Intermediate Test 4

Stellenbosch Camp 2021

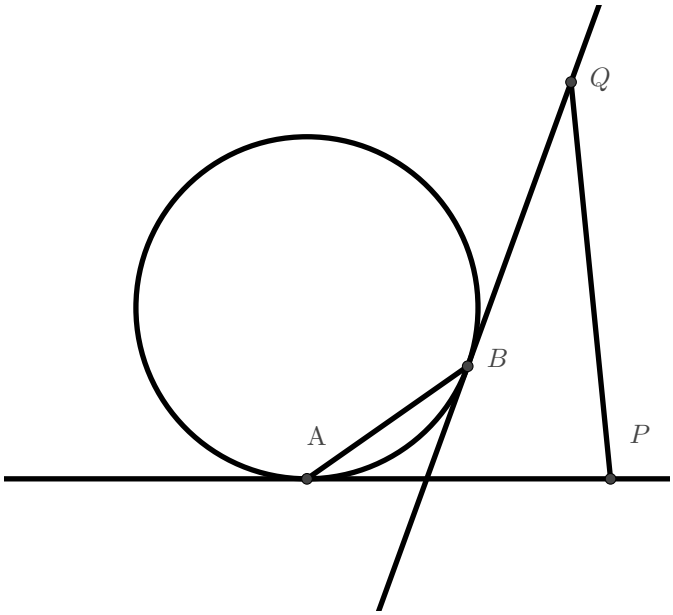
Time: $2\frac{1}{2}$ hours

1. Let $a, b, c, d, e, f \in \mathbb{R}$ satisfy the following two equations:

$$\frac{d}{a} + \frac{b}{e} = 1 \qquad \text{and} \qquad \frac{e}{b} + \frac{c}{f} = 1.$$

   Prove that $abc + def = 0$.

2. From points A and B on a circle, equal tangents AP and BQ are drawn, as shown. Prove that AB bisects PQ.



3. A *tetromino* is a rectangle with length 4 units and breadth 1 unit. Find all possible natural numbers n such that an $n \times n$ square grid can be exactly tiled by tetronimo tiles.

4. Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1} (p-k)!k! \equiv \frac{p-1}{2} \pmod{p}.$$

Please note that there is a second page.

5. Phil the physicist tries to come up with some equations describing the behaviour of a bucket of salt water as it is diluted by adding pure water. He uses an m kilogram bucket initially containing s kgs of salt and w kgs of water, before x kgs of extra water is added. He uses k as a positive proportionality constant. Phil realises that the following two expressions are important in his study.

$$\frac{ms+mw}{\frac{kms}{ms+mw}+1}+mx \qquad \text{and} \qquad \frac{ms+mw+mx}{\frac{kms}{ms+mw+mx}+1}.$$

Help Phil by determining which one of these expressions is always bigger than or equal to the other.

- Submit your solutions at https://forms.gle/T9HNgZgj8EhypBnR6
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.