Advanced Test 3 Solutions

Stellenbosch Camp 2022

1. For acute triangle $\triangle ABC$, a point Z interior to $\triangle ABC$ satisfies $ZB = ZC$. Suppose points X and Y lie outside $\triangle ABC$ such that $\triangle XAB \mathrel{|||} \triangle YCA \mathrel{|||} \triangle ZBC$. Prove that A, X, Y, Z are the vertices of a parallelogram.

   Solution:

$$\begin{aligned} &\angle ZCY = \angle ZCA + \angle ACY = \angle ZCA + \angle BCZ = \angle BCA \\ \implies &\frac{ZC}{BC} = \frac{YC}{AC} \qquad (\triangle ZCB \mathrel{|||} \triangle YCA). \end{aligned}$$

   Therefore $\triangle BCA \mathrel{|||} \triangle ZCY$ as the sides are proportional and they have a common angle. Similarly $\triangle BCA \mathrel{|||} \triangle BZX$.

   Since $BZ = ZC \implies \triangle BZX \equiv \triangle ZCY \implies XZ = YC = AY$ and $ZY = BX = XA$. Therefore $AXZY$ is a parallelogram.

2. For nonzero real numbers a, b, and c, show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

   if and only if two of a, b, and c are equal.

   Solution: If some pair of a, b, c are equal, we can assume without loss of generality that $a = b$ since the equation is symmetric. This yields:

$$1 + \frac{a}{c} + \frac{c}{a} = \frac{a}{c} + \frac{c}{a} + 1$$

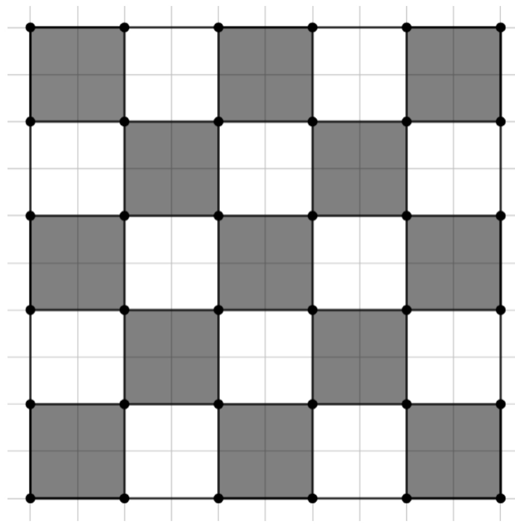
   Which is true. To show that some pair of a, b, c are equal given the expression, observe the following computation:

$$\begin{aligned} &\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \frac{a}{c} + \frac{c}{b} + \frac{b}{a} \\ &\quad \iff 0 = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - \frac{a}{c} - \frac{c}{b} - \frac{b}{a} \\ &\qquad\quad = \frac{a^2b + b^2c + c^2a}{abc} - \frac{a^2c + b^2a + c^2b}{abc} = \frac{a^2c + b^2a + c^2b - (a^2b + b^2c + c^2a)}{abc} \\ &\qquad\quad = \frac{abc + a^2c + b^2a + c^2b - (a^2b + b^2c + c^2a + abc)}{abc} \\ &\qquad\quad = \frac{(a-b)(b-c)(c-a)}{abc}. \end{aligned}$$

Thus the equation holds if and only if one of $a-b$, $b-c$, $c-a$ is zero, showing that some pair among a, b, c must be equal.

3. Can you tile a $10 \times 10 \times 10$ cube with $4 \times 1 \times 1$ blocks? (Standard tiling rules apply—cover the whole volume, no overlaps, etc.)

Solution: The answer is no. Consider the following colouring: there are 10 'layers' to the cube. Label these layers from 1 to 10. For layers 1,2,5,6,9,10, we colour the layer as follows:



For layers 3,4,7,8, we invert these colours. We note that each $4 \times 1 \times 1$ covers exactly 2 shaded blocks and 2 unshaded. But counting the total number of shaded blocks gives 504, while there are 496 unshaded blocks. Thus, we can not tile the shape as requested.

4. Consider an infinite sequence $(a_n)_{n=0}^{\infty}$ of positive integers such that for $n \geq 0$, we have that

$$a_{n+1} = a_n + b_n$$

where b_n is the last (units) decimal digit of a_n. Show that the sequence contains infinitely many powers of 2 if and only if a_0 is not a multiple of 5.

Solution: If a_0 is a multiple of 5, it will end in a 0 or 5 and be at least as big as 5. a_1 will then end in a 0, and the sequence becomes constant.

Assume a_0 is not a multiple of 5. a_1 will end in an even digit. We will now continue to cycle through the last digits $2 \to 4 \to 8 \to 6$, whose sum is 20. Thus, we will obtain all numbers of the form $a_1 + 20k$. We look at the last two digits of 2^n and note that they will repeat in the pattern

$$\mathbf{04}, 08, 16, 32, 64, 28, 56, 12, 24, 48, 96, 92, 84, 68, 36, 72, 44, 88, 76, 52, \mathbf{04}, 08, ...$$

If the second last digit of a_1 is even and the last digit is 4 or 8, then we will obtain all powers of 2 ending in 4 or 8. If the second last digit is odd and the last digit is 2 or 6, we will obtain all powers of 2 ending in 2 or 6.

If a_1 ends in a 4, and the second last digit is odd, then a_4 will be $a_1 + 18$, which will end in a 2 and have the second last digit odd. We will now obtain all numbers of the form $a_4 + 20k$, which again gives infinite powers of 2.

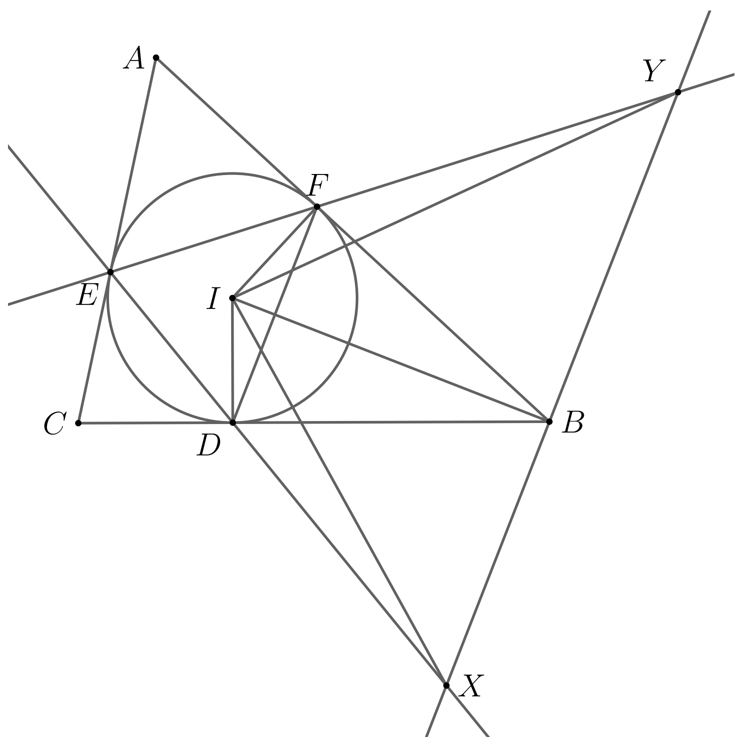
If a_1 ends in a 8 and the second last digit is odd, then $a_3 = a_1 + 14$ will end in a 2 and the second last digit will be odd. A similar argument applies.

If a_1 ends in a 2 and the second last digit is even, then $a_2 = a_1 + 2$ ends in a 4 and the second last digit is even.

If a_1 ends in a 6 and the second last digit is even, then $a_2 = a_1 + 6$ ends in a 2 and the second last digit is odd.

5. Let I be the incentre of triangle $\triangle ABC$. Let D, E and F be the points of tangency of the incircle of $\triangle ABC$ with BC, CA and AB, respectively. The line ℓ passes through B and is parallel to DF. The lines ED and EF intersect ℓ at the points X and Y. Prove that $\angle XIY$ is acute.

Solution:



Note that if $\triangle XIY$ is acute then $\angle XIY$ is acute. So the problem can be reduced to proving that

$$XI^2 + IY^2 > XY^2. \tag{1}$$

We notice that $IF = ID$ (radii) and $BF = BD$ (tangents), thus $DIFB$ is a kite, thus $IB \perp FD$, thus $IB \perp XY$ (corresponding angles, $DF \parallel XY$). We can now reduce our proof requirement (1)

$$\begin{aligned}
& XI^2 + IY^2 > XY^2 \\
\iff \quad & XB^2 + BI^2 + BI^2 + BY^2 > (XB + BY)^2 && \text{(Pythagoras)} \\
\iff \quad & XB^2 + BY^2 + 2BI^2 > XB^2 + 2XB \cdot BY + BY^2 \\
\iff \quad & BI^2 > XB \cdot BY. && (2)
\end{aligned}$$

Notice that $\angle BXD = \angle FDE = \angle AFE = \angle BFY$ (corresponding angles, tan-chord, vertically opposite respectively). Similarly we see $\angle BDX = \angle EDC = \angle EFD = \angle BYF$. Therefore $\triangle BDX \mathrel{|||} \triangle BYF$ (2 angles equal). This yields the equality of ratios

$$\frac{XB}{BD} = \frac{FB}{BY} \qquad \iff \qquad XB \cdot BY = FB \cdot BD = FB^2.$$

We deduce that

$$XB \cdot BY = FB^2 = BI^2 - IF^2 < BI^2.$$

So (2) is indeed satisfied.