Macroeconometrics

Lecture 11 Structural Vector Autoregressions

Tomasz Woźniak

Department of Economics
University of Melbourne

Structural VARs

Identification problem

Identification of the monetary policy shock using exclusion restrictions

Identification using exclusion restrictions

Other ways of identifying structural shocks

Compulsory readings:

Kilian & Lütkepohl (2017) Chapter 8: Identification by Short-Run Restrictions, Structural Vector Autoregressive Analysis

Useful readings:

Rubio-Ramírez, Waggoner & Zha (2010) Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference, Review of Economic Studies

Modeling Effects of Monetary Policy

Objectives.

- To introduce SVARs – a basic tool of empirical analyses
- To analyse the identification of the monetary policy shock
- To present the identification problem of SVARs

Learning outcomes.

- Understanding various forms of SVAR models
- Checking the identification of SVARs with exclusion restrictions
- Working with rotation and orthogonal matrices

Structural VARs

Structural VARs

$$B_0 y_t = b_0 + B_1 y_{t-1} + \cdots + B_p y_{t-p} + u_t$$
$$u_t | Y_{t-1} \sim iid(\mathbf{0}_N, I_N)$$

B_0 – $N \times N$ matrix of contemporaneous relationships also called structural matrix

It captures contemporaneous relationships between variables

u_t – $N \times 1$ vector of conditionally on Y_{t-1} orthogonal or independent structural shocks

Isolating these shocks allows us to identify dynamic effects of uncorrelated shocks on variables y_t

Structural Form (SF) model

The SVAR above is called a structural form model

Structural VARs

Premultiply the SVAR equation by B_0^{-1}

$$y_t = B_0^{-1} b_0 + B_0^{-1} B_1 y_{t-1} + \cdots + B_0^{-1} B_p y_{t-p} + B_0^{-1} u_t$$

to obtain a model in a form that uses the autoregressive parameters of the VAR

$$y_t = \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B_0^{-1} u_t$$

and a different formulation of the SF model

$$y_t = \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + B u_t$$

Structural VARs

$$
\begin{aligned}
y_t &= \mu_0 + A_1 y_{t-1} + \dots + A_p y_{t-p} + B u_t \\
u_t | Y_{t-1} &\sim iid(\mathbf{0}_N, I_N)
\end{aligned}
$$

$B = B_0^{-1}$ – contemporaneous effects matrix

It captures contemporaneous effects of shocks on variables y_t

$A_i = B_0^{-1} B_i$ – autoregressive slope coefficients for $i = 1, \dots, p$

$\mu_0 = B_0^{-1} b_0$ – a constant term

Structural VARs

Reduced Form (RF) representation

$$
\begin{aligned}
y_t &= \mu_0 + A_1 y_{t-1} + \cdots + A_p y_{t-p} + \epsilon_t \\
\epsilon_t | Y_{t-1} &\sim iid(\mathbf{0}_N, \Sigma)
\end{aligned}
$$

Either of the SF models lead to the same RF representation through various equivalence transformations

$$
\begin{aligned}
\epsilon_t &= B u_t = B_0^{-1} u_t \\
B_0 \epsilon_t &= u_t \\
\Sigma &= BB' = B_0^{-1} B_0^{-1\prime}
\end{aligned}
$$

These SF models have the same value of the likelihood function

Structural VARs

Observational Equivalence

Structural models that lead to exactly the same value of the likelihood function are called observationally equivalent

$$L(B_+, B_0|Y, X) = L(A, B|Y, X) = L(A, \Sigma|Y, X)$$

$$\underset{(N\times K)}{B_+} = \begin{bmatrix} b_0 & B_1 & \dots & B_p \end{bmatrix}$$
$$\underset{(N\times K)}{A} = \begin{bmatrix} \mu_0 & A_1 & \dots & A_p \end{bmatrix}$$

Identification problem

Identification problem

Estimation

To estimate an SF model utilize the information from an easy to estimate RF model and the parameter transformations

Given B_0 it is straightforward to compute autoregressive parameters by

$$B_i = B_0 A_i \text{ for } i = 1, \ldots, p$$
$$b_0 = B_0 \mu_0$$

Estimation of the structural matrix relies on the system of equations

$$\boldsymbol{\Sigma} = B_0^{-1} B_0^{-1\prime}$$

Identification problem

Problem 1. Insufficient information

$$\Sigma = B_0^{-1} B_0^{-1\prime}$$

Σ is a symmetric matrix and has $N(N+1)/2$ unique elements
– number of equations

B_0 has N^2 elements: the system has N^2 unknowns

B_0 and u_t are not identified

Identification problem

Problem 2. Identification up to a rotation matrix

Let $\tilde{B}_0 = QB_0$ where Q is an $N \times N$ orthogonal matrix such that $Q'Q = I_N$

$$\begin{aligned}\boldsymbol{\Sigma} &= \tilde{B}_0^{-1}\tilde{B}_0^{-1\prime} \\ &= (QB_0)^{-1}(QB_0)^{-1\prime} \\ &= B_0^{-1}Q^{-1}Q^{-1\prime}B_0^{-1\prime} \\ &= B_0^{-1}(Q'Q)^{-1}B_0^{-1\prime} \\ &= B_0^{-1}B_0^{-1\prime}\end{aligned}$$

Premultiplying the SF model by an orthogonal matrix Q does not change the value of the likelihood function – it leads to an observationally equivalent representation

SF models are often identified up to an orthogonal matrix that is a rotation matrix

Identification problem

Problem 2. Identification up to a rotation matrix

Premultiplying the SF model by a rotation matrix Q leads to observationally equivalent SF representation

$$L(QB_+, QB_0|Y, X) = L(A, BQ'|Y, X) = L(A, \Sigma|Y, X)$$

SF models are identified up to a rotation matrix

Various ways of identifying SVARs set the type of the rotation matrix

Orthogonal matrix

Let $\mathcal{O}(N)$ denote a set of $N \times N$ orthogonal matrices such that $Q \in \mathcal{O}(N)$

Properties.

$$
\begin{aligned}
QQ' &= Q'Q = I_N \\
Q_{[n\cdot]}Q'_{[n\cdot]} &= Q'_{[\cdot n]}Q_{[\cdot n]} = 1 \\
Q' &= Q^{-1} \\
\det(Q) &= \pm 1
\end{aligned}
$$

Rotation matrix

Definition.

A square matrix Q of order N is a rotation matrix if for given $r, s : r < s < N$

$$
\begin{aligned}
Q_{rr} &= Q_{ss} = \cos(x) \\
Q_{ii} &= 1 \text{ for } i = 1, \ldots, N \text{ and } i \neq r, s \\
Q_{sr} &= -\sin(x) \\
Q_{rs} &= \sin(x)
\end{aligned}
$$

and all other elements are zero. Other rotations are obtained by multiplying a sequence of rotation matrices.

Examples of rotation matrices.

$$
\begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \qquad \begin{bmatrix} \cos(x) & 0 & -\sin(x) \\ 0 & 1 & 0 \\ \sin(x) & 0 & \cos(x) \end{bmatrix}
$$

D – a diagonal matrix with ± 1 on the main diagonal

P – a permutation matrix with a single 1 in each column and row and zeros elsewhere

Identification of the monetary policy shock using exclusion restrictions

Identification of the monetary policy shock

$$\Sigma = B_0^{-1} B_0^{-1\prime}$$

At least $N(N-1)/2$ restrictions on B_0 are needed to identify the system

Impose exclusion (zero) restrictions to

- obtain the identification of the system: shocks u_t and matrix B_0
- assign shocks economic interpretation

Identification of the monetary policy shock

Monetary policy shock.
is often defined...

- as an unanticipated part of the monetary policy
- as an orthogonal shock to the monetary policy instrument
- as an orthogonal shock to the short-run nominal interest rate i_t
- through a Taylor's rule type relationship to the output gap $\tilde{y}_t$ and inflation's deviation from its target value π_t in which all of the variables are treated as endogenous

$$i_t = r^n + \phi_\pi \pi_t + \phi_y \tilde{y}_t + u_t^{(mp)}$$

  r^n is a natural rate of interest
- through a Taylor's rule using real output $rgdp_t$ and prices p_t

Identification of the monetary policy shock

To represent identifying restrictions consider a simplified system

$$B_0 y_t = u_t$$

Monetary policy shock.

$$\begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{bmatrix} p_t \\ rgdp_t \\ i_t \\ m_t \end{bmatrix} = \begin{bmatrix} u_{1.t}^{(as)} \\ u_{2.t}^{(ad)} \\ u_{3.t}^{(mp)} \\ u_{4.t}^{(md)} \end{bmatrix}$$

(as) – aggregate supply shock (ad) – aggregate demand shock
(mp) – monetary policy shock (md) – money demand shock

Shocks can be given economic interpretations thanks to the structure imposed on the model in the form of zero restrictions

Identification using exclusion restrictions

Based on Rubio-Ramírez, Waggoner & Zha (2010)
The material in this section presumes normalized systems

Identification using exclusion restrictions

Definitions.

A parameter point (B_+, B_0) is globally identified if and only if there is no other parameter point that is observationally equivalent.

A parameter point (B_+, B_0) is locally identified if and only if there is an open neighbourhood about (B_+, B_0) containing no other observationally equivalent parameter point.

A parameter point (B_+, B_0) is partially identified that is the nth equation is globally identified at the parameter point (B_+, B_0) if and only if there does not exist another observationally equivalent parameter point $(\tilde{B}_+, \tilde{B}_0)$ such that $B_{+[n\cdot]} \neq \tilde{B}_{+[n\cdot]}$ and $B_{0[n\cdot]} \neq \tilde{B}_{0[n\cdot]}$, where $X_{[n\cdot]}$ is the nth row of matrix X.

Identification using exclusion restrictions

General form of restrictions.

$$\mathbf{R}_n f\left(B_+, B_0\right) e_n = \mathbf{0}_{R\times 1} \qquad \text{for } n = 1, \dots, N$$

$f\left(B_+, B_0\right)$ – $R \times N$ matrix of functions of parameters to be restricted, e.g.:

$f\left(B_+, B_0\right) = B_0'$ – restrictions on contemporaneous relationships

$f\left(B_+, B_0\right) = B_0^{-1}$ – restrictions on contemporaneous effects

$\mathbf{R}_n$ – $R \times R$ matrix with ones and zeros such that $\text{rank}\left(\mathbf{R}_n\right) = r_n$

Assume that $r_1 \geq r_2 \geq \dots \geq r_N$

e_n – the nth column of I_N

Identification using exclusion restrictions

Example.

Consider the restrictions on the second row of B_0 from slide 21

$$f\left(B_+, B_0\right) = B_0'$$

$$e_2 = (0, 1, 0, 0)'$$

$$\mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\mathbf{R}_n B_0' e_n = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \\ b_{24} \end{bmatrix} = \begin{bmatrix} b_{23} \\ b_{24} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Identification using exclusion restrictions

Conditions for $f\left(B_{+}, B_{0}\right)$

admissible $f\left(QB_{+}, QB_{0}\right) = f\left(B_{+}, B_{0}\right) Q'$
continuously differentiable $\operatorname{rank}\left[f'\left(B_{+}, B_{0}\right)\right] = RN$
strongly regular see Rubio-Ramírez, Waggoner & Zha (2010)

Identification using exclusion restrictions

Rank conditions.

The identification results are stated as rank conditions for matrix:

$$\underset{(R+n)\times N}{\mathbf{M}_n[X]} = \begin{bmatrix} \mathbf{R}_n X \\ I_n \quad \mathbf{0}_{n\times(N-n)} \end{bmatrix} \quad \text{for } n = 1, \ldots, N$$

Identification using exclusion restrictions

The results below are the most useful for non-recursive identification patterns

Results.

Consider parameter point (B_+, B_0) with imposed zero restrictions. If $\mathbf{M}_n \left[f\left(B_+, B_0 \right) \right]$ is of rank N for $n = 1, \ldots, N$, then the SVAR is globally identified at the parameter point (B_+, B_0).

Consider parameter point (B_+, B_0) with imposed zero restrictions. If $\mathbf{M}_i \left[f\left(B_+, B_0 \right) \right]$ is of rank N for $i = 1, \ldots, n$, then the nth row of the SVAR is globally identified at the parameter point (B_+, B_0).

Identification using exclusion restrictions

Example.
Consider the restrictions on B_0 from slide 21

$$
\begin{array}{ccccc}
n = & 1 & 2 & 3 & 4 \\
\mathbf{R}_n = & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{M}_n(B_0') = & \begin{bmatrix} 0 & b_{22} & b_{32} & b_{42} \\ 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & 0 & b_{44} \\ 1 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & b_{33} & b_{43} \\ 0 & 0 & 0 & b_{44} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & b_{44} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & I_4 \\
\mathrm{rk}(\mathbf{M}_n(B_0')) = & 4 & 4 & 4 & 4
\end{array}
$$

The model is globally identified

Identification using exclusion restrictions

Exact identification.

The results below provide simplified analysis for triangular identification patterns

Definition.

The SVAR with zero restrictions is exactly identified if and only if, for almost any RF parameter point (A, Σ), there exists a unique structural parameter point (B_+, B_0) such that $\left(B_0^{-1}B_+, B_0^{-1}B_0^{-1\prime}\right) = (A, \Sigma)$

Rank condition.

The SVAR with zero restrictions is exactly identified if and only if $r_n = N - n$ for $n = 1, \ldots, N$.

Identification using exclusion restrictions

Example.

Consider the restrictions on B_0 from slide 21

$$
\begin{array}{lcccc}
n = & 1 & 2 & 3 & 4 \\
\mathbf{R}_n = &
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\text{rank}(\mathbf{R}_n) = & 3 & 2 & 1 & 0
\end{array}
$$

The model is exactly identified

Other ways of identifying structural shocks

Other ways of identifying structural shocks

- flexible exclusion restrictions, e.g., on long-run relationships
- sign restrictions
    - on contemporaneous effects
    - flexible sign restrictions
    - narrative sign restrictions
- using zero and sign restrictions
- using prior distributions
- using non-normal error terms
- using heteroskedastic error terms
- using instrumental variables
- using high-frequency data

Structural Vector Autoregressions

Structural models rely on economic theory that provides additional identifying information

Rank conditions provide necessary and sufficient conditions for global identification of SVARs with zero restrictions

Simple conditions guarantee global identification of triangular systems