Course Notes
for
MS4111
Discrete Mathematics 1

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CHAPTER 6 Functions

6.1 Functions

We begin this section with a definition:

Definition 6.1 *A function f from a set X to a set Y is a relation from X to Y satisfying the following conditions:*

1. *Domain of f=X;*
2. *If $(x, y_1) \in f$ and $(x, y_2) \in f$ then $y_1 = y_2$.*

Notation: If f is a function from X to Y, we will denote it by

$$f : X \longrightarrow Y.$$

Example 6.1

$$f = \{(1, a), (2, b), (3, a)\}$$

is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b\}$.

Example 6.2 *Let* $X = \{1, 2, 3, 4\}$ *and* $Y = \{a, b, c\}$*. Consider the relation*

$$\mathcal{R} = \{(1, a), (2, a), (3, b)\}.$$

$\mathcal{R}$ *is NOT a function from* X *to* Y *but it is a function from*

$$X' = \{1, 2, 3\}$$

to Y*.*

Example 6.3 *Let* $X = \{1, 2, 3, 4\}$ *and* $Y = \{a, b, c\}$*. Consider the relation*

$$\mathcal{R} = \{(1, a), (2, b), (3, c), (1, b)\}.$$

$\mathcal{R}$ *is NOT because* $(1, a) \in \mathcal{R}$ *and* $(1, b) \in \mathcal{R}$*, with* $a \neq b$*.*

Example 6.4 *Consider* $X = \{1, 2, 3, 4, 5\}$, $Y = \{0, 1, 2\}$, $f = \{(x, x \mod 3) | x \in X\}$ *(*$a \mod b$ *is the remainder when* a *is divided by* b*). We can list the elements of* f*:* $f = \{(1, 1), (2, 2), (3, 0), (4, 1), (5, 2)\}$*. We can conclude that* f *is a function as (by inspection) no 2 distinct ordered pairs have same first element.*

Note: If $f \subset X \times Y$ is a function from X to Y and we write

$$f : X \longrightarrow Y,$$

then for each $x \in X$ there is exactly one $y \in Y$ such that

$$(x, y) \in f.$$

We denote this unique value y with $f(x)$ i.e.

$$y = f(x),$$

in other words $y = f(x)$ is another way to say that $(x, y) \in f$.

Example 6.5 *Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Let f be the function from X to Y given by*

$$f = \{(1, a), (2, b), (3, a)\},$$

we can then write

$$f(1) = a; \quad f(2) = b; \quad f(3) = a.$$

Definition 6.2 *If $f : X \longrightarrow Y$ is a function, we call Y the codomain of f.*

Note: If If $f : X \longrightarrow Y$ is a function, then f is a relation from X to Y and we defined the range of f to be the set

$$\text{range of } f = \{y \in Y \mid (x, y) \in f, \quad \text{for some} \quad x \in X\}.$$

If we use the function notation, we can rewrite the range of f to be the set

$$\text{range of } f = \{y \in Y \mid y = f(x), \quad \text{for some} \quad x \in X\}.$$

For any function f from X to Y we will denote the range of f by $R(f)$.

Note: If $f : X \longrightarrow Y$ is a function (with domain X and codomain Y), then

$$R(f) \subseteq Y.$$

Definition 6.3 *The graph of f is the set $\{(x, f(x)) \mid x \in X\}$.*

6.1.1 Some properties of functions

Definition 6.4 *A function $f : X \longrightarrow Y$ is injective (or one-to-one) if for each $y \in Y$ there is at most one $x \in X$ such that*

$$f(x) = y.$$

The above definition is equivalent to the following one:

Definition 6.5 *A function $f : X \longrightarrow Y$ is injective (or one-to-one) if for any $x, x' \in X$ we have*

$$if \quad f(x) = f(x'), \quad \text{then} \quad x = x'.$$

Example 6.6 *Let* $X = \{1, 2, 3\}$ *and* $Y = \{a, b, c, d\}$*. Consider the function from* X *to* Y *given by*

$$f = \{(1, b), (3, a), (2, c)\}.$$

f *is a one-to-one function.*

Example 6.7 *Let* $X = \{1, 2, 3\}$ *and* $Y = \{a, b, c\}$*. Consider the function from* X *to* Y *given by*

$$f = \{(1, a), (2, b), (3, a)\}.$$

f *is NOT a one-to-one function since*

$$f(1) = f(3) = 1.$$

Definition 6.6 *A function $f : X \longrightarrow Y$ is surjective (or onto Y or an onto function) if*

$$R(f) = Y,$$

i.e. if

$$\forall y \in Y, \quad \exists x \in X \quad \text{such that} \quad y = f(x).$$

Example 6.8 *Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Consider the function from X to Y given by*

$$f = \{(1, a), (2, c), (3, b)\}.$$

f is one-to-one and onto Y.

Example 6.9 *Let* $X = \{1, 2, 3\}$ *and* $Y = \{a, b, c, d\}$. *Consider the function from* X *to* Y *given by*

$$f = \{(1, b), (3, a), (2, c)\}.$$

f *is one-to-one but it is NOT nto* Y.

Note: f *is onto* $Y' = \{a, b, c\}$.

Definition 6.7 *A function* $f : X \longrightarrow Y$ *is bijective or a bijection if it is injective and surjective.*

Example 6.10 *Let* $X = \{1, 2, 3\}$ *and* $Y = \{a, b, c\}$. *Consider the function from* X *to* Y *given by*

$$f = \{(1, a), (2, c), (3, b)\}.$$

f *is bijective because it is one-to-one and onto* Y.

6.1.2 Inverse of a function

Let $f : X \longrightarrow Y$ be a bijective function, i.e.

1) f is injective;

2) f is surjective.

Consider the inverse relation

$$f^{-1} = \{(y, x) \mid (x, y) \in f\}$$

Note: We want to show that if f is a bijection then f^{-1} is a function (from Y to X). To prove it, we have to show that

1) domain of f^{-1}=Y;

2) for any $y \in Y$ there is exactly one $x \in X$ such that $f^{-1}(y) = x$.

Proof.

1) Take $y \in Y$. We know that f is onto Y, i.e.

$$R(f) = Y,$$

therefore there exists $x \in X$ such that

$$(x, y) \in f$$

i.e.

$$(y, x) \in f^{-1}$$

and this proved part 1).

2) Take $y \in Y$ and suppose that

$$\begin{aligned} f^{-1}(y) &= x \\ f^{-1}(y) &= x', \quad \text{for} \quad x, x' \in X. \end{aligned}$$

This means that

$$(y, x) \in f^{-1}$$
$$(y, x') \in f^{-1},$$

i.e.

$$(x, y) \in f$$
$$(x', y) \in f,$$

BUT f is injective, therefore

$$x = x'$$

and this proved part 2). $\square$

Example 6.11 *Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Consider the function from X to Y given by*

$$f = \{(1, a), (2, c), (3, b)\}.$$

f is bijective and its inverse is the function

$$f^{-1} = \{(a, 1), (c, 2), (b, 3)\}.$$

Note: If a function f is bijective, then the inverse of f is a function and it is unique: we denote it with f^{-1}.

6.1.3 Composition of functions

If we consider two functions

$$f : X \longrightarrow Y \quad ; \quad g : Y \longrightarrow Z,$$

we can compose g with f and obtain a function $g \circ f$ from X to Z in the following way: if $x \in X$, apply f first

$$f(x) = y.$$

f applied to x determines a unique value $y \in Y$ and now we can apply g:

$$g(y) = g(f(x)) = z,$$

which gives a unique value $z \in Z$. The resulting function from X to Z is called the composition of g with f and it is denoted by $g \circ f$

$$g \circ f : X \longrightarrow Z,$$

defined by

$$(g \circ f)(x) = g(f(x)), \quad \text{for all} \quad x \in X.$$

Note: Functions are particular relations and the definition of the composition of two functions agree with the one given in the previous chapter about the composition if relations. It is left to the student as a useful exercise to check this.

Example 6.12 *Let* $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$ *and* $Z = \{x, y, z\}$. *Let* f *be a function (check it) from* X *to* Y *given by*

$$f = \{(1, a), (2, a), (3, c)\}$$

and let g *be a function (check it) from* Y *to* Z *given by*

$$g = \{(a, y), (b, x), (c, z)\}.$$

If we think at f *and* g *as relations, then the composite relation is*

$$g \circ f = \{(1, y), (2, y), (3, z)\},$$

which can be rewritten in function language as

$$\begin{aligned}(g \circ f)(1) &= y \\ (g \circ f)(2) &= y \\ (g \circ f)(3) &= z,\end{aligned}$$

which is correct in terms of the composition of function g with f

$$\begin{array}{ccc} f(1) = a & , & g(a) = y \\ f(2) = a & , & g(b) = x \\ f(3) = c & , & g(c) = z. \end{array}$$

6.1.4 Identity function

Let X be a set, we then define the function

$$I_X : X \longrightarrow X,$$

defined by

$$I_X(x) = x, \quad \text{for all} \quad x \in X,$$

or

$$I_X = \{(x, x) \mid x \in X\}.$$

Note: If it is clear from the context that we are referring to a certain set X, then we will denote the identity function on X simply by I.

Note: If we have a bijective function

$$f : X \longrightarrow Y,$$

with inverse

$$f^{-1} : Y \longrightarrow X,$$

where

$$f^{-1} = \{(y, x) \mid (x, y) \in f\},$$

then

$$\begin{aligned} f^{-1} \circ f &= \{(x, x) \mid x \in X\} = I_X \\ f \circ f^{-1} &= \{(y, y) \mid y \in Y\} = I_Y, \end{aligned}$$

in other words

$$f^{-1} \circ f : X \longrightarrow X$$
$$f \circ f^{-1} : Y \longrightarrow Y,$$

where

$$(f^{-1} \circ f)(x) = x, \quad \text{for all} \quad x \in X;$$
$$(f \circ f^{-1})(y) = y, \quad \text{for all} \quad y \in Y.$$

6.1.5 More operations on real functions

If $X = Y = \mathbb{R}$ and f is a function from $\mathbb{R}$ to $\mathbb{R}$ ($\mathbb{R}$ is the set of real numbers), we then say that f is a real function.

Definition 6.8 *If f and g are two real functions, we define the sum $f + g$ to be the function*

$$\begin{aligned} f+g &= \{(x, y_1 + y_2) \mid (x, y_1) \in f \quad \text{and} \quad (x, y_2) \in g\} \\ &= \{(x, f(x) + g(x)) \mid (x, f(x)) \in f \quad \text{and} \quad (x, g(x)) \in g\}. \end{aligned}$$

We can rewrite the above definition of $f + g$ in the more familiar notation of functions

$$(f+g)(x) = f(x) + g(x), \qquad \text{for all} \quad x \in X.$$

Definition 6.9 *If f a real function and $k \in \mathbb{R}$ (k is a constant), we define product of function f with a constant k, kf, to be the function*

$$\begin{aligned} kf &= \{(x, ky) \mid (x, y) \in f\} \\ &= \{(x, kf(x)) \mid (x, f(x)) \in f\}. \end{aligned}$$

We can rewrite the above definition of kf in the more familiar notation of functions

$$(kf)(x) = kf(x), \qquad \text{for all} \quad x \in X.$$

Definition 6.10 *If f and g are two real functions, we define the product fg to be the function*

$$
\begin{aligned}
fg &= \{(x, y_1y_2) \mid (x, y_1) \in f \quad \text{and} \quad (x, y_2) \in g\} \\
&= \{(x, f(x)g(x)) \mid (x, f(x)) \in f \quad \text{and} \quad (x, g(x)) \in g\}.
\end{aligned}
$$

We can rewrite the above definition of fg in the more familiar notation of functions

$$(fg)(x) = f(x)g(x), \qquad \text{for all} \quad x \in X.$$