Level Allocation Test
MEMO
For internal use only
Stellenbosch Camp 2021

The purpose of this test is to distinguish between students that should be part of the Intermediate stream, and those that may proceed to the Advanced stream. Each of the four topics is tested with five questions each. The philosophy behind each question is one of " if a candidate does not solve this, they should be in the Intermediate stream ", and " if a candidate does solve this, they may not find the Intermeidate stream to be enriching ". It is intended that any candidate obtaining a score ≤ 10 would likely benefit more from Intermediate, while a candidate obtaining > 10 can perhaps be placed in the Advanced stream. The applicability of this should be reviewed once the test has been written.

The use of jargon, and lack of definitions in questions is intentional as this test aims to assess a candidate's familiarity with certain terms or notation which one should be comfortable with in the Advanced stream.

1 Algebra

1. Let x, y and z be positive real numbers that satisfy $xy = 1$, $yz = 2$, $zx = 3$. What the value of xyz?

   A) $\sqrt{2}$ B) 2 C) $\boldsymbol{\sqrt{6}}$ D) 6 E) Impossible

2. Given that x satisfies $x + \frac{1}{x} = 10$, what is the value of $\frac{x^2-x+1}{x^2+1}$?

   A) $\frac{1}{100}$ B) $\boldsymbol{\frac{9}{10}}$ C) $\frac{91}{100}$ D) $\frac{99}{100}$ E) 1

3. How many roots (not necessarily unique) does the polynomial $P(x) = x^7 - 5x^3 + x + 1$ have in the complex numbers?

   A) Unknown B) 1 C) 3 D) 5 E) **7**

4. Given a sequence a_1, a_2, ..., a_{2021} satisfying $a_1 a_2 \cdots a_{2021} = 2021$, and $a_i \in \mathbb{R}^+ \ \forall i \in \{1, 2, \cdots, 2021\}$, what is the minimum of $\sum_{i=1}^{2021} a_i$?

   A) $\boldsymbol{2021^{\frac{2022}{2021}}}$ B) 2021 C) 1 D) $2021 + \frac{1}{2021}$ E) 0

5. Consider a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(x+y)^2 = f(x^2) + f(y^2)$ for all x, $y \in \mathbb{R}$. How many possible values are there for the expression $f(x) + f(y)$?

   A) 0 B) 1 C) 2 D) **3** E) 4

2 Combinatorics

1. Let $A = \{2, 3, 5, 7\}$ and $B = \{3, 4, 5, 6\}$, what is $|A \cap B|$?

   A) 0 B) **2** C) 4 D) 6 E) 8

2. Given a set of all of the 1×1 squares on an 8×8 chessboard (each square is assumed to be unique), how many subsets of this set do not contain any white squares?

A) 0 B) 32 C) **2^{32}** D) 32! E) 2^{64}

3. SAMF offers Liam five payment options (A-E) for his work at the camp. These will all be in the form of a bag full of R1 coins. Each bag is labelled with a set indicating that Liam can construct a bijection between the coins in the bag and that set. Which bag is the biggest?

A) $\mathbb{N}$ B) $\mathbb{Z}$ C) $\mathbb{Q}$ D) ϕ E) $\mathbb{R}$

4. What is the smallest natural number, n, such that any set of n integers will always contain two numbers whose sum or difference is a multiple of 10?

A) 5 B) 6 C) **7** D) 10 E) 11

5. The numbers 1, 2, ..., 2021 are written on a blackboard. Every minute, Aaron erases two numbers a and b and replaces them with the single number $|a - b|$. Which of the following could be the number he has on the board after 2020 minutes?

A) 156 B) **57** C) 48 D) 1024 E) 2020

3 Geometry

1. Let ABC and ABD be two triangles with $C \neq D$, B is the midpoint of segment CD, and $|AB| = |BC|$. What is the value of $\angle CAD$?

A) 72° B) 135° C) **90°** D) 108° E) Impossible

2. Let $ABCD$ be a square, and let M and N be the midpoints of BC and CD respectively. Let P be the intersection of AM and BN. What fraction of the square's area is $\triangle BPM$?

A) $\frac{1}{8}$ B) $\frac{1}{100}$ C) $\frac{1}{24}$ D) $\frac{1}{10}$ E) $\mathbf{\frac{1}{20}}$

3. Consider the following five statements:

- In non-degenerate triangle ABC, it is guaranteed that $|AB| < |BC| + |CA|$.
- The incentre, centroid, and circumcentre of a triangle are collinear.
- It is possible to divide an arbitrary angle into three equal parts using only a ruler and compass.
- In quadrilateral $ABCD$, it is always true that $|AB| \cdot |CD| + |AD| \cdot |BC| \geq |AC| \cdot |BD|$.
- It is possible to inscribe a circle inside any non-degenerate triangle, and the centre of this circle is the intersection of perpendicular bisectors of the sides of the triangle.

How many of these are correct?

A) 1 B) **2** C) 3 D) 4 E) 5

4. In triangle ABC, let H be the orthocenter. Let A', B', and C' be the reflections of H over lines BC, CA, and AB respectively. How many cyclic quadrilaterals are there that use only the points A, B, C, H, A', B', C'?

A) **15** B) 35 C) 0 D) 4 E) 7

5. Consider $\triangle ABC$ with $|AB| = 8$, $|BC| = 9$, and $|CA| = 6$. Let P be a point inside $\triangle ABC$ and let D, E, and F be the intersections of the extensions of AP, BP, and CP with BC, CA, and AB respectively. If $|AE| = 2$, and $|AF| = 3$, what is $\frac{|AP|}{|PD|}$?

A) $\mathbf{\frac{11}{10}}$ B) $\frac{3}{2}$ C) $\frac{9}{8}$ D) $\frac{14}{9}$ E) $\frac{5}{4}$

4 Number Theory

1. What is the largest positive integer that cannot be written as the sum of non-negative multiples of 11 and 13?

   A) **119** B) 93 C) 73 D) 2022 E) 128

2. How many natural numbers below 1001 are relatively prime to 1001?

   A) 168 B) **720** C) 852 D) 900 E) 1000

3. Consider the following four statements:

   - Every positive integer can be written as the sum of two squares.
   - If $a \mid b$ and $a \mid c$, then $b \mid c$.
   - Given a, $b \in \mathbb{N}$, we have that $\operatorname{lcm}(a, b) = \frac{ab}{\gcd(a,b)}$.
   - All prime numbers are odd.

   How many of these are correct?

   A) 0 B) **1** C) 2 D) 3 E) 4

4. $k \in \mathbb{N}$ is chosen such that $k^2 + 2016k + 2015$ is a perfect square. Which of the following is a valid value for k?

   A) 1544 B) 3471 C) 6190 D) **8585** E) 9381

5. What are the last two digits of $7^{7^{7^{7}}}$?

   A) 01 B) 07 C) 49 D) **43** E) 85