrm{disc}(\mu, \mathrm{d}x)$.

The Koksma–Hlawka inequality bounds the difference between the Haar integral and weighted average of a function on $G^\natural$ in terms of the discrepancy of the sequence and the variation of the function.

The following result is essential:

Theorem 2.8 (Koksma, Hlawka)**.** *Let G be as above. Let $f\colon G^\natural \to \mathbf{C}$ be such that $f\,\mathrm{d}x$ is a measure with bounded variation. Then*

$$\left| \boldsymbol{x}^C(f) - \int f \,\mathrm{d}x \right| \leqslant \mathrm{Var}(f)\,\mathrm{disc}(\boldsymbol{x}^C).$$

Proof. This is [Ökt99, Th. 3.2]. □

We will often use the soft version of this inequality. Namely, assume $\int f\,\mathrm{d}x = 0$. Then $|\boldsymbol{x}^C(f)| \ll_f \mathrm{disc}(\boldsymbol{x}^C)$ as $C \to \infty$. Here is another way of putting it. The sequence $f(\boldsymbol{x})$ has $|A_{f(\boldsymbol{x})}(C)| \ll_f \pi(C)\,\mathrm{disc}(\boldsymbol{x}^C)$.

3 Main results

Theorem 3.1. *Let $\boldsymbol{z} \in \mathbf{D}^\infty$. Then $L(\boldsymbol{z}, s)$ defines a holomorphic function on the region $\{\Re s > 1\}$. Moreover, on that region,*

$$\log L(\boldsymbol{z}, s) = \sum_{p^n} \frac{z_p^n}{n p^{ns}}.$$

Proof. Expanding the product for $L(\boldsymbol{z}, s)$ formally, we have

$$L(\boldsymbol{z}, s) = \sum_{n \geqslant 1} \frac{\prod_{p|n} z_p^{v_p(n)}}{n^s}.$$

An easy comparison with Riemann's zeta function tells us that the series expansion is holomorphic on $\{\Re s > 1\}$. By [Apo76, Th. 11.7], the product formula holds on the same region. The formula for $\log L(\boldsymbol{z}, s)$ comes from [Apo76, 11.9 Ex.2]. $\square$

Theorem 3.2. *Assume $A_{\boldsymbol{z}}(x) \ll x^{\alpha+\epsilon}$, $\alpha \in [\frac{1}{2}, 1]$. Then $\log L(\boldsymbol{z}, s)$ is holomorphic on $\{\Re > \alpha\}$.*

Proof. Split the sum for $\log L$ into two pieces:

$$\log L(\boldsymbol{z}, s) = \sum_p \frac{z_p}{p^s} + \sum_p \sum_{n \geqslant 2} \frac{z_p^n}{n p^{ns}}.$$

For each p, we have

$$\left| \sum_{n \geqslant 2} \frac{z_p^n}{n p^{ns}} \right| \leqslant \sum_{n \geqslant 2} p^{-n \Re s} = p^{-2\Re s} \frac{1}{1 - p^{-\Re s}}.$$

Elementary analysis gives

$$1 \leqslant \frac{1}{1 - p^{-\Re s}} \leqslant 2 + 2\sqrt{2},$$

so the second piece of $\log L(\boldsymbol{z}, s)$ converges absolutely when $\Re(s) > \frac{1}{2}$. By [Ten95, II.1 Th.10], our bound on $A_{\boldsymbol{z}}(x)$ yields the holomorphy of $\sum z_p p^{-s}$ on $\{\Re > \alpha\}$. $\square$

Corollary 3.3. *Let G be a compact connected semisimple Lie group, $\boldsymbol{x} \in G^{\natural, \infty}$ satisfy $\operatorname{disc}(\boldsymbol{x}^C, \mathrm{d}x) \ll C^{-\frac{1}{2}+\epsilon}$. Then for every $f \in C^{\mathrm{ae}}(G^\natural)^{\|\cdot\| \leqslant 1}$, $L_f(\boldsymbol{x}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$, and satisfies the Riemann Hypothesis, for all f bounded and almost-everywhere continuous with $\mu(f) = 0$.*

Proof. Koksma–Hlawka tells that if $\mu(f) = 0$, then $\boldsymbol{x}^C(f) \ll C^{-\frac{1}{2}+\epsilon}$. Thus the sequence $f(\boldsymbol{x})$ satisfies $A_{f(\boldsymbol{x})}(x) \ll x^{\frac{1}{2}+\epsilon}$, and the result follows from Theorem 3.2. $\square$

4 Strange L-functions over function fields

Let k be a finite field of characteristic p and cardinality q. Let $C_{/k}$ be a nice curve in the sense of Poonen (i.e., C is smooth, projective, and geometrically integral). Write $K = k(C)$ for the function field of C. Fix a non-empty open subset $U \subset C$ and a geometric point $\infty \in U(\bar{k})$. Fix a prime $l \neq p$ and an embedding $\overline{\mathbf{Q}_l} \hookrightarrow \mathbf{C}$.

Definition 4.1. An l-adic sheaf $\mathcal{F}$ on U is *good* if the following conditions hold.

1. $\mathcal{F}$ is pure of weight zero.
2. Let $G = \overline{\rho_{\mathcal{F}}(\pi_1(U_{\overline{k}}, \infty))}^{\mathrm{Zar}}$. Assume $\rho_{\mathcal{F}}(\pi_1(U, \infty)) \subset G(\overline{\mathbf{Q}}_l)$.

For any good sheaf $\mathcal{F}$, let $\mathrm{ST}(\mathcal{F})$ be a maximal compact subgroup of $G(\mathbf{C})$.For each $u \in U$, there is a well-defined conjugacy class $\theta(u) = \rho(\mathrm{fr}_u)^{\mathrm{ss}} \in \mathrm{ST}(\mathcal{F})^\natural$. For any $C > 0$, write

$$\boldsymbol{\theta}_{\mathcal{F}}^C = \frac{1}{\#\{u \in U : q_u \leqslant C\}} \sum_{q_u \leqslant C} \delta_{\theta(u)}.$$

Katz proves an equidistribution estimate for the $\theta(u)$'s.

Theorem 4.2. *Let σ be a non-trivial irreducible representation of* $\mathrm{ST}(\mathcal{F})$. *Then*

$$|\boldsymbol{\theta}_{\mathcal{F}}^C(\mathrm{tr}\,\sigma)| \ll_{\mathcal{F}} \dim(\sigma) C^{-\frac{1}{2}}.$$

Proof. This is [Kat88, p.39]. □

Now let $C^\natural(\mathrm{ST}(\mathcal{F}))$ be the space of functions $f \colon \mathrm{ST}(\mathcal{F})^\natural \to \mathbf{C}$ satisfying:

$$\|f\|^\natural = \sum_\sigma \dim(\sigma) |\widehat{f}(\sigma)| < \infty.$$

For such functions, we have:

$$|\boldsymbol{\theta}_{\mathcal{F}}^C(f) - \mu(f)| \ll_{\mathcal{F}} \|f\|^\natural C^{-\frac{1}{2}}.$$

Thus for any $f \in C^\natural(\mathrm{ST}(\mathcal{F}))$, the strange L-function $L_f(\boldsymbol{\theta}_{\mathcal{F}}, s)$ has analytic continuation to $\{\Re s > \frac{1}{2}\}$ and satisfies the Riemann Hypothesis.

5 Applications

Theorem 5.1. *Let $E_{/\mathbf{Q}}$ be a non-CM elliptic curve. Assume the Akiyama–Tanigawa conjecture for E. Then all the L-functions $L_f(\boldsymbol{\theta}, s)$ have analytic continuation to $\{\Re s > \frac{1}{2}\}$ and satisfy the Riemann Hypothesis. In particular, this holds for all $L(\mathrm{sym}^k E, s)$.*

Proof. Akiyama–Tanigawa conjecture that $\mathrm{disc}(\boldsymbol{\theta}^C, \mathrm{ST}) \ll C^{-\frac{1}{2}+\epsilon}$ implies the first by Corollary 3.3. The second part follows from the fact that any $L(\mathrm{sym}^k E, s)$ can be written as a product of L_f's. □

Theorem 5.2. *Let E_1, E_2 be two non-isogenous, non-CM elliptic curves over* $\mathbf{Q}$. *Assume the Akiyama–Tanigawa conjecture for the product $E_1 \times E_2$. Then for any $f \colon [0, \pi] \to \mathbf{C}$ that is not almost everywhere*

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