Sensitivity and block sensitivity

Definition: Given a boolean function $f:\{0,1\}^n \to \{0,1\}$. The local sensitivity $s(f,x)$ on the input x is defined as the number of indices i, such that $f(x) \neq f(x^{\{i\}})$, where $x^{\{i\}}$ is obtained by flipping the i-th bit of x. The sensitivity $s(f) := \max_{x\in\{0,1\}^n} s(f,x)$.

Example: the AND function $f(x_1,\dots,x_n) = x_1 \wedge \dots \wedge x_n$.
$s(f,\vec{0}) = 0, s(f,\vec{1}) = n, s(f) = n$.

Definition: The local block sensitivity $bs(f,x)$ on the input x is the maximum number of disjoint blocks $B_1,\dots,B_k$ of $[n]$, such that for each B_i, $f(x) \neq f(x^{B_i})$. Here x^{B_i} is the n-bit string obtained from x by flipping its coordinates in B_i. The block sensitivity $bs(f) := \max_{x\in\{0,1\}^n} bs(f,x)$.

Example: the AND function $f(x_1,\dots,x_n) = x_1 \wedge \dots \wedge x_n$.
$bs(f,\vec{0}) = 1, bs(f,\vec{1}) = n, s(f) = n$.

Obviously, $bs(f,x) \geq s(f,x)$ for every x and thus $bs(f) \geq s(f)$. But does there exist a function f, such that $bs(f) > s(f)$?

The Rubinstein function

Define $f:\{0,1\}^{n^2} \to \{0,1\}$ as

$$f(x_{11},\cdots,x_{nn}) = \bigvee_{i=1}^{n} g(x_{i1},\cdots,x_{in}),$$

where $g(x_1,\cdots,x_n) = 1$ if and only if $x_j = x_{j+1} = 1$ for some $1 \leq j \leq n-1$, and all other $x_k = 0$.

- $bs(f) \geq bs(f,\vec{0}) = \Omega(n^2)$.
- $s(f) = O(n)$.
  - **Case 1:** $f(x) = 0$

    Every row must output 0, there are at most two sensitive coordinates on each row, say when the row is

    $$0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0.$$

    So, $s(f,x) \leq 2n$.
  - **Case 2:** $f(x) = 1$

    If two rows output 1, $s(f,x) = 0$.

    If only one row outputs 1, $s(f,x) \leq n$.

A quick summary: $s(f) \leq bs(f)$, and Rubinstein's example shows that $bs(f)$ could be quadratic in $s(f)$.

Sensitivity Conjecture[Nisan Szegedy 1992]

For every boolean function f, $bs(f) \leq poly(s(f))$.

Two complexity measures s_1 and s_2 of boolean functions are polynomially related if $\exists C_1, C_2 > 0$, such that for every boolean f:

$$s_2(f)^{C_1} \le s_1(f) \le s_2(f)^{C_2}.$$

The following measures are polynomially related:

- Block sensitivity $bs(f)$.
- Decision tree complexity $D(f)$.
- Certificate complexity $C(f)$.
- Degree (as real polynomial) $deg(f)$.
- Approximate degree $\tilde{deg}(f)$.
- Randomized query complexity $R(f)$.
- Quantum query complexity $Q(f)$.

In some sense, sensitivity measures how "smooth" a boolean function is, with respect to the Hamming distance. Low sensitivity means more smooth. The Sensitivity Conjecture asserts that

- Computationally, "smooth" (low-sensitivity) functions are easy to compute in some of the simplest models like the deterministic decision tree model.
- Algebraically, such functions have low degree.

Bounds proven: (Exponential)

- $bs(f) = O(s(f)4^{s(f)})$. (Simon 1983)
- $bs(f) \le (e/\sqrt{2\pi})e^{s(f)}\sqrt{s(f)}$. (Kenyon, Kutin 2004)
- $bs(f) \le 2^{s(f)-1}s(f)$ (Ambainis, Bavarian, Gao, Mao, Sun, Zuo 2013)

Separations constructed: (Quadratic)

- $bs(f) = \frac{1}{2}s(f)^2$. (Rubinstein 1995)
- $bs(f) = \frac{1}{2}s(f)^2 + s(f)$. (Virza 2011)
- $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{2}s(f)$. (Ambainis, Sun 2011)

The Gotsman-Linial equivalence

Theorem (Gotsman, Linial 1992)

The following are equivalent for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- For any induced subgraph H of Q^n with $|V(H)| \neq 2^{n-1}$, we have

  $$\max\{\Delta(H), \Delta(Q^n - H)\} \ge h(n).$$

  where Δ is the maximum degree of a graph.

- For any boolean function f, we have $s(f) \ge h(deg(f))$.

Showing (i) for $h(n) = n^c$ if and only if Sensitivity conjecture.

The Gostman-Linial correspondence was established via an intermediate statement:

- For any boolean function g of full degree n, $s(g) \ge h(n)$.

The direction we care about is $(i) \Rightarrow (iii) \Rightarrow (ii)$.

Proof of $(i) \Rightarrow (iii)$:

- Suppose there exists $g : \{0,1\}^n \to \{-1,1\}$, with $s(g) < h(n)$, $deg(g) = n$. Let $p(x) = (-1)^{x_1+\cdots+x_n}$.
- Consider the induced subgraph H with vertex set

  $$V(H) = \{x : g(x) \cdot p(x) = 1\}.$$

- Obviously $s(g) = \max\{\Delta(H), \Delta(Q^n - H)\}$, and

$$(|V(H)| - |V(Q^n - H)|)/2^n = E[g(x)p(x)] = \hat{gp}(\phi) = \hat{g}([n]) \neq 0.$$

- The last inequality follows from $deg(g) = n$.

Proof of $(iii) \Rightarrow (ii)$:

Upper and lower bounds

Theorem (Chung, Furedi, Graham, Seymour 1988):

1. Q^n has a $(2^{n-1}+1)$-vertex induced subgraph of maximum degree $\lceil \sqrt{n} \rceil$. (quadratic separation)
2. Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n has maximum degree at least $(1/2 - o(1)) \cdot \log_2 n$. (exponential upper bound)

The main theorem

Theorem (H. 2019+):

- Every $(2^{n-1}+1)$-vertex induced subgraph of Q^n contains a vertex of degree at least $\sqrt{n}$.

Corollary:

- For every boolean function f, $s(f) \geq \sqrt{deg(f)}$

The **sensitivity conjecture** is true.

- $bs(f) \leq 2deg(f)^2$ (Nisan, Szegedy 1992)
  $bs(f) \leq deg(f)^2$ (Tal 2013)
  $bs(f) \leq \sqrt{2/3}deg(f)^2 + 1$ (Wellens 2020)
- These result imply $bs(f) = O(s(f)^4)$.

Proof of the main theorem

Principal Submatrix: Given a $n \times n$ matrix A, a principal submatrix of A is obtained by deleting the same set of rows and columns from A.

Lemma 1 (Cauchy's Interlace Theorem): Let A be a symmetric $n \times n$ matrix, and B be a $m \times m$ principle submatrix of A, for some $m < n$. If the eigenvalues of A are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and the eigenvalues of B are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, then for all $1 \leq i \leq m$,

$$\lambda_i \geq \mu_i \geq \lambda_{i+n-m}.$$

You can find the proof here

Lemma 2: We define a sequence of symmetric square matrices iteratively as follows,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then A_n is a $2^n \times 2^n$ matrix whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.

Proof of Lemma 2:

- We prove by induction that $A_n^2 = nI$.
  - For $n = 1$, $A_1^2 = I$.
  - Suppose the statement holds for $n-1$, that is $A_{n-1}^2 = (n-1)I$, then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = nI.$$

- Therefore, the eigenvalues of A_n are either $\sqrt{n}$ or $-\sqrt{n}$.
- Since $trace[A_n] = 0$, we know that A_n has exactly half of the eigenvalues being $\sqrt{n}$ and the rest being $-\sqrt{n}$.

Lemma 3: Suppose H is an m-vertex undirected graph, and A is a symmetric matrix whose entries are in $\{-1, 0, 1\}$ and whose rows and columns are indexed by $V(H)$, and whenever u and v are non-adjacent in H, $A_{u,v} = 0$. Then

$$\Delta(H) \geq \lambda_1 := \lambda_1(A).$$

Proof of Lemma 3:

- Suppose $\vec{v}$ is the eigenvector corresponding to λ_1, then $\lambda_1 \vec{v} = A\vec{v}$.
- WLOG, assume v_1 is the coordinate of $\vec{v}$ that has the largest absolute value. Then

$$|\lambda_1 v_1| = |(A\vec{v})_1| = \left|\sum_{j=1}^{m} A_{1,j} v_j\right| = \left|\sum_{j\sim 1} A_{1,j} v_j\right| \leq \sum_{j\sim 1} |A_{1,j}||v_1| \leq \Delta(H)|v_1|.$$

- Therefore $|\lambda_1| \leq \Delta(H)$.

Proof of the main theorem:

- Let A_n be the sequence of matrices defined in lemma 2. When we changing every (-1) entry of A_n to 1, we get exactly the adjacency matrix of Q^n, and thus A_n and Q_n satisfy the conditions in lemma 3.
- A $(2^{n-1}+1)$-vertex induced subgraph H of Q^n and the principal submatrix A_H of A_n naturally induced by H alse satisfy the conditions of lemma 3. So,

$$\Delta(H) \geq \lambda_1(A_H).$$

- From lemma 2, the eigenvalues of A_n are known to be

$$\sqrt{n}, \cdots, \sqrt{n}, -\sqrt{n}, \cdots, -\sqrt{n}.$$

- Note that A_H is a $(2^{n-1}+1) \times (2^{n-1}+1)$ submatrix of the $2^n \times 2^n$ matrix A_n. By Cauchy's Interlace Theorem,

$$\lambda_1(A_H) \geq \lambda_{1+2^n-2^{n-1}-1}(A_n) = \lambda_{2^{n-1}}(A_n) = \sqrt{n}.$$

- So, $\Delta(H) \geq \sqrt{n}$. complete our proof.