Advanced February Problem Set

Due: 11 February 2022

1. For which $z \in \mathbb{Z}$ does there exist a polynomial P with integer coefficients such that $P(2021) = 2022$ and $P(z) = z$?

2. Liam and Dylan play a game, alternating turns with Liam playing first. At the start, there is a pile of $1 + 2 + ... + 2022$ stones. On his turn, a player chooses a positive integer n between 1 and 2022 inclusive such that neither player has previously chosen this n. He then removes n stones from the pile and keeps them next to him. Once the pile is empty, the winner is the player with an even number of stones next to him. Who has the winning strategy?

3. Let ABC be a triangle. Let M, N, and P be points on sides BC, CA, and AB respectively. Let BN and CP intersect at D, CP and AM intersect at E, and AM and BN intersect at F. Let H be the centre of the circle passing through D, E, and F. Prove that $AH \perp BC$.

4. Consider the infinite sequence that starts with the six terms $1, 0, 1, 0, 1, 0$. From the seventh term onwards, each term is the units digit of the sum of the previous six terms. Show that the six numbers $0, 1, 0, 1, 0, 1$ never appear consecutively in the sequence. (In that order.)

5. Let $\mathbb{N}$ denote the set of positive integers. Consider the function $f : \mathbb{N} \times \mathbb{N} \to \{-1, 1\}$, defined as follows:

$$f(i,j) = \begin{cases} -1 & \text{if } i = 1 \text{ or } j = 1 \\ f(i-1,j) \cdot f(i,j-1) & \text{if } i > 1 \text{ and } j > 1. \end{cases}$$

   Determine the largest $i \leq 2021$ such that $f(i, 2021) = -1$.

6.

- Submit your solutions at https://forms.gle/Pv89v957obJMEAw26
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.

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