Multivariable calculus and differential forms

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Let U be a connected, simply-connected subset of the smooth manifold $\mathbf{R}^3$. We have the de Rham sheaf $\Omega^\bullet$ on U. Let $\mathcal{O}$ be the structure sheaf of U, and let $\mathcal{X} = (\Omega^1)^\vee$ be the sheaf of vector fields on U. We identify $\mathcal{X}$ with $\mathcal{O}^3$ in the usual way, i.e. $X = f\partial_x + g\partial_y + h\partial_z$ corresponds to $f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$. One defines maps

$$\begin{aligned}
\mathrm{grad} : \mathcal{O} \to \mathcal{X} \qquad & f \mapsto \nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} \\
\mathrm{curl} : \mathcal{X} \to \mathcal{X} \qquad & \mathbf{f} \mapsto \nabla \times \mathbf{f} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{pmatrix} = (\partial_x f_2 - \partial_y f_1)\mathbf{i} \\
\mathrm{div} : \mathcal{X} \to \mathcal{O} \qquad & \mathbf{f} \mapsto \nabla \cdot \mathbf{f} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3
\end{aligned}$$

The key fact is that we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & \mathcal{X} \arrow[r, "\mathrm{curl}"] \arrow[d, "(-)^\flat"] & \mathcal{X} \arrow[r, "\mathrm{div}"] \arrow[d, "\alpha"] & \mathcal{O} \arrow[d, "\beta"] \\
\mathcal{O} \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3
\end{tikzcd}$$

Since U is a Riemannian manifold, it comes with a metric, i.e. an isomorphism $\mathcal{O} \xrightarrow{g} \mathrm{Sym}^2\,\Omega^1$. It yields an isomorphism (the musical isomorphism) $(-)^\flat : \mathcal{X} \to \mathcal{X}^{\vee\vee} = \Omega^1$, given by $X^\flat(Y) = \langle X, Y\rangle = (X \otimes Y)(g)$.

The maps α and β are

$$\alpha : X \mapsto (Y \otimes Z \mapsto \langle X, Y \times Z\rangle)$$

d

$$\nabla \times X = \left({}^\star(dX^\flat)\right)^\sharp$$

1 General differential geometry

Let $(X, \mathcal{O})$ be a ringed topos, and $\mathcal{E}$ a locally free $\mathcal{O}$-module of finite type. General nonsense tells us that the natural map $(\mathcal{E} \otimes \mathcal{E})^\vee \to \hom(\mathcal{E}, \mathcal{E}^\vee)$ is an isomorphism. In particular, to any $g \in (\mathcal{E} \otimes \mathcal{E})^\vee$ we can associate the "musical map" $(-)^\flat : \mathcal{E} \to \mathcal{E}^\vee$. We call g non-degenerate if $(-)^\flat$ is an isomorphism.

Suppose we have a derivation $d : \mathcal{O} \to \Omega^1$, Ω^1 is locally free, and put $\mathcal{X} = (\Omega^1)^\vee$. Let $g \in (\mathcal{X} \otimes \mathcal{X})^\vee$ be a non-degenerate symmetric inner product. Put $\langle X, Y\rangle = g(X, Y)$. Since g is non-denerate,

Again, let $(X, \mathcal{O})$ be a ringed topos. The motivating problem is that if we define Ω^1_X in the obvious way, (i.e. as $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the kernel of $\mathcal{O} \otimes \mathcal{O} \to \mathcal{O}$, then Ω^1_X for X a smooth manifold doesn't agree with the "classical definition" in differential geometry. [This may not be the case.] One can define a sheaf (for any $\mathcal{O}$-module $\mathcal{M}$) $\mathcal{D}\mathit{er}(\mathcal{O}, \mathcal{M})$ in the usual manner. This agrees with the classical definition. However, $\mathcal{D}\mathit{er}(\mathcal{O}, \mathcal{O})^\vee$ is *not* necessarily isomorphic to Ω^1, if $(X, \mathcal{O})$ is a smooth manifold. A kludge is to put $\mathcal{X} = \mathcal{D}\mathit{er}(\mathcal{O}, \mathcal{O})$ for vector fields, and not worry directly about Ω^1.

1.1 Vector bundles and torsors

Let $(X, \mathcal{O})$ be a smooth manifold. Recall that a *vector bundle* on X is a locally free $\mathcal{O}$-module. Let $\mathsf{LF}(\mathcal{O})$ be the category of vector bundles on X.