Counterexamples related to the Sato–Tate conjecture

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Outline

Motivation and background

Discrepancy and Dirichlet series

Main theorem

Sketch of proof

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Sato–Tate conjecture

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Use discrepancy (Kolmogorov–Smirnov statistic).

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$$D_N = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta)\,\mathrm{d}\,\mathsf{ST}(\theta) \right|.$$

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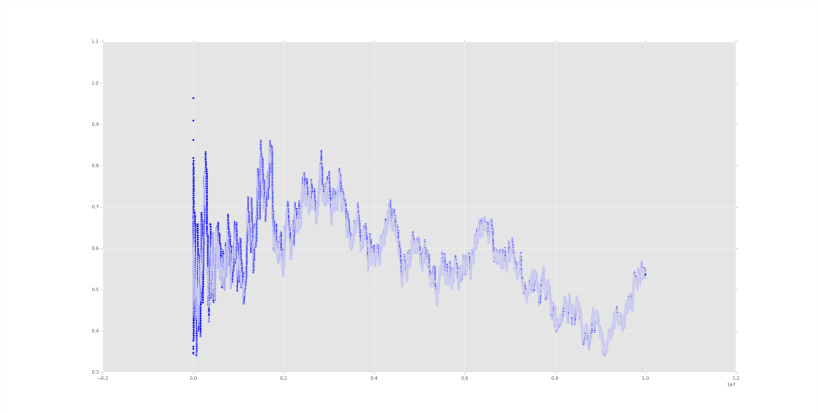
Computational evidence



$$\sqrt{\pi(N)} \cdot \mathrm{D}_N \text{ for } y^2 + y = x^3 - x,\ N \leqslant 10^7.$$

Related results

Theorem (Bucar–Kedlaya). Assume analytic continuation of $L(\mathrm{sym}^k E, s)$, GRH, and functional equation for all $k \geqslant 1$. Then $\mathrm{D}_N \ll N^{-\frac{1}{4}+\epsilon}$.

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Common ingredient. Erdös–Turán inequality: from a bound on $\left|\sum_{p\leqslant N} \mathrm{tr}\, \rho(x_p)\right|$ to a bound on D_N.

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Let $\epsilon > 0$. *Then there exists an infinitely ramified representation* $\rho\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_l)$ *such that* $\theta_p \in B_\epsilon(\pi/2)$ *for a density one set of primes.*

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Answer (Khare–Larsen–Ramakrishna). No!

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Answer. Yes! to Q1–Q5.

Discrepancy and Dirichlet series

Discrepancy

Definition

Let $\{\theta_p\}$ be a sequence in $[0,\pi]$, μ a measure on $[0,\pi]$. The *discrepancy* is

$$\mathrm{D}_N(\{\theta_p\},\mu) = \sup_{x\in[0,\pi]} \left| \frac{1}{\pi(N)} \sum_{p\leqslant N} 1_{[0,x)}(\theta_p) - \int 1_{[0,x)}(\theta)\,\mathrm{d}\mu(\theta) \right| .$$

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Fact. $\frac{\log N}{N} \ll D_N$. The *van der Corput sequence* achieves this.

Dirichlet series

Definition

For $k \geqslant 1$,

$$L(\operatorname{sym}^k \rho, s) = \prod_p \det\left(1 - \operatorname{sym}^k \left(\begin{smallmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{smallmatrix}\right) p^{-s}\right)^{-1}$$

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Example (Ramakrishna). $L_{\mathrm{sgn}}(s) = \prod_p \left(1 - \mathrm{sgn}(a_p)p^{-s}\right)^{-1}$.

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Theorem (M.)

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5. Fix $\alpha \in \left(0, \frac{1}{3}\right)$. The discrepancy will decay like $\pi(N)^{-\alpha}$.

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2. $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$. *(Yes to Q4.* $\log x$, $\log^{10^{10}} x$, $A^{-1}(x)$*)*
3. *For each unramified p, $a_p = \operatorname{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.*

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3. *For each unramified p, $a_p = \operatorname{tr} \rho(\mathrm{fr}_p) \in \mathbf{Z}$ and satisfies the Hasse bound.*
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If $\alpha \in \left(0, \frac{1}{3}\right)$*, there exists a sequence* $(x_2, x_3, x_5, \dots)$ *in* $[-1, 1]$ *such that* $|D_N - \pi(N)^{-\alpha}| \ll \pi(N)^{-1}$.

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Fix a finite set U of primes. Then there exists a finite set N of primes such that

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Given $\rho_n \colon G_{\mathbf{Q},R_n} \to \mathrm{GL}_2(\mathbf{Z}/I^n)$ and choices of $\operatorname{tr} \rho_{n+1}(\mathrm{fr}_p)$ $\pmod{I^{n+1}}$, need to add finite set N to R_n.

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Fact: constant in $\pi_{\mathrm{ram}(\rho)}(x) \ll h(x)$ only depends on $\bar{\rho}$.

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Make U_1 so large that for $p > \max U_1$, $l^2 < \log p$.

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$$\begin{aligned}\left|\sum_{p\leqslant N} U_k(\theta_p)\right| &= \left|\sum_{p_i,q_i\leqslant N} (U_k(\theta_{p_i}) + U_k(\theta_{q_i}))\right| \\ &\ll \sum_{n\leqslant N} n^{-\frac{1}{2}} \\ &\ll N^{\frac{1}{2}}.\end{aligned}$$

Consequences

If $f \in C([0,\pi])$, $f \circ \cos^{-1}\colon [-1,1] \to \mathbf{C}$ is Lipschitz, and $f(\pi - \theta) = -f(\theta)$, then $L_f(\rho, s)$ has a nonvanishing analytic continuation to $\Re > \frac{1}{2}$ (Riemann hypothesis).

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Can get equidistribution with respect to μ with non-continuous probability distribution functions.

Questions

Q1. Can Pande's results be strengthened to yield equidistribution?

Q2. If so, can the measure be specified?

Q3. Can the rate of convergence of empirical measures to the true measure be specified?

Q4. Can the growth of $\pi_{\mathrm{ram}(\rho)}(x)$ be controlled?

Q5. Can anything be said about the L-functions associated with ρ?

Conclusion & further questions

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3. Can we prove anything about D_N for CM elliptic curves?

Thank you!