Advanced Test 5

Stellenbosch Camp 2022

Time: 4 hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x^2+y^2)$$

for all $x, y \in \mathbb{R}$.

2. The game of Chomp is played on an $n \times n$ board by Dylan (on behalf of SA) and Fionn (on behalf of Ireland) as follows: Dylan moves first. On a players move, they must place an X on any square (i, j) which does not yet have an X on it, and they also fill in an X on any square above and to the right of that square which does not yet have an X on it. That is, any square (s, t) with $s \geq i$ and $t \geq j$ which does not yet have an X also gets an X filled in. The person who places the last X loses. Is there a winning strategy for either player?

3. An acute triangle $\triangle ABC$ has circumcircle Γ and orthocenter H. M is the midpoint of the segment BC. Let P be the point of the intersection of the line HM with the minor arc BC of Γ. Let $Q \neq B$ be the intersection of the line BH with Γ. Prove that $PQ = BC$.

4. Let p be a prime and a, b, and c be positive integers such that

$$p = a + b + c - 1 \quad \text{and} \quad p \mid a^3 + b^3 + c^3 - 1.$$

Prove that at least one of a, b, and c is equal to 1.

5. Find all polynomials $P(x)$ with real coefficients such that

$$(x-8)P(2x) = 8(x-1)P(x)$$

for all real x.

6. A square is dissected into n^2 rectangles, where $n \geq 2$, by $n-1$ horizontal lines and $n-1$ vertical lines. Prove that one may choose $2n$ of these rectangles such that for any two of the chosen rectangles, one can be put completely into the other (perhaps, after some rotation).