§2 Sequences

Definition

A sequence is an infinite list of real numbers.
(repetition of numbers is allowed).

Eg. i) $1, 2, 3, 4, 5, \ldots$ continue on ad infinitum in the obvious way.

ii) $1, 4, 9, 16, 25, 36, \ldots$

Definition (mathematical)

A real sequence is a function which

- domain is $\mathbb{N}$: set of positive integers
- range is $\mathbb{R}$.

E.G. i) $f : \mathbb{N} \to \mathbb{R}$, $f(1) = 1$, $f(2) = 2$, $f(3) = 3, \ldots$

ii) $f : \mathbb{N} \to \mathbb{R}$, $f(1) = 1$, $f(2) = 4$, $f(3) = 9, \ldots$

We can rewrite the previous examples as:

i) $f : \mathbb{N} \to \mathbb{R}$, $f(n) = n$, $\forall n \in \mathbb{N}$

ii) $f : \mathbb{N} \to \mathbb{R}$, $f(n) = n^2$, $\forall n \in \mathbb{N}$

($\forall$ means "for all")

Notation

In the case of sequences (which are particular kind of functions) we simplify the notation

$$f : \mathbb{N} \longrightarrow \mathbb{R} \quad , \quad f(n) = \dots$$

into:

$$(a_n)_{n\in\mathbb{N}} \quad , \quad a_n = \dots$$

E.G. i) $f : \mathbb{N} \to \mathbb{R}$, $f(n) = n$ becomes $(a_n)_{n\in\mathbb{N}}$, $a_n = n$

ii) $f : \mathbb{N} \longrightarrow \mathbb{R}$, $f(n) = n^2$ becomes $(a_n)_{n\in\mathbb{N}}$, $a_n = n^2$

Example: Define a sequence $(a_n)_{n\in\mathbb{N}}$ by:

$$a_n = 1 + \frac{1}{2^n} \quad \text{if } n \text{ is odd}$$

and

$$a_n = 2^n \quad \text{if } n \text{ is even}$$

This is usually written as:

$$a_n = \begin{cases} 1 + \frac{1}{2^n} , & n : \text{odd} \\ 2^n , & n : \text{even} \end{cases}$$

$a_0 = 2^0 = 1$

$a_1 = 1 + \frac{1}{2^1} = \frac{3}{2}$ $\qquad$ $a_3 = 1 + \frac{1}{2^3} = \frac{7}{8}$ $\qquad$ $a_{25} = 1 + \frac{1}{2^{25}}$

$a_2 = 2^2 = 4$ $\qquad$ $a_4 = 2^4 = 16$ $\qquad$ $a_{40} = 2^{40}$

Definition Unbounded sequence

A given sequence $(a_n)_{n\in\mathbb{N}}$ is said <u>unbounded</u>

if for any ~~real~~ positive real number L,

there exists $n \in \mathbb{N}$ such that

$$|a_n| > L$$

The sequence in the previous example is unbounded.

In fact, for any (large) L, there exist an even integer n such that:

$$|a_n| \underset{\uparrow}{=} 2^n > L$$

because we chose n even.

If $L = 1000$, we can take $n = 10$, $a_n = 2^{10} = 1024$

Definition: (Bounded sequence)

A given sequence $(a_n)_{n\in\mathbb{N}}$ is said bounded if there exists a positive real number C such that:

$$\forall n \in \mathbb{N} \qquad |a_n| < C$$

for any

E.G. $\quad (a_n)_{n\in\mathbb{N}}$, $a_n = \frac{1}{2^n}$

$$= \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \dots$$

$$= 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

For this sequence we can take $C = 2$ or ~~...~~ $C = \frac{3}{2}$ as a bound.

In fact the largest number in this sequence is 1.

E.G. $\quad (a_n)_{n\in\mathbb{N}}$, $a_n = \alpha x^n$ for ~~...~~ given $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$

(a_n) is bounded by $|\alpha|$ if $|x| \leq 1$

(a_n) is unbounded if $|x| > 1$

Geometric sequence

Definition Increasing sequence.

- A sequence $(a_n)_{n\in\mathbb{N}}$ is increasing if

$$\forall n \in \mathbb{N}, \quad a_{n+1} \geqslant a_n$$

- A sequence $(a_n)_{n\in\mathbb{N}}$ is strictly increasing if

$$\forall n \in \mathbb{N}, \quad a_{n+1} > a_n$$

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* An increasing sequence is a sequence in which successive terms are increasing.

Definition Decreasing sequence.

- A sequence $(a_n)_{n\in\mathbb{N}}$ is decreasing if

$$\forall n \in \mathbb{N}, \quad a_{n+1} \leq a_n$$

- A sequence $(a_n)_{n\in\mathbb{N}}$ is strictly decreasing if

$$\forall n \in \mathbb{N} \quad a_{n+1} < a_n$$

E.G. $(a_n)_{n\in\mathbb{N}}$, $a_n = 2^n$ is an increasing sequence

$$a_{n+1} = 2^{n+1} = 2 \cdot 2^n = \cancel{2a_n} \geqslant 2^n = a_n$$

E.G. $(a_n)_{n\in\mathbb{N}}$, $a_n = 1 - n$

$a_0 = 1$, $a_1 = 0$, $a_2 = 1-2 = -1$, $a_3 = 1-3 = -2 \ldots$

$$a_{n+1} = 1 - (n+1) = (1-n) - 1 = a_n - 1$$

$$a_{n+1} \leq a_n$$

$(a_n)_{n\in\mathbb{N}}$ is a (strictly) decreasing sequence.

Remark

A sequence can be neither increasing nor decreasing.

E.G. $(a_n)_{n\in\mathbb{N}}$, $a_n = (-1)^n$

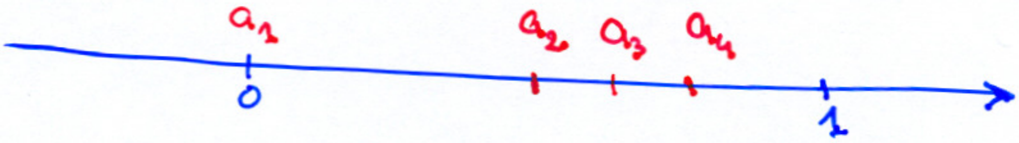
$a_0 = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = -1$, ...

Convergence of sequences

Ex $(a_n)_{n \in \mathbb{N}^*}$, $a_n = 1 - \frac{1}{n}$ $\left(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\right)$

set of strictly positive integers

$a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{3}{4}$, ...



As n gets larger, $1 - \frac{1}{n}$ is getting closer to 1.

* We say that $a_n \xrightarrow[n \to +\infty]{} 1$

  * read: " a_n tends to 1 as n goes to infinity"

* 1 is the limit of the sequence (a_n)

Interpretation:

We can view the limiting process as follows: the distance between the number $a_n = 1 - \frac{1}{n}$ and the limit 1 is becoming smaller and smaller as n becomes larger ($n \to +\infty$).

$$|a_n - 1| \longrightarrow 0 \quad \text{as } n \to +\infty$$

$$|a_n - 1| = \left|1 - \frac{1}{n} - 1\right| = \frac{1}{n}$$

Definition:

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$, ~~and we denote that~~

if $\quad |a_n - a| \xrightarrow[n \to +\infty]{} 0$.

In that case, we write

$$a_n \xrightarrow[n \to +\infty]{} a$$

Caution Not all sequences have ~~a limit~~. a real limit $a \in \mathbb{R}$.

E.G. $\quad (a_n)_{n\in\mathbb{N}}$, $a_n = 2^n$ does not converge.

If (a_n) converges to $a \in \mathbb{R}$,

then we can also say:

the limit of (a_n) is a.

Proposition:

If a sequence has a real limit (is convergent), then this limit is unique.

Elementary properties of sequences

1. Suppose (a_n) converges to $a \in \mathbb{R}$
(b_n) converges to $b \in \mathbb{R}$.

Define a new sequence (c_n) by: $c_n = a_n + b_n$

Then: (c_n) converges to $a+b$

2. Let (a_n) and (b_n) be as in 1

Define a new sequence (d_n) by $d_n = a_n \cdot b_n$

Then (d_n) converges to $a \cdot b$.

3. Let (a_n) and (b_n) be as in 1

Suppose also that $b_n \neq 0$, $\forall n \in \mathbb{N}$
and that $b \neq 0$.

Define a new sequence (e_n) by $e_n = \dfrac{a_n}{b_n}$

Then (e_n) converges to $\dfrac{a}{b}$.

⚠ In (3) we need both conditions:

$$\begin{cases} \cdot\ b_n \neq 0 \quad \forall n \in \mathbb{N} \\ \cdot\ b \neq 0 \end{cases}$$

In fact we can have a sequence (b_n) satisfying

$b_n \neq 0 \ \forall n \in \mathbb{N}$ and $b = 0$

In that case $\frac{a}{b} = \frac{a}{0}$ is not defined.

Ex. (b_n), $b_n = \frac{1}{n} \neq 0 \ \forall n \in \mathbb{N}$

and converges to $b = 0$

E.G. (a_n), $a_n = 1 + \frac{1}{n} \xrightarrow[n \to +\infty]{} 1 = a$

(b_n), $b_n = 1 - \frac{1}{2n} \xrightarrow[n \to +\infty]{} 1 = b$

$$c_n = a_n + b_n = \left(1 + \frac{1}{n}\right) + \left(1 - \frac{1}{2n}\right) = 2 + \frac{1}{2n} \xrightarrow[n \to +\infty]{} \underset{a+b\,=\,1+1}{\overset{2}{\|}}$$

$$d_n = a_n \cdot b_n = \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right) = 1 + \frac{1}{2n} - \frac{1}{2n^2} \xrightarrow[n \to +\infty]{} \underset{a \cdot b\,=\,1 \cdot 1}{\overset{1}{\|}}$$

$$e_n = \frac{a_n}{b_n} = \frac{1 + \frac{1}{n}}{1 - \frac{1}{2n}} = \frac{1 - \frac{1}{2n} + \frac{3}{2n}}{1 - \frac{1}{2n}} = 1 + \frac{3}{2n - 1} \xrightarrow[n \to +\infty]{} \underset{\frac{a}{b}\,=\,\frac{1}{1}}{\overset{1}{\|}}$$