Intermediate Test 1 Solutions

Stellenbosch Camp 2021

1. In rectangle $ABCD$, $|\overline{AB}| = 1$, $|\overline{BC}| = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. Lines AF and DH intersect at point X, and lines BF and CH intersect at point Y. What is the area of rectangle $XHYF$?

   Solution:

2. A permutable prime is a prime number which remains prime when its digits are rearranged in all possible ways. Show that a permutable prime cannot be a four digit number where all the digits are distinct.

   Solution: Note that no four-digit permutable prime can contain any of the digits $0, 2, 4, 5, 6, 8$ since any number ending in these digits is divisible by 2 or 5 and hence is not prime. Hence the four distinct digits must be $1, 3, 7, 9$. However, this is impossible since the number $1379 = 7 \times 197$ is not prime.

3. Show that, for all real numbers a, b, and c,

$$2021 \geq 4a + 18b + 88c - a^2 - b^2 - c^2.$$

   Solution: We complete the square to obtain:

$$\begin{aligned} 2021 - 4 - 81 - 1936 &\geq (-a^2 + 4a - 4) + (-b^2 + 18b - 81) + (-c^2 + 88c - 1936) \\ \iff 0 &\geq -(a-2)^2 - (b-9)^2 - (c-44)^2 \\ \iff (a-2)^2 + (b-9)^2 + (c-44)^2 &\geq 0 \end{aligned}$$

   This follows from the fact that all squares are nonnegative.

4. How many ways are there to choose 24 black squares from a standard chess board (with 32 black and 32 white squares alternating) such that exactly three squares are chosen in each row and in each column?

   Solution: Since each row and each solumn has 4 black squares, the question is equivalent to choosing 8 black squares such that each row and each column has one chosen square. Considering rows 1,3,5 and 7, we see there there are 4 options to choose the square in row 1, then 3 options in row 3, 2 options in row 5, and 1 in row 7, for a total of $4! = 24$ options. Similarly, there are 24 options to choose the squares in rows 2,4,6 and 8. This gives a total of $24 \times 24 = 576$ possibilities.

5. **Solution:**