Intermediate Test 2

Stellenbosch Camp 2022

Time: $2\frac{1}{2}$ hours

1. Let D be a point in the interior of $\triangle ABC$. Let M, N, P, Q be the midpoints of AC, AB, DC, DB respectively. Prove that the area of $ABDC$ is twice the area of $NMPQ$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(x) + f(y) = f(2x + y) - x$$

for all $x, y \in \mathbb{R}$.

3. Find all $m, n \in \mathbb{Z}$ satisfying the following equation:

$$m^3 + n^3 = (m + n)^2.$$

4. William and Beatrice take turns placing Kings on a $n \times m$ chessboard. Kings cannot be placed on any of the 8 adjacent squares of Kings of *differing* colour. With William playing first as white, and Beatrice playing second as black, who has the winning strategy?

5. In $\triangle ABC$ let $\angle C = 90°$, and let Γ be the circle with diameter AC. Define points D and E on Γ such that D is on BC and $DE \parallel AC$. Let P be the intersection of AE and BC. Prove that

$$PC \cdot BC = AC^2.$$