Advanced January Problem Set

Due: 14 January 2022

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all positive real numbers x and y,

$$f(x^{2020} + 2020y) = f(x^{2021} + 2021y) + f(x^{2022}).$$

2. We form all the subsets of $\{1, 2, 3, \ldots, 2021\}$ which do not contain two consecutive numbers. We then calculate the products of the numbers in each subset. Prove that the sum of the the squares of these products is equal to $2022! - 1$.

3. Let a, b, and c be positive real numbers such that $a + b + c = 3$. Show that

$$a^b b^c c^a \leq 1.$$

4. We call a solid tetrahedron that has all side lengths 1 a *unit tetrahedron*. Consider a unit tetrahedron T in space. We want to place more unit tetrahedrons around T such that each additional tetrahedron has one of its faces overlapping partially with one of the faces of T (the overlapping region must have positive surface area), and such that the interiors of the tetradehra are not allowed to overlap. What is the maximum number of additional unit tetrahedra one can place around T? (Points for this problem will be awarded according to how many tetrahedra you are able to place around T.)

5. For which prime numbers p do there exist infinitely many functions $f : \mathbb{N} \to \mathbb{N}$ satisfying:

   - $f(1) = p$,
   - $f(n) = 1$ for infinitely many n,
   - $f(2021) = 2021$, and

   - $f(mn) = \begin{cases} \operatorname{lcm}\{f(m), f(n)\} & \text{if } p \mid mn \\ \gcd\{f(m), f(n)\} & \text{if } p \nmid mn \end{cases}$.

6. Let P be a point interior to an acute-angled triangle ABC. Let D, E, and F be the feet of the perpendiculars from P to BC, CA, and AB respectively. Denote the perimeters of $\triangle DEF$, $\triangle AEF$, $\triangle BDF$, and $\triangle CDE$ with s_0, s_1, s_2, s_3. Prove that P is the circumcentre of $\triangle ABC$ if and only if

$$s_0 \leq \min\{s_1, s_2, s_3\}.$$

- Submit your solutions at https://forms.gle/Pv89v957obJMEAw26
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.