Senior Test 3

Stellenbosch Camp 2017

Time: $2\frac{1}{2}$ hours

1. Find all positive integers n with exactly 9 positive divisors such that all of those 9 divisors can be placed in a 3-by-3 table such that the products of the numbers in each row, each column and on each diagonal are all the same.

2. On the sides AD and BC of the rectangle $ABCD$ there are points M, N, and P, Q chosen correspondingly so, that $AM = MN = ND = BP = PQ = QC$. On the segment QC there is a chosen point X, that is different to the ends of the segment. Prove, that the perimeter of $\triangle ANX$ is greater than the perimeter of $\triangle MDX$.

3. There is a finite number of lamps in an electrical scheme. Some pairs of lamps are directly connected by a wire. Every lamp is lit either red or blue. With one switch all lamps that have a direct connection with a lamp of the other colour change their colour (from red to blue or vice versa). Prove that after some number of switches all lamps have the same colour as two switches before that.

4. Find all prime numbers p such that

$$\frac{p^2}{1 + \frac{1}{2} + \cdots + \frac{1}{p-1}}$$

is a perfect square.

5. Let $a, b, c > 0$ such that $abc = 1$. Prove that :

$$\frac{ab}{a^2 + b^2 + \sqrt{c}} + \frac{bc}{b^2 + c^2 + \sqrt{a}} + \frac{ca}{c^2 + a^2 + \sqrt{b}} \leq 1$$