Topics for A-exam

Daniel Miller

August 7, 2016

Contents

1 Algebraic number theory

1.1 Extensions of dedekind schemes

Recall that a *dedekind scheme* is an integral noetherian normal scheme of dimension 1. If X is a Dedekind scheme, then all the stalks $\mathscr{O}_{X,x}$ are one-dimensional regular local rings, i.e. discrete valuation rings. In particular, each closed point $x \in X$ induces a discrete valuation v_x on the field of fractions k of X.

If K/k is a finite separable extension, let Y be the normalization of X in K, i.e. the relative spectrum of the sheaf

$$U \mapsto \text{integral closure of } \mathscr{O}_X(U) \text{ in } K.$$

Then $p : Y \to X$ is a finite surjective morphism. Each fiber $p^{-1}(x)$ is a finite set, possibly with nilpotents.

1.2 Discrete valuation fields

Let k be a field with a discrete valuation $v : k^\times \twoheadrightarrow \mathbf{Z}$. Let $\mathfrak{o} = \{x \in k : v(x) \geqslant 0\}$ be the valuation ring of k, and let $\mathfrak{p} \subset \mathfrak{o}$ be the unique maximal ideal. Put $U = 1 + \mathfrak{p}$, and give $\mathfrak{p}$ and U filtrations by

$$\begin{aligned} \mathfrak{p}^r &= \text{the } r\text{-th power of } \mathfrak{p} \\ U^r &= 1 + \mathfrak{p}^r. \end{aligned}$$

There is a canonical isomorphism $\mathrm{gr}(\mathfrak{p}^\bullet) \xrightarrow{\sim} \mathrm{gr}(U^\bullet)$, given by $x \mapsto 1 + x$.

1.3 Henselian Fields

Let k be a field with a discrete valuation. One calls k *Henselian* if Hensel's lemma holds for $\mathfrak{o}_k$. Alternatively, one requires that valuations extend uniquely to algebraic extensions of k. Complete fields are Henselian.

One can give a reasonable description of the absolute Galois group G_k of a Henselian field k. Let κ be the residue field of k, and let $p \geqslant 0$ be the characteristic of κ. If K/k is a *finite* extension, then define for $r \geqslant 0$,

$$\mathrm{Gal}(K/k)_r = \ker\left(\mathrm{Gal}(K/k) \to \mathrm{Aut}_{\mathfrak{o}_k}(\mathfrak{o}_L/\mathfrak{p}^{r+1})\right).$$

There is a canonical embedding $\mathrm{gr}(\mathrm{Gal}(K/k)_\bullet) \hookrightarrow \mathrm{gr}(U_K^\bullet)$, given by $\sigma \mapsto \sigma\pi/\pi$, for $\pi \in \mathfrak{o}_K$ an arbitrary uniformizer. In particular, $\mathrm{Gal}(K/k)$ is solvable.

upper ramification numbering

1.4 Local fields

A *local field* is a locally compact topological field. Local fields are known to be finite extensions of either $\mathbf{F}_p((t))$ or $\mathbf{Q}_p$.

1.5 Global fields

1.6 Classical geometry of numbers

Our main reference is Chapter I, §5-7 of [Neu99]. Let k be a number field, and write $k_\infty = k_\mathbf{R} = k \otimes_\mathbf{Q} \mathbf{R}$. This is a finite étale $\mathbf{R}$-algebra isomorphic to $\mathbf{R}^r \times \mathbf{C}^s$, where r is the number of real places and s is the number of complex places of k. One gives k_∞ a standard measure (twice Lebesgue on copies of $\mathbf{C}$ and Lebesgue on copies of $\mathbf{R}$) under which the lattice $\mathfrak{o}$ has volume $\mathrm{vol}(\mathfrak{o}) = \mathrm{vol}(k_\infty/\mathfrak{o}) = |d_k|^{1/2}$. As a corollary, if $S \subset k_\infty$ is open, convex, and centrally symmetric, then $\mathrm{vol}(S) \geqslant \mathrm{vol}(\mathfrak{o})$ implies $S \cap \mathfrak{o} \neq 0$. The same type of theorem holds for any $\mathfrak{a} \subset \mathfrak{o}$, where $\mathrm{vol}(\mathfrak{a}) = [\mathfrak{o} : \mathfrak{a}]|d_k|^{1/2}$. Considering $S = \{a : |N(a)| \leqslant ?\}$ gives that for every nonzero ideal $\mathfrak{a} \subset \mathfrak{o}$, there is $a \in \mathfrak{a} \smallsetminus 0$ such that (there exist a for both of these)

$$|N_{k/\mathbf{Q}}(a)| \leqslant \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s |d_k|^{1/2}[\mathfrak{o} : \mathfrak{a}]$$
$$|N_{k/\mathbf{Q}}(a)| \leqslant \left(\frac{2}{\pi}\right)^s |d_k|^{1/2}[\mathfrak{o} : \mathfrak{a}].$$

There is a natural map $\log|\cdot| : k_\infty^\times$.

1.7 Adelic geometry of numbers

A key fact is that $\mathbf{I}_k^1/k^\times$ and $\mathbf{A}_k/k$ are compact.

Our main reference here is [PR94]. Let k be a number field, and G an algebraic group over k. Let S be a finite set of places of k, containing all the infinite places, and let $k_S = \prod_{v \in S} k_v$. It is a theorem of Borel (essentially Theorem 5.1 of [PR94]) that the double quotient $G(k)\backslash G(\mathbf{A}_k)/G(k_S)K$ is finite, whenever $K \subset G(\mathbf{A})$ is open compact.

Let $\mathrm{X}(G) = \hom_k(G, \mathbf{G}_m)$; this is a finite free $\mathbf{Z}$-module. There is a canonical homomorphism

$$c : G(\mathbf{A}) \to \mathrm{X}(G)^\vee_{\mathbf{R}} \qquad (g_v) \mapsto \chi \mapsto \prod_v |\chi(g_v)|_v.$$

Let $G(\mathbf{A})^1 = \ker(c)$. Then Theorem 5.6 of [PR94] tells us that $G(\mathbf{A})^1/G(k)$ is compact if and only if the semisimple part of G is anisotropic over k.

2 Class field theory

The main references are Chapter V of [Neu99] and Chapter VI of [CF86].

2.1 Local class field theory

These theorems are from chapter V of [Neu99]. Recall that if κ is a finite field, we regard $\mathbf{Z}$ as a subgroup of G_κ by letting $1 \in \mathbf{Z}$ correspond to the (arithmetic) Frobenius of κ. We normalize all valuations so that uniformizers have valuation 1.

Theorem 2.1.1. *Let k be a local field with residue field κ. Then there is a unique continuous homomorphism $r : k^\times \to G_k^{\mathrm{ab}}$ such that*

1. *the following diagram commutes:*

$$\begin{tikzcd} k^\times \arrow[r, "r"] \arrow[d, "v"] & G_k^{\mathrm{ab}} \arrow[d] \\ \mathbf{Z} \arrow[r, hook] & G_\kappa \end{tikzcd}$$

2. *for all finite abelian K/k, the map r induces an isomorphism $k^\times/N(K^\times) \xrightarrow{\sim} \mathrm{Gal}(K/k)$*

This homomorphism (called the reciprocity map*) induces an isomorphism $\widehat{k^\times} \xrightarrow{\sim} G_k^{\mathrm{ab}}$. Finally, a subgroup $G \subset k^\times$ is of the form $N(K^\times)$ for some finite K/k if and only if G is open and has finite index.*

Proof. This proof is inspired by that of Theorem 1.13 in [Mil13]. We only prove the uniqueness of r. Since $k^\times$ is topologically generated by uniformizers, it suffices to show that conditions determine $r(\pi)$ for any $\pi \in k^\times$ with $v(\pi) = 1$. For any such π, we get a decomposition $k^{\mathrm{ab}} = k^{\mathrm{ur}} \cdot k_\pi$, where k_π is the fixed field of $r(\pi)$. Since $r(\pi)$ has to act on k^{ur} as Frobenius, and (by definition), $r(\pi)$ acts trivially on k_π, the action of $r(\pi)$ on k^{ab} is determined. $\square$

This theorem is true even if k is archimedean. Just ignore part 1, and note that $G_{\mathbf{R}} = \mathbf{Z}/2$, which has no nontrivial automorphisms. In other words, there is a unique homomorphism $\mathbf{R}^\times \to G_{\mathbf{R}}$ inducing an isomorphism $\widehat{\mathbf{R}^\times} \xrightarrow{\sim} G_{\mathbf{R}}$.

2.2 Global class field theory

A good references is chapter VI of [Neu99]. For a global field k, write $\mathbf{A}_k$ for its ring of adeles, and define $C_k = \mathrm{GL}(1, \mathbf{A}_k) / \mathrm{GL}(1, k)$. For a place v of k, write r_v for the reciprocity map of k_v. Since the decomposition groups $D_v = G_{k_v}$ are well-defined up to conjugacy in G_k, the abelianized decomposition groups D_v^{ab} are well-defined as subgroups of G_k^{ab}.

Theorem 2.2.1. *There is a unique continuous homomorphism $r : C_k \to G_k^{\mathrm{ab}}$ such that*

1. *the following diagram commutes for all v:*

$$\begin{tikzcd} k_v^\times \arrow[r, hook] \arrow[d, "r_v"] & C_k \arrow[d, "r"] \\ D_v^{\mathrm{ab}} \arrow[r, hook] & G_k^{\mathrm{ab}} \end{tikzcd}$$

2. *if K/k is finite abelian, r induces an isomorphism $C_k / N(C_K) \xrightarrow{\sim} \mathrm{Gal}(K/k)$.*

This homomorphism (also called the reciprocity map*) induces an isomorphism $\widehat{C_k} \xrightarrow{\sim} G_k^{\mathrm{ab}}$. Finally, a subgroup $G \subset C_k$ is of the form $N(K^\times)$ for some finite K/k if and only if G is open and has finite index.*

3 Algebraic geometry

[Har77], [Sil09], and [EH00]

Zariski's main theorem

tangent space on affine / projective variety as special case of tangent sheaf for arbitrary morphism of schemes

3.1 Cartier divisors

For a ringed space X, let $\mathscr{M}$ be the sheaf of meromorphic functions, and put $\mathscr{D} = \mathscr{M}^\times / \mathscr{O}^\times$. This is a sheaf of abelian groups on X, called the *sheaf of Cartier divisors*. A global section of $\mathscr{D}$ is called a *Cartier divisor* on X. The (tautological) short exact sequence

$$1 \to \mathscr{O}^\times \to \mathscr{M}^\times \to \mathscr{D} \to 0$$

yields a long exact sequence in sheaf cohomology:

$$1 \to \Gamma(\mathscr{O}^\times) \to \Gamma(\mathscr{M}^\times) \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to \mathrm{H}^1(\mathscr{M}^\times) \to \cdots$$

If X is integral, $\mathscr{M}$ is flasque, so $\mathrm{H}^1(\mathscr{M}^\times) = 0$, whence $\mathrm{Pic}(X) = \mathrm{Div}(X) / \operatorname{div} \Gamma(\mathscr{M}^\times)$. If X is a one-dimensional scheme over S, then points $x \in X(S)$ yield Cartier divisors, denoted $\mathscr{I}(x)$, on X. If we think of $\mathscr{I}(x)$ as a sheaf, then it makes sense to talk about $\mathscr{I}^{-1}(x)$.

3.2 Algebraic groups and their Lie algebras

We would like to interpret the notion of a representation of an algebraic group over a field in terms of arbitrary ringed topoi. The main source here is [**?**]. We start with the following list of analogies.

classical	topos-theoretic
field k	base scheme S
finite-dimensional k-space V	coherent $\mathscr{O}_X$-module $\mathscr{M}$
$R \mapsto V_R = V \otimes_k R$	$(X \xrightarrow{f} S) \mapsto \Gamma(f^*\mathscr{M})$
$\operatorname{Spec}(k[V^\vee])$	$\mathbf{V}(\mathscr{M}) = \operatorname{Spec}(\mathscr{O}_S[\mathscr{M}^\vee])$
$\operatorname{GL}(V) : R \mapsto \operatorname{Aut}_R(V_R)$	$\operatorname{GL}(\mathscr{M}) : X \mapsto \operatorname{Aut}_{\Gamma(X)}(\Gamma(f^*\mathscr{M}))$
$\rho : G \to \operatorname{GL}(V)$	$\rho : G \to \operatorname{GL}(\mathscr{M})$
$G(R) \to \operatorname{Aut}_R(V_R)$	$G(X) \to \operatorname{Aut}_{\Gamma(X)}(\Gamma(f^*\mathscr{M}))$

An algebraic group G over a field k is *anisotropic* if it contains no k-split tori. If k is a non-algebraically closed local field, then G is anisotropic if and only if $G(k)$ is compact.

4 Geometry of curves

The main references for this section are Chapter IV of [Har77] and Chapter 7 of [Liu02]:

4.1 Divisors and invertible sheaves on curves

4.2 The Riemann-Roch theorem

5 Elliptic curves

The main reference on elliptic curves is [Sil09], especially chapters II-IV and VI-VIII. Chapter II mostly covers the geometry of curves.

elliptic curves over local fields

elliptic curves over global fields

5.1 The group law on an elliptic curve

If X is a curve over S, define

$$\operatorname{Pic}^1_{X/S}(T) = \operatorname{Pic}^1(X_T)/\operatorname{Pic}(T).$$

Theorem 5.1.1. *Let E be an elliptic curve over S. Then there is a natural isomorphism of functors $E \xrightarrow{\sim} \operatorname{Pic}^1_{E/S}$ given by $x \mapsto \mathscr{I}^{-1}(x)$. This induces the structure of an S-group on E, where $x + y + z = 0$ if and only if*

$$(\mathscr{I}(x) - \mathscr{I}(0)) + (\mathscr{I}(y) - \mathscr{I}(0)) + (\mathscr{I}(z) - \mathscr{I}(0)) = 0 \qquad \text{in } \operatorname{Pic}_{X/S}(T).$$

Proof. This is Theorem 2.1.2 of [KM85] □

5.2 Formal groups of elliptic curves

Let F be a formal group. By Proposition IV.4.2 of [Sil09], the invariant differential of F is $\omega = \frac{\partial F}{\partial X}(0, T)^{-1} dT$. Put $\log_F = \int \omega$. Then $\log_F : F \to \hat{\mathbf{G}}_a$ is an isomorphism over $\mathbf{Q}$. If F is defined over a mixed-characteristic complete discrete valuation ring and $v(p) > 0$, then for $r > v(p)/(p-1)$, $\log_F$ induces an isomorphism $F(\mathfrak{m}^r) \xrightarrow{\sim} \hat{\mathbf{G}}_a(\mathfrak{m}^r)$.

If E is an elliptic curve, let $\hat{E}$ denote the corresponding formal group. A nice theorem (IV.7.4 in [Sil09]) is that if $f : E_1 \to E_2$ is an isogeny of characteristic p elliptic curves, then $\deg_i(f) = p^{\operatorname{ht}\hat{f}}$, where $\deg_i$ denotes inseparable degree.

5.3 Elliptic curves over C

Let E be an elliptic curve over $\mathbf{C}$, which we will take to mean that E is a compact connected one-dimensional complex Lie group of genus one. Let $\mathfrak{e} = \operatorname{Lie} E$. The exponential map $\exp : \mathfrak{e} \to E$ is a surjective homomorphism of Lie groups, so we get a short exact sequence

$$0 \to \Lambda \to \mathfrak{e} \xrightarrow{\exp} E \to 0.$$

It is possible to go the other direction. Start with a lattice $\Lambda \subset \mathbf{C}$, and define

$$\begin{aligned} \wp_\Lambda(z) &= \sum_{\lambda \in \Lambda \smallsetminus 0} \left((z-\lambda)^{-2} - \lambda^{-2} \right) \\ G_k(\Lambda) &= \sum_{\lambda \in \Lambda \smallsetminus 0} \lambda^{-k}. \end{aligned}$$

Theorem 5.3.1. *Let $\Lambda \subset \mathbf{C}$ be a lattice. Let $g_2 = 60 G_2(\Lambda)$ and $g_3 = 140 G_6(\Lambda)$. Let E_Λ be the elliptic curve $y^2 = 4x^3 + g_2 x + g_3$. Then the map $(\wp_\Lambda : \wp'_\Lambda : 1) : \mathbf{C}/\Lambda \to E_\Lambda$ is an analytic isomorphism.*

Proof. This is Proposition VI.3.6 of [Sil09]. □

In fact, $\Lambda \mapsto E_\Lambda$ induces an equivalence of categories between lattices over $\mathbf{C}$ and elliptic curves over $\mathbf{C}$.

6 Representation theory

The main references are [Kna79] and [Don97].

6.1 Representations of reductive Lie groups

Following [Wal88], we say that a real Lie group G is *real reductive group* if it is a finite cover of the real points of a Zariski-closed subgroup of $\mathrm{GL}_n(\mathbf{C})$, defined over $\mathbf{R}$, which is closed under conjugate-transpose.

First, note that if G is an arbitrary locally compact group, $f \in C_c(G)$, and π is a continuous Banach representation of G, then we can put

$$\pi(f) = \int_G f(g) \pi(f)\, dg.$$

This is a representation of the algebra $C_c(G)$. We apply this construction to the special case where G be a real reductive group with maximal compact K. Let $\widehat{K}$ be the unitary dual of K. If $\tau \in \widehat{K}$, put $\alpha_\tau(k) = \dim(\tau) \operatorname{tr} \tau(k^{-1})$. For any representation π of K, set $\Pi_\tau = \pi(\alpha_\tau)$ for any $\tau \in \widehat{K}$. It turns out that Π_τ is a projection operator, and that

$$\pi = \widehat{\bigoplus_{\tau \in \widehat{K}}} \operatorname{im}(\Pi_\tau).$$

One says that π is *admissible* if each $\operatorname{im}(\Pi_\tau)$ is finite-dimensional (equivalently, if each τ has finite multiplicity in π). Call a representation π of G *admissible* if the restriction $\operatorname{res}_K \pi$ is admissible.

6.2 Decompositions of groups

Let G be a real reductive group with lie algebra $\mathfrak{g}$ and Cartan involution θ. Put

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} : \theta X = X\} \\ \mathfrak{p} &= \{X \in \mathfrak{g} : \theta X = -X\}. \end{aligned}$$

Then $\mathfrak{k}$ is the Lie algebra of a "canonical" maximal compact K of G. The *Cartan decomposition* of G is the fact that the map $\mathfrak{p} \times K \to G$, $(X, k) \mapsto \exp(X)k$, is a diffeomorphism. **Warning**: $\mathfrak{p}$ is *not* a subalgebra of $\mathfrak{g}$. The next decomposition is trickier. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra. For $\alpha \in \mathfrak{a}^\vee$, put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \operatorname{ad} Y(X) = \lambda(Y)X \text{ for all } Y \in \mathfrak{a}\}.$$

Let $R = \{\alpha \in \mathfrak{a}^\vee \smallsetminus 0 : \mathfrak{g}_\alpha \neq 0\}$. It is known that $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$. Let $\Delta^+ \subset R$ be some choice of positive roots, and put

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, inducing an isomorphism $G = KAN$ (at least, if everything is appropriately connected).

One interprets A and N as depending on $\mathfrak{p}$, so write $A_\mathfrak{p}$, $N_\mathfrak{p}$. Define

$$M = M_\mathfrak{p} = Z_K(A_\mathfrak{p}) = \{k \in K : ka = ak \text{ for all } a \in A_\mathfrak{p}\}.$$

Then $S = MAN$ is the "standard minimal parabolic." In the main example $G = \mathrm{SL}_2(\mathbf{R})$, we have

$$\begin{aligned}
\mathfrak{p} &= \left\{ \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right) : a, b \in \mathbf{R} \right\} = \left\langle \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) \right\rangle \\
\mathfrak{k} &= \left\langle \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \right\rangle = \operatorname{Lie} K = \operatorname{Lie} \left\{ k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) : 0 \leqslant \theta < 2\pi \right\} \\
\mathfrak{a} &= \left\langle \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right) \right\rangle = \operatorname{Lie} A = \operatorname{Lie} \left\{ \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) : a > 0 \right\} \\
\mathfrak{n} &= \left\langle \left(\begin{smallmatrix} & 1 \\ & \end{smallmatrix}\right) \right\rangle = \operatorname{Lie} N = \operatorname{Lie} \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \\
\mathfrak{m} &= 0 = \operatorname{Lie} M = \operatorname{Lie}(\pm 1)
\end{aligned}$$

One puts $\mathfrak{n}_+ = \mathfrak{n}$, $\mathfrak{n}_- = \sum_{\alpha \in -\Delta^+} \mathfrak{g}_\alpha$.

6.3 Induced representations

Let G be a real reductive group, and consider the canonical decomposition $G = KS = KMAN$. Suppose ρ is a unitary representation of S with space V_ρ. The induced space $\operatorname{ind}_S^G \rho$ is the completion of

$$\{f : G \to V_\rho : f(ks) = \rho(s^{-1}) f(k)\}$$

with respect to the norm

$$\|f\|^2 = \int_K |f(k)|^2 \, dk.$$

The action is $(\gamma \cdot f)(g) = f(\gamma^{-1} g)$. A function $f \in \operatorname{ind}_S^G \rho$ is determined by both its restrictions $f|_K$ and $f|_{N_-}$, so $\operatorname{ind}_S^G \rho$ can be realized as function spaces on both those subgroups of G.

6.4 Representations of $\mathrm{SL}_2(\mathbf{R})$

Let's start out with the standard families of unitary representations of $\mathrm{SL}_2(\mathbf{R})$. Each of these is defined as the completion of some space of smooth (or analytic) functions with respect to a specified norm.

Definition 6.4.1 (finite-dimensional: $\mathcal{F}_n$ for $n \geqslant 0$)**.** Let $\rho_0 : \mathrm{SL}_2(\mathbf{R}) \hookrightarrow \mathrm{GL}_2(\mathbf{C})$ be the inclusion morphism. Put $\rho_n = \operatorname{Sym}^n \rho_0$. The representation ρ_n can be realized in the space $\mathcal{F}_n$ of homogeneous polynomials $f \in \mathbf{C}[x, y]$ of degree n. A matrix $\gamma \in \mathrm{SL}_2(\mathbf{R})$ acts by $(\gamma f)(v) = f(\gamma^{-1} v)$. If we reinterpret $f \in \mathbf{C}[x, y]_n$ as an element of $\mathbf{C}[z]_{\leqslant n}$, then $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts by $(\gamma f)(z) = (bz + d)^n f\left(\frac{az+c}{bz+d}\right)$.

Theorem 6.4.2. *Every finite-dimensional irreducible representation of* $\mathrm{SL}_2(\mathbf{R})$ *is one of* $\{\mathcal{F}_n : n \geqslant 0\}$.

The finite-dimensional representations of $\mathrm{SL}_2(\mathbf{R})$ are *not* unitary, except for the trivial representation.

Definition 6.4.3 (discrete series: $\mathcal{D}_n^\pm$ for $n \geqslant 2$)**.** Let G act on the upper half-plane $\mathfrak{h}$ in the usual way: $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : z \mapsto \frac{az+b}{cz+d}$. The group G has a right action on functions on $\mathfrak{h}$:

$$(f \cdot \gamma)(z) = \frac{1}{(cz+d)^n} f\left(\frac{az+b}{cz+d}\right) = j(z,\gamma)^{-n} f(\gamma z).$$

Composing with the inverse gives us a left action: $(\gamma f)(z) = j(z,\gamma^{-1})^{-n} f(\gamma^{-1}z)$. The norm on $\mathcal{D}_n^+$ is

$$\|f\|^2 = \int_{\mathfrak{h}} |f(z)|^2 y^n \frac{dxdy}{y^2}.$$

To be precise: $\mathcal{D}_n^+ = \{f : \mathfrak{h} \to \mathbf{C} \text{ analytic} : \|f\| < \infty\}$.

Theorem 6.4.4. *The representations $\mathcal{D}_n^\pm$ are irreducible for $n \geqslant 1$.*

Proof. We only consider $\mathcal{D}_n^+$. Let $K = \mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbf{R})$ be the standard maximal compact subgroup. Its unitary representations are of the form $\chi_n : \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{in\theta}$ for $n \in \mathbf{Z}$. Let $\Pi_n = \Pi_{\chi_n}$. This is the projection operator on $\mathcal{D}_n^+$ defined by

$$(\Pi_n f)(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}(-z\sin\theta + \cos\theta)^{-n} f\left(\frac{z\cos\theta + \sin\theta}{-z\sin\theta+\cos\theta}\right) d\theta$$

The function $f_n(z) = (z+i)^{-n}$ is an eigenvector for Π_n, and an element of $\mathcal{D}_n^+$. Moreover, a computation in complex analysis shows that if $f \in \mathcal{D}_n^+$ has $f(i) \neq 0$, then $\Pi_n f \in \mathbf{C} \cdot f_n$. Using the transitivity of the action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathfrak{h}$, the result easily follows. $\square$

Definition 6.4.5 (principal series: $\mathcal{P}^{\pm,iv}$ for $v \in \mathbf{R}$)**.** The space underlying $\mathcal{P}^{\pm,iv}$ is $L^2(\mathbf{R})$. The action is

$$(\gamma \cdot f)(x) = \begin{cases} \frac{1}{|-bx+d|^{1+iv}} f\left(\frac{ax-c}{-bx+d}\right) & \text{if } + \\ \frac{\mathrm{sgn}(-bx+d)}{|-bx+d|^{1+iv}} f\left(\frac{ax-c}{-bx+d}\right) & \text{if } - \end{cases}$$

These are induced from unitary representations of $S = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)$. They can be realized as functions on K or N_-.

The only reducible principal series is $\mathcal{P}^{-,0} \simeq \mathcal{D}_1^+ \oplus \mathcal{D}_1^-$. There is a "nonunitary principal series" $\mathcal{P}^{k,z}$ for $k \in \mathbf{Z}$, $z \in \mathbf{C}$, where the space is $C_c^\infty(\mathbf{R})$, and the action is

$$(\gamma f)(x) = \frac{\pm(-bx+d)}{|-bx+d|^{1+z}} f\left(\frac{ax-c}{-bx+d}\right).$$

Definition 6.4.6 (complementary series $\mathcal{C}^s$ for $0 < s < 1$)**.** Let $0 < s < 1$. The space underlying $\mathcal{C}^s$ is the set of $f \in L^1_{\mathrm{loc}}(\mathbf{R})$ such that

$$\|f\|^2 = \int_{\mathbf{R}^2} \frac{f(x)\overline{f(y)}}{|x-y|^{1-u}} dxdy < \infty.$$

The action is what we would have with $\mathcal{P}^{+,u}$, i.e.

$$(\gamma f)(x) = \frac{1}{|-bx+d|^{1-u}} f\left(\frac{ax-c}{-bx+d}\right).$$

Definition 6.4.7 (limits of discrete series: $\mathcal{D}_1^\pm$)**.** The action here is the same as in the $\mathcal{D}_n^\pm$, but the norm is

$$\|f\|^2 = \sup_{y>0} \int_{\mathbf{R}} |f(x+iy)|^2 dx.$$

Theorem 6.4.8. *Every irreducible unitary representation of* $\mathrm{SL}_2(\mathbf{R})$ *is one of the following:*

- $\mathcal{D}_n^{\pm}$ *for* $n \geqslant 2$
- $\mathcal{D}_1^{\pm}$
- $\mathcal{P}^{\pm,iv}$ *for* $v \in \mathbf{R}$*, and* $v \neq 0$ *if* $-$.
- $\mathcal{C}^s$ *for* $0 < s < 1$.

The only isomorphisms between items in this list are:

$$\mathcal{P}^{+,iv} \simeq \mathcal{P}^{+,-iv}$$
$$\mathcal{P}^{-,iv} \simeq \mathcal{P}^{-,-iv}.$$

6.5 Representations of $\mathrm{GL}_2(\mathbf{R})$

This is discussed in section 2 of [Kna79]. Let $\mathrm{SL}_2^{\pm}(\mathbf{R}) = \{g \in \mathrm{GL}_2(\mathbf{R}) : \det g = \pm 1\}$, and consider $\mathrm{ind}_{\mathrm{SL}_2(\mathbf{R})}^{\mathrm{SL}_2^{\pm}(\mathbf{R})} \pi$ for irreducible unitary representations π of $\mathrm{SL}_2(\mathbf{R})$. We have

$$\mathrm{ind}_{\mathrm{SL}_2}^{\mathrm{SL}_2^{\pm}} (\mathcal{P}^{\pm,iv}) \simeq P^{\pm,iv} \oplus P^{\pm,iv}$$

for a canonical irreducible unitary representation $P^{\pm,iv}$ of $\mathrm{SL}_2^{\pm}(\mathbf{R})$. For $n \geqslant 2$, the representation $\mathrm{ind}_{\mathrm{SL}}^{\mathrm{SL}^{\pm}} (\mathcal{D}_n^{\pm})$ is irreducible unitary. These (also with the complementary series) exhaust the irreducible unitary representations of $\mathrm{SL}_2^{\pm}(\mathbf{R})$. The irreducible unitary representations of $\mathrm{GL}_2(\mathbf{R})$ are of the form

$$\left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(x)\pi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

for χ a unitary character of $\mathbf{R}^{+}$ and π a representation of $\mathrm{SL}_2^{\pm}(\mathbf{R})$.

6.6 Representations of $\mathrm{SL}_2(\mathbf{C})$

A good reference is II.4 of [Kna86].

Definition 6.6.1 (principal series $\mathcal{P}^{n,iv}$ for $n \in \mathbf{Z}$ and $v \in \mathbf{R}$)**.** The underlying space is $L^2(\mathbf{C})$. The action is, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$(\gamma f)(z) = |-bz+d|^{-2-iv}\left(\frac{-bz+d}{|-bz+d|}\right)^{-n} f\left(\frac{az-c}{-bz+d}\right).$$

Definition 6.6.2 (complementary series $\mathcal{C}^s$ for $0 < s < 2$)**.** As for $\mathrm{SL}_2(\mathbf{R})$, the space lives inside $L^1_{\mathrm{loc}}(\mathbf{C})$, has action just like $\mathcal{P}^{0,s}$:

$$(\gamma f)(z) = |-bz+d|^{-2-s} f\left(\frac{az-c}{-bz+d}\right).$$

The norm is

$$\|f\|^2 = \int_{\mathbf{C}^2} \frac{f(x)\overline{f(y)}}{|x-y|^{2-s}}\, dxdy.$$

Theorem 6.6.3. *Every irreducible unitary representation of* $\mathrm{SL}_2(\mathbf{C})$ *is one of the following:*

- $\mathcal{P}^{n,iv}$ *for* $n \in \mathbf{Z}$, $v \in \mathbf{R}$
- $\mathcal{C}^s$ *for* $0 < s < 2$

The only isomorphisms between items in this list are:

$$\mathcal{P}^{n,iv} \simeq \mathcal{P}^{-n,-iv}.$$

Proof. This is Theorem 16.2 of [Kna86]. $\square$

6.7 Representations of $\mathrm{GL}_2(\mathbf{C})$

It is claimed in the last paragraph of [Kna79] that every irreducible unitary representation of $\mathrm{GL}_2(\mathbf{C})$ is of the form

$$\left(\begin{smallmatrix} z & \\ & z \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(z)\pi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right),$$

for $z \in \mathbf{C}^\times$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{C})$, and χ a unitary character of $\mathbf{C}^\times$ that agrees with π on $\left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right)$.

References

[CF86] J. W. S. Cassels and A. Fröhlich, editors. *Algebraic number theory*, London, 1986. Academic Press, Inc. Reprint of the 1967 original.

[Don97] Robert W. Donley, Jr. Irreducible representations of $\mathrm{SL}(2, \mathbf{R})$. In *Representation theory and automorphic forms (Edinburgh, 1996)*, volume 61 of *Proc. Sympos. Pure Math.*, pages 51–59. 1997.

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Har77] Robin Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliiptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[Kna79] Anthony W. Knapp. Representations of $\mathrm{GL}_2(\mathbf{R})$ and $\mathrm{GL}_2(\mathbf{C})$. In *Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, Ore.), Part 1*, volume 33 of *Proc. Sympos. Pure Math.*, pages 87–91. 1979.

[Kna86] Anthony W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné.

[Mil13] J. S. Milne. Class field theory (v4.02), 2013. Available at `www.jmilne.org/math`.

[Neu99] Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissensschaften*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappencher.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[Wal88] Nolan R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.