

## 4 Volume of a $d$ -dimensional sphere

- a) Calculate the volume  $V_d(R)$  of a  $d$ -dimensional sphere with radius  $R$ . To do that write down the  $d$ -dimensional spherical coordinates and express the searched integral in those. In order to derive an expression for the  $d$ -dimensional spatial angle  $\Omega_d$  examine the  $d$ -dimensional Gauß-integral. Show that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\vec{x}^2} d^d x = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^d = \pi^{d/2} \quad (1)$$

and use its representation in spherical coordinates to derive the spherical angle  $\Omega_d$ .

- b) Peeling a multidimensional orange does not make sense. Argue why! Given is a  $d$ -dimensional orange with radius  $R$ . At which dimension  $d$  is the volume of an outer shell of the orange with a thickness of  $R \cdot 10^{-6}$  a million times bigger than the volume of the inner fruit?

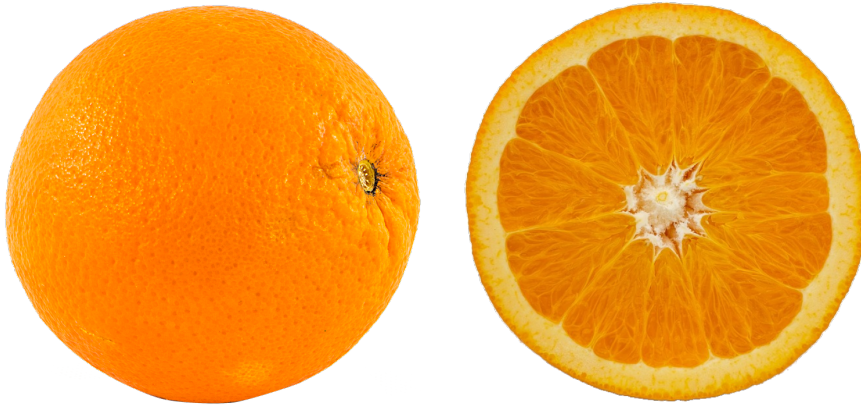


Figure 1: Sketch of a multidimensional orange ( $d = 3$ )<sup>1</sup>.

## 5 Ideal gas - phase space

Given is an ideal gas<sup>2</sup> with  $N$  identical, classical particles with the particle mass  $m$  which are confined in a square box of volume  $V$ . Each particle is characterized by its position  $\mathbf{x}$  and momentum  $\mathbf{p}$ . No external potential (like gravity or electric field) is applied.

- What is a proper description of a microstate of such a particle? How many degrees of freedom are there?

We want to fix another quantity namely the total energy  $E$ .

<sup>2</sup>Particles of an ideal gas are infinitely small (no collisions) and do not interact with each other (no attraction, no van der Waals force, etc.). Compare this concept with the generalization of a real gas (van der Waals gas equation).

- What does this restriction imply on the counting of microstates? How can you relate the total fixed energy  $E$  to the characterization of microstates? (Just calculate the total energy of a given microstate.).
- Write down the expression for the partition sum  $Z(N, V, E)$ . How are discrete ( $N$ ) and continuous scales ( $V, E$ ) are considered?

Since the condition  $E_{\text{microstate}} = E$  is hard to evaluate for a continuous variable we use a trick. Similar to the condition for the volume where the position of all microstates has to be within the volume, we want to count all microstates that have a lower energy than  $E$  using the condition  $E_{\text{microstate}} \leq E$ . We call this quantity volume of the phase space<sup>3</sup>  $F(N, V, E)$

$$F(N, V, E) := \int_0^E Z(N, V, E') dE'$$

and recognise that

$$Z(N, V, E) = \frac{\partial}{\partial E} F(N, V, E). \quad (2)$$

- Use the result for the  $n$ -dimensional sphere to evaluate  $F(N, V, E)$  and calculate  $Z(N, V, E)$ . Keep terms of order  $N$  or higher. Recall the result of Problem 4 of the multidimensional orange. How could you relate  $F$  and  $Z$ ?
- Calculate the entropy  $S(N, V, E)$  of the ideal gas in the box.

## 6 Gibb's paradox

Now assume we have one box which is separated in two volumes  $V_1$  and  $V_2$  which are separated by a virtual wall. The particles can move freely in the whole box which shall be in equilibrium, so the entropy is maximal which is realized by having equal particle densities in two virtual volumes<sup>4</sup>  $\frac{N_1}{V_1} = \frac{N_2}{V_2} =: n$  (for the proof see lecture notes Ch. 1.2.2). Since the densities are the same the average energy per particle is also the same  $\frac{E_1}{N_1} = \frac{E_2}{N_2} =: \tilde{E}$ .

From the lecture we know, that the total entropy of the system should be equal to the sum of the two virtual volumes (it should not make a difference how we combine microstates). Entropy as such should be an extrinsic quantity and as such additive. We want to check this now:

- Calculate the individual entropies  $S_1(N_1, V_1, E_1)$ ,  $S_2(N_2, V_2, E_2)$  and the total entropy  $S(N_1 + N_2, V_1 + V_2, E_1 + E_2)$  and express  $\Delta S = S - S_1 - S_2$ .
- Is the entropy increasing or decreasing when mixing those identical particles?

Experiments show that when mixing identical particles the entropy is not increasing. Theory tells us that the demanded additivity of entropy is violated. This result not gaining the same entropy is called **Gibbs paradox**.

<sup>3</sup>Try to explain the name.

<sup>4</sup>You can imagine the process as follows: Having the whole box in equilibrium take a snapshot of the system and count how many particles  $N_i$  are in which virtual volume  $V_i$ .

**Quantum correction:** One solution to this paradox can be given by quantum physics. Claiming that gas particles are indistinguishable will lead to an extra term that leads to an extensive entropy.

- Reformulate  $Z(N, V, E)$  and  $S(N, V, E)$  for indistinguishable particles and show that entropy is extensive:

$$S(\alpha N, \alpha V, \alpha E) = \alpha S(N, V, E).$$

Another important consequence from quantum physics is **Heisenberg's uncertainty principle** which states that position and momentum cannot be measured at the same time<sup>5</sup>. The uncertainty is quantified by  $\Delta x \Delta p \geq h$ <sup>6</sup> and gives the minimal phase space volume one can resolve. Therefore it is reasonable to divide the whole  $6N$  dimensional phase space volume by  $h^{3N}$ .

This principle comes into play when calculating the phase space.

- Use the quantum correction and the discretization of phase space to derive the entropy  $S(N, V, E)$  for the ideal gas. (Keep terms of the order  $N$ )

This equation is referred to as **Sackur-Tetrode equation**. Compare with the lecture notes (p. 87).

**Mixing entropy of different gases:** Assume the two volumes  $V_1$  and  $V_2$  contain two different gases (e.g. Helium and Neon) which are separated by a wall in the beginning. The densities shall be the same (equal pressure in both volumes).

- Calculate the mixing entropy when removing this wall using the correct quantum corrections.



Gerhard Dorn

<https://creativecommons.org/licenses/by/4.0/legalcode>

<sup>5</sup>When evaluating entropy which is proportional to the logarithm of number of microstates a particle can have, for a continuous scale, it diverges. Before quantizing phase space physicists used regularization techniques to overcome these singularities.

<sup>6</sup>With Planck's constant  $h = 6.62607015 \times 10^{-34} \text{ J} \cdot \text{s}$  (since 20th of May 2019 exact value).