March Intermediate Monthly Problem Set

Due: 31 March 2023

1. Can you enumerate a pentagon's edges and diagonals with natural numbers from 1 to 10 so that for every triangle having vertices which are vertices of the pentagon the sum of the numbers on its edges is the same?

2. Prove that the number $(a_1^2 + a_2^2 + \cdots + a_{12}^2)a_1^2a_2^2 \ldots a_{12}^2$ is divisible by 12 for all positive integers $a_1, a_2, \ldots, a_{12}$.

3. (a) Is it possible that the midpoints of four edges of a non-self-intersecting hexagon all belong to the same line?

   (b) Is it possible that the midpoints of five edges of a non-self-intersecting hexagon all belong to the same line?

4. Last summer 99 dwarfs used 99 houses, each one of them using a different house. This summer they have changed houses. Prove that each house can be coloured white, red or green so that each dwarf has used houses of different colours during these two summers.

5. Point O is in the interior of an equilateral triangle ABC. Prove that there exists a triangle with edges equal to OA, OB and OC.

6. Prove for each positive a, b, c that

$$1 < \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} < 2$$

- Submit your solutions at https://forms.gle/9EBuvypU7ppDmprt8.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.