Stellenbosch Camp December 2017
Intermediate Test 3
Solutions

1. Letting $x = y = 0$, we obtain $f(0) = 2f(0)$ and so $f(0) = 0$. Then letting $x = y$, we find:

$$f(0) = 2f(x) - 2x^2 \implies f(x) = x^2$$

$\forall x \in \mathbb{R}$. Now all that remains is to check that this is indeed a solution. Substituting $f(x) = x^2$ into the original equation gives:

$$f(x) + f(y) - 2xy = x^2 + y^2 - 2xy = (x - y)^2 = f(x - y)$$

which holds for all $x, y \in \mathbb{R}$, and so $f(x) = x^2$ is a solution (the only solution).

2. Let the feet of the altitudes from X onto BP and AP be Ea dn F respectively, and let the feet of the altitudes from Y onto DP and CP be G and H respectively. Then triangles XFP and YHP are similar since $\angle XPF = \angle YPH$ and $\angle XFP = 90° = \angle YHP$, but $XP = YP$ and so they are in fact congruent; hence $HP = FP$. Also, triangles BFP and DHP are similar since $\angle BFP = 90° = \angle DHP$ and $\angle BPF = \angle DPH$, but since $FP = HP$ they are in fact congruent and so $BP = DP$. Similarly $AC = CP$ and so the diagonals of $ABCD$ bisect each other; hence $ABCD$ is a parallelogram.

3.

4. Let

$$n = \prod_{i=1}^{k} p_i^{a_i}$$

be the prime decomposition of n. Then the number of divisors of n is equal to

$$(a_1 + 1)(a_2 + 1)(a_3 + 1) \cdots (a_k + 1).$$

This must be equal to 9, and so we can see that n must either be of the form $n = p^8$ for some prime number p, or of the form $n = p^2q^2$ for some distinct prime numbers p and q. It remains to show that both of these cases work.

For $n = p^8$, we have the following arrangement of its factors:

p^3	p^8	p
p^2	p^4	p^6
p^7	1	p^5

For $n = p^2q^2$, we have the following arrangement of its factors:

p^2q	q^2	p
1	pq	p^2q^2
pq^2	p^2	q

5.