project report

Fractional step method for the incompressible Navier-Stokes equations

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Contents

1 Equations

The incompressible Navier-Stokes equations are considered. Note that all variables are non-dimensional.

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u} \tag{2}$$

The density is constant and set to unity. The Reynolds number is defined as $\mathrm{Re} = \ell u_{\mathrm{ref}}/\nu$.

1.1 Derivation of the given equations

For the fractional step method, the velocity vectorfield is decomposed into a divergence-free contribution $\mathbf{u}_s$ and an irrotational contribution $\mathbf{u}_i$ (Helmholtz decomposition). In this project, a method by Kim and Moin [1] is used, which uses a preliminary velocity field $\mathbf{u}^*$. This preliminary vectorfield is not divergence-free and therefore does not satisfy the continuity equation. The pressure term in eq. 2 can be interpreted as a projection operator which transforms an arbitary vector field into an divergence free one [1]. Since the preliminary field does not need to be divergence free, this term is temporarily neglected. The time-discretized version of eq. 2 can then be written as

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \underbrace{\frac{1}{2}\left(3\mathbf{H}^n - \mathbf{H}^{n-1}\right)}_{\text{Adams-Bashforth } \mathcal{O}(\Delta t^2)} + \frac{1}{\mathrm{Re}} \underbrace{\frac{1}{2}\nabla^2\left(\mathbf{u}^* + \mathbf{u}^n\right)}_{\text{Crank-Nicholson } \mathcal{O}(\Delta t^2)} \tag{3}$$

where $\mathbf{H} = -(\mathbf{u} \cdot \nabla)\mathbf{u}$ denotes the advection terms.

In order to transform $\mathbf{u}^*$ into a divergence-free vector field, a correctional term $\nabla\phi$ has to be introduced, which substitutes the negeleted pressure term. Note that $\nabla\phi \neq \nabla p$. An implicit time-step from the preliminary velocity field to the velocity field at $n+1$ leads to the equation

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\left(\nabla\phi\right)^{n+1}. \tag{4}$$

The new velocity field satisfies the continuity equation ($\nabla \cdot \mathbf{u}^{n+1} = 0$). Rearranging eq. 4 and evaluating its divergence leads to the following poisson equation.

$$\nabla \cdot \mathbf{u}^{n+1} = \nabla \cdot \left(\mathbf{u}^* - \Delta t \left(\nabla\phi\right)^{n+1}\right)$$

$$\left(\nabla^2\phi\right)^{n+1} = \frac{1}{\Delta t} \nabla \cdot \mathbf{u}^* \tag{5}$$

Adding up eq. 4 to eq. 3 leads to the equation

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = -(\nabla\phi)^{n+1} + \frac{1}{2}\left(3\mathbf{H}^n - \mathbf{H}^{n-1}\right) + \frac{1}{\mathrm{Re}}\frac{1}{2}\nabla^2\left(\mathbf{u}^* + \mathbf{u}^n\right). \tag{6}$$

A direct discretization of the momentum equation with the same discretization schemes and an implicit scheme for the pressure leads to following equation:

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n}}{\Delta t} = -(\nabla p)^{n+1} + \frac{1}{2}\left(3\mathbf{H}^{n} - \mathbf{H}^{n-1}\right) + \frac{1}{\mathrm{Re}}\frac{1}{2}\nabla^2\left(\mathbf{u}^{n+1}+\mathbf{u}^{n}\right). \tag{7}$$

Subtracting eq. 6 from eq. 7 produces

$$(\nabla p)^{n+1} = (\nabla\phi)^{n+1} + \frac{1}{\mathrm{Re}}\frac{1}{2}\nabla^2\left(\mathbf{u}^{n+1}-\mathbf{u}^{*}\right). \tag{8}$$

Substituting $\mathbf{u}^{n+1}-\mathbf{u}^{*}$ with help of eq. 4 results in a correlation between p and ϕ.

$$p = \phi - \frac{1}{\mathrm{Re}}\frac{\Delta t}{2}\nabla^2\phi \tag{9}$$

1.2 Spatial discretization

Helmholtz equation

Equation 3 represents a Helmholtz equation in vector form. In order to discretize the equation, it is split up into its x- and y-component. For the purpose of readability, the derivation of the discretization is documented only for the x-component.

$$\begin{aligned}\frac{u^{*}-u^{n}}{\Delta t} = -\frac{1}{2}&\left(3\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]^{n} - \left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]^{n-1}\right)\\ &+\frac{1}{2}\frac{1}{\mathrm{Re}}\left(\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]^{*}+\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]^{n}\right)\end{aligned} \tag{10}$$

Discretizing eq. 10 with a central scheme on a staggered grid at the grid point $(i+\frac{1}{2}, j)$ produces

$$\begin{aligned}\frac{u^{*}_{i+\frac{1}{2},j}-u^{n}_{i+\frac{1}{2},j}}{\Delta t} = -\frac{1}{2}&\left[3\left(u^{n}_{i+\frac{1}{2},j}\cdot\frac{u^{n}_{i+\frac{3}{2},j}-u^{n}_{i-\frac{1}{2},j}}{2\Delta x}+v^{n}_{i+\frac{1}{2},j}\cdot\frac{u^{n}_{i+\frac{1}{2},j+1}-u^{n}_{i+\frac{1}{2},j-1}}{2\Delta y}\right)\right.\\ &\left.-\left(u^{n-1}_{i+\frac{1}{2},j}\cdot\frac{u^{n-1}_{i+\frac{3}{2},j}-u^{n-1}_{i-\frac{1}{2},j}}{2\Delta x}+v^{n-1}_{i+\frac{1}{2},j}\cdot\frac{u^{n-1}_{i+\frac{1}{2},j+1}-u^{n-1}_{i+\frac{1}{2},j-1}}{2\Delta y}\right)\right]\\ +\frac{1}{2}\frac{1}{\mathrm{Re}}&\left[\left(\frac{u^{*}_{i+\frac{3}{2},j}-2u^{*}_{i+\frac{1}{2},j}+u^{*}_{i-\frac{1}{2},j}}{\Delta x^2}+\frac{u^{*}_{i+\frac{1}{2},j+1}-2u^{*}_{i+\frac{1}{2},j}+u^{*}_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)\right.\\ &\left.+\left(\frac{u^{n}_{i+\frac{3}{2},j}-2u^{n}_{i+\frac{1}{2},j}+u^{n}_{i-\frac{1}{2},j}}{\Delta x^2}+\frac{u^{n}_{i+\frac{1}{2},j+1}-2u^{n}_{i+\frac{1}{2},j}+u^{n}_{i+\frac{1}{2},j-1}}{\Delta y^2}\right)\right].\end{aligned} \tag{11}$$

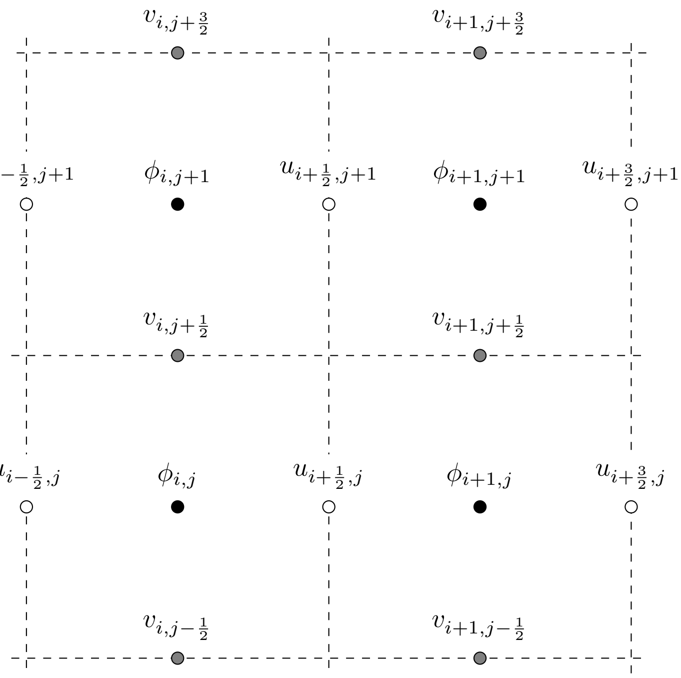

Figure 1: Staggered grid with scalar quantity ϕ located at the cell center and shifted velocity components.

The code is implemented on an uniform grid ($\Delta x = \Delta y$).
x-component:

$$\begin{aligned}
&\left(\frac{1}{\Delta t} + \frac{2}{\mathrm{Re}\,\Delta x^2}\right) u^*_{i+\frac{1}{2},j} - \frac{1}{2\,\mathrm{Re}\,\Delta x^2}\left(u^*_{i+\frac{3}{2},j} + u^*_{i-\frac{1}{2},j} + u^*_{i+\frac{1}{2},j+1} + u^*_{i+\frac{1}{2},j-1}\right) \\
&\quad = -\frac{1}{4\Delta x}\left[3\left(u^n_{i+\frac{1}{2},j}\cdot\left(u^n_{i+\frac{3}{2},j} - u^n_{i-\frac{1}{2},j}\right) + v^n_{i+\frac{1}{2},j}\cdot\left(u^n_{i+\frac{1}{2},j+1} - u^n_{i+\frac{1}{2},j-1}\right)\right)\right. \\
&\qquad \left. - \left(u^{n-1}_{i+\frac{1}{2},j}\cdot\left(u^{n-1}_{i+\frac{3}{2},j} - u^{n-1}_{i-\frac{1}{2},j}\right) + v^{n-1}_{i+\frac{1}{2},j}\cdot\left(u^{n-1}_{i+\frac{1}{2},j+1} - u^{n-1}_{i+\frac{1}{2},j-1}\right)\right)\right] \\
&\quad + \frac{1}{2\,\mathrm{Re}\,\Delta x^2}\left(u^n_{i+\frac{3}{2},j} + u^n_{i-\frac{1}{2},j} + u^n_{i+\frac{1}{2},j+1} + u^n_{i+\frac{1}{2},j-1} - 4u^n_{i+\frac{1}{2},j}\right) + \frac{1}{\Delta t}u^n_{i+\frac{1}{2},j}
\end{aligned} \tag{12}$$

y-component:

$$\begin{aligned}
&\left(\frac{1}{\Delta t} + \frac{2}{\mathrm{Re}\,\Delta x^2}\right) v^*_{i,j+\frac{1}{2}} - \frac{1}{2\,\mathrm{Re}\,\Delta x^2}\left(v^*_{i,j+\frac{3}{2}} + v^*_{i,j-\frac{1}{2}} + u^*_{i+1,j+\frac{1}{2}} + v^*_{i-1,j+\frac{1}{2}}\right) \\
&\quad = -\frac{1}{4\Delta x}\left[3\left(u^n_{i,j+\frac{1}{2}}\cdot\left(v^n_{i+1,j+\frac{1}{2}} - v^n_{i-1,j+\frac{1}{2}}\right) + v^n_{i,j+\frac{1}{2}}\cdot\left(v^n_{i,j+\frac{3}{2}} - v^n_{i,j-\frac{1}{2}}\right)\right)\right. \\
&\qquad \left. - \left(u^{n-1}_{i,j+\frac{1}{2}}\cdot\left(v^{n-1}_{i+1,j+\frac{1}{2}} - v^{n-1}_{i-1,j+\frac{1}{2}}\right) + v^{n-1}_{i,j+\frac{1}{2}}\cdot\left(v^{n-1}_{i,j+\frac{3}{2}} - v^{n-1}_{i,j-\frac{1}{2}}\right)\right)\right] \\
&\quad + \frac{1}{2\,\mathrm{Re}\,\Delta x^2}\left(v^n_{i+1,j+\frac{1}{2}} + v^n_{i-1,j+\frac{1}{2}} + v^n_{i,j+\frac{3}{2}} + v^n_{i,j-\frac{1}{2}} - 4v^n_{i,j+\frac{1}{2}}\right) + \frac{1}{\Delta t}v^n_{i,j+\frac{1}{2}}
\end{aligned} \tag{13}$$

Eq. 12 and eq. 13 contain the values $v_{i+\frac{1}{2},j}$ and $u_{i,j+\frac{1}{2}}$, which are not directly available and therefore have to be interpolated from the surrounding grid points.

$$\begin{aligned}
v_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{1}{2}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}}\right) + \mathcal{O}(\Delta x^2)\\
v_{i+\frac{1}{2},j-\frac{1}{2}} &= \frac{1}{2}\left(v_{i,j-\frac{1}{2}} + v_{i+1,j-\frac{1}{2}}\right) + \mathcal{O}(\Delta x^2)\\
v_{i+\frac{1}{2},j} &= \frac{1}{2}\left(v_{i+\frac{1}{2},j+\frac{1}{2}} + v_{i+\frac{1}{2},j-\frac{1}{2}}\right) + \mathcal{O}(\Delta x^2)
\end{aligned}$$

By combining these expressions, the following second order interpolation equation can be developed (same approach for $u_{i,j+\frac{1}{2}}$).

$$v_{i+\frac{1}{2},j} = \frac{1}{4}\left(v_{i,j+\frac{1}{2}} + v_{i+1,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} + v_{i+1,j-\frac{1}{2}}\right) \tag{14a}$$

$$u_{i,j+\frac{1}{2}} = \frac{1}{4}\left(u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2},j+1} + u_{i-\frac{1}{2},j} + u_{i-\frac{1}{2},j+1}\right) \tag{14b}$$

Poisson equation

Eq. 5 can be written as

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{15}$$

Using a central discretization scheme on a uniform staggered grid leads to the following discretized equation.

$$\frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{\Delta x^2} = \frac{1}{\Delta t}\left(\frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j} + v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{\Delta x}\right) \tag{16}$$

Note that the discretized terms for the partial derivations $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are also of second order, because they represent a central scheme evaluated at the position (i, j). Taking the time discretization into account, the equation can be rearranged to

$$\underbrace{\phi^{n+1}_{i+1,j} + \phi^{n+1}_{i-1,j} + \phi^{n+1}_{i,j+1} + \phi^{n+1}_{i,j-1} - 4\phi^{n+1}_{i,j}}_{\text{unknown variables}} = \frac{\Delta x}{\Delta t}\underbrace{\left(u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j} + v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}\right)}_{\text{known variables}}. \tag{17}$$

Corrector equation

The corrector equation (eq. 4) can be split into two components.

$$u^{n+1} - u^* = -\Delta t\frac{\partial\phi}{\partial x} \tag{18a}$$

$$v^{n+1} - v^* = -\Delta t\frac{\partial\phi}{\partial y} \tag{18b}$$

Discretizing the derivative term with a central scheme evaluated at $(i+\frac{1}{2},j)$ (eq. 18a) and $(i,j+\frac{1}{2})$ (eq. 18b) produces

$$u^{n+1}_{i+\frac{1}{2},j} = u^{*}_{i+\frac{1}{2},j} - \frac{\Delta t}{\Delta x}\left(\phi^{n+1}_{i+1,j} - \phi^{n+1}_{i,j}\right) \tag{19a}$$

$$v^{n+1}_{i,j+\frac{1}{2}} = \underbrace{v^{*}_{i,j+\frac{1}{2}} - \frac{\Delta t}{\Delta y}\left(\phi^{n+1}_{i,j+1} - \phi^{n+1}_{i,j}\right)}_{\text{known variables}} \tag{19b}$$

The discretized corrector equations can be solved directly for the unknown variable and are therefore easy to solve.

1.3 Matrix form

$$\left[\begin{array}{ccccccc:cccccc}
a & b & & \cdots & & & b & b & & & \cdots & & \\
b & a & b & & & & & & b & & & & \\
\vdots & & & \ddots & & & \vdots & \vdots & & & \ddots & & \vdots \\
 & & & & b & a & b & & & & & b & \\
b & & & \cdots & & b & a & & & & \cdots & & b \\
\hdashline
b & & & \cdots & & & & a & b & & \cdots & & b \\
 & b & & & & & & b & a & b & & & \\
 & & & \ddots & & & \vdots & & & & \ddots & & \vdots
\end{array}\right]$$

1.4 Subtasks

For the calculation of the next time step, various subtasks have to be executed. Thanks to the fractional step method, the problem is decoupled and no iteration is needed.

1. Solve the Helmholtz equations (eq. 12, 13) and calculate the preliminary velocity field $\mathbf{u}^*$. The equation correspond to a linear system of algebraic equations.
2. Solve the Poisson equation (eq. 17) to get the values for ϕ.
3. Calculate the new velocity field $\mathbf{u}^{n+1}$ using eq. 18a and eq. 18b.

For the perfomance of these steps, the velocity field at the current time step (n) and the previous time step ($n-1$) has to be available.

To do

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- discretized equations in matrix form with boundary conditions
- first time step (replace Adams-Bashforth)
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2 Stability analysis

In order to estimate the stability limit of the discretization scheme, a von Neumann stability analysis for a prototype PDE is performed. For this purpose, the one-dimensional advection-diffusion equation is considered.

$$u_{,t} = -cu_{,x} + \nu u_{,xx} \tag{20}$$

Applying the discretization schemes yields

$$\begin{aligned} u_i^{n+1} - u_i^n = &-\frac{\sigma_1}{4}\left[3\left(u_{i+1}^n - u_{i-1}^n\right) - \left(u_{i+1}^{n-1} - u_{i-1}^{n-1}\right)\right] \\ &+ \frac{\sigma_2}{2}\left[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} + u_{i+1}^n - 2u_i^n + u_{i-1}^n\right] \end{aligned} \tag{21}$$

with $\sigma_1 = c\frac{\Delta t}{\Delta x}$ and $\sigma_2 = \nu\frac{\Delta t}{\Delta x^2}$.

Fourier transformation and dividing by $e^{Ii\theta}$ leads to

$$\begin{aligned} \hat{u}^{n+1} - \hat{u}^n = &-\frac{\sigma_1}{4}\left[3\hat{u}^n\left(e^{I\theta} - e^{-I\theta}\right) - \hat{u}^{n-1}\left(e^{I\theta} - e^{-I\theta}\right)\right] \\ &+ \frac{\sigma_2}{2}\left[\hat{u}^{n+1}\left(e^{I\theta} + e^{-I\theta} - 2\right) + \hat{u}^n\left(e^{I\theta} + e^{-I\theta} - 2\right)\right]. \end{aligned} \tag{22}$$

Rearranging the equation and dividing by $\hat{u}^n$ produces

$$\underbrace{\frac{\hat{u}^{n+1}}{\hat{u}^n}}_{\equiv G} = \left[1 - \frac{3}{4}\sigma_1 \left(e^{I\theta} - e^{-I\theta}\right) + \frac{1}{2}\sigma_2 \left(e^{I\theta} + e^{-I\theta} - 2\right)\right] + \underbrace{\frac{\hat{u}^{n-1}}{\hat{u}^n}}_{\equiv G^{-1}} \left[\frac{1}{4}\sigma_1 \left(e^{I\theta} - e^{-I\theta}\right)\right]. \tag{23}$$

This equation can be simplified by making use if $e^{I\theta} = \cos(\theta) + I\sin(\theta)$ and $e^{-I\theta} = \cos(\theta) - I\sin(\theta)$.

$$G = \left[1 - \frac{3}{2}\sigma_1 I\sin(\theta) + \sigma_2\left(\cos(\theta) - 1\right)\right] + \frac{1}{G}\left[\frac{1}{2}\sigma_1 I\sin(\theta)\right] \tag{24}$$

Eq. 24 is a quadratic equation in respect of the amplification factor G.

To do

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- solving leads to unhandable equation
- partial analyzation of spatial and temporal discretization?
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References

[1] John Kim and Parviz Moin. Application of a fractional-step method to incompressible Navier-Stokes equations. *Journal of computational physics*, 59(2):308–323, 1985.