Test 3

April Camp 2023

Time: $4\frac{1}{2}$ hours

1. A convex quadrilateral $ABCD$ where $\angle DAB + \angle ABC < 180°$ is given on a plane. Let E be a point different from the vertices of te quadrilateral on the line segment AB such that the circumcircles of triangles AED and BEC intersect inside teh quadrilateral $ABCD$ at point F. Point G is defined so that $\angle DCG = \angle DAB$, $\angle CDG = \angle ABC$, and triangle CDG is located outside quadrilateral $ABCD$. Prove that the points E, F, and G are collinear.

2. During a mathematical contest, n contestants (where $n \geq 3$) wrote a paper containing 14 multiple choice problems. Suppose that:

   (i) for each problem, there is at least one contestant who solved it,

   (ii) for every three contestants, there is at least one problem which they all have solved, and

   (iii) no problem was solved by more than half of the contestants.

   Find the smallest value of n.

3. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers with the property that

$$a_{n+1}^2 + a_n a_{n+2} \leq a_n + a_{n+2}$$

   for all positive integers n. Show that $a_{2022} \leq 1$.

4. For each positive integer $i \leq 9$ and T, define $d_i(T)$ to be the total number of times the digit i appears when all multiples of 1829 between 1 and T inclusive are written out in base 10.

   Show that there are infinitely many positive integers T such that there are precisely two distinct values among $d_1(T), d_2(T), \ldots, d_9(T)$.