Intermediate Test 5

Stellenbosch Camp 2017

Time: 4 hours

1. Find all pairs (m, n) of positive integers which satisfy the equation

$$mn^2 = 100(n + 1).$$

2. Point M and N are chosen on the sides BC and CD of a square $ABCD$, respectively, so that $\angle MAN = 45^\circ$. Points P and Q are the intersections of the diagonal BD with AM and AN, respectively. Prove that P and Q lie on the circle with diameter MN.

3. Consider an 8-by-8 chessboard with 17 pieces placed on it in separate squares. Prove that there are three of these pieces which lie in three different rows and three different columns.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x + f(f(y))) = y + f(f(x)).$$

5. Find all primes p such that $\dfrac{2^{p-1} - 1}{p}$ is the square of an integer.

6. Let $ABCD$ be a convex quadrilateral with no parallel sides. Make a parallelogram on each two consecutive sides. Show that among these 4 new points, there is only one point inside quadrilateral $ABCD$.