January Monthly Problem Set

Due: 15 January 2018

1. Let p be a prime number such that $3p + 10$ is equal to the sum of the squares of six consecutive positive integers. Prove that $36 \mid p - 7$.

2. Let A and B be points on a circle k with centre O such that $AB > AO$. Let $C \neq A$ be the second intersection point of the angle bisector of $\angle OAB$ with k. Let $D \neq B$ be the intersection of the line AB with the circumcircle of OBC. Prove that $AD = AO$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$xf(x) - yf(y) = (x - y)(f(x + y) - xy)$$
for all real numbers x and y.

4. On a 2017x2017 board, some of the squares are occupied by a single cat; the rest of the squares are empty. The cats move, never leaving the board, according to the following rules: every second, each cat moves to a neighbouring square, either horizontally (to the square to its immediate left or right) or vertically (to the square below or above its current square); a cat which makes a horizontal move must move vertically on its next move, and a cat which makes a vertical move must move horizontally on its next move. Determine the smallest number of cats such that regardless of their initial positions and their chosen paths, we may ensure that two of them will eventually find themselves in the same square, at the same moment.

5. Each side of a regular dodecahedron detmines a plane in a unique way. These planes divide the space into a finite number of distinct parts. Determine the number of these parts.

   Remark: These planes or parts of them are not considered as parts of the space.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(xy - 1) + f(x)f(y) = 2xy - 1$$
for all $x, y \in R$.

7. Let Γ_1 be a circle with centre A and Γ_2 be a circle with centre B such that A lies on Γ_2. On Γ_2 there is a (variable) point P not lying on AB. A line through P is a tangent of Γ_1 at S, and it interesects Γ_2 again in Q, with Q and P lying on the same side of AB. A different line through Q is tangent to Γ_1 at T. Moreover, let M be the foot of the perpendicular to AB through P. Let N be the intersection of AQ and MT.

   Prove that N lies on a line independent of the position of P on Γ_2.

8. Let k be a positive integer, and let $s(n)$ denote the sum of the digits of n. Show that among all positive integers with k digits, there are as many numbers n satisfying $s(n) < s(2n)$ as there are numbers n satisfying $s(n) > s(2n)$.