Stellenbosch Camp December 2017
Intermediate Test 1
Solutions

1. The number of words with at least one a can be calculated by subtracting the number of words not containing an a from the total number of words that can be made. Since letters may be repeated, the total number of words possible is given by 26^5. Those not containing an a number 25^5. Thus there are $26^5 - 25^5$ words containing at least one a.

2. Let us assume for contradiction that 3 does not divide ab. Then neither a nor b can be divisible by 3, and hence $a^2 \equiv_3 1$ and $b^2 \equiv_3 1$. But then $8a^2 + 1 \equiv_3 0 \Rightarrow b^2 \equiv_3 0 \Rightarrow b \equiv_3 0$. This contradicts our initial assumption and so it must in fact be the case that $3 \mid ab$.

3. *Solution using AM-GM inequality*
   By the Arithmetic Mean-Geometric Mean inequality,

$$\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} \geq 3\sqrt[3]{\frac{a^2b^2c^2}{a^2b^2c^2}} = 3\sqrt[3]{1} = 3$$

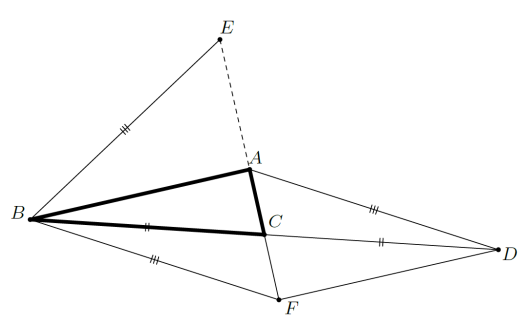
   Thus proving the inequality as required.

   Alternative solution using first principles
   Simplifying, the inequality becomes

$$\begin{aligned} \frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} &\geq 3 \\ \Leftrightarrow a^3 + b^3 + c^3 &\geq 3abc \\ \Leftrightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ac) &\geq 0 \end{aligned}$$

   Now $a + b + c \geq 0$, since a, b, c are positive real numbers. $a^2 + b^2 + c^2 - ab - bc - ac \geq 0 \Leftrightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$, which is true since squares are non-negative. Thus each bracket on the lefthand side is non-negative, hence the lefthand side is non-negative and the inequality is proved.

4. *EGMO 2013, Problem 1*
   Define F so that $ABFD$ is a parallelogram. Then E, A, C, F are collinear (as diagonals of a parallelogram bisect each other) and $BF = AD = BE$. Further, A is the midpoint of EF, since $AF = 2AC$, and thus AB is an altitude of the isosceles triangle EBF with apex B. Therefore $AB \perp AC$.



5. *EGMO 2013, Problem 2*
   The solution naturally divides into three different parts: we first obtain some bounds on m. We then describe the structure of possible dissections, and finally, we deal with the few remaining cases.

In the first part of the solution, we get rid of the cases with $m \leq 10$ or $m \geq 14$. Let $l_1, \ldots, l_5$ and $w_1, \ldots, w_5$ be the lengths and widths of the five rectangles. Then the rearrangement inequality yields the lower bound

$$\begin{aligned} & l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 \\ & \quad = \frac{1}{2}\left(l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 + w_1l_1 + w_2l_2 + w_3l_3 + w_4l_4 + w_5l_5\right) \\ & \quad \geq \frac{1}{2}\left(1 \cdot 10 + 2 \cdot 9 + 3 \cdot 8 + \cdots + 8 \cdot 3 + 9 \cdot 2 + 10 \cdot 1\right) = 110 \end{aligned}$$

and the upper bound

$$\begin{aligned} & l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 \\ & \quad = \frac{1}{2}\left(l_1w_1 + l_2w_2 + l_3w_3 + l_4w_4 + l_5w_5 + w_1l_1 + w_2l_2 + w_3l_3 + w_4l_4 + w_5l_5\right) \\ & \quad \leq \frac{1}{2}\left(1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + \cdots + 8 \cdot 8 + 9 \cdot 9 + 10 \cdot 10\right) = 192.5 \end{aligned}$$

As the area of the square is sandwiched between 110 and 192.5, the only possible candidates for m are 11, 12, and 13.

In the second part of the solution, we show that a dissection of the square into five rectangles must consist of a single inner rectangle and four outer rectangles that each cover one of the four corners of the square. Indeed, if one of the sides the square had three rectangles adjacent to it, removing these three rectangles would leave a polygon with eight vertices, which is clearly not the union of two rectangles. Moreover, since $m > 10$, each side of the square has at least two adjacent rectangles. Hence each side of the square has precisely two adjacent rectangles, and thus the only way of partitioning the square into five rectangles is to have a single inner rectangle and four outer rectangles each covering the four corners of the square, as claimed.

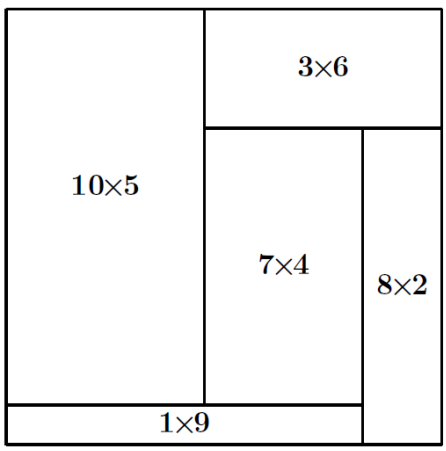
Let us now show that a square of size 12×12 cannot be dissected in the desired way. Let R_1, R_2, R_3 and R_4 be the outer rectangles (in clockwise orientation along the boundary of the square). If an outer rectangle has a side of length s, then some adjacent outer rectangle must have a side of length $12s$. Therefore, neither of $s = 1$ or $s = 6$ can be sidelengths of an outer rectangle, so the inner rectangle must have dimensions 1×6. One of the outer rectangles (say R_1) must have dimensions $10 \times x$, and an adjacent rectangle (say R_2) must thus have dimensions $2 \times y$. Rectangle R_3 then has dimensions $(12y) \times z$, and rectangle R_4 has dimensions $(12 - z) \times (12 - x)$. Note that exactly one of the three numbers x, y, z is even (and equals 4 or 8), while the other two numbers are odd. Now, the total area of all five rectangles is

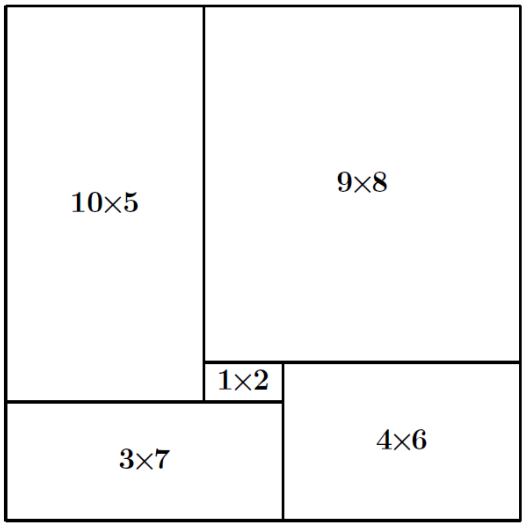
$$144 = 6 + 10x + 2y + (12 - y)z + (12 - z)(12 - x),$$

which simplifies to $(y - x)(z - 2) = 6$. As exactly one of the three numbers x, y, z is even, the factors $y - x$ and $z - 2$ are either both even or both odd, so their product cannot equal 6, and thus there is no solution with $m = 12$.

Finally, we handle the cases $m = 11$ and $m = 13$, which indeed are solutions. The corresponding rectangle sets are $10 \times 5, 1 \times 9, 8 \times 2, 7 \times 4$ and 3×6 for $m = 11$, and $10 \times 5, 9 \times 8, 4 \times 6, 3 \times 7$ and 1×2 for $m = 13$. These sets can be found by trial and error. The corresponding partitions are shown in the figure below.

Remark: The configurations for $m = 11$ and $m = 13$ given above are not unique.


3×6
10×5
7×4
8×2
1×9


10×5
9×8
1×2
3×7
4×6