Course Notes for MS4111 Discrete Mathematics 1

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CHAPTER 8 Natural numbers and proof by induction

8.1 Natural numbers

8.1.1 Axioms and inductive definition

The set of natural numbers

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

is an example of an inductive set defined by

1) a basic element: $0 \in \mathbb{N}$;

2) an inductive step: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

We say that he constructors of $\mathbb{N}$ are

i) the integer 0;

i) the successor function

$$S : \mathbb{N} \longrightarrow \mathbb{N}$$

defined by

$$S(n) = n + 1.$$

The inductive step 2) can therefore rephrased like

$$\text{if } n \in \mathbb{N}, \text{ then } S(n) \in \mathbb{N}.$$

The above axioms are part of the so-called PEANO'S POSTULATES, which guarantee that each $n \in \mathbb{N}$ is generated by a unique n':

$$S(n') = n.$$

In other words

$$S : \mathbb{N} \longrightarrow \mathbb{N} \quad \text{is} \quad \text{injective}.$$

8.1.2 Other numbers

$\mathbb{N} \subset \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers:

$$-5, 0, 7, 2, -3, \ldots$$

$\mathbb{Z} \subset \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers:

$$-3, \frac{2}{3}, \frac{4}{5}, 1, \frac{5}{2}, 3, \ldots$$

$\mathbb{Q} \subset \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

$\mathbb{R} setminus \mathbb{Q}$ is the set of irrational numbers:

$$\sqrt{2}, \pi, \ldots$$

8.2 Proof by induction

8.2.1 Principle of mathematical induction

Suppose that for each natural number n we have a statement $P(n)$.

Suppose that

i) P(1) is TRUE (basic step);

ii) If $P(i)$ is TRUE, for all $i = 1, \dots n$, then $P(n+1)$ is TRUE (inductive step).

The the principle of Mathematical Induction says that if i) and ii) are satisfied then:

$P(n)$ is TRUE for every natural number n.

Example 8.1 *Prove by induction that*

$$S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \quad \text{for} \quad n = 1, 2, \ldots.$$

Proof.

Basic step: for $n = 1$ we have

$$S_1 = \frac{1 \cdot (1+1)}{2} = 1,$$

i.e. the formula is verified for $n = 1$.

Suppose that the formula

$$S_i = \frac{i(i+1)}{2}$$

is TRUE for $i = 1, \ldots n$, in particular it is TRUE for $i = n$

$$S_n = \frac{n(n+1)}{2}.$$

We want to prove that it is TRUE also for $i = n+1$, i.e. that

$$S_{n+1} = \frac{(n+1)(n+2)}{2}$$

is TRUE. We have

$$\begin{aligned} S_{n+1} &= \underbrace{1+2+\cdots+n}_{S_n}+(n+1) = S_n + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

therefore the formula is TRUE also for $i = n+1$, therefore it is TRUE for all n. $\square$

Example 8.2 *Prove by mathematical induction that*

$$n! \geq 2^{n-1}, \quad \text{for} \quad n = 1, 2, \ldots.$$

Proof.

For $n = 1$ we have

$$1! = 1 = 2^0 = 2^{1-1},$$

therefore the formula is satisfied for $n = 1$.

Suppose that

$$i! \geq 2^{i-1}$$

is TRUE for $i = 1, \ldots n$, in particular it is true for $i = n$ i.e.

$$n! \geq 2^{n-1},$$

then

$$\begin{aligned}(n+1)! &= (n+1)n! \\ &\geq (n+1)2^{n-1} \\ &\geq 2 \cdot 2^{n-1} \\ &= 2^{n-1+1} = 2^n.\end{aligned}$$

Therefore the statement is TRUE also for $i = n + 1$, therefore it is TRUE for any n. $\square$

Note: In the two examples we just saw, we assumed

$$P(i) \quad \text{is} \quad \text{TRUE} \quad \text{for all} \quad i = 1, \dots n \tag{8.1}$$

and we proved that

$$P(n+1) \quad \text{is} \quad \text{TRUE} \quad \text{too.} \tag{8.2}$$

The inductive step (8.1) gives the STRONG FORMULATION OF THE MATHEMATICAL INDUCTION.

We actually deduced

$$P(n+1) \quad \text{is} \quad \text{TRUE}$$

by only using he statement

$$P(n) \quad \text{is} \quad \text{TRUE}.$$

There is indeed an equivalent formulation of the principle of mathematical induction which in which the strong inductive step is replace by the weak inductive step

$$P(n) \quad \text{is} \quad \text{TRUE}.$$

More precisely we have:

WEAK FORMULATION OF THE MATHEMATICAL INDUCTION

Suppose that for each natural number n we have a statement $P(n)$.

Suppose that

i) P(1) is TRUE (basic step);

ii) If $P(n)$ is TRUE, then $P(n+1)$ is TRUE (inductive step).

The the principle of Mathematical Induction says that if i) and ii) are satisfied then:

$$P(n) \text{ is TRUE for every natural number } n.$$

Note: Whether to use the strong or the weak formulation of the principle of mathematical induction, depends on the nature of the statement to prove.