April Monthly Problem Set

Due: 30 April 2018

1. For integers x and y, prove that the last two digits of $31x + 73y$ are 18 if and only if the last two digits of $33x + 39y$ are 74.

2. $ABCD$ is a rectangle, with points X and Y on sides AB and BC respectively. If the areas of triangles AXD, XBY and YCD are a, b and c, respectively, determine the area of triangle DXY in terms of a, b and c.

3. In how many ways can you place 16 rooks on a chessboard such that each row contains exactly 2 rooks and each column contains exactly 2 rooks?

4. Let $(F_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $F_1 = 1$, $F_2 = 3$ and

$$F_{n+1} = \sqrt{F_n F_{n+2} + (-1)^{n+1} \cdot 5} \qquad \text{for all} \quad n \in \mathbb{N}.$$

   Prove that F_n is an integer for all $n \in \mathbb{N}$.

5. Show that $\dfrac{a}{b} + \dfrac{b}{a} + \dfrac{c}{d} + \dfrac{d}{c}$ assumes infinitely many integer values for positive integers a, b, c, d such that a and b have no common divisors and c and d have no common divisors.

6. Let a set $S = \{x_1, x_2, \ldots, x_n\}$ of $n \geq 4$ real numbers be given. A subset $T = \{x_i, x_j, x_k, x_l\}$ of four distinct elements is called *strange* if the sum of the smallest and the largest elements in T is equal to the sum of the other two. What is the maximum number of strange subsets of S?

7. Let $\mathbb{Q}[x]$ denote the set of all polynomials with rational coefficients. Let $f(x) \in \mathbb{Q}[x]$ and let $h(x) = x^3 - 3x + 1$. Suppose $\alpha \in \mathbb{R}$ and $h(\alpha) = h(f(\alpha)) = 0$.

   (a) Show that $h(x)$ cannot be written as the product of two nonconstant polynomials in $\mathbb{Q}[x]$.

   (b) Show that α is not a root of any quadratic polynomial in $\mathbb{Q}[x]$.

   (c) Hence, or otherwise, show that $h(f^n(\alpha)) = 0$ for all $n \in \mathbb{N}$.

8. Let ABC be a triangle with incentre I, and let the incircle and side BC intersect at D. Prove that $\sin(\angle IAD) < 1/3$.

Email submission guidelines

- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as `CamScanner` to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as `PDFsam`.