February Advanced Monthly Problem Set

Due: 13 March 2023

1. Let ABC be a triangle with circumcentre O and circumradius R such that $\angle A = 45°$. Let BO (extended) meet AC at E and let CO (extended) meet AB at F. Find $BE \cdot CF$ in terms of R.

2. Let m and n be positive integers. Show that the complete bipartite graph $K_{m,n}$ has the same number of edges as its complement if and only if m and n are consecutive triangular numbers.*

3. Find all positive integers n such that $\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n}{3} \right\rfloor \cdot \left\lfloor \frac{n}{4} \right\rfloor = n^2$, where $\lfloor x \rfloor$ denotes the greatest integer not greater than x.

4. Let Γ_1 be a circle with a point O on it. Let Γ_2 be a circle centred at O and let P and Q be the intersection points of Γ_1 and Γ_2. Let Γ_3 be a circle which is externally tangent to Γ_2 at a point R and internally tangent to Γ_1 at a point S. Suppose that Q, R, and S are collinear. Let A and B be the other intersection points of PR and OR with Γ_1. Prove that QA is parallel to SB.

5. Let m and n be positive integers such that $\frac{m}{n} < \sqrt{23}$. Show that $\frac{m}{n} + \frac{3}{mn} < \sqrt{23}$.

6. Find all functions $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ such that for all $x, y, z \in \mathbb{Q}$ we have
$$f(x, y) + f(y, z) + f(z, x) = f(0, x + y + z).$$

7. There are 2023 students in a school. A group of students is called *friendly* if each pair of students in it are friends. There is at least one friendly group of 1012 students at the school, but there is no friendly group of 1013 students. Prove that there is at least one student such that every friendly group of 1012 students contains this student.

8. Consider vectors (m, n, p) in 3D space with integer coordinates. We say that two vectors point in the same direction if each one is a multiple of the other, and we call a vector *primitive* if it has the shortest length among vectors pointing in its direction. Let P be a primitive vector and consider the collection $\mathcal{D}$ of vectors with integer coordinates which are orthogonal to P. Consider pairs of vectors in $\mathcal{D}$ which do not point in the same direction, and the parallelograms created by such pairs. Prove that the smallest possible area of such a parallelogram is equal to the length of P.

- Submit your solutions at https://forms.gle/9EBuvypU7ppDmprt8.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.

*The definitions of the complete bipartite graph and the complement of a graph can be found on Wikipedia.