Senior Test 2

Stellenbosch Camp 2017

Time: $2\frac{1}{2}$ hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = x^2 - f(y^2)$$

for all $x, y \in \mathbb{R}$.

2. A snake trader has 2017 ravenous snakes with different tail lengths which he wants to place in a line from west to east, with each snake facing either east or west. The snakes are well-trained and will stay still once placed, but each will eat any snake it can see that has a shorter tail than it! (Each snake can see all the snakes in the direction in which it is facing.) Can the snakes be lined up such that no snake can see another snake with a shorter tail length? If so, how many ways are there to do this?

3. Let ABC be a triangle with circumcentre O. The points D, E, F lie in the interiors of the sides BC, CA, AB respectively, such that DE is perpendicular to CO and DF is perpendicular to BO. (By interior we mean, for example, that the point D lies on the line BC and D is between B and C on that line.) Let K be the circumcentre of triangle AFE. Prove that the lines DK and BC are perpendicular.

4. Let P be a monic cubic polynomial with roots a, b and c. If $P(1) = 91$ and $P(-1) = -121$, compute the maximum possible value of

$$\frac{ab + bc + ca}{abc + a + b + c}.$$

5. The numbers p and q are prime and satisfy

$$\frac{p}{p+1} + \frac{q+1}{q} = \frac{2n}{n+2}$$

for some positive integer n. Find all possible values of $q - p$.