Test 2

April Camp 2023

Time: $4\frac{1}{2}$ hours

1. There is a set M of 20 distinct real numbers. It is known that for any two numbers $a, b \in M$ with $a < b$ there exists a number $x \in M$ such that $a < -x < b$. How many positive numbers can be in M?

2. Find all positive integers a, b, and p, where p is prime, such that
$$a^4 + b^4 + a^2b^2 = p^3.$$

3. In the acute-angled triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E respectively. Points $X \neq A$ and $Y \neq A$ lie on circles ABD and ACE respectively such that $DA = DX$ and $EA = EY$. Prove that B, C, X, and Y are concyclic.

4. Let m and n be integers greater than 1, let X be a set with n elements, and let $X_1, \ldots, X_m$ be pairwise distinct non-empty subsets of X. A function $f : X \to \{1, 2, \ldots, n+1\}$ is called *nice* if there exists an index k such that
$$\sum_{x \in X_k} f(x) > \sum_{x \in X_i} f(x) \quad \text{for all } i \neq k.$$
Prove that the number of nice functions is at least n^n.