Advanced Test 3

Stellenbosch Camp 2021

Time: $2\frac{1}{2}$ hours

1. Point O is in the interior of an equilateral triangle ABC. Prove that there exists a triangle with edge lengths equal to OA, OB and OC.

2. Aaron attempts all six questions on the SAMO senior paper. For each question, his mark is an integer from 0 to 7. He never scores more points on a later question than on any earlier question. How many different possible sequences of six marks can he achieve?

3. Find all natural numbers n such that there exist integers $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ such that

$$\left(\sum_{k=1}^{n} a_i^2\right)\left(\sum_{k=1}^{n} b_i^2\right) - \left(\sum_{k=1}^{n} a_i b_i\right)^2 = n.$$

4. A polynomial $p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + 1$ has nonnegative coefficients and all of its roots are real numbers. Prove that $p(2) \geq 3^n$.

5. Let triangle ABC have circumcircle Γ. Let D be the intersection of the altitude from A and Γ, and let E, F be the feet of the altitudes from B and C to AC and AB respectively. Let H be the orthocenter of ABC and M the midpoint of AH. A line parallel to EF, passing through M, intersects AB and AC at P and Q respectively. Prove that $\angle PDF = \angle QDE$.

- Submit your solutions at https://forms.gle/T9HNgZgj8EhypBnR6
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.