Advanced Test 3

Stellenbosch Camp 2022

Time: $2\frac{1}{2}$ hours

1. For acute triangle $\triangle ABC$, a point Z interior to $\triangle ABC$ satisfies $ZB = ZC$. Suppose points X and Y lie outside $\triangle ABC$ such that $\triangle XAB \,|||\, \triangle YCA \,|||\, \triangle ZBC$. Prove that A, X, Y, Z are the vertices of a parallelogram.

2. For nonzero real numbers a, b, and c, show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

   if and only if two of a, b, and c are equal.

3. Can you tile a $10 \times 10 \times 10$ cube with $4 \times 1 \times 1$ blocks? (Standard tiling rules apply—cover the whole volume, no overlaps, etc.)

4. Consider an infinite sequence $(a_n)_{n=0}^{\infty}$ of positive integers such that for $n \geq 0$, we have that

$$a_{n+1} = a_n + b_n$$

   where b_n is the last (units) decimal digit of a_n. Show that the sequence contains infinitely many powers of 2 if and only if a_0 is not a multiple of 5.

5. Let I be the incenter of triangle $\triangle ABC$. Let D, E and F be the points of tangency of the incircle of $\triangle ABC$ with BC, CA and AB, respectively. The line ℓ passes through B and is parallel to DF. The lines ED and EF intersect ℓ at the points X and Y. Prove that $\angle XIY$ is acute.