February Monthly Problem Set

Due: 12 February 2018

1. Determine all positive integers a, b, p, where p is prime, satisfying the following equation:
$$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2}.$$

2. Let M be the midpoint of side AB of $\triangle ABC$ in which $BC > CA$. The perpendicular bisector of AB meets BC in P and AC extended in Q. Let R be the foot of the perpendicular from P onto AC and let S be the foot of the perpendicular from Q onto BC. Prove that M, R and S lie on a straight line.

3. Consider a deck of n cards with cards numbered from 1 to n. There is one card of each number, but the deck is shuffled, so the cards do not necessarily appear in numerical order. You are allowed to perform the following operation on the deck: You look at the top two cards of the deck, and place one of these two cards on the bottom of the deck, and the other card on top. Show that it possible, using only this operation some number of times, to sort the deck so that the cards appear in ascending order.

4. Determine all triples of real numbers (x, y, z) that simultaneously satisfy the equations:
$$\begin{aligned} (x^2 4x + 7)(y^2 + 6y + 14) &= 120 \\ (y^2 6y + 14)(z^+ 8z + 23) &= 336 \\ (z8z + 23)(x^+ 4x + 7) &= 96. \end{aligned}$$

5. Determine the number of sets $A = \{a_1, a_2, \ldots, a_{1000}\}$ of positive integers satisfying $a_1 < a_2 < \cdots a_{1000} \leq 2014$ for which the set
$$S = \{a_i + a_j \mid 1 \leq i, j \leq 1000,\ i + j \in A\}$$
is a subset of A.

6. Determine the largest integer C such that the inequality
$$(x_1^2 + 200) + (x_2^2 + 200) + \cdots + (x_k^2 + 200)$$
holds for all positive integers k and all k-tuples $(x_1, x_2, \ldots, x_k)$ of positive integers such that $x_1 + x_2 + \cdots + x_k = 100$.

7. Let ABC be a triangle with centre O. Let ω_1 and ω_2 be be circles both with centre O but with different sizes. Let ω_1 intersect AB in P and Q (with P closer to A) and ω_2 intersect AC in R and S (with R closer to A). Let RQ intersect BC at K and let PS intersect BC at L. Prove that $BK = CL$.

8. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for each pair of integers a and b, where $a < b$, there exists a prime number p and a nonnegative integer c such that

$$\frac{f(b) - f(a)}{b - a} = p^c.$$