Macroeconometrics

Lecture 16 Unobserved Component models

Tomasz Woźniak

Department of Economics
University of Melbourne

A simple Unobserved Component model

Identification and unconditional moments

UC model and Beveridge-Nelson decomposition

Useful readings:

Morley, Nelson, Zivot (2003) Why Are the Beveridge-Nelson and Unobserved-Components Decompositions of GDP so Different? Review of Economics and Statistics

Objectives.

- To present the elements of the unobserved component model
- To provide framework for the business-cycle decomposition
- To look at the identification of the parameters of UC models

Learning outcomes.

- Implementing decomposition into unit-root stationary and non-stationary component
- Specifying the cycle component
- Understanding stochastic trend through the Beveridge-Nelson decomposition

A simple Unobserved Component model

A simple Unobserved Component model

$$
\begin{aligned}
y_t &= \tau_t + \epsilon_t && \text{– measurement equation} \\
\tau_t &= \mu + \tau_{t-1} + \eta_t && \text{– state equation for } \tau_t \\
\alpha_p(L)\epsilon_t &= e_t && \text{– state equation for } \epsilon_t \\
\eta_t | Y_{t-1} &\sim iid\mathcal{N}\left(0, \sigma_\eta^2\right) && \text{– conditional distribution} \\
e_t | Y_{t-1} &\sim iid\mathcal{N}\left(0, \sigma_e^2\right) && \text{– conditional distribution}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_p(L) &= 1 - \alpha_1 L - \dots - \alpha_p L^p && \text{– lag polynomial} \\
\alpha_p(z) &= 0,\ \forall |z| > 1 \text{ and } z \in \mathbb{C} && \text{– stationarity restriction} \\
\sigma_{\eta e} &= \mathbb{C}ov[\eta_t, e_t] && \text{– potential correlation} \\
\theta &= \left(\mu, \alpha_1, \dots, \alpha_p, \sigma_\eta^2, \sigma_e^2\right) && \text{– parameters}
\end{aligned}
$$

A simple Unobserved Component model

y_t – an observation on a scalar random variable at time t

τ_t – trend component – unit-root non-stationary

ϵ_t – unit-root stationary component

η_t, e_t – Gaussian white noise error terms

μ – a drift parameter

Y_{t-1} – information set

A simple Unobserved Component model

Measurement equation.

$$y_t = \tau_t + \epsilon_t$$

Decomposes the process into a unit-root non-stationary and a unit-root stationary component

Contains measurements on variable y_t on the left-hand side of the equation

Includes latent processes τ_t and ϵ_t on the right-hand side

A simple Unobserved Component model

Trend component.

$$\tau_t = \mu + \tau_{t-1} + \eta_t$$
$$\eta_t | Y_{t-1} \sim iid\mathcal{N}\left(0, \sigma_\eta^2\right)$$

Trend process is specified as a Gaussian random walk with drift process

Captures the long-run forecast of y_t at an infinite horizon with the forecast origin t

The drift captures the deterministic time trend slope of y_t and is often specified to change over time to represent a structural break in the deterministic trend or allow y_t to have two unit roots. The extended specification is often given by

$$\tau_t = \mu_t + \tau_{t-1} + \eta_t$$
$$\mu_t = \mu_{t-1} + m_t$$
$$m_t | Y_{t-1} \sim iid\mathcal{N}\left(0, \sigma_m^2\right)$$

A simple Unobserved Component model

Unit-root stationary component.

$$\alpha_p(L)\epsilon_t = e_t$$

$$e_t | Y_{t-1} \sim iid\mathcal{N}\left(0, \sigma_e^2\right)$$

$$\alpha_p(z) = 0, \qquad \forall |z| > 1, \text{ and } z \in \mathbb{C}$$

Captures the deviations from the long-run trend

Exhibits persistence which allows to smoothen the trend

UC model is called a local level model when $p = 0$, or $p = 1$

ϵ_t **is called a cycle** if $p \geq 2$ when it can capture cyclical patterns in ϵ_t

A simple Unobserved Component model

Unobserved component model.

$$\begin{aligned} y_t &= \tau_t + \epsilon_t \\ \tau_t &= \mu + \tau_{t-1} + \eta_t \\ \alpha_p(L)\epsilon_t &= e_t \\ \eta_t | Y_{t-1} &\sim iid\mathcal{N}\left(0, \sigma_\eta^2\right) \\ e_t | Y_{t-1} &\sim iid\mathcal{N}\left(0, \sigma_e^2\right) \end{aligned}$$

Autoregressive model obtained by imposing $\sigma_\eta^2 = 0$

$$\begin{aligned} y_t - \mu t &= \epsilon_t \\ \alpha_p(L)\epsilon_t &= e_t \\ e_t | Y_{t-1} &\sim iid\mathcal{N}\left(0, \sigma_e^2\right) \end{aligned}$$

Identification and unconditional moments

Unconditional moments

Unobserved component model captures a single unit-root of y_t

Unconditional moments for y_t depend on time and the
variances can be shown to not be well-specified
– they take infinite values

Consider a model for the first differences Δy_t

$$\Delta y_t = \Delta \tau_t + \Delta \epsilon_t$$

$$\Delta \tau_t = \mu + \eta_t$$

$$\alpha_p(L) \Delta \epsilon_t = \Delta e_t$$

$$\Delta e_t | Y_{t-1} \sim \mathcal{N}\left(0, 2\sigma_e^2\right)$$

where $\Delta = 1 - L$

Unconditional moments

Rewrite the system as

$$\alpha_p(L)\Delta\epsilon_t = \Delta e_t$$

$$\Delta\epsilon_t = \Delta y_t - \mu - \eta_t$$

Plug in the second equation into the first one and reorganize elements

$$\alpha_p(L)(\Delta y_t - \mu) = \alpha_p(L)\eta_t + e_t - e_{t-1}$$

To obtain the ARIMA($p, 1, \max\{p, 1\}$) representation of the UC model

ARIMA(p,d,q) model is given by

$$\alpha_p(L)\Delta^d(y_t - \mu) = \beta_q(L)\varepsilon_t$$

where d is the integration order $y_t \sim I(d)$ and $\Delta^d = (1 - L)^d$

Unconditional moments

Unconditional mean.

$$\alpha_p(L)(\Delta y_t - \mu) = \alpha_p(L)\eta_t + \Delta e_t$$

$$\alpha_p(L)\mathbb{E}[\Delta y_t - \mu] = \alpha_p(L)\mathbb{E}[\eta_t] + \mathbb{E}[\Delta e_t] \overset{LIE}{=} 0$$

$$\mathbb{E}[\Delta y_t] = \mu$$

μ is the average growth rate of y_t over one period

Identification: matching autocovariances

$$\alpha_p(L)(\Delta y_t - \mu) = \alpha_p(L)\eta_t + \Delta e_t$$

LHS of the equation specifies a standard AR dynamics of $\Delta y_t - \mu$

LHS parameters $\alpha_1, \ldots, \alpha_p$ can be easily estimated

RHS of the equation specifies the MA($\max\{p,1\}$) component
$\beta_{\max\{p,1\}}(L) = 1 + \beta_1 L + \cdots + \beta_{\max\{p,1\}} L^{\max\{p,1\}}$

RHS of the equation is written as

$$rhs_t = \alpha_p(L)\eta_t + e_t - e_{t-1}$$

Derive the relationship between the parameters on the equation above to the unconditional autocovariances of rhs_t for two cases $p = 1$ and $p = 2$ to estimate $\sigma_\eta^2, \sigma_e^2, \sigma_{\eta e}$

Identification: matching autocovariances

Case 1: $p = 1$

$$rhs_t = \eta_t - \alpha_1 \eta_{t-1} + e_t - e_{t-1}$$

$$\gamma_s^r = \mathbb{E}[rhs_t \cdot rhs_{t-s}]$$

$s = 0$
$$\begin{aligned}\gamma_0^r &= \mathbb{E}\left[rhs_t^2\right] = \mathbb{E}\left[(\eta_t - \alpha_1 \eta_{t-1} + e_t - e_{t-1})^2\right] \\ &= \left(1 + \alpha_1^2\right)\sigma_\eta^2 + 2\sigma_e^2 + 2\left(1 + \alpha_1\right)\sigma_{\eta e}\end{aligned}$$

$s = 1$
$$\begin{aligned}\gamma_1^r &= \mathbb{E}\left[rhs_t \cdot rhs_{t-1}\right] \\ &= \mathbb{E}\left[(\eta_t - \alpha_1 \eta_{t-1} + e_t - e_{t-1})(\eta_{t-1} - \alpha_1 \eta_{t-2} + e_{t-1} - e_{t-2})\right] \\ &= -\alpha_1 \sigma_\eta^2 - \sigma_e^2 - \left(1 + \alpha_1\right)\sigma_{\eta e}\end{aligned}$$

$s \geq 2$
$$\gamma_s^r = 0$$

Identification: matching autocovariances

Case 1: $p = 1$

$$\begin{bmatrix} \gamma_0^r \\ \gamma_1^r \end{bmatrix} = \begin{bmatrix} 1+\alpha_1^2 & 2 & 2(1+\alpha_1) \\ -\alpha_1 & -1 & -(1+\alpha_1) \end{bmatrix} \begin{bmatrix} \sigma_\eta^2 \\ \sigma_e^2 \\ \sigma_{\eta e} \end{bmatrix}$$

Autocovariances on the LHS are features of data and the autoregressive parameter α_1

Variances and covariance on the RHS are the parameters of the UC model

The UC model is not identified and one identifying restriction is required

Identifying restriction is often chose to be $\sigma_{\eta e} = 0$ imposing the independence of the shocks of the model

Identification: matching autocovariances

Case 2: $p = 2$

$$rhs_t = \eta_t - \alpha_1\eta_{t-1} - \alpha_2\eta_{t-2} + e_t - e_{t-1}$$

$$\gamma_s^r = \mathbb{E}[rhs_t \cdot rhs_{t-s}]$$

$$\begin{aligned}\gamma_0^r &= \mathbb{E}\left[rhs_t^2\right] = \mathbb{E}\left[(\eta_t - \alpha_1\eta_{t-1} - \alpha_2\eta_{t-2} + e_t - e_{t-1})^2\right] \\ &= \left(1+\alpha_1^2+\alpha_2^2\right)\sigma_\eta^2 + 2\sigma_e^2 + 2\left(1+\alpha_1\right)\sigma_{\eta e}\end{aligned}$$

$$\begin{aligned}\gamma_1^r &= \mathbb{E}\left[rhs_t \cdot rhs_{t-1}\right] \\ &= \mathbb{E}\left[(\eta_t - \alpha_1\eta_{t-1} - \alpha_2\eta_{t-2} + e_t - e_{t-1})(\eta_{t-1} - \alpha_1\eta_{t-2} - \alpha_2\eta_{t-3} + e_{t-1} - e_{t-2})\right] \\ &= \alpha_1\left(\alpha_2 - 1\right)\sigma_\eta^2 - \sigma_e^2 - \left(1+\alpha_1-\alpha_2\right)\sigma_{\eta e}\end{aligned}$$

$$\begin{aligned}\gamma_2^r &= \mathbb{E}\left[rhs_t \cdot rhs_{t-2}\right] \\ &= \mathbb{E}\left[(\eta_t - \alpha_1\eta_{t-1} - \alpha_2\eta_{t-2} + e_t - e_{t-1})(\eta_{t-2} - \alpha_1\eta_{t-3} - \alpha_2\eta_{t-4} + e_{t-2} - e_{t-3})\right] \\ &= -\alpha_2\sigma_\eta^2 - \alpha_2\sigma_{\eta e}\end{aligned}$$

$$\gamma_s^r = 0 \text{ for } s \geq 3$$

Identification: matching autocovariances

Case 2: $p = 2$

$$\begin{bmatrix}\gamma_0^r\\ \gamma_1^r\\ \gamma_2^r\end{bmatrix} = \begin{bmatrix}1+\alpha_1^2+\alpha_2^2 & 2 & 2(1+\alpha_1)\\ \alpha_1(\alpha_2-1) & -1 & -(1+\alpha_1-\alpha_2)\\ -\alpha_2 & 0 & -\alpha_2\end{bmatrix}\begin{bmatrix}\sigma_\eta^2\\ \sigma_e^2\\ \sigma_{\eta e}\end{bmatrix}$$

Autocovariances on the LHS are features of data and

consistent estimates of parameters α_1, α_2 are functions of data only

Variances and covariance on the RHS are the parameters of the UC model

The UC model is identified

Restriction $\sigma_{\eta e} = 0$ is dispensable as it over identifies the model

UC model and Beveridge-Nelson decomposition

Beveridge-Nelson decomposition

Definition.

The BN estimate of trend for a time series integrated of order one, $y_t \sim I(1)$, is defined to be the limiting forecast as horizon goes to infinity adjusted for the mean rate of growth:

$$BN_t = \lim_{h \to \infty} \mathbb{E}_t \left[y_{t+h} - h\mu \right]$$

BN_t is a random walk with the same mean growth rate as the observed series

$y_t - BN_t$ the deviation from trend is a stationary process

Forecasting with UC model

One period ahead.

$$\begin{aligned}\mathbb{E}_t[y_{t+1}] &= \mathbb{E}_t[\tau_{t+1} + \epsilon_{t+1}] \\ &= \mathbb{E}_t[\mu + \tau_t + \eta_{t+1} + \alpha_p(L)^{-1} e_{t+1}] \\ &= \mu + \tau_t + \mathbb{E}_t[\alpha_p(L)^{-1} e_{t+1}]\end{aligned}$$

Two periods ahead.

$$\begin{aligned}\mathbb{E}_t[y_{t+2}] &= \mathbb{E}_t[\tau_{t+2} + \epsilon_{t+2}] \\ &= \mathbb{E}_t[2\mu + \tau_t + \eta_{t+2} + \eta_{t+1} + \alpha_p(L)^{-1} e_{t+2}] \\ &= 2\mu + \tau_t + \mathbb{E}_t[\alpha_p(L)^{-1} e_{t+2}]\end{aligned}$$

Forecasting with UC model

h **periods ahead.**

$$\begin{aligned} \mathbb{E}_t[y_{t+h}] &= \mathbb{E}_t[\tau_{t+h} + \epsilon_{t+h}] \\ &= \mathbb{E}_t[h\mu + \tau_t + \eta_{t+h} + \cdots + \eta_{t+1} + \alpha_p(L)^{-1} e_{t+h}] \\ &= h\mu + \tau_t + \mathbb{E}_t[\alpha_p(L)^{-1} e_{t+h}] \end{aligned}$$

UC model and Beveridge-Nelson decomposition

$$\lim_{h\to\infty} \mathbb{E}_t[y_{t+h} - h\mu] = \tau_t$$

τ_t is interpreted as a trend component in the BN sense

At the limit $\lim_{h\to\infty} \mathbb{E}_t[\alpha_p(L)^{-1} e_{t+h}] = 0$ as the stationarity condition holds and $h \gg p$

Unobserved Component models

- **decompose** the original time series into
  - **unit-root non-stationary trend** that is linked to the BN_t component
  - **unit-root stationary deviation from trend** that may exhibit cyclical patterns of persistence (if $p \geq 2$)
- **belong to a family** of state-space models
- **Its ARIMA representation** facilitates the interpretation and understanding of properties
- **are applied** in particular to modeling trend inflation and to estimation of output gap