Proving representability of deformation functors

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Let **F** be a finite field, W(**F**) its ring of Witt vectors, and $\mathfrak{Ar}_{W(\mathbf{F})}$ the category of finite local W(**F**)-algebras with residue field **F**. Let $\widehat{\mathfrak{Ar}}_{W(\mathbf{F})}$ be the opposite of the category of complete local noetherian W(**F**)-algebras with residue field **F**. For $R \in \widehat{\mathfrak{Ar}}_{W(\mathbf{F})}$, write Spf(R) for the corresponding object of $\widehat{\mathfrak{Ar}}_{W(\mathbf{F})}$.

Let G be a profinite action, $V_{\mathbf{F}}$ a finite-dimensional \mathbf{F} -vector space with continuous G-action. In proving the representability of the deformation functor $D_{V_{\mathbf{F}}}$, you use the following theorem of SGA:

Theorem 1 (SGA 3, VIIb, Thm. 1.4). Let $R \rightrightarrows X$ be an equivalence relation in $\widehat{\mathfrak{Ar}}_{\mathbf{W}(\mathbf{F})}^{\circ}$ such that the first projection is flat. Then the quotient of X by R exists. It represents the functor on points defined by the equivalence relation.

The problem is, this theorem is false.

Example 1. Consider the following group object in $\widehat{\mathfrak{Ar}}_{W(\mathbf{F})}^{\circ}$:

$$\widehat{\mathbf{G}}_{a/\mathbf{F}} = \operatorname{Spf}(\mathbf{F}[\![t]\!]), \qquad A \mapsto (\mathfrak{m}_A, +).$$

It admits a Frobenius endomorphism $\varphi \colon \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \to \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}}$, which on A-points is $\varphi(a) = a^p$. Let $\alpha_p = \ker(\varphi)$. Then

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \stackrel{\varphi}{\longrightarrow} \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \longrightarrow 0$$

is exact in the flat topology. Apply Theorem 1 to the equivalence relation

$$\alpha_p \times \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \rightrightarrows \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \times \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \qquad (a,b) \mapsto (b,a+b).$$

The quotient has coordinate ring

$$R = \{ f \in \mathbf{F} \llbracket t \rrbracket : f(t \otimes 1) = f(t \otimes 1 + 1 \otimes t) \text{ in } \mathbf{F} \llbracket t \rrbracket \otimes \mathbf{F} [t] / (t^p) \} = \mathbf{F} \llbracket t^p \rrbracket.$$

In other words, the quotient $\widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}}/\alpha_p \simeq \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}}$ via $\varphi \colon \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}} \to \widehat{\mathbf{G}}_{\mathbf{a}/\mathbf{F}}$. If Theorem 1 were true, the sequence

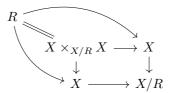
$$0 \longrightarrow \boldsymbol{\alpha}_p(A) \longrightarrow \mathfrak{m}_A \stackrel{\varphi}{\longrightarrow} \mathfrak{m}_A \longrightarrow 0$$

would be exact for all $A \in \mathfrak{Ar}_{W(\mathbf{F})}$. But this is clearly false.

So Theorem 1 as stated is false. The correct theorem (which does apply in your setting) is the following:

Theorem 2. Let $R \rightrightarrows X$ be an equivalence relation in $\widehat{\mathfrak{Ar}}_{W(\mathbf{F})}^{\circ}$ such that one projection is flat and one is smooth. Then the quotient of X by R exists. It represents the functor on points defined by the equivalence relation.

Proof. Let X/R be the quotient. It suffices to prove that for all $A \in \mathfrak{Ar}_{W(F)}$, the map $X(A) \to (X/R)(A)$ is surjective. It suffices to prove the existence of a section $X/R \to X$ of the projection map $X \to X/R$. The theorem in SGA tells us that $X \to X/R$ is a flat cover, so after base-change, the projection $X \to X/R$ is $X \to X \times_{X/R} X$, as in the following commutative diagram:



So $X \to X/R$ becomes smooth after an fppf base-change. By descent, $X \to X/R$ is itself smooth. In $\widehat{\mathfrak{Ar}}_{W(\mathbf{F})}$, smooth maps admit sections.

To summarize. You define a framed deformation functor $D_{V_{\mathbf{F}}}^{\square}$, and want the categorical quotient $D_{V_{\mathbf{F}}}^{\square}/\widehat{\mathrm{PGL}}_d$ to be the same as the "presheaf quotient" $D_{V_{\mathbf{F}}}$. The theorem in SGA does not give you this—you really need the fact that $\widehat{\mathrm{PGL}}_d$ is smooth.