Stellenbosch Camp December 2017
Intermediate Test 4
Solutions

1. Since $72 = 8 \cdot 9$, the number $\overline{a2017b}$ must satisfy divisibility rules for both 8 and 9. For 8, we must have that the number formed by the last 3 digits, $\overline{17b}$, is a multiple of 8. The only multiple of 8 between 170 and 179 is 176, fixing $b = 6$. Then for 9, the sum of the digits must be a multiple of 9. Since $2 + 0 + 1 + 7 = 10$, we must have $a + b = 8$ or 17 to bring the total up to 18 or 27 respectively. $17 - 6 = 11$ which is greater than 9 and so we must have that $a = 8 - 6 = 2$.

2. For a *catty* number with n-digits, there are two options for each digit (each digit may be either a 2 or a 3). Therefore, the number of n-digit catty numbers is simply 2^n. The total number of catty numbers with $k \leq n$ digits is then $S_n = \sum_{k=1}^{n} 2^k = 2^1 + 2^2 + \cdots 2^n$. To find a more compact expression for S_n, let us multiply S_n by 2 to get $2^2 + 2^3 + \cdots 2^{n+1}$. Subtracting S_n from $2S_n$ then gives $S_n = (2^2 + 2^3 + \cdots 2^{n+1}) - (2^1 + 2^2 + \cdots 2^n) = 2^{n+1} - 2$. $S_{10} = 2^{10+1} - 2 = 2046$ and so the 2047^{th} is the first *catty* number with 11 digits i.e. 22222222222. The 2048^{th} and 2049^{th} *catty* numbers are then 22222222223 and 22222222232 respectively, leaving the 2050^{th} as 22222222233.

3. Applying Cauchy in Engel-form to the left-hand side, we have for $x, y, z \in \mathbb{R}^+ \cup \{0\}$:

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \geq \frac{(1+1+1)^2}{(1+x)+(1+y)+(1+z)} = \frac{3^2}{3+(x+y+z)}$$

Using that $x + y + z \leq 3$, we have further that:

$$\frac{3^2}{3+(x+y+z)} \geq \frac{3^2}{3+3} = \frac{3}{2}$$

and so, as desired, we have:

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} \geq \frac{3}{2}$$

3. Let O be the centre of the circle Γ, and let the angle bisector of $\angle AXC$ meet Γ at the points S on the arc AC, and the point T on the arc BD. Since M and N are the midpoints of the chords AB and CD respectively, we have that $OM \perp AB$ and $ON \perp CD$. Thus $MXNO$ is a cyclic quadrilateral. We then have that $\angle MOX = \angle MNX$ (subtended by MX) $= \angle TXD$ (since $TX \parallel MN$) $= \angle MXT$ (angle bisector) $= \angle XMN$ (since $TX \parallel MN$) $= \angle XON$ (subtended by XN). Thus in triangles $\triangle MOX$ and $\triangle NOX$, we have that $\angle MOX = \angle XON$, $\angle XMO = \angle ONX = 90°$, and OX is common, and so $\triangle MOX \equiv \triangle NOX$, giving us that $OM = ON$. Thus in triangles $\triangle OBM$ and $\triangle OCN$, we have $\angle OMB = \angle ONC = 90°$, $OM = ON$, and $OB = OC$ (radii). Thus $\triangle OBM \equiv \triangle OCM$, and so $BM = CN$, giving us that $AB = 2MB = 2NC = CD$.

5. Since we want $2^{m^2} - 4$ to be a multiple of 7, we must have $2^{m^2} \equiv 4 \pmod 7$. To this end, let's look at powers of 2 (mod 7). Noticing that $2^3 = 8 \equiv 1 \pmod 7$, we have that: $2^{3k} = (2^3)^k \equiv (1)^k = 1 \pmod 7$, $2^{3k+1} = 2(2^{3k}) \equiv 2(1) = 2 \pmod 7$, and $2^{3k+2} = 2^2(2^{3k}) \equiv 4(1) = 4 \pmod 7$. We must therefore have that $m^2 = 3k + 2$ for some $k \in \mathbb{N}_0$. However, $(3k)^2 \equiv 0 \pmod 3$, $(3k+1)^2 \equiv 1 \pmod 3$ and $(3k+2)^2 \equiv 2^2 \equiv 1 \pmod 3$ and so squares can only ever be congruent to 0 or 1 (and not 2) modulo 3. Therefore, there does not exist an $m \in \mathbb{N}_0$ such that $7 | (2^{m^2} - 4)$.