Course Notes for MS4111 Discrete Mathematics 1

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CHAPTER 2 Propositional logic

2.1 Propositions

We start by giving the following definition.

Definition 2.1 *A proposition is a* ***sentence that is either true or false, but not both****.*

Example 2.1 *The only positive integers that divide* 7 *are* 1 *and* 7 *itself.*

The above sentence is a proposition and it is true.

Example 2.2 *The set of real numbers is finite.*

The above sentence is a proposition and it is false.

Example 2.3 *Sit down please.*

The above sentence is **not** a proposition: it is neither true nor false, it is a command!

We will denote propositions with lower case letters, such as

$$p,\, q,\, r \ldots$$

and we will use the notation

p: The only positive integers that divide 7 are 1 and 7 itself.

to define p to be the proposition ‘ The only positive integers that divide 7 are 1 and 7 itself.’

or

q: The set of real numbers is finite.

to define q to be the proposition ‘The set of real numbers is finite.’ and so on.

In ordinary speech and writing, we combine propositions using connectives as *and* and *or*. We can do the same thing in *logic*, by introducing the so called connectives.

2.1.1 Connectives and truth tables

We need to introduce "connectives" to combine propositions into compound propositions.

2.1.1.1 AND connective or conjunction

Definition 2.2 *Let p and q be two propositions. The conjunction of p and q, denoted by*

$$p \wedge q$$

is the proposition

$$p \quad \text{and} \quad q.$$

and it is defined by saying that the truth value of $p \wedge q$ is TRUE only when both pand q are TRUE (otherwise $p \wedge q$ is FALSE).

Definition 2.2 can be rewritten by making use of the so called truth table of $\wedge$.

Definition 2.3 *Given a compound proposition p made up of the individual propositions $p_1, \ldots p_n$, the truth table of p lists all possible combinations of truth values for $p_1, \ldots p_n$ and for each such combination lists the truth value of p.*

Truth tables allow unambiguous definitions in propositional logic.

Notation: Use T to represent the truth value TRUE and use F to represent the truth value FALSE.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 2.1: Truth Table for $\wedge$

The statement $p \wedge q$ is our first example of a "compound statement". This is called the *conjunction* of $p$and q.

Example 2.4

$$p : \text{Today is monday.}$$

$$q : \text{Today it is sunny.}$$

$p \wedge q$: Today is monday *and* today it is sunny.

2.1.1.2 OR connective or disjunction

Definition 2.4 *Let p and q be two propositions. The disjunction of p and q, denoted by*

$$p \vee q$$

is the proposition

$$p \quad \text{or} \quad q.$$

and it is defined by saying that the truth value of either $p \vee q$ is TRUE if either both por q are TRUE or both (the truth value of $p \vee q$ is FALSE if both pand q are false).

The disjunction $p \vee q$ is used in the inclusive sense, i.e. by saying that the truth value of $p \vee q$ is TRUE also in the case when both $p$and q are both TRUE. Definition 2.4 can be rewritten by making use of the so called truth table of $\vee$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 2.2: Truth Table for $\vee$

Remark 2.6 *$p \vee q$ is FALSE only when both pand q are FALSE. and is TRUE otherwise.*

2.1.1.3 NOT connective or negation

Definition 2.5 *Let p be a proposition. The negation of p, denoted by*

$$\bar{p}$$

is the proposition

$$\text{not } p.$$

and it is defined by saying that the truth value of not pis TRUE when pis FALSE and viceversa.

Definition 2.5 can be rewritten by making use of the truth table of ‘not’.

p	$\bar{p}$
T	F
F	T

Figure 2.3: Truth Table for NOT

2.1.1.4 Implication connective

Definition 2.6 *Let p and q be two propositions. The compound proposition*

$$p \Rightarrow q.$$

is called a conditional proposition and it is red

$$\text{if } p \text{ then } q.$$

The truth table of $p \Rightarrow q$ is the following

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.4: Truth Table for $\Rightarrow$

The proposition p is called the hypothesis and the proposition q is called the conclusion.

In ordinary English usage, we understand the statement '$p$implies q' to be true if both $p$and q are true and false if $p$is true and q is not.

What if $p$is false? This case is not usually of interest in ordinary conversation but for "implies" to be a connective, "A implies B" must have a definite truth value for *all* values of A & B.

We say that '$p$implies q" is 'vacuously true' if $p$is FALSE, irrespective of the truth value of q. Consider the following example:

Example 2.5 *'If every day is Christmas Day then I am Santa Claus.'*

The above proposition is TRUE!

We need to define the terms "necessary" and "sufficient" which are often used in mathematical discussions.

Definition 2.7 *We say that $p$is sufficient for q if $p \Rightarrow q$.*

Definition 2.8 *We say that $p$is necessary for q if $q \Rightarrow p$.*

In ordinary language, the hypothesis and the conclusion in a conditional proposition are normally related, **but in logic**, the hypothesis and the conclusion in a conditional proposition are not required to refer to the same subject matter.

Example 2.6

$p : 7 > 9$

q : Barack Obama is the new president of the United States.

Then we can consider

$p \Rightarrow q$: If $7 > 9$, then Barack Obama is the new president of the United States.

$p \Rightarrow q$ is a TRUE conditional proposition because the hypothesis $p$is FALSE and the conclusion q is TRUE ($p \Rightarrow q$ is actually TRUE independently from the truth value of q).

Example 2.7

$$p:\ 2<3$$
$$q:\ 5+3=8$$

We notice that pis FALSE and q is TRUE. Then we have:

$$p \Rightarrow q \quad \textit{is of type} \quad F \Rightarrow T \quad \textit{which is TRUE}$$
$$q \Rightarrow p \quad \textit{is of type} \quad T \Rightarrow F \quad \textit{which is FALSE.}$$

The above example shows that the conditional proposition $p \Rightarrow q$ can be TRUE while the conditional proposition $q \Rightarrow p$ is FALSE.

Definition 2.9 *We call the proposition $q \Rightarrow p$ the converse of the proposition $p \Rightarrow q$.*

Definition 2.10 *Given the conditional proposition*

$$p \Rightarrow q,$$

we call the contrapositive (or transposition) of $p \Rightarrow q$*, the proposition*

$$\bar{q} \Rightarrow \bar{p}.$$

2.1.1.5 Equivalence connective

Definition 2.11 *Let p and q be two propositions. The compound proposition*

$$p \Leftrightarrow q$$

is called a biconditional proposition and it is red

$$p \text{ if and only if } q.$$

The truth table of $p \Leftrightarrow q$ is the following

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 2.5: Truth Table for $\Leftrightarrow$

Definition 2.12 *Two propositions are equal or logically equivalent if they have the same truth value, irrespective of the truth values of their component propositions, if any.*

We use the symbol $\equiv$ instead of $=$ to represent equality for statements to remind ourselves that we are doing Statement Calculus, not Arithmetic.

Remark 2.7 $\Leftrightarrow$ is *a connective*, $\equiv$ is not.

Remark 2.8 *Connectives can be used to* combine *propositions into more complex propositions;* relational operators *like* $\Leftrightarrow$ *allow us to compare statements.*

2.1.2 Some important examples

1. **De Morgan Laws**

   Let p and q be two propositions, then

   (a) $\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$

   (b) $\overline{p \vee q} \equiv \overline{p} \wedge \overline{q}$.

   Proof. a) we want to prove that $\overline{p \wedge q}$ and $\overline{p} \vee \overline{q}$ have the same truth table.

p	q	$p \wedge q$	$\overline{p \wedge q}$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Figure 2.6: Truth Table for $\overline{p \wedge q}$

p	q	$\bar{p}$	$\bar{q}$	$\overline{p} \vee \overline{q}$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Figure 2.7: Truth Table for $\overline{p} \vee \overline{q}$

The above truth tables are the same, therefore $\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$.

b) we want to prove that $\overline{p \vee q}$ and $\overline{p} \wedge \overline{q}$ have the same truth table.

p	q	$p \vee q$	$\overline{p \vee q}$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Figure 2.8: Truth Table for $\overline{p \vee q}$

p	q	$\overline{p}$	$\overline{q}$	$\overline{p} \wedge \overline{q}$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure 2.9: Truth Table for $\overline{p} \wedge \overline{q}$

The above truth tables are the same, therefore
$\overline{p \vee q} \equiv \overline{p} \wedge \overline{q}$. $\square$

2. $p \Rightarrow q \equiv \overline{q} \Rightarrow \overline{p}$

Proof.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.10: Truth Table for $p \Rightarrow q$

p	q	$\overline{p}$	$\overline{q}$	$\overline{q} \Rightarrow \overline{p}$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Figure 2.11: Truth Table for $\overline{q} \Rightarrow \overline{p}$.

$\square$

3. $\overline{p \Rightarrow q} \equiv p \wedge \overline{q}$

Proof.

p	q	$p \Rightarrow q$	$\overline{p \Rightarrow q}$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Figure 2.12: Truth Table for $\overline{p \Rightarrow q}$

p	q	$\overline{q}$	$p \vee \overline{q}$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Figure 2.13: Truth Table for $p \vee \overline{q}$.

□

4. $p \equiv \overline{\overline{p}}$

Proof.

p	$\overline{p}$	$\overline{\overline{p}}$
T	F	T
F	T	F

Figure 2.14: Truth Table for $\overline{\overline{p}}$.

$\square$

5. $p \Rightarrow q \equiv \overline{p} \vee q$

We are going to give two proofs of the above statement.

Proof 1.

$$p \Rightarrow q \equiv \overline{\overline{p \Rightarrow q}} \equiv \overline{p \wedge \overline{q}} \equiv \overline{p} \vee \overline{\overline{q}} \equiv \overline{p} \vee q.$$

$\square$

Proof 2. Use truth tables.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.15: Truth Table for $p \Rightarrow q$

p	q	$\overline{p}$	$\overline{p} \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Figure 2.16: Truth Table for $\overline{p} \vee q$.

The truth tables of $p \Rightarrow q$ and $\overline{p} \vee q$ are the same therefore $p \Rightarrow q \equiv \overline{p} \vee q$.

$\square$

6. $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Proof.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 2.17: Truth Table for $p \Leftrightarrow q$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure 2.18: Truth Table for $(p \Rightarrow q) \wedge (q \Rightarrow p)$

$\square$

Note: We proved the following equivalences of propositions:

1. $p \Rightarrow q \equiv \overline{q} \Rightarrow \overline{p}$
2. $p \Rightarrow q \equiv \overline{p} \vee q$
3. $\overline{p \Rightarrow q} \equiv p \wedge \overline{q}$
4. $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

2.1.3 Tautologies and Contradictions

Definition 2.13 *A tautology is a statement which has the truth value TRUE (T) irrespective of the truth values of its component statements (if any.*

Example 2.8 $p \vee \overline{p}$ *is a tautology.*

p	$\overline{p}$	$p \vee \overline{p}$
T	F	T
F	T	T

Figure 2.19: Truth Table for $p \vee \overline{p}$

Example 2.9

$(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p})$ *is a tautology.*

p	q	$\overline{p}$	$\overline{q}$	$p \Rightarrow q$	$\overline{q} \Rightarrow \overline{p}$	$(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p})$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Figure 2.20: Truth Table for $(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p})$

Example 2.10

$(\overline{p \Rightarrow q}) \Leftrightarrow (p \wedge \overline{q})$ *is a tautology.*

p	q	$\overline{q}$	$p \Rightarrow q$	$\overline{p \Rightarrow q}$	$p \wedge \overline{q}$	$(\overline{p \Rightarrow q}) \Leftrightarrow (p \wedge \overline{q})$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

Figure 2.21: Truth Table for $(\overline{p \Rightarrow q}) \Leftrightarrow (p \wedge \overline{q})$

Remark 2.9 *If p, q are two propositions such that*

$$p \equiv q,$$

then p and q have the truth table, then

$$p \Leftrightarrow q \quad \text{is a} \quad \text{tautology}.$$

Definition 2.14 *A contradiction is a statement which has truth value FALSE (F) irrespective of the truth values of its component statements (if any).*

Example 2.11 *$p \wedge \overline{p}$ is a contradiction.*

p	$\overline{p}$	$p \wedge \overline{p}$
T	F	F
F	T	F

Figure 2.22: Truth Table for $p \wedge \overline{p}$

Example 2.12

$(p \Rightarrow q) \Leftrightarrow \overline{(\overline{q} \Rightarrow \overline{p})}$ *is a contradiction.*

p	q	$\overline{p}$	$\overline{q}$	$p \Rightarrow q$	$\overline{q} \Rightarrow \overline{p}$	$\overline{\overline{q} \Rightarrow \overline{p}}$	$(p \Rightarrow q) \Leftrightarrow \overline{(\overline{q} \Rightarrow \overline{p})}$
T	T	F	F	T	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	F
F	F	T	T	T	T	F	F

Figure 2.23: Truth Table for $(p \Rightarrow q) \Leftrightarrow \overline{(\overline{q} \Rightarrow \overline{p})}$

Remark 2.10 *If p, q are two propositions such that*

$$p \equiv q,$$

then p and q have the truth table, then

$$p \Leftrightarrow \overline{q} \quad \text{is a} \quad \text{contradiction.}$$

Remark 2.11 *The negation of any tautology is a contradiction; the negation of any contradiction is a tautology.*

Example 2.13

$$(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p}) \quad \text{is a} \quad \text{tautology;}$$

$$\overline{(p \Rightarrow q) \Leftrightarrow (\overline{q} \Rightarrow \overline{p})} \quad \text{is a} \quad \text{contradiction.}$$

2.1.4 Rules of Propositional Logic

Let us see here some properties of the connectives $\vee$ and $\wedge$ that are very similar to the rules in arithmetic of the operations $+$ and $\cdot$ (multiplication). We first recall the Rules of Arithmetic. If a, b, c are real numbers, then

1. **Commutative Law**

$$a + b = b + a \tag{2.1}$$

$$ab = ba \tag{2.2}$$

2. **Associative Law**

$$a + (b + c) = (a + b) + c \tag{2.3}$$

$$a(bc) = (ab)c \tag{2.4}$$

3. **Distributive Law**

$$a(b+c) = ab + ac \tag{2.5}$$

4. **Zero-One Law**

$$a\,0 = 0 \tag{2.6}$$

$$a\,1 = a \tag{2.7}$$

$$a + 0 = a \tag{2.8}$$

(There are, of course, other rules in Arithmetic, in particular those involving Division $\div$, but they are not of interest here.)

We now list the corresponding Laws for Propositional Logic, each of them must be checked by constructing a truth table.

Let p, q and r be three propositions, then

1. **Commutative Law**

$$p \vee q \equiv q \vee p \tag{2.9}$$

$$p \wedge q \equiv p \wedge q \tag{2.10}$$

2. **Associative Law**

$$p \vee (q \vee r) \equiv (p \vee q) \vee r \tag{2.11}$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \tag{2.12}$$

3. **Distributive Law**

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \tag{2.13}$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \tag{2.14}$$

4. **Identity Law**

If t is a tautology and f is a contradiction then

$$p \wedge t \equiv p \quad (2.15)$$

$$p \vee f \equiv p \quad (2.16)$$

5. **Complement Law**

If t is a tautology and f is a contradiction then

$$p \wedge \bar{p} \equiv f \quad (2.17)$$

$$p \vee \bar{p} \equiv t \quad (2.18)$$

Exercise 2.1 *Check the five Laws using truth Tables.*

The importance of the Laws are that they allow us to "do algebra" with statements, i.e. to manipulate logical expressions like $a \wedge (b \Rightarrow \overline{(c \Leftrightarrow a)})$ and simplify them where possible. The set of all propositions together with the three connectives "AND", "OR" and "NOT" is called a Boolean Algebra.

Note: If X is a universal set, then the power set $\mathcal{P}(X)$, together with the operations "$\cap$", "$\cup$" and "$'$" is a Boolean Algebra. Here we just replaced connectives "AND", "OR" and "NOT" with operations "$\cap$", "$\cup$" and "$'$" respectively. We also replaced $\equiv$ (equivalence) of propositional logic with $=$ (equality) of sets theory.