March Monthly Problem Set Solutions

1. Let $n = 2^{k(n)} \cdot t(n)$. Note that $r(n)$ divides $t(n)$, and so we can write $t(n) = s(n)r(n)$ for some natural number $s(n)$. The condition $n = 3t(n) + 5r(n)$ is then equivalent to

$$2^{k(n)}s(n)r(n) = 3s(n)r(n) + 5r(n),$$

or, equivalently,

$$2^{k(n)}s(n) = 3s(n) + 5.$$

If $k(n) \leq 1$, then we have that

$$2^{k(n)}s(n) \leq 2s(n) < 3s(n) + 5$$

since $s(n)$ is positive, and so we have no solutions in this case.

If $k(n) = 2$, then the equation is equivalent to $4s(n) = 3s(n) + 5$, which gives us that $s(n) = 5$. Since $s(n) = 5$ is an odd divisor of n, we then require that $r(n) \leq 5$, and so $r(n) \in \{3, 5\}$. Recalling that

$$n = 2^{k(n)}s(n)r(n) = 2^2 \cdot 5 \cdot r(n) = 20r(n),$$

we thus find the solutions $n = 60$ and $n = 100$.

Finally, if $k(n) \geq 3$, then we have that

$$3s(n) + 5 = 2^{k(n)}s(n) \geq 8s(n) = 3s(n) + 5s(n) \geq 3s(n) + 5,$$

and, since equality occurs, we must have that $k(n) = 3$ and $s(n) = 1$. Recalling that

$$n = 2^{k(n)}s(n)r(n) = 2^3 \cdot 1 \cdot r(n) = 8r(n),$$

and noting that $r(n)$ is a prime number, we thus find the family of solutions $n = 8p$ where p is a prime number.

All solutions are thus given by $n = 60$, $n = 100$, and $n = 8p$ where p is an odd prime number.

2. Let A, B, C, and D be the points $(-1, 0)$, $(0, 0)$, $(0, 1)$, and $(-1, 1)$ respectively. Let E be the point (e_x, e_y). Then G is the point $(-e_y, e_x)$. We then have that the midpoint of CG is the point $\frac{1}{2}(-e_y, 1 + e_x)$. Thus the slope of the median through B of triangle BCG is $-\frac{1+e_x}{e_y}$. The slope of the line AE is $\frac{e_y}{e_x+1}$. The product of the slopes of these lines is therefore -1, and so the lines are perpendicular.

3. Let the number of white triangles be w, and the number of black triangles be b. Each edge except for the edges of the decagon is part of exactly one white triangle, and each white triangle has three edges. It follows that the total number of edges is $3w + 10$. On the other hand, each edge (including those of the decagon) is part of exactly one black triangle, and each black triangle has three edges. Thus the total number of edges is also equal to $3b$. We thus have that $3w + 10 = 3b$, which is a contradiction since 10 is not divisible by 3.

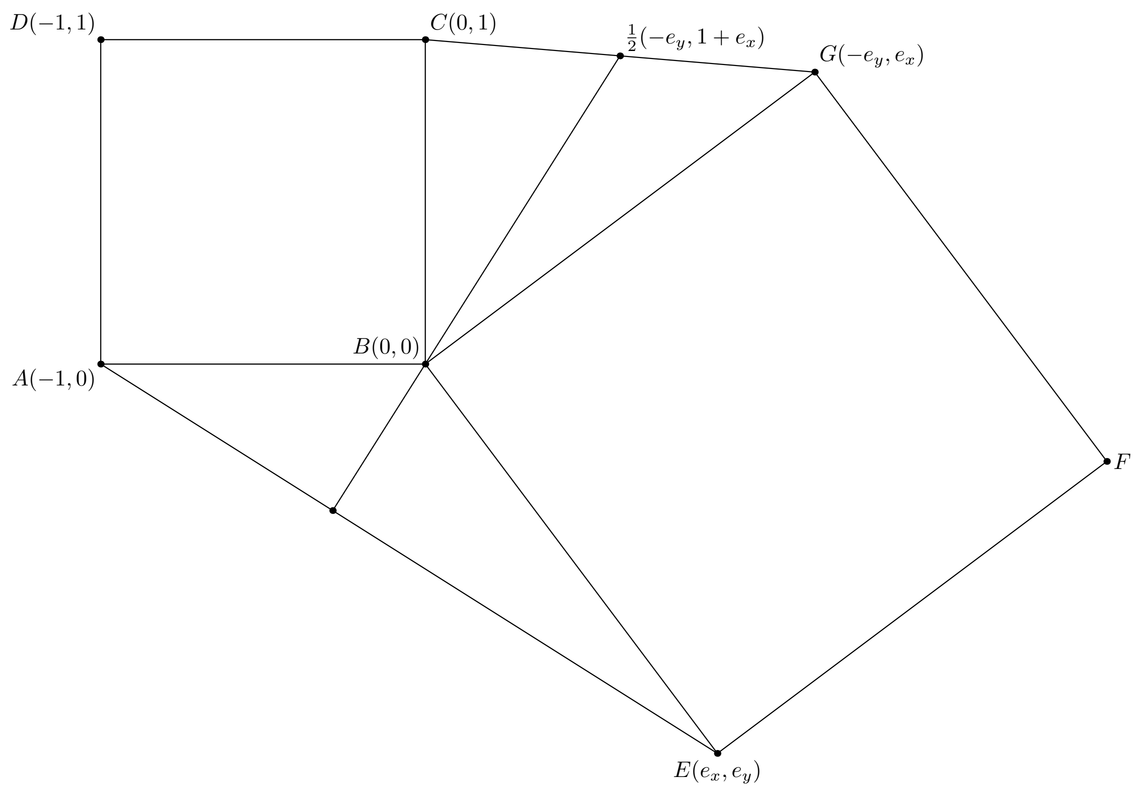

Figure 1: Problem 2

4. We claim that f is bounded below. Suppose not. Then there is some x such that $f(x) < \min\{f(0), g(0)\}$. Since $f(x) < f(0)$, we have that $x \neq 0$, and so

$$f(x) = \min\{f(x), f(0)\} = \max\{g(x), g(0)\} \geq g(0),$$

a contradiction. Let c be the greatest lower bound of the values of f.

A similar argument shows that g is bounded above. Let d be the least upper bound of the values of g. We will show that $c = d$.

For any $x \neq y$, we have that

$$c \leq \min\{f(x), f(y)\} = \max\{g(x), g(y)\} \leq d.$$

Thus $c \leq d$. Suppose that $c < d$. Since c is the greatest lower bound of the values of f, we have that d is not a lower bound, and so there is some x such that $c \leq f(x) < d$. Similarly, since $f(x)$ is not an upper bound for the values of g, there is some y such that $f(x) < g(y) \leq d$. If $y \neq x$, then we have that

$$f(x) \geq \min\{f(x), f(y)\} = \max\{g(x), g(y)\} \geq g(y) > f(x),$$

a contradiction. Otherwise, we have $y = x$, and so for any $z \neq x$, we have that

$$f(x) \geq \min\{f(x), f(z)\} = \max\{g(x), g(z)\} \geq g(x) > f(x),$$

which is again a contradiction. Thus we have that $c = d$.

Now suppose that $x \neq y$ are such that $f(x) > c$ and $f(y) > c$. The we have that

$$c < \min\{f(x), f(y)\} = \max\{g(x), g(y)\} \leq d = c,$$

a contradiction. It follows that there is at most one value of x such that $f(x) > c$. Similarly, there is at most one value of y such that $g(y) < c$. All solutions are thus of the form (f, g) where

$$f(x) = \begin{cases} c_0 & \text{if } x \neq d_0 \\ c_1 & \text{if } x = d_0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} c_0 & \text{if } x \neq d_1 \\ c_2 & \text{if } x = d_1 \end{cases}$$

for some constants c_0, c_1, c_2, d_0 and d_1 such that $c_1 \geq c_0$ and $c_2 \leq c_0$.

We now check that all such pairs of functions do satisfy the conditions in the problem. Suppose that f and g are as defined above. Consider any real numbers $x \neq y$. Since $x \neq y$, we can not have that $x = y = d_0$, and so either $f(x) = c_0$ or $f(y) = c_0$. Since $c_0 \leq c_1$, this implies that $\min\{f(x), f(y)\} = c_0$.

Similarly, we have that $\max\{g(x), g(y)\} = c_0$. We thus have that

$$\min\{f(x), f(y)\} = c_0 = \max\{g(x), g(y)\},$$

as required.

5. Consider an arbitrary box, and let $f(n)$ be the number of markers in the box at the start of the n^{th} step. (Before a marker is potentially added on that step.) Since we never remove a marker from the box, f is non-decreasing.

   We claim that $f(n) \geq n$ for all n. This is true for $n = 1$ since there is initially at least one marker in each box. Suppose that $f(n) \geq n$ for some n. If $f(n) = n$, then we add a marker to the box on the n^{th} step, and so $f(n+1) = n+1$. Otherwise, we have that $f(n+1) \geq f(n) \geq n+1$. The claim follows by induction. It follows that $f(n)$ is unbounded.

   Let p be a prime number larger than $f(1)$. Since $f(n)$ is unbounded, and f increases by at most 1 on each step, there is some m such that $f(m) = p$. Since p is not divisible by any of the numbers $m, m+1, \ldots, p-1$, we see that we do not add any markers to the box until the p^{th} step, and so $f(m) = f(m+1) = \cdots = f(p) = p$. It is then easy to see that $f(n) = n$ for all $n \geq p$.

   For each of the 100 boxes, we thus have that eventually the number of markers in the box is equal to the number of the step that we are on. It follows that, regardless of the initial distribution of the markers, there will eventually be the same number of markers in each box, and so Bruno can not achieve his goal.

6. We have that

$$\begin{aligned}
(n-1)^2 - 4\left(\sum_{1 \leq i < j \leq n} \sqrt{x_i x_j}\right)^2 &= \left((n-1)\sum_{1 \leq i \leq n} x_i\right)^2 - 4\left(\sum_{1 \leq i < j \leq n} \sqrt{x_i x_j}\right)^2 \\
&= \left((n-1)\sum_{1 \leq i \leq n} x_i - 2\sum_{1 \leq i < j \leq n} \sqrt{x_i x_j}\right)\left((n-1)\sum_{1 \leq i \leq n} x_i + 2\sum_{1 \leq i < j \leq n} \sqrt{x_i x_j}\right) \\
&= \left(\sum_{1 \leq i < j \leq n} (\sqrt{x_i} - \sqrt{x_j})^2\right)\left(\sum_{1 \leq i < j \leq n} (\sqrt{x_i} + \sqrt{x_j})^2\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality, this is greater than or equal to

$$\left(\sum_{1\le i<j\le n} |\sqrt{x_i}-\sqrt{x_j}||\sqrt{x_i}+\sqrt{x_j}|\right)^2 = \left(\sum_{1\le i<j\le n} |x_i - x_j|\right)^2,$$

and thus we have that

$$\frac{(n-1)^2}{4} \ge \left(\sum_{1\le i<j\le n} \sqrt{x_i x_j}\right)^2 + \frac{1}{4}\left(\sum_{1\le i<j\le n} |x_i - x_j|\right)^2,$$

as desired.

7. Let I be the incentre of $\triangle ABC$. Note that M is the intersection of CI with Γ, and N is the intersection of BI with Γ. Since BC is a diameter of Γ, we have that $\angle INC = \angle BNC = \angle BMC = \angle BMI = 90^\circ$. Radii are perpendicular to tangents, and so we have that $\angle IEC = \angle BFI = 90^\circ$. We see that $IENC$ and $BMFI$ are cyclic. It follows that $\angle ENB = \angle ENI = \angle ECI = \angle ICB = \angle MCB = \angle MNB$, and so E lies on MN. Similarly, we have that $\angle CMF = \angle IMF = \angle IBF = \angle CBI = \angle CBN = \angle CMN$, and so F also lies on MN.

8. We will show by induction on k that $k^m \equiv k^n \pmod{p}$ for all natural numbers k.

   This is clearly true for $k = 1$. Suppose that $k^m \equiv k^n \pmod{p}$ for some k.

   Since $\gcd(m, p-1) = 1$, there is some natural number s such that $ms \equiv 1 \pmod{p-1}$. We then have that $k^{ms} \equiv k \pmod{p}$. Let $a = k^s$ and $b = 1$. Since $(a^m + b^m)^n \equiv (a^n + b^n)^m \pmod{p}$, we have that

$$(k^{ms}+1)^n \equiv (k^{ns}+1)^m \pmod{p}.$$

   But we know that $k^{ms} \equiv k \pmod{p}$, and also that $k^{ns} \equiv (k^n)^s \equiv (k^m)^s \equiv k^{ms} \equiv k \pmod{p}$ since $k^m \equiv k^n \pmod{p}$ by the induction hypothesis. We thus have that

$$(k+1)^n \equiv (k+1)^m \pmod{p},$$

   as desired.

   Since $k^m \equiv k^n \pmod{p}$ holds for all k, it also holds when k is a primitive root of p, which immediately gives us that $m \equiv n \pmod{p-1}$, as requried.

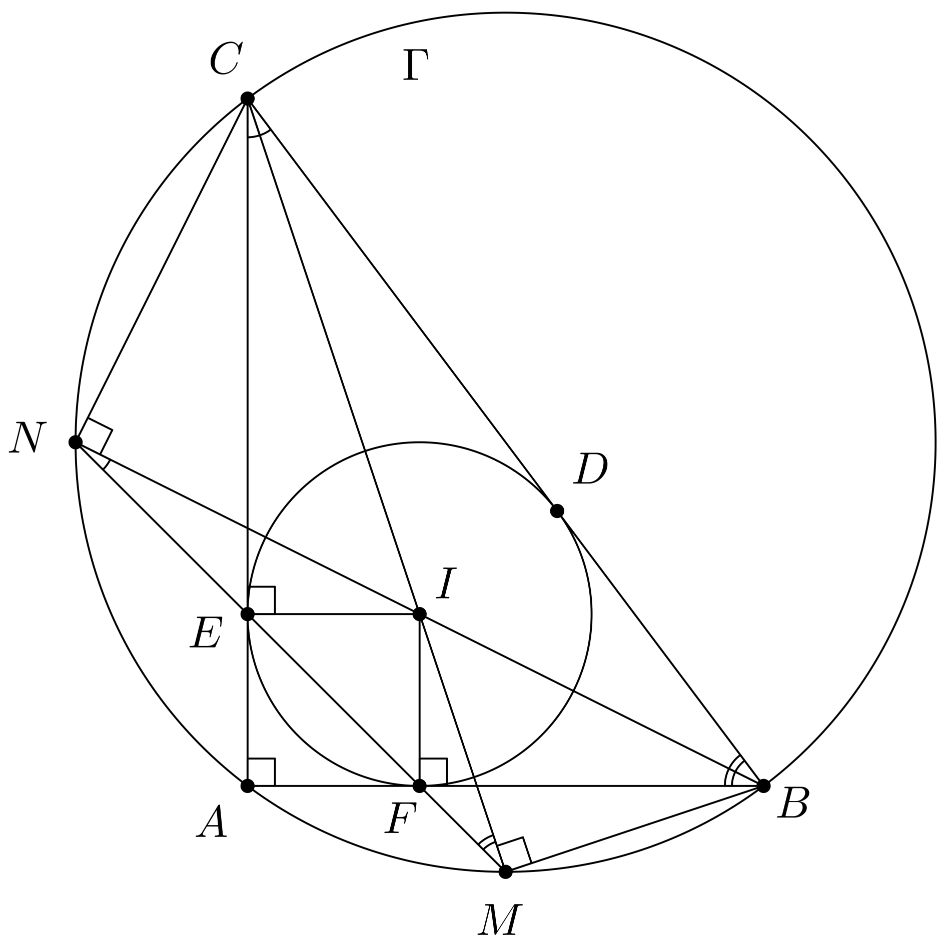

Figure 2: Problem 7