Advanced Test 4

Stellenbosch Camp 2022

Time: $2\frac{1}{2}$ hours

1. You have an $n \times n$ chessboard which starts with all its squares white. An operation involves choosing a row or column and flipping the colour of every square in this row or column between white and black. Is it possible that for all $0 \leq k \leq n^2$, you can do some sequence of operations and end up with k black squares?

2. Show that 2022×2^n can be written as the sum of three distinct non-zero squares for every natural number n.

3. A triple of positive real numbers (a, b, c) is called *good* if a, b, c are the side lengths of a non-degenerate triangle, and *cute* if a, b, c are the sides of an acute angled triangle. Prove that (a, b, c) is *good* if and only if $(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is *cute*.

4. The incircle of $\triangle ABC$ is tangent to BC, CA, and AB at D, E, and F respectively. P, Q, and R are the incentres of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ respectively. Prove that DP, EQ, and RS are concurrent.

5. Every point in space is coloured red, blue, or green. Prove that for at least one of the 3 colours, the set of distances between points of that colour contains all positive real numbers.