# Series

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence.

Let $S_n = a_0 + a_1 + \cdots + a_n \quad \forall n \in \mathbb{N}$

$S_n$ is "the sum of $a_k$'s for $k=0$ to $k=n$"

$(S_n)_{n\in\mathbb{N}}$ is a sequence.

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\begin{aligned}
S_0 &= a_0 \\
S_1 &= a_0 + a_1 \\
S_2 &= a_0 + a_1 + a_n \\
&\vdots \\
S_n &= a_0 + a_1 + a_2 + \cdots + a_n \\
S_{n+1} &= a_0 + a_1 + a_2 + \cdots + a_n + a_{n+1} \\
&\vdots
\end{aligned}
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**Notation:**

$$\underbrace{\sum_{k=0}^{n}}_{\substack{\text{sum of } a_k\text{'s} \\ \text{for } k=0 \text{ to } k=n}} a_k = a_0 + a_1 + \cdots + a_n$$

Ex. $$\sum_{k=0}^{10} k = 0+1+2+3+\dots+9+10$$

$$\sum_{k=0}^{5} k^2 = 0^2+1^2+2^2+3^2+4^2+5^2$$

$$\sum_{k=10}^{15} k^3 = 10^3+11^3+12^3+13^3+14^3+15^3$$

$$\sum_{k=n}^{m} a_k = a_n + a_{n+1} + \dots + a_{m-1} + a_m$$

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Ex. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ defined by

$$a_n = \frac{x^n}{n!}$$

where $x \in \mathbb{R}$ is a given real number.

$$a_0 = \frac{x^0}{0!} = \frac{1}{1} = 1 \qquad a_1 = \frac{x^1}{1!} = \frac{x}{1} = x$$

$$a_2 = \frac{x^2}{2!} = \frac{x^2}{1.2} = \frac{x^2}{2} \qquad a_3 = \frac{x^3}{3!} = \frac{x^3}{1.2.3} = \frac{x^3}{6}$$

$$a_n = \frac{x^n}{n!} = \frac{x^n}{1.2.3.\dots.(n-1).n}$$

Let $S_n(x) = \sum_{k=0}^{n} a_k$ , $\forall n \in \mathbb{N}$

$$S_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{(n-1)}}{(n-1)!} + \frac{x^n}{n!}$$

$$S_{n+1}(x) = S_n(x) + \frac{x^{n+1}}{(n+1)!}$$

For a fixed $x \in \mathbb{R}$, $\left(S_n(x)\right)_{n \in \mathbb{N}}$ forms a sequence.

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For any $x \in \mathbb{R}$, the sequence $\left(S_n(x)\right)_{n \in \mathbb{N}}$ converges and its limit is $e^x$.

We write: $\sum_{n=0}^{\infty} a_n(x) = e^x$

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In general, for a given sequence $(a_n)_{n \in \mathbb{N}}$, if the series $(S_n)$ converges to some limit $L$, where $S_n = \sum_{k=0}^{n} a_k$ , we write:

$$L = \sum_{k=0}^{\infty} a_k$$

## Criteria of convergence of series:

Let $(a_n)_{n\in\mathbb{N}}$ be a given sequence.

Define $(S_n)_{n\in\mathbb{N}}$ as the corresponding series.

$$S_n = \sum_{k=0}^{n} a_n .$$

① Suppose $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ exists

and is $\underline{\underline{L < 1}}$.

Then $(S_n)$ converges to a limit $\sum_{n=0}^{\infty} a_n$.

② Suppose $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ exists

and $\underline{\underline{L > 1}}$

Then $(S_n)$ is divergent

③ If $\lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then further investigation is needed to decide if $\sum a_n$ is convergent or divergent.

Ex: If we go back to the example where

$$a_n = \frac{x^n}{n!}, \text{ then}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{x^{n+1} \quad n!}{x^n \quad (n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x \quad n!}{n! \quad (n+1)}\right| = \left|\frac{x}{n}\right| \xrightarrow[n \to +\infty]{} 0 < 1$$

Then we are in case 1 and

$$\left(\sum a_n\right) \text{ converges to } e^x = \sum_{k=0}^{\infty} a_k$$

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Application: We can use this series to compute an approximate value of $e = e^1$ by computing:

$$e \simeq S_{10} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{10!}$$

approximately equal

**Ex** $a_n = (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ where $x \in \mathbb{R}$ is fixed.

**Ratio test:**

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1} \frac{x^{2(n+1)+1}}{\big(2(n+1)+1\big)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$$

$$= \left|\frac{x^2 \ (2n+1)!}{(2n+3)!}\right| = \frac{x^2}{(2n+2)(2n+3)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{(2n+2)(2n+3)} \xrightarrow[n \to +\infty]{} 0 < 1$$

Then $\left(\sum a_n\right)$ converges.

**Remark:**

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

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$$\cos(x) = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

# Recursive sequences:

Define the sequence $(a_n)$ as follows:

$$\begin{cases} a_0 = 1 \\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \quad \forall n \in \mathbb{N} \end{cases}$$

$\longrightarrow$ $a_1 = \frac{1}{2}\left(a_0 + \frac{2}{a_0}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$

$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = 1{,}4167$

$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = 1{,}4142$

$\vdots$

Suppose sequence $(a_n)$ is convergent and call $\ell$ its limit: $\ell = \lim\limits_{n \to +\infty} a_n$.

Then, since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$,

letting $n \to +\infty$, we get: $\ell = \frac{1}{2}\left(\ell + \frac{2}{\ell}\right)$

Then: $\frac{\ell}{2} = \frac{1}{\ell}$, i.e. $\ell^2 = 2$.

If in addition we can prove that $\ell \geq 0$ (which is true), we have $\ell = \sqrt{2}$.

The following ~~This~~ sequence defines a method to compute $\sqrt{p}$ : where $p>0$

$$(a_n) \quad , \quad \begin{cases} a_0 = 1 \\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n}\right) \quad \forall n \in \mathbb{N} \end{cases}$$