

# Elementary Cellular Automata as Multiplicative Automata

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## Software

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## Summary

Elementary cellular automata (ECA) are a set of simple binary programs in the form of truth tables called Wolfram codes that produce complex output when done repeatedly in parallel, and quaternions are frequently used to represent 3D space and its rotations in computer graphics. Both are well-studied subjects, this Java library puts them together in a new way. This project changes classical additive cellular automata into multiplicative automata ([Wolfram, 2002, p. 861](#)) via permutations, hypercomplex numbers, and pointer arrays. Valid solutions extend the binary ECA to complex numbers, produce a vector field, make an algebraic polynomial, and generate some very interesting fractals.

The code repository is at <https://github.com/dmcki23/MultiplicativeECA>.


## Statement of Need

Very loosely analogous to DeMorgan's law in Boolean algebra, the main algorithm produces several multiplicative versions of any given standard additive binary Wolfram code up to 32 bits and is written to support user supplied complex input at row 0 with choice of type of multiplication tables and partial product tables among other parameters. An algebraic polynomial of the automata that works with real and complex numbers is produced, and the hypercomplex 5-factor identity solution allows for the complex extension of any binary cellular automata. The GUI, though not required, allows for visual exploration of solutions with easy access to various parameters. The Java this is written in is designed to integrate well in other programs, such as Mathematica's JLink or Matlab, and is documented with Javadoc. The Cayley-Dickson and Fano construction libraries may be of value to the open source community as well.

There are other cellular automata implementations, Mathematica ([Inc., n.d.](#)), CellPyLib ([Antunes, 2021](#)), a JOSS Python project from three years ago, and books that cover related territory ([Ceccherini-Silberstein & Coornaert, 2023](#)). What sets this library apart is its focus on the conversion of existing binary cellular automata from additive to multiplicative and the extension to complex numbers. This is not intended to replace existing cellular automata utilities but to supplement them directly or indirectly by illustrating in open source a new kind of set of operations on a Wolfram code and its output. There are useful things you can build on this, like making Bloch spheres out of two layers of complex number output, modifying the multiplication tables by Gray code or k-cycle, and applying discrete Fourier transforms everywhere, that are out of scope and would be the subject of another paper. Some other scope issues are discussed in the readme.

37 **Functions**

38 Hypercomplex unit vector implementation

	Negative sign bit	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Complex	2's place	1	i	-1	-i												
Quaternions	4's place	1	i	j	k	-1	-i	-j	-k								
Octonions	8's place 	1	e1	e2	e3	e4	e5	e6	e7	-1	-e1	-e2	-e3	-e4	-e5	-e6	-e7

39

40 The Cayley-Dickson (CD) and Fano support classes are discussed in greater detail in the

41 readme and the documentation, they along with the Galois class provide sets of multiplication

42 tables to be compared with cellular automata. The CD multiplication implementation permutes

43 the steps of splitting and recombining hypercomplex numbers to increase the scope of the CD

44 equation,  $(a,b)x(c,d) = (ac - d * b, da + bc*)$ , where  $*$  is the conjugate. It verifies itself by

45 producing the symmetric group of its degree when interacting with other CD multiplications.

46 The Fano library octonions produce a triplet that is a linear match to the CD octonions as

47 triplets{0} when the up and down recursion factoradics are equal, and produce the triplet set

48 of John Baez's Fano plane as triplets{10}. (Baez, 2001).

49

50 The main algorithm uses a set of permutations operating on cellular automata input, each

51 permutation permuting the neighborhood, becoming a factor, with four kinds of multiplications.

52 The multiplication tables are input as 2D but used as N-D, where N=numFactors.

Table with 5 columns: Multiplications A, Multiplications B, Multiplications C, Multiplications D. Rows include Type, Size, Function, Scope, Produces, Data type, Base 2 sum of neighborhood, and N-th root in normalization.

53 Multiplications A, additive to multiplicative
54
55 r = specific Wolfram code
56
57 n = binary neighborhood = 1columnZero + 2columnOne+ 4columnTwo, points to its value in
58 r
59 h = hypercomplex unit vector from binary
60 H = inverse of h, binary value from hypercomplex unit vector
61 p = a permutation of the neighborhood
62 using hypercomplex multiplication, a valid permutation set produces:
63 WolframCode(r, n) = WolframCode(r, H(h(p(n)) h(p(n)) \* h(p(n)) ... numFactors) )
64
65 The first set of multiplications, column A, brute forces all possible sets of permutations on all
66 possible binary neighborhoods of the Wolfram code. A permutation in the set rearranges the

columns of the input neighborhood, these become a set of factors. A valid set of permutations is one that, for all possible input neighborhoods, the set of constructed factors using the permuted neighborhoods always multiplies out to a value that points to an equal value within the Wolfram code. The set of multiplication results is a pointer array that reproduces the original Wolfram code for every possible binary neighborhood.

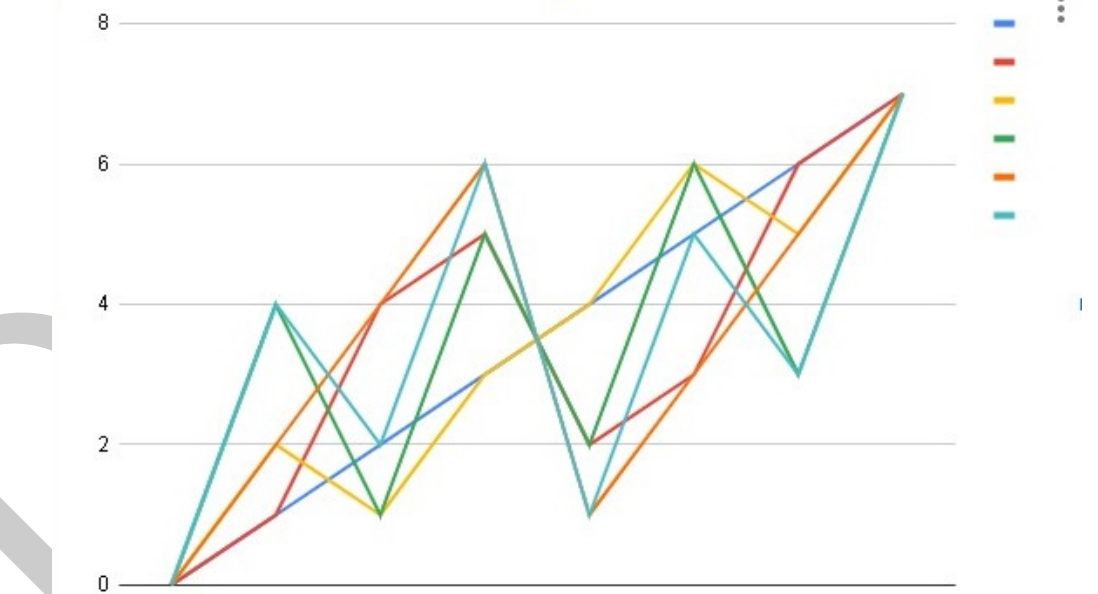
Identity solutions of 5 factors using all zero permutations exist for Wolfram codes up to 32 bits in this library using hypercomplex numbers and Galois addition. Galois multiplication takes a mix of numbers of factors to get the identity multiplication result array, there is a function in the GaloisField class that provides it. The factors constructed are a loose diagonal through the multidimensional multiplication table, starting at the origin and ending at the opposite corner while zig-zagging. The path lengths of each factor and the result are included in ValidSolution results.

Permutations of 3 bit neighborhoods

Permutation: 0, [0, 1, 2, 3, 4, 5, 6, 7]  
Permutation: 1, [0, 1, 4, 5, 2, 3, 6, 7]  
Permutation: 2, [0, 2, 1, 3, 4, 6, 5, 7]  
Permutation: 3, [0, 4, 1, 5, 2, 6, 3, 7]  
Permutation: 4, [0, 2, 4, 6, 1, 3, 5, 7]  
Permutation: 5, [0, 4, 2, 6, 1, 5, 3, 7]

Flattened path through a six dimensional multiplication table

Six factors, permutation set = {0,1,2,3,4,5}



Multiplications B and C apply a valid solution from the first set of multiplications to any given individual neighborhood with binary, non-negative real, and complex values. Multiplication B is the Cartesian product of the permuted neighborhoods, using a closed partial product table to generate a polynomial. Multiplication C does the binary sum of complex neighborhood, then multiplies as complex. Both B and C take the  $n$ -th root of the result, with  $n = \text{numColumns}$  and  $n = \text{numFactors}$ , respectively. Multiplications B and C both include a binary weighted sum of the neighborhood, same as the construction of the factors from A, though B and C use complex. B, as part of the normalization and C as the construction. Multiplication C is the permutation composition product. B, just before the normalization is a neighborhood of

94 multiplication results, with each column of it being a unit vector coefficient. This multiplication  
95 result neighborhood is permuted by the inverse of the permutation composition product to  
96 properly order the output vector.

97 Control Panel

ECA rule

Multiplication Table to use

Specific solution to use

Degree: 2 = quaternions, 3 = octonions, etc., if applicable

Number of factors to use

Number of rows in the ECA, 1 row = 3 bit neighborhood, 2 rows = 5 bit neighborhood

Partial product table, size = places x places

Keeps functions from running longer than the user want, in seconds

the calculate button produces all solutions for the chosen parameters

this button re-randomizes and displays the ECA rule with the particular solution number chosen

Deep search using above parameters

Width of random input 200

Number of factors in logic gate search

Logic gate, AND = 8, OR = 14, XOR = 6, etc

Logic gate solution:

Which multiplication table to use

Partial product table

Refresh logic gate solutions

Display specific logic gate solution

Search all logic gates for solutions and crossreference gates that have solutions in common

Table Display Degree, 2 = quaternions, 3 = octonions, 4 = sedonions

Cayley-Dickson permutation number, (cdz, \_\_), down in recursion

Cayley-Dickson permutation= number, (\_\_ , cdo), up in recursion

Fano plane octonions

Galois Field, Prime

Galois Field, Power

Length of permutations

Refresh permuted Cayley-Dickson solutions

Display tables with above parameters

Compare Fano-generated octonions with permuted Cayley-Dickson octonions

Compare permuted CD with permuted CD

Picks a random Wolfram code with 5 factors, identity solution

54

Permuted Cayley-Dickson

0

2

5

1

Galois addition, XOR, 3x3

30

Refresh

Display specific solution

Start deep search

5

6: XOR

XOR

Galois addition, XOR, 2x2

Refresh

Display specific solution

Deep logic gate search

2

0

0

0

2

1

4

Refresh

Display specific tables

Fano/CD Compare

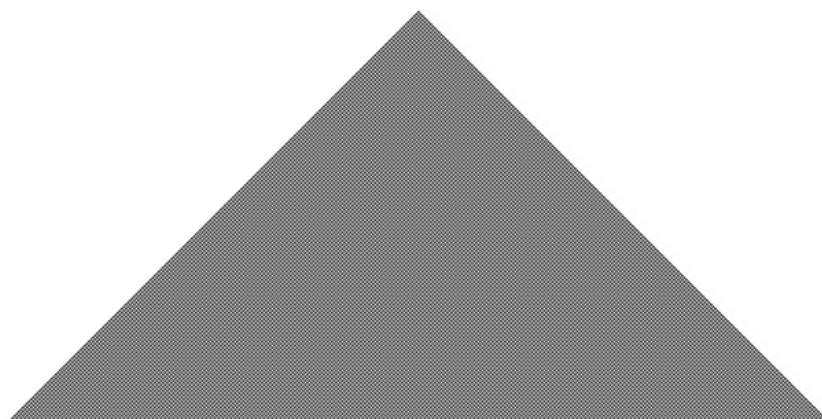
CD v CD

Random Wolfram Code

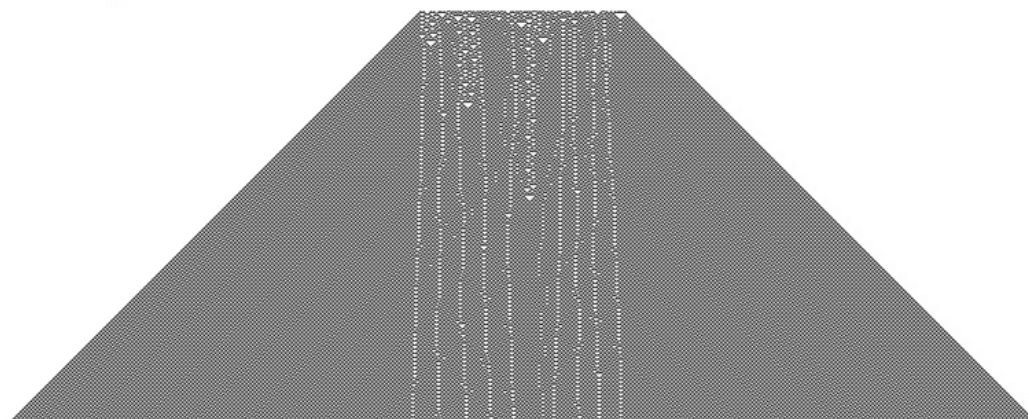
98  
99  
100 ECA 54, binary and non negative real



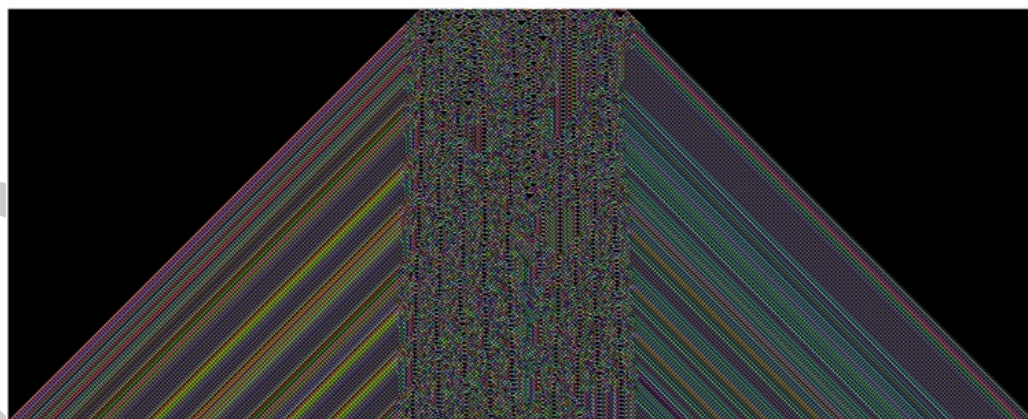
Single bit initial input:



Random initial input:



Same random initial input with solution applied to random  $(0,1)$  non negative real:



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102

103

ECA 54, solution parameters, including polynomial

```
ValidSolution
Wolfram code: [0, 1, 1, 0, 1, 1, 0, 0]
Permutation: 0 Permuted Axis: [0, 1, 2, 3, 4, 5, 6, 7]
times
Permutation: 0 Permuted Axis: [0, 1, 2, 3, 4, 5, 6, 7]
times
Permutation: 0 Permuted Axis: [0, 1, 2, 3, 4, 5, 6, 7]
times
Permutation: 0 Permuted Axis: [0, 1, 2, 3, 4, 5, 6, 7]
times
Permutation: 0 Permuted Axis: [0, 1, 2, 3, 4, 5, 6, 7]
times
-----
Equals: [0, 1, 2, 3, 4, 5, 6, 7]
Apply Wolfram code to multiplication result
Equals: [0, 1, 1, 0, 1, 1, 0, 0]
Original Wolfram code: [0, 1, 1, 0, 1, 1, 0, 0]

Permutation composition product: 0, inverse: 0

Multiplication table type: 0
2D multiplication table used:
[0, 1, 2, 3, 4, 5, 6, 7]
[1, 4, 7, 2, 5, 0, 3, 6]
[2, 3, 4, 5, 6, 7, 0, 1]
[3, 6, 1, 4, 7, 2, 5, 0]
[4, 5, 6, 7, 0, 1, 2, 3]
[5, 0, 3, 6, 1, 4, 7, 2]
[6, 7, 0, 1, 2, 3, 4, 5]
[7, 2, 5, 0, 3, 6, 1, 4]

numFactors: 5 numBits: 3

1*((a^5)*(b^0)*(c^0)) + 20*((a^3)*(b^1)*(c^1)) + 10*((a^2)*(b^3)*(c^0)) + 10*((a^2)*(b^0)*(c^3)) + 30*((a^1)*(b^2)*(c^2)) +
5*((a^0)*(b^4)*(c^1)) + 5*((a^0)*(b^1)*(c^4))

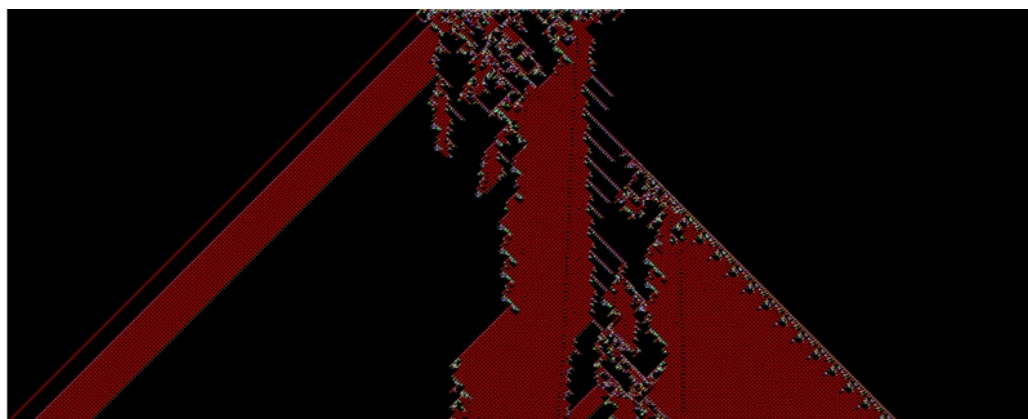
5*((a^4)*(b^1)*(c^0)) + 10*((a^3)*(b^0)*(c^2)) + 30*((a^2)*(b^2)*(c^1)) + 5*((a^1)*(b^4)*(c^0)) + 20*((a^1)*(b^1)*(c^3)) +
10*((a^0)*(b^3)*(c^2)) + 1*((a^0)*(b^0)*(c^5))

5*((a^4)*(b^0)*(c^1)) + 10*((a^3)*(b^2)*(c^0)) + 30*((a^2)*(b^1)*(c^2)) + 20*((a^1)*(b^3)*(c^1)) + 5*((a^1)*(b^0)*(c^4)) +
1*((a^0)*(b^5)*(c^0)) + 10*((a^0)*(b^2)*(c^3))
```

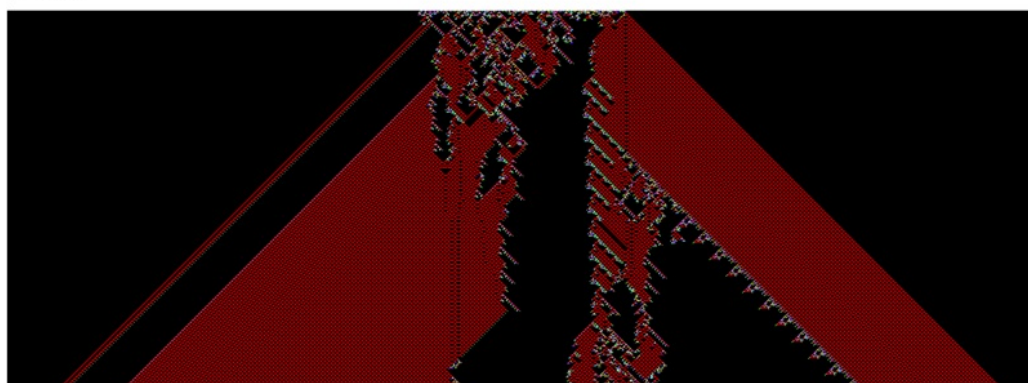
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106 ECA 54, solution output, complex



Complex part



107

108

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109

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