# Advanced Test 4 Solutions

## Stellenbosch Camp 2022

1. You have an $n \times n$ chessboard which starts with all its squares white. An operation involves choosing a row or column and flipping the colour of every square in this row or column between white and black. Is it possible that for all $0 \leq k \leq n^2$, you can do some sequence of operations and end up with $k$ black squares?

   **Solution:** First note that performing a flip on a row or column twice is the same as having not performed a flip at all. Secondly note that the order of flips do not matter. Let there be $p$ row-flips and $q$ column-flips, with $0 \leq p, q \leq n$. The number of values of $k$ we can achieve with a series of operations is less than or equal to the number of ordered pairs $(p, q)$, which we will define as $S_n$. The ordered restriction is due to $(p, q)$ and $(q, p)$ yielding the same number of black squares. If $S_n$ is less than the number of squares on the board, then there will exist some number of $k$ black squares we cannot achieve. Therefore we need that:

   $$S_n = \frac{(n+1)(n+2)}{2} \geq n^2 + 1$$

   Which is only true for $n \leq 3$. Checking the number of black squares for these cases with all the possible pairs $(p, q)$ shows that only the $n = 1$ case is possible.

2. Show that $2022 \times 2^n$ can be written as the sum of three distinct non-zero squares for every natural number $n$.

   **Solution:** Note that $1011 = 31^2 + 7^2 + 1^2$ and $2022 = 43^2 + 13^2 + 2^2$.

   If $n$ is odd, let $n = 2k + 1$. Then we have that

   $$2022 \times 2^n = 1011 \times 2^{2k+2} = \left(31^2 + 7^2 + 1^2\right) \times \left(2^{k+1}\right)^2 = \left(31 \times 2^{k+1}\right)^2 + \left(7 \times 2^{k+1}\right)^2 + \left(1 \times 2^{k+1}\right)^2.$$

   Similarly, if $n$ is even, let $n = 2k$. Then we have that

   $$2022 \times 2^n = 2022 \times 2^{2k} = \left(43^2 + 13^2 + 2^2\right) \times \left(2^k\right)^2 = \left(43 \times 2^k\right)^2 + \left(13 \times 2^k\right)^2 + \left(2 \times 2^k\right)^2.$$

3. A triple of positive real numbers $(a, b, c)$ is called *good* if $a, b, c$ are the side lengths of a non-degenerate triangle, and *cute* if $a, b, c$ are the sides of an acute angled triangle. Prove that $(a, b, c)$ is *good* if and only if $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is *cute*.

   **Solution:** Notice that if $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is *cute*, then $\sqrt{a}^2 + \sqrt{b}^2 > \sqrt{c}^2$ for all permutatios of $(a, b, c)$ (Pythagoras' inequality) so $(\sqrt{a}^2, \sqrt{b}^2, \sqrt{c}^2) = (a, b, c)$ is *good*.

   If $(a, b, c)$ is good we note that it is sufficient to prove that $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is *good* as Pythagoras' inequality would be trivialy satisfied. Use the Ravi substitution $a = x + y, b = y + z, c = z + x$ where $x, y, z > 0$.

Then

$$\begin{aligned}
\sqrt{a}+\sqrt{b} &= \sqrt{x+y}+\sqrt{y+z} \\
&= \sqrt{\left(\sqrt{x+y}+\sqrt{y+z}\right)^2} \\
&= \sqrt{\left(x+y+y+z+2\sqrt{(x+y)(y+z)}\right)} \\
&= \sqrt{\left(x+z+(2y+2\sqrt{(x+y)(y+z)})\right)} \\
&> \sqrt{x+z} \\
&= \sqrt{c}
\end{aligned}$$

Similarly, $\sqrt{a}+\sqrt{c}>\sqrt{b}$ and $\sqrt{b}+\sqrt{c}>\sqrt{a}$.

4. The incircle of $\triangle ABC$ is tangent to $BC, CA$, and $AB$ at $D, E$, and $F$ respectively. $P$, $Q$, and $R$ are the incenters of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ respectively. Prove that $DP$, $EQ$, and $RS$ are concurrent.

   **Solution:** Let the point $P'$ be the intersection of $AI$ and the incircle. Notice that $\angle AFP' = \angle P'EF = \angle P'FE$ (tan-chord, angles subtended by equal sides) therefore $FP'$ is the angle bisector of $\angle AFE$, so $P' = P$ is the incentre of $\triangle AFE$. Similarly we deduce $Q$ and $R$ are the intersections of $BI$ and $CI$ with the incircle respectively. Then $\angle PDF = \angle PDE$ (angles subtended by equal sides) so $DP$ is the angle bisector of $\angle FDE$. Similarly we deduce that $EQ$ and $FR$ are angle bisectors of $\angle DEF$ and $\angle EFD$ respectively. The three lines intersect at the incenter of $\triangle DEF$ and are therefore concurrent.

5. Every point in space is coloured red, blue, or green. Prove that for at least one of the 3 colours, the set of distances between points of that colour contains all positive real numbers.

   **Solution:** Assume for contradiction that none of the colours has this property. Let $x_R$ be a positive real number for which there are no two red points having this distance between them. Similarly, let $x_B$ and $x_G$ be such a real number for the colours blue and green respectively. Assume W.L.O.G. that $x_R \geq x_B \geq x_G$. Start by selecting a red point. If no such point exists, select a blue point. If no such point exists, we only have green points and we reach a clear contradiction. If a blue point does exist (but a red point does not), all points in a shell of radius $x_B$ around that point must be green. Then we have two green points on this shell with distance $x_G$ between them, since $x_G \leq x_B$, which is a contradiction.

   Finally, if our red point does exist, every point in a shell of radius $x_R$ around it must be either blue or green. Select a blue point on this shell. If no such point exists, this whole shell is green and there must exist two points on this shell with distance $x_G$ between them, a contradiction. If the blue point does exist, we consider the intersection of the shell around our blue point with radius $x_B$ and the shell around our red point with radius $x_R$. This intersection is a circle with radius at least $\frac{x_B}{\sqrt{2}}$, which means it has diameter larger than $x_B$. All points on this circle must be green, but there will exist two of them with distance $x_G$, a contradiction.