Intermediate Test 2

Stellenbosch Camp 2017

Time: $2\frac{1}{2}$ hours

1. Prove that there are an infinite number of solutions to $x^3 + y^3 = z^2$ over the positive integers.

2. Determine all triples (x, y, z) of positive real numbers that satisfy

$$\begin{aligned} 3\lfloor x \rfloor - \{y\} + \{z\} &= 20.3 \\ 3\lfloor y \rfloor + 5\lfloor z \rfloor - \{x\} &= 15.1 \\ \{y\} + \{z\} &= 0.9. \end{aligned}$$

   For a real number t, $\lfloor t \rfloor$ denotes the largest integer not greater than t, while $\{t\}$ denotes its fractional part, i.e. $\{t\} = t - \lfloor t \rfloor$. For example, if $t = 15.1$, then $\lfloor t \rfloor = 15$ and $\{t\} = 0.1$.

3. Let ABC be an acute-angled triangle with $\angle BAC = 75°$. Let P the midpoint of the side BC, and let M and N be the feet of the altitudes from vertices B and C respectively. Determine the angle $\angle MPN$.

4. A snake trader has 2017 ravenous snakes with different tail lengths which he wants to place in a line from west to east, with each snake facing either east or west. The snakes are well-trained and will stay still once placed, but each will eat any snake it can see that has a shorter tail than it! (Each snake can see all the snakes in the direction in which it is facing.) Can the snakes be lined up such that no snake can see another snake with a shorter tail length? If so, how many ways are there to do this?

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = x^2 - f(y^2)$$

for all $x, y \in \mathbb{R}$.