Stellenbosch Camp December 2017 Intermediate Test 4 Solutions

- 1. Since $72 = 8 \cdot 9$, the number $\overline{a2017b}$ must satisfy divisibility rules for both 8 and 9. For 8, we must have that the number formed by the last 3 digits, $\overline{17b}$, is a multiple of 8. The only multiple of 8 between 170 and 179 is 176, fixing b = 6. Then for 9, the sum of the digits must be a multiple of 9. Since 2 + 0 + 1 + 7 = 10, we must have a + b = 8 or 17 to bring the total up to 18 or 27 respectively. 17 6 = 11 which is greater than 9 and so we must have that a = 8 6 = 2.
- 2. For a catty number with n-digits, there are two options for each digit (each digit may be either a 2 or a 3). Therefore, the number of n-digit catty numbers is simply 2^n . The total number of catty numbers with $k \leq n$ digits is then $S_n = \sum_{k=1}^n 2^k = 2^1 + 2^2 + \cdots + 2^n$. To find a more compact expression for S_n , let us multiply S_n by 2 to get $2^2 + 2^3 + \cdots + 2^{n+1}$. Subtracting S_n from $2S_n$ then gives $S_n = (2^2 + 2^3 + \cdots + 2^{n+1}) (2^1 + 2^2 + \cdots + 2^n) = 2^{n+1} 2$. $S_{10} = 2^{10+1} 2 = 2046$ and so the 2047^{th} is the first catty number with 11 digits i.e. 222222222222. The 2048^{th} and 2049^{th} catty numbers are then 22222222222232 and 2222222222222232 respectively, leaving the 2050^{th} as 222222222233.
- 3. Applying Cauchy in Engel-form to the left-hand side, we have for $x, y, z \in \mathbb{R}^+ \cup \{0\}$:

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \ge \frac{(1+1+1)^2}{(1+x) + (1+y) + (1+z)} = \frac{3^2}{3 + (x+y+z)}$$

Using that $x + y + z \le 3$, we have further that:

$$\frac{3^2}{3 + (x + y + z)} \ge \frac{3^2}{3 + 3} = \frac{3}{2}$$

and so, as desired, we have:

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \ge \frac{3}{2}$$

- 3. Let O be the centre of the circle Γ , and let the angle bisector of $\angle AXC$ meet Γ at the points S on the arc AC, and the point T on the arc BD. Since M and N are the midpoints of the chords AB and CD respectively, we have that $OM \perp AB$ and $ON \perp CD$. Thus MXNO is a cyclic quadrilateral. We then have that $\angle MOX = \angle MNX$ (subtended by MX) = $\angle TXD$ (since $TX \parallel MN$) = $\angle MXT$ (angle bisector) = $\angle XMN$ (since $TX \parallel MN$) = $\angle XON$ (subtended by XN). Thus in triangles $\triangle MOX$ and $\triangle NOX$, we have that $\angle MOX = \angle XON$, $\angle XMO = \angle ONX = 90^{\circ}$, and OX is common, and so $\triangle MOX \equiv \triangle NOX$, giving us that OM = ON. Thus in triangles $\triangle OBM$ and $\triangle OCN$, we have $\angle OMB = \angle ONC = 90^{\circ}$, OM = ON, and OB = OC (radii). Thus $\triangle OBM \equiv \triangle OCM$, and so BM = CN, giving us that AB = 2MB = 2NC = CD.
- 5. Since we want $2^{m^2}-4$ to be a multiple of 7, we must have $2^{m^2}\equiv 4\pmod{7}$. To this end, let's look at powers of $2\pmod{7}$. Noticing that $2^3=8\equiv 1\pmod{7}$, we have that: $2^{3k}=(2^3)^k\equiv (1)^k=1\pmod{7}$, $2^{3k+1}=2(2^{3k})\equiv 2(1)=2\pmod{7}$, and $2^{3k+2}=2^2(2^{3k})\equiv 4(1)=4\pmod{7}$. We must therefore have that $m^2=3k+2$ for some $k\in\mathbb{N}_0$. However, $(3k)^2\equiv 0\pmod{3}$, $(3k+1)^2\equiv 1\pmod{3}$ and $(3k+2)^2\equiv 2^2\equiv 1\pmod{3}$ and so squares can only ever be congruent to 0 or 1 (and not 2) modulo 3. Therefore, there does not exist an $m\in\mathbb{N}_0$ such that $7|(2^{m^2}-4)$.