May Monthly Problem Set

Due: 31 May 2018

1. For a fixed integer k, determine all polynomials $f(x)$ with integer coefficients such that $f(n)$ divides $(n!)^k$ for every positive integer n.

2. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The distance between two cities is defined to be the least possible numbers of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

3. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

4. Let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation
$$f(f(f(n))) = f(n + 1) + 1$$
for all $n \in \mathbb{Z}_{\geq 0}$.

Email submission guidelines

- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as `CamScanner` to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as `PDFsam`.