Advanced Test 2 Solutions

Stellenbosch Camp 2022

1. William and Beatrice take turns placing Kings on a $n \times m$ chessboard. Kings cannot be placed on any of the 8 adjacent squares of Kings of *differing* colour. With William playing first as white, and Beatrice playing second as black, who has the winning strategy?

   Solution: William will always have a winning strategy. William's first move is to place his initial king on a square that either contains the center-point or has the center-point of the board on its border. William can then rotate Beatrice's moves 180° through the center-point of the board. Note that Beatrice cannot place a king in such a way that the reflection of her move is within the 8 adjacent squares of the placed king. Therefor William will always be able to make a move as any legal move Beatrice plays will also have to be legal for William to play on the reflection. As the game is forced to end in less than nm moves, William will place the last piece and have the winning strategy.

2. In $\triangle ABC$ let $\angle C = 90°$, and let Γ be the circle with diameter AC. Define points D and E on Γ such that D is on AB and $DE \parallel AC$. Let P be the intersection of AE and BC. Prove that

   $$PC \cdot BC = AC^2.$$

   Solution: Notice that the statement is true if $\triangle ABC \,|||\, \triangle PAC$, but $\angle PCB = \angle ACB$ so the problem is reduced to showing that $\angle ABC = \angle PAC$. We have that

   $$\begin{aligned} \angle ABC &= 90° - \angle CAB && (\angle's \text{ in } \triangle) \\ &= 90° - (180° - \angle ADE) && (\text{co-int } \angle's) \\ &= 90° - \angle ECA && (\text{opp } \angle's \text{ cyc. quad}). \end{aligned}$$

   But AC is the diameter of Γ so $\angle AEC = 90°$ by Thales' Theorem. Therefore

   $$\angle ABC = 90° - \angle ECA = \angle CAE = \angle PAC.$$

3. The given recurrence relation is equivalent to

   $$\frac{1}{a_n} = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}.$$

   It follows that (e.g. using induction)

   $$\frac{1}{a_n} = \frac{F_{n-2}}{\ell} + \frac{F_{n-1}}{m},$$

   where F_k is the k^{th} Fibonacci number.

   We thus have that

   $$a_{2022} = \frac{\ell m}{mF_{2020} + \ell F_{2021}}.$$

   We note that for any natural number k, we have that $\gcd(F_k, F_{k+1}) = 1$. (This can also be proven using induction.) By Bézout's Lemma, we know that there are infinitely many integers ℓ and m such that $\ell F_{2020} + mF_{2021} = \gcd(F_{2020}, F_{2021}) = 1$.

   It is clear that one of ℓ and m is positive, and the other is negative. For these values of ℓ and m we have that

   $$\frac{(-\ell)(-m)}{(-m)F_{2020} + (-\ell)F_{2021}} = \frac{\ell m}{-1}$$

   which is a positive integer. (Here we take $a_1 = -\ell$ and $a_2 = -m$.)

4. Evaluate the following expression for all positive integers n:

$$\binom{2n}{0} - \binom{2n-1}{1} + \binom{2n-2}{2} - \dots + (-1)^n \binom{n}{n}$$

Solution: Define:

$E_n = \binom{2n}{0} - \binom{2n-1}{1} + \binom{2n-2}{2} - \dots + (-1)^n \binom{n}{n}$ and $O_n = \binom{2n+1}{0} - \binom{2n}{1} + \binom{2n-1}{2} - \dots + (-1)^n \binom{n+1}{n}$.

Then, using the fact that $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$:

$$\begin{aligned}
E_n &= \binom{2n-1}{0} - \left(\binom{2n-2}{1} + \binom{2n-2}{0}\right) + \left(\binom{2n-3}{2} + \binom{2n-3}{1}\right) - \dots \\
&\qquad + (-1)^{n-1}\left(\binom{n}{n-1} + \binom{n}{n-2}\right) + (-1)^n \binom{n-1}{n-1} \\
&= \left(\binom{2n-1}{0} - \binom{2n-2}{1} + \binom{2n-3}{2} - \dots + (-1)^{n-1}\binom{n}{n-1}\right) \\
&\quad - \left(\binom{2n-2}{0} - \binom{2n-3}{1} + \dots + (-1)^{n-1}\binom{n}{n-2} + (-1)^n \binom{n-1}{n-1}\right) \\
&= O_{n-1} - E_{n-1}
\end{aligned}$$

A similar argument shows that $O_n = E_n - O_{n-1}$. Then:

$$E_n = O_{n-1} - E_{n-1} = E_{n-1} - O_{n-2} - E_{n-1} = -O_{n-2} = -(E_{n-2} - O_{n-3}) = -(O_{n-3} - E_{n-3} - O_{n-3}) = E_{n-3}.$$

Now, since $E_0 = 1, E_1 = 0, E_2 = -1$, we have:

$$E_i = \begin{cases} 1 & \text{if } i \bmod 3 = 0 \\ 0 & \text{if } i \bmod 3 = 1 \\ -1 & \text{if } i \bmod 3 = 2 \end{cases}.$$

5. Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^2y^2}{y^2(x+1)^2 + x^2 + x^2y^2} + \frac{y^2z^2}{z^2(y+1)^2 + y^2 + y^2z^2} + \frac{z^2x^2}{x^2(z+1)^2 + z^2 + z^2x^2} \le \frac{1}{2}.$$

Solution: Since $xyz = 1$, we can let $x = a/b$, $y = b/c$, and $z = c/a$ where $a, b, c > 0$. Then

$$\begin{aligned}
\frac{x^2y^2}{y^2(x+1)^2 + x^2 + x^2y^2} &= \frac{a^2/c^2}{(a+b)^2/c^2 + a^2/b^2 + a^2/c^2} = \frac{a^2b^2}{(a+b)^2b^2 + a^2c^2 + a^2b^2} = \frac{a^2b^2}{2a^2b^2 + 2ab^3 + b^4 + a^2c^2} \\
&\le \frac{a^2b^2}{2a^2b^2 + 2ab^3 + 2ab^2c} = \frac{a}{2a+2b+2c} \quad \text{since } b^4 + a^2c^2 \ge 2ab^2c \text{ by AM-GM.}
\end{aligned}$$

Similarly, for the other two terms we get

$$\frac{y^2z^2}{z^2(y+1)^2 + y^2 + y^2z^2} \le \frac{b}{2a+2b+2c} \quad \text{and} \quad \frac{z^2x^2}{x^2(z+1)^2 + z^2 + z^2x^2} \le \frac{c}{2a+2b+2c},$$

and so adding these together we get

$$\begin{aligned}
&\frac{x^2y^2}{y^2(x+1)^2 + x^2 + x^2y^2} + \frac{y^2z^2}{z^2(y+1)^2 + y^2 + y^2z^2} + \frac{z^2x^2}{x^2(z+1)^2 + z^2 + z^2x^2} \\
&\le \frac{a}{2a+2b+2c} + \frac{b}{2a+2b+2c} + \frac{c}{2a+2b+2c} = \frac{1}{2}
\end{aligned}$$

as desired.