L-functions of elliptic curves with complex multiplication

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Let $A_{/\mathbf{Q}}$ be an abelian variety with complex multiplication (over $\mathbf{Q}$!) by F. That is, if we write $\operatorname{End}^\circ(A) = \operatorname{End}_{\mathbf{Q}}(A) \otimes \mathbf{Q}$, then $F \simeq \operatorname{End}^\circ(A)$. Let p be a prime at which A is unramified. Then

$$\mathrm{fr}_p \in \operatorname{End}^\circ(A_{\mathbf{F}_p}) = \operatorname{End}^\circ(A) = F.$$

Moreover, for each prime l, the Galois representation $\rho_{A,l}$ takes values in $F_l^\times = (F \otimes \mathbf{Q}_l)^\times$. In fact, if we write O for the ring of integers of F, then $\rho_{A,l} \colon G_{\mathbf{Q}} \to O_l^\times$. Moreover, for each p, $\mathrm{fr}_p \in F^\times$ as a p-Weil number of weight 1, i.e. $|\mathrm{fr}_p| = \sqrt{p}$ under each embedding $F \hookrightarrow \mathbf{C}$.

We will define an algebraic Hecke character associated with E. For every unramified prime p, the Frobenius fr_p lives in $F^\times$. There is, for each $\sigma \colon F \hookrightarrow \mathbf{C}$, a weight-1 Hecke character $\chi_\sigma \colon \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{C}^\times$, such that

$$L^{\mathrm{alg}}(A, s) = \prod_{\sigma \colon F \hookrightarrow \mathbf{C}} L(s, \chi_\sigma).$$

Thus

$$L^{\mathrm{an}}(A, s) = L^{\mathrm{alg}}\left(A, s + \frac{1}{2}\right) = \prod_\sigma L\left(s + \frac{1}{2}, \chi_\sigma\right),$$

where $L(s + 1/2, \chi_\sigma) = L(s, \chi_\sigma \| \cdot \|^{-1/2})$, for $\| \cdot \| \colon \mathbf{A}^\times \to \mathbf{R}^+$ the adele norm.

The Sato–Tate group for A is the compact torus $\mathrm{R}_{F/\mathbf{Q}}\,\mathbf{G}_{\mathrm{m}}^{\mathrm{N}=1}(\mathbf{R}) = F_\infty^{\times,\mathrm{N}=1}$, which is isomorphic to $\prod_{\sigma\in\Phi} S^1$. The group of characters of $\mathrm{ST}(A)$ is generated by $\{\chi_\sigma\}$, so we know that if the Akiyama–Tanigawa conjecture for A is true, then each $L(s, \prod \chi_\sigma^{m_\sigma})$ satisfies the Riemann Hypothesis. My counterexample shows: the converse does not hold! Even if all the $L(s, \prod \chi_\sigma^{m_\sigma})$ satisfy the Riemann Hypothesis, it does not follow that Akiyama–Tanigawa holds.