Advanced Test 2

Stellenbosch Camp 2022

Time: $2\frac{1}{2}$ hours

1. William and Beatrice take turns placing Kings on a $n \times m$ chessboard. Kings cannot be placed on any of the 8 adjacent squares of Kings of *differing* colour. With William playing first as white, and Beatrice playing second as black, who has the winning strategy?

2. In $\triangle ABC$ let $\angle C = 90°$, and let Γ be the circle with diameter AC. Define points D and E on Γ such that D is on AB and $DE \parallel AC$. Let P be the intersection of AE and BC. Prove that
$$PC \cdot BC = AC^2.$$

3. Let $a_1, a_2, a_3, \ldots$ be a sequence of numbers defined by
   - $a_1 = l$
   - $a_2 = m$
   - $a_n = \dfrac{a_{n-1}a_{n-2}}{a_{n-1} + a_{n-2}}$ for all integers $n \geq 3$.

   Prove that the are infinitely many pairs of integers l and m such that a_{2022} is a positive integer.

4. Evaluate the following expression for all positive integers n:
$$\binom{2n}{0} - \binom{2n-1}{1} + \binom{2n-2}{2} - ... + (-1)^n \binom{n}{n}$$

5. Let x, y, and z be positive real numbers such that $xyz = 1$. Prove that
$$\frac{x^2y^2}{y^2(x+1)^2 + x^2 + x^2y^2} + \frac{y^2z^2}{z^2(y+1)^2 + y^2 + y^2z^2} + \frac{z^2x^2}{x^2(z+1)^2 + z^2 + z^2x^2} \leq \frac{1}{2}.$$

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