Monthlies Pool

Stellenbosch Camp 2017

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- Consider a deck of n cards with cards numbered from 1 to n. There is one card of each number, but the deck is shuffled, so the cards do not necessarily appear in numerical order. You are allowed to perform the following operation on the deck: You look at the top two cards of the deck, and place one of these two cards on the bottom of the deck, and the other card on top. Show that it possible, using only this operation some number of times, to sort the deck so that the cards appear in ascending order.

- Given 100 infinitely large boxes with finitely many markers in each of them, the following procedure is carried out: At step 1 one adds one marker in every box. At step 2 one marker is added in every box containing an even number of markers. At step 3 one marker is added in every box in which the number of markers is divisible by 3, and so on. Before the process starts Bruno wants to distribute several markers in the boxes such that there is at least one marker in each box and the following holds: after any number of steps there exist two boxes containing a different number of markers. Is it possible for Bruno to do this?

- Determine all positive integers a, b, p, where p is prime, satisfying the following equation:

$$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2}.$$

- Determine all pairs of positive integers (m, n) for which

$$(m+n)^3 \mid 2n(3m^2+n^2)+8.$$

- For a positive integer that is not a power of 2, define $t(n)$ as the greatest odd divisor of n and $r(n)$ as the smallest odd divisor of n greater than 1. Determine all positive integers n which are not a power of 2 and satisfies

$$n = 3t(n) + 5r(n).$$

- Let k be a positive integer, and let $s(n)$ denote the sum of the digits of n. Show that among all positive integers with k digits, there are as many numbers n satisfying $s(n) < s(2n)$ as there are numbers n satisfying $s(n) > s(2n)$.

- For a certain value of n, the numbers 4^n and 5^n start with the same digit. Prove that this digit must be either a 2 or a 4.

- Let Γ_1 be a circle with centre A and Γ_2 be a circle with centre B such that A lies on Γ_2. On Γ_2 there is a (variable) point P not lying on AB. A line through P is a tangent of Γ_1 at S, and it intersects Γ_2 again in Q, with Q and P lying on the same side of AB. A different line through Q is tangent to Γ_1 at T. Moreover, let M be the foot of the perpendicular to AB through P. Let N be the intersection of AQ and MT.

  Prove that N lies on a line independent of the position of P on Γ_2.

- $\triangle ABC$ is right-angled with $\angle A = 90°$ and circumcircle Γ. The incircle of $\triangle ABC$ touches BC at D, CA at E and AB at F. Let M be the midpoint of arc AB of Γ not containing C and let N be the midpoint of arc AC of Γ not containing B. Prove that E, F, M and N lie on a straight line.

- Let M be the midpoint of side AB of $\triangle ABC$ in which $BC > CA$. The perpendicular bisector of AB meets BC in P and AC extended in Q. Let R be the foot of the perpendicular from P onto AC and let S be the foot of the perpendicular from Q onto BC. Prove that M, R and S lie on a straight line.

- Determine the number of sets $A = \{a_1, a_2, \ldots, a_{1000}\}$ of positive integers satisfying $a_1 < a_2 < \cdots a_{1000} \leq 2014$ for which the set

  $$S = \{a_i + a_j \mid 1 \leq i, j \leq 1000,\ i + j \in A\}$$

  is a subset of A.

- Given positive real a, b, c such that $2abc + ab + bc + ca = 1$, prove the inequality

  $$\frac{a+1}{(a+1)^2 + (b+1)^2} + \frac{b+1}{(b+1)^2 + (c+1)^2} + \frac{c+1}{(c+1)^2 + (a+1)^2} \leq 1$$

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- Find all prime numbers a and b such that

  $$20a^3 - b^3 = 1$$