A relation between Mirković-Vilonen cycles and modules over preprojective algebra of Dynkin quiver of type ADE

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Abstract

The irreducible components of the variety of all modules over the preprojective algebra and MV cycles both index bases of the universal enveloping algebra of the positive part of a semisimple Lie algebra canonically. To relate these two objects Baumann and Kamnitzer associate a cycle in the affine Grassmannian for a given module. It is conjectured that the ring of functions of the T-fixed point subscheme of the associated cycle is isomorphic to the cohomology ring of the quiver Grassmannian of the module. We give a proof of this conjecture for a very special case. We also mention the relation to the reduceness conjecture.

1 Introduction

Let $\mathfrak{g}$ be simply-laced semisimple finite dimensional complex Lie algebra. There are several modern constructions of irreducible representation of $\mathfrak{g}$. In this paper we consider two models which realize the crystal of the positive part $U(\mathfrak{n})$ of $U(\mathfrak{g})$. One is by using the geometry of affine Grassmannian more precisely the MV cycles and the other is by using Lusztig nilpotent variety Λ of representations of preprojective algebra of the quiver Q corresponding to $\mathfrak{g}$. In [BK12], Kamnitzer showed that the moment map image of MV cycles, which is called MV polytopes, are in bijection with MV cycles and in [Kam07] he showed that under this bijection, the crystal structure constructed from MV cycles[BFG06] coincides with the one using MV polytope[Kam07] via bijection to Lusztig datum.

Baumann and Kamnitzer [BK12] studied the relations between Λ and MV polytopes and realized MV polytopes from the quiver representation of preprojective algebra of Q. They associate an MV polytope $P(M)$ to a generic module $M \in \Lambda$ and construct a bijection between the set of irreducible components of Λ and MV polytopes compatible with respect to crystal structures. Then Kamnitzer and Knutson launched a program towards geometric construction of the MV cycle $X(M)$ in terms of a module M over the preprojective algebra.

Here we consider a version by Kamnitzer, Knutson and Mirkovic: There exists a cycle $X(M)$ associated to any generic module M, such that the ring of functions $\mathcal{O}(X(M)^T)$ on the T fixed point subscheme of the cycle $X(M)$ is isomorphic to $H^*(Gr^{\Pi}(M))$, the cohomology ring of the quiver Grassmannian of M.

In this paper we use a naive definition of $X(M)$ and construct a map from $\mathcal{O}(X(M)^T)$ to $H^*(Gr^{\Pi}(M))$ and prove it is isomorphism for the case when M is a representation of Q. We will present a counterexample for this version of $X(M)$. We have difficulty to understand the cohomology ring for general generic M.

We can not prove if the Chern classes generate the cohomology ring nor give any counterexample. Also we can not make precise conjecture regarding the full relations of these generators in general. When the M is the direct sum of injective simple modules, the quiver Grassmannian $Gr_e(M)$ specializes to the Lagrangian quiver variety [ST11]. In the finite ADE case the cohomology ring is proved to be isomorphic to the function ring on T-fixed point of transverse slice Gr^{λ}_{μ} in affine Grassmannian. In type A, It is proved that $X(M)$ is reduced hence $X(M)^T_{\mu} \cong Gr^{\lambda}_{\mu}$ so we get a explicit formula of $H^*(\mathfrak{L}(V, W)$. In type D/E, we do not know the reduceness of $X(M)$ but we expect that the map Φ is isomorphic (which means we have found all relations between Chern classes of quiver variety) could be proved using dimension estimate. We also expect that this could be generalized to other cases other than finite ADE.

In section two we recall the definition of MV cycles, quivers, preprojective algebra and Lusztig's nilpotent variety and state the conjecture precisely.

In chapter three we describe the ring of functions on the T fixed point subscheme of the intersection of closures of certain semi-infinite orbits (which is called "cycle" in this paper). A particular case of these intersections is a scheme theoretic version of MV cycles. We realize these cycles as the affine Grassmannian with a certain condition Y.

In section four we construct the map Ψ from $\mathcal{O}(X(M)^T)$ to $H^*(Gr^{\Pi}(M))$. Here, Ψ maps certain generators of $\mathcal{O}(X(M)^T)$ to Chern classes of tautological bundles over $Gr^{\Pi}(M)$. So we need to check that the Chern classes satisfy the relations of generators of $\mathcal{O}(X(M)^T)$. We reduce this problem to the case where the quiver is $1 \longrightarrow 2$. In this case we have a torus action on $Gr^{\Pi}(M)$ so we could use localization in equivariant cohomology theory (GKM theory).

In chapter five I will prove Ψ is an isomorphism in the case when M is a representation of Q of type A.

In chapter six I will explain how this conjecture is related to the reduceness conjecture studied in ?

2 Statement of the conjecture

2.1 Notation

Let G be a simply-laced semisimple group over complex numbers. Let I be the set of vertices in the Dynkin diagram of G. In this paper I will work over base field $k = \mathbf{C}$. We fix a Cartan subgroup T of G and a Borel subgroup $B \subset G$. Denote by N the unipotent radical of B. Let $\varpi_i, i \in I$ be the fundamental weights. Let X_*, X^* be the cocharacter, character lattice and $\langle\ ,\ \rangle$ be the pairing between them. Let W be the Weyl group. Let e and w_0 be the unit and the longest element in W. Let α_i and $\check{\alpha}_i$ be simple roots and coroots. Let $\Gamma = \{w\varpi_i, w \in W, i \in I\}$. Γ is called the set of chamber weights.
Let d be the formal disc and d^* be the punctured formal disc. The ring of formal Taylor series is the ring of functions on the formal disc, $\mathcal{O} = \{\sum_{n\geq 0} a_n t^n\}$. The ring of formal Laurent series is the ring of functions on the punctured formal disc, $\mathcal{K} = \{\sum_{n\geq n_0} a_n t^n\}$.
For X a variety, let $\mathrm{Irr}(X)$ be the set of irreducible components of X.

2.2 MV cycles and polytopes

For a group G, let $G_{\mathcal{K}}$ be the affine group of G and $G_{\mathcal{O}}$ the disc group of G. We define affine grassmannian $\mathcal{G}(G)$ as the left quotient $G_{\mathcal{O}} \setminus G_{\mathcal{K}}$ and view $\mathcal{G}(G)$ as an ind-scheme [BD91], [Zhu16]. An MV cycle is a certain finite dimensional subscheme in $\mathcal{G}(G)$. For a cocharacter $\lambda \in X_*(T)$, we denote the point $t^\lambda G(\mathcal{O})$ in $\mathcal{G}(G)$ by L_λ. For $w \in W$, let $N^w = wNw^{-1}$ and $S^w_\lambda = L_\lambda N^w_{\mathcal{K}}$. This orbit is an ind-subscheme of $\mathcal{G}(G)$ and is called semi-infinite orbit since it is of infinite dimension and codimension in $\mathcal{G}(G)$.
An irreducible component of $\overline{S^e_0 \bigcap S^{w_0}_\lambda}$ is called an MV cycle of weight λ. Kamnitzer [Kam05] describes them as follows:

Theorem 1 ([Kam05])**.** *Given a collection of integers $(M_\gamma)_{\gamma\in\Gamma}$, if it satisfies edge inequalities, and certain tropical relations, put $\lambda_w = \sum_i M_{w\varpi_i} w\check{\alpha}_i$. Then $\overline{\bigcap_{w\in W} S^w_{\lambda_w}}$ is an MV cycle, and each MV cycle arises from this way for the unique data $(M_\gamma)_{\gamma\in\Gamma}$.*

The data $(M_\gamma)_{\gamma\in\Gamma}$ determines a pseudo-Weyl polytope. It is called an MV polytope if the corresponding cycle $\overline{\bigcap_{w\in W} S^w_{\lambda_w}}$ is an MV cycle. MV polytopes are in bijection with MV cycles. Using this description, Kamnitzer [Kam07] reconstruct the crystal structure for MV cycles.

Proposition 1 ([Kam07])**.** *MV polytopes have a crystal structure isomorphic to $B(\infty)$.*

2.3 Objects on the quiver side

Let $Q = \{I, E\}$ be a Dynkin quiver of type ADE, where I is the set of vertices and E is the set of edges. We double the edge set E by adding all the opposite edges. For $a = i \to j$, let $a^* = j \to i$ and set $E^* = \{a^* | a \in E\}$. Define $s(a) = i, t(a) = j$ and $\epsilon(a) = 1$ when $a \in E$, $\epsilon(a) = -1$, when $a \in E^*$. Let $H = E \bigsqcup E^*$ and $\overline{Q} = \{I, H\}$. The preprojective algebra Π_Q of Q is defined as quotient of the path algebra by a certain ideal

$$\Pi_Q = k\overline{Q} / < \sum_{a \in H} \epsilon(a) a a^* > .$$

We often omit the subscript Q and just write Π when there is no confusion. A Π_Q-module is the data of an I graded vector space $\bigoplus_{i \in I} M_i$ and linear maps $\phi_a : M_{s(a)} \to M_{t(a)}$ for each $a \in H$ satisfying the preprojective relations

$$\sum_{a \in H, t(a) = i} \epsilon(a) \phi_a \phi_{a*} = 0.$$

Given a dimension vector $d \in \mathbf{N}^I$, define $\Lambda(d)$ to be the variety of all representations of Π on M for $M_i = k^{d_i}$.

Proposition 2 ([KS$^+$97], [BK12])**.** *Irr(Λ) has a crystal structure of $B(\infty)$.*

2.4 A conjectural relation between MV cycles and modules over the preprojective algebra

Baumann and Kamnitzer found an isomorphism between the crystal structure of Irr(Λ) and MV polytopes. For each $\gamma \in \Gamma$, they define constructible funtion $D_\gamma : \Lambda(d) \to \mathbf{Z}_{\geq 0}$[1]. For any $M \in \Lambda(d)$, the collection $(D_\gamma)_{\gamma \in \Gamma}$ satisfies certain edge inequalities hence determines a polytope which we denote by $P(M)$.

Theorem 2 ([BK12])**.** *When M is generic, $P(M)$ is an MV-polytope and for $d = (d_i)_{i \in I}$ this gives a map from Irr($\Lambda(d)$) to the set of MV polytopes of weight $\sum_{i \in I} d_i \alpha_i$. This map is a bijection compatible with the crystal structures.*

We have MV-cycles (in bijection with MV-polytopes) as the geometric object on the affine Grassmannian side. In order to upgrade the relations geometrically, Kamnitzer-Knutson consider the quiver Grassmannian on the quiver side.

The quiver Grassmannian $Gr^{\Pi}(M)$ of a Π-module M is defined as the moduli of submodules of M. Note that a $k\overline{Q}$-submodule of M is automatically a Π-module. So we will often write $Gr^(M)$ instead of $Gr^{\Pi}(M)$.

It is a subscheme of the moduli of k-vector subspaces of M which is product of usual Grassmannian $\prod_{i \in I} Gr(M_i)$. Here we will only consider $Gr^{\Pi}(M)$ with

[1] $\Lambda(d)$ and D_γ do not depend on the direction of the edges in E.

its reduced structure, and actually just as a topological space. As the case of usual Grassmannian, the quiver Grassmannian $Gr^{\Pi}(M)$ is the disjoint union of Grassmannians of different dimension vectors. Denote $Gr^{\Pi}_e(M)$ by the moduli of submodule N of M of dimension vector e, we have $Gr^{\Pi}_e(M) \subset \prod_{i\in I} Gr_{e_i}(M_i)$. Given a module $M \in \Lambda(d)$, form the subscheme[2] $X(M)=\bigcap_{w\in W} \overline{S^w_{\lambda_w}}$, where $\lambda_w = \sum_{i\in I} -D_{-w\varpi_i}(M) w\check{\alpha}_i$. The Cartan T acts on $S^w_{\lambda_w}$ by multiplication, hence it also acts on the closure and the intersection $X(M)$.

Conjecture 1. *The ring of functions on the T-fixed point subscheme of $X(M)$ is isomorphic to the cohomology ring of the quiver Grassmannian of M*

$$\mathcal{O}(X(M)^T) \xrightarrow[\sim]{\Psi} H^*(Gr^{\Pi}(M)).$$

More precisely, $X(M)^T$ is disjoint union of finite schemes $X(M)^T_\nu$ supported at L_ν, $\nu \in X_*(T)$ and we can further identify two sides for each connected component

$$\mathcal{O}(X(M)^T_\nu) \xrightarrow[\sim]{\Psi} H^*(Gr^{\Pi}_e(M)), \text{ where } e_i = (\nu, \varpi_i).$$

We define $X(M)$ as a scheme theoretic intersection of closures while MV-cycles have been defined as varieties (closure of intersections). It is possible that $X(M)$ is reducible when $P(M)$ is an MV-polytope. Indeed in an example [section ?] when the closure of intersections does not coincide with intersection of closures the conjecture fails.

3 The T fixed point subscheme of the cycle

We introduce some notation first. It is known that the T-fixed point subscheme of the affine Grassmannian of a reductive group G is the affine Grassmannian of the Cartan T of G, i.e, $\mathcal{G}(G)^T = \mathcal{G}(T)$. We identify T with I copies of the multiplicative group by $T \xrightarrow[\sim]{\prod \varpi_i} G_m^I$ and this gives $\mathcal{G}(T) \xrightarrow[\sim]{\prod \varpi_i} \mathcal{G}(G_m)^I$.

For $\mathcal{G}(G_m)$, we have

$$\begin{aligned} \mathcal{G}(G_m) &= G_m(\mathcal{O}) \setminus G_m(\mathcal{K}) && (1)\\ &= \{\text{unit} \in \mathcal{O}\} \setminus \{\text{unit} \in \mathcal{K}\} && (2)\\ &= t^{\mathbf{Z}} \cdot K_- && (3)\end{aligned}$$

where K_- is called the negative congruence subgroup (of G_m). For a commutative ring R, the R-points of K_- can be described as:

$$K_-(R) = \{a = (1 + a_1 t^{-1} + ... + a_m t^{-m}) | a_i \text{ is nilpotent in } R\}.$$

[2]We will call it cycle in this paper.

We define the degree function from $K_-(R)$ to $\mathbf{Z}_{\geq}$: $\deg(a) = m$ if $a_m \neq 0$.

Then $(\bigcap \overline{S^w_{\lambda_w}})^T$ is a subscheme of $\mathcal{G}(G)^T \cong (t^{\mathbf{Z}} \cdot K_-)^{|I|}$.

Theorem 3. *Let $(\lambda_w)_{w\in W}$ be a collection of cocharacters such that $\lambda_v \geq_w \lambda_w$[3] for all $w \in W$ in which case we know ([Kam05]) that $(\lambda_w)_{w\in W}$ determines a pseudo-Weyl polytope. The integers $A_{w\varpi_i}$ are well defined by $A_{w\varpi_i} = (\lambda_w, w\varpi_i)$. The R-points of $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$ is the subset of R-point of $(t^{\mathbf{Z}} \cdot K_-)^{|I|}$ containing elements $(t^{(\nu,\varpi_i)}a_i) \in \prod(t^{\mathbf{N}} \cdot K_-)^{|I|}|$ subject to the degree relations:*

$$deg(\Pi_{i\in I} a_i^{(\gamma,\check{\alpha_i})}) \leq -A_\gamma + \sum(\gamma,\nu) \text{ for all } \gamma \in \Gamma\}.$$

Proof. We define affine Grassmannian with a condition Y and list the facts we need. For details, see [Mir17]. Let G acts on scheme Y and y be a point in Y. Denote the stack quotient by Y/G. Then $\mathcal{G}(G,Y)$ is the moduli of maps of pairs from (d,d^*) to $(Y/G,y)$. When Y is a point we recover $\mathcal{G}(G)$. In general, $\mathcal{G}(G,Y)$ is the subfunctor of $\mathcal{G}(G)$ subject to a certain extension condition:

$$\mathcal{G}(G,Y) = G_{\mathcal{O}} \setminus \{g \in G_{\mathcal{K}} \mid d^* \xrightarrow{g} G \xrightarrow{o} Y \text{ extends to } d\}, \text{ where } o(g) = gy.$$

We can realize semi-infinite orbits and their closures as follows:

- $\mathcal{G}(G, G/N) = S_0$, where G acts G/N by left multiplication.
- $\mathcal{G}(G,(G/N)^{aff}) = \overline{S_0}$, where "aff" means affinization.
- $\mathcal{G}(G\times T,(G/N)^{aff})_{red} = \bigsqcup \overline{S_\lambda}$, where"red" means the reduced subscheme. Here T acts on G/N by left multiplication with the inverse and this extends to an action on $(G/N)^{aff}$.
- $$\mathcal{G}(G \times \prod_{w\in W} T_w, \prod_{w\in W}(G/N^w)^{aff}) = \bigsqcup_{(\lambda_w)_{w\in W}} (\bigcap_{w\in W} \overline{S^w_{\lambda_w}}),$$

  where we denote a copy of T corresponding to $w \in W$ by T_w.

A single cycle $\bigcap_{w\in W} \overline{S^w_{\lambda_w}}$ can be written as the fiber product:

$$\bigcap_{w\in W} \overline{S^w_{\lambda_w}} = \mathcal{G}(G \times \prod_{w\in W} T_w, \prod(G/N^w)^{aff}) \times_{\mathcal{G}(\prod_{w\in W} T_w)} (t^{\lambda_w})_{w\in W}.$$

In this fiber product, the morphism for the first factor is the second projection and the morphism for the second factor is the inclusion of the single point $t^{\underline{\lambda}} = (t^{\lambda_w})_{w\in W}$.
For a reductive group G, we have $\mathcal{G}(G,Y)^T = \mathcal{G}(T,Y)$, where T is the Cartan

[3]This notation is used in [Kam05], $\lambda_v \geq_w \lambda_w$ whenever $w^{-1}\lambda_v \geq w^{-1}\lambda_w$.

of G. So, the T fixed point subscheme is

$$(\bigcap_{w\in W}\overline{S^w_{\lambda_w}})^T=\mathcal{G}(T\times\prod_{w\in W}T_w,\prod(G/N^w)^{aff})\times_{\mathcal{G}(\prod_{w\in W}T_w)}t^{\underline{\lambda}}.$$

In terms of the above extension condition, this fiber product is:

$$(\bigcap_{w\in W}\overline{S^w_{\lambda_w}})^T=T(\mathcal{O})\backslash\{g\in T_{\mathcal{K}},\text{ such that } d^*\xrightarrow{g,t^{\underline{\lambda}}}T\times T^W\to\prod(G/N^w)^{aff}\text{ extends to } d\}$$

This is the $T(\mathcal{O})$ quotient of the set of all $g\in T_{\mathcal{K}}$, such that

$$d^*\xrightarrow{g,t^{\lambda_w}}T\times T_w\to(G/N^w)^{aff}\text{ extends to } d \text{ for all } w\in W.$$

For $\gamma\in W\cdot\varpi_i\subset\Gamma$, we fix weight vectors v_γ in the weight space $(V_{\varpi_i})_\gamma$ of V_{ϖ_i}. For each $w\in W$, we embed G/N^w into $\bigoplus_{i\in I}V_{\varpi_i}$ by $g\mapsto(g\cdot v_{w\varpi_i})_{i\in I}$. Under this embedding, $(G/N^w)^{aff}$ is a closed subscheme in $\bigoplus_{i\in I}V_{\varpi_i}$.

For $g\in T_{\mathcal{K}}$, $w\in W$, the composition $y_w(g)$ of the map :

$$d^*\xrightarrow{g,t^{\lambda_w}}T\times T_w\to G/N^w\hookrightarrow\bigoplus V_{\varpi_i}$$

is

$$y_w(g)=(g\cdot(t^{\lambda_w})^{-1})\sum_{i\in I}v_{w\varpi_i}=\sum_{i\in I}(w\varpi_i(g\cdot t^{-\lambda_w}))v_{w\varpi_i}.$$

This map extends to d when for each $i\in I$, the coefficient of $v_{w\varpi_i}$ is in the ring of Tayler series $\mathcal{O}$. The coefficient of $v_{w\varpi_i}$ is

$$\begin{aligned}w\varpi_i(g\cdot t^{-\lambda_w})&=w\varpi_i(g)\cdot w\varpi_i(t^{-\lambda_w})=w\varpi_i(g)\cdot t^{-(w\varpi_i,\ \lambda_w)}\\&=w\varpi_i(g)t^{-A_{w\varpi_i}}=\gamma(g)z^{-A_\gamma}\text{ where }\gamma=w\varpi_i.\end{aligned}$$

It follows that

$$(\bigcap_{w\in W}\overline{S^w_{\lambda_w}})^T=T(\mathcal{O})\setminus\{g\in T(\mathcal{K});\gamma(g)t^{-A_\gamma}\in\mathcal{O}\text{ for all }\gamma\in\Gamma\}.$$

and the description of the R-points of $(\bigcap\overline{S^w_{\lambda_w}})^T_\nu$ in the theorem follows when we identify $\mathcal{G}(T)\xrightarrow{\prod\varpi_i}\mathcal{G}(G_m)^I=(t^{\mathbf{Z}}\cdot K_-)^I$.

$\square$

Corollary 1. *The ring of functions on $\mathcal{O}((\bigcap\overline{S^w_{\lambda_w}})^T_\nu)$ is generated by a_{ij}'s and b_{ik}'s, for $i\in I$. The relations are degree conditions:*

$$deg(\prod_{i\in I^+_\gamma}a_i^{\gamma_i{}^+}\prod_{i\in I^-_\gamma}b_i^{\gamma_i{}^-})\le(\gamma,\nu)-A_\gamma$$

for each $\gamma\in\Gamma$ and conditions $a_ib_i=1$ for each i in I.

Proof. For an R-point $(t^{(\nu,\varpi_i)}a_i)_{i\in I}$ of $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$, let us write $a_i = 1 + a_{i1}t^{-1} + \cdots + a_{im}t^{-m}$. When $\gamma = \varpi_i$, the degree inequality is $\deg(a_i) \le (\varpi_i,\nu) - A_{\varpi_i}$. We can take the coefficients a_{ij} to be the coordinate functions on $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$. Since $\deg(a_i) \le (\varpi_i,\nu) - A_{\varpi_i}$, there are finitely many a_{ij} which generate the ring of functions on $\mathcal{O}((\bigcap \overline{S^w_{\lambda_w}})^T_\nu)$. The inverse of a_i is computed in K_- as $a_i^{-1} = 1 + \sum_{s\ge 0}(-1)^i(a_{i1}t^{-1} + \cdots + a_{im}t^{-m})^s$. Let b_{ik} be the coefficient of t^{-k} in a_i^{-1}. These coefficients are polynomials of a_{ij}s. Set $b_i = 1 + \sum_k b_{ik}t^{-k} = a_i^{-1}$, We add b_{ij}'s as generators and also add the relations $a_ib_i = 1$ for $i \in I$ which eliminate all b_{ij}'s. For $\gamma \in \gamma$, let $\gamma_i = (\gamma, \check{\alpha}_i)$. Denote by I^+_γ the subset of I containing all i such that γ_i is positive and by I^-_γ containing all i γ_i is negative. Set $\gamma_i^+ = \gamma_i$ when γ_i is positive and $\gamma_i^- = -\gamma_i$ when γ_i negative. Then the remaining relation

$$deg(\Pi_{i\in I}a_i^{(\gamma,\check{\alpha}_i)}) \le -A_\gamma + \sum(\gamma,\nu) \text{ for all } \gamma \in \Gamma.$$

is equivalent to the coefficient of the term t^{-1} to the power $-A_\gamma + \sum(\gamma,\nu) + 1$ in $(\Pi_{i\in I}a_i^{(\gamma,\check{\alpha}_i)})$ is 0, hence the corollary follows. $\square$

4 Construction of the map Ψ from functions to cohomology

4.1 Map Ψ

For $M \in \Lambda(d)$, to apply corollary 1 to $X(M)$, we set $A_\gamma = -D_{-\gamma}(M)$. Then

$$\mathcal{O}(X(M)^T_\nu) = k[a_{ij}, b_{ik}]/I(M).$$

where $I(M)$ is the ideal generated by the degree conditions:

$$deg(\prod_{i\in I^+_\gamma}(a_i)^{\gamma_i^+}\prod_{i\in I^-_\gamma}(b_i)^{\gamma_i^-}) \le (\gamma,\nu) + D_{-\gamma}(M)$$

for each $\gamma \in \Gamma$ and the conditions $a_ib_i = 1$ for each i in I. The conjecture $\mathcal{O}(X(M)^T_\nu) \cong H^*(Gr^\Pi_e(M))$, where $e_i = (\nu, \varpi_i)$, is now equivalent to

$$k[a_{ij}, b_{ik}]/I(M) \cong H^*(Gr^\Pi_e(M)).$$

The quiver Grassmannian $Gr^\Pi_e(M))$ is a subvariety of $\prod_{i\in I} Gr_{e_i}(M_i)$ and we have on each $Gr_{e_i}(M_i)$ the tautological subbundle S_i and quotient bundle Q_i. We pull back S_i and Q_i to $\prod_{i\in I} Gr_{e_i}(M_i)$ and denote their restrictions on $Gr^\Pi_e(M))$ still by S_i and Q_i by abusing notion. For a rank n bundle E, denote the Chern class by $c(E)$ and the i^{th} Chern class $c_i(E)$, where $c(E) = 1 + c_1(E) + \cdots + c_n(E)$. We want to define the map

$$\Psi : \mathcal{O}(X(M)^T_\nu) \to H^*(Gr^\Pi_e(M)), \text{ where } e_i = (\nu, \varpi_i),$$

by mapping the generators a_{ij} to $c_j(S_i)$ and b_{ij} to $c_j(Q_i)$.

Theorem 4. *The map Ψ described above is well defined.*

4.2 Two lemmas

For the proof, we need two lemmas. Lemma 1 is the special case of theorem 4 when Q is the quiver $1 \to 2$ and M is a kQ-module.

Lemma 1. *Let Q be the quiver $1 \to 2$ and M be $\mathbf{C}^{d_1} \xrightarrow{\phi} \mathbf{C}^{d_2}$. On $X = Gr_e^{\Pi}(M)$, we have $c_i(S_2 \oplus Q_1) = 0$ when $i > e_2 - e_1 + dim(ker\phi)$.*

Let $\phi_{ij} : M_i \to M_j$ be the composition of ϕ_a where a travels over the unique no going-back path which links i and j. Let $M_\gamma = \oplus_{i \in I_\gamma^-} M_i^{\gamma^-} \xrightarrow{\phi_\gamma = \oplus \phi_{ij}} \oplus_{i \in I_\gamma^+} M_i^{\gamma^+}$ be the module over $k(1 \to 2)$.

Lemma 2. *For a Π-module M and any chamber weight γ, we have*

$$dim(ker\phi_\gamma) = D_{-\gamma}(M).$$

Lemma 2 is a property of D_γ and will be proved in the appendix.

4.3 Proof of theorem 4 from lemmas in § 4.2

Proof of theorem 4. We prove the theorem can be reduced to lemma 1.

For each $\gamma \in \Gamma$, we have to prove the degree inequalities carry over to Chern classes:

$$\Psi\Big(\prod_{i \in I_\gamma^+} t_i^{\gamma_i^+} \prod_{i \in I_\gamma^-} s_i^{\gamma_i^-}\Big) = \prod_{i \in I_\gamma^+} c(S_i)^{\gamma_i^+} \prod_{i \in I_\gamma^-} c(Q_i)^{\gamma_i^-} \leq D_{w_0\gamma}(M) + (\nu, \gamma).$$

Define a map Φ from $Gr^{\Pi}(M)$ to $Gr^{k(1 \to 2)}(M_\gamma)$: for $N \in Gr^{\Pi}(M)$, $\Phi(N) = \oplus_{i \in I_\gamma^-} N_i \xrightarrow{\phi_\gamma} \oplus_{i \in I_\gamma^+} N_i$. We have

$$\Phi^*(c(S_2)c(Q_1)) = c(\Phi^*(S_2))c(\Phi^*(Q_1)) = c(\oplus_{i \in I_\gamma^+} S_i^{\gamma_i^+})c(\oplus_{i \in I_\gamma^-} Q_i^{\gamma_i^-})$$

$$= \prod_{i \in I_\gamma^+} c(S_i)^{\gamma_i^+} \prod_{i \in I_\gamma^-} c(Q_i)^{\gamma_i^-}.$$

Apply lemma 1 to M_γ we have

$$deg(c(Q_1)c(S_2)) \leq dimker(\phi_\gamma) + \sum_{i \in I_\gamma^+} \gamma_i e_i - \sum_{i \in I_\gamma^-} \gamma_i e_i$$

$$= dimker(\phi_\gamma) + \sum_{i \in I} \gamma_i e_i = dimker(\phi_\gamma) + (\gamma, \nu).$$

Then the theorem follows by lemma 2. □

We know devote to the proof of lemma 1. Chern class vanishes in certain degree when the bundle contains a trivial bundle of certain degree but the desired trivial bundle in $Q_1 \oplus S_2$ does not exist. The idea is to pass to T-equivariant cohomology. Over X^T which is just a union of isolated points we will decompose $Q_1 \oplus S_2$ into the sum of the other two bundles E_1 and E_2 pointwisely, where E_1 will play the role of trivial bundle. Although there is no bundle over X whose restriction is E_2, there exist T-equivariant cohomology class in $H_T^*(X)$ whose restriction on X^T is the T-equivariant Chern class of E_2.

4.4 Recollection of GKM theory

We first recall some facts in T-equivariant cohomoloy theory. We follow the paper [Tym05]. Denote a n-dimensional torus by T, topologically T is homotopic to$(S^1)^n$. Take ET to be a contractible space with a free T-action. Define BT to be the quotient ET/T. The diagonal action of T on $X \times ET$ is free, since the action on ET is free. Define $X \times_T ET$ to be the quotient $(X \times ET)/T$. We define the equivariant cohomology of X to be

$$H_T^*(X) = H^*(X \times_T ET).$$

When X is a point and $T = G_m$,

$$H_T^*(X) = H^*(\mathrm{pt} \times_T ET) = H^*(ET/T) = H^*(BT) = H^*(\mathbf{CP}^\infty) \cong k[t].$$

When $T = (S^1)^n$,

$$H^*(pt) = k[t_1, \cdots, t_n] \cong S(\mathfrak{t}^*). \tag{4}$$

So we can identify any class in $H^*(pt)$ as a function on the lie algebra $\mathfrak{t}$ of T. The map $X \to pt$ allows us to pull back each class in $H_T^*(\mathrm{pt})$ to $H_T^*(X)$, so $H_T^*(X)$ is a module over $H_T^*(\mathrm{pt})$.

Fix a projective variety X with an action of T. We say that X is equivariantly formal with respect to this T-action if $E^2 = E^\infty$ in the spectral sequence associated to the fibration $X \times_T ET \longrightarrow BT$.

When X is equivariantly formal with respect to T, the ordinary cohomology of X can be reconstructed from its equivariant cohomology. Fix an inclusion map $X \to X \times_T ET$, we have The kernel of i is $\sum_{s=1}^n t_s \cdot H_T^*(X)$, where t_s is the generator of $H_T^*(pt)$ (see (4)) and we view it as an element in $H_T^*(X)$ by pulling back the map $X \to pt$. Also i is surjective so $H^*(X) = H_T^*(X)/ker(i)$.

If in addition X has finitely many fixed points and finitely many one-dimensional orbits, Goresky, Kottwitz, and MacPherson show that the combinatorial data encoded in the graph of fixed points and one-dimensional orbits of T in X implies a particular algebraic characterization of $H_T^*(X)$.

Theorem 5 (GKM, see [Tym05], [GKM97])**.** *Let X be an algebraic variety with a T-action with respect to which X is equivariantly formal, and which has finitely many fixed points and finitely many one-dimensional orbits. Denote the one-dimensional orbits $O_1, \ldots, O_m$. For each i, denote the the T-fixed points of O_i by N_i and S_i and denote the stabilizer of a point in O_i by T_i. Then the map $H_T^*(X) \xrightarrow{l} H_T^*(X^T) = \oplus_{p_i \in X^T} H_T^*(p_i)$ is injective and its image is*

$$\left\{ f = (f_{p_1}, \ldots, f_{p_m}) \in \bigoplus_{\text{fixed pts}} S(t^*) : f_{N_i}|_{\mathfrak{t}_i} = f_{S_i}|_{\mathfrak{t}_i} \text{ for each } i = 1, \ldots, m \right\}.$$

Here $\mathfrak{t}_i$ is the lie algebra of T_i.

4.5 Proof of lemma 1

Proof. For M given by $\mathbf{C}^{d_1} \xrightarrow{\phi} \mathbf{C}^{d_2}$ and a choice of $e = (e_1, e_2)$, denote $X = Gr_{\mathbf{e}}(M)$. First, we define a torus action on X. Let $I = ker\phi$. Choose a basis $e_1, e_2, \cdots, e_s$ of I and extend it to a basis $e_1, \cdots, e_s, e_{s+1}, \cdots, e_t$ of M_1. Let J be span$\{e_{s+1}, \cdots, e_t)\}$ so the image of J is span $\{f_{s+1}, \cdots, f_t\}$. We extend the basis $\{f_i = \phi(e_i)\}$ of the image of J to a basis $(f_{s+1}, \cdots, f_t, f_{t+1}, \cdots, f_r)$ of M_2. Let K=span$\{f_{t+1}, ...f_r\}$. we have $M_1 = I \oplus J$ and $M_2 = \phi(J) \oplus K$.

Let $\mathcal{I} = \{1, \cdots, s\}$, $\mathcal{J} = \{s+1, \cdots, t\}$ and $\mathcal{L} = \{t+1, \cdots, r\}$. Let tori $T_I = G_m^{\mathcal{I}}, T_J = G_m^{\mathcal{J}}, T_L = G_m^{\mathcal{L}}$ act on $I, J \cong \phi(J)$, K by multiplication compotentwisely (For instance, T_I acts on I by $(t_1, \cdots, t_s) \sum a_i e_i = \sum a_i t_i e_i$ and on J, K trivially). Hence they act on $M_1 = I \oplus J$ and $M_2 = \phi(J) \oplus K$. This induces an action of $T = T_I \times T_J \times T_K$ on $Gr_{\mathbf{e}}(M)$. By lemma ? in section ?, $Gr_{\mathbf{e}}(M)$ is paved by affines so by proposition ? it has odd cohomology vanishing therefore the spectral sequence associated to the fibration $X \times_T ET \longrightarrow BT$ converges at E^2 and X is equivariantly formal.

Denote by f the forgetful map $H_T^*(X) \xrightarrow{f} H^*(X)$. From §4.4 we have $ker(f) = \sum_1^{dimT} t_s H_T^*(X)$. Since $c^i(S_2 \oplus Q_1) = f(c_T^i(S_2 \oplus Q_1))$, it suffices to prove $c_T^i(S_2 \oplus Q_1)) \in ker(f)$ when $i > e_2 - e_1 + dimI$.

To use GKM theorem, we need to know the one dimensional orbits and T-fixed points of X.

First, we see what X^T is. For a point $p = (V_1, V_2)$ in X, in order to be fixed by T, V_1 and V_2 need to be spanned by some of basis vectors e_i and f_i. For a subset S of $\mathcal{I}$ (resp. $\mathcal{J}$), we denote by e_S (resp. f_S) the span $\{e_i | i \in S\}$ (resp.

span$\{f_i|i \in S\}$). The T-fixed points in X consist of all $V = (V_1, V_2)$, such that $V_1 = e_{A \bigcup B}, V_2 = f_{C \bigcup D}$, for some $A \subset I, B \subset C \subset J$ and $D \subset K$.

For any point $p = (V_1, V_2)$ in X^T, let $V_1 = e_{A \bigcup B}, V_2 = f_{C \bigcup D}$. Over p, $Q_1 = (I \oplus J)/e_{A \bigcup B}$ is isomorphic to $e_{(\mathcal{I} \setminus A) \oplus (\mathcal{J} \setminus B)}$ (The restriction of a T-equivariant bundle to a T-fixed point is just a T-module). So over X^T, we can decompose $S_2 \oplus Q_1$ as follows:

$$S_2 \oplus Q_1 \cong e_{(\mathcal{I} \setminus A) \oplus (\mathcal{J} \setminus B)} \oplus f_{(C \bigcup D)} = (e_{(\mathcal{I} \setminus A) \oplus (C \setminus B)} \oplus f_D) \oplus (e_{\mathcal{J} \setminus C} \oplus f_C).$$

Denote the bundle over X^T whose fiber over each point p is $e_{(\mathcal{I} \setminus A) \oplus (C \setminus B)} \oplus f_D)$ by E_1 and the bundle over X^T whose fiber over p is $e_{(\mathcal{J} \setminus C)} \oplus f_C$ by E_2.

We now use localization. Denote by l the map $H_T^*(X) \xrightarrow{l} H_T^*(X^T) = \oplus_{p \in X^T} H^*(p)$. From GKM theory l is injective, so the condition $c_T^i(S_2 \oplus Q_1)) \in ker(f)$ is equivalent to $l(c_T^i(S_2 \oplus Q_1)) \in l(ker(f))$. We have

$$l(ker(f)) = l(\sum_{s=1}^{dimT} t_s H_T^*(X)) = \sum_{s=1}^{dimT} \underbrace{(t_s, \cdots, t_s)}_{\text{the number of T-fixed points in X}} l(H_T^*(X)). \tag{5}$$

By functorality of Chern class, $l(c_T^i(S_2 \oplus Q_1)) = c_T^i(S_2|_{X^T} \oplus Q_1|_{X^T}))$.
We compute the T-equivariant Chern class over X^T. For[4] each p,

$$c_T^p(S_2 \oplus Q_1) = c_T^p(E_2 \oplus E_1) = \sum_i c_T^{p-i}(E_1) c_T^i(E_2) = \sum_{i \geq 1} c_T^{p-i}(E_1) c_T^i(E_2).$$

The last equality holds since $c_T^p(E_2) = 0$ when $p > dimE_2 = dimI + e_2 - e_1$. Now to show $c_T^p(S_2 \oplus Q_1) \in l(ker(f))$, It suffices to show that $c_T^{p-i}(E_1) c_T^i(E_2) \in l(ker(f))$, for any i. The action of T on E_2 is actually the same on each T-fixed point. And at each point, $c_T^i(E_2)$ is the i^{th} elementary symmetric polynomial of $t_s, 1 \leq s \leq dimT$. So by (5), it suffices to show that $c_T^i(E_1) \in l(H_T^*(X))$.

Now we will see what 1-dimensional orbits are. Take an orbit O_i, in order to be 1 dimensional its closure must contain two fixed points. Let $\overline{O_i} = O_i \bigcup \{N_i\} \bigcup \{S_i\}$, where $N_i = (e_{A \bigcup B}, f_{C \bigcup D})$ and $S_i = (e_{A' \bigcup B'}, f_{C' \bigcup D'})$ are the fixed points. O_i is one dimensional whenever either $A \bigcup B$ and $A' \bigcup B'$ differ by one element with $C \bigcup D = C' \bigcup D'$ or $C \bigcup D$ and $C' \bigcup D'$ differ by one element with $A \bigcup B = A' \bigcup B'$. In the first case, we have some $s \in A \bigcup B$ and $s' \in A' \bigcup B'$, such that $A \bigcup B \setminus s = A' \bigcup B' \setminus s'$.

Notice that the annihilator for the lie algebra $\mathfrak{t}_i$ in $S(t^*)$ is generated by $t_s - t_{s'}$, so by theorem 5, the condition along O_i for an element $h \in H_T^*(X^T)$ to be in $im(l)$ is

$$(t_s - t_{s'}) \mid (h_{N_i} - h_{S_i}).$$

[4]We always denote S and Q but indicate over which space we are considering.

But we have

$$c_T(E_1)|_{N_i} - c_T(E_1)|_{S_i} = (1+t_{s'}) \prod_{i \in \mathcal{I} \bigcup C \setminus (A \bigcup B) \setminus \{s'\}} (1+t_i) - (1+t_s) \prod_{i \in \mathcal{I} \bigcup C \setminus (A' \bigcup B') \setminus \{s\}} (1+t_i).$$

Note that $\mathcal{I} \bigcup C \setminus (A \bigcup B) \setminus \{s'\} = \mathcal{I} \bigcup C \setminus (A' \bigcup B') \setminus \{s\}$, so $t_s - t_{s'}$ divides $c_T(E_1)|_{N_i} - c_T(E_1)|_{S_i}$. We conclude that $c_T^i(E_1) \in l(H_T^*(X))$.
The other case is similar. $\square$

5 Analogue to Springer theory

We will view $Gr_{\mathbf{e}}(M)$ as graded version of the fixed point subvariety of Grassmannian under a nilpotent operator.

Lemma 3. *Let $V = \oplus M_i$ be the underlying vector space and $\phi = \oplus_{a \in H} \phi_a$ be the nilpotent operator on V. Let $e = \sum e_i$. Then the map*

$$Gr_{\mathbf{e}}(M) \xrightarrow{i} Grass^{\phi}(e, V)$$
$$(N_i)_{i \in I} \mapsto \bigoplus (N_i).$$

is injective. Moreover, denote the image of i by $Grass_I^{\phi}(e, V)$. It consists of all the subspace V' such that there exists $N_i \subset M_i$ of dimension e_i such that $V' = \oplus N_i$.

Proposition 3. *The operator ϕ is nilpotent in finite ADE case[Lus90]. The rest of the lemma follows directly.*

Our approach is to view $Gr_{\mathbf{e}}(M)$ as a graded version of (partical flag)[5] Springer fiber since the latter was studied extensively[Tan82, DCP81]. But difficulty occurs when the normal basis of ϕ is not contained in a single M_i. In order to say more results, we specialize to case where the normal basis of ϕ is contained in a single M_i in the next section. This is actually very restrictive. For example, for the quiver $1 \longleftarrow 2 \longrightarrow 3$, take module $M = (1, 1, 1)$ and this can not fit into this case. The author learned that using technique in algebraic representation theory, one could prove $Gr_{\mathbf{e}}(M)$ has affine paving for M being a kQ-module [IEFR18].

[5] also called Steinberg variety

6 The map Ψ is an isomorphism for a special type of M in type A

6.1 Affine paving

Definition 1 ([Tym07] 2.2)**.** *We say a space X is paved by affines if X has an order partition into disjoint $X_1, X_2, \cdots$ such that each finite union $\bigcup_{i=1}^{j} X_i$ is closed in X and each X_i is an affine space.*

A space with an affine paving has odd cohomology vanishing.

Proposition 4 ([Tym07], 2.3)**.** *Let $X = \bigcup X_i$ be a paving by a finite number of affines with each X_i homeomorphic to $\mathbf{C}^{d_i}$. The cohomology groups of X are given by $H^{2k}(X) = \bigoplus_{\{i \in I \ \mid \ d_i = k\}} \mathbf{Z}$.*

The next lemma will be used several times.

Lemma 4. *Suppose X is paved by X_i's. Let $Y \subset X$ be a subspace. if for each i, $Y_i = X_i \bigcap Y$ is $\emptyset$ or affine then $Y = \bigcup Y_i$ is an affine paving.*

Proof. $\bigcup_{i \leq j} Y_i = \bigcup_{i \leq j} (X_i \bigcap Y) = (\bigcup_{i \leq j} X_i) \bigcap Y$ is closed in Y since $\bigcup_{i \leq j} X_i$ is closed in X. $\square$

6.2 Affine paving of $Gr_{\mathbf{e}}(M)$ for special M

Lemma 5. *Let Q be equioriented type A quiver. Let M_1 be a kQ-module and M_2 be a $k\overline{Q}$-module so $M = M_1 \oplus M_2$ is a Π-module. Then the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ is paved by affines for any dimension vector e.*

Before we give the proof , let us introduce some notations about Schubert decomposition of Grassmannian, following [Shi85]. let V be an n-dimensional vector space over a field k. Let d be an integer such that $1 \leq d < n$. Let V^d be the direct sum of d copies of V and $\wedge^d V$ be the d-th alternating product of V. Let

$$\pi : V^d \to \wedge^d V$$

be the morphism defined by $(v_1, \cdots, v_d) \to v_1 \wedge \cdots \wedge v_d$. Fix a basis $\{e_1, \cdots, e_n\}$ of V. Then we can identify V^d with the set of all $d \times n$ matrices over k by

$$(v_1, \cdots, v_d) \mapsto (x_i(j))_{1 \leq i \leq d, 1 \leq j \leq n},$$

where $v_i = \Sigma_{1 \leq j \leq n} x_i(j) e_j, x_i(j) \in k$, namely $x_i(j)$ is the j-th coordinate of vector v_i. Let $\mathbf{P}(\wedge^d V)$ be the projective variety associated to $\wedge^d V$ and p be the projection

$$\wedge^d V - 0 \xrightarrow{p} \mathbf{P}(\wedge^d V).$$

We have $\mathrm{Grass}(d, V) = p(\pi V^d - \{0\})$. Put

$$\mathfrak{I} = \{\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbf{Z}^d; 1 \leq \alpha_1 < \cdots < \alpha_d \leq n\}.$$

For $\alpha = (\alpha_1, \cdots, \alpha_d)$ in $\mathfrak{I}$, put

$$D_\alpha = \{(x_i(j)) \in V^d; x_i(j) = 0 \text{ for } j < \alpha_i \ (1 \leq i \leq d)\},$$

$$C_\alpha = \{(x_i(j)) \in D_\alpha;\ x_i(\alpha_j) = \delta_{ij} \ (1 \leq i, j \leq d)\}.$$

Then $S_\alpha = p\pi C_\alpha$ is the Schubert cell and $p(\pi D_\alpha - \{0\})$ is the Schubert variety, which is the closure of S_α. We have Schubert decomposition

$$Grass(d, V) = \bigsqcup_{\alpha \in \mathfrak{I}} S_\alpha,$$

which is an affine paving. Let λ be an ordered partition of n, i.e, an ordered sequence $(\lambda_1, \cdots, \lambda_r)$ of positive integers such that $\lambda_1 + \cdots + \lambda_r = n$. We represent λ by a Young diagram with rows consisting of $\lambda_1, \cdots, \lambda_r$ squares respectively.

Definition 2. *Fix a Young diagram λ with n squares. Let d be an integer such that $1 \leq d < n$. A d-tableau is a Young diagram of type λ whose d squares are distinguished by* $\blacksquare$.

Fill in the squares of λ with the numbers $1, 2, \cdots, n$ in the order from the left column to the right and for each column from up to down. For example, if $\lambda = (4, 3, 2)$, we have $\begin{array}{|c|c|c|c|}\hline 7 & 4 & 2 & 1 \\ \hline 8 & 5 & 3 \\ \cline{1-3} 9 & 6 \\ \cline{1-2}\end{array}$. Now we could identify $\alpha = (\alpha_1, \cdots, \alpha_d)$ in I with the d-tableau of type λ whose $\alpha_1, \cdots, \alpha_d$-th squares are $\blacksquare$. An nilpotent operator ϕ can be represented by a Young diagram $\lambda(\phi)$ using its Jordan normal basis $\{e_1, \cdots, e_n\}$. Explicitly, for a Young diagram numbered above, we define ϕ corresponding to λ by $\phi e_i = e_j$ if λ contains $\boxed{i}\boxed{j}$ and $\phi e_i = 0$ if $\boxed{i}$ lies on the most left column and vice versa. We often omit (ϕ) in $\lambda(\phi)$ when we consider a fixed ϕ (M).

Definition 3. *A d-tableau is said to be semistandard if every square on the left position to* $\blacksquare$*on the same row is* $\blacksquare$.

Denote by $SST(\lambda)$ consisting of all the semistandard tableau of type λ.

Lemma 6. *[Shi85, lemma 1.8] Let S_α^ϕ be the subspace invariant under ϕ, i.e*

$$S_\alpha^\phi = \{W \in S_\alpha; \phi W \subset W\},$$

where S_α is the Schubert cell corresponding to α. Then S_α^ϕ is nonempty if and only if α is semistandard. So we have

$$Grass(V, d)^\phi = \bigsqcup_{\alpha \in SST(\lambda)} S_\alpha^\phi.$$

Now we want to understand S_α^ϕ when α is semistandard. For that purpose, the notion initial number was introduced.

Definition 4. *[Shi85, 1.9] For a semistandard d-tableau $\alpha = (\alpha_1, \cdots, \alpha_i, \cdots, \alpha_d)$ of type λ, we say i is an initial number of α if the square on the right side of $\boxed{\alpha_i}$ is not $\blacksquare$.*

Denote by C'_α the inverse image of S_α of the map $V^d \xrightarrow{p\circ\pi} \mathbf{P}(\wedge^d V)$. Note that to analyze S_α we choose a subset C_α of C'_α which under $p\circ\pi$ maps isomorphic to S_α. In order to analyze S_α^ϕ, we choose another subset of C'_α that maps isomorphic to S_α^ϕ.

Lemma 7. *[Shi85, lemma 1.10] For a semistandard d-tableau α, let C_α^ϕ consists $(v_1, \cdots, v_d) \in C'_\alpha$ such that for when i is an initial number, the vector v_i satisfies the condition that $x_i(\alpha_j) = \delta_{ij}$ and for l not initial, it satisfies the condition $v_l = \phi v_k$ whenever α contains $\begin{array}{|c|c|}\hline \alpha_l & \alpha_k \\ \hline\end{array}$, namely v_j is determined by successively applying ϕ on v_i for α_j appearing in the row where α_i is initial. Then the morphism $p\circ\pi$ induces an isomorphism*

$$C_\alpha^\phi \cong S_\alpha^\phi.$$

Hence the decomposition in the previous lemma

$$Grass(V,d)^\phi = \bigsqcup_{\alpha\in SST(\lambda)} S_\alpha^\phi$$

is an affine paving by lemma ?.

Proof. For an element w in S_α^ϕ, we can always choose its preimage $(v_1, \cdots v_d)$ of $p\circ\pi$ such that v_i satisfy $x_i(\alpha_j) = \delta_{ij}$ when α_i is initial. The condition that w is ϕ-invariant and $w \in C'_\alpha$ determines w as described once these v_i were chosen. $\square$

Remark 1. *Note that element in C'_α is not always in C_α. For example, when α is $\begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & \ \\ \hline \blacksquare & \ \\ \cline{1-2}\end{array}$, (remember the number indexing basis is $\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 \\ \cline{1-2}\end{array}$) we have $\alpha_1 = 2$, $\alpha_2 = 4$, $\alpha_3 = 5$. let $v_1 = e_2 + e_3$, then $v_2 = e_4 + e_5$, violating $x_5(\alpha_4) = 0$, so (v_1, v_2, v_3) is not in C_α. So we have to make a change to the statement to relax the condition that $(v_1, \cdots, v_d) \in C_\alpha$ to C'_α.*

Proof of lemma 5. By Lemma 3, it suffices to show $S_\alpha^\phi \bigcap Grass_I^\phi(e, V)$ is affine for $\alpha \in SST(\lambda)$.

Take $x \in S_\alpha^\phi \bigcap Grass_I^\phi(e, V)$, such that $x = v_1 \wedge \cdots \wedge v_d$ where $v_i, 1 \le i \le d$ satisfy conditions in lemma 7. By our basis choice, there exists $t(i)$ such that $e_{\alpha_i} \in M_{t(i)}$.

Since $span(v_1, ..., v_d)$ is a direct sum of some $N_k \subset M_k$, applying projection Pr_j from V to M_j, we have $Pr_j(v_i) \in span(v_1, ..., v_d)$ for any j. Write $v_i =$

$e_{\alpha_i} + \sum\limits_{r \notin \alpha} a_r e_r$. For $e_r \notin V_{t(i)}$, say $e_r \in V_j$, if $a_r \neq 0$, we have $0 \neq Pr_j(v_i) \in span(v_1, ..., v_d)$, contradict to the fact that the element in $span(v_1, ..., v_d)$ must have nonzero coefficient of some e_{α_m} for $1 \leq m \leq d$. So we have $v_i \in M_{t(i)}$.

By our assumption, there exist t_j such that $v_j \in M_{t_j}$ for any j once $v_i \in M_{t_i}$ for initial i. Write $v_i = e_{\alpha_i} + \sum x_{ij} e_j$, the condition $v_i \in M_{t_i}$ is that $x_{ij} \in \mathbf{C}$ for $e_j \in M_{t_i}$ and $x_{ij} = 0$ otherwise. For other j not initial, v_j is determined by v_i. We examine when $span(v_1, \cdots, v_d)$ is ϕ-invariant. Write $v_i = e_{\alpha_i} + \sum\limits_{e_j \in M_{t_i}} x_{ij} e_j$. If ϕe_{α_i} and ϕe_j are not in the same M_l, using the same argument for proving $v_i \in M_{t(i)}$ for ϕv_i, we see $x_{ij} = 0$. If ϕe_{α_i} and ϕe_j are in the same M_l, there is no condition for x_{ij}. Denote $S_\alpha^\phi \bigcap Gr_e^\Pi(M)$ by $S_{\alpha,\mathbf{e}}^\phi$. Denote by $SST(\lambda, \mathbf{e})$ the set consisting of $\alpha \in SST(\lambda)$ and has e_i blocks with basis vector in M_i. Hence $S_{\alpha,\mathbf{e}}^\phi$ is an affine space and it is nonempty when $\alpha \in SST(\lambda, \mathbf{e})$. By lemma ? , the decomposition

$$Gr_{\mathbf{e}}(M) = \bigcup_{\alpha \in SST(\lambda, \mathbf{e})} S_{\alpha,\mathbf{e}}^\phi$$

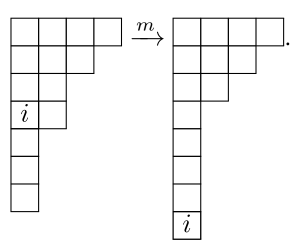
is an affine paving.

□

6.3 Affine paving of its complement

Lemma 8. *(a) Denote $Y = \prod Gr_{e_i}(k^{d_i})$ and $X = Gr_e^\Pi(M)$. Then $Y \setminus X$ is paved by affines.*
(b) Ψ is surjective.

Proof. (a). Let a be the number where the Young diagram of ϕ_Y has a^{th} row as the first row from the bottom that does not have one block. For example, in the left diagram, $a = 4$.



Move the left most block i of the a^{th} row to the bottom in the diagram of ϕ_Y and move other blocks in a^{th} row one block left. Note that applying m changes the order of blocks. For example,

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \xrightarrow{m} \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The order on left is 4231 and on the right is 4321. We reorder the tableau $m(\lambda)$ to make it the same order as λ and still denote the map by m. Recall that the order of blocks changes C_α also C'_α. Define ϕ' be the operator of V that corresponds to the $m(\lambda)$. Let M' be the corresponding module and X' be $Gr_{\mathbf{e}}(M')$. We still denote by m the map moving i of young tableau α in λ. Note that $m(\alpha)$ is then a young tableau in λ. The map m induces an inclusion map $X \xrightarrow{m} X'$.

We now prove that $X' \setminus m(X)$ is paved by affines. By lemma 4, we have $X = \bigsqcup_{\alpha \in SST(\lambda,\mathbf{e})} S^{\phi}_{\alpha,\mathbf{e}}$ and $X' = \bigsqcup_{\alpha \in SST(\lambda',\mathbf{e})} S^{\phi'}_{\alpha,\mathbf{e}}$.

If α contains block i, α is semi-standard in λ implies α' is semi-standard in λ'. If α does not contain block A , α also does not contain any block in that row, so α is still semi-standard in λ'.

So $m(SST(\lambda,\mathbf{e})) \subset SST(\lambda',\mathbf{e})$ and moreover $m(S^{\phi}_{\alpha,\mathbf{e}}) = S^{\phi'}_{m(\alpha),\mathbf{e}}$ when $\alpha \in SST$ and α does not contain configuration like $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 2 \\ \hline\end{array}$.[6] But in the latter case, we still could reorder it to make the identity hold.

Hence $X' \setminus m(X) = \bigsqcup_{\alpha \in SST(\lambda',\mathbf{e}) \setminus m(SST(\lambda,\mathbf{e}))} S^{\phi'}_{\alpha,e}$ is paved by affines. We can do this procedure step by step until X' becomes Y, so (1) follows.

(b). By lemma 1, X is paved. By part(a) , we have the homology map from X to Y is injective hence Ψ is surjective (see 2.2 in[Tym07]). $\square$

6.4 The map Φ is isomorphism in this special case

We now devote to prove the two sides of Ψ have the same dimension as k-vector spaces. Denote the ideal in Conj? by $I(M,e)$ and the ring by $R(M,e)$.

Lemma 9. *In the special case as lemma ?, the dimension of two sides of Ψ have the inequality: dim $R(M,e) \leq \chi(Gr_{\mathbf{e}}(M))$, where χ is the Euler character, in this case the number of affine cells.*

We could assume that there exist i such that $e_i \in M_1$.
Let λ' be the young diagram removing the block of e_{i1} from the original one and M' be the corresponding module. Let λ'' be the young diagram removing the row of e_i and M'' be the corresponding module. Apply lemma ?, for $\alpha \in SST(\lambda,\mathbf{e})$, there are two cases. One is α contains block e_i. Then α must contain the whole row of it. Let h be the number of blocks that the row contains. We

[6] The number i means the basis vector of that block is in M_i. In the case $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 2 \\ \hline\end{array}$, $S^{\phi}_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array},(1,1,0)}$ is a point but for $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$, $S^{\phi}_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array},(1,1,0)}$ is affine line.

have

$$|\{\alpha \in SST(\lambda, \mathbf{e}) \mid \alpha \text{ contains the row}\}| = |SST(\lambda', \mathbf{e} - \sum_{1 \le i \le h} \alpha_i)|.$$

The other case is that α does not contain the block e_i and we have

$$|\{\alpha \in SST(\lambda, \mathbf{e}) \mid \alpha \text{ does not contain block } e_i\}| = |SST(\lambda'', \mathbf{e})|.$$

Lemma 10. $R(M,e)/ < b_{1(d_1-e_1)} > \cong R(M', e)$.

Lemma 11. *$b_{1(d_1-e_1)}R(M,e)$ is a module over $R(M'', e - \sum_{i \in I} \alpha_i)$.*

Proof of lemma 5. By lemma? and the argument above, we have

$$\chi(Gr_{\mathbf{e}}(M)) = \chi(Gr_e(M')) + \chi(Gr_{e-\sum \alpha_i}(M'')).$$

We count the dim of $R(M,e)$ by dividing it into two parts.

$$dim R(M,e) = dim b_{1(d_1-e_1)}R(M,e) + dim R(M,e)/b_{1(d_1-e_1)}R(M,e).$$

By lemma 7 and 8, since $b_{1(d_1-e_1)}R(M,e)$ is acyclic,

$$dim b_{1(d_1-e_1)}R(M,e) \le dim R(M'', e - \sum \alpha_i).$$

So $dim R(M,e) = dim b_{1(d_1-e_1)}R(M,e)) + dim R(M,e)/b_{1(d_1-e_1)}R(M,e)$ $\le dim R(M'', e - \sum \alpha_i) + dim R(M', e) = \chi(Gr_e(M') + \chi(Gr_{(e-\sum \alpha_i)}(M'')) = \chi(Gr_e^{\Pi}(M))$. $\square$

Proof of lemma 7. Recall $I(M) = \deg \prod_{i \in I+} (a_i)^{\gamma_i} \prod_{i \in I-} (b_i)^{\gamma_i} \le (\gamma, \nu) + D_{-\gamma}(M)$ We denote $v(M, \gamma, e) = (\gamma, \nu) + D_{-\gamma}(M)$. The difference between $I(M)$ and $I(M)'$ only occurs when $\gamma = -\varpi_1$. In this case $v(M', -\varpi_1, e) = v(M, -\varpi_1, e) - 1$. The degree of s_1 goes down by by 1, meaning we have one more vanishing condition which is $b_{1(d_1-e_1)} = 0$. $\square$

Proof of lemma 8. In order to define a module structure on $b_{1(d_1-e_1)}R(M,e)$, we lift the element in $R(M'', e - \sum_{i \in I} \alpha_i)$ to $R(M,e)$ (since the former is a quotient of the latter) and let it act on $b_{1(d_1-e_1)}R(M,e)$ by multiplication. We need to check it is independent of the choice of the lift, i.e

$$b_{1(d_1-e_1)}I(M'', e'') \subset I(M,e), \tag{6}$$

Denote the module corresponding to i^{th} row P. We have $M = M'' \oplus P$. Then $v(M, \gamma, e) - v(M'', \gamma, \sum_{i \in I} \alpha_i) = v(P, \gamma, \sum_{i \in I} \alpha_i)$.
Compute and check all the chamber weights, we have $v(P, \gamma, \sum_{i \in I} \alpha_i) = 0 \text{ or } 1$ and is 0 when $\gamma_1 = 1$ and When it is 0 (6) automatically holds so we only need to check the cases when it is 1. And in this case the difference between $I(M,e)$ and $I(M'', e'')$ is the degree $v(M, \gamma, e) - 1 = v(M'', \gamma, e'')$ part of $< \prod_{i \in I+} (a_i)^{\gamma_i} \prod_{i \in I-} (b_i)^{\gamma_i} >$ for $\gamma \in \Gamma$.

- Case $\gamma_1 = -1$. Let one of the monomial of an polynomial in $v(M,\gamma,e)-1$ part of $\prod_{i\in I_\gamma^+}(a_i)^{\gamma_i}\prod_{i\in I_\gamma^-}(b_i)^{\gamma_i}$ be $a_{1j}k$. Then

$$\begin{aligned} b_{1(d_1-e_1)}a_{1j}k &= \\ -\sum_{p+q=d_1-e_1+j} b_{1p}a_{1q}k &= \\ -b_{1(d_1-e_1+j-q)}\sum_{q>j} a_{1q}k. \end{aligned}$$

  Then index appearing in the sum is from q bigger than j so the degree becomes $v(M,\gamma,e)$ so (6) holds.

- Case $\gamma_1 = 0$. Examine all the cases where $v(P,\gamma,\sum_{i\in I}\alpha_i)$ is 1 , we have $\gamma - \varpi_1 \in \Gamma$. Let one of the monomial of an polynomial of degree $v(M,\gamma,e)-1$ part of $< \prod_{i\in I+}(a_i)^{\gamma_i}\prod_{i\in I-}(b_i)^{\gamma_i} >$ for $\gamma\in\Gamma$ be k. We want to show $a_{1(d_1-e_1)}k$ is in the degree $v(M,\gamma-\varpi_1,e)$ part of $< \prod_{i\in I+}(a_i)^{\gamma_i}\prod_{i\in I-}(b_i)^{\gamma_i} >$ for $\gamma-\varpi_1\in\Gamma$. So we need $v(M,\gamma,e)+d_1-e_1 \geq v(M,\gamma-\varpi_1,e)+1$, which is equivalent to

$$D_{-\gamma}(M) \geq D_{-(\gamma-\varpi_1)}(M) - 1. \tag{7}$$

  Since M contain P the image of $\Phi_{\gamma-\varpi_1}$ is at least 1 dimensional bigger than the image of Φ_γ. By lemma 2, (7) holds.

□

Theorem 6. *The map Ψ is an isomorphism.*

Proof. by lemma ?, Ψ is surjective so dim $k[a_{ij},b_{ik}]/I(M,e)) \geq \chi(Gr_{\mathbf{e}}(M))$ and by lemma ? this is an equality so the theorem follows. □

7 Relation to reduceness conjecture

In section 3, we defined $\mathcal{G}(G,Y)$ as moduli of maps of between pairs form (d,d^*) to $(G/Y,pt)$. This is actually a local version of (fiber at a closed point c) the global affine Grassmannian with a condition Y to a curve C, $\mathcal{G}^C(G,Y)$. To a curve C, define $\mathcal{G}^C(G,Y)$ over the ran space $\mathcal{R}_C$ with the fiber at $E\in\mathcal{R}_C$:

$$\mathcal{G}^C(G,Y)_E =^{def} map[(C,C-E),(G/Y,pt)].$$

Denote the map from $\mathcal{G}^C(G,Y)$ to $\mathcal{R}_C$ remembering the singularities by π.

One can ask if π is (ind) flat for any G and (Y,pt). The case we are concerned is when $G' = G\times\prod_w T_w$ and $Y=\prod_w(G/N^w)^{aff}$. Let $c\in C, \underline{\lambda_w},\underline{\mu_w}\in X_*(T)^W$. In particular, we restrict $\mathcal{G}^C(G',Y)$ to $C\times c$ and denote the image under projection from $\mathcal{G}(G'$ to $\mathcal{G}(G)$ by X. We have X is a closed subscheme of $Gr_{G,X\times c}$. Explicitly, an R-point of $Gr_{G,X\times c}$ consists of the following data

- $x : specR \to C$. Let Γ_x be the graph of x. Let Γ_c be the graph of the constant map taking value c.
- β a G-bundle on spec$R \times C$.
- A trivialization $\eta : \beta_0 \xrightarrow{\eta} \beta$ defined on $specR \times C - (\Gamma_x \bigcup \Gamma_c)$.

An R-point of X over $C \times c$ consists of an R-point of $Gr_{G,X\times c}$ subject to the condition: For every $i \in I$, the composition

$$\eta_i : \beta_0 \times^G V(\varpi_i) \to \beta \times^G V(\varpi_i) \to \beta \times^G V(\varpi_i) \otimes \mathcal{O}(\langle\gamma, \lambda_w\rangle \cdot \Gamma_x + \langle\gamma, \mu_w\rangle \cdot \Gamma_c).$$

is regular on all of $specR \times C$.

We can show the fiber over a closed point other than c is $\bigcap \overline{S^w_{\lambda_w}} \times \bigcap \overline{S^w_{\mu_w}}$ and the fiber over c is $\bigcap \overline{S^w_{\lambda_w+\mu_w}}$.

Corollary 2 (Given the conjecture)**.** *The T-fixed point subscheme of this family is flat.*

We need a lemma.

Lemma 12. *For* Π*-module* M, N*, we have*

$$\chi(Gr_g(M \oplus N) = \sum_{d+e=g} \chi(Gr_d(M))\chi(Gr_e(N)).$$

Proof. The proof is the same as the case where M, N are kQ-modules [CC04, prop 3.6]. $\square$

Proof of Corollary 2.

$$dim\mathcal{O}((\overline{\bigcap S^w_{\lambda_w+\mu_w}}^T)_\nu) = dimH^*(Gr^\Pi_e(M))$$
$$= \sum_{e_1+e_2=e} dimH^*(Gr^\Pi_{e_1}(M_1)dimH^*(Gr^\Pi_{e_2}(M_2)$$
$$= \sum_{\nu_1+\nu_2=\nu} \mathcal{O}(\overline{\bigcap S^w_{\lambda_w}}^T_{\nu_1}) \cdot \mathcal{O}(\overline{\bigcap S^w_{\mu_w}}^T_{\nu_2})$$
$$= dim\mathcal{O}(\bigsqcup_{\nu_1+\nu_2=\nu} \overline{\bigcap S^w_{\lambda_w}}^T_{\nu_1} \times \overline{\bigcap S^w_{\mu_w}}^T_{\nu_2} = dim(\overline{\bigcup S^w_{\lambda_w}}^T \times \overline{\bigcap S^w_{\lambda_w}}^T)_\nu.$$

$\square$

Conjecture 2. *T-fixed subschemes flatness imply flatness.*

We take $\lambda_w = -w_0\lambda + w\lambda$, then $\overline{\bigcap S^w_{\lambda_w}} = \overline{Y^\lambda}$. In this case the conjecture is proved to be true. This flatness is mentioned in [KMW16] remark 4.3 and will reduce the proof of reduceness of $\overline{Y^\lambda}$ to the case when λ is ϖ_i for each $i \in I$.

8 A counter example

9 Appendix

Lemma 13. *Set $s_0 = id$. For any Chamber weight $w\gamma = \varpi_i$, we can take $w = s_n \cdots s_1$ such that*

$$\langle s_k \cdots s_1 s_0 \gamma, \alpha_{k+1} \rangle \leq 0.$$

Proof. If γ is not dominant , which means there exits some t such that $\langle \gamma, \alpha_t \rangle \leq 0$, we apply s_t to γ and let $\gamma_1 = s_t\gamma$. Do the same if γ_1 is not dominant and we get γ_2 and so on. We get $\gamma \leq \gamma_1 \leq \gamma_2 \cdots$. Since all γ_j is in the W-orbits of γ we have finitely of them and this procedure must terminate at some N where γ_N is dominant. Since γ_N is chamber weight so it must be some fundamental weight.

□

Proof of lemma 2. By previous lemma and the lemma holds for all fundamental weights, it suffices to show that if the lemma holds for β , it holds for $s_i\beta$ when $\langle \gamma, s_i \rangle \leq 0$. Let $\gamma = s_i\beta$. Apply prop 4.1 in [BK12]. Then $D_\gamma(M) = D_{s_i(s_i\gamma)}(M) = D_{s_i\gamma}(\Sigma_i M)$, where Σ_i is the reflection functor defined in section 2.2 in [BK12].
Let $A = \{j \mid j$ is adjacent to i, $j \in I\}$ and $M_A = \oplus_{s\in A} M_s$. The i^{th} component of $\Sigma_i M$ is the kernel of the map ξ (Still see section 2.2 in [BK12] for the definition of ξ) from M_A to M_i.

Recall we denote by I_γ^+ the subset of I containing all i such that $\langle \gamma, \check{\alpha_i} \rangle$ is positive and by I_γ^- containing all i $\langle \gamma, \check{\alpha_i} \rangle$ is negative. Denote $A_+ = \{j \mid j$ is adjacent to i, $j \in I_\gamma^+\}$ and $A_- = A \setminus A_+$. For a multiset S, let $M_S = \oplus M_s^{m(s)}$. Regarding I_γ^- as a multiset by setting $m(i) = \gamma_i^-$, we can rewrite $\oplus_{i\in I_\gamma^-} M_i^{\gamma^-}$ as $M_{I_\gamma^-}$, similarly $\oplus_{i\in I_\gamma^+} M_i^{\gamma^+}$ as $M_{I_\gamma^+}$.

When $\langle \gamma, \alpha_i \rangle = -1$, Apply this lemma to weight β and module $\Sigma_i M$. We have $I_\beta^+ = I_{s_i\gamma}^+ = (I_\gamma^+ \setminus A_+) \bigcup \{i\}$ and $I_\beta^- = I_{s_i\gamma}^- = (I_\gamma^- \setminus \{i\}) \bigcup A_-$ as multisets. Therefore $D_{s_i\gamma}(\Sigma_i M)$ is the dimension of the kernel the natural map (which is ϕ_β) from $M_{I_\gamma^+ \setminus A_+} \oplus ker(M_A \xrightarrow{\xi} M_i)$ to $M_{I_\gamma^- \setminus \{i\}} \oplus M_{A_-}$. This is equal to the dimension of the kernel of the natural map from $M_{I_\gamma^+ \setminus A_+} \oplus ker(M_{A_+} \xrightarrow{\xi} M_i)$ to $M_{I_\gamma^+ \setminus \{i\}}$, which is just $ker(M_{I_\gamma}^+ \xrightarrow{\phi_\gamma} M_{I_\gamma^+})$ hence lemma holds for γ.

The case when $\langle \gamma, \alpha_i \rangle = -2$ is similar.

□

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