Counterexamples related to the Sato–Tate conjecture

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Outline

Motivation and background

Discrepancy and Dirichlet series

Main theorem

Idea of the proof

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Sato–Tate Conjecture

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Use discrepancy (Kolmogorov–Smirnov statistic).

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Theorem (Bucar–Kedlaya). Assume analytic continuation of $L(\operatorname{sym}^k E, s)$, GRH, and functional equation for all $k \geqslant 1$. Then $\mathrm{D}_N \ll N^{-\frac{1}{4}+\epsilon}$.

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Common ingredient. Erdös–Turán–Koksma inequality: from a bound on $\left|\sum_{p\leqslant N} \mathrm{tr}\, \rho(x_p)\right|$ to a bound on D_N.

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Answer (Khare–Larsen–Ramakrishna). No!

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Answer. Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Discrepancy

Definition

Let $\{\theta_p\}$ be a sequence in $[0, \pi]$, μ a measure on $[0, \pi]$. The *discrepancy* is

$$\mathrm{D}_N(\{\theta_p\}, \mu) = \sup_{x \in [0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p \leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta) \, \mathrm{d}\mu(\theta) \right|.$$

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Fact. $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

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For $k \geqslant 1$,

$$L(\operatorname{sym}^k \rho, s) = \prod_p \det\left(1 - \operatorname{sym}^k \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix} p^{-s}\right)^{-1}$$

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Example (Ramakrishna). $L_{\mathrm{sgn}}(s) = \prod_p \left(1 - \operatorname{sgn}(a_p) p^{-s}\right)^{-1}$.

Dirichlet series—basic facts

Theorem (M.)

If $\left|\sum_{p\leqslant N} f(\theta_p)\right| \ll N^{\alpha+\epsilon}$, *then* $L_f(s)$ *admits a nonvanishing analytic continuation to* $\Re > \alpha$.

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5. Fix $\alpha \in \left(0, \frac{1}{3}\right)$. The discrepancy will decay like $\pi(N)^{-\alpha}$.

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2. $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$. *(Yes to Q4.* $\log x$, $\log^{10^{10}} x$, $A^{-1}(x)$*)*
3. *For each unramified p, $a_p = \operatorname{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.*

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Idea of the proof

Overview

1. Prove the theorem for a_p not coming from a Galois representation.
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If $\alpha \in \left(0, \frac{1}{3}\right)$*, there exists a sequence* $(x_2, x_3, x_5, \dots)$ *in* $[-1, 1]$ *such that* $|D_N - \pi(N)^{-\alpha}| \ll \pi(N)^{-1}$*.*

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Fix a finite set U of primes. Then there exists a finite set N of primes such that

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Given $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/I^n)$ and choices of $\operatorname{tr} \rho_{n+1}(\mathrm{fr}_p)$ (mod I^{n+1}), need to add finite set N to R_n.

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Fact: constant in $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$ only depends on $\bar{\rho}$.

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Consequences

If $f \in C([0,\pi])$, $f \circ \cos^{-1} \colon [-1,1] \to \mathbf{C}$ is Lipschitz, and $f(\pi - \theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

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Can get equidistribution with respect to μ with non-continuous probability distribution functions.

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the growth of $\pi_{\mathrm{ram}(\rho)}(x)$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Conclusion & further questions

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Thanks!