Advanced Test 5

Stellenbosch Camp 2021

Time: $4\frac{1}{2}$ hours

1. Let ABC be a triangle with $AB = AC$, and let D be the midpoint of BC. Let E be the reflection of D across AC. Let F be the point on AB such that $FE||BC$. Prove that $AB \perp FC$.

2. Let A be a natural number with 2^n digits, all of which are equal. Show that A has at least n distinct prime factors.

3. Let A and B be two points in the plane with distance 1 between them. Find the maximum length of a path from A to B, comprising at most n straight line segments, with the property that at every point along the path (not only at the endpoints of the segments) as one moves from A to B the distance to B is reducing.

4. At an $n+1$ day long maths camp with n students, some pairs of students start the camp as friends with each other. On the i^{th} evening ($1 \leq i \leq n$), the i^{th} student hosts a mafia game and invites all their friends at the camp (whether they started the camp as friends or became friends on an earlier evening). At any mafia game, participants have such a good time that each pair of students at the game become friends with each other. On evening $n+1$, one of the n students in the camp hosts one final mafia game and invites all their friends. Prove that no new friendships are formed in this final mafia game.

5. Let ABC be a triangle with circumcircle Γ. The line tangent to Γ at A meets BC at D. Let the circumcircle of $\triangle ACD$ be Ω. The tangents to Ω at A and C meet at T. The lines DT and AB meet at M. Prove that M is the midpoint of AB.

6. Find all prime numbers p and q such that

$$a^{3pq} \equiv a \pmod{3pq}$$

for all integers a.

- Submit your solutions at https://forms.gle/T9HNgZgj8EhypBnR6
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.