Equidistribution, discrepancy, and the analytic properties of Dirichlet series

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Outline

Background

The Sato–Tate conjecture

Breaking the Akiyama–Tanigawa converse

Generalizations

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Elliptic curves

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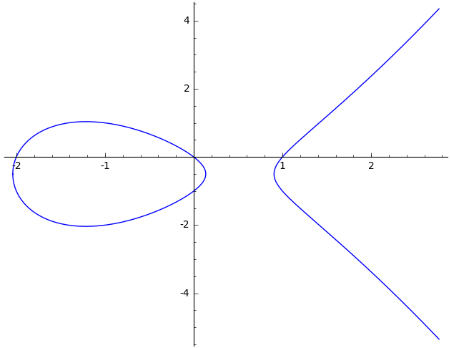
Geometric structure of $E(\mathbf{C})$

Our example



$$E : y^2 = x^3 - 3024x + 46224$$

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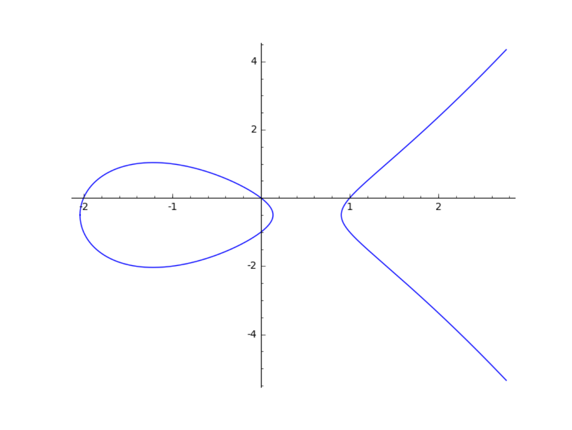

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Where is ∞?

Initial data

p	2	3	5	7	11	13	...	999999929	999999937
$\#E(\mathbf{F}_p)$	1	2	3	3	8	11	...	999950222	1000031072

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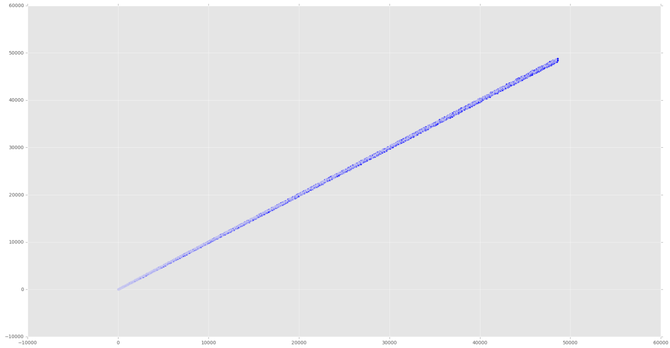
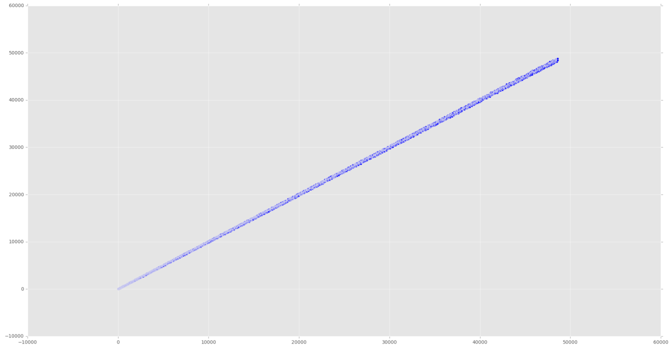
Look at more data...



$\#E(\mathbf{F}_p)$ as a function of p.



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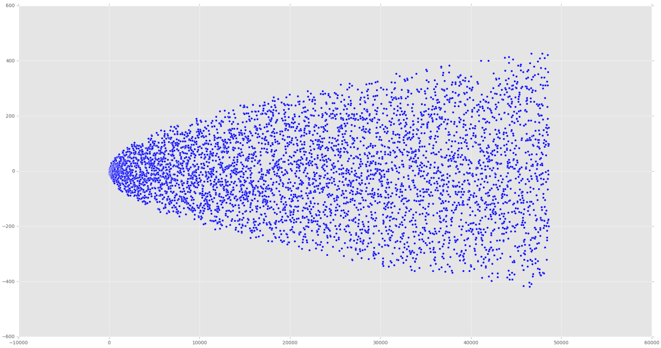
How does the error term behave?

Hasse bound



$a_p(E) := p + 1 - \#E(\mathbf{F}_p)$ as a function of p.

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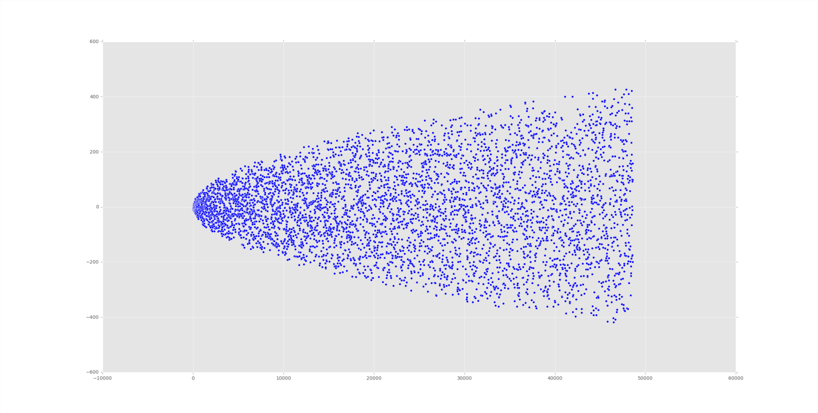

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Intuition: why is a_p small? (and how small is it?)

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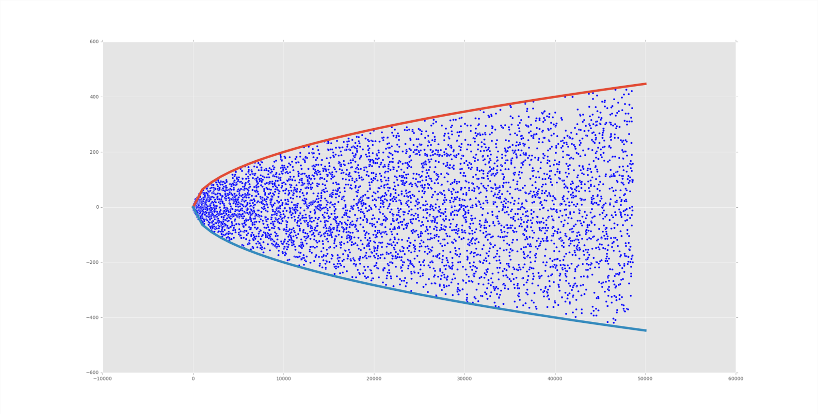

$a_p(E)$ vs. $\pm 2\sqrt{p}$.

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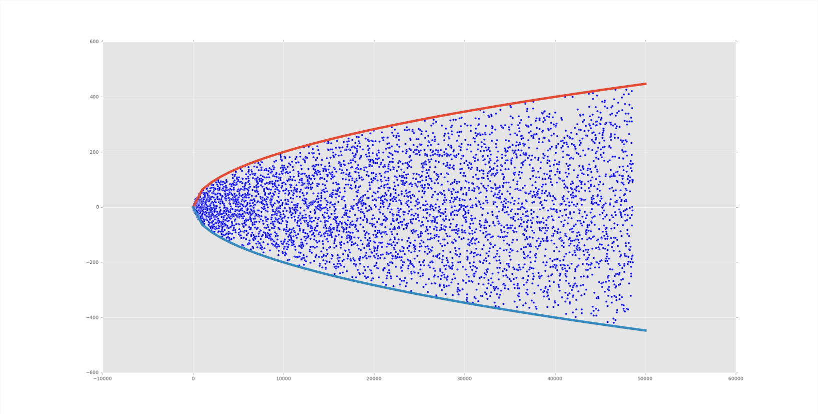

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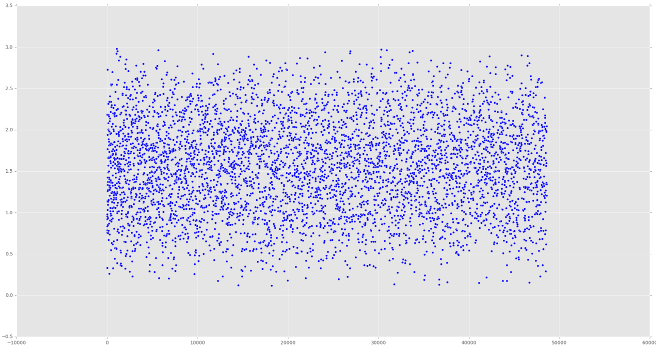
Perhaps we should normalize?

Satake parameters



$\theta_p = \cos^{-1}\left(\frac{a_p}{2\sqrt{p}}\right)$ as a function of p.

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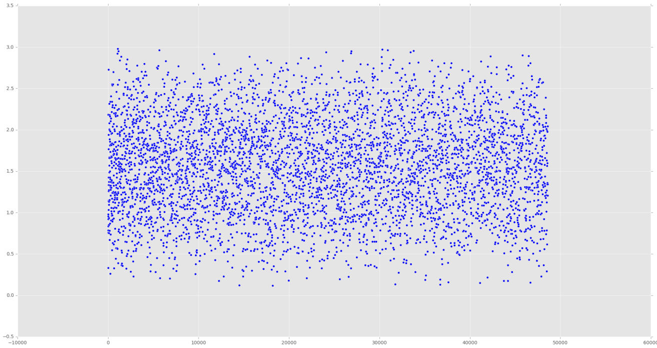

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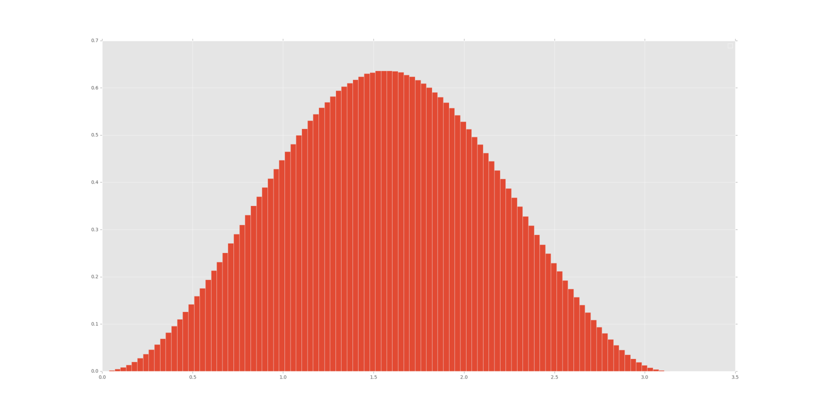
Look at the statistics of $\{\theta_p\}$.

Their statistics



Histogram of $\{\theta_p\}_{p \leqslant 10^9}$.

Their statistics

Histogram with graph of $\mathrm{ST}(\theta) = \frac{2}{\pi}\sin^2(\theta)$.

Their statistics

Histogram with graph of $\mathrm{ST}(\theta) = \frac{2}{\pi}\sin^2(\theta)$.

Some kind of convergence happening. . .

The Sato–Tate conjecture

Some definitions

Two cumulative distribution functions:

$$\mathsf{cdf}_N(x) = \frac{\#\{p \leqslant N : \theta_p \leqslant x\}}{\#\{p \leqslant N\}}$$

$$\mathsf{cdf}_{\mathsf{ST}}(x) = \int_0^x \mathsf{ST}(x)\,\mathrm{d}x = \frac{x - \sin(x)\cos(x)}{\pi}$$

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Other ways to measure distabce between distributions?

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Key idea: Koksma–Hlawka inequality.

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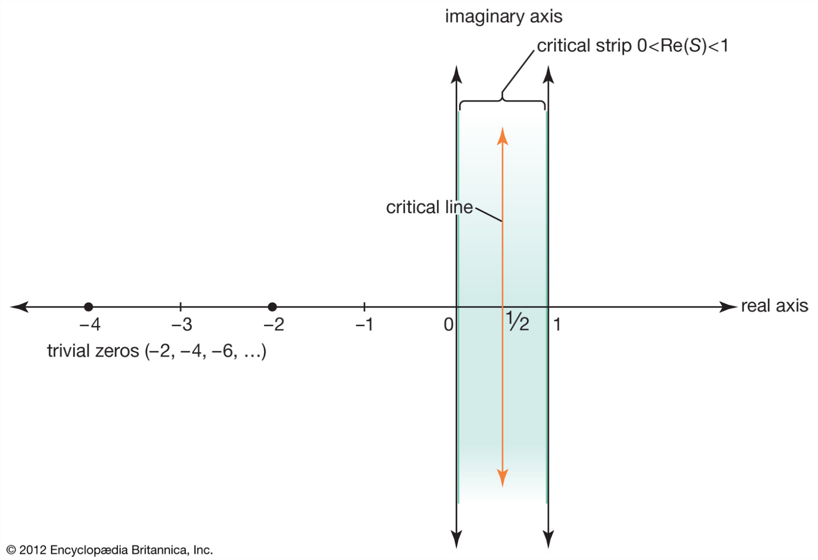
(Standing assumption: $\int f \cdot \mathrm{ST} = 0$.)

L-functions on the complex plane



Intuition for Akiyama–Tanigawa conjecture

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A–T $\Rightarrow$ RH. Is the converse true? No!

Breaking the Akiyama–Tanigawa converse

What is needed?

Construct a sequence $\{\theta_p\}$ such that

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Construct a sequence $\{\theta_p\}$ such that

1. Sums of the form $\sum_{p \leqslant N} f(\theta_p)$ have "good bounds" like $O(\sqrt{N})$.
2. The discrepancy $\mathrm{disc}_{\{\theta_p\}}(N)$ is *not* $O(N^{-\frac{1}{2}})$.

Key idea

Choose an angle θ, and let let $\theta_n = n\theta \mod \pi$. Then

$$\left|\sum_{n\leqslant N} e^{2\pi i m\theta_n}\right| = O\left(\frac{1}{|e^{2\pi i\theta} - 1|}\right)$$

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Corollary

If f is a smooth function, then

$$\left|\sum_{n \leqslant N} f(\theta_n)\right| = O_f(1).$$

Two degrees of freedom

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Also, we can control the discrepancy of the sequence $\{\theta_p\}$ via an *irrationality exponent*.

Diophantine approximation

Definition

The *irrationality exponent* of x is the largest η such that

$$\left| x - \frac{p}{q} \right| < q^{-\eta}$$

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Theorem

There are x with arbitrary irrationality exponent > 2.

Putting things together

Theorem

For any $\eta \in (-1/2, 0)$, *there exists a sequence* $\{\theta_p\}$ *such that*

$$L(\{\theta_p\}, s) = \prod_p \frac{1}{(1 - e^{i\theta_p} p^{-s})(1 - e^{-i\theta_p} p^{-s})}$$

satisfies the Riemann Hypothesis, but for which

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Problem: this sequence $\{\theta_p\}$ is uniformly distributed, not ST-distributed.

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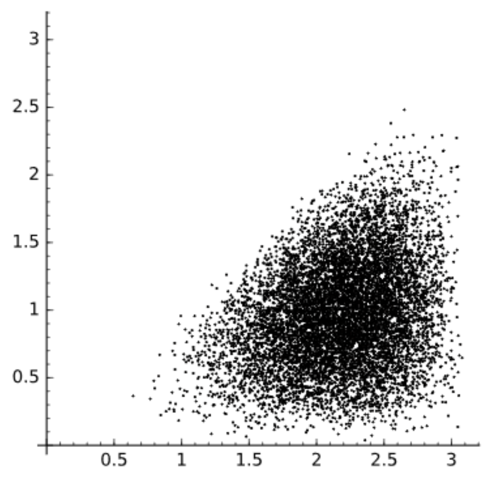
Solution: tweak them so they do, then prove everything still works.

Generalizations

From $y^2 = x^3 + ax + b$ to $y^2 = x^5 - 1$:



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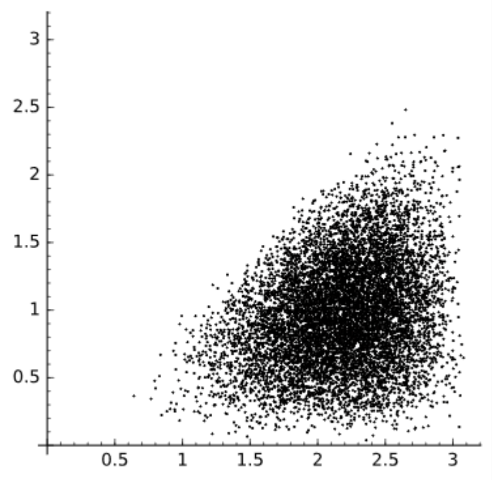

Higher-dimensional counterexamples, Galois representations.

Questions?

Further reading

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