Sub-sigma Fields as Information

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Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-sigma field $\mathcal{A} \subset \mathcal{F}$. The sub-sigma field $\mathcal{A}$ is often thought of, intuitively, as containing a subset of the information in $\mathcal{F}$[1]. For example, we might think of $\mathbb{P}(B|\mathcal{A})$ as the probability of the event B given the information in $\mathcal{A}$.

Also recall that an event B is independent of a sub-sigma field $\mathcal{A}$ if $\mathbb{P}(B|A) = \mathbb{P}(B)$ for all $A \in \mathcal{A}$. So, again intuitively, the information in $\mathcal{A}$ does not tell us anything about the probability of event B occurring. Now a counter-example:

Consider a probability space on the unit interval, $\Omega = [0, 1]$. Let $\mathcal{G}$ be the sigma-field of all countable sets and sets whose complement is countable. So each set in $\mathcal{G}$ has measure 0 or 1 and so is independent of each event in $\mathcal{F}$. However, notice that $\mathcal{G}$ also contains all the singleton events in $\mathcal{F}$ (those sets which contain only a single $\omega \in \Omega$). So knowing which of the events in $\mathcal{G}$ occurred is equivalent to knowing exactly which $\omega \in \Omega$ occurred! So in one sense, $\mathcal{G}$ contains no information about $\mathcal{F}$ (it is independent of it), and in another sense it contains all the information in $\mathcal{F}$.

This example is taken from Billingsley example 4.6 [1].

[1] We mean 'information' in the casual sense, not in the sense of entropy.

References

[1] P. Billingsley, *Probability and Measure.* Wiley Series in Probability and Statistics, Wiley, 1995.