1. Markov Transition Operator

Recall that

$$\begin{aligned}\mathcal{E}[f] = \frac{1}{2}\mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}\left[(f(\mathbf{u})-f(\mathbf{v}))^2\right] &= \frac{1}{2}\mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}\left[(f(\mathbf{u}))^2\right] + \frac{1}{2}\mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}\left[(f(\mathbf{v}))^2\right] - \mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}[f(\mathbf{u})f(\mathbf{v})] \\ &= \mathop{\mathbb{E}}_{\mathbf{u}\sim\pi}\left[(f(\mathbf{u}))^2\right] - \mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}[f(\mathbf{u})f(\mathbf{v})] \\ &= ||f||_2 - \mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}[f(\mathbf{u})f(\mathbf{v})].\end{aligned}$$

We can write

$$\mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}[f(\mathbf{u})f(\mathbf{v})] = \mathop{\mathbb{E}}_{\mathbf{u}\sim\pi}\mathop{\mathbb{E}}_{\mathbf{v}\sim\mathbf{u}}[f(\mathbf{u})f(\mathbf{v})] = \mathop{\mathbb{E}}_{\mathbf{u}\sim\pi}\left[f(\mathbf{u})\mathop{\mathbb{E}}_{\mathbf{v}\sim\mathbf{u}}[f(\mathbf{v})]\right]$$

Now we can define a new function $Kf : V \to \mathbb{R}$, for $u \in V$, $Kf(u) := \mathop{\mathbb{E}}_{\mathbf{v}\sim u}[f(\mathbf{v})]$.

Fact: $K(f+g) = Kf + Kg$.

Proof: For all $u \in V$,

$$\begin{aligned}K(f+g)(u) &= \mathop{\mathbb{E}}_{\mathbf{v}\sim u}[(f+g)(\mathbf{v})] \\ &= \mathop{\mathbb{E}}_{\mathbf{v}\sim u}[(f(\mathbf{v})+g(\mathbf{v})] \\ &= \mathop{\mathbb{E}}_{\mathbf{v}\sim u}[f(\mathbf{v})] + \mathop{\mathbb{E}}_{\mathbf{v}\sim u}[g(\mathbf{v})] \\ &= Kf(u) + Kg(u).\end{aligned}$$

Definition: The operator K is the Markov Transition Operator(Matrix) for G.

$K_{u,v} = \frac{1}{deg(u)}$ if $(u,v) \in E$; $K_{u,v} = 0$ else. So, K is the adjacency matrix A, normalized so that row-sums are 1.

If G is a d-regular graph then $K = \frac{1}{d}A$, and K is symmetric $K^{\mathrm{T}} = K$.

Fact: For $f, g : V \to \mathbb{R}$, $\langle f, Kg\rangle = \mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}[f(\mathbf{u})g(\mathbf{v})]$.

Proof:

$$\begin{aligned}\langle f, Kg\rangle &= \mathop{\mathbb{E}}_{\mathbf{u}\sim\pi}[f(\mathbf{u})\cdot Kg(\mathbf{u})] \\ &= \mathop{\mathbb{E}}_{\mathbf{u}\sim\pi}\left[f(\mathbf{u})\cdot\mathop{\mathbb{E}}_{\mathbf{v}\sim\mathbf{u}}[g(\mathbf{v})]\right] \\ &= \mathop{\mathbb{E}}_{\mathbf{u}\sim\pi}\mathop{\mathbb{E}}_{\mathbf{v}\sim\mathbf{u}}[f(\mathbf{u})\cdot g(\mathbf{v})] \\ &= \mathop{\mathbb{E}}_{\mathbf{v}\sim\mathbf{u}}[f(\mathbf{u})\cdot g(\mathbf{v})]\end{aligned}$$

Corollary: $\langle f, Kg\rangle = \langle Kf, g\rangle$, which implies K is self-adjoint.

Suppose $S \subseteq V, T \subseteq V$, and $f = 1_S, g = 1_T$, then
$\langle f, Kg\rangle = \mathop{\mathbb{E}}_{\mathbf{u}\sim\mathbf{v}}[1_S(\mathbf{u})1_T(\mathbf{v})] = \mathop{Pr}_{\mathbf{u}\sim\mathbf{v}}[\mathbf{u}\in S \wedge \mathbf{v} \in T]$.

Let ρ be any distribution on vertices, we do:

1. $u \sim \rho$.
2. 1 random step $u \to v$.

If ρ' is the distribution of v then $\rho K = \rho'$.

Corollary: $\pi K = \pi$.

Claim: $K^2 = K \circ K$ operator as $K^2 f(u) = \underset{u \to w \text{ 2 step}}{\mathbb{E}}[f(w)]$

Proof: Given f, let $g = Kf$, then $K^2 f = K(Kf) = Kg$.

For all $u \in V$:

$$(K^2 f)(u) = Kg(u) = \underset{v \sim u}{\mathbb{E}}[g(v)] = \underset{v \sim u}{\mathbb{E}}[Kf(v)] = \underset{v \sim u}{\mathbb{E}}[\underset{w \sim v}{\mathbb{E}}[f(w)]] = \underset{u \to w \text{ 2 step}}{\mathbb{E}}[f(w)].$$

Corollary: $\forall t \in \mathbb{N}$, $(K^t f)(u) = \underset{u \to w \text{ t step}}{\mathbb{E}}[f(w)]$, even when $t = 0$.

2. The Laplacian

Now, we consider the quadratic form $\mathcal{E}[f]$.

$$\begin{aligned}\mathcal{E}[f] = \frac{1}{2}\underset{u \sim v}{\mathbb{E}}\left[(f(u) - f(v))^2\right] &= \langle f, f\rangle - \underset{u \sim v}{\mathbb{E}}[f(u)f(v)] \\ &= \langle f, f\rangle - \underset{u \sim \pi}{\mathbb{E}}\underset{v \sim u}{\mathbb{E}}[f(u)f(v)] \\ &= \langle f, f\rangle - \underset{u \sim \pi}{\mathbb{E}}\left[f(u)\underset{v \sim u}{\mathbb{E}} f(v)\right] \\ &= \langle f, f\rangle - \underset{u \sim \pi}{\mathbb{E}}[f(u) \cdot Kf(u)] \\ &= \langle f, f\rangle - \langle f, Kf\rangle \\ &= \langle f, (I - K)f\rangle\end{aligned}$$

Definition: $L = I - K$ is the (normalized) laplacian operator for G.

For a d-regular graph G, $L = I - \frac{1}{d}A = \frac{1}{d}(dI - A)$.

I.e. $Lf : V \to \mathbb{R}$, $Lf(u) = f(u) - \underset{v \sim u}{\mathbb{E}}[f(v)]$.

Let $S \subseteq V$, $f = 1_S$, then

- $\langle f, Lf\rangle = \mathcal{E}[f] = \underset{u \sim v}{Pr}[u \in S, v \notin S]$.
- $\langle f, f\rangle = \underset{u \sim \pi}{\mathbb{E}}[f(u)^2] = \underset{u \sim \pi}{Pr}[u \in S]$.

$\frac{\langle f, Lf\rangle}{\langle f, f\rangle} = \underset{u \sim v}{Pr}[v \notin S | u \in S] = Pr[\text{Pick a random } u \in S \text{ do 1 step, that you get out of } S] \in [0, 1]$
.

Definition: $\frac{\langle f, Lf\rangle}{\langle f, f\rangle}$ is the conductance $\Phi(S)$.

We want to max $\mathcal{E}[f] = \langle f, Lf\rangle = \frac{1}{2}\underset{u \sim v}{\mathbb{E}}\left[(f(u) - f(v))^2\right]$ subject to
$||f||_2^2 = \langle f, f\rangle = \underset{u \sim \pi}{\mathbb{E}}[f(u)^2] = 1$.

A maximizer exists call it $\varphi : V \to \mathbb{R}$.

Claim: $L\varphi = \lambda\varphi$ for some λ.

Proof: Assume for the sake of contradiction φ and $L\varphi$ are not parallel. Suppose ψ in a unit vector and orthogonal to φ, and let $\epsilon \neq 0$ be small real number. Let $f = \frac{1}{\sqrt{1+\epsilon^2}}(\varphi + \epsilon \cdot \psi)$, we have $||f||_2^2 = \frac{1}{1+\epsilon^2}||\varphi + \epsilon \cdot \psi||_2^2 = \frac{1}{1+\epsilon^2}||\varphi||_2^2 + \epsilon^2 \cdot ||\psi||_2^2 = 1$ by Pythagorus.

$$\begin{aligned}\langle f, Lf \rangle &= \frac{1}{1+\epsilon^2} \langle \varphi + \epsilon \cdot \psi, L\varphi + \epsilon \cdot L\psi \rangle \\ &= \frac{1}{1+\epsilon^2} \left(\langle \varphi, L\varphi \rangle + 2\epsilon \langle \psi, L\varphi \rangle + O(\epsilon^2) \right) \\ &> \langle \varphi, L\varphi \rangle.\end{aligned}$$

If we choose $|\epsilon|$ small enough.

Corollary: $\mathcal{E}[\varphi] = \langle \varphi, L\varphi \rangle = \langle \varphi, \lambda\varphi \rangle = \lambda \langle \varphi, \varphi \rangle = \lambda \in [0, 2]$.

Fact: $\mathop{\mathbb{E}}_{u\sim\pi} [\varphi(u)] = 0$ and $\langle \varphi, \mathbb{1} \rangle = 0$.

Proof of the Fact: Assume $\mathbb{E}[\varphi(u)] \neq 0$, define $f = \varphi - \mathbb{E}[\varphi(u)]$. Notice that $\mathcal{E}[f] = \mathcal{E}[\varphi]$ does not change but $||f||_2^2 = \langle \varphi - \mathbb{E}[\varphi(u)], \varphi - \mathbb{E}[\varphi(u)] \rangle = ||\varphi||_2^2 - \mathbb{E}[\varphi(u)]^2 < ||\varphi||_2^2 = 1$. Let $f' = \frac{f}{||f||_2}$, $||f'||_2^2 = 1$ and $\mathcal{E}[f'] = \frac{1}{||f||_2^2}\mathcal{E}[f] > \mathcal{E}[f] = \mathcal{E}[\varphi]$, which is a contradiction since φ is the maximizer.

Now, we consider Max $\langle f, Lf \rangle$ subject to $||f||_2^2 = 1$ and $f \perp \varphi$. using the same argument above, the maximizing vector φ' having $L\varphi' = \lambda'\varphi'$ for some $\lambda' \leq \lambda$ and $\mathop{\mathbb{E}}_{u\sim\pi} [\varphi'(u)] = 0$ ($\langle \varphi', \mathbb{1} \rangle = 0$). Notice that $L\mathbb{1} = 0$.

Theorem (Spectral Theorem): Given a graph G, there exists orthogonal functions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ with $\varphi_0 \equiv \mathbb{1}$ and real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ with $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2$ such that $L\varphi_i = \lambda_i\varphi_i$.