Spectral Graph Theory	May 25, 2021

The Quadratic Form and Standard Random Walk

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1 The Labelling Function

We consider the undirected graph $G = (V, E)$ and it satisfies the following properties:

- The graph is finite.
- Multiple parallel edges and self-loops are allowed.
- vertices of degree 0 are not allowed.

For simplicity, maybe we assume G is regular.

We can label the vertice set V by real numbers:

$$f : V \to \mathbb{R} \equiv \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix}$$

For example, f can be temperature, voltage, coordinate or $0-1$ indicator of $S \subseteq V$.

Remark that we can add or scalar multiply this function.

$$(f+g)(x) = f(x) + g(x),$$

$$c \cdot f(x) = f(c \cdot x).$$

So, $\{f : V \to \mathbb{R}\}$ is a vector space with dimension $n = |V|$.

2 Key to SGT: The Quadratic Form

Definition 1 *The quadratic form is defined to be*

$$\mathcal{E}[f] := \frac{1}{2} \mathop{\mathbb{E}}_{u \sim v} \left[(f(\boldsymbol{u}) - f(\boldsymbol{v}))^2 \right]$$

Where $u \sim v$ denotes we choose a uniform random edge $(u, v) \in E$.

From the definition, we have some facts about the quadratic form.

- $\mathcal{E}[f] \geq 0$.
- $\mathcal{E}[c \cdot f] = c^2 \cdot \mathcal{E}[f]$.

- $\mathcal{E}[f+c] = \mathcal{E}[f]$.

Intuitively, The quadratic form is small if and only if f's value don't vary much along edges.

For example, if we take $S \subseteq V$ and $f = 1_S$ (the indicator function):

$$f(v) = \begin{cases} 1, & \text{if } v \in S \\ 0, & \text{if } v \notin S \end{cases}$$

Then we have:

$$\begin{aligned} \mathcal{E}[f] &= \frac{1}{2} \cdot \mathop{\mathbb{E}}_{u \sim v}[(1_s(u) - 1_s(v))^2] \\ &= \frac{1}{2} \cdot \mathop{\mathbb{E}}_{u \sim v}[1_{\{(u,v) \text{ cross the cut } (S, \bar{S})\}}] \\ &= \frac{1}{2} \cdot \{\text{fraction of edges on } \partial S\} \\ &= \Pr_{u \sim v}[u \to v \text{ is stepping out of } S] \end{aligned}$$

3 Standard Random Walk

Next, we define a distribution over V. To choose a random vertex

- choose a uniform random edge (u, v) (direct).
- output u.

We denote this distribution by π.

Fact 2 $\pi[u]$ *is proportional to deg(u) and*

$$\pi(u) = \frac{deg(u)}{2|E|}.$$

If G is regular, π is a uniform distribution on V.