Advanced Test 1 Solutions

Stellenbosch Camp 2022

1. Where p is a prime and $n \in \mathbb{N}$, find all solutions to the equation
$$p^2 = 2^n + 1.$$

   Solution: The only solution is $p = 3$ and $n = 3$.

   Clearly $p > 2$. Then
$$\begin{aligned} p^2 &= 2^n + 1 \\ \iff p^2 - 1 &= 2^n \\ \iff (p-1)(p+1) &= 2^n \end{aligned}$$
   so that both $p - 1$ and $p + 1$ are powers of two. Since at least one of $p - 1$ and $p + 1$ is not divisible by 4, it has to be equal to 2.

   Now $p + 1 = 2$ does not give a prime, and $p - 1 = 2$ yields that $p = 3$ and $n = 3$ as the only solution.

2. There are at least 3 people at a party. All of them have an even number of friends, where friendship is mutual. Show that there are 3 of them who each have the same number of friends.

   Solution: If there are $2n$ people, then they can have $0, 2, ...$ or $2n-2$ friends. There are thus n options for how many friends each person can have. If there is someone with 0 friends, there cannot be 2 people with $2n-2$ friends and if there is someone with $2n-2$ friends there cannot be 2 people with 0 friends. We now consider 2 cases:

   (a) There is a number of friends that no one has (either 0 or $2n - 2$). Then there are $n - 1$ options for how many friends people have, and $2n$ people. Thus, there must be 3 people that have the same number of friends (by pigeonhole principle).

   (b) There is one person who has 0 friends and one person who has $2n - 2$. Then there are $2n - 2$ other people, and $n - 2$ options for how many friends they have. Again, there must now be 3 people that have the same number of friends.

   If instead there are $2n + 1$ people, they can have $0, 2, ...$ or $2n$ friends. There are thus $n + 1$ options for how many friends each person can have. If someone has 0 friends, no one can have $2n$. Then we have n options with $2n + 1$ people, so there must be 3 people that have the same number of friends.

3. Let ABC be an acute-angled triangle. Let D, E, and F be the feet of the perpendiculars from A, B, and C onto BC, CA, and AB respectively. The incircle of triangle DEF touches EF, and DF at X and Y respectively. Prove that XY is parallel to AB.

   Solution: Notice that the orthocenter, H, of $\triangle ABC$ is the in incenter of $\triangle DEF$. Note that $XF = YF$ (common tangents), therefore $FH \perp XY$ (angle bisector of isosceles $\triangle \perp base$), but $FH \perp AB$ as H is the orthocenter, therefore, $AB||XY$ (corresponding angles).

4. Given positive real numbers a, b, and c such that $a + b + c = 3$ and $a^2 + b^2 + c^2 = 3$. Find the value of
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

Solution: We claim that $a = b = c = 1$ so that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} = 3$.
Squaring yields

$$\begin{aligned} a + b + c &= 9 \\ \implies (a+b+c)^2 &= 3 \\ \implies a^2 + b^2 + c^2 + 2ab + 2bc + 2ca &= 9 \\ \implies ab + bc + ca &= 3 \end{aligned}$$

Now suppose WLOG that $a \geq b \geq c$ so that the rearrangement inequality gives that

$$a \cdot a + b \cdot b + c \cdot c = a^2 + b^2 + c^2 \geq ab + bc + ca$$

with equality holding if and only if $a = b = c$. Thus $a = b = c = 1$ as required.

5. For every positive integer n, define

$$r(n) = (n \bmod 1) + (n \bmod 2) + \cdots + (n \bmod n),$$

where by $n \bmod m$ we denote the remainder of n on division by m. Show that there are infinitely many positive integers k such that $r(k) = r(k+1)$.

Solution: We show that $r(4^i - 1) = r(4^i)$ for $i \in \mathbb{N}$. Consider $(k \bmod a)$ compared to $(k + 1 \bmod a)$. Either $(k + 1 \bmod a) = (k \bmod a) + 1$, or $(k + 1 \bmod a) = 0$. When $k = 4^i - 1$, this second case only happens for $a = 2^j$ where $0 \leq j < 2i$, in which case $(k \bmod a) = 2^j - 1$. Using our comparison, and noting that $(n \bmod n) = 0$, we come up with the following difference between $r(4^i - 1)$ and $r(4^i)$:

$$\sum_{\alpha=1}^{4^i-1} 1 - \sum_{\alpha=0}^{2i-1} 1 - \sum_{\alpha=0}^{2i-1} (2^\alpha - 1) = \sum_{\alpha=1}^{4^i-1} 1 - \sum_{\alpha=0}^{2i-1} 2^\alpha = 4^i - 1 - [2^{2i} - 1] = 0.$$

(All of the numbers you can have as a modulus, minus the ones where you won't increase by 1, minus how much you would decrease by for each of these). Hence, $r(4^i - 1) = r(4^i)$ for $i \in \mathbb{N}$.