Test 5: PAMO

April Camp 2023

Time: $4\frac{1}{2}$ hours

1. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all rational x and y,

$$f(x + y) = f(x) + f(y) + 2xy.$$

2. Let $P_1, P_2, \dots, P_{2022}$ be distinct points on a circle and let $Q_1, Q_2, \dots, Q_{2022}$ be distinct points on another circle. Given that P_n, Q_n, P_{n+1}, and Q_{n+1} are concyclic for all $n \in \{1, 2, \dots, 2021\}$, show that P_1, Q_1, P_{2022} and Q_{2022} are also concyclic.

3. A finite non-empty set T of positive real numbers is called *charismatic* if T has at least two elements and for any two distinct elements a and b of T, at least one of a^b and b^a is an element of T. How many elements can a charismatic set contain?

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