January Advanced Monthly Problem Set

Due: 20 January 2023

1. We say that a non-empty set A consisting of positive integers is *complete* if for any positive integers a and b such that $a+b \in A$, the number ab also lies in A (the numbers a and b are not required to be distinct or to belong to A). Find all complete sets.

2. Dylan has 11 metal pieces indistinguishable in appearance. He knows that their weights in some order are $1, 2, \ldots, 11$ kg. He also has a bag that breaks if it contains more than 11 kg (it will not break if it contains exactly 11 kg). Liam knows the weight of each piece and he wants to convince Dylan beyond a shadow of a doubt about which piece has the 1 kg weight. By a 'move', Liam can put several pieces into Dylan's bag to demonstrate that the bag is still not broken (he may not break the bag!). Find the least number of moves Liam needs to convince Dylan.

3. We say that an infinite sequence $a_1, a_2, \ldots$ is a *binary sequence* if each a_i is equal to 1 or to -1. Find all binary sequences such that

$$\max\{|a_1|, |2a_1 + a_2|, |3a_1 + 2a_2 + a_3|, |4a_1 + 3a_2 + 2a_3 + a_4|, \ldots\} = 1.$$

4. Let $n \geq 5$ be an odd positive integer. Let P be a point in the same plane as the convex n-gon $A_1A_2\cdots A_n$, satisfying the following conditions:

   - $\angle PA_1A_2 = \angle PA_2A_3 = \cdots = \angle PA_{n-1}A_n = \angle PA_nA_1$
   - $\angle PA_1A_3 = \angle PA_2A_4 = \cdots = \angle PA_{n-1}A_1 = \angle PA_nA_2$
   - for each $1 \leq i \leq n$, P and A_i are on opposite sides of the line $A_{i-1}A_{i+1}$.

   Show that $PA_1 = PA_2 = \cdots = PA_n$ and $A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n = A_nA_1$.

5. Let $n \geq 3$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2} \quad \text{and} \quad x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}.$$

   Prove that $x_1 = x_2 = \cdots = x_n = 1$.

6. Suppose that m and n are natural numbers such that $m > n^{n-1}$, and the numbers $m+1, m+2, \ldots, m+n$ are all composite. Prove that there are *distinct* primes $p_1, p_2, \ldots, p_n$ such that $m+k$ is divisible by p_k for each k from 1 to n.

7. Oh no! The triangle you drew on your piece of paper was too large and none of the vertices are on the piece of paper! Is it possible through ruler and compass (no additional sheets of paper!) to find the orthocentre of your oversized triangle?

   It is given that the triangle's orthocentre is contained within the sheet of paper, and that a segment of each side of the triangle is visible on the piece of paper.

8. Find all pairs of positive integers (m, n) for which there exists a set A such that for all positive integers x and y, we have that if $|x - y| = m$, then at least one of the numbers x or y belongs to the set A, while if $|x - y| = n$ then at least one of the numbers x or y does not belong to the set A.

- Submit your solutions at https://forms.gle/9EBuvypU7ppDmprt8
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.