

Stellenbosch Camp December 2017
Intermediate Test 5
Solutions

1. Find all pairs (m, n) of positive integers which satisfy the equation

$$mn^2 = 100(n + 1).$$

Note that n and $n + 1$ differ by 1, and so are coprime. Therefore n^2 must divide 100. Whichever value n takes, m can be found as $\frac{(n+1) \cdot 100}{n^2}$. So (m, n) can take only the following solutions: $(200, 1)$; $(75, 2)$; $(24, 5)$; $(11, 10)$.

2. Point M and N are chosen on the sides BC and CD of a square $ABCD$, respectively, so that $\angle MAN = 45^\circ$. Points P and Q are the intersections of the diagonal BD with AM and AN , respectively. Prove that P and Q lie on the circle with diameter MN .

$\angle PAN = \angle PDN = 45^\circ$ (diagonal BD), so $APND$ is a cyclic quadrilateral, since the angles subtended by chord PN are equal. Thus $\angle ANP = \angle PDA = 45^\circ$ (AP common chord), and $\angle APN = 90^\circ$ ($\triangle APN$). Then $\angle MPN = 90^\circ$ (straight line), so P is subtended by diameter MN in circle MNP . By symmetry, the same must apply for point Q , using the cyclic quadrilateral $BMQA$. Therefore P and Q both lie on the circle with diameter MN .

3. Consider an 8-by-8 chessboard with 17 pieces placed on it in separate squares. Prove that there are three of these pieces which lie in three different rows and three different columns.

We consider partitioning the chessboard into 8 diagonal sets. Specifically, if we label some two squares as (i_1, j_1) and (i_2, j_2) , then these two squares are in the same set if and only if $i_1 - j_1 \equiv i_2 - j_2 \pmod{8}$. Since this divides the chessboard into 8 sets of 8 squares each, by pigeonhole principle, this implies one such set contains at least 3 pieces, since $17 = 8 \cdot 2 + 1$. As no two pieces in the same set lie in the same row or column, this completes the proof.

4. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x + f(f(y))) = y + f(f(x)).$$

Solution 1: First, let us prove that f is injective. Assume $f(x) = f(y)$. Applying f on either side, adding x and then applying f again gives: $f(x + f(f(x))) = f(x + f(f(y)))$. Applying the condition in the problem statement to both sides then gives:

$$x + f(f(x)) = f(x + f(f(x))) = f(x + f(f(y))) = y + f(f(x))$$

and so $x + f(f(x)) = y + f(f(x))$ which implies $x = y$, and so we have f injective. f is also surjective since fixing x in $f(x + f(f(y)))$, we can reach all $y \in \mathbb{R}$. So it is bijective.

Now letting $x = y = 0$, we have: $f(f(f(0))) = f(f(0))$ which from bijectivity (i.e. we can cancel f 's), implies that $f(0) = 0$. Letting $y = 0$, we get: $f(x + f(f(0))) = f(f(x))$. Using that $f(0) = 0$, this simplifies to $f(x) = f(f(x))$. Finally, from bijectivity, we get $x = f(x)$. All that remains is to check this is indeed a solution. Subbing $f(x) = x$ into the original equation gives:

$$f(x + f(y)) = f(x + y) = x + y = y + x = y + f(x) = y + f(f(x))$$

and so $f(x) = x$ is a solution.

Solution 2: From the problem statement we have:

$$f(x + f(f(y))) = y + f(f(x)) \quad (1)$$

Swapping x and y around, we get:

$$f(y + f(f(x))) = x + f(f(y)) \quad (2)$$

Noticing that the right-hand side of (2) is the same as the left-hand side argument in (1), we take f 's on both sides and apply (1):

$$f(f(y + f(f(x)))) = f(x + f(f(y))) = y + f(f(x))$$

Since $z = y + f(f(x))$ may take on any \mathbb{R} value, the above implies that $f(f(z)) = z$, $\forall z \in \mathbb{R}$. Then, taking $x = 0$ in (1) and using $f(f(z)) = z$, we get:

$$f(y) = f(f(f(y))) = f(0 + f(f(y))) = y + f(f(0)) = y + 0 = y$$

and so once again, $f(y) = y$ is a candidate for a solution. The check follows as in the previous solution.

5. Find all primes p such that $\frac{2^{p-1} - 1}{p}$ is the square of an integer.

Note, p cannot be 2 since $(2^{2-1} - 1)/2 = 1/2$ is not an integer. Therefore, we must have p odd. In order for $\frac{2^{p-1}-1}{p}$ to be a square, we must also have that $2^{p-1} - 1 = pa^2$ for some $a \in \mathbb{N}$.

We can factorize the left-hand side as a difference of squares: $2^{p-1} - 1 = (2^{\frac{p-1}{2}} + 1)(2^{\frac{p-1}{2}} - 1)$. $2^{\frac{p-1}{2}} + 1$ and $2^{\frac{p-1}{2}} - 1$ are relatively prime since their gcd must divide their difference (i.e. 2) but they are both odd (due to $p \geq 3$). So, in fact, we must have that one bracket is a square and the other is a square times p . For $k, q \in \mathbb{N}$, we have two cases:

Case 1: $2^{\frac{p-1}{2}} + 1 = k^2$, $2^{\frac{p-1}{2}} - 1 = pq^2$

We can write $2^{\frac{p-1}{2}} = k^2 - 1 = (k-1)(k+1)$. Both $k-1$ and $k+1$ must then be powers of 2 which differ by 2. The only such numbers are 2 and 4. This sets $k = 3$ and further implies that $\frac{p-1}{2} = 3$ and so $p = 7$. Check: $2^{7-1} - 1 = 64 - 1 = 7 \cdot 3^2$ and so $\frac{2^{7-1}-1}{7}$ is indeed a square.

Case 2: $2^{\frac{p-1}{2}} - 1 = k^2$, $2^{\frac{p-1}{2}} + 1 = pq^2$ Let's look at $2^{\frac{p-1}{2}} - 1 = k^2$ modulo 4. All powers of 2 are 0 (mod 4) apart from $2^0 = 1 \equiv 1$ and $2^1 = 2 \equiv 2$ (mod 4). Since, squares may only be 0 or 1 (mod 4), we must have $\frac{p-1}{2} = 0$ or 1 in order for $2^{\frac{p-1}{2}} - 1 \equiv 0$ or 1 (mod 4) respectively. We cannot have $\frac{p-1}{2} = 0$ since then $p = 1$ which is not prime. However, $\frac{p-1}{2} = 1$ gives $p = 3$ which is prime. Check: $2^{3-1} - 1 = 2^2 - 1 = 3 \cdot 1^2$ and so $\frac{2^{3-1}-1}{3}$ is indeed a square.

Therefore, there are two solutions, namely $p = 3$ and $p = 7$.

6. Let $ABCD$ be a convex quadrilateral with no parallel sides. Make a parallelogram on each two consecutive sides. Show that among these 4 new points, there is only one point inside quadrilateral $ABCD$.

Let $\angle A, \angle B, \angle C$ and $\angle D$ denote the interior angles of quadrilateral $ABCD$ at vertices A, B, C and D respectively, and let P_A denote the vertex opposite A in the parallelogram formed by D, A and B , and similarly for P_B, P_C and P_D .

Now for P_A to lie inside the quadrilateral $ABCD$, we must have that $\angle ABP_A < \angle ABC = \angle B$ and $\angle ADP_A < \angle ADC = \angle D$. However, since ABP_AD is a parallelogram, $\angle ABP_A = \angle ADP_A = 180^\circ - \angle A$ and so P_A lies inside the quadrilateral if and only if $\angle A + \angle B > 180^\circ$ and $\angle A + \angle D > 180^\circ$, and similarly for P_B, P_C and P_D .

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