Intermediate January Problem Set

Due: 14 January 2022

1. n teams participate in a round robin soccer tournament, in which each team plays each other team exactly once. Each team starts the tournament with 0 points. A team gains 3 points for every match they win, and 1 point for every match they draw. However, they lose 1 point for every match they lose. At the end of the tournament, the sum of the number of points each team has is 110. Find all possible values of n.

2. A natural number n is said to be *close to prime* if, for some prime numbers p and q, $n = pq$. What is the largest possible natural number k such that there is a sequence of k consecutive natural numbers, each of which is close to prime?

3. One root of $mx^2 - 10x + 3 = 0$ is two thirds of the other root. What is the sum of the roots?

4. Let D be a point on side AB of an acute-angled triangle ABC. The line through C perpendicular to AC meets the line AB at E. Let D' be the reflection of D across AC. The line CD' meets AB at F. Prove that
$$\frac{DE}{EF} = \frac{CD}{CE}.$$

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^{2020} + 2020y) = f(x^{2021} + 2021y) + f(x^{2022})$$
for all $x, y \in \mathbb{R}$.

6. We form all the subsets of $\{1, 2, 3, \ldots, 2021\}$ which do not contain two consecutive numbers. We then calculate the products of the numbers in each subset. Prove that the sum of the the squares of these products is equal to $2022! - 1$.

- Submit your solutions at https://forms.gle/Pv89v957obJMEAw26
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.

```
            _ _ _
        ,="`/ / /'=.
       / / / / / / /'.
      /_/_/_/_/_/ / / \     _.='/
   .-' - _ - _ -`"-,/ /\  .'_.='/
  / -_ - _ - _ - _ -\/.'|/_.=`/
 / @    _="`} _- _- _\.=/_. =";
/     -"_="} - _  - _ -=_-_"=;
\-.   '=._}          ,._--_=_;
 `-._     ._        /;' \ `"=.;
     `"\`;-.}_ _ _.;\ \/ \'=._\
        \ =.}\ \ \ \ \'   '._=_.\
         \_}`=._\_\.'`       '=.\
```