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6 Vectors and Matrices

Data is frequently arranged in **arrays**, that is, sets whose elements are indexed by one or more subscripts. Frequently, a one-dimensional array is called a **vector** and a two-dimensional array is called a **matrix**.(The dimension, in this case, denotes the number of subscripts). Vectors are the exact same as linear arrays.

6.1 Introduction to Vectors

By a vector u, we mean the list of numbers $a_1, a_2, ..., a_n$. Such a vector is denoted by

$$u = \langle a_1, a_2, ..., a_n \rangle.$$

The numbers a_i are called the **components** or entries of u. If all the $a_i = 0$, then u is called the **zero vector**. Two such vectors, u and v, are **equal** if

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they have the same number of components *and* corresponding components are equal.

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6.2 Vector Operations

Consider two arbitrary vectors u and v with the same number of components, say

$$u = \langle a_1, a_2, ..., a_n \rangle \qquad \text{and} \qquad v = \langle b_1, b_2, ..., b_n \rangle$$

Definition 6.1 (Sum of two vectors) *The* **sum** *of u and v, written $u + v$, is the vector obtained by adding corresponding components from u and v; that is*

$$u + v = \langle a_1 + b_1, a_2 + b_2, ..., a_n + b_n \rangle$$

It is possible to multiply a vector by a real number k. (A real number k can be referred to as a **scalar** as distinct from a vector).

Definition 6.2 (Scalar Multiplication) *The* **scalar product** *of a scalar k and the vector u, written ku, is the vector obtained by multiplying each*

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component of u by k; that is

$$ku = \langle ka_1, ka_2, ..., ka_n \rangle$$

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We also define

$$-u = (-1)u$$

and

$$u - v = \langle a_1 - b_1, a_2 - b_2, ..., a_n - b_n \rangle$$

Definition 6.3 (Dot product) *The **dot product** or inner product of two vectors $u = \langle a_1, a_2, ..., a_n \rangle$ and $v = \langle b_1, b_2, ..., b_n \rangle$ is denoted and defined by*

$$u.v = a_1 b_1 + a_2 b_2 + ... + a_n b_n.$$

The dot product of two vectors is a scalar.

Definition 6.4 (Magnitude of a vector) *The **magnitude** or **norm** or **length** of a vector u is denoted and defined by*

$$|u| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$$

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Example 6.1 *Let $u = \langle 2, 3, -4 \rangle$ and $v = \langle 1, -5, 8 \rangle$. Find $u + v, 5u, 2u - 3v, u.v$ and $|v|$.*

Solution:

$$\begin{aligned}
u + v &= \langle 2 + 1, 3 + (-5), -4 + 8 \rangle = \langle 3, -2, 4 \rangle \\
5u &= 5\langle 2, 3, -4 \rangle = \langle 5.2, 5.3, 5.(-4) \rangle = \langle 10, 15, -20 \rangle \\
2u - 3v &= 2\langle 2, 3, -4 \rangle - 3\langle 1, -5, 8 \rangle = \langle 4, 6, -8 \rangle - \langle 3, -15, 24 \rangle = \langle 1, 21, -32 \rangle \\
u.v &= 2.1 + 3.(-5) + -4(8) = 2 - 15 - 32 = -45 \\
|v| &= \sqrt{1^2 + (-5)^2 + 8^2} = \sqrt{90}
\end{aligned}$$

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Sometimes a list of numbers is written vertically rather than horizontally, and the list is called a **column vector**. In this context, the above horizontally written vectors are called **row vectors**. The previous stated operations for row vectors work the exact same for column vectors.

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Exercise

Let $u = \langle 1, 4, -3 \rangle$ and $v = \langle -2, 1, 4 \rangle$. Calculate $v.u$

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6.3 The angle between two vectors

The angle, θ, between two vectors, u and v, can be calculated by using the following formula:

$$\cos\theta = \frac{u.v}{|u||v|}$$

Example 6.2 *Find the angle between the vectors $a = \langle 2, 2, -1\rangle$ and $b = \langle 5, -3, 2\rangle$.*

Solution: Since

$$|a| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \text{ and } |b| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since $a.b = 2(5) + 2(-3) + (-1)(2) = 2$

we have,

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$$\cos\theta = \frac{a.b}{|a||b|} = \frac{2}{3\sqrt{38}}$$

So the angle between a and b is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 1.46 \text{ radians}$$

(or $84°$ in **degree measure**.)

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6.4 Introduction to Matrices

A **matrix** A is a rectangular array of numbers usually represented in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The m horizontal lists of numbers are called the **rows** of A, and the n vertical lists of numbers are the **columns** of A. Thus the element, a_{ij}, called the **ij entry**, appears in row i and column j.

A matrix with m rows and n columns is called an **m by n** matrix, written $(m \times n)$. Another way of stating this is to say that the matrix is of **order** $(m \times n)$.

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Two matrices are **equal** if

- they are of the same order, and
- their corresponding entries are equal.

A matrix whose entries are all zero is called a **zero matrix** and will usually be denoted by 0.

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6.5 Matrix Operations

If two matrices are to be added or subtracted then they must be of the same order. If A and B are two matrices of the same order then $A + B$ and $A - B$ are found by adding and subtracting the corresponding elements. If for example, A and B were $(m \times n)$ matrices, then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \ldots & a_{2n} + b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \ldots & a_{mn} + b_{mn} \end{bmatrix}$$

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Example 6.3 *If* $A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & -2 \\ -3 & 5 \end{bmatrix}$, *find* $A + B$.

Solution: $A + B = \begin{bmatrix} 1+0 & 3+(-2) \\ 4+(-3) & -3+5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

If A and B are $(m \times n)$ matrices, then

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \ldots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \ldots & a_{2n} - b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \ldots & a_{mn} - b_{mn} \end{bmatrix}$$

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The **product** of the matrix A by the scalar k, written kA, is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

The **transpose** of a matrix A, written A^T, is got by interchanging the rows and columns of A.

Example 6.4 *If* $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & -3 & 2 \end{bmatrix}$ *then* $A^T = \begin{bmatrix} 2 & 1 \\ -1 & -3 \\ 3 & 2 \end{bmatrix}$

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Theorem 6.1 *Let A, B and C be matrices of the same order, and let k and c be scalars. Then:*

- $(A + B) + C = A + (B + C)$
- $A + 0 = 0 + A$
- $A + (-A) = 0$
- $A + B = B + A$
- $k(A + B) = kA + kB$
- $(k + c)A = kA + cA$
- $(kc)A = k(cA)$
- $1A = A$

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Theorem 6.2 *Let A and B be matrices and k be a scalar. Then, whenever the sum and products are defined:*

- $(A+B)^T = A^T + B^T$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$
- $(A^T)^T = A$

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The product of matrices A and B, written AB, is somewhat complicated. Two matrices A and B can only be multiplied if the number of columns of A is equal to the number of rows of B. If A is a $(m \times n)$ matrix and B is a $(n \times p)$ matrix, then A and B can be multiplied and the resulting product matrix will be of order $(m \times p)$.

$$(m \times n) \times (n \times p) = (m \times p)$$

To determine the **product matrix** use the following rule: If A is a $(m \times n)$ matrix and B is a $(n \times p)$ matrix then the produce AB is a $(n \times p)$ matrix whose ij-entry is obtained by multiplying the ith row of A by the jth column of B. This is just calculating several vector dot products.

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Example 6.5 *If*

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}$$

the **product matrix** AB *is defined by the following procedure:*

$$AB = \begin{bmatrix} \boxed{4 \quad 3 \quad 2} \\ \boxed{5 \quad 1 \quad 6} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -1 \end{matrix}} & \boxed{\begin{matrix} 4 \\ 0 \\ 2 \end{matrix}} \end{bmatrix}$$

We simply calculate the dot products of each row in the first matrix with each column in the second.

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Row 1, Column 1

$$\begin{bmatrix} \boxed{4 \quad 3 \quad 2} \\ 5 \quad 1 \quad 6 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -1 \end{matrix}} & \begin{matrix} 4 \\ 0 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{9} & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Row 1, Column 2

$$\begin{bmatrix} \boxed{4 \quad 3 \quad 2} \\ 5 \quad 1 \quad 6 \end{bmatrix} \begin{bmatrix} \begin{matrix} 2 \\ 1 \\ -1 \end{matrix} & \boxed{\begin{matrix} 4 \\ 0 \\ 2 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 9 & \boxed{20} \\ \cdot & \cdot \end{bmatrix}$$

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Row 2, Column 1

$$\begin{bmatrix} 4 & 3 & 2 \\ \boxed{5 \quad 1 \quad 6} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -1 \end{matrix}} & \begin{matrix} 4 \\ 0 \\ 2 \end{matrix} \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ \boxed{5} & \cdot \end{bmatrix}$$

Row 2, Column 2

$$\begin{bmatrix} 4 & 3 & 2 \\ \boxed{5 \quad 1 \quad 6} \end{bmatrix} \begin{bmatrix} \begin{matrix} 2 \\ 1 \\ -1 \end{matrix} & \boxed{\begin{matrix} 4 \\ 0 \\ 2 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ 5 & \boxed{32} \end{bmatrix}$$

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Exercise

If $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix}$, find BA and AB.

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6.6 Translations

A **translation** is applied to an object to reposition it along a straight-line path from one coordinate location to another. We translate a two-dimensional point by adding *translation distances*, t_x and t_y, to the original coordinate position (x, y) to move the point to a new position $(x^{'}, y^{'})$.

The translation distance pair (t_x, t_y) is called a **translation vector**.

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We can express the translation as a single matrix equation by using column vectors to represent coordinate positions and the translation vector:

$$P = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad P^{'} = \begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix}, \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}.$$

This allows us to write the two-dimensional translation equations in the matrix form:

$$P^{'} = P + T$$

Example 6.6 *Translate the point* $(3, 2)$ *2 units to the left.*

Solution:

$$P = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad T = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

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therefore

$$P^{'} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

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6.7 Rotation about the Origin

If we rotate the point with coordinates $(1, 0)$ anti-clockwise by the angle θ about the origin it moves to a point with coordinates $(\cos\theta, \sin\theta)$. (basic trigonometry)

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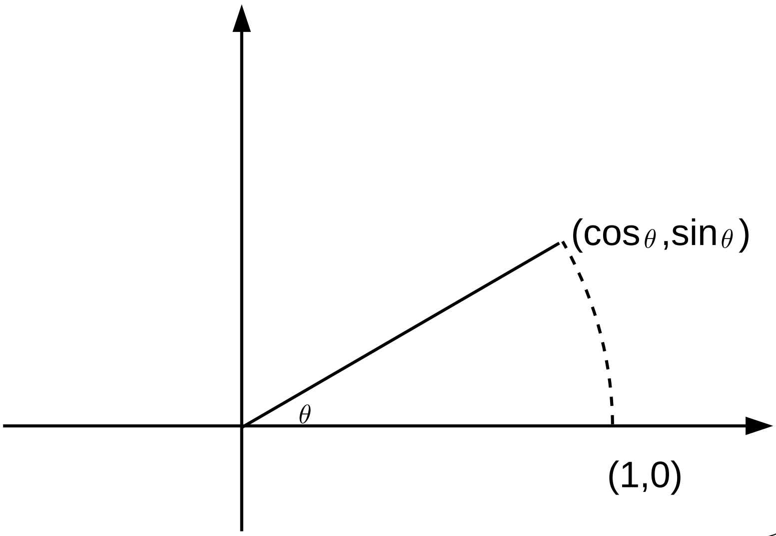

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Similarly, the point $(0, 1)$ moves to the point with coordinates $(-\sin\theta, \cos\theta)$ when rotated anti-clockwise through the angle θ.

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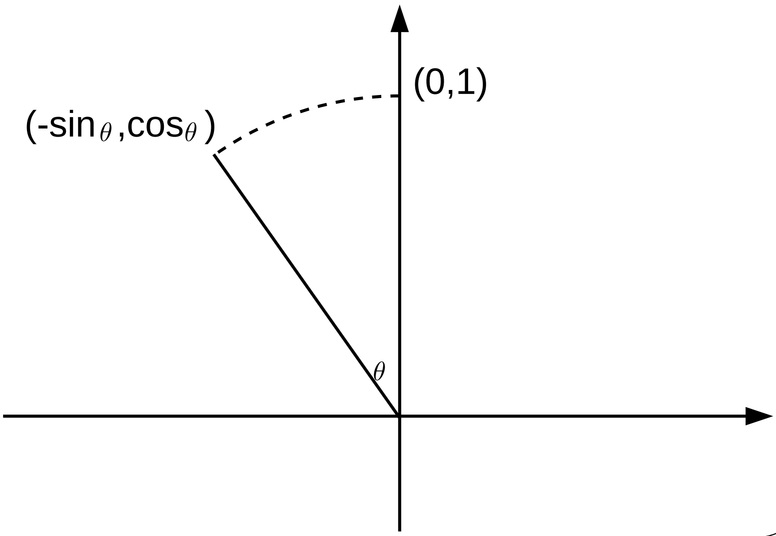

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In general, a point with "coordinates" $\begin{bmatrix} x \\ y \end{bmatrix}$ when rotated about the origin, by an angle θ, in an anti-clockwise direction results in a new point with "coordinates"

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 6.7 *Rotate the point* $(2, 3)$ *about the origin through an angle of* $\frac{\pi}{4}$.

Solution: The rotated point will be

$$\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cos\frac{\pi}{4} - 3\sin\frac{\pi}{4} \\ 2\sin\frac{\pi}{4} + 3\cos\frac{\pi}{4} \end{bmatrix}$$

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$$= \begin{bmatrix} 2(\frac{1}{\sqrt{2}}) - 3(\frac{1}{\sqrt{2}}) \\ 2(\frac{1}{\sqrt{2}}) + 3(\frac{1}{\sqrt{2}}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix}$$

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6.8 Rotation about another Point

If you wish to rotate an object around a point other than the origin, then the following three steps will explain the procedure.

Step 1: Translate the object so that the point of translation is moved to the origin.

Step 2: Rotate the relocated object as normal around the origin.

Step 3; Undo the translation in Step 1 to return the newly rotated object to its "original" location.

Exercise

Rotate the line segment connecting the point $(1, 1)$ to $(3, 3)$ about the point $(1, 1)$ through and angle of $\frac{\pi}{2}$.

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