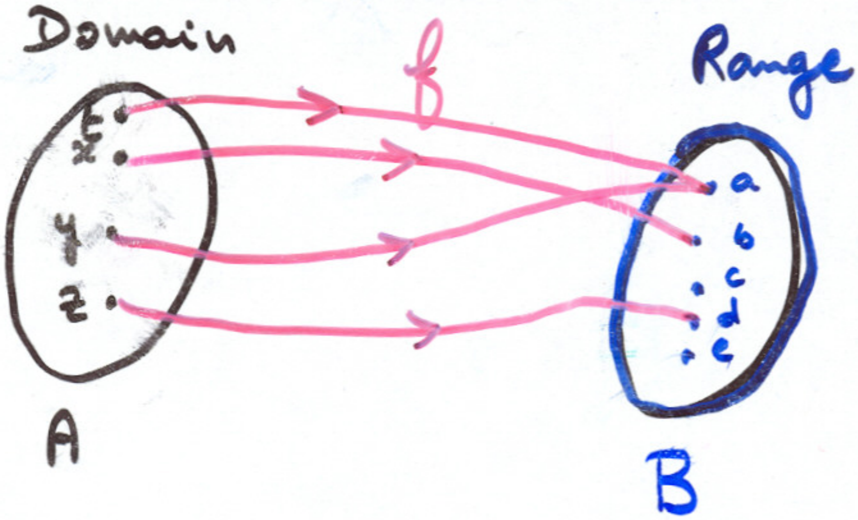
§1 Functions



Sofiane.soussi@ul.ie

D2-036

f : is called a function

A : is a set called "the domain of f"

B : is a set called "the range of f"

Notation:

$$\begin{cases} f(x) = b \quad (*) \\ f(y) = a \\ f(z) = d \end{cases}$$

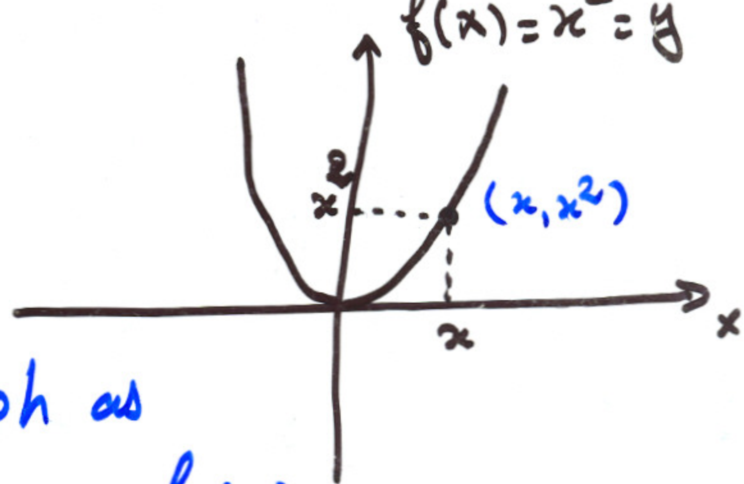
Note For (function) f to be (well) defined, there must be an arrow leaving each point in the domain.

Range points don't have to have arrows arriving at all of them.

Examples:

Age, surface of countries, temp of days...

Graphs as representation of functions:



Can think of a graph as follows in context of example (*)

$$\text{Graph}(f) = \{(x,b),(y,a),(z,d)\}$$

In general a function is defined by specifying:

i) A (domain)

ii) B (range)

iii) Rule $f(x) = \ldots$ for all points x in domain

Most of time $A = B = \mathbb{R}$

where $\mathbb{R}$ = Set of real numbers, or else some subset of $\mathbb{R}$.

e.g. $[-1,2] = \{x \mid -1 \leq x \leq 2\}$

Another important domain:

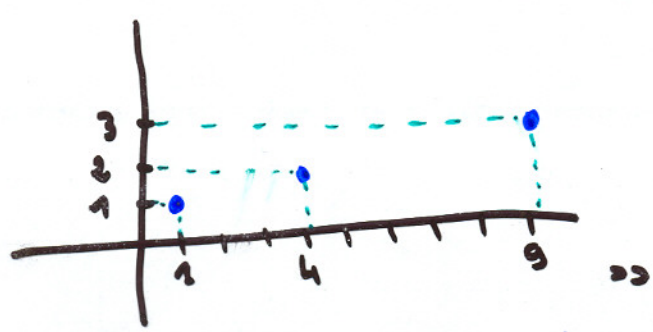
$\mathbb{Z}$: set of integers

<u>Example:</u> $A = \{1, 4, 9\}$

$B = \{1, 2, 3\}$

We set f as: $f(1) = 1$, $f(4) = 2$, $f(9) = 3$

Then: $\text{Graph}(f) = \{(1,1), (4,2), (9,3)\}$



This function f comes from the rule

$$f(x) = \sqrt{x}$$

From pt of view of a graph, a fct. is defined by specifying:

- the domain (A)
- the range (B)
- the pairs $(x, f(x))$

⚠ In graph (f) each pair $(x, f(x))$ appearing covers every point in the domain exactly once.

$$\{(1, \tfrac{1}{2}), (2,3), (1,2), (4,7), \ldots\}$$

could not be the pairs of graph(f) for a function, because it states:

$$f(1) = \tfrac{1}{2} \quad \& \quad f(1) = 2$$

which would lead to:

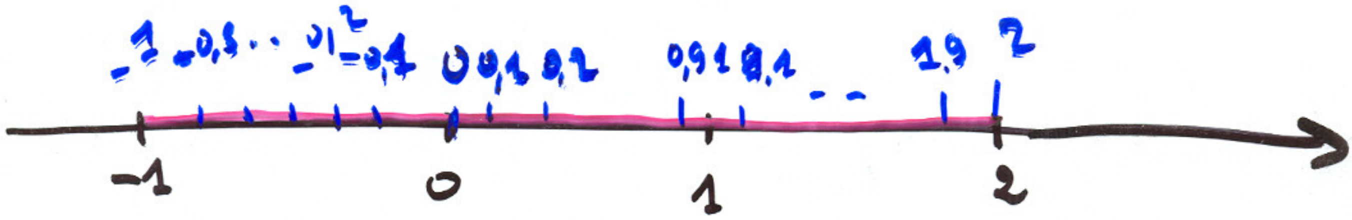
$$\tfrac{1}{2} = f(1) = 2$$

$$\tfrac{1}{2} = 2 \ ? \quad \text{not true!}$$

Let $A = [-1, 2]$

and suppose we have range B and some rule for f.

How to represent this function on a computer?



Break interval up into smaller intervals (of size 0.1) and use end-points of small intervals as a "discrete approximation" of A.

Then define a (new) fct which has old values at these end-points.

$\{-1.0, -0.9, -0.8, \ldots, -0.1, 0, 0.1, \ldots, 1.9, 2.0\}$

This is really a new function because it has a <u>new domain</u> (approx. version of old domain)

Its range and rule are as before

Note: We could have chosen Range of f^{n}

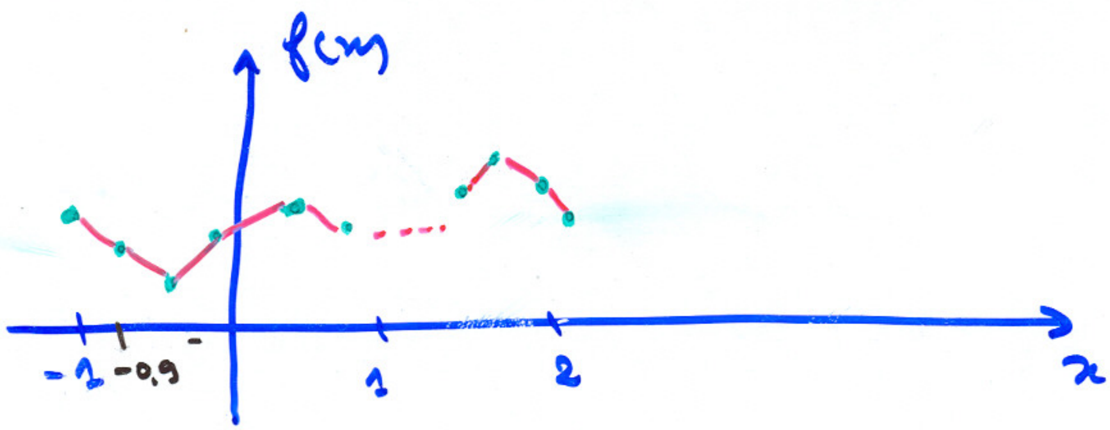
$$B = \{+1, -1\}$$

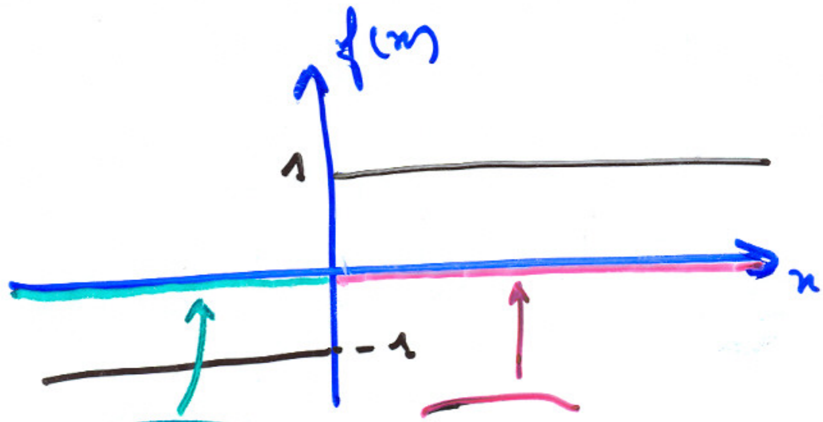
or

$$B = \mathbb{R}$$

The last representation is not an accurate representation of the fct, and what's at issue here is the fact that the function is "not continuous at the origin (0)", i.e. the fct jumps from -1 to $+1$ at 0.

One way to recognize this "discontinuity" at 0 is the fact that "the limit of $f(x)$ as $x \longrightarrow 0$ from left does not equal limit of $f(x)$ as $x \longrightarrow 0$ from right.

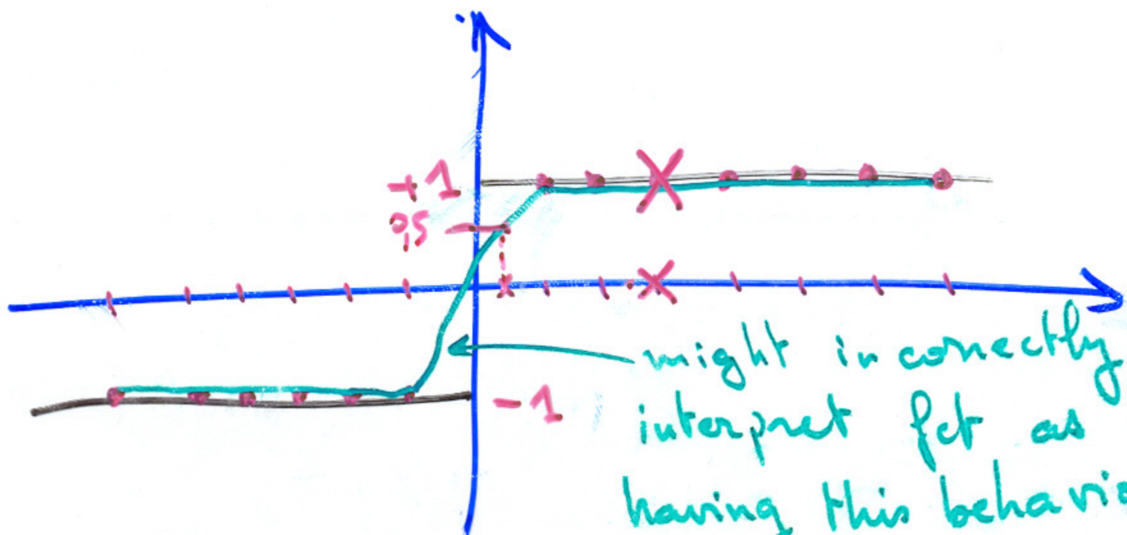




Domain(f) = $(-\infty, 0) \cup (0, +\infty)$

does not include 0

i.e. Domain = all real values except 0.



might incorrectly interpret fct as having this behaviour.

If A is domain of a fct

If B is range of a fct

If $f(x) = x^2+1$ for all $x \in A$

↑ rule

Then we summarize with notation:

$$f : A \longrightarrow B \,, \quad f(x) = x^2+1$$

for $x \in A$.

The above graph does not define a fct

$f : \mathbb{R} \to \mathbb{R}$

because it does not assign a value for $x = 0$.

We will come back to the notion of continuity and discontinuity when we've dealt with sequences.

Surjectivity & injectivity

(onto)

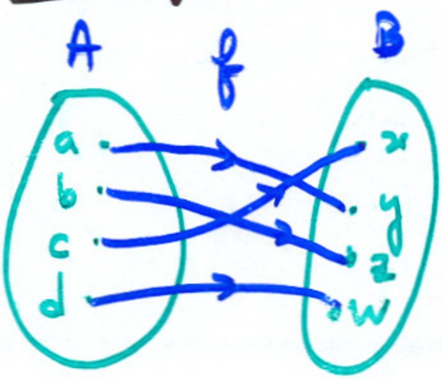
Surjectivity

Suppose f is a function given by

$$f : A \longrightarrow B \ , \ f(x) = \cdots$$

We say f is surjective (onto) if given <u>any</u> $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Examples:

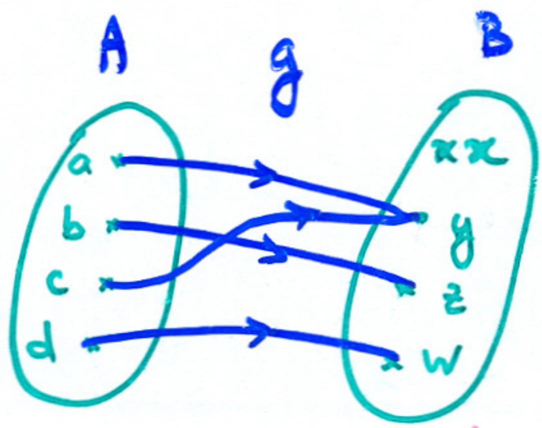


f surjective

x, $f(c) = x$

y, $f(a) = y$

z, - -



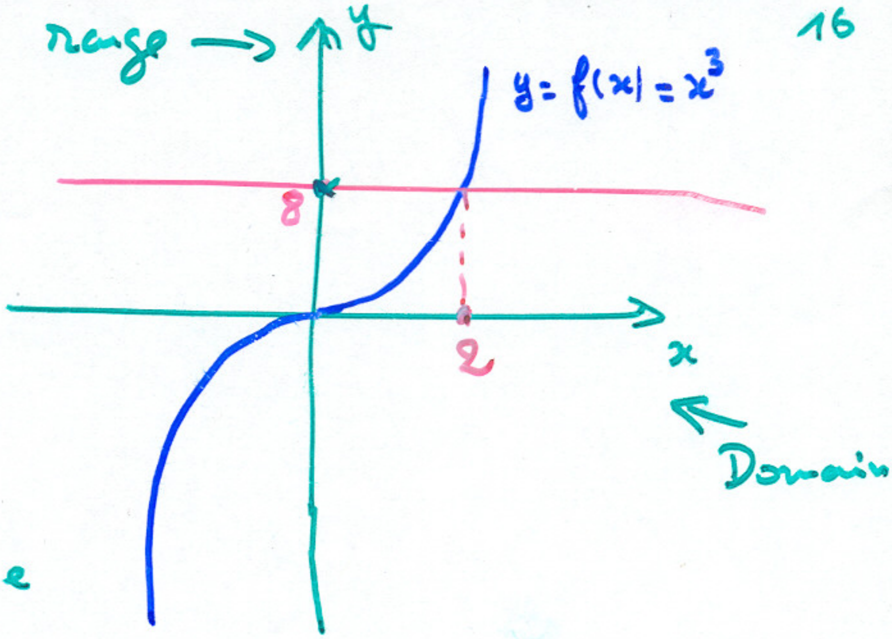
g not surjective

there is no $t \in A$ s.t. $f(t) = x$

More examples:

- $f : \mathbb{R} \to \underline{\underline{\mathbb{R}}}, f(x) = x^3$

  range



$y = 8$

Draw a horizontal line of height y and see if it intersects the graph,

if the answer is "yes" for any y then

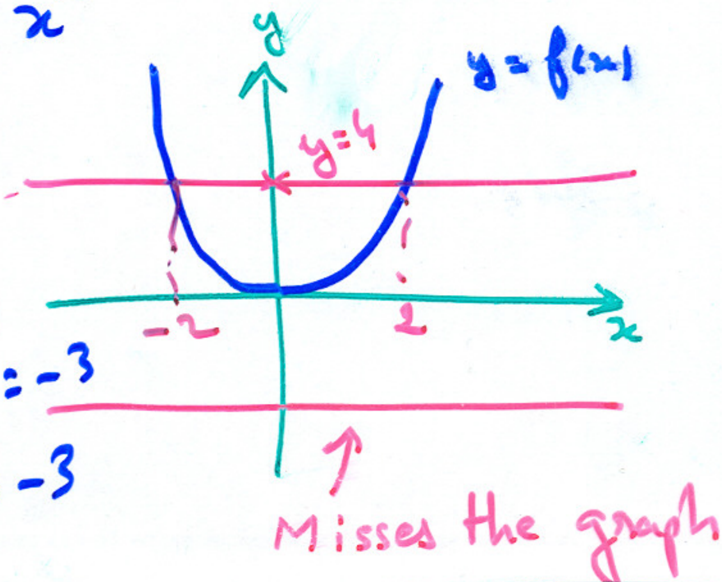
f is surjective.

- $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$

$y, \ \exists ?\ x$ s.t. $f(x) = y$

$-3, \ \exists ?\ x \in \mathbb{R}$ s.t. $f(x) = -3$

$\exists ?\ x \in \mathbb{R}$ s.t. $x^2 = -3$



$$\boxed{f : \mathbb{R} \longrightarrow [0, +\infty), \ f(x) = x^2 \quad \text{is surjective.}}$$

$\mathbb{R}^+$

$y \in \mathbb{R}^+$, $y \geq 0$, $\exists ?\ x \in \mathbb{R}$, s.t. $f(x) = y$, $x^2 = y$ → answer is yes!

$x = \sqrt{y}$

or $x = -\sqrt{y}$.

Injectivity

Suppose f is a function given by

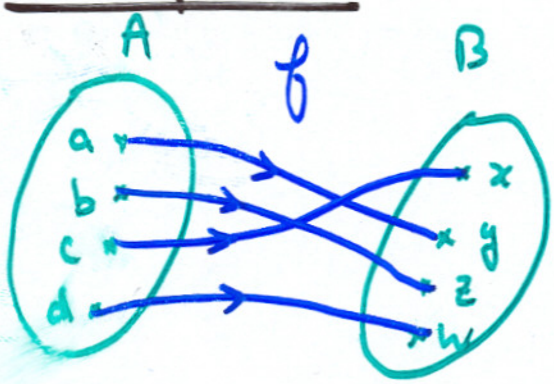
$$f : A \longrightarrow B, \quad f(x) = \ldots$$

We say f is injective if :

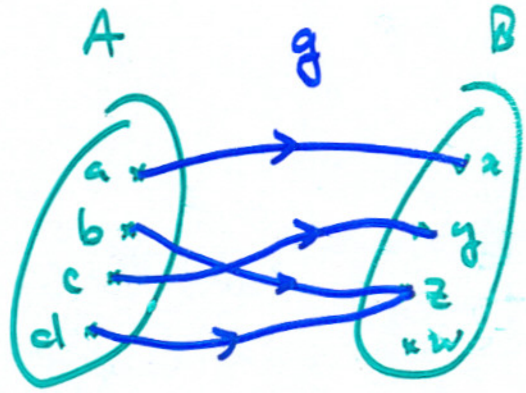
given <u>any</u> $a, b \in A$ such that <u>$a \neq b$</u>

then $f(a) \neq f(b)$.

Examples



f is injective



g is not injective.

$\cdot \neq \cdot$ and $f(\cdot) = f(\cdot)$

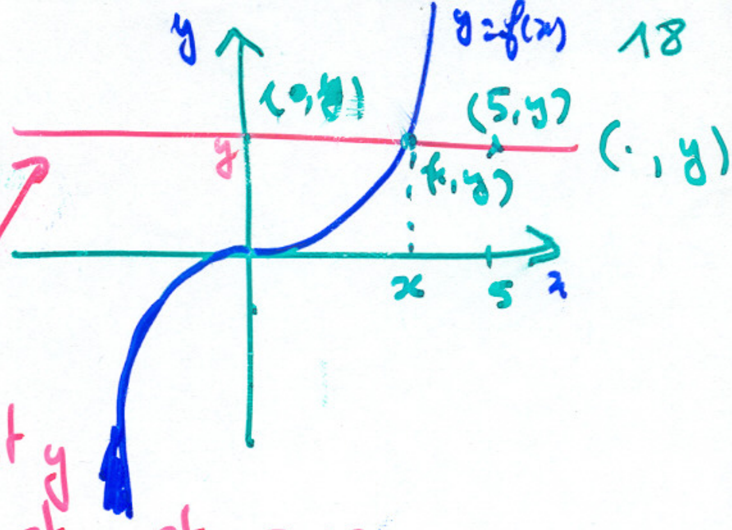
Other definition

$f : A \longrightarrow B$, $f(x) = \ldots$ is injective if

given $b \in B$, there exits <u>at most</u> one $a \in A$ such that $f(a) = b$.

Examples

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$



line of height y cuts the graph at most once.

From point of view of algebra :

$$y = x^3 \Rightarrow x = y^{\frac{1}{3}} = \sqrt[3]{y}$$

$y^{\frac{1}{3}}$ is the unique solution of equation $x^3 = y$

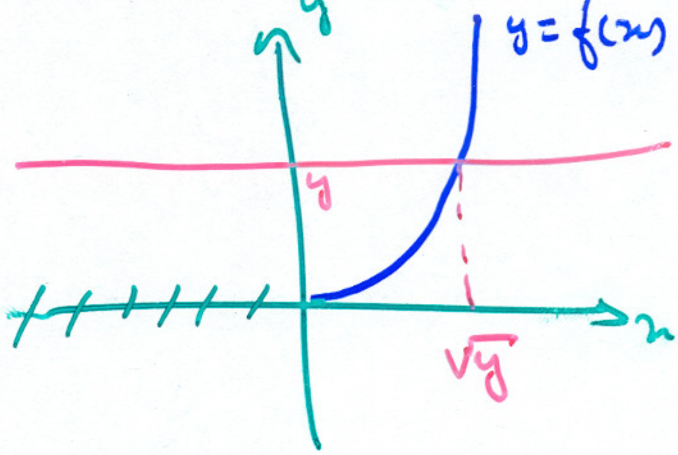
$$\left(y^{\frac{1}{3}}\right)^3 = y$$

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

f not injective

From point of view of algebra:
There are 2 solutions to equation
$x^2 = y$: $-\sqrt{y}$ and $\sqrt{y}$

- $f : [0, +\infty) \longrightarrow \mathbb{R}$, $f(x) = x^2$



line of height y cuts the graph in at most 1 point.

f is injective.

Equation $x^2 = y$, $y > 0$ has only one solution $x \in [0, +\infty)$ that is $x = \sqrt{y}$.

Homework:

① Which of the following functions are surjective?

i) $f : [-1, 1] \to \mathbb{R}$, $f(x) = e^x$ $(0, +\infty) = \mathbb{R}_+^*$

ii) $f : \mathbb{R} \to \{x \mid x > 0\}$, $f(x) = e^x$

iii) $f : \mathbb{R} \to \{x \mid x \geqslant 0\}$, $f(x) = e^x$

② Does $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin^{-1}(x)$ define a function?

③ Give your own examples of functions which are

i) injective but not surjective

ii) surjective but not injective

Bijectivity

Definition

A function which is both injective and surjective is called **bijective** or **1-1**.

Remark

Such a function sets up a 1-1 correspondance between points in domain and range.

i.e.

- Given any $a \in A$, there is a unique $b \in B$ s.t. $f(a) = b$
- Given any $b \in B$, there is a unique $a \in A$ s.t. $f(a) = b$.