1. Eigenvalues

Theorem (Spectral Theorem): Given a graph G, there exists orthogonal functions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ with $\varphi_0 \equiv \mathbb{1}$ and real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ with $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 2$ such that $L\varphi_i = \lambda_i \varphi_i$.

Remember that $L = I - K$, K has same eigenvectors as L, and eigenvalues $k_i = 1 - \lambda_i$.

$$-1 \leq k_{n-1} \leq \cdots \leq k_1 \leq k_0 = 1.$$

Given $f : V \to \mathbb{R}$, we can uniquely express as a linear combination of all φ_i.

$$f = \hat{f}(0)\varphi_0 + \hat{f}(1)\varphi_1 + \cdots + \hat{f}(n-1)\varphi_{n-1},$$

where $\hat{f}(i) \in \mathbb{R}$ for all i.

Multiply by L, we have

$$\begin{aligned} Lf &= \lambda_0 \hat{f}(0)\varphi_0 + \lambda_1 \hat{f}(1)\varphi_1 + \cdots + \lambda_{n-1}\hat{f}(n-1)\varphi_{n-1} \\ &= \lambda_1 \hat{f}(1)\varphi_1 + \cdots + \lambda_{n-1}\hat{f}(n-1)\varphi_{n-1} \end{aligned}$$

I.e. $\hat{Lf}(i) = \lambda_i \hat{f}(i)$.

If $g = \hat{g}(0)\varphi_0 + \hat{g}(1)\varphi_1 + \cdots + \hat{g}(n-1)\varphi_{n-1}$, then $\langle f, g \rangle = \sum\limits_{0 \leq i \leq n-1} \hat{f}(i)\hat{g}(i)$.

Corollary:

- $||f||_2^2 = \langle f, f \rangle = \sum\limits_{0 \leq i \leq n-1} \hat{f}(i)^2$.
- $\mathop{\mathbb{E}}\limits_{u \sim \pi}[f(u)] = \langle f, \mathbb{1} \rangle = \langle f, \varphi_0 \rangle = \hat{f}(0)$.
- $Var[f(u)] = ||f||_2^2 - \mathbb{E}[f(u)]^2 = \sum\limits_{0 < i \leq n-1} \hat{f}(i)^2$.
- $\mathcal{E}[f] = \langle f, Lf \rangle = \sum\limits_{0 < i \leq n-1} \lambda_i \hat{f}(i)^2$.

If $Var[f] = 1$,

$$\mathcal{E}[f] = \sum_{0 < i \leq n-1} \lambda_i \hat{f}(i)^2 \geq \lambda_1 \sum_{0 < i \leq n-1} \hat{f}(i)^2 = \lambda_1,$$

the equation holds when $f = \varphi_1$. Indeed, $\min\limits_{f} \left\{ \frac{\mathcal{E}[f]}{Var[f]} \right\} = \lambda_1$.

2. Conductance and Sparse-Cut

Recall: For $S \subseteq V$, the conductance is $\Phi(S) = \mathop{Pr}\limits_{u \sim v}[v \notin S | u \in S] = \frac{\mathcal{E}[\mathbb{1}_S]}{Vol(S)} = \frac{\mathcal{E}[\mathbb{1}_S]}{\mathbb{E}[\mathbb{1}_S]}$.

The "Sparse-Cut" problem is to determine $\Phi_G = \min\limits_{S} \{\Phi(S)\}$, where $0 < vol(S) \leq \frac{1}{2}$.

Consider the following equation

$$\frac{\mathcal{E}[\mathbb{1}_S]}{Var[\mathbb{1}_S]} = \frac{\mathcal{E}[\mathbb{1}_S]}{Var[\mathbb{1}_S]} = \frac{\mathcal{E}[\mathbb{1}_S]}{\mathbb{E}[\mathbb{1}_S^2] - \mathbb{E}[\mathbb{1}_S]^2} = \frac{\mathcal{E}[\mathbb{1}_S]}{\mathbb{E}[\mathbb{1}_S](1 - \mathbb{E}[\mathbb{1}_S])} = \frac{\mathcal{E}[\mathbb{1}_S]}{vol(S) \cdot vol(\bar{S})}.$$

$\frac{1}{2} \leq vol(\bar{S}) < 1$, so $\Phi(S) < \frac{\mathcal{E}[\mathbb{1}_S]}{vol(S) \cdot vol(\bar{S})} \leq 2 \cdot \Phi(S)$. Indeed $\Phi_G < \min_{S, \bar{S} \neq \phi} \left\{ \frac{\mathcal{E}[\mathbb{1}_S]}{Var[\mathbb{1}_S]} \right\} \leq 2 \cdot \Phi_G$.

From last section, we have $\lambda_1 \leq \min_{S, \bar{S} \neq \phi} \left\{ \frac{\mathcal{E}[\mathbb{1}_S]}{Var[\mathbb{1}_S]} \right\}$, put them all, we have the following corollary:

Corollary: $\Phi_G \geq \frac{1}{2}\lambda_1$.

Cheeger's Inequality: $\Phi_G \leq const \cdot \sqrt{\lambda_1}$.

3. Mixing Time

λ_1 "large" ($\lambda_1 \geq \epsilon$) $\Rightarrow$ "fast mixing of random walk".

Recall that K has

- eigenvector $\varphi_0, \ldots, \varphi_{n-1}$ with $\varphi \equiv \mathbb{1}$, and
- eigenvalues $1 = \kappa_0 \geq \kappa_1 \geq \cdots \geq \kappa_{n-1} \geq -1$ and $\kappa_{n-1} = -1$ if and only if G is bipartite.

$\kappa_1 = 1 - \lambda_1 \leq 1 - \epsilon \Rightarrow$ Fast mixing.

Main Theorem: Say $|\kappa_i| \leq 1 - \epsilon$ for all $i > 0$. Then for any worst-case distribution ρ_0 on V, if $u_0 \sim \rho_0$, and $u_0 \to u_1 \to \cdots \to u_t$ is Standard Random Walk, $t \geq const \cdot \frac{\ln(n)}{\epsilon}$, then ρ_t is "very close" to π.

"ρ is close to π" if $f(u) = \frac{\rho[u]}{\pi[u]} \approx 1$ for all $u \in V$, notice $f : V \to \mathbb{R}^{\geq 0}$.

Closeness: $d_{\chi^2}(\rho, \pi) := \mathbb{E}[(f(u) - 1)^2]$.

Fact: For any ρ, $\mathbb{E}[f(u)] = 1$. so, $\hat{f}(0) = 1$.

Proof of the fact: $\mathbb{E}[f(u)] = \sum_u \pi[u] \cdot f(u) = \sum_u \pi[u] \cdot \frac{\rho[u]}{\pi[u]} = \sum_u \rho[u] = 1$.

Lemma:
$d_{\chi^2}(\rho, \pi) := \mathbb{E}[(f(u) - 1)^2] = \mathbb{E}[(f(u) - \mathbb{E}[f(u)])^2] = Var[f(u)] = \sum_{0 < i \leq n-1} \hat{f}(i)^2$.

Proof idea of the main theorem:

Instead of taking $\rho_0, \rho_1, \ldots, \rho_t$ we take $f_0, f_1, \ldots, f_t$.

Say

$$\begin{aligned} f_0 &= \hat{f}(0)\varphi_0 + \hat{f}(1)\varphi_1 + \cdots + \hat{f}(n-1)\varphi_{n-1} \\ &= \mathbb{1} + \hat{f}(1)\varphi_1 + \cdots + \hat{f}(n-1)\varphi_{n-1} \end{aligned}$$

But what is f_1?

Claim: $f_1 = Kf_0$.

Proof of the Claim: For any $u \in V$,

- $f_1(u) = \frac{\rho_1[u]}{\pi[u]} = \frac{\sum_v \rho_0[v] \cdot K_{v,u}}{\pi[u]} = \sum_v \frac{2|E| \cdot \rho_0[v]}{deg(v) \cdot deg(u)}$.
- $Kf_0(u) = \sum_v K_{u,v} f_0(v) = \sum_v \frac{1}{deg(u)} \cdot \frac{\rho_0[v]}{\pi[v]} = \sum_v \frac{2|E| \cdot \rho_0[v]}{deg(u) \cdot deg(v)}$.

Now, we have

$$f_1 = Kf_0 = \mathbb{1} + \kappa_1 \hat{f}(1)\varphi_1 + \cdots + \kappa_{n-1}\hat{f}(n-1)\varphi_{n-1}.$$

Apply K t times, we have

$$f_t = K^t f_0 = \mathbb{1} + \kappa_1^t \hat{f}(1)\varphi_1 + \cdots + \kappa_{n-1}^t \hat{f}(n-1)\varphi_{n-1}.$$

The closeness of ρ_t and π is

$$\begin{aligned} d_{\chi^2}(\rho_t, \pi) = Var[f_t] &= \sum_{1 \le i \le n-1} \kappa_i^{2t} \hat{f}(i)^2 \\ &\le \max\{\kappa_i^{2t}\} \cdot \sum_{1 \le i \le n-1} \hat{f}(i)^2 \\ &= \max\{\kappa_i^{2t}\} \cdot d_{\chi^2}(\rho_0, \pi) \\ &\le (1-\epsilon)^{2t} \cdot d_{\chi^2}(\rho_0, \pi) \le exp(-2t\epsilon) \cdot d_{\chi^2}(\rho_0, \pi). \end{aligned}$$

The "worst" ρ_0 of form $\rho_0[u_0] = 1$ for one $u_0 \in V$, in this case $f_0(u) = \frac{1}{\pi[u_0]}$ if $u = u_0$ and $f_0(u) = 0$ else.

$$d_{\chi^2}(\rho_0, \pi) = Var[f(u)] \le \mathbb{E}[f_0^2] = \pi[u_0] \cdot \frac{1}{\pi[u_0]^2} = \frac{1}{\pi[u_0]} \le 2|E|.$$

Say G is regular, $\frac{1}{\pi[u_0]} = \frac{2|E|}{d} = n.$

So, if we set $t \ge \ln(n)/\epsilon$, $d_{\chi^2}(\rho_t, \pi) \le exp(-2t\epsilon) \cdot n \le n^{-2} \cdot n = 1/n.$