Advanced Test 1

Stellenbosch Camp 2022

Time: $2\frac{1}{2}$ hours

1. Where p is a prime and $n \in \mathbb{N}$, find all solutions to the equation

$$p^2 = 2^n + 1.$$

2. There are at least 3 people at a party. All of them have an even number of friends, where friendship is mutual. Show that there are 3 of them who each have the same number of friends.

3. Let ABC be an acute-angled triangle. Let D, E, and F be the feet of the perpendiculars from A, B, and C onto BC, CA, and AB respectively. The incircle of triangle DEF touches EF, and DF at X and Y respectively. Prove that XY is parallel to AB.

4. Given positive real numbers a, b, and c such that $a+b+c=3$ and $a^2+b^2+c^2=3$. Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

5. For every positive integer n, define

$$r(n) = (n \bmod 1) + (n \bmod 2) + \cdots + (n \bmod n),$$

where by $n \bmod m$ we denote the remainder of n on division by m. Show that there are infinitely many positive integers k such that $r(k) = r(k+1)$.