Intermediate Test 3

Stellenbosch Camp 2017

Time: $2\frac{1}{2}$ hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - y) = f(x) + f(y) - 2xy$$

   for all $x, y \in \mathbb{R}$.

2. In a quadrilateral $ABCD$ the intersection of the diagonals is called P. Point X is the orthocentre of PAB. (The orthocentre of a triangle is the point where the three altitudes of the triangle intersect.) Point Y is the orthocentre of triangle PCD. Suppose that X lies inside triangle PAB and Y lies inside triangle PCD. Moreover, suppose that P is the midpoint of the line segment XY. Prove that $ABCD$ is a parallelogram.

3. There are four distinct stations in the toy factory at the North Pole, called A, B, C and D, and there are conveyor belts between some pairs of stations. There is at most one conveyor belt between each pair of stations, and each conveyor belt operates in only one direction. When a toy is travelling on a conveyor belt, it reaches the next station, continues on any one of the conveyor belts going out from that station, and stops if there aren't any.

   Rudolph the Red-nosed Reindeer wants to arrange the conveyor belts between some of the stations and place a toy on a conveyor belt in such a way that the toy can keep on moving forever. In how many ways can he do this?

4. Find all positive integers n with exactly 9 positive divisors such that all of those 9 divisors can be placed in a 3-by-3 table such that the products of the numbers in each row, each column and on each diagonal are all the same.

5. On the sides AD and BC of the rectangle $ABCD$ there are points M, N, and P, Q chosen correspondingly so, that $AM = MN = ND = BP = PQ = QC$. On the segment QC there is a chosen point X, that is different to the ends of the segment. Prove, that the perimeter of $\triangle ANX$ is greater than the perimeter of $\triangle MDX$.