Matthew Donaldson OPT. HW 2.

1. $\min f(\underline{x}) = (x_1 - 1)^2 + (x_2 - 1)^2$ S.T. $h(x_1, x_2) = x_1 + x_2 - 4 = 0$

$L(x, u) = (x_1 - 1)^2 + (x_2 - 1)^2 + u(x_1 + x_2 - 4) = 0$

1). $\frac{\partial L}{\partial x_1} = 2(x_1 - 1) + u = 0$

2). $\frac{\partial L}{\partial x_2} = 2(x_2 - 1) + u = 0$

3) $\frac{\partial L}{\partial u} = x_1 + x_2 - 4 = 0$

Let eq. 1) = eq. 2). $\rightarrow 2(x_1 - 1) + u = 2(x_2 - 1) + u$

$x_1 - 1 = (x_2 - 1)$

$x_1 = x_2$

from This equality it can be seen That only values for x_1, x_2 TO Satisfy equation 3) is:

$\boxed{x_1 = x_2 = 2 \; ; \; u = -2}$

2. $\min f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$ S.T. $g_1(x) = x_1 + x_2 - 4 \leq 0$

$g_2(x) = -x_1 + 2 \leq 0.$

$L(x, U) = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + S_1^2) + u_2(-x_1 + 2 + S_2^2)$

1). $\frac{\partial L}{\partial x_1} = 2(x_1 - 1) + u_1 - u_2 = 0$

2). $\frac{\partial L}{\partial x_2} = 2(x_2 - 1) + u_1 = 0$

3). $\frac{\partial L}{u_1} = x_1 + x_2 - 4 + S^2 = 0$

4). $\frac{\partial L}{\partial u_2} = -x_1 + 2 + S_2^2 = 0$

5). $\frac{\partial L}{\partial S_1} = 2S_1 u_1 = 0$

6). $\frac{\partial L}{\partial S_2} = 2S_2 u_2 = 0$

<u>Case 1: $S_1 = 0 \quad S_2 = 0$</u>

from equ. 4). $-X_1 + 2 = 0 \rightarrow X_1 = 2$
from equ. 3). $X_1 + X_2 - 4 = 0 \rightarrow X_2 = 2$

now Checking U_1, U_2:
equ. 2). $2(X_2 - 1) + U_1 = 0 \rightarrow U_1 = -2$ ← not allowed for inequality constraints.

<u>Case 2: $S_1 = 0 \quad U_2 = 0$</u>

from equ. 1, 2. $X_1 = X_2$
from equ. 3 : $X_1 + X_2 - 4 = 0 \quad X_1 = X_2 = 2.$

Checking Slack Variables (S_2):
from equ. 4). $-X_1 + 2 + S_2^2 = 0 \rightarrow S_2^2 = 0$ ✓
Checking Lagrange multiplier U_1: equ. 2; $2(X_2 - 1) + U_1 = 0 \rightarrow U_1 = -2$ X.

<u>Case 3: $S_2 = 0 \quad U_1 = 0$</u>
from equ. 4). $-X_1 + 2 = 0 \rightarrow X_1 = 2$
from equ 2). $2(X_2 - 1) = 0 \rightarrow X_2 = 1$
Checking Slack Variable (S_1): equ 3. : $2 + 1 - 4 + S_1^2 = 0 \rightarrow S_1^2 = 1$ ✓
Checking Lagrange multiplier U_2: equ. 1: $2(2-1) - U_2 = 0 \rightarrow U_2 = 2$ ✓

<u>Case 4. $U_1 = U_2 = 0$</u>
From equ. 1, 2 it yields $X_1 = X_2 = 1$.

from equ. 4 $-1 + 2 + S_2^2 = 0 \Rightarrow S_2^2 = -1$ ← bad.

$$\boxed{\text{Solution: } X_1 = 2,\ X_2 = 1 \quad U_1 = 0 \quad U_2 = 2 \quad S_1^2 = 1 \quad S_2^2 = 0}$$

3. Min $f(x) = (x_1-3)^2 + (x_2-3)^2$ $\quad h(x) = x_1 - 3x_2 - 1 = 0$

$g_1(x) = x_1 + x_2 - 4 \le 0.$

$L(x,u,v) = (x_1-3)^2 + (x_2-3)^2 + v(x_1 - 3x_2 - 1) + u(x_1 + x_2 - 4 + S_1^2) = 0$

1). $\frac{\partial L}{\partial x_1} = 2(x_1-3) + V + u = 0$

2). $\frac{\partial L}{\partial x_2} = 2(x_2-3) - 3V + u = 0$

3). $\frac{\partial L}{\partial V} = x_1 - 3x_2 - 1 = 0$

4). $\frac{\partial L}{\partial u} = x_1 + x_2 - 4 + S_1^2 = 0$

5). $\frac{\partial L}{\partial S} = 2Su = 0$

$\boxed{\text{Solution: } x_1 = 3.25 \quad x_2 = 0.75 \quad S = 0 \quad u = -1.75 \quad V = 1.25}$

Case $S=0$:

This makes equ. 4: $x_1 + x_2 - 4 = 0$; eq 3: $x_1 = 1 + 3x_2$

Plug 3 into 4: $1 + 3x_2 + x_2 - 4 = 0 \rightarrow 4x_2 - 3 = 0 \rightarrow \boxed{x_2 = 0.75}$

now: equ 4 $\rightarrow$ $x_1 + 3/4 - 4 = 0 \rightarrow x_1 = 3.25$

Set equ. 1 = equ. 2:

$2(x_1-3) + V + \cancel{u} = 2(x_2-3) - 3V + \cancel{u}$

$0.5 + V = -4.5 - 3V$

$-5.0 = -4V \rightarrow V = 1.25$

look @ equ 1: $2(3.25-3) + 1.25 + u = 0 \rightarrow u = -1.75$

Case 2: $u=0$

3) $\rightarrow x_1 = 1 + 3x_2$ $\quad$ 1) $\rightarrow 2(x_1-3) + V = 0$ $\quad$ 2) $2(x_2-3) - 3V = 0$

3·equ 1 + equ 2 $\rightarrow$ $6(x_1-3) + 3V = 0$ $\rightarrow 6(x_1-3) + 2(x_2-3) = 0$

$+ 2(x_2-3) - 3V = 0$ $\quad 6x_1 - 18 + 2x_2 - 6 = 0$

$6(1+3x_2) + 2x_2 = 24 \rightarrow x_2 = 0.9 \xrightarrow{eq.3} x_1 = 3.7$

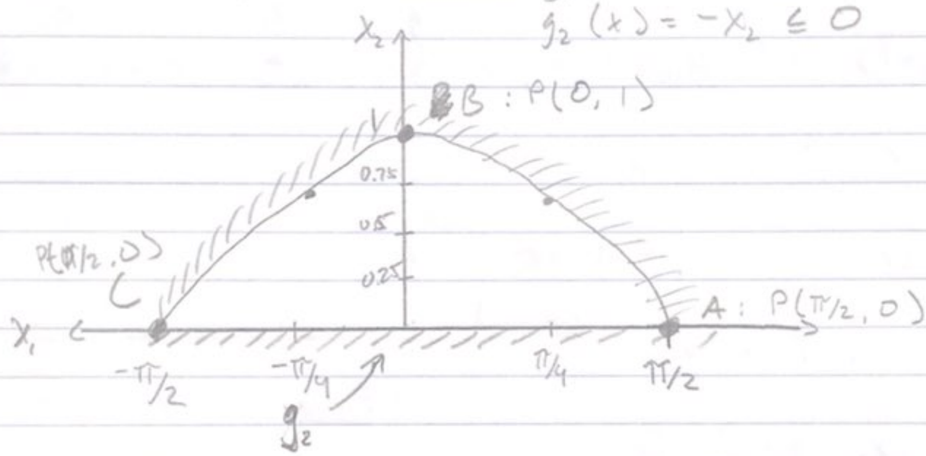
$6 + 18x_2 + 2x_2 = 24$

$S^2 = -.6$ bad

Solution:

4. min $f(x) = -(x_1^2 + x_2^2)$ $g_1(x) = \cos x_1 + x_2 \le 0$
$g_2(x) = -x_2 \le 0$

i).



ii). by inspection possible K-T points are A, B, C

Point A: $f(\pi/2, 0) = -((\pi/2)^2 + 0^2) \cong -2.467$
Point C: $f(-\pi/2, 0) = -((-\pi/2)^2 + 0^2) \cong -2.467$
Point B: $f(0, 1) = -(0^2 + 1^2) = -1$

Finding $\vec{u}$: $\frac{\partial f}{\partial x_i} = \sum_1^2 u_i^* \frac{\partial g_i}{\partial x_i} \rightarrow \nabla f = u_1 \nabla g + u_2 \nabla g$

So $\nabla f(x) = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix} = u_1 \begin{bmatrix} \sin x_1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Point A: $\begin{bmatrix} -\pi \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$u_1 = -\pi$, $u_2 = \pi$

Point C: $\begin{bmatrix} \pi \\ 0 \end{bmatrix} = u_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$u_1 = -\pi$ $u_2 = \pi$

Point B: $\begin{bmatrix} 0 \\ -2 \end{bmatrix} = u_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

equ. 1 $0 = u_1(0) + u_2(0)$ or $-2 = u_1(1) + u_2(-1)$.
$u_1 = -(2 + u_2) \rightarrow$ inc.
$\rightarrow$

(iii): $Q = d^T \nabla^2 L(x^*) d$

$$\cancel{\nabla} L = -(x_1^2 + x_2^2) + u_1(-\cos x_1 + x_2 + s_1^2) + u_2(-x_2 + s_2^2)$$

$$\nabla L = \begin{bmatrix} -2x_1 + u_1 \sin x_1 \\ -2x_2 + u_1 - u_2 \\ -\cos x_1 + x_2 + s_1^2 \\ -1x_2 + s_2^2 \\ 2u_1 s_1 \\ 2u_2 s_2 \end{bmatrix}$$

$$\nabla^2 L = \begin{array}{c} \\ \\ x_1 \\ u_1 \\ u_2 \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & u_1 & u_2 & s_1 & s_2 \end{array} \\ \begin{bmatrix} -2 + u_2 \cos x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 & 0 & 0 \\ -\sin x_1 & 1 & 0 & 0 & 2s_1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2s_2 \\ 0 & 0 & 2s_1 & 0 & 2u_1 & 0 \\ 0 & 0 & 2s_2 & 0 & 2u_2 & 0 \end{bmatrix} \end{array}$$

<u>note</u>: $s_1, s_2 = 0$ since on the constraint for $x^* = (\pm\pi/2, 0)$

So $Q = d^T \begin{bmatrix} \pm\pi & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 & 0 & 0 \\ \pm 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\pi & 0 \\ 0 & 0 & 0 & 0 & 2\pi & 0 \end{bmatrix} d$

Where $d \to$ any feasible direction.

5. $f(x) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$

$\nabla f(x) = \begin{bmatrix} 3x_1^2 + 12x_2^2 + 10x_1 \\ 24x_1x_2 + 4x_2 + 3 \end{bmatrix}$ $\quad Hf(x) = \begin{bmatrix} 6x_1 + 10 & 24x_2 \\ 24x_2 & 24x_1 + 4 \end{bmatrix}$

One way to solve is by principle minors:

Where $|a_{11}| \geq 0$ and $|Hf(x)| \geq 0$ for $f(x)$ to be convex.

So $6x_1 + 10 \geq 0 \rightarrow \underline{x_1 \geq -5/3}$

$\begin{vmatrix} 6x_1+10 & 24x_2 \\ 24x_2 & 24x_1+4 \end{vmatrix} = (6x_1+10)(24x_1+4) - (24x_2)^2$

$= [\text{Plugging in } x_1] = 0 - (24x_2)^2 \geq 0 \rightarrow x_2 \geq 0$

One set $(x_1 \geq -5/3,\ x_2 \geq 0)$

Can also solve for eigen values:

$|Hf(x) - \lambda I| = 0 \rightarrow \begin{vmatrix} 6x_1+10-\lambda & 24x_2 \\ 24x_2 & 24x_1+4-\lambda \end{vmatrix}$

$= (6x_1+10-\lambda)(24x_1+4-\lambda) - (24x_2)^2$

$= 144x_1^2 + 24x_1 - 6x_1\lambda + 240x_1 + 40 - 10\lambda - 24x_1\lambda - 4\lambda + \lambda^2 - (24x_2)^2$

$= \lambda^2 + \lambda(-6x_1 - 10 - 24x_1 - 4) + 144x_1^2 + 264x_1 + 40 - 576x_2^2 = 0$

$= \lambda^2 - (30x_1 + 14)\lambda + 144x_1^2 + 264x_1 + 40 - 576x_2^2$

Using quadratic equation yields:

$$\lambda = \frac{30x_1 + 14 \pm \sqrt{(30x_1+14)^2 - 4(144x_1^2 + 264x_1 + 40 - 576x_2^2)}}{2}$$

$\rightarrow$

$$\lambda = 15x_1 + 7 \pm \sqrt{\frac{(30x_1+14)^2 - 4(144x_1^2 + 264x_1 + 40 - 576x_2^2)}{4}}$$

now focusing on inside the quadratic:

$$\frac{(30x_1+14)^2 - 4(144x_1^2 + 264x_1 + 40 - 576x_2^2)}{4}$$

$$= \frac{900x_1^2 + 840x_1 + 196}{4} - 144x_1^2 - 264x_1 - 40 + 576x_2^2$$

$$= 225x_1^2 + 210x_1 + 49 - 144x_1^2 - 264x_1 - 40 + 576x_2^2$$

$$= 81x_1^2 - 54x_1 + 9 + 576x_2^2$$

$= 9(9x_1^2 - 6x_1 + 1 + 64x_2^2) \leftarrow$ plug this back into square root.

$$\lambda = 15x_1 + 7 \pm 3\sqrt{9x_1^2 - 6x_1 + 1 + 64x_2^2}$$

for $\lambda \geq 0$ need the part under the square root ≥ 0. So

evaluating that. $9x_1^2 - 6x_1 + 1 + 64x_2^2 \geq 0$

$$x_1 = 6 \pm \sqrt{36 - 4(9)(1+64x_2^2)} = 6 \pm \sqrt{36(1 - 1 + 64x_2^2)}$$

$= 6 \pm \sqrt{36(64x_2^2)} \rightarrow$ so $x_2 \geq 0$. for $\sqrt{\ }$ to be "+."

Not sure where to go from here?

Do I plug x_1's equation in? or graph?

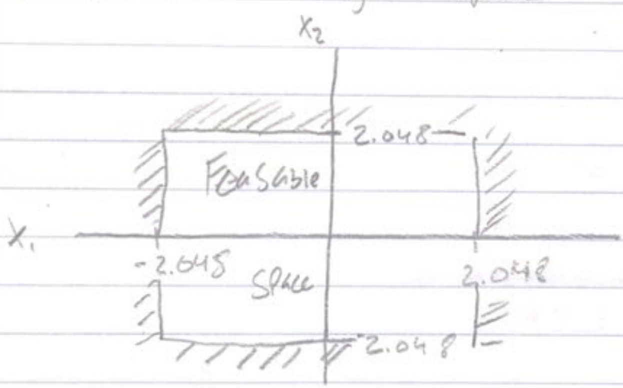
$(0, \infty)$ $x_1 \in (0, 6)$

6. PloT is with MATLab Code

AS Can be seen from The graph when you Change The right hand Side OF The Constraints it will Limit The minimum Solution To your Obj. function.

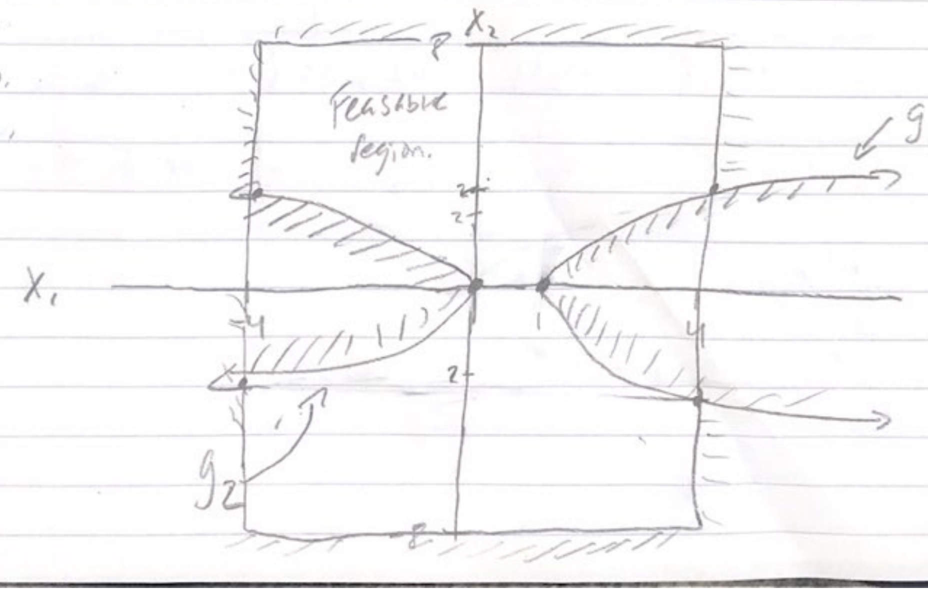
7. a. Solver found minimum point To be (1, 1) $f(x^*) = 3.714 \cdot 10^{-12}$
Code is attached

b. X1 - X2 Design Space



8. a. Solver in MATLab found minimum point To be (1.0134, 0.2702)
$f(x^*) = -3.0676$: MATLab Code attached.

b.



9. 1. F
2. T
3. F
4. T
5. F
6. T
7. T
8. T
9. F
10. T
11. F
12. T
13. F
14. F
15. F