Absolute continuity and Fourier coefficients

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Let T be a compact connected abelian Lie group, i.e. a torus. Let $\mathfrak{t} = \operatorname{Lie} T$, and $\widehat{T} = \hom(T, S^1)$ the (discrete) group of characters of T.

We start with the baby example: $T = \mathbf{R}/(2\pi\mathbf{Z})$, which we identify with the set $[-\pi, \pi)$. The Lie algebra is $\mathfrak{t} = \mathbf{R}$ and $\exp(X) = X$. Here $\widehat{T} = \mathbf{Z}$, with $m(x) = e^{imx}$. The action of $\mathfrak{t}$ on $C^\infty(T)$ via differential operators is:

$$(X \cdot f)(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(x + tX)\bigg|_{t=0} = X \frac{\mathrm{d}f}{\mathrm{d}x}(x).$$

Moreover, note that the Fourier coefficients, defined as:

$$\widehat{f}(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{imx} \, \mathrm{d}x$$

behave with respect to the action of $\mathfrak{t}$ as:

$$\begin{aligned}
\widehat{X \cdot f}(m) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (X \cdot f)(x) e^{imx} \, \mathrm{d}x \\
&= \frac{X}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{imx} \, \mathrm{d}x \\
&= \frac{X}{2\pi} \int_{-\pi}^{\pi} (f(x) e^{imx})' - imf(x) e^{imx} \, \mathrm{d}x \\
&= -\frac{imX}{2\pi} \int_{-\pi}^{\pi} f(x) e^{imx} \, \mathrm{d}x \\
&= -imX \widehat{f}(m).
\end{aligned}$$

Better, suppose that f is a function on T which is absolutely continuous, in the sense that f is continuous, and there is $g \in L^1(T)$ such that

$$f(x) = \int_{-\pi}^{x} g(t) \, \mathrm{d}t.$$

(Even if we may require $f(x) = f(-x)$, we don't necessarily require this for g.)

Then there is an easy computation:

$$\begin{aligned}\widehat{f}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{x} g(t)\,\mathrm{d}t e^{imx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{t}^{\pi} g(t)e^{imx}\,\mathrm{d}x\mathrm{d}t \\ &= \dots\end{aligned}$$