March Monthly Problem Set

Due: 12 March 2018

1. For a positive integer that is not a power of 2, define $t(n)$ as the greatest odd divisor of n and $r(n)$ as the smallest odd divisor of n greater than 1. Determine all positive integers n which are not powers of 2 and satisfy
$$n = 3t(n) + 5r(n).$$

2. $ABCD$ is a square. At B, another square $BEFG$ is drawn so that $\angle ABE$ is obtuse. Prove that the median through B of triangle BCG is perpendicular to AE.

3. A hexagon is partitioned into triangles, each coloured either black or white, such that:
   (a) Two triangles either share a common side (and then they are of different colours) or a common vertex, or else they have no points in common.
   (b) Every side of the hexagon is also a side of one of the black triangles.

   Prove that a decagon cannot be partitioned in this way.

4. Determine all pairs of functions (f, g) from the set of real numbers to itself such that
$$\min(f(x), f(y)) = \max(g(x), g(y))$$
for all real numbers x and y with $x \neq y$.

5. Given 100 infinitely large boxes with finitely many markers in each of them, the following procedure is carried out: At step 1 one adds one marker in every box. At step 2 one marker is added in every box containing an even number of markers. At step 3 one marker is added in every box in which the number of markers is divisible by 3, and so on. Before the process starts Bruno wants to distribute several markers in the boxes such that there is at least one marker in each box and the following holds: after any number of steps there exist two boxes containing a different number of markers. Is it possible for Bruno to do this?

6. Let $n > 1$ be an integer. Prove that the inequality
$$\frac{(n-1)^2}{4} \geq \left(\sum_{1 \leq i < j \leq n} \sqrt{x_i x_j} \right)^2 + \left(\sum_{1 \leq i < j \leq n} |x_i - x_j| \right)^2$$
holds for all nonnegative real numbers $x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \cdots + x_n = 1$.

7. $\triangle ABC$ is right-angled with $\angle A = 90^\circ$ and circumcircle Γ. The incircle of $\triangle ABC$ touches BC at D, CA at E and AB at F. Let M be the midpoint of arc AB of Γ not containing C and let N be the midpoint of arc AC of Γ not containing B. Prove that E, F, M and N lie on a straight line.

8. Let m, n be positive integers and $p > 3$ a prime number such that $\gcd(m, p-1) = \gcd(n, p-1) = 1$, and assume that $(a^m + b^m)^n \equiv (a^n + b^n)^m \pmod{p}$ for all integers a and b.

   Prove that $m \equiv n \pmod{p-1}$.