

**Stellenbosch Camp December 2017**  
**Senior Test 4**  
**Solutions**

1. Since we want  $2^{m^2} - 4$  to be a multiple of 7, we must have  $2^{m^2} \equiv 4 \pmod{7}$ . To this end, let's look at powers of 2 (mod 7). Noticing that  $2^3 = 8 \equiv 1 \pmod{7}$ , we have that:  $2^{3k} = (2^3)^k \equiv (1)^k = 1 \pmod{7}$ ,  $2^{3k+1} = 2(2^{3k}) \equiv 2(1) = 2 \pmod{7}$ , and  $2^{3k+2} = 2^2(2^{3k}) \equiv 4(1) = 4 \pmod{7}$ . We must therefore have that  $m^2 = 3k + 2$  for some  $k \in \mathbb{N}_0$ . However,  $(3k)^2 \equiv 0 \pmod{3}$ ,  $(3k+1)^2 \equiv 1 \pmod{3}$  and  $(3k+2)^2 \equiv 2^2 \equiv 1 \pmod{3}$  and so squares can only ever be congruent to 0 or 1 (and not 2) modulo 3. Therefore, there does not exist an  $m \in \mathbb{N}_0$  such that  $7|(2^{m^2} - 4)$ .
2. Consider colouring the board in a checkerboard pattern, with colours black and white. Since 2017 is odd, there cannot be same number of black squares as white squares. As such a described permutation moves every desk on a black square to that of a white square and vice versa, this implies no such permutation is possible.
3. Let  $O$  be the centre of the circle  $\Gamma$ , and let the angle bisector of  $\angle AXC$  meet  $\Gamma$  at the points  $S$  on the arc  $AC$ , and the point  $T$  on the arc  $BD$ . Since  $M$  and  $N$  are the midpoints of the chords  $AB$  and  $CD$  respectively, we have that  $OM \perp AB$  and  $ON \perp CD$ . Thus  $MXNO$  is a cyclic quadrilateral. We then have that  $\angle MOX = \angle MNX$  (subtended by  $MX$ ) =  $\angle TXD$  (since  $TX \parallel MN$ ) =  $\angle MXT$  (angle bisector) =  $\angle XMN$  (since  $TX \parallel MN$ ) =  $\angle XON$  (subtended by  $XN$ ). Thus in triangles  $\triangle MOX$  and  $\triangle NOX$ , we have that  $\angle MOX = \angle XON$ ,  $\angle XMO = \angle ONX = 90^\circ$ , and  $OX$  is common, and so  $\triangle MOX \cong \triangle NOX$ , giving us that  $OM = ON$ . Thus in triangles  $\triangle OBM$  and  $\triangle OCN$ , we have  $\angle OMB = \angle ONC = 90^\circ$ ,  $OM = ON$ , and  $OB = OC$  (radii). Thus  $\triangle OBM \cong \triangle OCN$ , and so  $BM = CN$ , giving us that  $AB = 2MB = 2NC = CD$ .
4. To find all *interesting* numbers, note that, if  $f(x) = -x$  for all  $x \in \mathbb{R}$ , we have:

$$f(x) - f(x+y) = y = y^1$$

Hence  $n = 1$  is interesting. Conversely, if  $n$  is interesting, we set  $x = 0$  which yields  $f(0) - f(y) = y^n$ . Letting  $x = y$ , we obtain  $f(y) - f(2y) = y^n$ . Summing these two equations gives us  $f(0) - f(2y) = 2y^n$ . We therefore have:

$$(2y)^n = f(0) - f(2y) = 2y^n$$

If  $y = 1$ , this implies  $2^n = 2$ , hence  $n = 1$ . Therefore the only interesting number is  $n = 1$ .

To find all *beautiful* numbers, note that letting  $f(x) = 0$  for  $x \in \mathbb{R}$  satisfies the inequality for all even  $n$ . We now assume  $n$  is odd. Hence:

$$\begin{aligned} f(x+y) - f(x) &= f(x+y) - f(x+y+(-y)) \leq (-y)^n = -y^n \\ \implies f(x) - f(x+y) &\geq y^n \\ \implies f(x) - f(x+y) &= y^n \end{aligned}$$

Hence, if  $n$  beautiful and odd, then  $n$  must be interesting and so  $n = 1$ . Thus, all beautiful numbers are  $n = 1$  and even  $n$ .

5. Let us consider the general case where there are  $S$  scientists altogether, and we want any subset of  $M$  scientist not to be able to open the lock, but any subset of  $M + 1$  scientists to be able to open the lock.

Given a set of  $M$  scientists out of the  $S$  scientists (we call it an  $M$ -subset), they are missing a key for some lock.

Moreover two such distinct subsets have a scientist not common to both and thus their union has more then  $M$  scientists. If two such  $M$ -subsets are missing the same key then they're union is missing that key, the union has more then  $M$  scientists and thus we have a contradiction.

We define a multimap as follows, for each lock we define its preimage to be the  $M$ -subset which is missing a key for that lock, as described above there cannot be more then one  $M$ -subset missing that key (A priori we may have locks which no  $M$ -subset is missing a key for, so nothing will map to them). Now if some  $M$ -subset maps to more then one lock then we can throw away all but one lock, the  $M$ -subset will still not be able to open the safe because he is missing a lock. So we end up with an injection from the collection of  $M$ -subsets to the set of locks. Thus we have at least  $N = \binom{S}{M}$  locks.

Now we ask if we need more locks then this. Suppose there are more then  $N$  locks. WLOG assume our injection from above maps to the first  $N$  locks labeled 1 to  $N$ . So each  $M$ -subset is missing a key for one of the first  $N$  locks. Now consider the  $(N + 1)$ -th lock which we call  $L$ , consider the collection  $C$  of  $M$ -subsets not having the key for  $L$ .

Imagine we throw away lock  $L$  and all its keys. Any subset of the scientists which could open the safe before can still open it as we just removed a lock. The  $M$ -subsets still cannot open the safe because they are missing some key from the first  $N$  keys. Thus the  $(N + 1)$ -th key is redundant, By a similar argument any lock labeled with a number  $j$   $N$  is redundant.

Thus  $\binom{S}{M}$  is the minimum sufficient number of locks. For our case, we have  $S = 11$  and  $M = 5$ . Thus, the number of locks we need is  $\binom{11}{5} = 462$ .