Sailing into the wind, or faster than the wind

23 March, 2009 in expository, non-technical | Tags: Newton's laws of motion, sailing, tacking, velocity

One of the more unintuitive facts about sailing is that it is possible to harness the power of the wind to sail in a direction *against* that of the wind or to sail with a speed faster than the wind itself, even when the water itself is calm. It is somewhat less known, but nevertheless true, that one can (in principle) do both at the same time – sail against the wind (even *directly* against the wind!) at speeds faster than the wind. This does not contradict any laws of physics, such as conservation of momentum or energy (basically because the reservoir of momentum and energy in the wind far outweighs the portion that will be transmitted to the sailboat), but it is certainly not obvious at first sight how it is to be done.

The key is to exploit all three dimensions of space when sailing. The most obvious dimension to exploit is the windward/leeward dimension – the direction that the wind velocity v_0 is oriented in. But if this is the only dimension one exploits, one can only sail up to the wind speed $|v_0|$ and no faster, and it is not possible to sail in the direction opposite to the wind.

Things get more interesting when one also exploits the crosswind dimension perpendicular to the wind velocity, in particular by tacking the sail. If one does this, then (in principle) it becomes possible to travel up to double the speed $|v_0|$ of wind, as we shall see below.

However, one still cannot sail against to the wind purely by tacking the sail. To do this, one needs to not just harness the power of the wind, but also that of the water *beneath* the sailboat, thus exploiting (barely) the third available dimension. By combining the use of a sail in the air with the use of sails in the water – better known as keels, rudders, and hydrofoils – one can now sail in certain directions against the wind, and at certain speeds. In most sailboats, one relies primarily on the keel, which lets one sail against the wind but not directly opposite it. But if one tacks the rudder or other hydrofoils as well as the sail, then in fact one can (in principle) sail in arbitrary directions (including those directly opposite to v_0), and in arbitrary speeds (even those much larger than $|v_0|$), although it is quite difficult to actually achieve this in practice. It may seem odd that the water, which we are assuming to be calm (i.e. traveling at zero velocity) can be used to increase the range of available velocities and speeds for the sailboat, but we shall see shortly why this is the case.

If one makes several simplifying and idealised (and, admittedly, rather unrealistic in practice) assumptions in the underlying physics, then sailing can in fact be analysed by a simple two-dimensional geometric model which explains all of the above statements. In this post, I would like to describe this mathematical model and how it gives the conclusions stated above.

— One-dimensional sailing —

Let us first begin with the simplest case of one-dimensional sailing, in which the sailboat lies in a one-dimensional universe (which we describe mathematically by the real line $\mathbb{R}$). To begin with, we will ignore the friction effects of the water (one might imagine sailing on an iceboat rather than a sailing boat). We assume that the air is blowing at a constant velocity $v_0 \in \mathbb{R}$, which for sake of argument we shall take to be positive. We also assume that one can do precisely two things with a sailboat: one can either *furl* the sail, in which case the wind does not propel the sailboat at all, or one can *unfurl* the sail, in order to exploit the force of the wind.

When the sail is furled, then (ignoring friction), the velocity v of the boat stays constant, as per Newton's first law. When instead the sail is unfurled, the motion is instead governed by Newton's second law, which among other things asserts that the velocity v of the boat will be altered in the direction of the net force exerted by the sail. This net force (which, in one dimension, is purely a drag force) is determined not by the true wind speed v_0 as measured by an observer at rest, but by the apparent wind speed $v_0 - v$ as experienced by the boat, as per the (Galilean) principle of relativity. (Indeed, Galileo himself supported this principle with a thought-experiment on a ship.) Thus, the sail can increase the velocity v when $v_0 - v$ is positive, and decrease it when $v_0 - v$ is negative. We can illustrate the effect of an unfurled sail by the following vector field in velocity space:

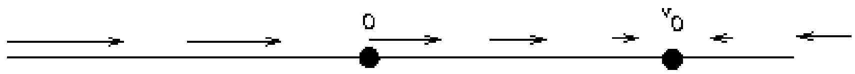

Figure 1. The effect of a sail in one dimension.

The line here represents the space of all possible velocities v of a boat in this one-dimensional universe, including the rest velocity 0 and the wind velocity v_0. The vector field at any given velocity v represents the direction the velocity will move in if the sail is unfurled. We thus see that the effect of unfurling the sail will be to move the velocity of the sail towards v. Once one is at that speed, one is stuck there; neither furling nor unfurling the sail will affect one's velocity again in this frictionless model.

Now let's reinstate the role of the water. Let us use the crudest example of a water sail, namely an anchor. When the anchor is raised, we assume that we are back in the frictionless situation above; but when the anchor is dropped (so that it is dragging in the water), it exerts a force on the boat which is in the direction of the apparent velocity $0 - v$ of the water with respect to the boat, and which (ideally) has a magnitude proportional to square of the apparent speed $|0 - v|$, thanks to the drag equation. This gives a second vector field in velocity space that one is able to effect on the boat (displayed here as thick blue arrows):

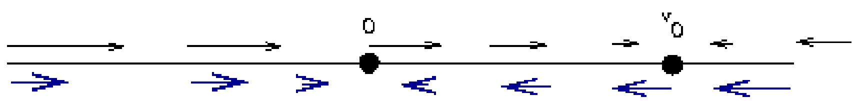

Figure 2. The effects of a sail and an anchor in one dimension.

It is now apparent that by using either the sail or the anchor, one can reach any given velocity between 0 and v_0. However, once one is in this range, one cannot use the sail and anchor to move faster than v_0, or to move at a negative velocity.

— Two-dimensional sailing —

Now let us sail in a two-dimensional plane $\mathbb{R}^2$, thus the wind velocity v_0 is now a vector in that plane. To begin with, let us again ignore the friction effects of the water (e.g. imagine one is ice yachting on a two-dimensional frozen lake).

With the square-rigged sails of the ancient era, which could only exploit drag, the net force exerted by an unfurled sail in two dimensions followed essentially the same law as in the one-dimensional case, i.e. the force was always proportional to the relative velocity $v_0 - v$ of the wind and the ship, thus leading to the black vector field in the figure below:

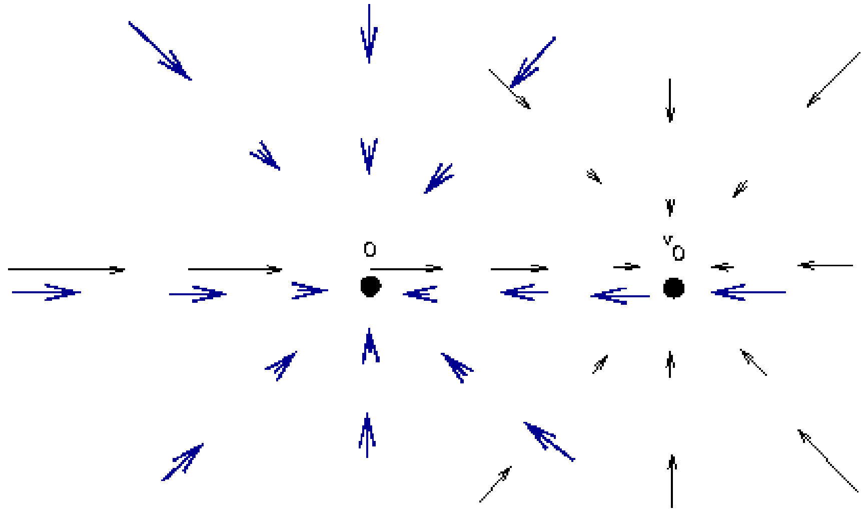

Figure 3. The effects of a pure-drag sail (black) and an anchor (blue) in two dimensions.

We thus see that, starting from rest $v = 0$, the only thing one can do with such a sail is move the velocity v along the line segment from 0 to v_0, at which point one is stuck (unless one can exploit water friction, e.g. via an anchor, to move back down that line segment to 0). No crosswind velocity is possible at all with this type of sail.

With the invention of the curved sail, which redirects the (apparent) wind velocity $v_0 - v$ to another direction rather than stalling it to zero, it became possible for sails to provide a lift force which is essentially perpendicular to the (apparent) wind velocity, in contrast to the drag force that is parallel to that velocity. (Not co-incidentally, such a sail has essentially the same aerofoil shape as an airplane wing, and one can also explain the lift force via Bernoulli's principle.)

[Despite the name, the lift force is not a vertical force in this context, but instead a horizontal one; in general, lift forces are basically perpendicular to the orientation of the aerofoil providing the lift. Unlike airplane wings, sails are vertically oriented, so the lift will be horizontal in this case.]

By setting the sail in an appropriate direction, one can now use the lift force to adjust the velocity v of a sailboat in directions perpendicular to the apparent wind velocity $v_0 - v$, while using the drag force to adjust v in directions parallel to this apparent velocity; of course, one can also adjust the velocity in all intermediate directions by combining both drag and lift. This leads to the following vector fields, displayed in red:

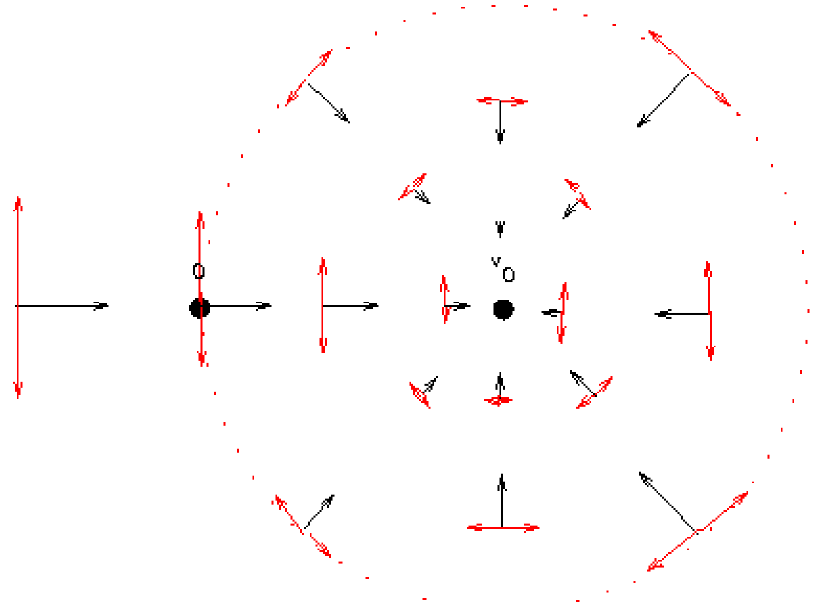

Figure 4. The effect of a pure-drag sail (black) and a pure-lift sail (red) in two dimensions. The disk enclosed by the dotted circle represents the velocities one can reach from these sails starting from the rest velocity $v = 0$.

Note that no matter how one orients the sail, the apparent wind speed $|v_0 - v|$ will decrease (or at best stay constant); this can also be seen from the law of conservation of energy in the reference frame of the wind. Thus, starting from rest, and using only the sail, one can only reach speeds in the circle centred at v_0 with radius $|v_0|$ (i.e. the circle in Figure 4); thus one cannot sail against the wind, but one can at least reach speeds of twice the wind speed, at least in principle. (In practice, friction effects of air and water, such as wave making resistance, and the difficulty in forcing the sail to provide purely lift and no drag, mean that one cannot quite reach this limit, but it has still been possible to exceed the wind speed with this type of technique.)

— Three-dimensional sailing —

Now we can turn to three-dimensional sailing, in which the sailboat is still largely confined to $\mathbb{R}^2$ but one can use both air sails and water sails as necessary to control the velocity v of the boat. (Some boats do in fact exploit the third dimension more substantially than this, e.g. using sails to vertically lift the boat to reduce water drag, but we will not discuss these more advanced sailing techniques here.)

As mentioned earlier, the crudest example of a water sail is an anchor, which, when dropped, exerts a pure drag force in the direction of $0 - v$ on the boat; this is displayed as the blue vector field in Figure 3. Comparing this with Figure 4(which is describing all the forces available from using the air sail) we see that such a device does not increase the range of velocities attainable from a boat starting at rest (although it does allow a boat moving with the wind to return to rest, as in the one-dimensional setting). Unsurprisingly, anchors are not used all that much for sailing in practice.

However, we can do better by using other water sails. For instance, the keel of a boat is essentially a water sail oriented in the direction of the boat (which in practice is kept close to parallel to v, e.g. by use of the rudder, else one would encounter substantial (and presumably unwanted) water drag and torque effects). The effect of the keel is to introduce significant resistance to any lateral movement of the boat. Ideally, the effect this has on the net force acting on the boat is that it should orthogonally project that force to be parallel to the direction of the boat (which, as stated before, is usually parallel to v). Applying this projection to the vector fields arising from the air sail, we obtain some new vector fields along which we can modify the boat's velocity:

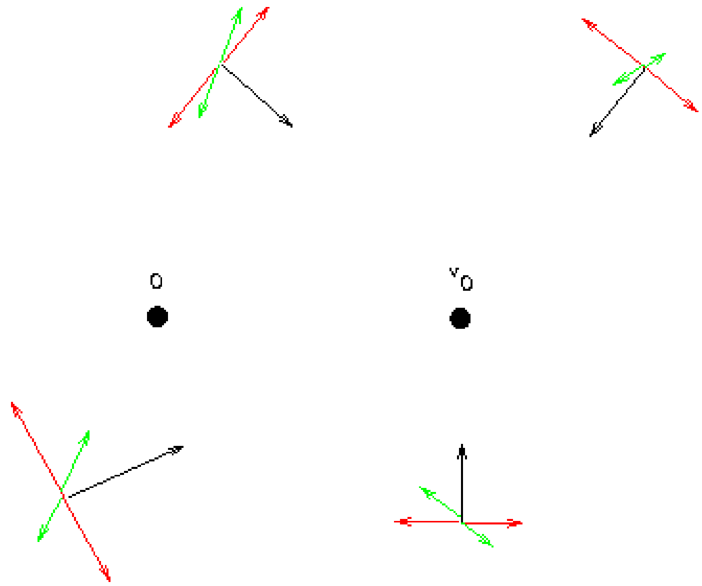

Figure 5. The effect of a pure-drag sail (black), a pure-lift sail (red), and a pure-lift sail combined with a keel (green). Note that one now has the ability to shift the velocity v away from both 0 and v_0 no matter how fast one is already traveling, so long as v is not collinear with 0 and v_0.

In particular, it becomes possible to sail against the wind, or faster than the wind, so long as one is moving at a non-trivial angle to the wind (i.e. v is not parallel to v_0 or $-v_0$).

What is going on here is as follows. By using lift instead of drag, and tacking the sail appropriately, one can make the force exerted by the sail be at any angle of up to $90°$ from the actual direction of apparent wind. By then using the keel, one can make the net force on the boat be at any angle up to $90°$ from the force exerted by the sail. Putting the two together, one can create a force on the boat at any angle up to $180°$ from the apparent wind speed – i.e. in any direction other than directly against the wind. (In practice, because it is impossible have a pure lift force free of drag, and because the keel does not perfectly eliminate all lateral forces, most sailboats can only move at angles up to about $135°$ or so from the apparent wind direction, though one can then create a net movement at larger angles by tacking and beating. For similar reasons, water drag prevents one from using these methods to move too much faster than the wind speed.)

In theory, one can also sail at any desired speed and direction by combining the use of an air sail (or aerofoil) with the use of a water sail (or hydrofoil). While water is a rather different fluid from air in many respects (it is far denser, and virtually incompressible), one could in principle deploy hydrofoils to exert lift forces on a boat perpendicular to the apparent water velocity $0 - v$, much as an aerofoil can be used to exert lift forces on the boat perpendicular to the apparent wind velocity $v_0 - v$. We saw in the previous section that if the effects of drag on the aerofoil could somehow be ignored, then one could use lift to alter the velocity v along a circle centred at the true wind speed v_0; similarly, if the effects of drag on the hydrofoil could also be ignored (e.g. by planing), then one could alter the velocity v along a circle centred at the true water speed 0. By alternately using the aerofoil and hydrofoil, one could in principle reach arbitrarily large speeds and directions, as illustrated by the following diagram:

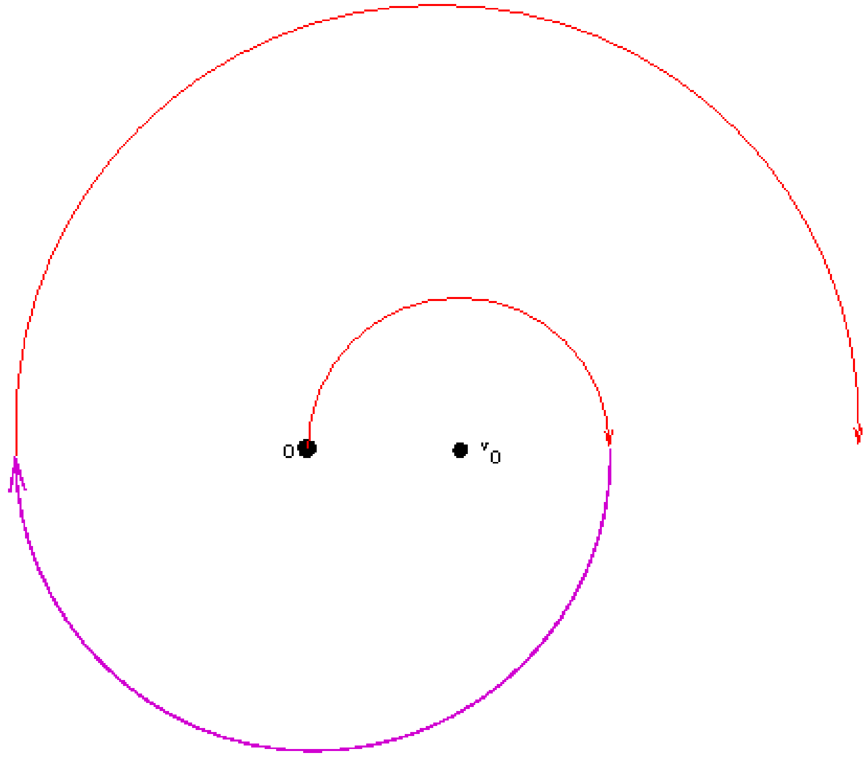

Figure 6. By alternating between a pure-lift aerofoil (red) and a pure-lift hydrofoil (purple), one can in principle reach arbitrarily large speeds in any direction.

I do not know however if one could actually implement such a strategy with a physical sailing vessel.

It is reasonable (in light of results such as the Kutta-Joukowski theorem) to assume that the amount of lift provided by an aerofoil or hydrofoil is linearly proportional to the apparent wind speed or water speed. If so, then some basic trigonometry then reveals that (assuming negligible drag) one can use either of the above techniques to increase one's speed at what is essentially a constant rate; in particular, one can reach speeds of $n|v_0|$ for any $n > 0$ in time $O(n)$. On the other hand, as drag forces are quadratically proportional to apparent wind or water speed, one can *decrease* one's speed at an very rapid rate simply by dropping anchor; in fact one can drop speed from $n|v_0|$ to $|v_0|$ in *bounded* time $O(1)$ no matter how large n is! (This fact is the time-reversal of the well-known fact that the Riccati ODE $u' = u^2$ blows up in finite time.) These appear to be the best possible rates for acceleration or deceleration using only air and water sails, though I do not have a formal proof of this fact.