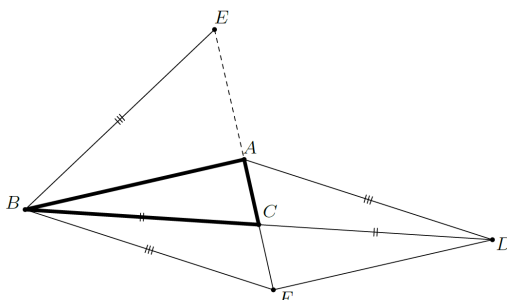


Senior Test 1 Solutions

1. *EGMO 2013, Problem 1*

Define F so that $ABFD$ is a parallelogram. Then E, A, C, F are collinear (as diagonals of a parallelogram bisect each other) and $BF = AD = BE$. Further, A is the midpoint of EF , since $AF = 2AC$, and thus AB is an altitude of the isosceles triangle EBF with apex B . Therefore $AB \perp AC$.

2. *EGMO 2013, Problem 2*

The solution naturally divides into three different parts: we first obtain some bounds on m . We then describe the structure of possible dissections, and finally, we deal with the few remaining cases.

In the first part of the solution, we get rid of the cases with $m \leq 10$ or $m \geq 14$. Let l_1, \dots, l_5 and w_1, \dots, w_5 be the lengths and widths of the five rectangles. Then the rearrangement inequality yields the lower bound

$$\begin{aligned} & l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 \\ &= \frac{1}{2} (l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 + w_1l_1 + w_2l_2 + w_3l_3 + w_4l_4 + w_5l_5) \\ &\geq \frac{1}{2} (1 \cdot 10 + 2 \cdot 9 + 3 \cdot 8 + \cdots + 8 \cdot 3 + 9 \cdot 2 + 10 \cdot 1) = 110 \end{aligned}$$

and the upper bound

$$\begin{aligned} & l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 \\ &= \frac{1}{2} (l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 + w_1l_1 + w_2l_2 + w_3l_3 + w_4l_4 + w_5l_5) \\ &\leq \frac{1}{2} (1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + \cdots + 8 \cdot 8 + 9 \cdot 9 + 10 \cdot 10) = 192.5 \end{aligned}$$

As the area of the square is sandwiched between 110 and 192.5, the only possible candidates for m are 11, 12, and 13.

In the second part of the solution, we show that a dissection of the square into five rectangles must consist of a single inner rectangle and four outer rectangles that each cover one of the four corners of the square. Indeed, if one of the sides the square had three rectangles adjacent to it, removing these three rectangles would leave a polygon with eight vertices, which is clearly

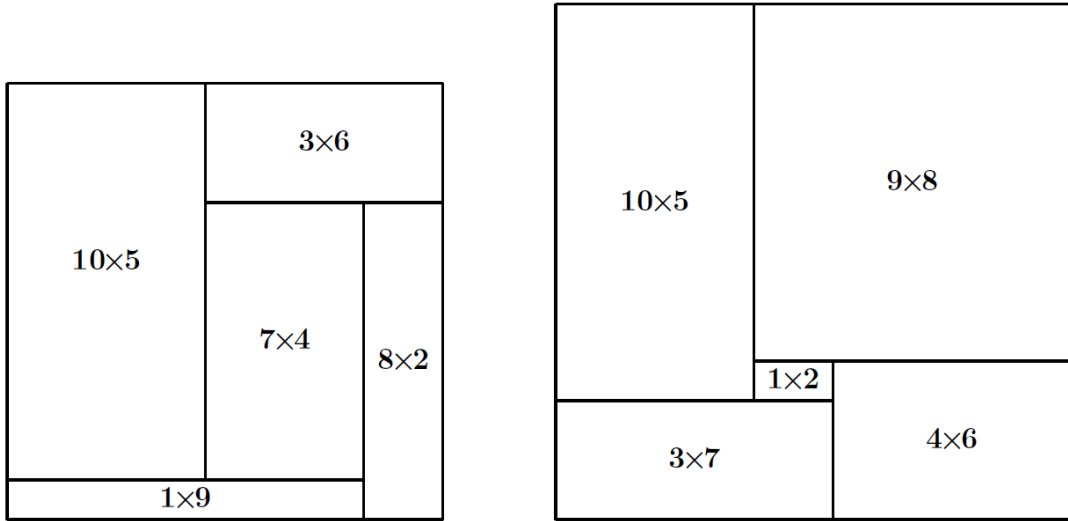
not the union of two rectangles. Moreover, since $m > 10$, each side of the square has at least two adjacent rectangles. Hence each side of the square has precisely two adjacent rectangles, and thus the only way of partitioning the square into five rectangles is to have a single inner rectangle and four outer rectangles each covering the four corners of the square, as claimed.

Let us now show that a square of size 12×12 cannot be dissected in the desired way. Let R_1, R_2, R_3 and R_4 be the outer rectangles (in clockwise orientation along the boundary of the square). If an outer rectangle has a side of length s , then some adjacent outer rectangle must have a side of length $12s$. Therefore, neither of $s = 1$ or $s = 6$ can be sidelengths of an outer rectangle, so the inner rectangle must have dimensions 1×6 . One of the outer rectangles (say R_1) must have dimensions $10 \times x$, and an adjacent rectangle (say R_2) must thus have dimensions $2 \times y$. Rectangle R_3 then has dimensions $(12y) \times z$, and rectangle R_4 has dimensions $(12 - z) \times (12 - x)$. Note that exactly one of the three numbers x, y, z is even (and equals 4 or 8), while the other two numbers are odd. Now, the total area of all five rectangles is

$$144 = 6 + 10x + 2y + (12 - y)z + (12 - z)(12 - x),$$

which simplifies to $(y - x)(z - 2) = 6$. As exactly one of the three numbers x, y, z is even, the factors $y - x$ and $z - 2$ are either both even or both odd, so their product cannot equal 6, and thus there is no solution with $m = 12$.

Finally, we handle the cases $m = 11$ and $m = 13$, which indeed are solutions. The corresponding rectangle sets are $10 \times 5, 1 \times 9, 8 \times 2, 7 \times 4$ and 3×6 for $m = 11$, and $10 \times 5, 9 \times 8, 4 \times 6, 3 \times 7$ and 1×2 for $m = 13$. These sets can be found by trial and error. The corresponding partitions are shown in the figure below.



Remark: The configurations for $m = 11$ and $m = 13$ given above are not unique.

3. We consider two cases: $a \geq 1$ and $a < 1$. For $a \geq 1$, we have:

$$(a^n + 1)^{\sqrt[n]{a^n + 1}} \geq (a^n + 1)a = a^{n+1} + a \geq a^{n+1} + 1$$

Taking both sides to the n th power gives us the desired inequality for $a \geq 1$. For $a < 1$, noting that $a^n + 1 > a^{n+1} + 1$, we have:

$$(a^n + 1)^{n+1} > (a^{n+1} + 1)^{n+1} > (a^{n+1} + 1)^n$$

which yields the desired inequality.

4. *EGMO 2013, Problem 4*

Denote the three consecutive integers by $x - 1$, x , and $x + 1$, so that

$$(x - 1)^5 + a \equiv 0 \pmod{b}, \quad x^5 + a \equiv 0 \pmod{b}, \quad (x + 1)^5 + a \equiv 0 \pmod{b}. \quad (1)$$

By computing the differences of the equations in (1) we get

$$A := (x + 1)^5 - (x - 1)^5 = 10x^4 + 20x^2 + 2 \equiv 0 \pmod{b}, \quad (2)$$

$$B := (x + 1)^5 - x^5 = 5x^4 + 10x^3 + 10x^2 + 5x + 1 \equiv 0 \pmod{b}. \quad (3)$$

Adding the first and third equation in (1) and subtracting twice the second equation yields

$$C := (x + 1)^5 + (x - 1)^5 - 2x^5 = 20x^3 + 10x \equiv 0 \pmod{b}. \quad (4)$$

Next, (2) and (4) together yield

$$D := 4xA - (2x^2 + 3)C = -22x \equiv 0 \pmod{b}. \quad (5)$$

Finally we combine (3) and (5) to derive

$$22B + (5x^3 + 10x^2 + 10x + 5)D = 22 \equiv 0 \pmod{b}.$$

As the positive integer b divides 22, we are left with the four cases $b = 1, b = 2, b = 11$ and $b = 22$.

If b is even (i.e. $b = 2$ or $b = 22$), then we get a contradiction from (3), because the integer $B = 2(5x^3 + 5x^2) + 5(x^4 + x) + 1$ is odd, and hence not divisible by any even integer.

For $b = 1$, it is trivial to see that a polynomial of the form $P(n) = n^5 + a$, with a any positive integer, has the desired property.

For $b = 11$, we note that

$$\begin{aligned} n &\equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{11} \\ \implies n^5 &\equiv 0, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1 \pmod{11} \end{aligned}$$

Hence a polynomial of the form $P(n) = (n^5 + a)/11$ has the desired property if and only if $a \equiv \pm 1 \pmod{11}$. This completes the proof.

5. Meh...