Test 4

April Camp 2023

Time: $4\frac{1}{2}$ hours

1. Prove that there are infinitely many positive integers which can't be expressed as $a^{d(a)}+b^{d(b)}$ where a and b are positive integers and $d(n)$ denotes the number of positive divisors of a positive integer n.

2. Determine the largest real number k such that the inequality
$$\left(k+\frac{a}{b}\right)\left(k+\frac{b}{c}\right)\left(k+\frac{c}{a}\right) \leq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$
holds for all positive real numbers a, b, and c.

3. In each square of a garden shaped like a 2022×2022 board, there is initially a tree of height 0. A gardener and a lumberjack take turns playing the following game, with the gardener taking the first turn:

   - The gardener chooses a square in the garden. Each tree on that square and all the surrounding squares (of which there are at most eight) then becomes one unit taller.
   - the lumberjack then chooses four different squares on the board. Each tree of positive height on those squares then becomes one unit shorter.

   We call a tree *majestic* if its height is at least 10^6. Determine the largest number K such that the gardener can ensure that there are eventually K majestic trees in the garden, no matter how the lumberjack plays.

4. Let ABC be an acute-angled triangle with $AC > BC$, let O be its circumcentre, and let D be a point on the segment BC. The line through D perpendicular to BC intersects the lines AO, AC, and AB at W, X, and Y respectively. The circumcircles of AXY and ABC intersect again at $Z \neq A$. Prove that if $OW = OD$, then DZ is tangent to the circle AXY.