Senior Test 5

Stellenbosch Camp 2017

Time: 4 hours

1. Find all primes p such that $\dfrac{2^{p-1}-1}{p}$ is the square of an integer.

2. Consider an 8-by-8 chessboard with 17 pieces placed on it in separate squares. Prove that there are three of these pieces which lie in three different rows and three different columns.

3. Let ABC be a right-angled isosceles triangle with $\angle BAC = 90^\circ$ and $AB = AC$. Let I be the incentre of $\triangle ABC$ and P be the intersection of BI and AC. Let C' be the midpoint of AB and let X be the intersection of IC' and AC. Prove that X is the midpoint of CP.

4. Each brick of a set has 5 holes in a horizontal row. We can either place pins into individual holes or brackets into two neighboring holes. No hole is allowed to remain empty. We place n such bricks in a row in order to create patterns running from left to right, in which no two brackets are allowed to follow another, no three pins may be in a row, and no bracket can cross over between consecutive bricks. How many such patterns of bricks can be created?

5. Let $f : \mathbb{N} \to \mathbb{N}$ be such that
$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$
for all $w, x, y, z \in \mathbb{N}$. Prove that $f(n!) \geq n!$ for all $n \in \mathbb{N}$.

6. The number 6 is written on the blackboard. At the n^{th} step, the integer k on the board is replaced by $k + \gcd(n, k)$. Prove that at each step, the number on the blackboard increases either by 1 or by a prime number.

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