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Combinatorics
Assigment 09
11/10/2014

Please select two or three sections of the book that you have already read, and re-read those sections carefully. It is best if you pick something you feel uncertain about. Write down 3 bullet points describing 3 ideas in the reading that you found interesting.

Things I found interesting:

- Given the extremely formulaic nature of finding closed form for recusive fuctions, I'm incrediably excited about the idea of automating this process with an algorithm. The natural pitfalls would be finding generating functions for vrious standard inputs (such as sin, cosine, exponential, polynomials, binonials, etc) and a robust partial fraction engine. This seems doable and incrediably powerful!
- It is creepily beautiful how naturally combinatorics finds its way into egf and ops in natural ways. Other than just the difference by n!, I want to explore the combinatorial differences and implications of these definitions.
- I think its cool how you can apply the multiplcation table property for multiplaying GF's in a combinatorial fashion.

Shahriari 1.2.7 [(a)]
Let $f_0, f_2, \cdots f_n, \cdots$ denote the Fibbonaci sequence. For each of the following, conjecture a general foruma and then prove it:

(a) $f_0 + f_2 + f_4 + \cdots + f_{2n}$

(b) $f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2$

(a) $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$
Proof by Induction
Base Case $(n = 1)$ $f_0 + f_2 = 1 + 1 = f_{2\times 1+1} = 3 - 1$✓
Inductive Step: Assume $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$
to show $f_0 + f_2 + f_4 + \cdots + f_{2n+2} = f_{2n+3}$
$f_0 + f_2 + f_4 + \cdots + f_{2n+2} = f_0 + f_2 + f_4 + \cdots + f_{2n} + f_{2n+2} = f_{2n+1} - 1 + f_{2n+2} = f_{2n+3} - 1 = f_{2(n+1)+1} - 1$✓

(b) $f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$
Proof by Induction

Base Case $n = 1$ $f_0^2 + f_1^2 = 1 + 1 = f_1 \times f_2 = 2\checkmark$
Inductive Step: Assume $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$ WTS $\sum_{k=0}^{n+1} f_k^2 = f_{n+1} f_{n+2}$

$$\sum_{k=0}^{n+1} f_k^2 = f_{n+1}^2 + \sum_{k=0}^{n} f_k^2 = f_{n+1}^2 + f_n f_{n+1} = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}\checkmark$$

■

Shahriari 1.2.10
We have an 8×8 chess board with 64 squares. We want to tile this board with pieces that are made up of three squares as shown in the book. The three squares on the piece are identical to the squares on the board, and we want each piece of cover exactly three squares of the board. So we cut a piece out a the square Can we now tile the board with 21 pieces? Does the answer depend on which piece? Can you generalize and prove it?

Note: I thought the wording of this problem was incrediably ambiguous, as to whether the problem allowed for rotations. Therefore, I solved the problem either way and will let the graders ascertain which case is the truism
Case 1: You cannot rotate the tiles.
Denote the rows and columns of the chessboard as $\{1, 2, \cdots, 8\}$ such that the weight of the i, jth entry is x^{i+j} for $i, j \in \{1, 2, \cdots, 8\}$. By Multiplication Table, the total area of all 64 squares is

$$(\frac{1 - x^8}{1 - x})^2$$

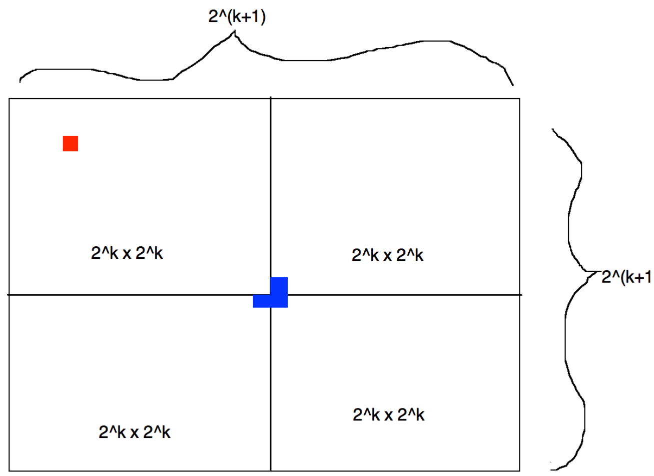
Considering the geometry of the tile, it is clear that the weighting of an arbitrary tile is $x^k(1 + x + x^2)$ for $k \in \{1, \cdots, 8\}$. Let $1 + x + x^2 = 0 \implies x = (-1)^{2/3}$. Plugging this is into the total weighting of the chessboard yeilds $\mathfrak{C} = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ (by Wolfram Alpha). Thus, this total weight minus an the weight of the square we will remove must be equal to zero for the tesselation to hold:

$$\mathfrak{C} - x^m = 0 \text{ for some } m \in \{0, \cdots, 16, \} \implies \frac{-1}{2} + \frac{\sqrt{3}}{2}i - (-1)^{2m/3} = 0$$

But there does not exist an m in the domain which satisfies the condtion. Because if $m\%3 = 1, (-1)^{2m/3} = (-1)^{2/3}$. If $m\%3 = 2, (-1)^{2m/3} = -\sqrt[3]{-1}$. If $m\%3 = 3, (-1)^{2m/3} = 1$. Thus it does not tesselate.
Case 2: You can rotate the tiles.
You *can* place the tiles on a $2^k \times 2^k$ board for $k \in \mathfrak{Z}$, and the square you remove doesn't matter.
Proof by induction
Base case ($n = 2$). Consider a 2×2 board. Cleary this board can be tesselated by simply removing one of the 4 squares and adding the necessary tile to complete the board.
Inductive Step: Assume you can place tiles on a $2^k \times 2^k$ board to show you can place tiles on a $2^{k+1} \times 2^{k+1}$ board.

$$2^{k+1} \times 2^{k+1} = 2(2^k) \times 2(2^k)$$

Consider the parititioning of the $2^{k+1} \times 2^{k+1}$ board into 4 equal parts, such that each is a $2^k \times 2^k$ board. Now choose the square to remove arbitrarily from the $2^{k+1} \times 2^{k+1}$ board. We will designate the square to be removed to be the upper right sub-$2^k \times 2^k$ board without loss of generality. So the inductive step now implies that this upper-right quadrant can be tesselated correctly. Now, take a tile and place it such that it occupies one square from



each of the remaining quadrants as shown above. Now by inductive hypothesis, these will tesslate as well. ✓ ■

Shahriari 1.3.1
We are given the following recurrence relation. Find the values of a_n for small n. Make a conjecture and prove by induction.

$$a_1 = 2, a_n = 3a_{n-1} + 2$$

$$a_1 = 2, a_2 = 8, a_3 = 2, a_n = 3^n - 1$$

Proof by Induction
Base Case $n = 1$ $a_1 = 2 = 3^1 - 1 = 2\checkmark$
Inductive Step: Assume $a_n = 3^n - 1$ to show $a_{n+1} = 3^{n+1} - 1$

$$a_{n+1} = 3a_n + 2 = 3 \times (3^n - 1) + 2 = 3^{n+1} - 1\checkmark$$

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Shahriari 1.3.2
Given $f(0) = -3$, find and prove a closed formula for $f(n)$

Embarrisingly for a combinatorics class and highly illuminating of my proclivities for computational and applied math, I found the following equation scouring R:

$$\boxed{f(n) = 2 \times 3^n - 5}$$

Proof by Induction
Base Case $n = 0$ $f(0) = -3 = 2(1) - 5\checkmark$
Inductive Step: Assume $f(n) = 2 \times 3^{n-1} - 5$ to show $f(n+1) = 2 \times 3^n - 5$

$$f(n+1) = 3f(n) + 10 = 3 \times (2 \times 3^{n-1} - 5) + 10 = 2 \times 3^n - 5\checkmark$$

■

Shahriari 1.3.4
We have n dollars. Every day we buy exactly one of the following products: Mustard (1 dollar), Mint (2 dollars), Marjoram (2 dollars). Let $f(n)$ be the number of possible ways of spending all the money. Which one(s) of the following are true:

(a) $f(n) = 2f(n-1) + f(n-3)$;

(b) $f(n) = f(n-1) + \frac{n-1}{2}[3 + (-1)^n]$;

(c) $f(n) = f(n-1) + 2f(n-2)$;

(d) $f(n) = 2f(n-1) - f(n-2)$;

Combinatorial Argument Consider the case where we have n dollars. On the first day, we can either buy Mustard, Mint or Marjoram. If we buy Mustard, then the choices for the remaining days is simply $f(n-1)$. If we buy Mint or Marjoram, then the choices for the remaining days is simply $f(n-2)$. Thus $f(n) = f(n-1) + 2f(n-2)$ (which is c). We write the first four terms of the sequence through brute force. Notice our recurrence holds.

$$\{1, 3, 5, 11, 27, \cdots\}$$

We leverage proofs by contradiction to show (a) and (d) are not true.
a) True $\implies f(5) = 2f(4) + f(2) \implies 27 = 25 \rightarrow\leftarrow$ a) not true
b) True $\implies f(5) = f(4) + \frac{4}{2}[3 + (-1)^5] \implies 27 = 15 \rightarrow\leftarrow$ b) not true
d) True $\implies f(4) = 2f(3) - f(2) \implies 11 = 7 \rightarrow\leftarrow$ d) not true

■

Shahriari 1.3.8
Let $h(0) = 1$ and let $h(n)$ be the number of sequences

$$a_1, a_2, \cdots, a_n$$

with the condition that each a_i is 0,1 or 2 for $i = 1, \cdots, n$ and if both 0 and 1 occur in the sequence then the first 0 occurs before the first 1.

(a) Find $h(1)$ and $h(2)$.

(b) Find a recurrence relation for $h(n)$

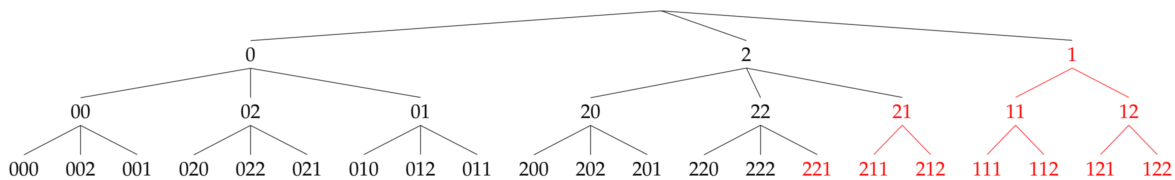
(c) Find a explicit formula

(a) $h(1) = 3, h(2) = 8, h(3) = 21, h(4) = 56, \cdots$

(b) If the sequence is all 1 and 2, you can only append a 1 or 1 to the end, (unless it is all 2's). there are $2^{n-1} - 1$ of those cases. Otherwise, of which there is $h(n-1) - 2^{n-1} + 1$ cases, you multiply be three because you can apped a 0,1 or 2

$$h(n) = 2(2^{n-1} - 1) + 3 \times (h(n-1) - 2^{n-1} + 1) = 3h(n-1) - 2^{n-1} + 1$$

(c) Below is the diagram representing the pattern, my friend, Brutus Forz, and I stared at this pattern to derive the following closed form solution:



$$h(n) = \boxed{2^n + \frac{1}{2}(3^n - 1)}$$

∎

Shahriari 1.3.9
Let h_n denote the number of ways of filling a $2 \times n$ array with 1×2 dominoes. Find a recurrence relation for h_n, as well as h_1, h_2. Use these to find h_8.

Through an intense battle with graph paper, me and my friend, Brutus Forz, noticed that this sequence is just the Fibonacci numbers with differing starting conditions: $h_1 = 1, h_2 = 2, h_3 = 3, \cdots$. Thus $h_8 = f_0 = 34$. ∎

Shahriari 9.1.1
Determine in closed form the gf for the sequence of cubes

$$0, 1, 3, \cdots, n^3, \cdots$$

Note the following facts:

$$\binom{n}{0} = 1$$
$$\binom{n+1}{1} = n+1$$
$$\binom{n+2}{2} = \frac{n^2 + 3n + 2}{2}$$
$$\binom{n+3}{3} = \frac{n^3 + 6n^2 + 11n + 6}{6}$$

We use these to express n^3 as a linear combination of these choices:

$$\begin{aligned}
n^3 &= 6\frac{n^3+6n^2+11n+6}{6} - 12\frac{n^2+3n+2}{2} - 7(n+1) + 13 \\
&= 6\binom{n+3}{3} - 12\binom{n+2}{2} - 7\binom{n+1}{1} + 13\binom{n}{0} \\
\sum_{n\geq 0} n^3x^n &= 6\sum_{n\geq 0}\binom{n+3}{3}x^n - 12\sum_{n\geq 0}\binom{n+2}{2}x^n - 7\sum_{n\geq 0}\binom{n+1}{1}x^n + 13\sum_{n\geq 0}\binom{n}{0}x^n \\
&= \boxed{\frac{6}{(1-x)^4} - \frac{12}{(1-x)^3} - \frac{7}{(1-x)^2} + \frac{13}{1-x}}
\end{aligned}$$

■

Shahriari 9.1.2 Find the gf for the sequence

$$a_0, a_1, \cdots, a_, \cdots$$

where $a_0 = 3$ and

$$a_n + 2a_{n-1} = n + 3$$

Let

$$G(x) = \sum_{n\geq 0} a_n x^n, H(x) = \sum_{n\geq 0}(n+3)x^n$$

$$\begin{array}{rcccc}
G(x) = & a_0 + a_1x + & a_2x^2 + \cdots + & a_nx^n \\
H(x) = & n + nx + & nx^2 + \cdots + & nx^n \\
F(x) = & 1 + x + & x^2 + \cdots + & x^n
\end{array}$$

$$\begin{array}{rcccc}
G(x) = & 3-2x & +9x^2 + \cdots & +a_nx^n \\
2xG(x) = & 0+6x & -4x^2 + \cdots & 2a_{n-1}x^n \\
-H(x) = & -3-4x & -5x^2 - \cdots & -(n+3)x^n
\end{array}$$

$$G(x) + 2xG(x) - H(x) = 0 \implies G(x) = \frac{H(x)}{1+2x}$$

$$H(X) = \sum_{n\geq 0}(n+3)x^n = \sum_{n\geq 0} nx^n + \sum_{n\geq 0} 3x^n = \frac{x}{(1-x)^2} + \frac{3}{1-x} = \frac{3-2x}{(1-x)^2}$$

$$G(X) = \frac{3-2x}{(1-x)^2(1+2x)}$$

through gnarly partial fractions yield:

$$= \frac{16}{9}\frac{1}{1+2x} + \frac{8}{9}\frac{1}{1-x} + \frac{1}{3}\frac{1}{(1-x)^2}$$

$$= \sum_{n\geq 0} \frac{16}{9}(-2)^n x^n + \frac{8}{9}x^n + \frac{1}{3}\binom{n+1}{n}x^n$$

$$\Longrightarrow a_n = \boxed{\frac{16}{9}(-2)^n + \frac{8}{9} + \frac{1}{3}(n+1)}$$

■

Shahriari 9.1.3

Use GFs to find an explicit formula for $f(n)$ if $f(0) = 0, f(1) = 1$ and $f(n) = 2f(n-2)$ for $n \geq 2$.

Let

$$G(x) = \sum_{n\geq 0} f(n)x^n$$

$$\begin{aligned} G(x) &= \quad 0 + 1x \quad && + f(2)x^2 + \cdots + \quad && f(n)x^n \\ -2x^2G(x) &= \quad -0 - 0 \quad && -2f(0)x^2 - \cdots \quad && -2f(n-2)x^n \end{aligned}$$

$$(1-2x^2)G(x) = x$$

$$G(x) = \frac{x}{1-2x^2} = \frac{1}{2} \times \frac{x}{(\sqrt{2}-2x)(\sqrt{2}+2x)} = \frac{1}{2} \times (\frac{A}{(\sqrt{2}-2x)} + \frac{B}{\sqrt{2}+2x})$$

gnarly partial fractions yield $A = \frac{1}{4}$ and $B = \frac{-1}{4}$

$$= \frac{1}{8\sqrt{2}} \times (\frac{1}{1-\sqrt{2}x} - \frac{1}{1+\sqrt{2}x}) =$$

$$= \boxed{\frac{\sqrt{2}^n - (-\sqrt{2})^n}{8\sqrt{2}}}$$

■

Shahriari 9.1.4

Use GFs to find an explicit formula for $f(n)$ if $f(0) = 1, f(1) = -2$ and $f(n) = 5f(n-1) - 6f(n-2)$ for $n \geq 2$.

Let

$$G(x) = \sum_{n\geq 0} f(n)x^n$$

$$\begin{array}{rllll} G(x) = & f(0)+f(1)x & +f(2)x^2+\cdots+ & f(n)x^n \\ -5xG(x) = & -0-5f(0)x & -5f(1)x^2+\cdots & -5f(n-1)x^n \\ 6x^2G(x) = & +0+0 & 6f(2)x^2+\cdots & 6f(n-2)x^n \end{array}$$

$$(1-5x+6x^2)G(x) = 1-7x$$

$$\begin{aligned} G(x) &= \frac{1-7x}{1-5x+6x^2} = \frac{1-7x}{(3x-1)(2x-1)} = \frac{A}{3x-1} + \frac{B}{2x-1} \\ &\text{gnarly partial fractions yield } A = 4 \text{ and } B = -5 \\ &= \frac{4}{3x-1} - \frac{5}{2x-1} = \frac{-4}{1-3x} + \frac{5}{1-2x} \\ &= \boxed{-4\times 3^n + 5\times 2^n} \end{aligned}$$

■

Shahriari 9.1.5 Use GFs to find an explicit formula for $f(n)$ if $f(0) = -1, f(1) = 0$ and $f(n) = 8f(n-1) - 16f(n-2)$ for $n \geq 2$.

Let

$$G(x) = \sum_{n\geq 0} f(n)x^n$$

$$\begin{array}{rllll} G(x) = & f(0)+f(1)x & +f(2)x^2+\cdots+ & f(n)x^n \\ -8xG(x) = & -0-8f(0)x & -8f(1)x^2+\cdots & -8f(n-1)x^n \\ 16x^2G(x) = & +0+0 & 16f(2)x^2+\cdots & 16f(n-2)x^n \end{array}$$

$$(1-8x+16x^2)G(x) = 8x-1$$

$$\begin{aligned} G(x) &= \frac{8x-1}{1-8x+16x^2} = \frac{8x-1}{(4x-1)^2} = \frac{A}{(4x-1)^2} + \frac{B}{4x-1} \\ &\text{gnarly partial fractions yield } A = 1 \text{ and } B = 2 \\ &= \frac{1}{(4x-1)^2} + \frac{2}{4x-1} = \frac{1}{(1-4x)^2} - \frac{2}{1-4x} \\ &= 4^n[\binom{n+1}{n} - 2] \\ &= \boxed{4^n(n-1)} \end{aligned}$$

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1.x.1 Martian hares are hermaphroditic. Each mature Martian hare produces one leveret during each breeding cycle. Each leveret takes one breeding cycle to mature into a fully grown hare and then lives for ever. Starting with a single Martian martion leveret at the beginning of the first breeding cycle, show that there are 4807526976 hares (including leverets) at the end of the 47th breeding cycle.

Me and my friend, Brutus Forz, found the values for the start of the sequence : $\{1,1,2,3,5,8,13,\cdots\}$ This along with the argument that the number of leverets at a given time n is just the total number of hares at $n-2$ (since all hares at $n-2$ will be adult by $n-1$ and produce a leveret at n) and the number of adults at n is simply the number at total hares at $n-1$ (since they will all be adult at n, yields the following recurrence relation:

$$f(n) = f(n-1) + f(n-2)$$

which is the Fibonacci sequence. It has been shown that the closed form the the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$$

Note however that this closed form starts at $f_0 = 0$ but out sequence is intitialized at $f_0 = 1$, thus we must evaluate f_{48}, which is indeed 4807526976. ∎

1.x.2
Jovian hares are hermaphroditic. Each mature Jovian hare produces one leveret during each breeding cycle. Each leveret takes two breeding cycles to mature into a fully grown hare and then lives for ever. Let (h_n) denote the number of hares, including leverets at the end of the nth breeding cycle. Write down a recurrence relation and initial conditions for the sequence (h_n). Evaluate h_{47}.

The number of adult Jovian hares on day n is equal to the total hares on day $n-3$, since each hare needs 2 days to become an adult. The number of leverets on day n is equal to the total number of hares on $n-1$ because . Thus we get the following recurrence relation:

$$j_n = j_{n-1} + j_{n-3}$$

where the first three terms $j_{1,2} = 1$ and $j_3 = 2$. Using the conditions and my main homey `R`, we get the total number of hares at the end of the 47th breeding cycle is 38789712 ∎

1.x.3 The recurrence relation $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ can be used to extend the Al Karaji triagnle upwards to negative valyes of n, if we insist $\binom{n}{k} = 0$ when k is negative. Complete the values of $\binom{n}{k}$ for $n \in \{-4,-3,-2,-1\}$ and $k \in \{0,1,2,3,4,5\}$.

-4	0	1	-4	10	-20	35	-56
-3	0	1	-3	6	-10	15	-21
-2	0	1	-2	3	-4	5	-6
-1	0	1	-1	1	-1	1	-1
0	0	1	0	0	0	0	0
1	0	1	1	0	0	0	0
2	0	1	2	1	0	0	0
3	0	1	3	3	1	0	0
$^{n}\!\!\!\;_{,k}$	-1	0	1	2	3	4	5

∎