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5 Graph Theory

5.1 Introduction

One of the first people to experiment with graph theory was a man by the name of Euler (pronounced “oiler”)(1707-1783). He attempted to solve the problem of crossing seven bridges onto an island without using any of them more than once.

From that point on, the study of graphs has been applied to a large number of real world problems. Today, graphs are all around us. They are used in many diverse industries, from urban planning, to shipping lanes, to computer networks such as the Internet.

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5.2 Terminology

A **graph** consists of a nonempty set of points or **vertices** or **nodes**, and a set of **edges** or **arcs** that link together the vertices. A simple real world example of a graph would be an airport flight network. The airports are the vertices and the flight paths between them are the edges connecting any two vertices.

The relationship between the vertices and the edges in a graph can be thought of as a function. This function associated with each edge e an unordered pair $a - b$ of vertices called the endpoints of e.

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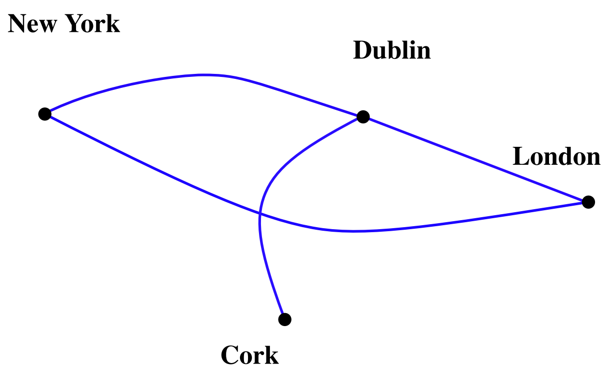

Figure 5.1: Graph of Flight Network between Airports

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The vertices u and v are said to be **adjacent** if there is an edge e connecting the two of them. In such a case, u and v are called the **endpoints** of e, and e is said to **connect** u and v.

A **loop** in a graph is an edge connecting a node to itself.

Two edges are **parallel** if they have common endpoints.

An **isolated node** is a node which is not the end point of any edge.

A graph is called **simple** if it has no loops or parallel edges. Otherwise it is called a **multigraph**.

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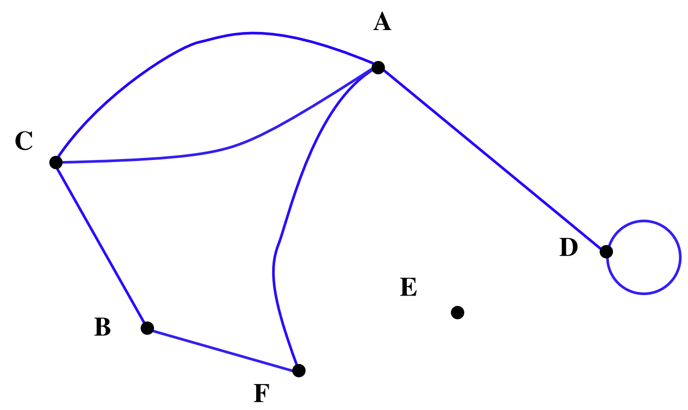

Figure 5.2: An example of a Multigraph

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The **degree** of a vertice v, written deg(v), is the number of edge ends at that vertice.

Example 5.1 *What is the degree of the vertices of the graph in Figure 5.2.*

Solution: deg(A) = 4; deg(B) = 2; deg(C) = 3; deg(D) = 3; deg(E) = 0; deg(F) = 2;

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A **path** from vertice v_0 to vertice v_k is a sequence

$$v_0, e_0, v_1, e_1, v_2, e_2, ..., v_{k-1}, e_{k-1}, v_k$$

of vertices and edges where for each i, the endpoints of edge e_i are $v_i - v_{i+1}$.

The **length** of a path is the number of edges it contains; if an edge is used more than once, it is counted each time it is used.

A graph is **connected** if there is a path from any vertice to any other vertice.

A **cycle** in a graph is a path from some vertice v_0 back to v_0, where no edge appears more than once in the path sequence, v_0 is the only vertice appearing more than once, and v_0 occurs only at the ends.

A graph with no cycles is called **acyclic**.

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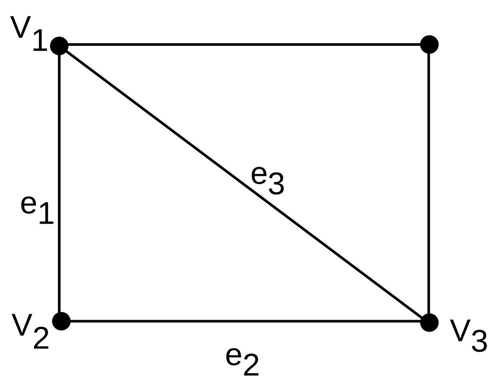

Figure 5.3: An example of a Cycle

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$V_1, e_1, V_2, e_2, V_3, e_3, V_1$ is an example of a cycle.

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A graph can take on many forms: directed or undirected. A **directed graph** is one in which the direction of any given edge is defined. Conversely, in an **undirected graph** you can move in both directions between vertices. The edges can also be **weighted** or **unweighted**.

Finally, a **complete graph** is a graph in which every pair of vertices is adjacent e.g. a triangle.

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5.3 Complete graphs

The following figure represents the simple, complete graphs with 1,2,3 and 4 vertices. The simple, complete graph with n vertices is denoted K_n.

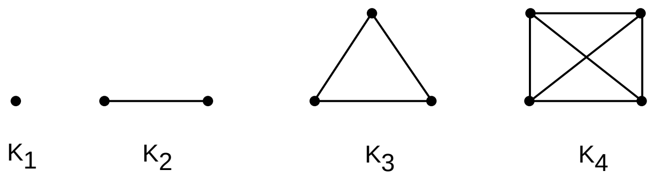

Figure 5.4: K_1, K_2, K_3 and K_4

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Exercise

Draw K_5

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Now consider the graph in Figure 5.5 on the next slide. It is not a complete graph because it is not true that every vertice is adjacent to every other vertice. However, the vertices can be divided into two disjoint sets, $\{1, 2\}$ and $\{3, 4, 5\}$, such that any two vertices chosen from the same set are not adjacent but any two vertices chosen one from each set are adjacent. Such a graph is a **bipartite complete graph.**

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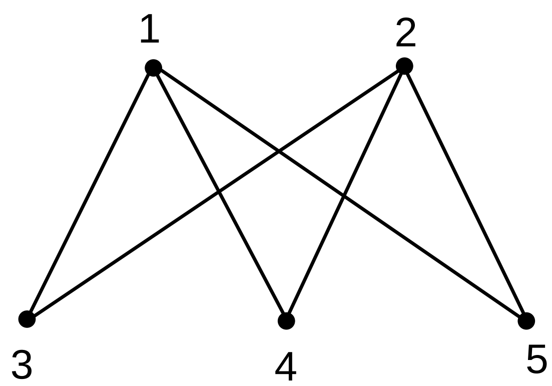

Figure 5.5: An example of a bipartite graph

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Definition 5.1 (Bipartite Complete Graph) *A graph is a* **bipartite complete graph** *if its vertices can be partitioned into two disjoint nonempty sets V_1 and V_2 such that two vertices x and y are adjacent if and only if $x \in V_1$ and $y \in V_2$. If $|V_1| = m$ and $|V_2| = n$, such a graph is denoted $K_{m,n}$.*

Therefore, the graph in Figure 5.5 is $K_{2,3}$.

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Exercise

Draw $K_{3,3}$

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5.4 Isomorphic Graphs

Often, two graphs may look completely different on paper, but are essentially the same from a mathematical point of view. Take for example the two graphs on the following slide. These graphs are the same — they have the same vertices, the same edges and the same edge-to-endpoint function. If we relabel the vertices and edges of the graph in Figure 5.6(a) by the following mappings, the graphs would be the same:

$$f_1 \quad : \quad \begin{aligned} 1 &\to a \\ 2 &\to c \\ 3 &\to b \\ 4 &\to d \end{aligned} \qquad\qquad f_2 \quad : \quad \begin{aligned} a_1 &\to e_2 \\ a_2 &\to e_1 \end{aligned}$$

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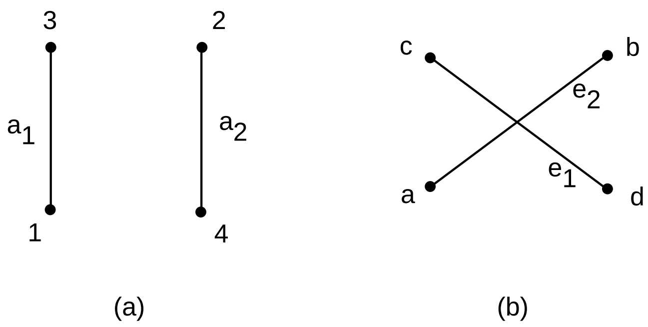

Figure 5.6: Isomorphic graphs

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Structures that are the same except for relabeling are called **isomorphic** structures. To show that two structures are the isomorphic, we must produce a relabeling (one-to-one, onto mappings between the elements of the structures) and then show that the important properties of the structures are preserved under the relabeling.

In the case of graphs,the elements are vertices and edges. The “important property” in a graph is which edges connect which vertices.

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We can use (V_1, E_1, g_1) and (V_2, E_2, g_2) to represent two graphs. V represents the vertices, E the edges and g the rule linking edges with vertices.

Definition 5.2 (Isomorphic Graph) *Two graphs (V_1, E_1, g_1) and (V_2, E_2, g_2) are* **isomorphic** *if there are bijections $f_1 : V_1 \to V_2$ and $f_2 : E_1 \to E_2$ such that for each edge $a \in E_1$, $g_1(a) = x - y$ if and only if $g_2[f_2(a)] = f_1(x) - f_1(y)$.*

Exercise

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Give an example of two isomorphic graphs and list the bijections which establish the isomorphism.

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It is not always easy to establish if 2 graphs are isomorphic or not. An exception is the case where the graphs are simple. In this case, we just need to check if there is a bijection $f : V_1 \to V_2$ which preserves adjacent vertices (i.e. if v_1, v_2 are adjacent in graph 1, then $f(v_1)$, $f(v_2)$ must be adjacent in graph 2).

If the graphs are not simple, we need more sophisticated methods to check for when two graphs are isomorphic. However, it is often straightforward to show that two graphs are **not** isomorphic. You can do this by showing **any** of the following seven conditions are true.

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(i) The two graphs have different numbers of vertices.

(ii) The two graphs have different numbers of edges.

(iii) One graph has parallel edges and the other does not.

(iv) One graph has a loop and the other does not.

(v) One graph has a vertice of degree k (for example) and the other does not.

(vi) One graph is connected and the other is not.

(vii) One graph has a cycle and the other has not.

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5.5 Planar Graphs

Definition 5.3 (Planar Graph) *A* **planar graph** *is one which can be represented (possibly after mapping it to an isomorphism) so that its edges intersect only at vertices.*

Designers of integrated circuits want all components in one layer of a chip to form a planar graph so that no connections cross.

The key word in the definition of a planar graph is that it **can** be drawn in a certain way.

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Example 5.2 *Recall what K_4 looks like. K_4 has edges which intersect at none-vertex locations. Therefore in its original state K_4 is not planar - but K_4 is isomorphic to the graph below, which is planar. Therefore K_4 is planar.*

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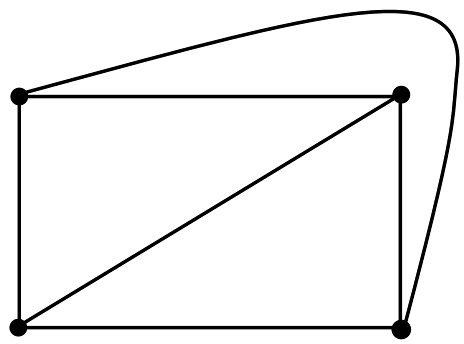

Figure 5.7: A K_4 planar graph

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One fact about planar graphs was discovered by the Swiss mathematician Euler. A simple, connected, planar graph (when represented in its planar form, with no edges crossing) divides the plane into a number of regions, including totally enclosed regions and one infinite exterior region. Euler observed a relationship between the number n of nodes(vertices), the number a of arcs(edges), and the number r of regions in such a graph. This relationship is know as **Euler's formula**:

$$n - a + r = 2$$

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Example 5.3 *Does Euler's Formula hold true for the following graph ?*

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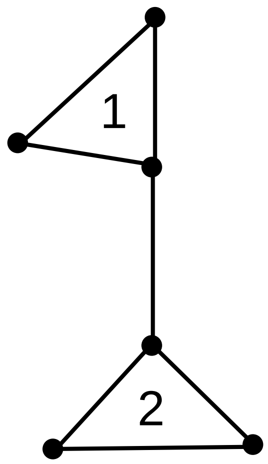

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Solution: We represent by the numbers $1, 2$ and 3 the regions of the graph — 1 and 2 being the enclosed regions and 3 being the infinite exterior region. It is clear to see that there are 6 vertices and 7 edges in this graph. It is also easy to check that this is a simple, connected, planar graph and so Euler's formula should hold.

Euler's formula states:

$$n - a + r = 2,$$

for this graph we have

$$\begin{aligned} 6 - 7 + 3 &= 2, \\ \Rightarrow 2 &= 2. \end{aligned}$$

Therefore Euler's formula holds.

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Theorem 5.1 *For a simple, connected, planar graph with n vertices and a edges:*

1. *If the planar representation divides the plane into r regions, then*

$$n - a + r = 2$$

2. *If $n \geq 3$, then*

$$a \leq 3n - 6$$

3. *If $n \geq 3$ and there are no cycles of length 3, then*

$$a \leq 2n - 4$$

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Exercise

Is K_5 planar ?

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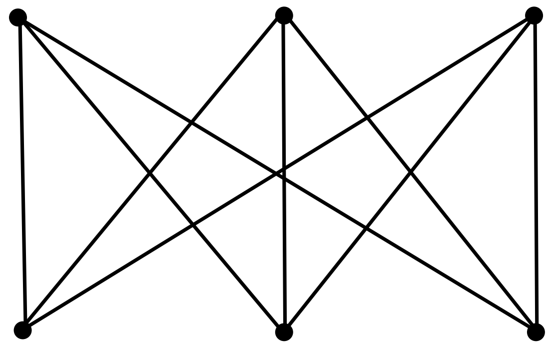
Comment 5.1 *Consider the graph $K_{3,3}$.*



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$n = 6$ and $a = 9$. Therefore using the equation in the last theorem,

$$a \leq 3n - 6$$

we get

$$9 \leq 3(6) - 6$$

which is true.

So $K_{3,3}$ is simple and connected and the inequality $a \leq 3n - 6$ is true but $K_{3,3}$ is actually **not** *planar (accept this).*

The inequality is a necessary condition for planar graphs which are simple and connected. Unfortunately it is not sufficient to conclude that just because the inequalities hold and the graph is simple and connected that it will be planar.

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5.6 Computer Representation of Graphs

We can see that the major advantage of a graph is its visual representation of information. What if we wanted to store a graph in digital form ? We could store it as a digital image of the graph, but this takes a lot of space. What we need to store are the essential data that are part of the graph — what the vertices are and which vertices have connecting edges. From this information a visual representation could be reconstructed if desired. The computer representation method that we shall look at is called an adjacency matrix.

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Suppose a graph has n vertices, numbered $n_1, n_2, n_3, ..., n_n$. Having ordered the nodes, we can form an $n \times n$ matrix where entry ij is the number of edges between vertices n_i and n_j. This matrix is called the **adjacency matrix** of the graph with respect to this ordering.

Example 5.4 *Write down the adjacency matrix of the following graph.*

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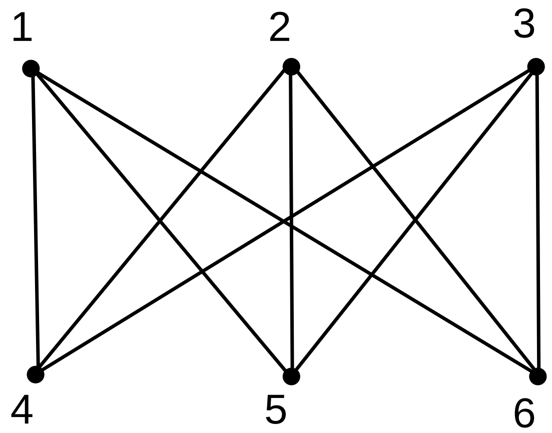

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Solution: *Adjacency Matrix (A) is*

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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The adjacency matrix is symmetric, which will be true for any undirected graph — if there are p edges between n_i and n_j then are certainly p edges between n_j and n_i. The symmetry of the matrix means that only elements on or below the main diagonal need to be stored. Therefore, all the information contained in the graph is contained in the “lower triangular” section of the matrix, and the graph could be reconstructed from this array.

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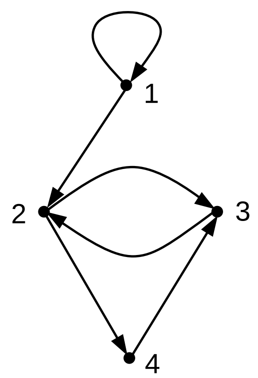
Exercise

Construct the adjacency matrix for the following directed graph.



Comment 5.2 *The adjacency matrix for a directed graph is not symmetric.*

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5.7 Trees and their Representations

Definition 5.4 (Tree) *A* **tree** *is an acyclic, connected graph with one node designated as the* **root** *of the tree.*

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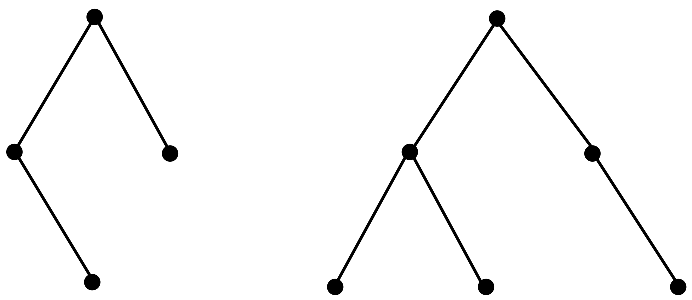

Figure 5.8: Two examples of trees

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A tree can also be defined recursively. A single node(vertex) is a tree (with that node as its root). If $T_1, T_2, ..., T_t$ are disjoint trees with roots $r_1, r_2, ..., r_t$, the graph formed by attaching a new node r by a single edge to each $r_1, r_2, ..., r_t$ is a tree with root r. The roots $r_1, r_2, ..., r_t$ are **children** of r, and r is a **parent** of $r_1, r_2, ..., r_t$ (just like a family tree).

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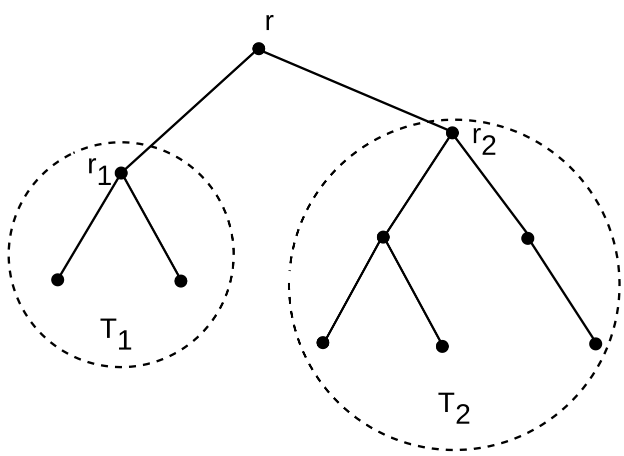

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Definition 5.5 (Depth of a node) *The* **depth of a node** *in a tree is the length of the path from the root to the node; the root itself has depth zero.*

Because a tree is a connected graph, there is a path from the root to any other node in the tree; because the tree is acyclic, that path is unique.

Definition 5.6 (Depth of the tree) *The* **depth(height) of a tree** *is the maximum depth of any node in the tree; in other words, it is the length of the longest path from the root to any node.*

A node with no children is called a **leaf** of the tree; all non-leaves are **internal nodes**.

Definition 5.7 (Binary tree) *A* **binary tree** *is a tree where each node has at most two children. A* **full binary tree** *is a tree where all nodes have exactly two children and all leaves are at the same depth.*

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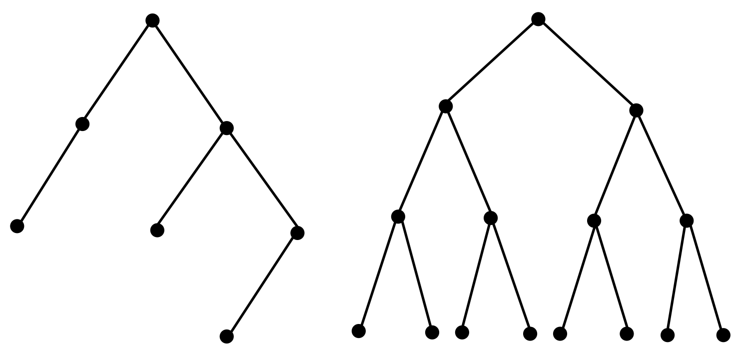

Figure 5.9: A binary tree on the left and a full binary tree of height 3 on the right.

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Comment 5.3 *A tree with n vertices has $n-1$ edges.*

Algebraic expressions involving binary operations can be represented by labeled binary trees. The leaves are labeled as operands, and the internal nodes are labeled as binary operations. For any internal node, the binary operation of its label is preformed on the expressions associated with its left and right subtrees. The binary tree below represents the algebraic expression $(2 + x) - (3 * y)$.

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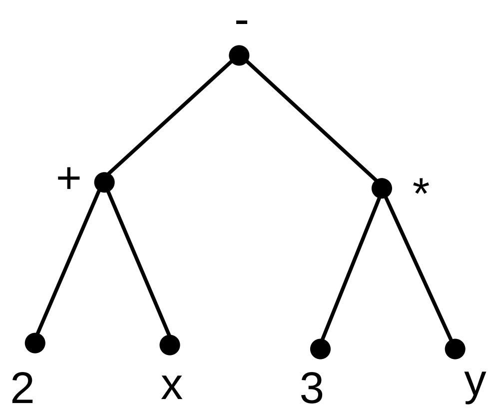

Exercise

What is the expression tree for $(2+x)^2 - (1-x)$?

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5.8 Eulerian and Hamiltonian Circuits

The following problem, often referred to as the bridges of Königsberg problem, as first solved by Euler in the eighteenth century. The problem was rather simple — the town of Königsberg consists of two islands and seven bridges. Is it possible, by beginning anywhere and ending anywhere, to walk through the town by crossing all seven bridges but not crossing any bridge twice?

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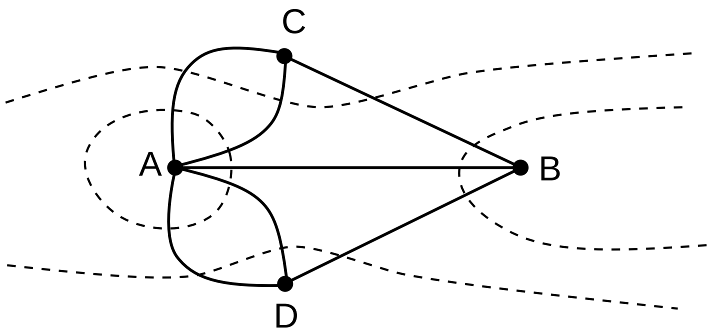

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We will first present some definitions and then present a theorem that Euler used to show that it is in fact impossible to walk through the town and transverse all the bridges only once.

Definition 5.8 (Eulerian trail) *An* **Eulerian trail** *is a trail that visits every edge of the graph once and only once. It can end on a vertex different from the one on which it began. A graph of this kind is said to be* **traversable**.

Definition 5.9 (Eulerian Circuit) *An* **Eulerian circuit** *is an Eulerian trail that is a circuit. That is, it begins and ends on the same vertex.*

Definition 5.10 (Eulerian Graph) *A graph is called* **Eulerian** *when it contains an Eulerian circuit.*

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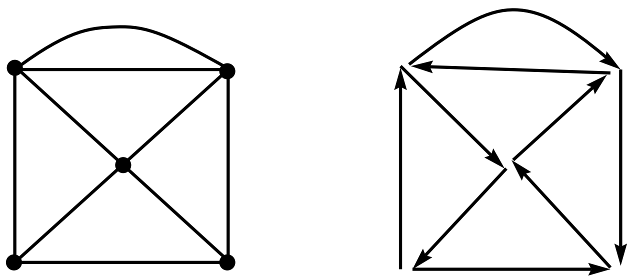

Figure 5.10: An example of an Eulerian trial. The actual graph is on the left with a possible solution trail on the right.

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A vertex is **odd** if its degree is odd and **even** if its degree is even.

Theorem 5.2 *An Eulerian trail exists in a connected graph if and only if there are either no odd vertices or two odd vertices.*

For the case of no odd vertices, the path can begin at any vertex and will end there; for the case of two odd vertices, the path must begin at one odd vertex and end at the other.

Comment 5.4 *Any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex.*

From this we can see that it is not possible to solve the bridges of Königsberg problem because there exists within the graph more than 2 vertices of odd degree.

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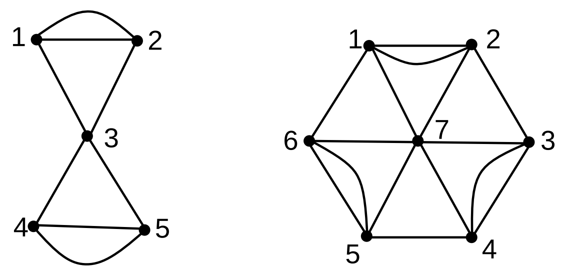
Exercise

Are either of the following graphs traversable?



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Definition 5.11 (Hamiltonian Circuit) *A* **Hamiltonian circuit** *in a graph is a closed path that visits every vertex in the graph exactly once. (Such a closed loop must be a cycle.)*

Comment 5.5 *A Hamiltonian circuit ends up at the vertex from where it started.*

Named after the nineteenth-century Irish mathematician Sir William Rowan Hamilton(1805-1865). This is often referred to as the traveling salesman or postman problem.

Definition 5.12 (Hamiltonian Graph) *If a graph has a Hamiltonian circuit, then the graph is called a* **Hamiltonian graph**.

Comment 5.6 *An Eulerian circuit traverses every edge in a graph exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex in a graph exactly once but may repeat edges.*

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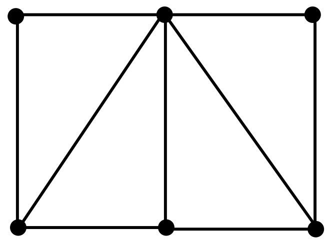

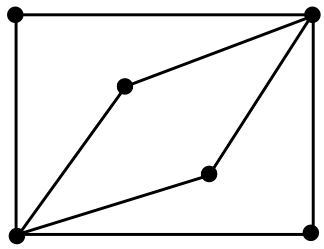

Figure 5.11: On the left a graph which is Hamiltonian and non-Eulerian and on the right a graph which is Eulerian and non-Hamiltonian.

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Exercise

Is the following graph Hamiltonian or Eulerian or both?