Fourier analysis on the Boolean Function

- We consider the boolean cube $\{-1,1\}^n$ with the uniform measure,
- and study (balanced) Boolean functions $f:\{-1,1\}^n \to \{-1,1\}$.

Definition

- The influence of coordinate $i \in [n]$ is defined as

$$\mathbf{I}_i[f] = \Pr_x[f(x) \neq f(x \oplus e_i)]$$

- The total influence of f is $\mathbf{I}[f] = \mathbf{I}_1[f] + \cdots + \mathbf{I}_n[f]$.

key question

- Suppose $f:\{-1,1\}^n \to \{-1,1\}$ has a small total influence, $\mathbf{I}[f] \leq K$ what can be said about its structure?
- Appears in TCS, learning theory, extremal combinatorics, sharp threshold...

Theorem of KKL and Friedgut

Examples

- Dictatorship: $f(x) = x_1$.
- Juntas: $f(x) = g(x_1, \ldots, x_k)$ for some $g:\{-1,1\}^K \to \{-1,1\}$.

Theorem[Kahn-Kalai-Linial]

- If $\mathbf{I}[f] \leq K$, then there exists $i \in [n]$ such that $\mathbf{I}_i[f] \geq e^{-O(K)}$. (i.e. f resembles as a dictatorship, can it be strengthened?)

Theorem[Friedgut]

- If $\mathbf{I}[f] \leq K$, then f essentially depends on $e^{O(K)}$ coordinates: there exists $S \subseteq [n]$ of size $e^{O(K)}$, and $g:\{0,1\}^S \to \{-1,1\}$ such that $f(x) = g(x_S)$, for $1-\epsilon$ fraction of x's.

Both theorems only meaningful for $K = o(\log n)$. "the logarithmic barrier"

Basics of Fourier analysis

Definition [Inner Product]

- Let $f, g:\{-1,1\}^n \to \mathbb{R}$. Define $\langle f, g\rangle = \mathbb{E}_x[f(x)g(x)]$.

Definition [Characters]

- For each $S \subseteq [n]$, define $\chi_S:\{-1,1\}^n \to \{-1,1\}$, $\chi_S(x) = \Pi_{i \in S} x_i$.

- Fact: $\{\chi_S\}_{S \subseteq [n]}$ is an orthonormal basis for the space of real-valued functions.

Definition [Fourier Decomposition]

- Write $f:\{-1,1\}^n \to \mathbb{R}$ according to the basis of characters: $f(x) = \sum_{S \subseteq [n]} \hat{f}(S)\chi_S(x)$.
- Coefficients are given by $\hat{f}(S) = \langle f, \chi_S\rangle$.

Fact [Parseval/Plancherel]

- For $f, g:\{-1,1\}^n \to \mathbb{R}$ we have that

$$\langle f, g \rangle = \sum_{s \subseteq [n]} \hat{f}(S)\hat{g}(S).$$

- Particular case of interest: $f = g$ is boolean (+1,-1 valued), getting we have that

$$1 = \langle f, f \rangle = \sum_{S \subseteq [n]} \hat{f}(S)^2$$

i.e. $(\hat{f}(S)^2)_{S \subseteq [n]}$ is a distribution.

The Fourier Entropy Conjecture

Morally, If $\mathbf{I}[f] \leq K$, then the Fourier spectrum of f is concentrated on $e^{O(K)}$ coefficients

Conjecture

- There exists an absolute constant $C > 0$, such that $H_{shannon}[\hat{f}^2] \leq C \cdot \mathbf{I}[f]$ for all $f : \{-1, 1\}^n \to \{-1, 1\}$, where $H_{shannon}[\hat{f}^2] = \sum_S \hat{f}(S)^2 \cdot \log\left(1/\hat{f}(S)^2\right)$.

The Fourier Min-Entropy Conjecture

- $H_\infty[\hat{f}^2] \leq C \cdot \mathbf{I}[f]$.
- Clearly weaker than FEC; stronger than [KKL]
- Yet (just as) wide open!

Fourier analytic formulas for influences

Lemma

- Let $f : \{-1, 1\}^n \to \{-1, 1\}$. Then $\mathbf{I}_i[f] = \sum_{i \in S} \hat{f}(S)^2$.

Corollary

- Let $f : \{-1, 1\}^n \to \{-1, 1\}$. Then $\mathbf{I}[f] = \sum_S |S|\hat{f}(S)^2$.

Remark

- Note: function $f : \{-1, 1\}^n \to \{-1, 1\}$ satisfying $\mathbf{I}[f] \leq K$ are approximated by low degree polynomials.
- Indeed, by the corollary and markov's inequality $\sum_{|S| \geq K/\epsilon} \hat{f}(S)^2 \leq \epsilon$.
  - $K \geq \mathbf{I}[f] = \sum_S |S|\hat{f}(S)^2 \geq \sum_{|S| \geq K/\epsilon} (K/\epsilon)\hat{f}(S)^2$,
  - $\sum_{|S| \geq K/\epsilon} \hat{f}(S)^2 \leq \epsilon$.

Main result

Theorem [KKLMS]

- For all boolean, balanced functions $f : \{-1, 1\}^n \to \{-1, 1\}$ it holds that

$$\sum_S \hat{f}(S)^2 \cdot \log\left(1/\hat{f}(S)^2\right) \leq C \cdot \sum_S |S| \log(|S| + 1)\hat{f}(S)^2$$

Theorem [KKLMS]

- For all boolean, balanced functions $f : \{-1, 1\}^n \to \{-1, 1\}$, $\forall T \in \mathbb{N}$,

$$\sum_{|S| \leq T} \hat{f}(S)^2 \cdot \log\left(1/\hat{f}(S)^2\right) \leq C \cdot \sum_{|S| \leq T} |S| \log\left(|S|+1\right) \hat{f}(S)^2 + C \cdot \mathbf{I}[f].$$

Hypercontractivity

Hypercontractivity [Bonami, Bcker, Gross]

- If $p : \{-1,1\}^n \to \mathbb{R}$ multilinear of degree d, then $||p||_4 \leq \sqrt{3}^d \cdot ||p||_2$, where $||p||_q = E[|p(x)|^q]^{1/q}$.

Fact

- Suppose $p : \{-1,1\}^n \to \{-1,0,1\}$ satisfies $Pr[p(x) \neq 0] = \delta$. Then $deg(p) \geq \Omega(\log(1/\delta))$.

Proof

- $||p||_4 = E[|p(x)|^4]^{1/4} = \delta^{1/4}$.
- $||p||_2 = E[|p(x)|^2]^{1/2} = \delta^{1/2}$.
- By hypercontractivity: $||p||_4 \leq \sqrt{3}^{deg(p)} ||p||_2$, this implies that $deg(p) \geq \Omega(\log(1/\delta))$.

(Robust) Fact

- Suppose $p : \{-1,1\}^n \to \{-1,0,1\}$ satisfies $Pr[p(x) \neq 0] = \delta$. Then $\sum_{|S| \leq 0.1 \log(1/\delta)} \hat{p}(S)^2 \leq \delta^{9/8} \ll \delta$.

Proof

- Define the low degree part $g(x) = \sum_{|S| \leq 0.1 \log(1/\delta)} \hat{p}(S) \chi_S(x)$. Then by Plancherel $\sum_{|S| \leq 0.1 \log(1/\delta)} \hat{p}(S)^2 = \langle g, p \rangle$.
- Using Holder and Hypercontractivity $\langle g, p \rangle \leq ||g||_4 ||p||_{4/3} \leq 3^{deg(g)/2} ||g||_2 ||p||_{4/3}$.
- By Parseval, $||g||_2 \leq ||p||_2$.
- $||p||_2 = E[|p(x)|^2]^{1/2} = \delta^{1/2}$.
- $||p||_{4/3} = E[|p(x)|^{4/3}]^{3/4} = \delta^{3/4}$.
- $\left< g,p \right> \leq 3^{deg(g)/2} ||p||_2||p||_{4/3} \leq 3^{1/20 \cdot \log (1/\delta)} \delta^{5/4} \leq e^{(\log 3)/20 \cdot \log (1/\delta)} \delta^{5/4} $ $= \delta^{-(\log 3)/20} \delta^{5/4} \leq \delta^{-1/10} \delta^{5/4} \leq \delta^{-1/8} \delta^{5/4} = \delta^{9/8}$.

Theorem[Kahn-Kalai-Linial]

- If $\mathbf{I}[f] \leq K$, then there exists $i \in [n]$ such that $\mathbf{I}_i[f] \geq e^{-O(K)}$.

Proof of KKL theorem:

- Assume for contradiction that $I_i[f] \leq e^{-100K}$ for all i.
- For each $i \in [n]$, define $\partial_i f : \{-1,1\}^n \to \{-1,0,1\}$ by $\partial_i f(x) = \frac{f(x) - f(x \oplus e^i)}{2}$; note that

$$\partial_i f(x) = \sum_{i \in S} \hat{f}(S) \chi_S(x)$$

- Note that $Pr[\partial_i f(x) \neq 0] = \mathbf{I}_i[f] \leq e^{-100K}$. By the robust fact we get that

$$\sum_{i \in S, |S| \leq 10K} \hat{f}(S)^2 = \sum_{|S| \leq 10K} \hat{\partial_i f}(S)^2 \leq \mathbf{I}_i[f]^{9/8} \leq e^{-100K/8} \mathbf{I}_i[f].$$

- Summing over i gives us that $\sum_{|S|\leq 10K}\hat{f}(S)^2 \leq \sum_{|S|\leq 10K}|S|\hat{f}(S)^2 \leq e^{-100K/8}\sum_i \mathbf{I}_i[f] \leq e^{-100K/8}K \leq 0.1$.
- $\sum_{|S|>10K}\hat{f}(S)^2 \geq 0.9$.
- $\mathbf{I}[f] = \sum_S |S|\hat{f}(S)^2 \geq \sum_{|S|>10K}|S|\hat{f}(S)^2 > \sum_{|S|>10K} 10K\cdot\hat{f}(S)^2 \geq 9K > K$.
- This is a contradiction.

On The Fourier Entropy Conjecture

It's all about correlation inequalities.

Definition: Suppose $g : \{-1,1\} \to \mathbb{R}$, $i \in [n]$, define $\mathbf{I}_i[g] = E_x(\frac{g(x)-g(x\oplus e_i)}{2})^2$.

Lemma: Suppose $f : \{-1,1\}^n \to \{-1,1\}$ is a balanced boolean function, $d \in \mathbb{N}$ and $\mathbf{I}_i[f^{\leq d}] \leq \delta$, then

$$\langle f, f^{\leq d}\rangle \leq \delta^{1/8}\cdot e^{O(d)}\cdot \mathbf{I}[f].$$

The Dream: try to reduce the degree of $f^{\leq d}$ prior to applying hypercontractivity.

Definition: Suppose $f : \{-1,1\}^n \to \mathbb{R}$, $A \subseteq [n]$, $z \in \{-1,1\}^{\bar{A}}$, then $f_{\bar{A}\to z} : \{-1,1\}^A \to \mathbb{R}$ s.t. $f_{\bar{A}\to z}(y) = f(y,z)$.

- $f^{\leq d}(x) = \sum_{|S|\leq d}\hat{f}(S)\prod_{i\in S\cap A}x_i \cdot \prod_{i\in S\cap\bar{A}}x_i$.
- $f^{\leq d}_{\bar{A}\to z}(x) = \sum_{|S|\leq d}\hat{f}(S)\prod_{i\in S\cap\bar{A}}z_i\prod_{i\in S\cap A}x_i$. (the characters from χ_S to $\chi_{S\cap A}$).

Suppose we want to reduce the degree of $f^{\leq d}$ to $v \ll d$. Choose A randomly by including each $i \in [n]$ w.p. $\frac{v}{d}$. Then $|S| \leq d$ implies that $|S\cap A| \lesssim v$.

Assumption: $deg(f_{\bar{A}\to z}) \sim v$.

Lemma: Suppose $f : \{-1,1\}^n \to \{-1,1\}$, $g : \{-1,1\}^n \to \mathbb{R}$ with $deg(g) \leq d$ and $\mathbf{I}_i[f^{\leq d}] \leq \delta$. Then

$$\langle f, g\rangle \leq \delta^{1/8}\cdot e^{O(d)}\cdot \mathbf{I}[f,g],$$

where $\mathbf{I}[f,g] = \sum_{i\in[n]}\sqrt{\mathbf{I}_i[f]\mathbf{I}_i[g]} \leq \sqrt{\mathbf{I}[f]\mathbf{I}[g]}$.

Lemma: Suppose $f : \{-1,1\}^n \to \{-1,1\}$, $g : \{-1,1\}^n \to \mathbb{R}$ with $deg(g) \leq d$ and $\delta > 0$, then

$$\langle f, g\rangle \leq (\frac{1}{\delta})^d \max_S |\hat{f}(S)|\cdot|\hat{g}(S)| + \delta^{1/8}\cdot e^{O(d)}\cdot \mathbf{I}[f,g].$$