Course Notes
for
MS4111
Discrete Mathematics 1

R. Gaburro

CHAPTER 3 Predicate Logic

3.1 Predicates and free variables

In chapter 2 we studied the so called Propositional Logic. Propositional logic does not provide enough instruments to deal with mathematical statements and proofs: we need to generalize the concept of proposition by introducing statements which contain variables.

Example 3.1 *The expression*

$$x \geq 1$$

contains the variable x and the truth value of it depends on the value of x, for example it is true if $x = 3$ and it is false if $x = 0$. This is an example of a (logical) predicate.

Definition 3.1 *Let $p(x)$ be a statement with variable x and let D be a set. p is a predicate or propositional function (with respect to D) if for each $x \in D$, $p(x)$ is a proposition. We call D the domain of discourse of D.*

NOTE: Predicates can involve several variables! Let us see some examples of predicates.

Example 3.2 *1) $p(x)$: x is an honest man.*

2) $q(x, y)$: x is a man capable of doing job y.

3) $r(x, y, z)$: The product of number x with number y is z.

Remark 3.12 *A predicate $p(x)$ in one variable is often expressing a property of x.*

Remark 3.13 *A predicate $q(x, y)$ in two variables is often expressing a relation between x and y.*

Remark 3.14 *A predicate containing no variables is a proposition.*

Remark 3.15 *Whenever we replace the variables of a predicate with constants, the predicate becomes a proposition.*

The variables involved in a predicate are called free variables.

3.2 Predicates and bound variables: quantifiers

3.2.1 Universal quantifier

Definition 3.2 *Given a predicate $p(x)$, consider the statement*

$$\text{for every } x, \quad p(x). \tag{3.1}$$

The above statement is called the universally quantified statement and it is denoted by

$$\forall x, \quad p(x), \tag{3.2}$$

where the symbol $\forall$ is called the universal quantifier. (3.1) *(or* (3.2)*) is defined in the following way: it is true if $p(x)$ is true for every x and it is false if $p(x)$ is false for at least one x.*

(3.2) *is red:*

for every x, $p(x)$

for any x, $p(x)$

for all x, $p(x)$

Example 3.3 $p(x)$*: x is an honest man.*

The universally quantified statement is

$\forall x, \quad p(x)$*: "For all x, x is an honest man", or, "Every man is honest".*

Remark 3.16 *The universally quantified statement*

$$\forall x, \quad p(x)$$

is false if $p(x)$ is false for at least one x. A value x that makes $p(x)$ false is called a counterexample to

$$\forall x, \quad p(x).$$

Exercise 3.1 *The statement*

$$\forall x \in \mathbf{R}, \quad x^2 - 1 > 0$$

is false since if $x = 1$, proposition

$$1^2 - 1^2 > 0$$

is false. The value 1 is a counterexample to the statement

$$\forall x \in \mathbf{R}, \quad x^2 - 1 > 0.$$

3.2.2 Existential quantifier

Definition 3.3 *Given a predicate $p(x)$, consider the statement*

$$\text{for some } x, \quad p(x). \tag{3.3}$$

The above statement is called the existentially quantified statement and it is denoted by

$$\exists x, \quad p(x), \tag{3.4}$$

where the symbol $\exists$ is called the existential quantifier. (3.3) *(or* (3.4)*) is defined in the following way: it is true if $p(x)$ is true for at least one x and it is false if $p(x)$ is false for every x.*

(3.4) *is red:*

for some x, $p(x)$

for at least one x, $p(x)$

there exists x such that, $p(x)$

Example 3.4 *$p(x)$: x is an honest man.*

The existentially quantified statement is

$\exists x, \quad p(x)$: "There exists at least one x, such that x is an honest man" or, "There exists at least one honest man".

Remark 3.17 *By introducing quantifiers in a predicate $p(x)$, the variable x becomes silent and we say that x becomes a bound variable. Moreover the predicate becomes a proposition.*

Note: If there is more than one quantifier in a predicate, then the order of the quantifiers is important!

Example 3.5 *Consider the predicate*

$q(x,y)$: *x is a man capable of doing job y.*

The quantified statement

$$\forall y,\ \exists x : \quad q(x,y)$$

is red "For any job y, there is a man x who is capable of doing job y".

The quantified statement

$$\exists x,\ \forall y : \quad q(x,y)$$

is red "There is a man x that is capable of doing any job y".

In general, given a predicate $p(x)$, we can restrict the range of variation of the free variable to the domain of discourse D

$$\forall x \in D, \quad p(x)$$

and the above universally quantified statement is true if $p(x)$ is true for all $x \in D$ and it is false if $p(x)$ is false for at least one $x \in D$. We can similarly define

$$\exists x \in D, \quad p(x)$$

which is true if $p(x)$ is true for at least one $x \in D$ and it is false if $p(x)$ is false for all $x \in D$.

3.3 Rules for negating quantifiers

3.3.1 Generalized De Morgan Laws (for logic)

The rules for negating quantifiers is given by the following important theorem.

Theorem 3.1 ***Generalized De Morgan Laws (for logic)***

a)
$$\overline{\left(\forall x,\ p(x)\right)} \equiv \exists x,\ \overline{p(x)}$$

b)
$$\overline{\left(\exists x,\ p(x)\right)} \equiv \forall x,\ \overline{p(x)}$$

Proof.

a) We want to prove that

$$\overline{\left(\forall x,\ p(x)\right)} \quad \text{and} \quad \exists x,\ \overline{p(x)}$$

have the same truth values.

If $\overline{\left(\forall x,\ p(x)\right)}$ is TRUE, then $\forall x,\ p(x)$ is FALSE, then $p(x)$ is FALSE for at least one x, then $\overline{p(x)}$ is TRUE for at least one x, i.e. $\exists x,\ \overline{p(x)}$ is TRUE.

If $\overline{\left(\forall x,\ p(x)\right)}$ is FALSE, then $\forall x,\ p(x)$ is TRUE, then it is NOT TRUE that $\exists x,\ \overline{p(x)}$, i.e. $\exists x,\ \overline{p(x)}$ is FALSE.

b) We want to prove that

$$\overline{\left(\exists x,\ p(x)\right)} \quad \text{and} \quad \forall x,\ \overline{p(x)}$$

have the same truth values.

If $\overline{\left(\exists x,\ p(x)\right)}$ is TRUE, then $\exists x,\ p(x)$ is FALSE, then $p(x)$ is FALSE for every x, then $\overline{p(x)}$ is TRUE for every x, i.e. $\forall x,\ \overline{p(x)}$ is TRUE.

If $\overline{\left(\exists x,\ p(x)\right)}$ is FALSE, then $\exists x,\ p(x)$ is TRUE, then it is NOT TRUE that $p(x)$ is false for any x, i.e. $\forall x,\ \overline{p(x)}$ is FALSE. $\square$

3.3.2 Negation of quantified statement restricted to D

The Generalized De Morgan Laws (for logic) can be easily adapted to the case in which the quantified statements are restricted to a domain of discourse D in the following way

a′) $$\overline{\left(\forall x \in D,\ p(x)\right)} \equiv \exists x \in D,\ \overline{p(x)}$$

b′) $$\overline{\left(\exists x \in D,\ p(x)\right)} \equiv \forall x \in D,\ \overline{p(x)}$$

3.3.3 Final remarks on the Generalized De Morgan Laws

Remark 3.18 *Given n propositions*

$$p_1,\ p_2, \ldots,\ p_n$$

consider the compound proposition

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n. \tag{3.5}$$

(3.5) *is TRUE when p_i is true, for every $i = 1, \ldots, n$. The universally quantified proposition*

$$\forall x,\ p(x) \tag{3.6}$$

generalizes (3.5) *because it is TRUE when $p(x)$ is true for any x.*

Remark 3.19 *Given n propositions*

$$p_1,\, p_2, \ldots,\, p_n$$

consider the compound proposition

$$p_1 \vee p_2 \vee \cdots \vee p_n. \tag{3.7}$$

(3.5) *is TRUE when there exists i, with $1 \leq i \leq n$ such that p_i is true. The existentially quantified proposition*

$$\exists x,\ p(x) \tag{3.8}$$

generalizes (3.7) *because it is TRUE when $p(x)$ is true for some x.*

Remark 3.20 *From the De Morgan Laws for logic we have*

$$\overline{p_1 \wedge p_2 \wedge \cdots \wedge p_n} \equiv \overline{p}_1 \vee \overline{p_2} \vee \cdots \vee \overline{p_n}. \tag{3.9}$$

From the generalized De Morgan Laws we have

$$\overline{\forall x,\ p(x)} \equiv \exists x,\ \overline{p(x)} \tag{3.10}$$

and because of remarks 3.18 and 3.19 it makes sense to call (3.10) *Generalized De Morgan Law (it generalizes* (3.9)*).*

In the same way, if we compare the De Morgan Law for logic

$$\overline{p_1 \vee p_2 \vee \cdots \vee p_n} \equiv \overline{p}_1 \wedge \overline{p_2} \wedge \cdots \wedge \overline{p_n}. \tag{3.11}$$

with the generalized De Morgan law

$$\overline{\exists x,\ p(x)} \equiv \forall x,\ \overline{p(x)}, \tag{3.12}$$

it makes sense to call (3.12) *Generalized De Morgan Law because of remarks 3.18 and 3.19 (it generalizes* (3.11)*).*

3.4 Revision Exercises

Exercise 3.2 *Prove that*

$$\forall x \in \mathbf{R}, \quad x > 1 \Rightarrow x + 1 > 1 \tag{3.13}$$

is true.

Answer. Let us denote

$$p(x) : x > 1 \Rightarrow x + 1 > 1.$$

We want to prove that $p(x)$ is true for any real number x.

1) If $x > 1$ is false then $p(x)$ is true because of the false hypothesis (independently of the truth value of $x + 1 > 1$).

2) If $x > 1$ is true, then we have

$$x + 1 > x > 1$$

therefore $x + 1 > 1$ is true, therefore

$$p(x) : x > 1 \;\Rightarrow\; x + 1 > 1$$

is true because of type $T \Rightarrow T$.

Therefore we proved that $p(x)$ is true for any real number x, therefore

$$\forall x \in \mathbf{R}, \quad x > 1 \;\Rightarrow\; x + 1 > 1$$

is true. $\square$

Exercise 3.3 *Prove that*

$$\exists \text{positive integer} n, \quad n \text{is prime} \Rightarrow n{+}1, n{+}2, n{+}3, n{+}4 \quad \text{are not prime.} \tag{3.14}$$

is true.

Answer. Let us denote

$$p(n) : n \text{ is prime } \Rightarrow\ n+1, n+2, n+3, n+4 \quad \text{are not prime}$$

We just need to find a positive integer n for which $p(n)$ is true.
Take

$$n = 23,$$

we obtain the proposition

$$p(23):\ 23 \text{ is prime} \quad \Rightarrow \quad 24, 25, 26, 27 \text{ are not prime},$$

which is a true proposition since it is of type $T \Rightarrow T$. $\square$

Note: $p(n)$ in the above exercise is not true for every values of n, if we take for example $n = 2$, we obtain

$$p(2) : 2 \text{ is prime} \quad \Rightarrow \quad 3, 4, 5, 6 \text{ are not prime},$$

which is of type $T \Rightarrow F$ and therefore it is false.

Exercise 3.4 *Prove that*

$$\exists\, x \in \mathbf{R}, \quad \frac{1}{x^2+1} > 1. \tag{3.15}$$

is false.

Answer. If we denote

$$p(x): \ \frac{1}{x^2+1} > 1,$$

we want to prove that $p(x)$ is false for any $x \in \mathbf{R}$, i.e. we want to prove that

$$\frac{1}{x^2+1} > 1$$

is false for any $x \in \mathbf{R}$, i.e. we want to prove that $\overline{p(x)}$ is true for any $x \in \mathbf{R}$ which means that

$$\frac{1}{x^2+1} \leq 1$$

is true for any $x \in \mathbf{R}$. In other words we are going to prove that the following statement

$$\forall x \in \mathbf{R}, \quad \frac{1}{x^2+1} \leq 1 \tag{3.16}$$

is true. Let x be a real number, then $x^2 \geq 0$, therefore

$$x^2 + 1 \geq 1,$$

therefore

$$\frac{1}{x^2+1} \leq 1,$$

which proves that statement (3.16) is true and therefore that the statement

$$\exists\, x \in \mathbf{R}, \quad \frac{1}{x^2+1} > 1$$

is false. $\square$

Note: In the above exercise we had $p(x) : \frac{1}{x^2+1} > 1$ and we proved that

$$\exists x,\ p(x)$$

is false by proving that

$$\forall x,\ \overline{p(x)}$$

is true: this method is correct because of the Generalized De Morgan Laws

$$\overline{\exists x,\ p(x)} \equiv \forall x,\ \overline{p(x)},$$

therefore to prove that $\exists x,\ p(x)$ is false, we proved that its negation $\overline{\exists x,\ p(x)}$ is true by making use of the Generalized De Morgan Law and proving that $\forall x,\ \overline{p(x)}$ is true.

Exercise 3.5 *Prove that*

$$\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, \quad x + y = 0. \tag{3.17}$$

is true.

Answer. If we denote

$$p(x, y) :\ x + y = 0,$$

then we want to prove that

$$\exists y \in \mathbf{R}, \quad p(x, y)$$

is true for all $x \in \mathbf{R}$. Take a general $x \in \mathbf{R}$ and we want to prove that

$$p(x, y)$$

is true for at least one $y \in \mathbf{R}$. If we take

$$y = -x,$$

then

$$x + y = x - x = 0$$

and we found one value of y for which $p(x, y)$ is true, i.e.

$$\exists y \in \mathbf{R}, \quad p(x, y)$$

is true and we proved it by taking a general $x \in \mathbf{R}$, which proves that

$$\exists y \in \mathbf{R}, \quad p(x, y)$$

is true for all $x \in \mathbf{R}$, i.e.

$$\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, \quad x + y = 0$$

is true. $\square$

Exercise 3.6 *Prove that*

$$\exists y \in \mathbf{R}, \forall x \in \mathbf{R}, \quad x + y = 0. \tag{3.18}$$

is false.

Answer. If we denote

$$p(x, y) : \ x + y = 0,$$

then we can prove that

$$\exists y \in \mathbf{R}, \ \forall x \in \mathbf{R}, \quad p(x, y)$$

is false by proving that its negation is true, but its negation can be expressed by making use of the Generalized De Morgan Law by

$$\begin{aligned}
\overline{\exists y,\, \forall x, \quad p(x,y)} &\equiv \forall y, \overline{\forall x, \quad p(x,y)} \\
&\equiv \forall y, \exists x,\ \overline{x+y=0} \\
&\equiv \forall y, \exists x,\ x+y \neq 0.
\end{aligned}$$

We therefore want to prove that the statement

$$\forall y, \exists x,\ x+y \neq 0$$

is true, i.e. that the statement

$$\exists x,\ x+y \neq 0$$

is true for every y. Take a general $y \in \mathbf{R}$ and we want to prove that

$$x+y \neq 0,$$

for at least a value of $x \in \mathbf{R}$. If we take

$$x = 1 - y,$$

we obtain

$$x + y = 1 - y + y = 1 \neq 0,$$

therefore we proved that

$$x + y \neq 0$$

for at least one value of x given by $x = 1 - y$, i.e. we proved that

$$\exists x,\ x + y \neq 0$$

is true and it is true for any value $y \in \mathbf{R}$, i.e.

$$\forall y, \exists x,\ x + y \neq 0$$

is true. $\square$

Exercise 3.7 *Prove that*

$$\forall x \in \mathbf{R},\ \forall y \in \mathbf{R}, \quad x^2 < y^2 \ \Rightarrow\ x < y. \tag{3.19}$$

is false.

Answer. If we denote

$$p(x,y):\ x^2 < y^2 \ \Rightarrow\ x < y,$$

we want to prove that

$$\overline{\forall x \in \mathbf{R},\, \forall y \in \mathbf{R}, \quad p(x,y)}$$

is true, i.e.

$$\exists x \in \mathbf{R}, \exists y \in \mathbf{R}, \quad \overline{p(x,y)}$$

is true. In other words we want to find a counterexample: take

$$x = 1, \qquad y = -2$$

and with the above choices we have that $1 < 4$ is true and $1 < -2$ is false, therefore

$$1 < 4 \;\Rightarrow\; 1 < -2$$

is false because of type $T \Rightarrow F$, therefore

$$\overline{1 < 4 \;\Rightarrow\; 1 < -2}$$

is true for the values $x = 1$ and $y = -2$. $\square$

Exercise 3.8 *Prove that*

$$\forall x \in \mathbf{R},\ \exists y \in \mathbf{R},\quad x^2 < y^2 \;\Rightarrow\; x < y. \tag{3.20}$$

is true.

Answer. If we denote

$$p(x,y):\ x^2<y^2\ \Rightarrow\ x<y,$$

we want to prove that

$$\exists y\in\mathbf{R},\quad p(x,y)$$

is true for all $x\in\mathbf{R}$. Take a general $x\in\mathbf{R}$, we want to prove that

$$p(x,y)$$

is true for at least one value of $y\in\mathbf{R}$. If we take $y=0$, then we obtain

$$p(x,0):\ x^2<0\ \Rightarrow\ x<0,$$

which is of type $F \Rightarrow T$ and therefore it is true. We proved that

$$\exists y \in \mathbf{R},\ p(x,y)$$

is true (by taking $y = 0$) and this is true for any $x \in \mathbf{R}$, therefore we proved that

$$\forall x \in \mathbf{R},\ \exists y \in \mathbf{R}, \quad p(x,y)$$

is true. $\square$