June Monthly Problem Set

Due: 30 May 2018

1. (a) Are there 5 circles in the plane such that each circle passes through exactly 3 centers of other circles?
   (b) Are there 6 circles in the plane such that each circle passes through exactly 3 centers of other circles?
2. Suppose all of the 2018 integers lying in between (and including) 1 and 2018 are written on a blackboard. Suppose we choose exactly 1009 of these numbers and circle each one of them. By the score of such a choice, we mean the square of the difference between the sum of the circled numbers and the sum of the non-circled numbers. What is the average scores over all possible choices for 1009 numbers?
3. At a mathematical competition n students work on 6 problems each one with three possible answers. After the competition, the Jury found that for every two students the number of the problems, for which these students have the same answers, is 0 or 2. Find the maximum possible value of n.
4. Is there is a permutation $a_1, a_2, \cdots, a_{6666}$ of the numbers $1, 2, \cdots, 6666$ with the property that the sum $k + a_k$ is a perfect square for all $k = 1, 2, \cdots, 6666$?
5. Two circles ω_1 and ω_2, centered at O_1 and O_2, respectively, meet at points A and B. A line through B intersects ω_1 again at C and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE intersects the circle ω through AO_1O_2 at F. Prove that the length of segment EF is equal to the diameter of ω.
6. Let a sequence of real numbers $a_0, a_1, a_2, \cdots$ satisfy the following condition for all sufficiently large $m \in \mathbb{N}$:

$$\sum_{n=0}^{m} a_n \cdot (-1)^n \cdot \binom{m}{n} = 0.$$

   Show that there exists a polynomial P such that $a_n = P(n)$ for all $n \geq 0$.
7. Let n be a positive integer with the following property: $2^n - 1$ divides a number of the form $m^2 + 81$, where m is a positive integer. Find all possible values of n.
8. In triangle ABC with incentre I, let M_A, M_B and M_C by the midpoints of BC, CA and AB respectively, and H_A, H_B and H_C be the feet of the altitudes from A, B and C to the respective sides. Denote by ℓ_b the line being tangent to the circumcircle of triangle ABC and passing through B, and denote by ℓ_b' the reflection of ℓ_b in BI. Let P_B by the intersection of M_AM_C and ℓ_b, and let Q_B be the intersection of H_AH_C and ℓ_b'. Defined $\ell_c, \ell_c', P_C, Q_C$ analogously. If R is the intersection of P_BQ_B and P_CQ_C, prove that $RB = RC$.

Email submission guidelines

- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as `CamScanner` to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as `PDFsam`.