Stellenbosch Camp December 2017
Senior Test 4
Solutions

1. Since we want $2^{m^2} - 4$ to be a multiple of 7, we must have $2^{m^2} \equiv 4 \pmod 7$. To this end, let's look at powers of 2 (mod 7). Noticing that $2^3 = 8 \equiv 1 \pmod 7$, we have that: $2^{3k} = (2^3)^k \equiv (1)^k = 1 \pmod 7$, $2^{3k+1} = 2(2^{3k}) \equiv 2(1) = 2 \pmod 7$, and $2^{3k+2} = 2^2(2^{3k}) \equiv 4(1) = 4 \pmod 7$. We must therefore have that $m^2 = 3k + 2$ for some $k \in \mathbb{N}_0$. However, $(3k)^2 \equiv 0 \pmod 3$, $(3k + 1)^2 \equiv 1 \pmod 3$ and $(3k + 2)^2 \equiv 2^2 \equiv 1 \pmod 3$ and so squares can only ever be congruent to 0 or 1 (and not 2) modulo 3. Therefore, there does not exist an $m \in \mathbb{N}_0$ such that $7|(2^{m^2} - 4)$.

2. Consider colouring the board in a checkerboard pattern, with colours black and white. Since 2017 is odd, there cannot be same number of black squares as white squares. As such a described permutation moves every desk on a black square to that of a white square and vice versa, this implies no such permutation is possible.

3. Let O be the centre of the circle Γ, and let the angle bisector of $\angle AXC$ meet Γ at the points S on the arc AC, and the point T on the arc BD. Since M and N are the midpoints of the chords AB and CD respectively, we have that $OM \perp AB$ and $ON \perp CD$. Thus $MXNO$ is a cyclic quadrilateral. We then have that $\angle MOX = \angle MNX$ (subtended by MX) $= \angle TXD$ (since $TX \parallel MN$) $= \angle MXT$ (angle bisector) $= \angle XMN$ (since $TX \parallel MN$) $= \angle XON$ (subtended by XN). Thus in triangles $\triangle MOX$ and $\triangle NOX$, we have that $\angle MOX = \angle XON$, $\angle XMO = \angle ONX = 90°$, and OX is common, and so $\triangle MOX \equiv \triangle NOX$, giving us that $OM = ON$. Thus in triangles $\triangle OBM$ and $\triangle OCN$, we have $\angle OMB = \angle ONC = 90°$, $OM = ON$, and $OB = OC$ (radii). Thus $\triangle OBM \equiv \triangle OCM$, and so $BM = CN$, giving us that $AB = 2MB = 2NC = CD$.

4. To find all *interesting* numbers, note that, if $f(x) = -x$ for all $x \in \mathbb{R}$, we have:

$$f(x) - f(x + y) = y = y^1$$

Hence $n = 1$ is interesting. Conversely, if n is interesting, we set $x = 0$ which yields $f(0)-f(y) = y^n$. Letting $x = y$, we obtain $f(y) - f(2y) = y^n$. Summing these two equations gives us $f(0) - f(2y) = 2y^n$. We therefore have:

$$(2y)^n = f(0) - f(2y) = 2y^n$$

If $y = 1$, this implies $2^n = 2$, hence $n = 1$. Therefore the only interesting number is $n = 1$.

To find all *beautiful* numbers, note that letting $f(x) = 0$ for $x \in \mathbb{R}$ satisfies the inequality for all even n. We now assume n is odd. Hence:

$$\begin{aligned} & f(x + y) - f(x) = f(x + y) - f(x + y + (-y)) \leq (-y)^n = -y^n \\ \implies \quad & f(x) - f(x + y) \geq y^n \\ \implies \quad & f(x) - f(x + y) = y^n \end{aligned}$$

Hence, if n beautiful and odd, then n must be interesting and so $n = 1$. Thus, all beautiful numbers are $n = 1$ and even n.

5. Let us consider the general case where there are S scientists altogether, and we want any subset of M scientist not to be able to open the lock, but any subset of $M+1$ scientists to be able to open the lock.

   Given a set of M scientists out of the S scientists (we call it an M-subset), they are missing a key for some lock.

   Moreover two such distinct subsets have a scientist not common to both and thus their union has more then M scientists. If two such M-subsets are missing the same key then they're union is missing that key, the union has more then M scientists and thus we have a contradiction.

   We define a multimap as follows, for each lock we define its preimage to be the M-subset which is missing a key for that lock, as described above there cannot be more then one M-subset missing that key (A priori we may have locks which no M-subset is missing a key for, so nothing will map to them). Now if some M-subset maps to more then one lock then we can throw away all but one lock, the M-subset will still not be able to open the safe because he is missing a lock. So we end up with an injection from the collection of M-subsets to the set of locks. Thus we have at least $N = \binom{S}{M}$ locks.

   Now we ask if we need more locks then this. Suppose there are more then N locks. WLOG assume our injection from above maps to the first N locks labeled 1 to N. So each M-subset is missing a key for one of the first N locks. Now consider the $(N+1)$-th lock which we call L, consider the collection C of M-subsets not having the key for L.

   Imagine we throw away lock L and all its keys. Any subset of the scientists which could open the safe before can still open it as we just removed a lock. The M-subsets still cannot open the safe because they are missing some key from the first N keys. Thus the $(N+1)$-th key is redundant, By a similar argument any lock labeled with a number ¿ N is redundant.

   Thus $\binom{S}{M}$ is the minimum sufficient number of locks. For our case, we have $S = 11$ and $M = 5$. Thus, the number of locks we need is $\binom{11}{5} = 462$.