Course Notes
for
MS4111
Discrete Mathematics 1

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CHAPTER 5 Relations

5.1 Relations

We start by defining the cartesian product of two sets.

Definition 5.1 *Let X, Y be two sets. The cartesian product of X and Y is defined by the set of ordered pairs*

$$X \times Y = \{(x, y) \mid x \in X \quad \text{and} \quad y \in Y\}$$

Example 5.1

$$\begin{aligned} X &= \{1, 2, 3\} \\ Y &= \{1, 5\}, \end{aligned}$$

then

$$X \times Y = \{(1,1), (1,5), (2,1), (2,5), (3,1), (3,5)\}.$$

What is $Y \times X$?

$$Y \times X = \{(1,1),(1,2),(1,3),(5,1),(5,2),(5,3)\}.$$

Note:

$$X \times Y \neq Y \times X$$

Definition 5.2 *Let A be a set. We will denote by $|A|$ the number of elements of set A.*

Note:

$$|X \times Y| = |X||Y|.$$

In general, if we have n sets

$$X_1, X_2, \ldots, X_n$$

we can define the cartesian product as the set of ordered n-tuples

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) | x_i \in X_i, \quad \text{for} \quad i = 1, \ldots, n\}.$$

Definition 5.3 *A (binary) relation $\mathcal{R}$ from a set X to a set Y is a subset of the cartesian product $X \times Y$*

$$\mathcal{R} \subset X \times Y.$$

If $(x, y) \in \mathcal{R}$, we write

$$x\mathcal{R}y$$

and we say that x is related to y.

Note: If $X = Y$ we call $\mathcal{R}$ a (binary) relation on X.

Definition 5.4 *If $\mathcal{R}$ is a relation from X to Y, the set*

$$\{x \in X \mid (x, y) \in \mathcal{R}, \quad \text{for some} \quad y \in Y\}$$

is called the domain of $\mathcal{R}$.

The set

$$\{y \in Y \mid (x, y) \in \mathcal{R}, \quad \text{for some} \quad x \in X\}$$

is called the range of $\mathcal{R}$.

Note: A relation $\mathcal{R}$ can be given

1) by specifying the ordered pairs that belong to $\mathcal{R}$;

2) by giving a rule for membership in $\mathcal{R}$.

Example 5.2 *Consider the two sets*

$$X = \{2,3,4,5\} \quad ; \quad Y = \{3,4,5,6\}$$

and the relation from X to Y given by the ordered pairs

$$\mathcal{R} = \{(2,4),(2,6),(3,3),(3,6),(4,4)\}.$$

What are the domain and the range of $\mathcal{R}$?

$$Domain\, of\; \mathcal{R} = \{2,3,4\}$$

$$Range\, of\; \mathcal{R} = \{3,4,6\}.$$

The above relation can be also given by the following characterization:

$$(x, y) \in \mathcal{R} \quad \text{if} \quad x \quad \text{divides} \quad y.$$

Example 5.3 *Let $X = \{1, 2, 3, 4\}$ and $\mathcal{R}$ the relation on X defined by*

$$(x, y) \in \mathcal{R} \quad \text{if} \quad x \leq y, \quad x, y \in X.$$

We can list the elements of $\mathcal{R}$

$$\mathcal{R}\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.$$

$$Domain = \{1, 2, 3, 4\} = X$$

$$Range = \{1, 2, 3, 4\} = X$$

Note: A relation $\mathcal{R}$ on a set X can be represented by the so-called digraph: see examples given in the class or the book for this topic.

5.1.1 Some properties of Relations

Definition 5.5 *A relation $\mathcal{R}$ on X is reflexive if*

$$(x, x) \in \mathcal{R}, \quad \text{for any} \quad x \in X.$$

Example 5.4 *Let $X = \{1, 2, 3, 4\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

$\mathcal{R}$ is a reflexive relation.

Example 5.5 *Let $X = \{a, b, c, d\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(a,a), (b,c), (c,b), (d,d)\}.$$

$\mathcal{R}$ *is NOT a reflexive relation since*

$$(b,b) \notin \mathcal{R} \quad \text{and} \quad (c,c) \notin \mathcal{R}.$$

Definition 5.6 *A relation $\mathcal{R}$ on X is symmetric if for all $x, y \in X$ we have*

$$\text{if} \quad (x,y) \in \mathcal{R}, \quad \text{then} \quad (y,x) \in \mathcal{R}.$$

Example 5.6 *Let $X = \{1,2,3,4\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.$$

$\mathcal{R}$ *is NOT a symmetric relation because for example*

$$(1,3) \in \mathcal{R} \quad \text{but} \quad (3,1) \notin \mathcal{R}.$$

Example 5.7 *Let* $X = \{a, b, c, d\}$ *and* $\mathcal{R}$ *the relation on* X *given by*

$$\mathcal{R} = \{(a,a), (b,c), (c,b), (d,d)\}.$$

$\mathcal{R}$ *is a symmetric relation since*

$$(b,c) \notin \mathcal{R} \quad \text{and} \quad (c,b) \notin \mathcal{R}.$$

Definition 5.7 *A relation* $\mathcal{R}$ *on* X *is antisymmetric if for all* $x, y \in X$ *we have*

$$\text{if} \quad (x,y) \in \mathcal{R}, \quad \text{and} \quad x \neq y \quad \text{then} \quad (y,x) \notin \mathcal{R}.$$

Example 5.8 *Let* $X = \{1, 2, 3, 4\}$ *and* $\mathcal{R}$ *the relation on* X *given by*

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

$\mathcal{R}$ *is an antisymmetric relation.*

Example 5.9 *Let* $X = \{a, b, c, d\}$ *and* $\mathcal{R}$ *the relation on* X *given by*

$$\mathcal{R} = \{(a,a), (b,c), (c,b), (d,d)\}.$$

$\mathcal{R}$ *is NOT an antisymmetric relation since*

$$(b,c) \notin \mathcal{R} \quad \text{and} \quad (c,b) \notin \mathcal{R}.$$

Note: If a relation $\mathcal{R}$ on X has no members of the form (x, y), with $x \neq y$, then $\mathcal{R}$ is antisymmetric because the proposition

$$\text{if} \quad (x, y) \in \mathcal{R}, \quad \text{and} \quad x \neq y \quad \text{then} \quad (y, x) \notin \mathcal{R}$$

is always true since the hypothesis

$$[(x, y) \in \mathcal{R}] \wedge [x \neq y]$$

is false.

Example 5.10 *Let $X = \{a, b, c\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(a, a), (b, b), (c, c)\}.$$

$\mathcal{R}$ is an antisymmetric relation. $\mathcal{R}$ is also reflexive and symmetric.

Note: Antisymmetric does not mean not symmetric. The example above shows us that a relation can be symmetric and antisymmetric at the same time.

Definition 5.8 *A relation $\mathcal{R}$ on X is transitive if for all $x, y, z \in X$ we have*

$$\text{if} \quad (x, y) \in \mathcal{R}, \quad \text{and} \quad (y, z) \in \mathcal{R}, \quad \text{then} \quad (x, z) \in \mathcal{R}.$$

Example 5.11 *Let $X = \{1, 2, 3, 4\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

$\mathcal{R}$ is transitive relation.

Example 5.12 *Let $X = \{a, b, c\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(a, a), (b, b), (c, c)\}.$$

$\mathcal{R}$ is NOT a transitive relation since

$$(b, c) \in \mathcal{R} \quad \text{and} \quad (c, b) \in \mathcal{R}, \quad \text{but} \quad (b, b) \notin \mathcal{R}.$$

5.1.1.6 Partial and total orders

Definition 5.9 *We say that a relation $\mathcal{R}$ on X is a partial order (on X) if it is reflexive, antisymmetric and transitive.*

Example 5.13 *Let* $X = \{1, 2, 3, 4\}$ *and* $\mathcal{R}$ *the relation on* X *given by*

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

$\mathcal{R}$ *is a partial order.*

Definition 5.10 *If* $\mathcal{R}$ *is a partial order on* X*, we sometime write*

$$x \leq y$$

to denote $x\mathcal{R}y$ *and we say that* x *and* y *are comparable.*

Definition 5.11 *Let $\mathcal{R}$ is a partial order on X. If $x, y \in X$ and we have*

$$x \not\preceq y \quad \text{and} \quad y \not\preceq x,$$

we say that x and y are incomparable.

Definition 5.12 *Let $\mathcal{R}$ be a partial order on X. If every pair of elements in X are comparable, we call $\mathcal{R}$ a total order (on X).*

In example 5.13, $\mathcal{R}$ is also a total order.

Example 5.14 *Let us consider the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and the relation on $\mathbb{N}$*

$$x\mathcal{R}y \quad \text{if} \quad x \leq y.$$

$\mathcal{R}$ is a total order because

1) if $x, y \in \mathbb{N}$, then either $x \leq y$ or $y \leq x$ i.e. any pair of elements in $\mathbb{N}$ is comparable;

2) $x \leq x$, for all $x \in \mathcal{N}$;

3) if $x \leq y$ and $x \neq y$, then $y \neq x$ ($\mathcal{R}$ is antisymmetric);

4) if $x \leq y$ and $y \leq z$, then $x \leq z$ ($\mathcal{R}$ is transitive).

5.1.1.7 Inverse of a relation

Definition 5.13 *Let X and Y be two sets and $\mathcal{R}$ a relation from X to Y. The inverse of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is the relation from Y to X defined by*

$$\mathcal{R}^{-1} = \{(y, x) \mid (x, y) \in \mathcal{R}\}.$$

Example 5.15 *Let* $X = \{2, 3, 4\}$ *and* $Y = \{3, 4, 5, 6\}$. *Let* $\mathcal{R}$ *be a relation from* X *to* Y *defined by*

$$(x, y) \in \mathcal{R} \quad \text{if} \quad x \quad \text{divides} \quad y.$$

Let us list the elements of $\mathcal{R}$*:*

$$\mathcal{R} = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}.$$

The inverse of $\mathcal{R}$ *is therefore*

$$\mathcal{R}^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}.$$

5.1.1.8 Composition of two relations

Let X, Y and Z be three sets. Let $\mathcal{R}_1$ be a relation from X to Y and $\mathcal{R}_2$ a relation from Y to Z.

Definition 5.14 *The composition of $\mathcal{R}_2$ with $\mathcal{R}_1$, denoted by $\mathcal{R}_2 \circ \mathcal{R}_1$, is the relation from X to Z defined by*

$$\mathcal{R}_2 \circ \mathcal{R}_1 = \{(x,z) | (x,y) \in \mathcal{R}_1 \quad \text{and} \quad (y,z) \in \mathcal{R}_2 \quad \text{for some} \quad y \in Y\}.$$

Example 5.16 *Let $X = \{1,2,3\}$, $Y = \{2,4,6,8\}$ and $Z = \{u,s,t\}$. Let $\mathcal{R}_1$ a relation from X to Y defined by*

$$\mathcal{R}_1 = \{(1,2),(1,6),(2,4),(3,4),(3,6),(3,8)\}$$

and let $\mathcal{R}_2$ *be a relation from* Y *to* Z *defined by*

$$\mathcal{R}_2 = \{(2, u), (4, s), (4, t), (6, t), (8, u)\}.$$

The composition of $\mathcal{R}_2$ *with* $\mathcal{R}_1$ *is the relation given by*

$$\mathcal{R}_2 \circ \mathcal{R}_1 = \{(1, u), (1, t), (2, s), (2, t), (3, s), (3, t), (3, u)\}.$$

5.1.2 Equivalence relations

We start by giving the following definition.

Definition 5.15 *A relation $\mathcal{R}$ on a set X is an equivalence relation if it is reflexive, symmetric and transitive.*

Example 5.17 *Let $X = \{1, 2, 3, 4, 5\}$ and consider the relation $\mathcal{R}$ on X given by*

$$\begin{aligned}\mathcal{R} = \quad & \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),\\ & (3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\}.\end{aligned}$$

1) $\mathcal{R}$ is reflexive:

$$(1,1) \in \mathcal{R},\ (2,2) \in \mathcal{R},\ (3,3) \in \mathcal{R},\ (4,4) \in \mathcal{R},\ (5,5) \in \mathcal{R}.$$

2) $\mathcal{R}$ is symmetric:

$$\begin{aligned}
(1,3) \in \mathcal{R} &\quad \text{and} \quad (3,1) \in \mathcal{R} \\
(1,5) \in \mathcal{R} &\quad \text{and} \quad (5,1) \in \mathcal{R} \\
(2,4) \in \mathcal{R} &\quad \text{and} \quad (4,2) \in \mathcal{R} \\
(3,5) \in \mathcal{R} &\quad \text{and} \quad (5,3) \in \mathcal{R}
\end{aligned}$$

3) $\mathcal{R}$ is transitive:

$$
\begin{array}{l}
(1,1) \in \mathcal{R}\,,\ (1,3) \in \mathcal{R} \quad \text{and} \quad (1,3) \in \mathcal{R} \\
(1,1) \in \mathcal{R}\,,\ (1,5) \in \mathcal{R} \quad \text{and} \quad (1,5) \in \mathcal{R} \\
(1,3) \in \mathcal{R}\,,\ (3,1) \in \mathcal{R} \quad \text{and} \quad (1,1) \in \mathcal{R} \\
(1,3) \in \mathcal{R}\,,\ (3,3) \in \mathcal{R} \quad \text{and} \quad (1,3) \in \mathcal{R} \\
(1,3) \in \mathcal{R}\,,\ (3,5) \in \mathcal{R} \quad \text{and} \quad (1,5) \in \mathcal{R} \\
\ldots\ldots \text{and so on} \ldots\ldots
\end{array}
$$

Therefore $\mathcal{R}$ is an equivalence relation.

Example 5.18 *Let* $X = \{1, 2, 3, 4\}$ *and* $\mathcal{R}$ *the relation on* X *defined by*

$$x\mathcal{R}y \quad \text{if} \quad x \leq y.$$

Let us list the element of $\mathcal{R}$

$$\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.$$

1) $\mathcal{R}$ *is obviously reflexive;*

2) $\mathcal{R}$ *is NOT symmetric:*

$$(1,2) \in \mathcal{R} \quad \text{but} \quad (2,1) \notin \mathcal{R};$$

3) $\mathcal{R}$ *is transitive. This can be easily checked by using the definition of* $\mathcal{R}$*:*

$$\text{if} \quad (x,y) \in \mathcal{R} \quad \text{and} \quad (y,z) \in \mathcal{R}$$

then, by definition of $\mathcal{R}$*, we have*

$$x \leq y \quad \text{and} \quad y \leq z,$$

therefore

$$x \leq z \quad \text{i.e.} \quad (x, z) \in \mathcal{R}.$$

$\mathcal{R}$ *is NOT and equivalence relation because it is not symmetric.*

Example 5.19 *Let* $X = \{a, b, c, d\}$ *and consider the relation* $\mathcal{R}$ *given by*

$$\mathcal{R} = \{(a, a), (b, c), (c, b), (d, d)\}.$$

1) $\mathcal{R}$ *is NOT reflexive:*

$$(b, b) \notin \mathcal{R};$$

2) $\mathcal{R}$ is symmetric:

$$(b, c) \in \mathcal{R} \quad \text{and} \quad (c, b) \in \mathcal{R};$$

3) $\mathcal{R}$ is NOT transitive:

$$(b, c) \in \mathcal{R} \quad \text{and} \quad (c, b) \in \mathcal{R}, \quad \text{but} \quad (b, b) \notin \mathcal{R}.$$

Therefore $\mathcal{R}$ is NOT an equivalence relation.

Example 5.20 *Let $X = \{a, b, c\}$ and $\mathcal{R}$ the relation on X defined by*

$$\mathcal{R} = \{(a, a), (b, b), (c, c)\}.$$

Check that $\mathcal{R}$ is an equivalence relation.

Example 5.21 *Let $X = \{a, b, c\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}.$$

1) $\mathcal{R}$ is reflexive:

$$(a,a) \in \mathcal{R},\ (b,b) \in \mathcal{R},\ (c,c) \in \mathcal{R};$$

2) $\mathcal{R}$ is symmetric:

$$(a,b) \in \mathcal{R} \quad \text{and} \quad (b,a) \in \mathcal{R};$$

3) $\mathcal{R}$ is transitive:

$$\begin{array}{l} (a,b) \in \mathcal{R}\,;\;\; (b,a) \in \mathcal{R} \quad \text{and} \quad (a,a) \in \mathcal{R} \\ (a,b) \in \mathcal{R}\,;\;\; (b,b) \in \mathcal{R} \quad \text{and} \quad (a,b) \in \mathcal{R} \\ (b,a) \in \mathcal{R}\,;\;\; (a,a) \in \mathcal{R} \quad \text{and} \quad (b,a) \in \mathcal{R} \end{array}$$

Therefore $\mathcal{R}$ is an equivalence relation.

5.1.2.9 Partitions and equivalence classes

Definition 5.16 *If X is a set and $\mathcal{R}$ an equivalence relation on X, for each $a \in X$ we define the equivalence class of a to be the set*

$$[a] = \{x \in X \mid x\mathcal{R}a\}.$$

The idea id that if we consider the set

$$S = \{[a] \mid a \in X\}$$

then the elements of S are subsets of X such that the union of all of them gives X itself and each pair of sets of type $[a]$ is either disjoint or the same set. We therefore say that S is a partition of X.

Example 5.22 *In the example where we considered the set $X = \{a, b, c\}$ and the equivalence relation R on X given by*

$$\mathcal{R} = \{(a, a), (b, b), (c, c), (a, b), (b, a)\},$$

the equivalence classes of $\mathcal{R}$ *are*

$$\begin{aligned} [a] &= \{a, b\} = [b] \\ [c] &= \{c\} \end{aligned}$$

Example 5.23 *Let* $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *and* $\mathcal{R}$ *the relation on* X *defined by*

$$x \mathcal{R} y \quad \text{if} \quad 3 \, \text{divides} \, x - y.$$

It can be shown that $\mathcal{R}$ *is an equivalence relation on* X *(left as an exercise). What are the equivalence classes of* $\mathcal{R}$*?*

$$
\begin{aligned}
[1] &= \{x \in X \mid 3 \quad \text{divides} \quad x-1\} = \{1,4,7,10\} = [4] = [7] = [10] \\
[2] &= \{x \in X \mid 3 \quad \text{divides} \quad x-2\} = \{2,5,8\} = [5] = [8] \\
[3] &= \{x \in X \mid 3 \quad \text{divides} \quad x-3\} = \{3,6,9\} = [6] = [9]
\end{aligned}
$$

Therefore there are three equivalence classes i.e. [1], [2], [3].

5.2 Matrix representation of relations

We will explain how to represent a relation with a matrix by giving some examples.

Example 5.24 *Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$. Let $\mathcal{R}$ be the relation from X to Y given by*

$$\mathcal{R} = \{(1, b), (1, d), (2, c), (2, c), (3, c), (3, b), (4, a)\}.$$

The matrix of the relation $\mathcal{R}$ relative to the orderings

$$\text{ordering of X}: \; 1, 2, 3, 4 \quad \text{and} \quad \text{ordering of Y}: \; a, b, c, d$$

is given by

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

i.e. entry ij (row i and column j) of the matrix is defined to be 1 *if* $(i, j) \in \mathcal{R}$ *and* 0 *if* $(i, j) \notin \mathcal{R}$.

Note: The matrix that represents a relation $\mathcal{R}$ from a set X to a set Y depends on the orderings chosen for X and for Y.

If we change the orderings of the above example in the following way

$$\text{ordering of } \mathrm{X}: \ 2,3,4,1 \quad \text{and} \quad \text{ordering of } \mathrm{Y}: \ d,b,a,c,$$

we obtain the following matrix

$$\begin{array}{cc} & \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

which is different from the previous one.

5.2.1 Matrix of a relation on a set X

Example 5.25 *Let* $X = \{a, b, c, d\}$ *and* $\mathcal{R}$ *a relation on* X *given by*

$$\mathcal{R} = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}.$$

The matrix of $\mathcal{R}$ *relative to the ordering*

$$\text{ordering of } X : \quad a, b, c, d$$

is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \quad \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note: The matrix of a relation on a set X is always a square matrix.

The matrix of a relation from a set X to a set Y depends on the orderings of X and Y; in particular a matrix on a set X depends on the ordering of X. Nevertheless there are some properties of the relation on a set X that can be spotted by looking at the matrix which represents the relation and these properties do not depend on the ordering of X considered. We list these properties in the following notes.

Note 1: Let $\mathcal{R}$ be a relation on a set X and A a matrix representing $\mathcal{R}$ in some ordering. Then $\mathcal{R}$ is reflexive if and only if A has 1's on the main diagonal.

Note 2: Let $\mathcal{R}$ be a relation on a set X and A a matrix representing $\mathcal{R}$ in some ordering. Then $\mathcal{R}$ is symmetric if and only if

$$[A]_{ij} = [A]_{ji} \qquad \text{for all} \quad i, j.$$

Note 3: Let $\mathcal{R}$ be a relation on a set X and A a matrix representing $\mathcal{R}$ in some ordering. Then $\mathcal{R}$ is antisymmetric if and only if

$$\text{if} \quad [A]_{ij} = 1 \quad \text{for } i \neq j, \qquad \text{then} \quad [A]_{ji} = 0.$$

Note 4: There is no simple way to test whether a relation $\mathcal{R}$ on a set X is transitive by examining the matrix A relative to $\mathcal{R}$ in some ordering.

Example 5.26 *Let* $X = \{a, b, c\}$ *and* $\mathcal{R}$ *the relation on* X *given by*

$$\mathcal{R} = \{(a, a), (b, b), (c, c)\}.$$

The matrix of $\mathcal{R}$ *relative to the ordering*

$$\text{ordering of } X: \quad a, b, c,$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} a & b & c \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

By looking at the above matrix we can say that $\mathcal{R}$ is symmetric and antisymmetric.

Example 5.27 *Let $X = \{1, 2, 3, 4\}$ and $\mathcal{R}$ the relation on X given by*

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

The matrix of $\mathcal{R}$ relative to the ordering

$$\text{ordering of } X: \quad 1, 2, 3, 4$$

is

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

by looking at the above matrix we can say that $\mathcal{R}$ is antisymmetric but it is no symmetric. The matrix is also reflexive.

5.2.2 Matrix multiplication and composition of relations

Let us consider the following example.

Example 5.28 *Consider the following three sets*

$$\begin{aligned} X &= \{1, 2, 3\} \\ Y &= \{a, b\} \\ Z &= \{x, y, z\}. \end{aligned}$$

Let $\mathcal{R}_1$ *a relation from* X *to* Y *given by*

$$\mathcal{R}_1 = \{(1, a), (2, b), (3, a), (3, b)\}$$

and $\mathcal{R}_2$ *a relation from* Y *to* Z *given by*

$$\mathcal{R}_2 = \{(a, x), (a, y), (b, y), (b, z)\}.$$

Let us consider the orderings

$$\text{ordering of}\, X : 1, 2, 3; \quad \text{ordering of}\, Y : a, b; \quad \text{ordering of}\, Z : x, y, z.$$

The matrices of $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *are therefore respectively*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad ; \quad A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

If we multiply A_1 *with* A_2

$$A_1 A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

which is almost the matrix of $\mathcal{R}_2 \circ \mathcal{R}_1$.

Note: *The ik-entry of* $A_1 A_2$ *is not zero if and only if* $(i, k) \in \mathcal{R}_2 \circ \mathcal{R}_1$.

Therefore the matrix of $\mathcal{R}_2 \circ \mathcal{R}_1$ *is*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We give the following theorem which summarizes what we just saw in the above example.

Theorem 5.1 *Let X, Y, Z be three sets. Let $\mathcal{R}_1$ be a relation from X to Y and $\mathcal{R}_2$ a relation from Y to Z. Choose orderings for X, Y, Z and write the matrices of $\mathcal{R}_1$ and of $\mathcal{R}_2$ relative to these orderings, let us denote then by A_1 and A_2 respectively. Then the matrix of $\mathcal{R}_2 \circ \mathcal{R}_1$ is obtained by replacing each non zero term in the matrix $A_1 A_2$ by 1.*