Intermediate Test 3 Solutions

Stellenbosch Camp 2022

1. Given positive real numbers a, b, and c such that $a^2 + b^2 + c^2 = 26$ and $ab + bc + ca = 19$, find the value of $a + b + c$.

   Solution:
$$(a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca) = 26 + 2 \cdot 19 = 8^2$$
$$\implies a + b + c = 8.$$

2. Find all positive integers a and b such that
$$a! + b! = 3^n - 1$$
   where n is a positive integer.

   Solution: W.L.O.G. let $a \geq b$.

   Case 1: $a, b \geq 3$
   Note that $3 \mid a!$ and $3 \mid b!$, meaning $3 \mid a! + b!$, but 3 does not divide the RHS. Therefore there are no solutions for this case.

   Case 2: $b = 1$
$$\begin{aligned} a! + 1 &= 3^n - 1 \\ \implies \qquad a! &= 3^n - 2 \end{aligned}$$
   if $a \geq 3$ there are no solutions as 3 will divide the RHS but not the LHS. Checking $a < 3$ gives the solution $a = 1, b = 1, n = 1$.

   Case 3: $a \geq 3, b = 2$
$$\begin{aligned} a! + 2 &= 3^n - 1 \\ \implies \qquad a! &= 3^n - 3 \\ \implies \qquad a! &= 3(3^{n-1} - 1) \end{aligned}$$
   if $a \geq 6$ there are no solutions as 9 will divide the RHS but not the LHS. Checking $a < 6$ gives the solutions: $a = 3, b = 2, n = 2$ and $a = 4, b = 2, n = 3$.

3. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

   Solution: Let $a_0, a_1, \ldots, a_n$ be the number of doors you can see in room i with a_0 being the number of outside doors. Notice that every door is counted twice so that $\sum_{i=0}^{n} a_i$ is even. Now since each door has an even number of rooms, we have that $\sum_{i=1}^{n} a_i$ is even so that a_0 is even as required.

4. For acute triangle $\triangle ABC$, a point Z interior to $\triangle ABC$ satisfies $ZB = ZC$. Suppose points X and Y lie outside $\triangle ABC$ such that $\triangle XAB \mid\mid\mid \triangle YCA \mid\mid\mid \triangle ZBC$. Prove that A, X, Y, Z are the vertices of a parallelogram.

Solution:

$$\angle ZCY = \angle ZCA + \angle ACY = \angle ZCA + \angle BCZ = \angle BCA$$

$$\frac{ZC}{BC} = \frac{YC}{AC} \qquad (\triangle ZCB \;|||\; \triangle YCA).$$

Therefore $\triangle BCA \;|||\; \triangle ZCY$ as the sides are in proportion and they share a common angle. Similarly $\triangle BCA \;|||\; \triangle BZX$.

Since $BZ = ZC \implies \triangle BZX \equiv \triangle ZCY \implies XZ = YC = AY$ and $ZY = BX = XA$. Therefore $AXZY$ is a parallelogram.

5. For nonzero real numbers a, b, and c, show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

if and only if two of a, b, and c are equal.

Solution: If some pair of a, b, c are equal, we can assume without loss of generality that $a = b$ since the equation is symmetric. This yields:

$$1 + \frac{a}{c} + \frac{c}{a} = \frac{a}{c} + \frac{c}{a} + 1$$

Which is true. To show that some pair of a, b, c are equal given the expression, observe the following computation:

$$\begin{aligned}
\frac{a}{b} + \frac{b}{c} + \frac{c}{a} &= \frac{a}{c} + \frac{c}{b} + \frac{b}{a} \\
\iff \frac{a^2b + b^2c + c^2a}{abc} &= \frac{a^2c + b^2a + c^2b}{abc} \\
\iff 0 &= \frac{a^2c + b^2a + c^2b - (a^2b + b^2c + c^2a)}{abc} \\
\iff 0 &= \frac{abc + a^2c + b^2a + c^2b - (a^2b + b^2c + c^2a + abc)}{abc} \\
\iff 0 &= \frac{(a-b)(b-c)(c-a)}{abc}.
\end{aligned}$$

Thus one of $a - b$, $b - c$, $c - a$ must be zero, showing that some pair among a, b, c must be equal.