Course Notes
for
MS4111
Discrete Mathematics 1

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CHAPTER 7 Recurrence Relations

7.1 Sequences and Recurrence Relations

Definition 7.1 *A sequence is a function*

$$\begin{aligned} f: \quad & \mathbb{N} \longrightarrow \mathbb{R} \\ & n \mapsto f(x) = a_n \end{aligned}$$

We can think at a sequence as an ordered list of real numbers $a_0, a_1, a_2, \ldots$. We usually denote it by

$$\{a_n\}_n$$

Definition 7.2 *Given a sequence $\{a_n\}_n$, a recurrence relation for $\{a_n\}_n$ is an equation that relates a_n to some of its predecessors $a_0, a_1, \ldots, a_{n-1}$ i.e.*

$$a_n = f(a_0, a_1, \ldots, a_{n-1}).$$

Initial conditions for the sequence $\{a_n\}_n$ are explicitly given values for a number of the terms of the sequence.

Recurrence relations are equations which express the "general term" a_n of a sequence $a_1, a_2, \cdots$, in terms of one or more of its predecessors. The recurrence equation can be used to calculate successive terms in the sequence given (say) the values of a_1 and a_2. Often the recurrence relation is the result of the mathematical analysis of some problem (as we will see below and in what follows). Simply calculating successive terms in the sequence can be useful but often we seek to calculate a "closed form" or explicit

formula for the general term a_n in terms of n. The most general form of interest is $a_n = f(a_{n-1}, a_{n-2}, \cdots, a_1)$. However equations as complex as (for example)
$a_n = \log(a_{n-1} + a_{n-2}\sin^2(a_{n-1}a_{n-3})) + a_{n-4}$ can rarely be solved in closed form. The best we could do for this example would be to calculate a_5 given a_1, a_2, a_3, a_4; then calculate a_6 in terms of a_2, a_3, a_4, a_5; etc.

Example 7.1

$$a_n = a_{n-1} + 3, \qquad n \geq 2 \tag{7.1}$$

$$a_1 = 5 \tag{7.2}$$

Then the first few elements of the sequence are

$$a_1 = 5; \quad a_2 = a_1 + 3 = 5 + 3 = 8, \quad a_3 = a_2 + 3 = 8 + 3 = 11, \ldots$$

Note: By using 7.1 and (7.2) we can compute any term of the sequence.

Problem: Given a recurrence relation (describing a sequence), we want to find an explicit formula for the $n-th$ term a_n in terms of n: we call this formula the closed form of the recurrence relation.

Example 7.2 *The Fibonacci sequence is defined by the recurrence relation*

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 3$$

with initial conditions

$$f_1 = 1 \quad ; \quad f_2 = 2.$$

Therefore we obtain

$$f_1 = 1; \quad f_2 = 2; \quad f_3 = f_2 + f_1 = 2 + 1 = 3;$$
$$f_4 = f_3 + f_2 = 3 + 2 = 5; \ \ldots$$

7.1.1 Different types of recurrence relations

Definition 7.3 *A recurrence relation*

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_1)$$

is linear if

$$a_n = \alpha_{n-1}a_{n-1} + \alpha_{n-2}a_{n-2} + \cdots + \alpha_1 a_1$$

Example 7.3 $a_n = 3a_{n-1}^2 + 4a_{n-3}$ *is non-linear.*

Example 7.4 $a_n = 3a_{n-1} + 4a_{n-3} + 7$ *is linear.*

Example 7.5 *The fibonacci sequence* $f_n = f_{n-1} + f_{n-2}$ *is linear.*

Definition 7.4 *A recurrence relation is of order k if*

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$$

i.e. if k is the number of steps back in the sequence we need to go in order to calculate a_n.

Example 7.6 *The $a_n = 4a_{n-1} - 5 + 7a_{n-7}$ is a recurrence relation of order seven.*

Example 7.7 *The $a_n = 2a_{n-1} + 1 + 5a_{n-2}^2$ is a recurrence relation of order 2.*

Definition 7.5 *A recurrence relation*

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_1)$$

has constant coefficients if each term on the RHS of the above equation only involves terms in the sequence and constants (numbers).

Example 7.8 $a_n = 3\frac{a_{n-2}^4}{\ln(a_{n-3})} + 4a_{n-6}$ *has constant coefficients.*

Example 7.9 $a_n = 2^n a_{n-1} + 4n a_{n-2}$ *does not have constant coefficients.*

Example 7.10 *The recurrence relation* $a_n = 3a_{n-1} - 5a_{n-4}$ *is linear, with constant coefficients.*

Definition 7.6 *A recurrence relation*

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_1)$$

is homogeneous if every term on the RHS of the above equation involves one or more term in the sequence. Otherwise it is said to be inhomogeneous.

Example 7.11 $a_n = a_{n-3}^3 sin^2(a_{n-4}) + 7a_{n-2}^3$ *is homogeneous.*

Example 7.12 $a_n = a_{n-3}^3 sin^2(a_{n-4}) + 7a_{n-2}^3 + 7n^2 + 4$ *is inhomogeneous.*

Example 7.13 *The fibonacci sequence* $f_n = f_{n-1} + f_{n-2}$ *is linear, of second order, with constant coefficients.*

Note: Solving a recurrence relation will mean to find its closed form.

7.2 Derivation of a Recurrence Relation

Consider the famous "Tower of Hanoi" problem. It may be posed as follows. Start with n heavy disks stacked on the left-handmost of three posts in decreasing order of size (smallest at the top). The disks are to be moved (one at a time) from Post 1 to Post 3; while obeying the rule that larger disk may not be place on top of a smaller one. It is not immediately obvious how long it takes to perform this task for n disks — in units of time where one unit is the time required to carry one disk from one post to another.

The solution to the problem leads to the idea of an "algorithm".

Definition 7.7 *An algorithm is an unambiguous sequence of operations which solve a specified problem in a finite time.*

We need to specify an algorithm to solve the Tower of Hanoi problem. This is easily done for small values of n — it rapidly becomes complicated for (say) $n > 3$. The insight which leads to a simple algorithm is the idea of "recursion".

Definition 7.8 (Recursion) *Informally, a recursive algorithm is one which refers to itself.*

Consider the following recursive algorithm for the Tower of Hanoi problem:

To solve the n-disk problem:

- A: Move the top $n - 1$ disks to the middle Post.
- B: Move the biggest (base) disk to its final destination (RH Post)
- C: Move the top $n - 1$ disks from middle Post to RH Post.

Note: Note that the procedure or algorithm is not explicit: we describe the procedure for n-disks in terms of the procedure for the $(n-1)$-disks problem.

Exercise 7.1 *Persuade yourself that the above algorithm works by trying it for (say)* $n = 2, 3$.

Let

$$a_n = \text{time (number of moves)} \text{ taken to solve the } n\text{-disks problem.}$$

Therefore $a_1 = 1$ and $a_2 = 3$. We can now derive the recurrence relation for this algorithm. Clearly $a_n = T_A + T_B + T_C$ (where T_A, T_B, T_C are the times taken to accomplish steps A,B,C of the algorithm respectively. We must therefore have:

$$a_n = a_{n-1} + 1 + a_{n-1}$$

(substituting for T_A, T_B, T_C). Therefore we have:

$$a_n = 2a_{n-1} + 1 \tag{7.3}$$

with initial condition

$$a_1 = 1.$$

All that remains is to find a solution to the , i.e. to find an expression for a_n in terms of n . Want a_n as a function of n (closed form).

7.3 Solving recurrence relations

We will consider two general solution techniques; the iteration technique and the substitution technique. The first has the advantage of being applicable to any, although it may not always yield a closed-form solution. The second is only applicable to linear, homogeneous, second-order with constant coefficients recurrence relations; but will always yield a closed form solution. (It can be extended to inhomogeneous — as we will see.)

7.3.1 Iteration Technique

This technique is applicable to any recurrence relation and it consists simply of repeatedly substituting for $a_{n-1}, a_{n-2}, \cdots$ in the RHS of the recurrence relation

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_1).$$

In other words, given the recurrence relation

$$a_n = f(a_{n-1}, a_{n-2}, \cdots, a_1)$$

substitute for

$$a_{n-1} = f(a_{n-2}, a_{n-3}, \cdots, a_1),$$

$$a_{n-2} = f(a_{n-3}, a_{n-4}, \cdots, a_1),$$

etc. We try then to guess the general pattern for a_n (if any) by studying the formulas obtained for a_n in the first few substitutions. After we guess a formula for a_n we check it by substituting it into the original recurrence relation. This is an example of inductive reasoning.

Example 7.14 *Consider the Tower of Hanoi , Eq. 7.3. We will apply the above technique.*

$$a_n = 2a_{n-1} + 1, \qquad n \geq 2$$
$$a_1 = 1.$$

We have

$$
\begin{aligned}
a_n &= 2a_{n-1} + 1 \\
&= 2(2a_{n-2} + 1) + 1 \\
&= 2^2 a_{n-2} + 2 + 1 \\
&= 2^2(2a_{n-3} + 1) + 2 + 1 \\
&= 2^3 a_{n-3} + 2^2 + 2 + 1 \\
&= 2^3(2a_{n-4} + 1) + 2^2 + 2 + 1 \\
&= 2^4 a_{n-4} + 2^3 + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} a_{n-(n-1)} + 2^{n-2} + \cdots + 2^2 + 2 + 1 \\
&= 2^{n-1} a_1 + 2^{n-2} + \cdots + 2^2 + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1
\end{aligned}
$$

We finally obtain

$$a_n = 1 + 2 + 2^2 + ...2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1. \qquad (7.4)$$

Therefore $a_n = 2^n - 1$

Equation (7.4) is derived by using the following useful general formula

$$1 + a + a^2 + ...a^{n-1} = \frac{a^n - 1}{a - 1}.$$

Check that the derived formula (7.4) satisfies the recurrence relation — ie that $a_n = 2a_{n-1} + 1$ and the initial condition $a_1 = 1$.

We have

$$a_1 = 2^1 - 1 = 2 - 1 = 1.$$

We also have

$$\begin{aligned}
a_n &= 2^n - 1 \\
&= 2^{(n-1+1)} - 1 \\
&= 2 \cdot 2^{n-1} - 1 \\
&= 2(a_{n-1} + 1) - 1 \\
&= 2a_{n-1} + 2 - 1 \\
&= 2a_{n-1} + 1.
\end{aligned}$$

Therefore the formula $a_n = 2^n - 1$ satisfies the recurrence relation and the initial condition.

7.3.2 Substitution technique

This technique is only applicable to linear, homogeneous, second-order recurrence relation with constant coefficients i.e. a recurrence relation of type

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \tag{7.5}$$

where c_1 and c_2 are constants (numbers).

We start with the following result.

Theorem 7.1 *If S_n and R_n are both solutions to (7.5), then, for any choice of A, B (constants),*

$$AS_n + BR_n$$

is a solution to (7.5).

Proof: We have

$$
\begin{aligned}
S_n &= c_1 S_{n-1} + c_2 S_{n-2} \\
R_n &= c_1 R_{n-1} + c_2 R_{n-2}
\end{aligned}
$$

and if we consider $a_n = AS_n + BR_n$ we have

$$
\begin{aligned}
a_n &= AS_n + BR_n \\
&= A(c_1 S_{n-1} + c_2 S_{n-2}) + B(c_1 R_{n-1} + c_2 R_{n-2}) \\
&= c_1 (AS_{n-1} + BR_{n-1}) + c_2 (AS_{n-2} + BR_{n-2}) \\
&= c_1 a_{n-1} + c_2 a_{n-2}. \qquad \square
\end{aligned}
$$

Note: Given the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

if S_n and R_n are two solutions, then the general solution (i.e. the set of all the possible solutions) of the recurrence relation is given by

$$a_n = AS_n + BR_n,$$

for any pairs of constants A and B.

7.3.3 Method for solving by substitution

Given the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

we want to find two solutions for it in order to find then the general solution a_n. Try a simple substitution for a_n which contains a parameter to be determined. In other words we use a trial solution or "ansatz" which satisfies the equation for some choice of a free parameter.

In our work, the trial solution will usually take the form

$$a_n = t^n,$$

where t is to be determined. So

$$
\begin{aligned}
a_n &= c_1 a_{n-1} + c_2 a_{n-2} \\
t^n &= c_1 t^{n-1} + c_2 t^{n-2} / : t^{n-2}
\end{aligned}
$$

Dividing across by t^{n-2}, we find that t must satisfy the following quadratic equation

$$t^2 - c_1 t - c_2 = 0.$$

When we solve the quadratic, as usual three cases arise:

1. Two real distinct roots.
2. Equal real roots.
3. Two distinct complex roots.

We consider each case separately.

7.3.4 Two Distinct Real Roots

This is the most straightforward case. We will use the Fibonacci example to illustrate this case.

Example 7.15 *The Fibonacci sequence*

$$a_n = a_{n-1} + a_{n-2}.$$

Method: We use the trial solution

$$a_n = t^n,$$

where t is to be determined. So

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \\ t^n &= t^{n-1} + t^{n-2} / : t^{n-2} \end{aligned}$$

for all $n \geq 3$.. Dividing across by t^{n-2}, we find that t must satisfy the following quadratic equation

$$t^2 - t - 1 = 0.$$

Therefore

$$t = \frac{1 \pm \sqrt{5}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} = t_1 \\ \frac{1-\sqrt{5}}{2} = t_2 \end{cases}$$

We therefore have two solutions

$$t_1^n = \left(\frac{1+\sqrt{5}}{2}\right)^n \quad ; \quad t_2^n = \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Therefore the general solution to

$$a_n = a_{n-1} + a_{n-2}$$

is

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

If we impose the initial condition

$$a_1 = 1 \quad ; \quad a_2 = 2$$

we obtain

$$\begin{cases} 1 = a_1 = At_1 + Bt_2 \quad (n = 1) \\ 2 = a_2 = At_1^2 + +Bt_2^2 \quad (n = 2) \end{cases}$$

i.e.

$$\begin{cases} 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \\ 2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 \end{cases}$$

If we solve the above system for A and B, we obtain

$$
\begin{aligned}
A &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) \\
B &= -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)
\end{aligned}
$$

therefore the particular solution to the recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

that satisfies the initial conditions

$$a_1 = 1 \quad ; \quad a_2 = 2$$

is given by

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Note: Note hat there are no constants appearing in the particular solution.

7.3.5 Equal Real Roots

Given the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

we try the substitution

$$a_n = t^n,$$

which leads to the quadratic equation

$$t^2 - c_1 t - c_2 = 0.$$

Suppose the quadratic equation only has the unique solution

$$t = r.$$

Theorem 7.2 *The general solution to* $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ *for the case when the associated quadratic equation*

$$t^2 - c_1 t - c_2 = 0$$

has a unique real solution $t = r$

is of the form

$$a_n = At^n + Bn\,r^n.$$

Proof. We want to show that r^n and $n\,r^n$ are two solutions to the recurrence relations. We already know that r^n is a solution to it, we then want to show that $n\,r^n$ is a solution too.

If $t = r$ is the unique real solution to

$$t^2 - c_1 t - c_2 = 0,$$

we have

$$t^2 - c_1 t - c_2 = (t - r)^2 = t^2 - 2rt + r^2,$$

Therefore

$$\begin{aligned} c_1 &= 2r; \\ c_2 &= -r^2, \end{aligned}$$

therefore, substituting $R_n = nr^n$ in the recurrence relation, we have

$$
\begin{aligned}
& c_1\left[(n-1)r^{n-1}\right] + c_2\left[(n-2)r^{n-2}\right] \\
&= 2r\left[(n-1)r^{n-1}\right] - r^2\left[(n-2)r^{n-2}\right] \\
&= 2(n-1)r^n - (n-2)r^n \\
&= (2n-2-n+2)r^n = n\,r^n.
\end{aligned}
$$

$\square$

So the general solution for the Single Real Root case is

$$a_n = Ar^n + Bnr^n. \tag{7.6}$$

Example 7.16

$$
\begin{aligned}
& a_n = 4a_{n-1} - 4a_{n-2}; \\
& a_1 = 1, a_2 = 3.
\end{aligned}
$$

The substitution $a_n = t^n$ leads to the quadratic equation

$$t^2 - 4t + 4 = 0$$

i.e.

$$(t-2)^2 = 0,$$

whose only solution is $r = 2$. The general solution is

$$a_n = A2^n + Bn2^n.$$

If we want to find the particular solution that satisfies also the initial conditions

$$a_1 = 1 \quad ; \quad a_2 = 3$$

we solve the system

$$\begin{cases} 1 = a_1 = A \cdot 2 + B \cdot 1 \cdot 2 & (n = 1) \\ 3 = a_2 = A 2^2 + B \cdot 2 \cdot 2^2 & (n = 2) \end{cases}$$

i.e.

$$\begin{cases} 1 = 2A + 2B \\ 3 = 4A + 8B \end{cases}$$

and by solving the above system for A and B we obtain

$$A = B = \frac{1}{4}.$$

Therefore the particular solution satisfying the given recurrence relation and the initial conditions is

$$a_n = \frac{1}{4}\,2^n + \frac{1}{4}\,n\,2^n = 2^{n-2}(n+1).$$

7.3.6 Two Complex Roots

Given the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

we try the substitution

$$a_n = t^n,$$

which leads to the quadratic equation

$$t^2 - c_1 t - c_2 = 0.$$

Suppose the quadratic equation has the two complex solutions

$$
\begin{aligned}
t_1 &= a + ib \\
t_2 &= a - ib.
\end{aligned}
$$

If we denote

$$D = \sqrt{a^2 + b^2} \quad ; \quad \tan\theta = \frac{b}{a},$$

then the general solution to the recurrence relation is

$$a_n = D^n \left(A\cos(n\theta) + B\sin(n\theta)\right).$$

Example 7.17 *Consider the recurrence relation*

$$a_n = 2a_{n-1} - 5a_{n-2}$$

with initial conditions

$$a_1 = 1 \quad ; \quad a_2 = 3.$$

The usual substitution $a_n = t^n$ *yields the quadratic equation*

$$t^2 - 2t + 5 = 0$$

whose roots are

$$1 \pm 2i.$$

Therefore

$$D = \sqrt{1^2 + 2^2} = \sqrt{5} \quad ; \quad \tan\theta = \frac{2}{1} = 2,$$

therefore the general solution to the recurrence relation is

$$a_n = (\sqrt{5})^n \, (A\cos(n\theta) + B\sin(n\theta)) = 5^{n/2} \, (A\cos(n\theta) + B\sin(n\theta))$$

(There is no need to numerically evaluate θ at this stage.)

To find the particular solution that satisfies also the initial conditions

$$a_1 = 1 \quad ; \quad a_2 = 3,$$

we need to solve the following system for A and B

$$
\begin{aligned}
1 &= \sqrt{5}(A\cos\theta + B\sin\theta) \quad (n=1)\\
3 &= 5(A\cos 2\theta + B\sin 2\theta) \quad (n=2)
\end{aligned}
$$

We are not going to solve this system here.

7.3.7 Substitution technique: Inhomogeneous Recurrence Relations

We now know how to solve linear homogeneous order 2 with constant coefficients; i.e. of form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$.

We will now extend the technique to inhomogeneous recurrence relations of form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n),$$

where f is an arbitrary given function of n.

The technique for homogeneous doesn't work — as if we substitute $a_n = t^n$ in such a recurrence relation we get $t^n = c - 1t^{n-1} + c - 2t^{n-2} + f(n)$, which cannot be solved for t in general without t depending on n. The extended technique can be stated by the following steps.

The general solution to

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$$

can be found by the following steps:

1) Find the general solution h_n to the homogeneous equation

$$h_n = c_1 h_{n-1} + c_2 h_{n-2}.$$

3) Find the particular solution p_n to

$$p_n = c_1 p_{n-1} + c_2 p_{n-2} + f(n).$$

3) The general solution a_n to

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$$

is given by

$$a_n = h_n + p_n.$$

7.3.7.10 Choice of the particular solution p_n

1) Suppose we have the inhomogeneous recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n),$$

where $f(n)$ is a polynomial in n. In particular, suppose that

$$f(n) = an^2 + bn + c$$

Therefore we have

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + an^2 + bn + c.$$

Note: For a trial of a particular solution p_n to satisfy this equation, p_n it must be at least a quadratic in n i.e.

$$p_n = kn^2 + ln + m$$

.

In general the trial solution p_n must be a polynomial of degree at least equal to that of $f(n)$.

Example 7.18 *Consider the recurrence relation*

$$a_n = 5a_{n-1} - 6a_{n-2} + 9$$

- *First we solve the homogeneous* $h_n = 5h_{n-1} - 6h_{n-2}$. *The solution is* $h_n = A2^n + B3^n$.*(Check.)*

- *Now we need a particular solution of the full equation. We try the simplest possible choice:* $p_n = Kk$ — *a constant. Substituting the the yields*

$$k = 5k - 6k + 9,$$

i.e.

$$k = \frac{9}{2}.$$

Therefore

$$p_n = \frac{9}{2}.$$

- *Therefore the general solution of the inhomogeneous relation is*

$$a_n = h_n + p_n = A\,3^n + B\,2^n + \frac{9}{2}.$$

If the initial conditions for

$$a_n = 5a_{n-1} - 6a_{n-2} + 9$$

are for example

$$a_1 = 1 \quad ; \quad a_2 = -1,$$

then the particular solution satisfying them is given by solving the following system for A and B

$$\begin{cases} 1 = 3A + 2B + \frac{9}{2} \\ -1 = 9A + 4B + \frac{9}{2}, \end{cases}$$

which leads to

$$A = \frac{1}{2} \quad ; \quad B = -\frac{5}{2}.$$

- *Therefore the particular solution is*

$$a_n = \frac{1}{2}\, 3^2 - \frac{5}{2}\, 2^n + \frac{9}{2}.$$

2) Suppose we have the inhomogeneous recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n),$$

where $f(n)$ involves $\cos(n)$ or $\sin(n)$. For example, consider the $f_n = 2\cos(n) - 7\sin(n)$.

Rule: In this case the most general needed p_n takes the form

$$p_n = a\cos n + b\sin n.$$

Example 7.19

$$a_n = 2a_{n-1} - 3a_{n-2} + 2\cos(n) - 7\sin(n).$$

7.4 Some working problems

1) The problem may be "degenerate" and the rule for finding p_n for inhomogeneous relations we just saw may not work, e.g.

$$a_n = 4a_{n-1} - 3a_{n-2} + 15.$$

If we try $p_n = k$. Then k must satisfy $k = 4k - 3k + 15$. So $k = k + 15$ and therefore $0 = 15$. The problem is degenerate as $h_n = A3^n + B(1)^n = A3^n + B$.

Try $p_n = kn + l$. Then

$$kn + l = 4[k(n-1) + l] - 3[k(n-2) + l] + 15$$

Solve for k, l and show that the general solution is

$$a_n = A3^n + B - \frac{15}{2}n$$

2) Find the general solution to

$$a_n = -8a_{n-1} - 16a_{n-2} + 3n$$

Answer:

$$a_n = A(-4)^n + B\,n(-4)^n + \frac{3}{25}n + \frac{24}{125}.$$

3) Population growth 1

Suppose the population of a county is 1000 at time (year) $n = 0$ and that the population increases from year $n-1$ to year n of 10% of the size at time $n-1$. Write the recurrence relation and an initial condition that define the deer population at year n and solve it.

Answer:

$$a_0 = 1000$$

and

$$a_n = a_{n-1} + 10\% a_{n-1} = a_{n-1} + (0.1)a_{n-1} = (1.1)a_{n-1},$$

i.e.

$$\begin{aligned} a_n &= 1.1a_{n-1} \\ a_0 &= 1000. \end{aligned}$$

We can try to solve it by iteration:

$$\begin{aligned} a_n &= 1.1a_{n-1} \\ &= 1.1(1.1a_{n-2}) = (1.1)^2 a_{n-2} \\ &= \cdots = (1.1)^n a_{n-n} = (1.1)^n a_0 = 1000(1.1)^n. \end{aligned}$$

Therefore the particular solution to our problem is the exponential population growth

$$a_n = 1000(1.1)^n.$$

4) Population growth 2

Suppose the population of a county is 200 at time (year) $n = 0$ and 220 at year $n = 1$. Suppose that the population increases from year $n - 1$ to year n twice than from year $n - 2$ to $n - 1$. Write the recurrence relation and an initial condition(s) that define the deer population at year n and solve it.

Answer:

$$a_n = 3a_{n-1} - 2a_{n-2}$$
$$a_0 = 200 \quad ; \quad a_1 = 220.$$

And the particular solution to the problem is

$$a_n = 20(2^n) + 180,$$

which is again an exponential growth.

7.5 Recursive Algorithm

A recurrence relation that defines a sequence can be directly converted to a recursive algorithm to compute the sequence.

Example 7.20 *Find the amount of money at the end of n years assuming an initial amount of* 1000 *invested at an interest rate of* 12% *compounded annually.*

a) The recurrence relation is

$$a_n = a_{n-1} + 0.12a_{n-1} = (1.12)a_{n-1}$$

with initial condition

$$a_0 = 1000.$$

The above recurrence relation can be solved iteration (exercise).

b) The recursive algorithm is

$$Input: \quad n, \; the\, number\, of\, years$$

$$Output: \quad the\, ampunt\, of\, money\, at\, the\, end\, of\, n\, years$$

1. *procedure* $compound\_interest(n)$
2. *if* $n = 0$ *then*
3. *return* (1000)
4. *return* $(1.12 * compound\_interest(n-1))$
5. *end* $compound\_interest.$

The algorithm is a direct translation of the recurrence relation and the initial condition obtained above that define the sequence $a_0, a_1, \ldots$.