

April Monthly Problem Set Solution

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7. (a) The intermediate value theorem confirms that h has three real roots. By the rational root test, the only possible rational roots of $h(x)$ are ± 1 , but they are not. Hence α , $f(\alpha)$ and the third root of $h(x)$ are all irrational; and $h(x)$ is irreducible in $\mathbb{Q}[x]$.
(b) Suppose for a contradiction that α is a root of a quadratic $g(x) \in \mathbb{Q}[x]$. Then we can write

$$h(x) = g(x) \cdot q(x) + r(x)$$

with $r(x)$ a linear member of $\mathbb{Q}[x]$. Then

$$0 = h(\alpha) = g(\alpha) \cdot q(\alpha) + r(\alpha) = r(\alpha)$$

and so $r(\alpha) = 0$. Since $r(x) \in \mathbb{Q}[x]$ and α is irrational, we must have that r is constantly 0. Thus $h(x) = g(x) \cdot q(x)$, a contradiction to the irreducibility of $h(x)$.

Alternative:

If α is the root of a quadratic, then we can write $\alpha = r + \sqrt{s}$ where $r, s \in \mathbb{Q}$ and s is not a perfect square. The other root of the quadratic, $r - \sqrt{s}$, will be denoted $\bar{\alpha}$.

We argue that both α and $\bar{\alpha}$ are roots of h . For this, note that since $(r + \sqrt{s})^3 - 3(r + \sqrt{s}) + 1 = 0$, we have (by separating rational and irrational parts) that $r^3 + 3rs - 3r + 1 = 0$ and $3r^2 + s - 3 = 0$. It follows that $(r - \sqrt{s})^3 - 3(r - \sqrt{s}) + 1 = 0$.¹

This gives a contradiction, because it follows that the quadratic is a divisor of h .

- (c) Suppose inductively that $h(f^n(\alpha)) = 0$. We show that $h(x)$ is a factor of $h(f^n(x))$. We can write

$$h(f^n(x)) = h(x) \cdot s(x) + g(x)$$

¹All we have done here is reinvent a little of the theory of conjugate surds. The conjugate surd of a number of the form $\alpha = r + \sqrt{s}$, where $r, s \in \mathbb{Q}$ and s is not a perfect square, is defined to be $\bar{\alpha} = r - \sqrt{s}$. Given any polynomial $p(x) \in \mathbb{Q}[x]$, if $p(\alpha) = 0$, then $p(\bar{\alpha}) = 0$.

This is quite similar to the well known fact that the complex roots of a polynomial with real coefficients occur in complex conjugate pairs.

with $s(x)$, $g(x)$ members of $\mathbb{Q}[x]$ with the degree of $g(x)$ less than or equal to 2. Then

$$0 = h(f^n(\alpha)) = h(\alpha) \cdot s(\alpha) + g(\alpha)$$

and so $g(\alpha) = 0$. From the previous paragraph g then cannot be a quadratic, and clearly cannot be linear, and so we get that g is constantly 0.

Hence $h(f^n(x)) = h(x) \cdot s(x)$, and so by substitution $h(f^{n+1}(x)) = h(f(x)) \cdot s(f(x))$. Thus $h(f^{n+1}(\alpha)) = h(f(\alpha)) \cdot s(f(\alpha)) = 0$, completing the inductive step.

8.