Intermediate Test 1 Solutions

Stellenbosch Camp 2022

1. **Solution:** Note that the tile in the same column as the 1×1 square will have to be covered by an **L**-shaped tile to form a 2×2 square. We can then consider the amount of ways to tile the 2×5 board with a 2×2 square and two **L**-shaped tiles. Note that the square will have to be placed on the board such that the amount of tiles to the left and right of it are multiples of 3, leaving two possible locations for the square on the edges of the board. The resulting 2×3 rectangle can be tiled in two ways with the remaining **L**-shaped tiles. Considering the 4 rotations of the 2×2 square as unique tilings, the 2 locations the square can be, as well as the 2 ways the resulting rectangle can be tiled, there are 16 possible tilings.

2. Let the altitude from point A meet BC at D, and let this length be h. Note that in $\triangle ADB$, AB is the hypotenuse of a right angled triangle, meaning that $h = AD \leq AB = 1$. The area of $\triangle ABC$ is $\frac{1}{2}AD.BC = \frac{1}{2}h \leq \frac{1}{2}.1$. This is achieved in the degenerate case where $D = B$ and $\angle ABC = 90°$.

3. Given an acute angled triangle with sides of lengths a, b and c. Prove that
$$a^2 + b^2 > c^2.$$
Solution: Consider an acute-angled triangle ABC such that the perpendicular from A intersects BC at D. Let $AB = a$, $BC = b$ and $CA = c$. Notice that $AB^2 = BD^2 + DA^2$ by Pythagoras' Theorem. and $BC^2 = (BD + DC)^2 = BD^2 + 2BD \times DC + DC^2$. Thus
$$AB^2 + BC^2 = DA^2 + DC^2 + 2BD \times DC + 2BD^2.$$
Moreover, since $CA^2 = DA^2 + DC^2$ by Pythagoras' Theorem. We have
$$AB^2 + BC^2 = CA^2 + 2BD \times DC + 2BD^2 > CA^2.$$
Therefore $a^2 + b^2 = AB^2 + BC^2 > CA^2 = c^2$.

4. Where p is a prime and $n \in \mathbb{N}$, find all solutions to the equation
$$p^2 = 2^n + 1.$$
Solution: The only solution is $p = 3$ and $n = 3$.
Clearly $p > 2$. Then
$$\begin{aligned} p^2 &= 2^n + 1 \\ \iff p^2 - 1 &= 2^n \\ \iff (p-1)(p+1) &= 2^n \end{aligned}$$
so that both $p - 1$ and $p + 1$ are powers of two. Since at least one of $p - 1$ and $p + 1$ is not divisible by 4, it has to be equal to 2.
Now $p + 1 = 2$ does not give a prime, and $p - 1 = 2$ yields that $p = 3$ and $n = 3$ as the only solution.

5. There are at least 3 people at a party. All of them have an even number of friends, where friendship is mutual. Show that there are 3 of them who each have the same number of friends.

   Solution: If there are $2n$ people, then they can have $0, 2, ...$ or $2n-2$ friends. There are thus n options for how many friends each person can have. If there is someone with 0 friends, there cannot be 2 people with $2n-2$ friends and if there is someone with $2n-2$ friends there cannot be 2 people with 0 friends. We now consider 2 cases:

   (a) There is a number of friends that no one has (either 0 or $2n-2$). Then there are $n-1$ options for how many friends people have, and $2n$ people. Thus, there must be 3 people that have the same number of friends (by pigeonhole principle).

   (b) There is one person who has 0 friends and one person who has $2n-2$. Then there are $2n-2$ other people, and $n-2$ options for how many friends they have. Again, there must now be 3 people that have the same number of friends.

   If instead there are $2n+1$ people, they can have $0, 2, ...$ or $2n$ friends. There are thus $n+1$ options for how many friends each person can have. If someone has 0 friends, no one can have $2n$. Then we have n options with $2n+1$ people, so there must be 3 people that have the same number of friends.