Advanced Test 1

Stellenbosch Camp 2021

Time: $2\frac{1}{2}$ hours

1. How many ways are there to choose 24 black squares from a standard chess board (with 32 black and 32 white squares alternating) such that exactly three squares are chosen in each row and in each column?

2. Triangle ABC has $AC = BC$ and is right-angled at C. Points K, L are chosen on side AC and points M, N on side BC. It turns out that K and L divide the side AC into three equal parts. Similarly, M and N divide BC into three equal parts. Prove that there exists precisely one point P interior to ABC such that $\angle KPL = \angle MPN = 45°$.

3. If a, b, and c are real numbers in $(0, 4)$, prove that at least one of the numbers

$$\frac{1}{a} + \frac{1}{4-b}, \quad \frac{1}{b} + \frac{1}{4-c}, \quad \text{and} \quad \frac{1}{c} + \frac{1}{4-a}$$

   is greater than or equal to 1.

4. Find all natural numbers n such that every prime divisor of $n^6 - 1$ is also a prime divisor of $(n^2 - 1)(n^3 - 1)$.

5. A 30×30 grid is covered by 1×2 non-overlapping rectangular pieces (placed horizontally or vertically) until no more rectangular pieces can be added. (The pieces are placed in arbitrary locations on the grid, but each piece covers two of the cells in the grid.) Prove that when no more rectangles can be added, at least 300 small rectangles have been placed.

- Submit your solutions at https://forms.gle/T9HNgZgj8EhypBnR6
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.