Complexification

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2 September 2016

So far we have only considered real Lie groups, even though some of our examples, like $\mathrm{GL}(n, \mathbf{C})$, seem like they should be "complex" objects in some way. So, without further ado, we make a definition.

Definition 1. A *complex Lie group* is a group which is also a complex manifold, such that all the group operations are analytic maps.

With this definition, it is easy to see that $\mathrm{GL}(n, \mathbf{C})$ is a complex Lie group, as is $\mathrm{Sp}(2n, \mathbf{C})$. Clearly, every complex Lie group can be made into a real Lie group in a trivial way, by forgetting the complex structure. That is, we have a kind of map (a functor, for those in the know)

$$(-)^{\mathrm{real}} \colon \{\text{complex Lie groups}\} \to \{\text{real Lie groups}\}.$$