Advanced Test 4

Stellenbosch Camp 2021

Time: $2\frac{1}{2}$ hours

1. Let p be an odd prime. Prove that

$$\sum_{k=1}^{p-1}(p-k)!k! \equiv \frac{p-1}{2} \pmod{p}.$$

2. Phil the physicist tries to come up with some equations describing the behaviour of a bucket of salt water as it is diluted by adding pure water. He uses an m kilogram bucket initially containing s kgs of salt and w kgs of water, before x kgs of extra water is added. He uses k as a positive proportionality constant. Phil realises that the following two expressions are important in his study:

$$\frac{ms+mw}{\frac{kms}{ms+mw}+1}+mx \qquad \text{and} \qquad \frac{ms+mw+mx}{\frac{kms}{ms+mw+mx}+1}.$$

   Help Phil by determining which one of these expressions is always greater than or equal to the other.

3. Let $ABCD$ be a convex quadrilateral with an inscribed circle whose centre is O. Let W and X be the feet of the perpendiculars from A onto OB and OD respectively. Let Y and Z be the feet of the perpendiculars from C onto OB and OD respectively. Prove that W, X, Y, and Z are concyclic.

4. Six points in space are given so that the pairwise distances between them are all different. Consider the triangles formed by the edges between the points. Prove that the shortest side in one of these triangles is at the same time the longest side in some other triangle.

5. Find all functions $f : \mathbb{N}_0 \to \mathbb{Z}$ such that

$$f(a)f(b)f(c)f(d) = f\left(a^2+b^2+c^2+d^2\right)$$

   for all $a, b, c, d \in \mathbb{N}_0$.

- Submit your solutions at https://forms.gle/T9HNgZgj8EhypBnR6
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.