Computer Mathematics II
MA4402

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Chapter 1

Functions

Exercise 1

Let A be the set of irish residents and B the set of all possible PPS numbers (7 digits and 1 letter).

(i) Can you define a function from A to B?

(ii) Can you define a function from B to A?

Exercise 2

Are the following functions well defined?

(i) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 + 1.$

(ii) $f : \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2 + 1.$

(iii) $f : \mathbb{R} \to [1, +\infty), \quad f(x) = x^2 + 1.$

(iv) $f : \mathbb{R} \to (1, +\infty), \quad f(x) = x^2 + 1.$

(v) $f : [-1, 1] \to \mathbb{R}, \quad f(x) = \sqrt{x}.$

(vi) $f : [-1, 1] \to \mathbb{R}, \quad f(x) = \sqrt{|x|}.$

(vii) $f : [-1, 1] \to [0, 1], \quad f(x) = \sqrt{|x|}.$

(viii) $f : [-1, 1] \to (0, 1), \quad f(x) = \sqrt{|x|}.$

Exercise 3

Which of the following functions are injective?

(i) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 + 1.$

(ii) $f : \mathbb{R} \to [1, +\infty), \quad f(x) = x^2 + 1.$

(iii) $f : [0, +\infty) \to \mathbb{R}, \quad f(x) = x^2 + 1.$

(iv) $f : [0, +\infty) \to [0, +\infty), \quad f(x) = x^2 + 1.$

(v) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = e^{2x}.$

(vi) $f : \mathbb{R} \to (0, +\infty), \quad f(x) = e^{2x}.$

(vii) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \sin(x).$

(viii) $f : [-\pi, \pi] \to \mathbb{R}, \quad f(x) = \sin(x).$

(ix) $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}, \quad f(x) = \sin(x).$

(x) $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1], \quad f(x) = \sin(x).$

Exercise 4

Which of the previous functions are surjective?

Exercise 5

Which of the previous functions are bijective?

Exercise 6

Give your own examples of functions which are:

(i) injective but not surjective.

(ii) surjective but not injective.

(iii) bijective.

Is it possible to have a function that is neither injective nor surjective? Illustrate your answer by way of an example.

Chapter 2

Sequences

Exercise 1

Compute the 6 first terms u_0, u_1, u_2, u_3, u_4, u_5 of the following seqences:

(i) $u_n = \frac{2n^2-n}{n+2}$

(ii) $u_n = -n^2 + 100n$

(iii) $u_n = (-1)^n n^2$

(iv) $u_n = 2^n - 3^n$

(v) $u_n = \frac{n}{n+1}$

(vi) $u_n = \frac{n+2}{n+1}$

Exercise 2

Find whether the previous sequences are increasing, decreasing, or neither increasing nor decreasing. Give a proof or a counter example to justify your answer.

Exercise 3

Let $(u_n)_{n\in\mathbb{N}}$ be sequence defined as follows:

$$\left\{ \begin{array}{l} u_0 = 2 \\ u_{n+1} = 6 - u_n \quad \forall n \in \mathbb{N} \end{array} \right\}$$

(i) Compute u_k for $k = 1, 2, ..., 6$.

(ii) Compute u_{100}.

(iii) Prove that $\forall n \in \mathbb{N}$, we have

$$u_{n+2} = u_n$$

(iv) Redefine the sequence $(u_n)_{n\in\mathbb{N}}$ without using a recursive relation.

Exercise 4

Determine the sign of $u_{n+1} - u_n$ for the following sequences and then precise whether the sequence is increasing or decreasing.

(i) $u_n = \frac{3+5n}{6} - 1$

(ii) $u_n = n^2$

(iii) $u_n = n^2 + 4n$

(iv) $u_n = \frac{2n+1}{3n-1}$

(v) $u_n = \left(\frac{5}{4}\right)^n$

(vi) $u_n = -\left(\frac{5}{4}\right)^n$

(vii) $u_n = -\frac{3}{n+1}$

(viii) $u_n = -n^2 + 3$

Chapter 3

Series

Exercise 1

(i) What is the difference between a sequence and a series?

(ii) Is a sequence a series?

(iii) Is a series a sequence?

Exercise 2

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence.

(i) Is the series defined by $\sum_{n=0}^{\infty} a_n$ necessarlly convergent?

(ii) Can you give a condition on the bound of the sequence $(a_n)_{n\in\mathbb{N}}$ so that the series $\sum_{n=0}^{\infty} a_n$ is convergent?

Exercise 3

(i) Show that the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

defines a convergent series for all $x \in \mathbb{R}$. Note this series defines $\sin(x)$.

(ii) Use the series defined in the previous question to estimate the value of $\sin(\pi/6)$.

Exercise 4

(i) Show that the series defined by the sequence $\left(\frac{x^n}{n!}\right)_{n\in\mathbb{N}}$ is convergent. Note this series defines e^x.

(ii) Use the series defined in the previous question to estimate the value of e^2 correct to 3 decimal places.

Exercise 5

(i) Explain why $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| > 1$. What happens if $|x| = 1$?

(ii) Does the series $\sum_{n=0}^{\infty} n!x^n$ converge? Why?

Exercise 6

Let p a positive number. Define the sequence $(a_n)_{n\in\mathbb{N}}$ by:

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{p}{a_n^2}\right), \quad \forall n \in \mathbb{N}. \end{cases}$$

(i) Assuming the above recursively defines a convergent sequence to a positive limit, what is its limit?

(ii) Use this series to estimate $\sqrt[3]{5}$ to two decimal places.

Chapter 4

Numerical methods

Exercise 1

(i) Find the slope of the tangent to the curve $y = x^2 - 3x + 2$ when $x = 1$.

(ii) Find the equation of the tangent at this point.

Exercise 2

Using mostly derivative information, sketch the graph of

$$f(x) = x^3 - 4x^2 + x + 6$$

Note that we need information about the function such as critical points, roots... To find the roots, we remark that $x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$.

Exercise 3

(i) Consider the same $f(x)$ as in the previous question. Use the right initial guess x_0 to estimate the root at $x = -1$, $x = 2$, and $x = 3$.

(ii) Are there any choices of x_0 for which the method fails to find a root?

Exercise 4

(i) Using derivative information, sketch the graph of the function

$$f : \mathbb{R} \longmapsto \mathbb{R}, \qquad f(x) = x^3 - 5x^2 + 8x - 3.$$

(find critical points etx...)

(ii) Use Newton's method to approximate the root(s) of this function. Note, we can sue the graph in (i) to determine approximate value(s) of our initial guess(es) x_0.

(iii) Sketch another graph of the function incorporating the root(s) obtained in (ii).

Exercise 5

When using the Newton-Raphson method of root finding, suppose our initial guess x_0 is lucky and x_0 is a root of f, which means $f(x_0) = 0$. What happens to the next approximation x_1 and later approximations?

Chapter 5

Graph theory

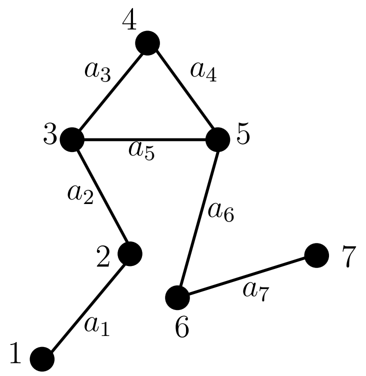
Exercise 1

Sketch a graph having nodes $\{1, 2, 3, 4, 5\}$, arcs $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ and with rule:

$$g(a_1) = 3 - 4, \quad g(a_2) = 1 - 2, \quad g(a_3) = 3 - 4, \quad g(a_4) = 1 - 1$$
$$g(a_5) = 2 - 3, \quad g(a_6) = 1 - 5, \quad g(a_7) = 5 - 5.$$

Exercise 2

Consider the follwing graph



(i) Is it simple?

(ii) Is it complete?

(iii) Is it connected?

(iv) Find a path from node 1 to node 6.

(v) Are there any cycles in the graph?

(vi) Is it possible to remove an arc so the resulting graph is a tree?

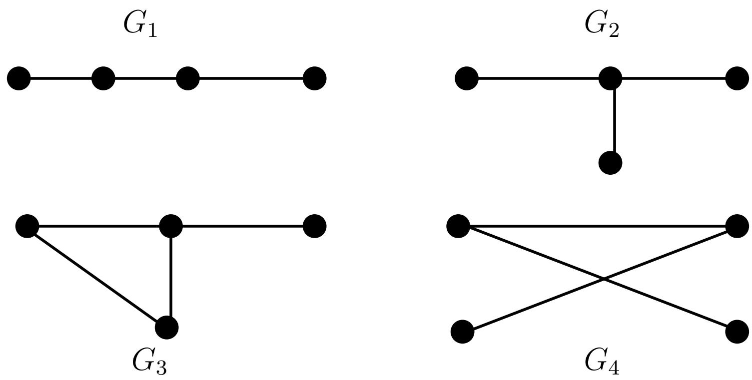
(vii) Is it possible to remove an arc so the resulting graph is not connected?

Exercise 3

Find a connected graph that is not complete.

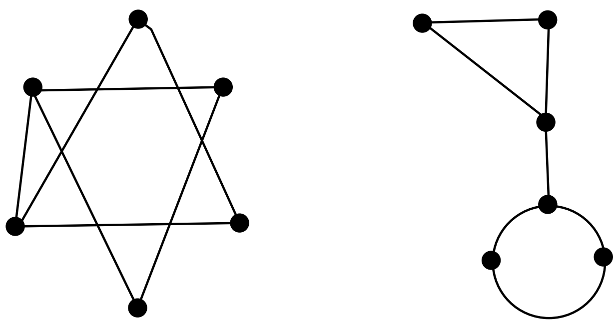
Exercise 4

Are any of the following graphs isomorphic to each other?



Exercise 5

Construct an isomorphism between the following graphs:

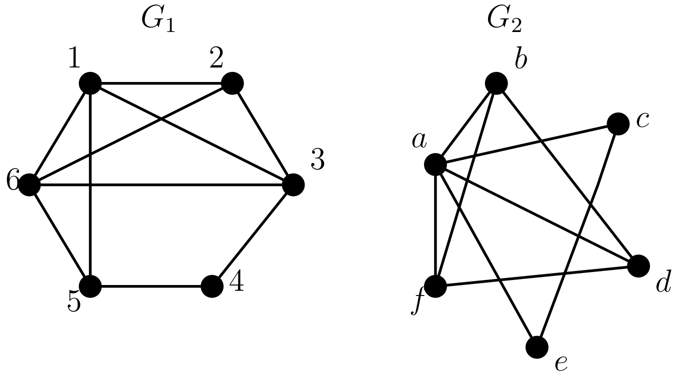


G_1 G_2

Exercise 6

(i) Draw the graphs : K_4, $K_{1,3}$, $K_{3,4}$.

(ii) Redraw the following as planar graphs and verify Euler's formula for each of them.



(iii) How many edges must be drawn to obtain a connected planar graph with 7 nodes and 7 regions?

Exercise 7

(i) Draw all non-isomorphic trees with 5 nodes.

(ii) A football tournament is played with 9 teams. We denote these teams by T_i, $i = 1, \ldots 9$. We design the tournament so that in order for team T_i for $i = 1, \ldots 8$ to win, they must play i games. Model such a situation with a tree and determine how many games must the team T_9 play in order to win the tournament.

(iii) How many leaves are in a binary tree with 5 interior nodes?

(iv) Draw a tree to represent the following algebraic expressions:

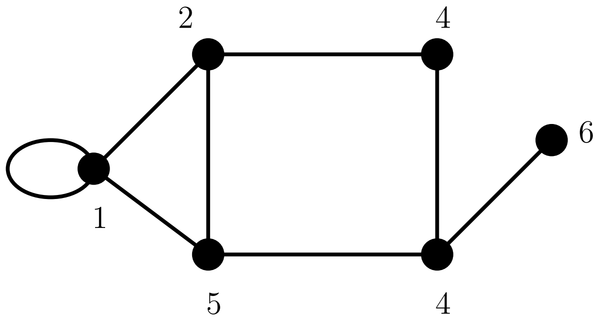
a) $(2 + x)^2 * ((2 - y)/(7 + x))$.

b) $((3 + z) * ((x - y) + 4)) - x^2$

Exercise 8

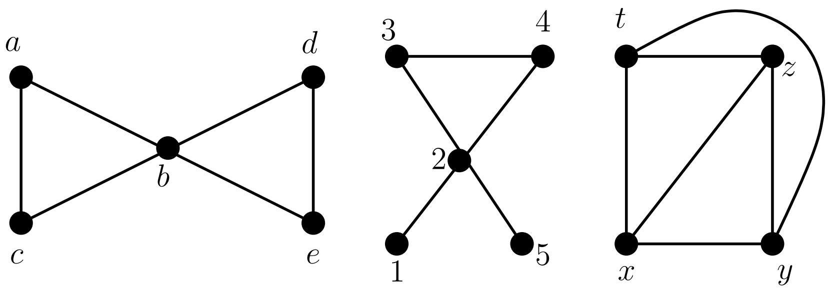
(i) Construct the adjacency matrix for the following graph.

(ii) Suppose we consider a simple graph. What can we say about its adjacency matrix?



Exercise 9

Find an Euler Circuit or an Euler path in each of the graphs below or say that neither exist.

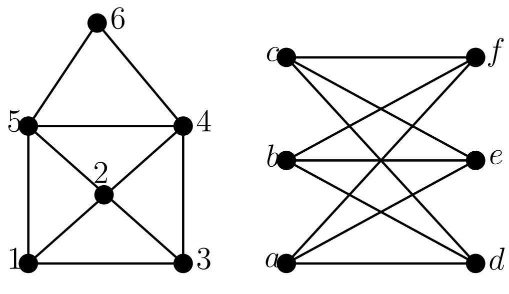


G_1 G_2 G_3

Exercise 10

(i) Show the complete graph K_4 is Hamiltonian.

(ii) Is there a Hamiltonian circuit in the following graphs, if not do they have a Hamiltonian path?



G_1 $K_{3,3}$

Chapter 6

Linear algebra

Exercise 1

Consider the following matrices.

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 8 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 4 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad F = \begin{pmatrix} 7 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}, \quad G = \begin{pmatrix} 3 & 4 \\ 9 & 5 \end{pmatrix}.$$

Calculate the following sums and products if possible:

1. $3C$
2. $2A - G$
3. $E + 3F$
4. $C + A$
5. AG
6. AC
7. CA
8. AB
9. BD
10. EF
11. FE

Exercise 2

Find the transpose of all matrices in the previous exercise.

Exercise 3

Show that

1. $(AG)^T = G^T A^T$
2. $(AC)^T = C^T A^T$
3. $(AB)^T = B^T A^T$
4. $(EF)^T = F^T E^T$

Exercise 4

1. Sketch the following three vectors.
$$u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$
2. Find the length of each of these vectors.
3. Fing the angle between each pair of vectors.

Exercise 5

Find the length and midpoint of the line segments:

1. with end points $(5, -7)$ and $(8, -11)$.
2. whose endpoints are defined by the vectors $\begin{pmatrix} -1//12 \end{pmatrix}$ and $\begin{pmatrix} 11//7 \end{pmatrix}$.

Exercise 6

1. Translate the line segment with endpoints $(5, -7)$, $(8, -11)$ three units up and one unit to the left.
2. Find the length of this new line segment. Is it the same as in exercise 2, (a)?

Exercise 7

Rotate the line segment with endpoints $(0, 0)$, $(3, 3)$ anti-clockwise by $\pi/4$ radians $(45°)$ abouth the origin.

Exercise 8

Rotate the line segment with endpoints $(2, 2)$, $(3, 3)$ anti-clockwise about the endpoint $(2, 2)$ by $\pi/2$ radians. (Note: first you must translate the line segment so the endpoin $(2, 2)$ is at the origin, then perform the rotation, and then reverse the translation.)

Exercise 9

Consider the line segment with endpoints $(2, 2)$, $(4, 6)$.

1. Find its length.
2. Rotate the line segment $\pi/2$ radians anti-clockwise about its midpoint.
3. Find the length of this new line segments. Is it the same?

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