# February Monthly Problem Set Solutions

1. The given equation is equivalent to
$$p(a^2+b^2) = a^2b^2.$$
We note that $p \mid a^2b^2$, and so either $p \mid a$ or $p \mid b$. Without loss of generality, let $p \mid a$, so that $a = pk$ for some natural number $k$. The equation then becomes
$$a^2+b^2 = pk^2b^2.$$
We thus have that $b^2 \mid a^2+b^2$, and so $b^2 \mid a^2$, which gives us that $b \mid a$. Let $a = mb$ for some natural number $m$. The equation then becomes
$$p(m^2+1)b^2 = m^2b^4,$$
or equivalently
$$p(m^2+1) = m^2b^2.$$
Since $\gcd(m^2, m^2+1) = 1$, this implies that $m^2 \mid p$, and so $m = 1$. This gives us that $a = b$, and so the equation simplifies to
$$2p = b^2,$$
which implies that $a = b = p = 2$. This is indeed a solution since
$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}.$$

2. Since $\angle PRQ = \angle PSQ = 90°$, we have that $P$, $R$, $S$, and $Q$ lie on the circle with diameter $PQ$. Let $O$ be the midpoint of $PQ$, so that $O$ is the centre of this circle. Since $O$ lies on the perpendicular bisector of $AB$, we have that $OA = OB$, and so $A$ and $B$ have equal power with respect to this circle. It follows that $BP \times BS = AR \times AQ$. By applying Power of a Point in this circle to point $C$, we also have that $CQ \times CR = CP \times CS$.

   By Menelaus's Theorem applied to triangle $\triangle ABC$ and line $MPQ$, we have that
$$\frac{AM}{MB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$
   Using that
$$\frac{BP}{QA} = \frac{RA}{BS}$$
   and
$$\frac{CQ}{PC} = \frac{SC}{CR},$$
   we thus have that
$$\frac{AM}{MB} \cdot \frac{RA}{BS} \cdot \frac{SC}{CR} = -1.$$

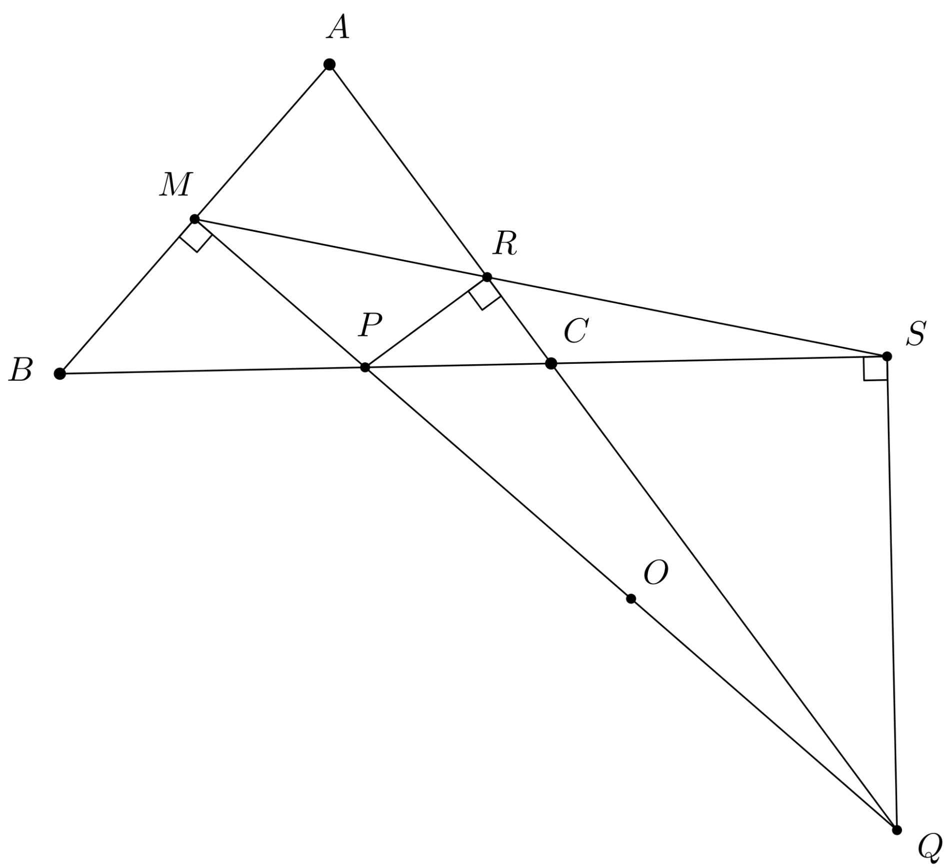
Since

$$\frac{AM}{MB} = 1$$

this is equivalent to

$$\frac{AM}{MB} \cdot \frac{BS}{SC} \cdot \frac{CR}{RA} = -1,$$

and so by Menelaus's Theorem, we have that $M$, $R$, and $S$ are collinear.




3. We will show by induction on $k$ that for $0 \leq k \leq n$, it is possible to use the allowed move to arrange the deck such that the cards from 1 to $k$ are in ascending order at the bottom of the deck. This clearly implies the desired result.

   The claim is vacuously true for $k = 0$. Now suppose that we have the cards from 1 to $k$ in ascending order at the bottom of the deck. Repeatedly draw two cards, and place either one at the bottom of the deck, until one of the two cards that we draw is $(k + 1)$. At this point, place $(k + 1)$ at the top of the deck, and the other card on the bottom. Now repeatedly draw two cards, and place $(k + 1)$ at the top of the deck and the other card at the bottom until the card that we place is the bottom is card number $k$. At this point, we have $(k+1)$ at the top of the deck, and the cards from 1 to $k$ in ascending order at the bottom of the deck. Now draw two cards, and place $(k + 1)$ at the bottom of the deck and the other card at the top so that we have the cards from 1 to $(k + 1)$ at the bottom of the deck in ascending order as desired.

4. Note that

$$(x^2 - 4x + 7)(x^2 + 4x + 7) = (x^2 + 7)^2 - 16x^2 = x^4 - 2x^2 + 49 = (x^2 - 1)^2 + 48.$$

Similarly,

$$(y^2 - 6y + 14)(y^2 + 6y + 14) = (y^2 - 4)^2 + 180,$$

and

$$(z^2 - 8z + 23)(z^2 + 8z + 23) = (z^2 - 9)^2 + 448.$$

Thus multiplying the given equations together, we obtain that

$$\left((x^2 - 1)^2 + 48\right)\left((y^2 - 4)^2 + 180\right)\left((z^2 - 9)^2 + 448\right) = 120 \cdot 336 \cdot 96.$$

Now the left hand side is greater than or equal to $48 \cdot 180 \cdot 448$, which conveniently happens to be equal to $120 \cdot 336 \cdot 96$. Since equality holds, we must have that

$$x^2 - 1 = y^2 - 4 = z^2 - 9 = 0.$$

If $x = 1$, then the equation

$$(x^2 - 4x + 7)(y^2 + 6y + 14) = 120$$

gives us that

$$y^2 + 6y + 14 = 30$$

and so $y = 2$. The equation

$$(y^2 - 6y + 14)(z^2 + 8z + 23) = 336$$

then gives us that

$$z^2 + 8z + 23 = 56$$

and of $z = 3$ and $z = -3$, only $z = 3$ satisfies this equation.

If, on the other hand, $x = -1$, then the equation

$$(x^2 - 4x + 7)(y^2 + 6y + 14) = 120$$

gives us that

$$y^2 + 6y + 14 = 10,$$

and neither $y = 2$ nor $y = -2$ satisfies this equation.

Thus the only solution to the given system of equations is given by $(x, y, z) = (1, 2, 3)$.

5. We will first show that there is no $k \leq 1000$ such that $1000 < a_k \leq 2000$. Suppose that $a_k = 1000 + m$ where $1 \leq m \leq 1000$. Then we have that $1000 + m = a_k \in A$, and so $a_{1000} + a_m \in A$. But $a_{1000} + a_m > a_{1000}$, which is a contradiction since $a_{1000}$ is the largest element of $A$.

   Note that at most 14 elements of $A$ can be elements of $\{2001, 2002, \ldots, 2014\}$, and so at least $1000 - 14 = 986$ are at most 1000.

   Let $k$ be the largest natural number such that $a_k \leq 1000$. Our previous observation gives us that $k \geq 986$, and so $1000 - k \leq 14$. We claim that $a_k = k$. Clearly we have that $a_k \geq k$ since the sequence is increasing. Suppose that $a_k > k$. Let $a_k = k + m$ where $1 \leq m \leq 1000 - k \leq 14$. Since $k + m = a_k \in A$, we have that $a_k + a_m \in A$. But $m \leq 14 < k$, so $a_m < a_k \leq 1000$, and thus $a_k + a_m < 1000 + 1000 = 2000$. Thus we must have that $a_k + a_m \leq 1000$, and so $a_k + a_m \leq a_k$, a contradiction. Thus $a_k = k$, and so $a_i = i$ for all $i < k$.

Thus the set $A$ must be of the form

$$\{1, 2, 3, \ldots, k\} \cup B$$

where $k = 1000 - |B|$, and $B$ is some subset of $\{2001, 2002, \ldots, 2014\}$.

We claim that all sets of this form do satisfy the condition in the problem. Suppose that $A$ is of the given form, and that $i$ and $j$ are such that $1 \leq i, j \leq 1000$, and $i + j \in A$. Since $i + j \leq 2000$, we must have that $i+j \leq k$. Thus $i \leq k$ and $j \leq k$, and so $a_i = i$ and $a_j = j$. Thus $a_i+a_j = i+j \in A$, as required.

It follows that the number of sets $A$ which satisfy the condition in the problem is equal to the number of subsets of $\{2001, 2002, \ldots, 2014\}$, which is equal to $2^{14}$.

6. By [insert your favourite inequality here], we know that

$$k(x_1^2 + x_2^2 + \cdots + x_k^2) \geq (x_1 + x_2 + \cdots + x_k)^2 = 10000.$$

We thus have that

$$(x_1^2 + 200) + (x_2^2 + 200) + \cdots + (x_k^2 + 200) \geq \frac{10000}{k} + 200k \geq 2\sqrt{2 \times 10^6} = 2000\sqrt{2},$$

where the last inequality follows from AM-GM. Since $\sqrt{2} > 1.414$, we have that

$$(x_1^2 + 200) + (x_2^2 + 200) + \cdots + (x_k^2 + 200) > 2828.$$

Now $x_i^2$ has the same parity as $x_i$, so we have that

$$(x_1^2 + 200) + (x_2^2 + 200) + \cdots + (x_k^2 + 200)$$

is an even integer, and hence is at least equal to 2830. Thus the inequality holds for $C = 2830$.

To see that 2830 is the best possible value for $C$, let

$$x_1 = x_2 = \cdots = x_5 = 14 \qquad \text{and} \qquad x_6 = x_7 = 15.$$

Then

$$x_1 + x_2 + \cdots + x_7 = 5 \times 14 + 2 \times 15 = 100,$$

and

$$(x_1^2 + 200) + (x_2^2 + 200) + \cdots + (x_7^2 + 200) = 5 \times 14^2 + 2 \times 15^2 + 7 \times 200 = 2830.$$

7. We make use of directed line segments.

Since $A$ and $B$ are an equal distance from $O$, they have equal power with respect to $\omega_1$, and so we have that $AP \times AQ = BP \times BQ$. Similarly, since $A$ and $C$ are an equal distance from $O$, they have equal power with respect to $\omega_2$, and so we have that $AR \times AS = CS \times CR$.

Applying Menelaus's Theorem to triangle $\triangle ABC$ and line $RQK$, we have that

$$\frac{AQ}{QB} \cdot \frac{BK}{KC} \cdot \frac{CR}{RA} = -1.$$

Applying Menelaus's Theorem to triangle $\triangle ABC$ and line $PSL$ gives us that

$$\frac{AP}{PB} \cdot \frac{BL}{LC} \cdot \frac{CS}{SA} = -1.$$

Multiplying these together gives us that

$$\frac{AP \times AQ}{PB \times QB} \cdot \frac{BL \times BK}{LC \times KC} \cdot \frac{CS \times CR}{SA \times RA} = 1.$$

But we know that

$$\frac{AP \times AQ}{PB \times QB} = \frac{CS \times CR}{SA \times RA} = 1.$$

and so we have that

$$BL \times BK = LC \times KC.$$

We thus have that

$$BK(BC + CL) = LC(KB + BC)$$

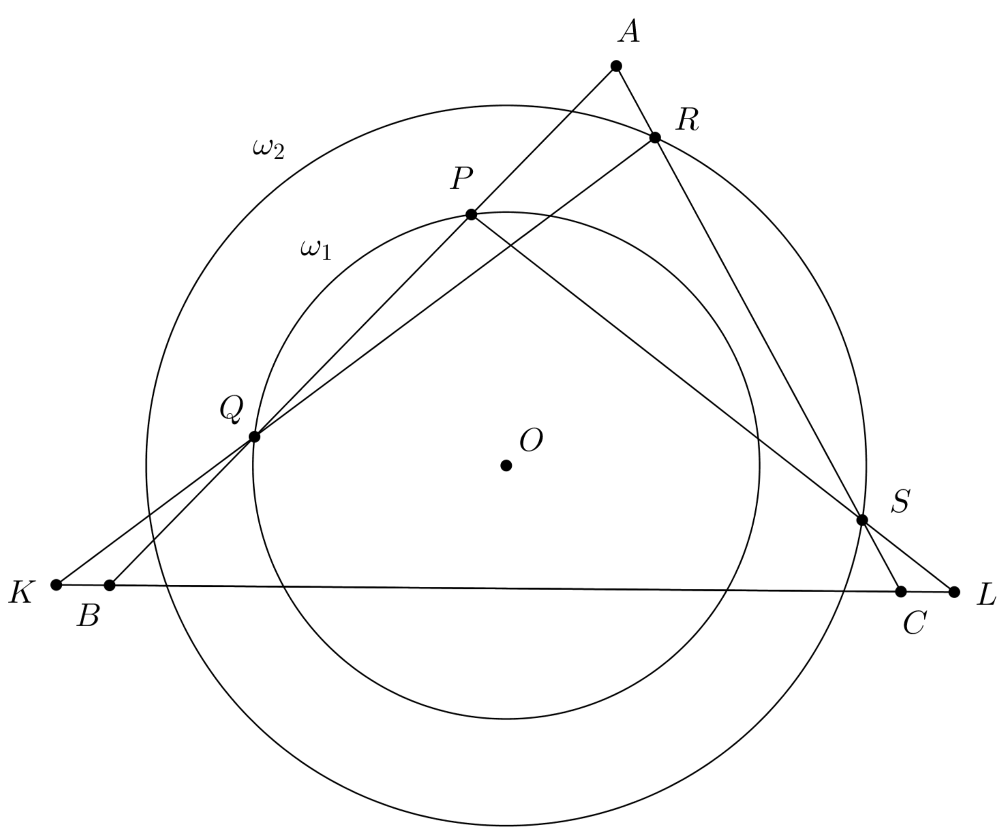
and so

$$BK \times BC + BK \times CL = BK \times CL + LC \times BC$$

giving us that

$$BK = LC$$

as desired.



8. Note that the function $f$ satisfies the condition if and only if the function $n \mapsto f(n) - f(0)$ does, and so we may assume that $f(0) = 0$.

   Also note that exchanging $a$ and $b$ does not alter the value of

$$\frac{f(b) - f(a)}{b - a},$$

   and so we may replace the condition that $a < b$ with $a \neq b$.

   Let $a = 0$. Then we have that for all $b \neq 0$ that

$$\frac{f(b)}{b}$$

   is a power of a prime number. Let $g$ be the function defined by $f(n) = n \cdot g(n)$ for all $n$, and $g(0) = 1$. Our previous observation then gives us that $g(n)$ is a power of a prime for all $n$.

   Suppose that there is some $k \neq 0$ such that $g(k) \neq 1$. Let $g(k) = p^c$ where $p$ is a prime number, and let $k = p^s \cdot t$ where $\gcd(p, t) = 1$.

   I claim that for all $m > c + s$, that we have that

$$g\left(p^m + k\right) = p^c.$$

   Indeed, the condition on $f$ and the definition of $g$ gives us that

$$\frac{\left(p^m + k\right) g\left(p^m + k\right) - k \cdot p^c}{p^m} = \frac{f\left(p^m + k\right) - f(k)}{\left(p^m + k\right) - k}$$

   is a power of a prime.

   Since $p^{c+s}$ divides both $p^m$ and $k \cdot p^c$, this gives us that

$$p^{c+s} \mid k \cdot g\left(p^m + k\right) = p^s \cdot t \cdot g\left(p^m + k\right),$$

   which implies that

$$p^c \mid t \cdot g\left(p^m + k\right),$$

   and since $\gcd(p, t) = 1$, this gives us that

$$p^c \mid g\left(p^m + k\right),$$

   and so

$$g\left(p^m + k\right) = p^{c+r}$$

   for some non-negative integer $r$. (Since $g$ is always a power of a prime.)

   We thus have that

$$\frac{p^{c+r}\left(p^m + k\right) - p^{s+c} \cdot t}{p^m}$$

   is a power of a prime.

   If $r > 0$, then we note that since $k$ and $p^m$ are both divisible by $p^s$, we have that

$$p^{c+r}\left(p^m + k\right)$$

is divisible by $p^{s+c+1}$, and since $\gcd(p,t)=1$, this implies that the largest power of $p$ which divides

$$p^{c+r}\left(p^m+k\right)-p^{s+c}\cdot t$$

is exactly $p^{s+c}$. But $p^m>p^{s+c}$, so

$$\frac{p^{c+r}\left(p^m+k\right)-p^{s+c}\cdot t}{p^m}$$

is not an integer, which is a contradiction. Thus we have that $r=0$, and so

$$g\left(p^m+k\right)=p^c.$$

We thus have that there are arbitrarily large values of $m$ such that $g(m)=p^c$.

Now consider any $n\neq 0$. For any $m$ such that $g(m)=p^c$, we have that

$$n-m\mid f(n)-f(m)=ng(n)-m\cdot p^c,$$

and so

$$m-n\mid ng(n)-m\cdot p^c+(m-n)\cdot p^c=n(g(n)-p^c).$$

Since this holds for arbitrarily large values of $m$, we must have that

$$n(g(n)-p^c)=0,$$

and so $g(n)=p^c$. We thus have that $f(n)=n\cdot p^c$ for all $n$, and we can check that this is a valid solution.

What we have shown is that if there is some $k$ such that $g(k)=p^c\neq 1$, then $f(n)=n\cdot p^c$ for all $n$. The other possibility is that instead $g(k)=1$ for all $k$. This then gives us that $f(n)=n$ for all $n$, and we can check that this is also a valid solution.

Remembering that we have been dealing with the function $n\mapsto f(n)-f(0)$ rather than $f$, we see that all solutions to the problem are of the form

$$f(n)=p^c\cdot n+d$$

for some prime number $p$, some non-negative integer $c$, and some integer $d$.