Stellenbosch Camp December 2017
Senior Test 2
Solutions

1. Letting $x = y = 0$ yields $f(0) = 0$. For $y = 0$, we obtain $f(x^2) = x^2$ which implies $f(x) = x$ for all $x \geq 0$. For $x = 0$, we obtain $f(-y^2) = -f(y^2) = -y^2$, which implies $f(x) = x$ for all $x \leq 0$. Hence $f(x) = x$ for all $x \in \mathbb{R}$ which can easily be checked to be a solution.

2. **Base case, $n = 1$:** The snake can either face west or east, and hence can be lined up in 2 ways.

   Clearly the snakes can always be lined up so that no snake can see another snake with a shorter tail length, since for example the snake trader can always line them up facing west in descending order. The number of suitable arrangements for n snakes is 2^n, which we prove using induction.

   Inductive hypothesis, $n = k$: Assume that for $n = k$ snakes, they can be lined up in 2^k ways.

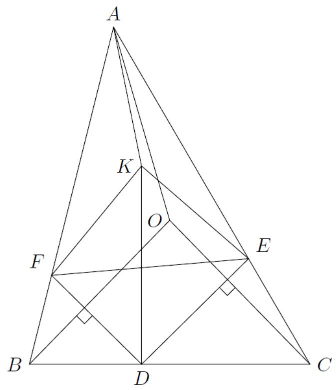
   Inductive step, $n = k + 1$: Since all snakes have distinct tail lengths, we can find and remove the tallest snake from the $k + 1$ snakes. The other k snakes can be lined up in 2^k ways, using our inductive hypothesis. The remaining snake can only be inserted on the far sides of any line of the k snakes, since if it is inserted between any two snakes it must be taller than both of them, and hence no matter which way it faces it would have a snake to eat. Similarly, on either the west or the east edge, the snake must face away from the other snakes, else again it would see a shorter snake. So there are 2 ways to add the last snake, and thus the total arrangements of snakes numbers 2^{k+1}.

   Thus by induction, n snakes can be lined up in 2^n ways, so for 2017 snakes there are 2^{2017} arrangements.

3. Let l_C be the tangent at C to the circumcircle of $\triangle ABC$. As $CO \perp l_C$, the lines DE and l_C are parallel. Now we find that

$$\angle CDE = \angle(BC, l_C) = \angle BAC$$

   hence the quadrilateral $BDEA$ is cyclic. Analogously, we find that the quadrilateral $CDFA$ is cyclic. As we now have $\angle CDE = \angle A = \angle FDB$, we conclude that the line BC is the external angle bisector of $\angle EDF$. Furthermore, $\angle EDF = 180° - 2\angle A$. Since K is the circumcentre of $\triangle AEF$, $\angle FKE = 2\angle FAE = 2\angle A$. So $\angle FKE + \angle EDF = 180°$, hence K lies on the circumcircle of $\triangle DEF$. As $|KE| = |KF|$, we have that K is the midpoint of the arc EF of this circumcircle. It is well known that this point lies on the internal angle bisector of $\angle EDF$. We conclude that DK is the internal angle bisector of $\angle EDF$. Together with the fact that BC is the external angle bisector of $\angle EDF$, this yields that $DK \perp BC$, as desired.



4. Let $P(x) = x^3 + \alpha x^2 + \beta x + \gamma$ where $\alpha, \beta, \gamma \in \mathbb{R}$. The condition that $P(1) = 91$ then becomes $1 + \alpha + \beta + \gamma = 91$ and $P(-1) = -121$ becomes $-1 + \alpha - \beta + \gamma = -121$. Adding the two equations yields $2\alpha + 2\gamma = -30$, hence $\alpha + \gamma = -15$.

   By Vieta's formulae, we have: $\alpha = -(a+b+c), \beta = ab+bc+ca$ and $\gamma = -abc$. Hence $abc + a + b + c = -\alpha - \gamma = 15$ and $ab + bc + ca = \beta = (1 + \alpha + \beta + \gamma) - (\alpha + \gamma) - 1 = 91 + 15 - 1 = 105$. Thus

   $$\frac{ab + bc + ca}{abc + a + b + c} = \frac{105}{15} = 7$$

   which is, indeed, the *only* value that the fraction can attain.

5. Rearranging the equation, $2qn(p+1) = (n+2)(2pq+p+q+1)$. The left hand side is even, so either $n+2$ or $p+q+1$ is even, so either p =2 or $q = 2$ since p and q are prime, or n is even.

   If $p = 2$, $6qn = (n+2)(5q+3)$, so $(q-3)(n-10) = 36$. Considering the divisors of 36 for which q is prime, we find the possible solutions (p, q, n) in this case are $(2, 5, 28)$ and $(2, 7, 19)$ (both of which satisfy the equation).

   If $q = 2$, $4n(p+1) = (n+2)(5p+3)$, so $n = pn + 10p + 6$, a contradiction since $n < pn$, so there is no solution with $q = 2$.

   Finally, suppose that $n = 2k$ is even. We may suppose also that p and q are odd primes. The equation becomes $2kq(p+1) = (k+1)(2pq+p+q+1)$. The left hand side is even and $2pq+p+q+1$ is odd, so $k+1$ is even, so $k = 2l+1$ is odd. We now have

   $$q(p+1)(2l+1) = (l+1)(2pq+p+q+1)$$

   or equivalently

   $$lq(p+1) = (l+1)(pq+p+1)$$

   Note that $q|pq+p+1$ if and only if $q|p+1$. Furthermore, because $(p, p+1) = 1$ and q is prime, $(p+1, pq+p+1) = (p+1, pq) = (p+1, q) > 1$ if and only if $q|p+1$.

   Since $l, l+1)$, we see that, if $q \not| p+1$, then $l = pq+p+1$ and $l+1 = q(p+1)$, so $q = p+2$ (and $(p, p+2, 2(2p^2+6p+3))$ satisfies the original equation). In the contrary case, suppose $p+1 = rq$, so $l(p+1) = (l+1)(p+r)$, a contradiction since $l < l+1$ and $p+1 \leq p+r$.

   Thus the possible values of $q-p$ are $2, 3$ and 5.

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