Equidistribution for function fields

Daniel Miller

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We mostly follow chapter 3 of [Kat88].

Let k be a function field of characteristic p. Let $\mathbf{F}_q$ be the field of constants of k. We interpret k as the function field of a unique (up to isomorphism) smooth proper geometrically connected curve C over $\mathbf{F}_q$. Let $S \subset C$ be a finite set of closed points, and let $U = C \smallsetminus S$. We pick a generic point η of U, and let $\bar{\eta}$ be the corresponding geometric point. The group $\pi_1(U, \bar{\eta})$ is the Galois group of the maximal extension of k unramified outside S, i.e. $\pi_1(U, \bar{\eta}) = G_{k,S}$.

Following [Del77, 2.2.4], a *lisse* $\overline{\mathbf{Q}_\ell}$*-sheaf* on U corresponds (uniquely) to a continuous representation $\rho : \pi_1(U, \bar{\eta}) \to \mathrm{GL}(n, E_\lambda)$ for a finite extension $E_\lambda/\mathbf{Q}_\ell$. We will start with such a sheaf. In other words, we have a continuous representation $\rho : G_k \to \mathrm{GL}(n, E_\lambda)$ that is unramified outside S. Let k^g be the function field of $U_{\overline{\mathbf{F}_q}}$; note that there is a natural embedding $G_{k^g} \hookrightarrow G_k$, corresponding to the sequence:

$$1 \to \pi_1^g(U) \to \pi_1^a(U) \to G_{\mathbf{F}_q} \to 1,$$

where $\pi_1^g(U) = \pi_1(U_{\overline{\mathbf{F}_q}}, \bar{\eta})$ is the *geometric fundamental group* of U, and $\pi_1^a(U) = \pi_1(U, \bar{\eta})$ is the *arithmetic fundamental group* of U.

Let $G \subset \mathrm{GL}(n)_{\overline{\mathbf{Q}_\ell}}$ be the Zariski closure of $\rho(\pi_1^g)$. After fixing an embedding $\overline{\mathbf{Q}_\ell} \hookrightarrow \mathbf{C}$, we regard G as a algebraic group over $\mathbf{C}$. As such, $G(\mathbf{C})$ is a complex Lie group.

It is easy to construct an example. Let A be a d-dimensional abelian variety over k, and let $\rho = \rho_{A,\ell} : G_k \to \mathrm{GL}(2d, \mathbf{Q}_\ell)$ be the associated ℓ-adic representation. If, for example $A = E$ is an elliptic curve without complex multiplication, then for all ℓ, we have $G = \overline{\rho_{E,\ell}(\pi_1^g)} = \mathrm{GL}(2)$.

Let K be a maximal compact subgroup of $G(\mathbf{C})$. Katz claims that restriction induces an equivalence of categories

$$\mathsf{Rep}_{\mathbf{C}}^{\mathrm{alg}}(G) \xrightarrow{\sim} \mathsf{Rep}_{\mathbf{C}}^{\mathrm{top}}(K).$$

$$\left| \sum_{\deg(v) \mid n} \deg(v) \operatorname{tr} \psi \left(\theta(v)^{n/\deg v} \right) \right| \leqslant \left(2g - 2 + N - \sum r_i \right) \dim(\psi) q^{n/2}$$

References

[Del77] Pierre Deligne. *Cohomologie étale*, volume 569 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$.

[Kat88] Nicholas M. Katz. *Gauss sums, Kloosterman sums, and monodromy groups*, volume 116 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1988.