Course Notes
for
MS4111
Discrete Mathematics 1

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CHAPTER 10 Introductory combinatorics

10.1 Permutations

Suppose we have n distinct elements

$$x_1, x_2, \ldots, x_n.$$

Question: In how many ways can we order $x_1, x_2, \ldots, x_n$?

x_1 can be ordered in n ways;

x_2 can be ordered in $n-1$ ways;

x_3 can be ordered in $n-2$ ways;

$\vdots$

$\vdots$

x_n can be ordered in 1 way.

Therefore the total number of orderings is

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1 = n!$$

Definition 10.1 *A permutation of $x_1, \ldots, x_n$ (n distinct elements) is an ordering of the elements $x_1, \ldots, x_n$.*

Note: There are $n!$ different permutation os n elements (say $x_1, \ldots, x_n$).

Example 10.1 *How many permutations of*

$$A, B, C$$

can we have? Show all the permutations.

Answer: We have three elements

$$A, B, C$$

therefore there are

$$3! = 3 \cdot 2 = 6 \quad \text{different permutations.}$$

The 6 permutations are

$$A, B, C; \quad A, C, B; \quad B, A, C; \quad B, C, A; \quad C, A, B; \quad C, B, A.$$

Example 10.2 *How many permutations of the letters*

$$A \;\; B \;\; C \;\; D \;\; E \;\; F$$

contain the substring DEF?

Answer: We want to construct permutations of

$$A \;\; B \;\; C \;\; D \;\; E \;\; F$$

that contain DEF by permuting four tokens

$$DEF \;\; A \;\; B \;\; C,$$

therefore the number of permutations are

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

Example 10.3 *How many permutations of the letters*

$$A\ \ B\ \ C\ \ D\ \ E\ \ F$$

contain the letters DEF together in any order?

Answer:

STEP 1: Select an ordering of

$$D\ E\ F.$$

This can be done in $3! = 6$ ways.

STEP 2: Once we select an order for $D\ E\ F$, then (see previous example) we can construct $4! = 24$ permutations of

$$A\ \ B\ \ C\ \ D\ \ E\ \ F$$

containing the given ordering of STEP 1.

Therefore, the total number of permutations is

$$6 \cdot 24 = 144.$$

Definition 10.2 *If*

$$x_1, \dots, x_n$$

are n (distinct) elements, an r-permutation of $x_1, \dots, x_n$ is an ordering of r elements of $x_1, \dots, x_n$. We will denote by P(n,r) the number of r-permutations of a set of n elements.

Theorem 10.1 *The number of r-permutations of a set of n (distinct) elements is*

$$P(n, r) = n(n-1)(n-2)\dots(n-r+1), \qquad r \leq n.$$

Proof.

If we have n elements

$$x_1, x_2, dots, x_n,$$

let us denote (for simplicity) by $x_1, x_2, \dots x_k$ the k elements chosen out of $x_1, \dots x_n$.

x_1 can be selected in n ways;

x_2 can be selected in $n - 1$ ways;

x_3 can be selected in $n - 2$ ways;

$\vdots$

$\vdots$

x_r can be selected in $n - (r - 1) = n - r + 1$ ways.

Therefore

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1). \qquad \square$$

Note:

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots 2\cdot 1}{(n-r)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

Therefore we have

$$P(n,r) = \frac{n!}{(n-r)!} \tag{10.1}$$

Example 10.4 *Find the number of 2-permutations of the set*

$$X = \{a, b, c\}$$

and show the 2-permutations.

Answer: By formula (10.1) we have

$$P(3,2) = \frac{3!}{(3-2)!} = \frac{6}{1} = 6.$$

The 2-permutations of X are

$$ab; \quad ba; \quad ac; \quad ca; \quad bc; \quad cb$$

10.2 Combinations

Definition 10.3 *Given a set of n (distinct) elements*

$$X = \{x_1, \ldots n_n\},$$

an r-combination of X is an unordered selection of r elements of X (i.e. a selection of a subset of X of r elements). We will denote by $C(n,r)$ the number of r-combinations of a set of n (distinct elements).

Theorem 10.2 *The number of r-combinations of a set of n elements is*

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!}, \qquad r \le n.$$

Proof. The proof is very straight forward:

$P(n, r)$ is the number if ways to chose k elements out of n with order;

$C(n, r)$ is the number if ways to chose k elements out of n with NO order;

$r!$ is the number of orderings of r elements;

therefore

$$C(n, r) = \frac{P(n, r)}{r!},$$

but $P(n, r) = \frac{n!}{(n-r)!}$,

therefore

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!\,r!}. \qquad \square$$

Example 10.5 *In how many ways can we select a committee of 3 out of 10 persons?*

Answer: A committee is an unordered group of people, therefore

$$C(10,3) = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = 120.$$

Example 10.6 *In how many ways can we select a committee of 2 women and 3 men from a group of 5 women and 6 men?*

Answer: We can select 2 women from a group of 5 for the committee in the following way

$$C(5,2) = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!} = 10$$

and we can select 3 men from a group of 6 for the committee in the following way

$$C(6,3) = \frac{6!}{(6-3)!\,3!} = \frac{6!}{3!\,3!} = 20.$$

Therefore the total committee can be made in

$$10 \cdot 20 \qquad \text{ways.}$$