March Intermediate Monthly Solutions 2023

1. Suppose that such an enumeration is possible, and let the common edge sum equal S. Note that every edge is part of three different triangles, and that there are $\binom{5}{3} = 10$ different triangles. Hence the sum of the numbers on the edges of each triangle must equal $10S$, but every number from 1 to 10 is counted thrice, so we have
$$10S = 3(1 + 2 + \cdots + 10) = 3(55) = 165,$$
which gives $S = 16.5$. Since S must be an integer, this is a contradiction. Hence such an enumeration is not possible.

2. Note that $a^2 \equiv_4 1$ if a is odd, and $a^2 \equiv_3 1$ if a is not divisible by 3.

   Firstly, if any a_i is even, then a_i^2 is divisible by 4. If all the a_i are odd, then $a_1^2 + a_2^2 + \cdots + a_{12}^2 \equiv_4 1 + 1 + \cdots + 1 \equiv_4 0$, so the given product is divisible by 4 in both cases.

   Next, if any a_i is divisible by 3, then so is the given product. If no a_i is divisible by 3, then $a_1^2 + \cdots + a_{12}^2 \equiv_3 1 + 1 + \cdots + 1 \equiv_3 0$, so the given product is divisible by 3 in both cases.

   Finally, since 3 and 4 are relatively prime, the given product is divisible by $3 \times 4 = 12$.

3. Suppose this is possible, and let ℓ be the line containing the 5 midpoints of the sides, and suppose WLOG that ℓ is horizontal. Note that the vertices of a particular edge must be equidistant from ℓ, since ℓ contains its midpoint. This holds for all 5 of the edges, so all 6 vertices of the hexagon are equidistant from ℓ, lying alternately above and below ℓ. In particular, this implies that the midpoint of the sixth edge also lies on ℓ.

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   Now consider the leftmost vertex above ℓ and the leftmost vertex below ℓ. Every edge of the hexagon intersects ℓ; the vertex above ℓ must thus connect to two vertices below ℓ and the vertex below ℓ must connect to two vertices above ℓ. This leads to at least three edges, two of which must intersect, as in the diagram.

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4. Construct a *cycle* of houses as follows: pick any house as the first house and suppose that, inductively, the n^{th} house has been chosen. Let house $n + 1$ be the house which is occupied by the dwarf which occupied the n^{th} house last summer. At some point this sequence must start repeating to form a cycle. If not all the houses have been used up, pick a random house, and construct a new cycle using the above process.

   In this way, all the houses are divided up into several cycles. We now colour the cycles as follows: colour the first house in each cycle green, and colour the remaining houses in each cycle alternately red and white.

   Thus, any dwarf who lived in a green house last summer would be in a red or white house this summer. Similarly, dwarfs who lived in a white house, will move to a red or green house, and dwarfs who lived in a red house will move to a white or green house.

5. Suppose WLOG that the equilateral triangle has side length 1. Note that since the line segments OA, OB and OC lie entirely inside the triangle ABC, each one of these lengths are less than 1. Also, using the triangle inequality in triangle AOB we get
$$AO + BO > AB = 1 > CO$$

Similarly, $BO + CO > AO$ and $AO + CO > BO$. Hence the segments AO, BO and CO satisfy all three triangle inequalities, which implies that they form a triangle.

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6. For the first inequality, note that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} > \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1.$$

For the second inequality, note that

$$\begin{aligned} \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} &= 1 - \frac{b}{a+b} + 1 - \frac{c}{b+c} + 1 - \frac{a}{c+a} \\ &= 3 - \left(\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}\right). \end{aligned}$$

We obtain

$$\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} > 1$$

in a similar fashion to the first inequality, and so

$$3 - \left(\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}\right) < 3 - 1 = 2.$$