Intermediate Test 5

Stellenbosch Camp 2022

Time: 4 hours

1. Two baseball teams are playing each other in a best of 7 match, with each game having a definite winner (no draws). The teams are equally matched so that each game has an equal probability of either team winning. Another game will not be played once any team has won 4 out of the 7 games. Prove that the probability that the game will end in 6 games is the same as the probability that the game will end in 7 games.

2. Let n be a positive integer. There are n points in the plane such that any three of them form a triangle of area 1. Prove that all the points lie inside a triangle of area 4.

3. Let x, y and z be positive real numbers. Show that

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

4. The center of the circumcircle of the acute triangle ABC is O, and the circumcircle of ABO meets BC and AC at P and Q ($P \neq B$). Show that the extension of the line CO is perpendicular to PQ.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x^2+y^2)$$

for all $x, y \in \mathbb{R}$.

6. The game of Chomp is played on an $n \times n$ board by Dylan (on behalf of SA) and Fionn (on behalf of Ireland) as follows: Dylan moves first. On a player's move, they must place an X on any square (i, j) which does not yet have an X on it, and they must also fill in an X on any square above and to the right of that square which does not yet have an X on it. That is, any square (s, t) with $s \geq i$ and $t \geq j$ which does not yet have an X also gets an X filled in. The person who places the last X loses. Is there a winning strategy for either player?