Stellenbosch Camp December 2017
Senior Test 5
Solutions

1. In order for $\frac{2^{p-1}-1}{p}$ to be a square, we must have that $2^{p-1}-1 = pa^2$ for some $a \in \mathbb{N}$. In fact, since we can factorize this as a difference of squares: $2^{p-1}-1 = (2^{\frac{p-1}{2}}+1)(2^{\frac{p-1}{2}}-1)$, we must have that one bracket is a square and the other is a square times p. For $k, q \in \mathbb{N}$, we have two cases:

   <u>Case 1:</u> $2^{\frac{p-1}{2}}+1 = k^2$, $2^{\frac{p-1}{2}}-1 = pq^2$

   We can write $2^{\frac{p-1}{2}} = k^2-1 = (k-1)(k+1)$. Both $k-1$ and $k+1$ must then be powers of 2 which differ by 2. The only such numbers are 2 and 4. This sets $k=3$ and further implies that $\frac{p-1}{2} = 3$ and so $p=7$. Check: $2^{7-1}-1 = 64-1 = 7 \cdot 3^2$ and so $\frac{2^{7-1}-1}{7}$ is indeed a square.

   <u>Case 2:</u> $2^{\frac{p-1}{2}}-1 = k^2$, $2^{\frac{p-1}{2}}+1 = pq^2$

   Let's look at $2^{\frac{p-1}{2}}-1 = k^2$ modulo 4. All powers of 2 are $0 \pmod 4$ apart from $2^0 = 1 \equiv 1$ and $2^1 = 2 \equiv 2 \pmod 4$. Since, squares may only be 0 or $1 \pmod 4$, we must have $\frac{p-1}{2} = 0$ or 1 in order for $2^{\frac{p-1}{2}}-1 \equiv 0$ or $1 \pmod 4$ respectively. We cannot have $\frac{p-1}{2} = 0$ since then $p=1$ which is not prime. However, $\frac{p-1}{2} = 1$ gives $p=3$ which is prime. Check: $2^{3-1}-1 = 2^2-1 = 3 \cdot 1^2$ and so $\frac{2^{3-1}-1}{3}$ is indeed a square.

   Therefore, there are two solutions, namely $p=3$ and $p=7$.

2. We consider partitioning the chessboard into 8 diagonal sets. Specifically, if we label some two squares as (i_1, j_1) and (i_2, j_2), then these two squares are in the same set if and only if $i_1 - j_1 \equiv i_2 - j_2 \pmod 8$. Since this divides the chessboard into 8 sets of 8 squares each, by pigeonhole principle, this implies one such set contains at least 3 pieces, since $17 = 8 \cdot 2 + 1$. As no two pieces in the same set lie in the same row or column, this completes the proof.

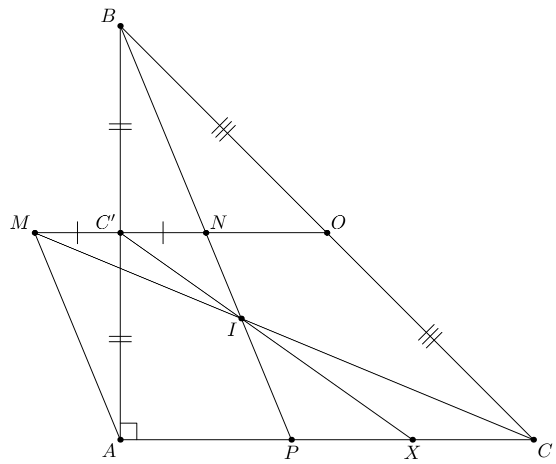
3. Let O be the midpoint of BC. As ABC is right-angled, we have O the centre of the circle through ABC. Let N be the intersection of BP and OC' and let M be the intersection of CI and OC'. Note that OC' is parallel to AC by the midpoint theorem. Hence $\angle OMC = \angle ACM = \angle OCM$ as CI is the angle bisector of $\angle ACB$. Hence, $OM = OC$ which implies O lies of the circumcircle of ABC. As $AMCB$ cyclic, we have $\angle AMC = \angle ABC = 45°$.

   Note that $\angle AMC' = \angle AMC + \angle CMO = 45° + 22.5° = 67.5°$. Also, $\angle BNC' = 90° - \angle C'BN = 67.5°$. Since $AC' = C'N$ this implies $\triangle AMC'$ and $\triangle BNC'$ congruent, hence C' is the midpoint of MN.

   The claim that X is the midpoint of PC then follows since $\triangle MIN$ is similar to $\triangle CIP$ and $C'IX$ collinear.

4. *Austrian Mathematical Olympiad 2011, Final Round, part 2, day 1*
   Since 3 pins (P) or 2 brackets (B) may not lie in a row, they may not do so on an individual brick. This means that there are only three different types of brick: PBPP (type A); PPBP (type B); BPB (type C). Naming the number of possible patterns of n bricks with a brick of type A at the end a_n, and analogously b_n and c_n for types B and C, the number we wish to determine is $s_n = a_n + b_n + c_n$. Due to the restrictions on the bricks, we see that $a_{n+1} = b_n + c_n$, $b_{n+1} = c_n$ and $c_{n+1} = a_n + b_n$ with starting values $a_1 = b_1 = c_1 = 1$. This yields:

$$s_{n+1} = s_n + (b_n + c_n) = s_n + (a_{n-1} + b_{n-1} + c_{n-1}) = s_n + s_{n-1}$$



with $s_1 = 3$ and $s_2 = 5$. We see that the resulting sequence s_n us simply the Fibonacci sequence starting from the fourth element, and so $s_n = F_{n+3}$.

5. We first prove that f is injective. Let $a, b \in \mathbb{N}$ such that $f(a) = f(b)$. Letting $w = x = y = 1$ and $z = a$ and $z = b$ for the LHS and RHS respectively, we obtain:

$$\begin{aligned} f(a) = f(b) &\implies f(f(f(a)))f(f(f(a))) = f(f(f(b)))f(f(f(b))) \\ &\implies a^2 f(f(1))f(1) = b^2 f(f(1))f(1) \\ &\implies a = b \end{aligned}$$

as $f(f(1))f(1) \in \mathbb{N}$. Hence f is injective. Setting $z = 1$ and $w = f(f(1))$, we obtain:

$$\begin{aligned} f(f(f(1)))f(f(f(1))xf(yf(1))) &= f(xf(y))f(f(f(1))) \\ \implies \quad f(f(f(1))xf(yf(1))) &= f(xf(y)) \\ \implies \quad f(f(1))xf(yf(1)) &= xf(y) \end{aligned}$$

where the last line is obtained due to injectivity. We now let $f(1) = c$ and set $x = 1$. The last equation therefore yields:

$$f(c)f(yc) = f(y)$$

For $y = 1$, we get: $f(c)f(c) = f(1) = c$. For $y = c$, we get: $f(c)f(c^2) = f(c)$ which implies $f(c^2) = 1$. For $y = c^2$, we get: $f(c)f(c^3) = f(c^2) = 1$ which implies $f(c) = f(c^3) = 1$ since the only product of two natural numbers which gives 1 is $1 \cdot 1$. By injectivity, we obtain: $c = c^3$, hence $f(1) = c = 1$.

Letting $z = y = 1$ in the original equation now gives us:

$$f(wx) = f(x)f(w)$$

Hence, f is multiplicative, which implies:

$$f(n!) = f(1)f(2)f(3)\dots f(n)$$

As f is injective, each of the factors on the right hand side must be a distinct natural number. Since the product of n distinct natural numbers is at least $n!$, we obtain $f(n!) \geq n!$ which completes the proof.

6. Let a_n be the number on the blackboard after the n^{th} step and let $i_n = a_n - a_{n-1}$ be the increase at the n^{th} step. We claim that for any n, either $i_n = 1$ or $a_n = 3n$. We shall prove this by induction.

   Note that we have $i_1 = i_2 = i_3 = i_4 = 1$ and $i_5 = 5 \neq 1$ but $a_5 = 15$. Hence the base case is proven. Let's assume that for some natural n we have $i_n \neq 1$ and $a_n = 3n$. Let k be the least natural number such that $i_{n+k} \neq 1$. Note that

$$i_{n+k} = \gcd(n+k, i_{n+k-1}) = \gcd(n+k, 3n+k-1) = \gcd(n+k, 2k+1)$$

   If $2k+1$ is prime, we hence obtain $i_{n+k} = 2k+1$ prime and $a_{n+k} = 3n+k-1+2k+1 = 3(n+k)$. This would prove the induction hypothesis, which would hence solve the problem.

   We are now only left to prove $2k+1$ is prime. Assume for contradiction $2k+1$ is not prime. Consider any prime divisor $p|\gcd(n+k, 2k+1)$. We have

$$p|2k+1, p \neq 2k+1 \quad \Longrightarrow \quad p \leq \frac{2k+1}{3} \quad \Longrightarrow \quad p < k$$

   Now consider

$$i_{n+k-p} = \gcd(n+k-p, 3n+k-p-1) = \gcd(n+k-p, 2k+1-2p)$$

   Now, since $p|n+k, 2k+1$, we have $p|\gcd(n+k-p, 2k+1-2p)$ which implies $i_{n+k-p} \neq 1$ contradicting the minimality of k.

   Hence $2k+1$ is prime, which completes the proof.

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