Exam 2, Editing time: 90 minutes

1 Ideal Bose gas near Bose–Einstein condensation [4P]

Given is an ideal Bose gas with N particles of mass m in a box with volume V at the temperature T in a thermal equilibrium in d dimensions.

Which of the following parameters have to be increased (+), decreased (–) or have not to be changed at all (0) to effect the following changes:

a) Which change of the given parameters could lead to a Bose–Einstein condensation (**increase** the probability of having a BEC)

b) The chemical potential shall **increase** (not being in BEC phase)

c) The specific heat shall **increase** in the BEC phase.

d) The specific heat shall **increase** not being in the BEC phase.

Change of state:	V	T	λ	Dim d	N
a) + get a BEC					
c) + of μ (not in BEC phase)					
b) + of c_V (in BEC phase)					
c) + of c_V (not in BEC phase)					

2 True or false [3P]

Which of the following statements are correct. **Correct the false statements!**

a) In the Debye model we assume all oscillators to have the same frequency.

b) The critical temperature for ideal non–interacting Bose gases increases with mass (is a BEC for Argon more likely than for Helium?).

c) Below the critical temperature the chemical potential is constant.

d) In the Debye model (3D) the density of frequencies $D(\omega)$ is proportional to $D(\omega) \approx \omega^2$.

e) The Einstein model reproduces the low–temperature behaviour of the specific heat correctly whereas the high temperature behaviour is wrong.

f) There is no Bose–Einstein condensation for a non-interacting Bose gas in 1D or 2D.

3 Answer the questions [4P]

1. Consider a system of 3 non-interacting particles. Denoting partition functions of each particle by Z_1, Z_2 and Z_3, express the total partition function of the system for
   a) distinguishable particles,
   b) indistinguishable particles.
2. In which situations does quantum statistics become important?
3. Write down the Fermi-Dirac distribution function and sketch (**draw!**) the average occupation number n_ε as a function of the single particle energy $\frac{\varepsilon}{\mu}$ for:
   a) temperature T = 0.
   b) small temperature T.
   c) very large temperature T. Which classical theorem is reproduced?
4. What happens to an ideal quantum gas in the limit temperature T $\rightarrow$ 0:
   a) if the particles are fermions?
   b) if the particles are bosons?

4 Gas-Solid phases [4P]

Consider molecules with the same mass m coexist in gas-solid phases, at the temperature T.
Gas phase: First consider the **indistinguishable ideal non–interacting** gas molecules. Assume to have N_g gas molecules in the volume V_g.

a) Calculate the grand partition function Ξ_g for the gas molecules, where the thermal wavelength is $\lambda = h\beta^{1/2}/\sqrt{2\pi m}$, and the chemical potential is μ_g.
Hint: Start to calculate the canonical partition function for one particle ($H_g = \frac{p^2}{2m}$), think about what the canonical partition function for N indistinguishable particles is and construct the grandcanonical partition function. Remember the series expansion $\sum_n \frac{x^n}{n!} = \exp(x)$.

You should end up with $\Xi_g = \exp\left(\frac{V_g}{\lambda^3}\exp(\beta\mu_g)\right)$.

b) Calculate and show that the average number of the gas molecules $\langle N_g\rangle$ is equal to

$$\langle N_g\rangle = \frac{V_g}{\lambda^3}\exp\beta\mu_g$$

Calculate the average energy U_g. For U_g, express it in terms of $\langle N_g\rangle$ and k_BT.

Now consider the molecules that are in the solid phase and **distinguishable** with N_s the number of molecules and V_s the volume. They shall be described as three-dimensional harmonic oscillators with the frequency ω. (i.e. the Einstein solid model with the potential $m\omega^2q^2/2$ and the canonical partition sum for one particle $Z_c(1) = \frac{1}{2\sinh(\beta\hbar\omega/2)} \approx \frac{1}{\beta\hbar\omega}$)

c) Calculate the grand partition function Ξ_s for the solid molecules where the chemical potential is μ_s.

d) Calculate and show that the average number of the solid molecules $\langle N_s \rangle$ is equal to

$$\langle N_s \rangle = \frac{\exp(\beta\mu_s)(\frac{2\pi}{\beta h\omega})^3}{1 - \exp(\beta\mu_s)(\frac{2\pi}{\beta h\omega})^3}$$

Calculate the average energy U_s. For U_s, express it in terms of $\langle N_s \rangle$ and $k_B T$.

Consider the molecules coexist in a gas-solid phase.

e) Calculate and express the density $\rho_g = \langle N_g \rangle / V_g$ of the gas molecules in terms of $\langle N_s \rangle, \beta, h, \omega$ and λ (chemical potential balance $\exp \beta\mu_g = \exp \beta\mu_s$).