January Monthly Problem Set

Due: 15 January 2018

1. *JMMO 2017 Question 1*

   Let the smallest of the six consecutive integers be n. Then we have that

   $$\begin{aligned} 3p + 10 &= n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2 + (n+5)^2 \\ &= n^2 + (n^2 + 2n + 1) + (n^2 + 4n + 4) + (n^2 + 6n + 9) + (n^2 + 8n + 16) + (n^2 + 10n + 25) \\ &= 6n^2 + 30n + 55, \end{aligned}$$

   and so we have that $p = 2n^2 + 10n + 15$. We thus wish to show that $p - 7 = 2n^2 + 10n + 8$ is divisible by 36. Note that

   $$2n^2 + 10n + 8 = 2(n+1)(n+4).$$

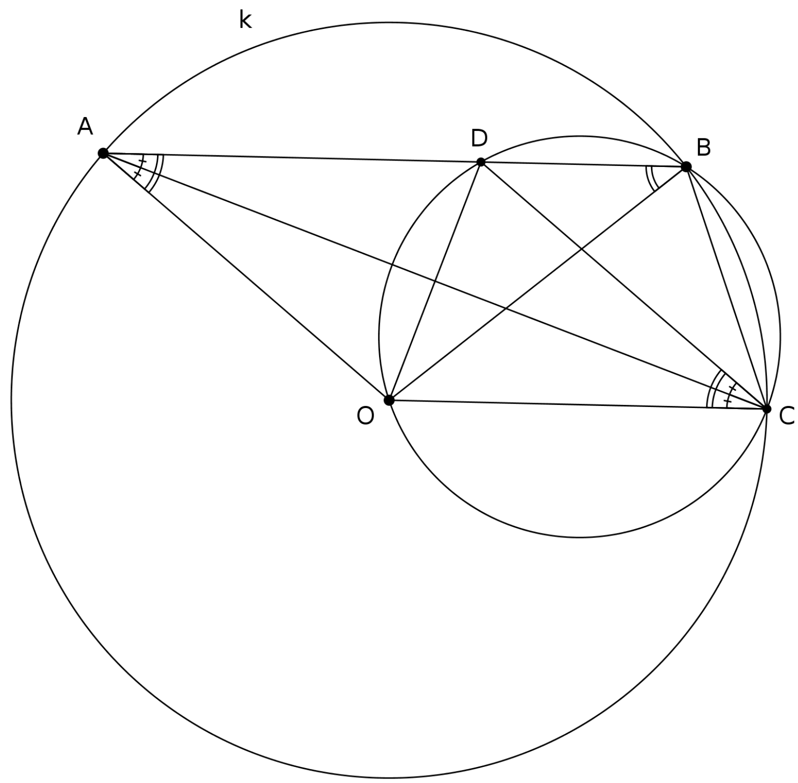
   Since exactly one of $(n+1)$ and $(n+4)$ is even, we have that $p-7$ is divisible by 4. To show that it is also divisible by 9, it is enough to show that each of $(n+1)$ and $(n+4)$ is divisible by 3, or, equivalently, that n is congruent to 2 modulo 3.

   Suppose that n is not congruent to 2 modulo 3. If n is divisible by 3, then clearly so is $p = 2n^2 + 10n + 15$, and hence we must have that $p = 3$. But then $3p + 10 = 19$ is not a sum of six consecutive squares, a contradiction. If, instead, $n \equiv 1 \pmod 3$, then we have that

   $$p = 2n^2 + 10n + 15 \equiv 2 \cdot 1 + 10 \cdot 1 + 15 \equiv 0 \pmod 3$$

   and so we again must have that $p = 3$, which is again a contradiction. Thus $n \equiv 2 \pmod 3$, as required.

2. Since $OC = OA$ (radii), we have that $\angle ACO = \angle OAC = \angle CAB$. Since $OA = OB$, we then have that $2\angle ACO = \angle OAB = \angle ABO$, and since $DBCO$ is a cyclic quadrilateral, this is equal to $\angle DCO$. It follows that $\angle DCA = \angle ACO$, and so in triangles $\triangle OAC$ and $\triangle DAC$, we have that $\angle OAC = \angle CAD$, $\angle DCA = \angle ACO$, and AC is common, and so $\triangle OAC \cong \triangle DAC$, giving us that $AD = AO$.



3. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$g(x) = f(x) - x^2 - ax - b,$$

where $a = f(1) - f(0) - 1$, and $b = f(0)$. Note that this gives us that $g(0) = g(1) = 0$.

The functional equation then becomes

$$\begin{aligned}
xg(x) &+ x^3 + ax^2 + bx - yg(y) - y^3 - ay^2 - by \\
&= (x-y)(g(x+y) + (x+y)^2 + ax + ay + b - xy) \\
&= (x-y)g(x+y) + a(x^2-y^2) + b(x-y) + (x-y)(x^2+xy+y^2) \\
&= (x-y)g(x+y) + a(x^2-y^2) + b(x-y) + x^3 - y^3,
\end{aligned}$$

which is equivalent to

$$xg(x) - yg(y) = (x-y)g(x+y) \tag{1}$$

for all real numbers x and y.

Letting $y = -x$, we get that

$$xg(x) + xg(-x) = 2xg(0) = 0$$

and so for $x \neq 0$, we have that $g(-x) = -g(x)$. Since $g(0) = 0$, this also holds for $x = 0$. This gives us that

$$xg(x) - yg(y) = xg(x) + yg(-y)$$

for all real numbers x and y.

Replacing y with $-y$, in equation 1, we find that

$$xg(x) + yg(-y) = (x+y)g(x-y).$$

We see that

$$(x+y)g(x-y) = (x-y)g(x+y)$$

for all real numbers x and y. Letting $x = y+1$, we see that

$$0 = (2y+1)g(1) = g(2y+1)$$

for all real numbers y, and so we see that g must be identically 0. It follows that

$$f(x) = x^2 + ax + b$$

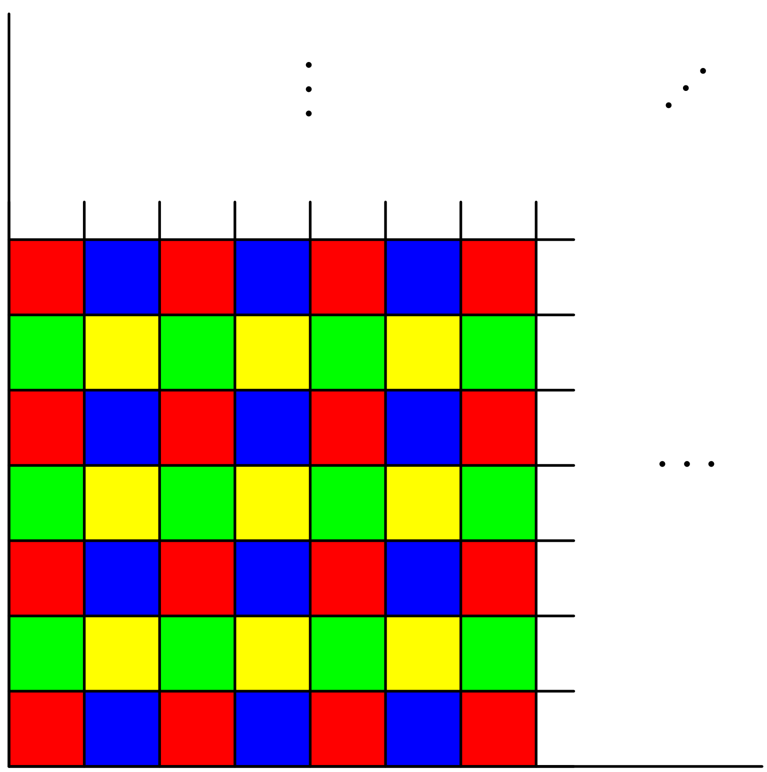
for all real numbers x.

All functions of this form do indeed satisfy the functional equation, since if $f(x) = x^2 + ax + b$, then we have that

$$\begin{aligned} xf(x) - yf(y) &= x^3 + ax^2 + bx - y^3 - ay^2 - by \\ &= (x^3 - y^3) + a(x^2 - y^2) + b(x-y) \\ &= (x-y)(x^2 + xy + y^2 + a(x+y) + b) \\ &= (x-y)((x+y)^2 - xy + a(x+y) + b) \\ &= (x-y)(f(x+y) - xy) \end{aligned}$$

as required.

4. Colour the board so that the odd numbered rows are coloured alternately red and blue, and the even numbered rows are coloured alternately green and yellow, as in the figure below.



Notice that after every two seconds, any cat that is in a red square will end up in a yellow square and vice-versa.

Suppose that no two cats are ever on the same square at the same moment. Then our previous observation implies that, at every point in time, the number of cats on red squares is at most equal to the number of yellow squares, as otherwise, by the pigeon-hole principle, two of these cats would end up on the same yellow square after two seconds.

We also note that after every second, any cat that is on a blue or green square will be on a red or yellow square. Of those, the number that will be on red squares is at most equal to the number of yellow squares, and so we conclude that, at any point in time, the number of cats on blue or green squares is at most twice the number of yellow squares.

Putting these observations together, the total number of cats on the board is at most equal to 4 times the number of yellow squares, which is equal to $4 \times 1008^2 = 2016^2$.

Thus if we have $2016^2 + 1$ cats on the board, then we are guaranteed that two of them will eventually find themselves on the same square at the same time. To see that this is the smallest number of cats necessary, it is enough to show that we can place 2016^2 cats on the board, and coordinate their moves so that no two of them are ever on the same square at the same time. To do this, place a cat on each of the squares on the lower left 2016×2016 sub-board. Then have all of the cats move up, then right, then down, and then left, and repeat this sequence of moves indefinitely.

5.

6. Let $y = 0$ in the functional equation. We obtain that

$$f(-1) + f(x)f(0) = -1.$$

If $f(0) \neq 0$, then this would imply that f is constant, but constant functions do not satisfy the functional equation. Thus we must have that $f(0) = 0$, from which we obtain that $f(-1) = -1$.

Substituting xy for x, and 1 for y in the functional equation, we get that

$$f(xy)f(1) = 2xy - 1 - f(xy - 1) = f(x)f(y). \tag{2}$$

Now let $x = y = 1$ in the functional equation. This gives us that $f(1)^2 = 1$, and so either $f(1) = 1$ or $f(1) = -1$. We consider the two cases separately.

Case 1: $f(1) = 1$

In this case, equation 2 gives us that $f(xy) = f(x)f(y)$.

Letting $y = 1$ in the functional equation, we see that

$$f(x - 1) + f(x) = 2x - 1. \tag{3}$$

Replacing x with $x + 1$ in equation 3, we also have that

$$f(x + 1) + f(x) = 2x + 1. \tag{4}$$

Adding equations 3 and 4, we find that

$$f(x + 1) + f(x - 1) = 4x - 2f(x)$$

while multiplying them gives us that

$$f(x+1)f(x-1)+f(x)(f(x-1)+f(x+1)+f(x)^2=4x^2-1. \tag{5}$$

We know that $f(x+1)f(x-1)=f(x^2-1)$, and the functional equation tells us that

$$f(x^2-1)+f(x)^2=2x^2-1.$$

Equation 5 thus becomes

$$f(x)(4x-2f(x))=2x^2$$

which simplifies to

$$(f(x)-x)^2=0.$$

Thus we have that $f(x)=x$ for all x. We check that this satisfies the functional equation. If $f(x)=x$ for all x, we have that

$$f(xy-1)+f(x)f(y)=xy-1+xy=2xy-1,$$

as required.

Case 2: $f(1)=-1$

In this case, equation 2 gives us that $f(xy)=-f(x)f(y)$.
Let $y=1$ in the functional equation. This gives us that

$$f(x-1)-f(x)=2x-1. \tag{6}$$

Replacing x with $x+1$ in equation 6, we get that

$$f(x+1)-f(x)=-(2x+1). \tag{7}$$

Adding equations 6 and 7 gives us that

$$f(x-1)+f(x+1)=2f(x)-2$$

while multiplying them gives us that

$$f(x-1)f(x+1)-f(x)(f(x-1)+f(x+1))+f(x)^2=1-4x^2. \tag{8}$$

We know that $f(x-1)f(x+1)=-f(x^2-1)$, and the functional equation tells us that

$$f(x^2-1)=2x^2-1-f(x)^2.$$

Equation 8 thus becomes

$$f(x)^2-2x^2+1-f(x)(2f(x)-2)+f(x)^2=1-4x^2,$$

which simplifies to

$$2f(x)=-2x^2.$$

Thus we have that $f(x)=-x^2$ for all real numbers x. We check that this satisfies the function equation. If $f(x)=-x^2$ for all x, we have that

$$f(xy-1)+f(x)f(y)=-(xy-1)^2+(-x^2)(-y^2)=-x^2y^2+2xy-1+x^2y^2=2xy-1,$$

as required.

All solutions to the functional equation are thus given by the functions defined by $f(x) = x$ for all real numbers x, and $f(x) = -x^2$ for all real numbers x.

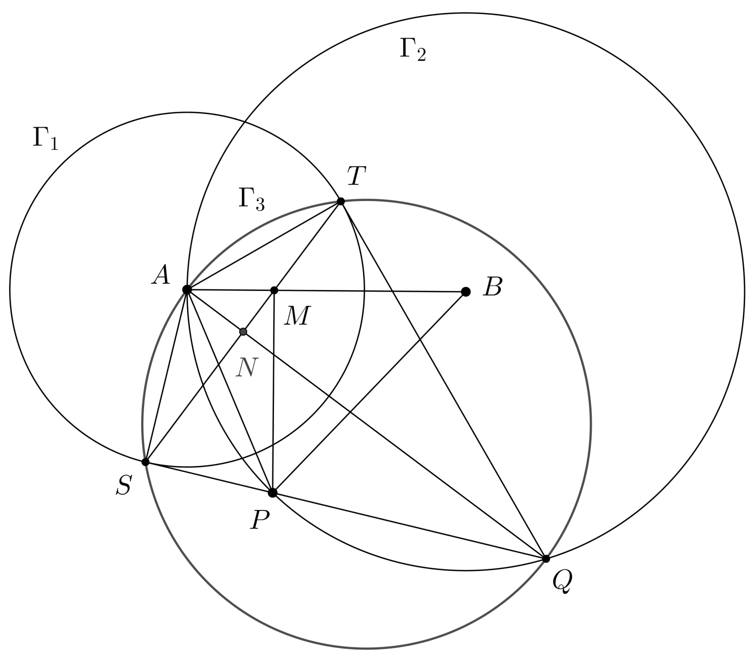
7. We use directed angles to avoid considering two separate cases depending on whether P lies between S and Q, or Q lies between S and P.

   We note that since $\angle QSA = \angle ATQ = 90°$, we have that $ASQT$ is cyclic, and so $\angle QST = \angle QAT$. Since triangles $\triangle TAQ$ and $\triangle SAQ$ have three corresponding sides equal, we have that $\triangle TAQ \cong \triangle SAQ$, and so

$$\angle QAT = \angle SAQ = 90° - \angle AQS = 90° - \frac{1}{2}\angle ABP = \angle PAB = \angle PAM.$$

   Since $\angle PSA = \angle AMP = 90°$, we have that $ASPM$ is cyclic, and so $\angle PAM = \angle PSM = \angle QSM$. We thus have that $\angle QST = \angle QSM$, and so T, M, and S are collinear, and so N is the intersection of the lines AQ and ST.

   Let Γ_3 be the circumcircle of $ASQT$. Then N lies on ST, which is the radical axis of Γ_1 and Γ_3, and N lies on AQ, which is the radical axis of Γ_2 and Γ_3. Thus N is the radical center of Γ_1, Γ_2, and Γ_3, and hence N lies on the radical axis of Γ_1 and Γ_2, which is a line that does not depend on P.



8. Suppose that n is a natural number with at most k digits. Let $m = 10^k - 1 - n$, and note that m is also a natural number with at most k digits.

   Since there are no carries involved in the substraction $10^k - 1 - n$, we have that $s(m) = s(10^k - 1 - n) = s(10^k - 1) - s(n) = 9k - s(n)$. (This is because each digit in $10^k - 1 - n$ is just the difference of the corresponding digits in $10^k - 1$ and in n.)

Similarly, since there are no carries involved in the subtraction $2(10^k - 1) - 2n$, we have that $s(2m) = s(2(10^k - 1) - 2n) = s(2(10^k - 1)) - s(2n) = 9k - s(2n)$.

We thus have that $s(m) - s(2m) = -(s(n) - s(2n))$, and so $s(n) < s(2n)$ if and only if $s(m) > s(2m)$, and $s(n) > s(2n)$ if and only if $s(m) < s(2m)$.

We see that there is a bijection between the set of natural numbers n with at most k digits such that $s(n) < s(2n)$, and the set of natural numbers n with at most k digits such that $s(n) > s(2n)$.

We now claim that, in fact, there is an equal number of natural numbers n with *exactly* k digits such that $s(n) < s(2n)$ as there are natural numbers n with *exactly* k digits such that $s(n) > s(2n)$.

Let $a(k)$ denote the number of natural numbers n with at most k digits such that $s(n) < s(2n)$ and let $b(k)$ denote the number of natural numbers n with at most k digits such that $s(n) > s(2n)$. Then the number of natural numbers with exactly k digits such that $s(n) < s(2n)$ is $a(k) - a(k-1)$. Similarly, the number of natural numbers n with exactly k digits such that $s(n) > s(2n)$ is $b(k) = b(k-1)$. But $a(k) = b(k)$ and $a(k-1) = b(k-1)$, and so these quantities are equal, as claimed.