Course Notes for MS4111 Discrete Mathematics 1

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CHAPTER 4 Proofs

4.1 Application of Predicate Logic to Proofs

A mathematical system is made by axioms, definitions, undefined terms and statements to be proved (theorems, lemmas and corollaries). A proof is an argument that establishes the truth of a theorem (or a lemma or a corollary). LOGIC is a tool for the analysis of proofs. In this section we will describe two type of proofs and we will use logic in the next section to analyze whether they are valid and invalid arguments. The two type of proofs we will consider are the direct proof and the proof by contradiction. We start with some considerations.

Theorems are often of the form

Theorem 4.1

$$\text{For all } x, \qquad \text{if } p(x) \text{ then } q(x). \tag{4.1}$$

is true.

In symbols

$$\forall x, \qquad p(x) \;\Rightarrow\; q(x). \tag{4.2}$$

is true.

(4.1) or (4.2) is true if the conditional proposition

$$p(x) \;\Rightarrow\; q(x) \tag{4.3}$$

is true for all x in the domain of discourse.

Note: If $p(x)$ is false, then (4.2) is true (false hypothesis), therefore we only need to consider the case when $p(x)$ is true.

4.1.1 Direct Proof

To prove (4.2) we assume that x is an arbitrary element of the domain of discourse. We assume that $p(x)$ is true and we want to prove that $q(x)$ is true too (by making also use of axioms, definitions and previously defined theorems).

Example 4.1 *Prove by direct proof that:*

For all real numbers d, d_1, d_2 and x,
if $d = \min\{d_1, d_2\}$ and $x \leq d$, then $x \leq d_1$ and $x \leq d_2$.

Answer. We want to prove that

$$\forall d \in \mathbb{R},\ \forall d_1 \in \mathbb{R},\ \forall d_2 \in \mathbb{R},\ \forall x \in \mathbb{R},$$
$$(d = \min\{d_1, d_2\}) \wedge (x \leq d) \ \Rightarrow\ (x \leq d_1) \wedge (x \leq d_2)$$

is true.

We assume

$$(d = \min\{d_1, d_2\}) \wedge (x \leq d)$$

is true and we want to prove that

$$(x \leq d_1) \wedge (x \leq d_2)$$

is true too. By definition of the minimum of two numbers, min, we have

$$d = \min\{d_1, d_2\} \ \Rightarrow\ (d \leq d_1) \wedge (d \leq d_2)$$

is true, therefore

$$d = \min\{d_1, d_2\} \wedge (x \leq d) \ \Rightarrow\ (d \leq d_1) \wedge (d \leq d_2) \wedge (x \leq d)$$

is true too, therefore $(d \leq d_1) \wedge (d \leq d_2) \wedge (x \leq d)$ is true, therefore

$$(d \leq d_1) \wedge (d \leq d_2) \wedge (x \leq d) \;\Rightarrow\; (d \leq d_1) \wedge (d \leq d_2)$$

is true, therefore $(d \leq d_1) \wedge (d \leq d_2)$ is true.

$\square$

4.1.2 Proof by Contradiction

We want to prove the following theorem.

Theorem 4.2

$$\text{For all } x, \qquad \text{if } p(x) \text{ then } q(x). \tag{4.4}$$

is true.

The method of contradiction consists in taking a general x and assuming that $p(x)$ is true and $q(x)$ is false i.e. assuming that

$$p(x) \wedge \overline{q(x)} \quad \text{is true.}$$

The method consists then in deriving a contradiction

$$r(x) \wedge \overline{r(x)}.$$

Idea behind the proof of contradiction: To prove by contradiction that

$$p \Rightarrow q,$$

we assume that $p \wedge \overline{q}$ is true and of course that p is true and we want to derive that

$$p \wedge \overline{q} \ \Rightarrow \ r \wedge \overline{r}$$

is a true statement, but $r \wedge \overline{r}$ is a contradiction (it is always false), therefore

$$p \wedge \overline{q} \quad \text{must be false},$$

therefore $\overline{q}$ must me false (because p is true) and therefore q must be true.

Example 4.2 *Prove by contradiction that the following statement is true:*

For all real numbers x and y, if $x + y \geq 2$, then either $x \geq 1$ or $y \geq 1$.

Proof. Let us denote

$$\begin{aligned} p(x,y) &: \quad x + y \geq 2 \\ q(x) &: \quad x \geq 1 \\ q(y) &: \quad y \geq 1. \end{aligned}$$

Let x and y be arbitrary real numbers, we want to prove by contradiction that the statement

$$p(x,y) \Rightarrow q(x) \vee q(y)$$

is true. Assume that

$$p(x, y) \quad \text{is} \quad \text{true (T)} \quad \text{and} \quad q(x) \vee q(y) \quad \text{is} \quad \text{false (F)},$$

i.e

$$p(x, y) \wedge \overline{q(x) \vee q(y)} \quad \text{is} \quad \text{T}$$

i.e

$$p(x, y) \wedge \left(\overline{q(x)} \wedge \overline{q(y)}\right) \quad \text{is} \quad \text{T}$$

i.e

$$(x + y \geq 2) \wedge (x < 1) \wedge (y < 1) \quad \text{is} \quad \text{T}$$

therefore

$$(x+y \geq 2) \wedge (x<1) \wedge (y<1) \Rightarrow (x+y \geq 2) \wedge (x+y<2) \quad \text{is} \quad \mathrm{T}.$$

$$(x+y \geq 2) \wedge (x+y<2) \quad \text{is a} \quad \text{contradiction i.e.} \quad F,$$

therefore

$$(x+y \geq 2) \wedge (x<1) \wedge (y<1) \quad \text{is} \quad \mathrm{F}$$

therefore

$$(x<1) \wedge (y<1) \quad \text{is} \quad \mathrm{F}$$

i.e

$$(x \geq 1) \vee (y \geq 1) \quad \text{is} \quad \mathrm{T}$$

and

$$x + y \geq 2 \;\Rightarrow\; (x \geq 1) \vee (y \geq 1) \quad \text{is} \quad \mathrm{T}.$$

$\square$

Note: In the example above we proved that

$$\forall x, \quad p(x) \Rightarrow q(x)$$

is T by assuming $\overline{q(x)}$ T and deriving the contradiction

$$p(x) \wedge \overline{p(x)}.$$

This special case of proof by contradiction is called Proof by Contrapositive. The name comes from the fact that we basically proved that

$$\overline{q(x)} \Rightarrow \overline{p(x)} \quad \text{is} \quad T$$

to prove that

$$p(x) \Rightarrow q(x) \quad \text{is} \quad T.$$

4.2 Valid and Invalid Arguments

Note: In constructing a proof, we must make sure that the arguments used are valid: what does this mean? Let us consider the following example.

Example 4.3 *Consider the following statements (that we will call hypotheses or premises):*

The book "Discrete Mathematics" is either in my bag or on the table.

"Discrete Mathematics" is a maths book.

There is no maths book on the table.

Assuming that the above hypotheses are true, it is reasonable to conclude with the statement (that we will call conclusion):

The book "Discrete Mathematics" is in my bag.

Note: This process of drawing a conclusion from a sequence of propositions (hypotheses) is called deducting reasoning.

Note: A deductive argument consists of hypotheses and a conclusion. Many proofs in Mathematics and Computer Science are deducting arguments.

Any argument has the form

$$\text{If} \quad p_1 \text{ and } p_2 \text{ and } \ldots \text{ and } p_n, \quad \text{then} \quad q$$

or, in symbols

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \;\Rightarrow\; q \tag{4.5}$$

Definition 4.1 *Argument* (4.5) *is valid when:*

$$\text{if} \quad p_1 \wedge p_2 \wedge \cdots \wedge p_n \quad \text{is} \quad \text{T}, \quad \text{then} \quad q \quad \text{is} \quad \text{T}.$$

In a valid argument sometimes we say that the conclusion follows from the hypotheses.

Definition 4.2 *Argument* (4.5) *is invalid when:*

if $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is T, then we cannot say that q is T.

Notation: We will denote argument (4.5) symbolically with

$$
\begin{array}{l}
p_1 \\
p_2 \\
\vdots \\
p_n \\
\hline
\therefore q
\end{array}
$$

Example 4.4 Direct Proof: *it is given by the argument*

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Is the Direct Proof a valid argument? There are two ways to check it:

1) Suppose $p \Rightarrow q$ and p are TRUE. Then q must be TRUE, otherwise $p \Rightarrow q$ would be FALSE. Therefore the argument is valid.

2) Another way to check if the argument is valid is to look at the following truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

We notice from the truth table that when both $p \Rightarrow q$ and p are T, q is also T, therefore the argument is valid.

Example 4.5 Proof by Contradiction: *it is given by the argument*

$$p \wedge \overline{q} \Rightarrow r \wedge \overline{r}$$

$$p$$

$$\therefore q$$

Is the Proof by Contradiction a valid argument?

We are going to check it by looking only at the following truth table:

p	q	r	$\overline{q}$	$\overline{r}$	$p \wedge \overline{q}$	$r \wedge \overline{r}$	$(p \wedge \overline{q}) \Rightarrow (r \wedge \overline{r})$
T	T	T	F	F	F	F	T
T	T	F	F	T	F	F	T
T	F	T	T	F	T	F	F
T	F	F	T	T	T	F	F
F	T	T	F	F	F	F	T
F	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T
F	F	F	T	T	F	F	T

We notice form the above truth table that when both

$(p \wedge \overline{q}) \Rightarrow (r \wedge \overline{r})$ and p are T, q is also T, therefore the argument is valid.

We conclude this section with the following example.

Example 4.6 *Consider the argument*

If 4=5, then I am Santa Claus.

I am Santa Claus.

∴ 4=5

Represent it symbolically and determine whether the argument is valid.

Answer. Denote

$$\begin{aligned} p &: \quad 4 = 5 \\ q &: \quad I\ am\ Santa\ Claus. \end{aligned}$$

The argument is therefore symbolically

$$p \Rightarrow q$$

$$q$$

$$\therefore p$$

1) Suppose $p \Rightarrow q$ and q are TRUE. Then q can be either TRUE or FALSE. Therefore the argument is invalid.

2) Another way to check if the argument is valid is to look at the following truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T