2017 Stellenbosch Mathematics Camp
Senior Algebra Problem Set

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1 Polynomials

1. Factorise the following polynomials:
   (a) $a^n - b^n \quad$ where $n \in \mathbb{N}$
   (b) $a^{2n+1} + b^{2n+1} \quad$ where $n \in \mathbb{N}$
   (c) $a^4 + 4b^4$
   (d) $a^3 + b^3 + c^3 - 3abc$
   (e) $x^{10} + x^5 + 1$
   (f) $x^{10} + x^8 + x^6 + x^4 + x^2 + 1$
   (g) $x^5 + x + 1$

2. (a) Find a polynomial with integral coefficients, such that a root of the polynomial is $\sqrt{2} + \sqrt{3} + \sqrt{5}$.
   (b) Find a polynomial with integral coefficients, such that a root of the polynomial is $\sqrt[3]{2} + \sqrt[3]{3} + \sqrt[3]{5}$

3. Prove that if coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then the roots of the equation cannot be rational numbers.

4. Let a, b, c be three non-zero integers. It is known that the sums $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{b}{a} + \frac{c}{b} + \frac{a}{c}$ are integers. Find these sums.

5. Let P be a cubic monic polynomial with roots a, b, and c. If $P(1) = 91$ and $P(-1) = -121$, compute the maximum possible value of
$$\frac{ab + bc + ca}{abc + a + b + c}.$$

6. Let $p(x) = a_n x^n + ... + a_0$ be a polynomial with integer coefficients. If a_0, a_n, and $p(1)$ are odd, prove $p(x)$ has no rational roots.

7. A polynomial P has integer coefficients with $P(3) = 4$ and $P(4) = 3$.
   (a) Prove there are no *integer* solutions to $P(x) = x$.
   (b) What about non-integer solutions? How many can there be?

8. Let $p(x)$ be a polynomial with integer coefficients with $\deg(p) > 1$. If all its n roots (not necessarily all distinct) lie in the interval $(0, 1)$, prove that its leading coefficient is larger than 2^n.

9. Find all polynomials $P \in \mathbb{R}[x]$ such that $P(4x^3 + 3x) = P(x)P(2x)^2$ for all $x \in \mathbb{R}$.

10. Let $p(x)$ be a monic fourth degree polynomial with integer coefficients such that $p(p(1)) = 21$, $p(-1) = 0$, and $|p(x)| \geq 85$ for all $|x| \geq 3$. Find $p(6)$.

11. Let c be any rational number. Prove that the polynomial $p(x) = x^3 - 3cx^2 - 3x + c$ has at most one rational root.

12. Let $n \in \mathbb{N}$. How many pairs of $P(x), Q(x)$ with $P(x), Q(x) \in \mathbb{R}[x]$ and $\deg(P) > \deg(Q)$ satisfy $[P(x)]^2 + [Q(x)]^2 = x^{2n} + 1$?

2 Inequalities

1. Let a, b, c be positive real numbers. Show that:

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 3$$

2. If x and y are positive reals, prove that

$$x^2\sqrt{\frac{x}{y}} + y^2\sqrt{\frac{y}{x}} \geq x^2 + y^2$$

3. Let x, y, z be non-negative numbers such that $x + y + z \leq 3$. Prove that

$$\frac{2}{1+x} + \frac{2}{1+y} + \frac{2}{1+z} \geq 3$$

4. For any positive real numbers a, b, c show that the following inequality holds

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

5. Let $a \in \mathbb{R}^+$, $n \in \mathbb{N}$. Prove that:

$$(a^n + 1)^{n+1} \geq (a^{n+1} + 1)^n$$

6. Prove that for all real $a, b, c \geq 0$,

$$(a + b + c)^5 \geq 81abc(a^2 + b^2 + c^2)$$

7. Let a, b, c be three positive real numbers. Show that

$$\frac{a+b+3c}{3a+3b+2c} + \frac{a+3b+c}{3a+2b+3c} + \frac{3a+b+c}{2a+3b+3c} \geq \frac{15}{8}$$

8. Let a, b, c be the sides of a triangle with $abc = 1$. Find the maximum area of the triangle.

9. Let a, b, c be real numbers such that $0 < a \leq b \leq c$. Prove that:

$$(a + b)(c + a)^2 \geq 6abc$$

10. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that:

$$a + b + c - 2abc \leq \sqrt{2}$$

11. Prove that in any acute-angled triangle ABC we have:

$$\frac{a+b}{\cos C} + \frac{b+c}{\cos A} + \frac{c+a}{\cos B} \geq 4(a+b+c)$$

12. Let $a, b, c > 0$ such that $abc = 1$. Prove that :

$$\frac{ab}{a^2 + b^2 + \sqrt{c}} + \frac{bc}{b^2 + c^2 + \sqrt{a}} + \frac{ca}{c^2 + a^2 + \sqrt{b}} \leq 1$$

13. Let $0 < a < b$ and $x_i \in [a, b]$.Prove that

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

14. Let a, b, c be positive real numbers. Prove that:

$$\frac{a^2}{\sqrt{bc}} + \frac{b^2}{\sqrt{ca}} + \frac{c^2}{\sqrt{ab}} + a + b + c \geq 2\left(\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} + \sqrt{c^2 - ca + a^2}\right)$$

3 Functional Equations

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = x^2 - f(y^2)$$

for all $x, y \in \mathbb{R}$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - y) = f(x) + f(y) - 2xy$$

for all $x, y \in \mathbb{R}$.

3. Determine whether there exists an injective function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) - [f(x)]^2 \geq \frac{1}{4}$$

for all $x \in \mathbb{R}$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) - f(xy) = x + y$$

for all $x, y \in \mathbb{R}$

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(y + z) + f(z + x)) \geq 3.f(x + 2y + 3z)$$

for all $x, y, z \in \mathbb{R}$

6. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f\left(\frac{x+1}{x-2}\right)+2f\left(\frac{x-2}{x+1}\right)=x$$

for all $x\in\mathbb{R}, x\neq -1, x\neq 2$

7. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$xf(y)-yf(x)=f\left(\frac{y}{x}\right)$$

for all $x\neq 0, y\in\mathbb{R}$

8. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(xy)\leq xf(y)$$

for all $x,y\in\mathbb{R}$.

9. Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that:

$$f(xy)=f(x)f(y)-f(x+y)+1$$

for all $x,y,\in\mathbb{Q}$ and $f(1)=2$

10. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$(x-1)f\left(\frac{x+1}{x-1}\right)-f(x)=x$$

for all $x\in\mathbb{R}, x\neq 1$.

11. Find all functions : $f:\mathbb{R}^+\to\mathbb{R}^+$ such that:

$$f(1+xf(y))=yf(x+y)$$

for all $x,y\in\mathbb{R}^+$

12. An integer n is called *interesting* if there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)-f(x+y)=y^n\quad\forall x,y\in\mathbb{R}$$

An integer n is called *beautiful* if there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)-f(x+y)\leq y^n\quad\forall x,y\in\mathbb{R}$$

Find all *interesting* numbers and *beautiful* numbers.

13. Let $f:\mathbb{N}\to\mathbb{N}$ be a function such that, for any $w,x,y,z\in\mathbb{N}$,

$$f(f(f(z)))f(wxf(yf(z)))=z^2f(xf(y))f(w).$$

Show that $f(n!)\geq n!$ for every positive integer n.

14. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^{2017}+(f(y))^{2017})=(f(x))^{2017}+y^{2017}$$

for all $x,y\in\mathbb{R}$

15. Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 1$$

for all $n \in \mathbb{N}$.

16. Given $k > 0$ with k an *odd* integer, find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + k$$

for all $n \in \mathbb{N}$.

17. Show that for all integers $a, b > 1$ there is a function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $f(a \cdot f(n)) = b \cdot n$ for all positive integer n.

18. Given a fixed $a > 0$, find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(a) = 1$ and

$$f(x)f(y) + f\left(\frac{a}{x}\right) f\left(\frac{a}{y}\right) = 2f(xy)$$

for all positive reals x, y.

19. Suppose $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a function that satisfies $f(1) = 1$ and

$$f(n+1) = \begin{cases} f(n) + 2 & \text{if } n = f(f(n) - n + 1), \\ f(n) + 1 & \text{otherwise} \end{cases}$$

   (a) Prove that $f(f(n) - n + 1)$ is either n or $n + 1$.
   (b) Determine f.

20. Find all $f : \mathbb{R} \to \mathbb{R}$ with f continuous, such that:

$$f(x+y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}$$

for all $x, y \in \mathbb{R}$

21. Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$x(f(x+1) - f(x)) = f(x),$$

for all $x \in \mathbb{R}$ and

$$|f(x) - f(y)| \leq |x - y|,$$

for all $x, y \in \mathbb{R}$.

22. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(x+y) \geq (f(x))^2 + xy$$

for any $x, y \in \mathbb{R}$