Stellenbosch Camp December 2017
Senior Test 3
Solutions

1.

2.

3.

4. Note that if $p=2$ then the given fraction yields 4, which is indeed a perfect square. We now assume $p>2$. By Bertrand's postulate, there exists a prime q, such that $\frac{p-1}{2}<q\leq p-1$. We now consider writing the denominator as a single fraction, where:

$$1+\frac{1}{2}+\cdots+\frac{1}{p-1}=\frac{M}{1\cdot 2\cdot 3\cdot\ldots(p-1)} \tag{1}$$

where

$$M=\sum_{i=1}^{p-1}\prod_{k=1,k\neq i}^{p-1}k$$

Noting that q divides all terms in M except for one, we have that $q\not| M$. Also q divides the denominator of (1) exactly once. Hence, the exponent of q in the original fraction is 1, which proves it cannot be a perfect square of a *rational* number if $p>2$.

Hence, only $p=2$ satisfies the desired property.

5. *Solution 1:* Note that we have $a^2+b^2+\sqrt{c}\geq 2ab+\sqrt{c}=\frac{2}{c}+\sqrt{c}$. Hence, we have

$$\sum_{\text{cyc}}\frac{ab}{a^2+b^2+\sqrt{c}}\leq\sum_{\text{cyc}}\frac{\frac{1}{c}}{\frac{2}{c}+\sqrt{c}}=\sum_{\text{cyc}}\frac{1}{2+c\sqrt{c}}$$

Reducing to a common denominator, we have that:

$$\begin{aligned}\sum_{\text{cyc}}\frac{1}{2+c\sqrt{c}}&=\frac{(2+b\sqrt{b})(2+c\sqrt{c})+(2+c\sqrt{c})(2+a\sqrt{a})+(2+a\sqrt{a})(2+b\sqrt{b})}{(2+a\sqrt{a})(2+b\sqrt{b})(2+c\sqrt{c})}\\&=\frac{4\cdot 3+(ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca})+4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})}{1+2(ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca})+4(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+8}\end{aligned} \tag{2}$$

It now remains to prove that (2) ≤ 1, where we now use AM-GM:

$$ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}\geq 3\sqrt[3]{a^2b^2c^2\sqrt{a^2b^2c^2}}=3$$

which proves that:

$$\sum_{\text{cyc}}\frac{1}{2+c\sqrt{c}}\leq 1$$

This completes the proof.

Solution 2: As before, we begin with $a^2+b^2+\sqrt{c} \geq 2ab+\sqrt{c} = 2ab+\sqrt{abc^2} = \sqrt{ab}\left(2\sqrt{ab}+c\right)$. Hence, we have

$$\sum_{\text{cyc}} \frac{ab}{a^2+b^2+\sqrt{c}} \leq \sum_{\text{cyc}} \frac{\sqrt{ab}}{2\sqrt{ab}+c} = \frac{1}{2} - \frac{c}{4\sqrt{ab}+2c}$$

and so it remains to show that

$$\sum_{\text{cyc}} \frac{c}{2\sqrt{ab}+c} \geq 1.$$

But by Cauchy's Inequality in Engel form, the left-hand side is

$$\sum_{\text{cyc}} \frac{(\sqrt{c})^2}{2\sqrt{ab}+c} \geq \frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{a+b+c+2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ca}} = 1,$$

and so the inequality is proved.