

Senior Test 4

Stellenbosch Camp 2017

Time: $2\frac{1}{2}$ hours

1. Does there exist a positive integer m such that $2^{m^2} - 4$ is divisible by 7?
2. There are 2017^2 desks arranged in 2017×2017 square. Is there a permutation of the desks that leaves each desk in an adjacent position (up, down, right or left) with respect to its original position?
3. Let Γ be a circle with two chords AB and CD which intersect at point X inside Γ . Let M and N be the midpoints of AB and CD respectively. Show that if MN is parallel to the angle bisector of $\angle AXC$, then $AB = CD$.
4. A positive integer n is called *interesting* if there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) - f(x + y) = y^n \quad \forall x, y \in \mathbb{R}.$$

A positive integer n is called *beautiful* if there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) - f(x + y) \leq y^n \quad \forall x, y \in \mathbb{R}.$$

Find all interesting positive integers and all beautiful positive integers.

5. Eleven scientists want to have a cabinet built where they will keep some top secret work. They want multiple locks installed, with keys distributed in such a way that if any six scientists are present then they can open all the locks, but if only five are present then they cannot open all the locks. Each key can open exactly one lock. What is the minimum number of locks on the cabinet that would satisfy these conditions?

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