Stellenbosch Camp December 2017
Intermediate Test 5
Solutions

1. *Find all pairs (m, n) of positive integers which satisfy the equation*

$$mn^2 = 100(n + 1).$$

Note that n and $n+1$ differ by 1, and so are coprime. Therefore n^2 must divide 100. Whichever value n takes, m can be found as $\frac{(n+1)\cdot 100}{n^2}$. So (m, n) can take only the following solutions: $(200, 1)$; $(75, 2)$; $(24, 5)$; $(11, 10)$.

2. *Point M and N are chosen on the sides BC and CD of a square $ABCD$, respectively, so that $\angle MAN = 45°$. Points P and Q are the intersections of the diagonal BD with AM and AN, respectively. Prove that P and Q lie on the circle with diameter MN.*

$\angle PAN = \angle PDN = 45°$ (diagonal BD), so $APND$ is a cyclic quadrilateral, since the angles subtended by chord PN are equal. Thus $\angle ANP = \angle PDA = 45°$ (AP common chord), and $\angle APN = 90°$ ($\triangle APN$). Then $\angle MPN = 90°$ (straight line), so P is subtended by diameter MN in circle MNP. By symmetry, the same must apply for point Q, using the cyclic quadrilateral $BMQA$. Therefore P and Q both lie on the circle with diameter MN.

3. *Consider an 8-by-8 chessboard with 17 pieces placed on it in separate squares. Prove that there are three of these pieces which lie in three different rows and three different columns.*

We consider partitioning the chessboard into 8 diagonal sets. Specifically, if we label some two squares as (i_1, j_1) and (i_2, j_2), then these two squares are in the same set if and only if $i_1 - j_1 \equiv i_2 - j_2 \pmod 8$. Since this divides the chessboard into 8 sets of 8 squares each, by pigeonhole principle, this implies one such set contains at least 3 pieces, since $17 = 8 \cdot 2 + 1$. As no two pieces in the same set lie in the same row or column, this completes the proof.

4. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,*

$$f(x + f(f(y))) = y + f(f(x)).$$

Solution 1: First, let us prove that f is injective. Assume $f(x) = f(y)$. Applying f on either side, adding x and then applying f again gives: $f(x + f(f(x))) = f(x + f(f(y)))$. Applying the condition in the problem statement to both sides then gives:

$$x + f(f(x)) = f(x + f(f(x))) = f(x + f(f(y))) = y + f(f(x))$$

and so $x + f(f(x)) = y + f(f(x))$ which implies $x = y$, and so we have f injective. f is also surjective since fixing x in $f(x + f(f(y)))$, we can reach all $y \in \mathbb{R}$. So it is bijective.

Now letting $x = y = 0$, we have: $f(f(f(0))) = f(f(0))$ which from bijectivity (i.e. we can cancel f's), implies that $f(0) = 0$. Letting $y = 0$, we get: $f(x + f(f(0))) = f(f(x))$. Using that $f(0) = 0$, this simplifies to $f(x) = f(f(x))$. Finally, from bijectivity, we get $x = f(x)$. All that remains is to check this is indeed a solution. Subbing $f(x) = x$ into the original equation gives:

$$f(x + f(y)) = f(x + y) = x + y = y + x = y + f(x) = y + f(f(x))$$

and so $f(x) = x$ is a solution.

Solution 2: From the problem statement we have:

$$f(x + f(f(y)) = y + f(f(x)) \tag{1}$$

Swapping x and y around, we get:

$$f(y + f(f(x)) = x + f(f(y)) \tag{2}$$

Noticing that the right-hand side of (2) is the same as the left-hand side argument in (1), we take f's on both sides and apply (1):

$$f(f(y + f(f(x)))) = f(x + f(f(y)) = y + f(f(x))$$

Since $z = y + f(f(x))$ may take on any $\mathbb{R}$ value, the above implies that $f(f(z)) = z$, $\forall z \in \mathbb{R}$. Then, taking $x = 0$ in (1) and using $f(f(z)) = z$, we get:

$$f(y) = f(f(f(y))) = f(0 + f(f(y)) = y + f(f(0)) = y + 0 = y$$

and so once again, $f(y) = y$ is a candidate for a solution. The check follows as in the previous solution.

5. *Find all primes p such that $\dfrac{2^{p-1}-1}{p}$ is the square of an integer.*

   In order for $\frac{2^{p-1}-1}{p}$ to be a square, we must have that $2^{p-1} - 1 = pa^2$ for some $a \in \mathbb{N}$. In fact, since we can factorize this as a difference of squares: $2^{p-1} - 1 = (2^{\frac{p-1}{2}} + 1)(2^{\frac{p-1}{2}} - 1)$, we must have that one bracket is a square and the other is a square times p. For $k, q \in \mathbb{N}$, we have two cases:

   <u>Case 1:</u> $2^{\frac{p-1}{2}} + 1 = k^2$, $2^{\frac{p-1}{2}} - 1 = pq^2$

   We can write $2^{\frac{p-1}{2}} = k^2 - 1 = (k-1)(k+1)$. Both $k-1$ and $k+1$ must then be powers of 2 which differ by 2. The only such numbers are 2 and 4. This sets $k = 3$ and further implies that $\frac{p-1}{2} = 3$ and so $p = 7$. Check: $2^{7-1} - 1 = 64 - 1 = 7 \cdot 3^2$ and so $\frac{2^{7-1}-1}{7}$ is indeed a square.

   <u>Case 2:</u> $2^{\frac{p-1}{2}} - 1 = k^2$, $2^{\frac{p-1}{2}} + 1 = pq^2$

   Let's look at $2^{\frac{p-1}{2}} - 1 = k^2$ modulo 4. All powers of 2 are $0 \pmod 4$ apart from $2^0 = 1 \equiv 1$ and $2^1 = 2 \equiv 2 \pmod 4$. Since, squares may only be 0 or $1 \pmod 4$, we must have $\frac{p-1}{2} = 0$ or 1 in order for $2^{\frac{p-1}{2}} - 1 \equiv 0$ or $1 \pmod 4$ respectively. We cannot have $\frac{p-1}{2} = 0$ since then $p = 1$ which is not prime. However, $\frac{p-1}{2} = 1$ gives $p = 3$ which is prime. Check: $2^{3-1} - 1 = 2^2 - 1 = 3 \cdot 1^2$ and so $\frac{2^{3-1}-1}{3}$ is indeed a square.

   Therefore, there are two solutions, namely $p = 3$ and $p = 7$.

6. *Let $ABCD$ be a convex quadrilateral with no parallel sides. Make a parallelogram on each two consecutive sides. Show that among these 4 new points, there is only one point inside quadrilateral $ABCD$.*

   Let $\angle A, \angle B, \angle C$ and $\angle D$ denote the interior angles of quadrilateral $ABCD$ at vertices A, B, C and D respectively, and let P_A denote the vertex opposite A in the parallelogram formed by D, A and B, and similarly for P_B, P_C and P_D.

   Now for P_A to lie inside the quadrilateral $ABCD$, we must have that $\angle ABP_A < \angle ABC = \angle B$ and $\angle ADP_A < \angle ADC = \angle D$. However, since ABP_AD is a parallelogram, $\angle ABP_A = \angle ADP_A = 180° - \angle A$ and so P_A lies inside the quadrilateral if and only if $\angle A + \angle B > 180°$ and $\angle A + \angle D > 180°$, and similarly for P_B, P_C and P_D.

However, $\angle A + \angle B + \angle C + \angle D = 360°$ and (since no two sides of $ABCD$ are parallel) none of $\angle A + \angle B$, $\angle B + \angle C$, $\angle C + \angle D$ or $\angle D + \angle A$ is equal to $180°$. Moreover, $\angle A + \angle B$ is less (greater) than $180°$ exactly as $\angle C + \angle D$ is greater (less) than $180°$ and $\angle B + \angle C$ is less (greater) than $180°$ exactly as $\angle D + \angle A$ is greater (less) than $180°$. Hence it follows that for exactly one of A, B, C and D (let's call it X) P_X lies inside $ABCD$.

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