## Navier - Stokes Equations

- The Navier-Stokes equations in differential form can be derived by using an infinitesimal control volume to yield,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) + \rho g_x$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + \rho g_y$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

To write the equations in a compact form, it is more convenient to write the terms using Einstien notation and tensor notation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i$$

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}(\rho u_j H) = \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \frac{\partial}{\partial x_j}(u_i \tau_{ij})$$

- where indices that repeat must be expanded from 1 through 3.

- where $u_i$ denotes the velocity component $\vec{u} = [u_1, u_2, u_3]^T$ and $x_i$ stands for the coordinate direction.

- Tensor notation,

① First-order tensors :

$$x_i = [x_1, x_2, x_3]^T = [x, y, z]^T$$
$$u_i = [u_1, u_2, u_3]^T = [u, v, w]^T = \vec{V}$$

② Second-order tensors. Consist of nine component and can be written as 3×3 matrices.

$$u_i u_j \equiv \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix}$$

$$\bar{\bar{\tau}} = \tau_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

③ Einstien summation notation

$$u_i u_i = u_1u_1 + u_2u_2 + u_3u_3$$

$$\frac{\partial}{\partial x_j}(\rho u_j) = \frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) + \frac{\partial}{\partial x_3}(\rho u_3)$$
$$= \nabla \cdot (\rho \vec{V})$$

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \vec{V}$$

- The components of the viscous stress tensor $\tau_{ij}$ is defined by,

$$\tau_{ij} = 2\mu S_{ij} + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

using Stokes hypothesis. $\lambda + \frac{2}{3}\mu = 0$
and is termed the bulk viscosity,
where $\lambda$ = Second viscosity coefficient
and $\mu$ = dynamic viscosity.

- $S_{ij}$ is the strain tensor and are given by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- In the energy equation, the total energy $E$ is defined as

$$E = e + \frac{1}{2} u_i u_i$$

and the total enthalpy $H$ is defined as

$$H = h + \frac{1}{2} u_i u_i$$

In integral form,

$$\frac{\partial}{\partial t}\int_{\Omega}\rho \, d\Omega + \oint_{\delta\Omega}\rho(\vec{V}\cdot\vec{n})\,dS = 0$$

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\vec{V}\,d\Omega + \oint_{\delta\Omega}\rho\vec{V}(\vec{V}\cdot\vec{n})\,dS = \int_{\Omega}\rho\vec{f_e}\,d\Omega - \int_{d\Omega}p\vec{n}\,dS + \int_{\delta\Omega}(\bar{\bar{\tau}}\cdot\vec{n})\,dS$$

$$\frac{\partial}{\partial t}\int_{\Omega}\rho E\,d\Omega + \oint_{\delta\Omega}\rho H(\vec{V}\cdot\vec{n})\,dS$$

$$= \oint_{\delta\Omega}k(\nabla T\cdot\vec{n})\,dS + \int_{\Omega}(\rho\vec{f_e}\cdot\vec{V} + \dot{q}_h)\,d\Omega + \oint_{\delta\Omega}(\bar{\bar{\tau}}\cdot\vec{V})\cdot\vec{n}\,dS$$

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## Reynolds Averaging

- The approach is based on the decomposition of the flow variables into a mean and a fluctuating part.
- The averaging process gives rise to new terms which is interpreted as "apparent" stress gradients.
- Additional equations must be added to the new set of "averaged Navier–Stokes" equations to close the system of equations.
- These new additional equations are called turbulence models.

- There are two ways to average the equations; first, the classical Reynolds averaging; second, mass-weighte[d] averaging suggested by Favre. If density fluctuations can be neglected, the two formulations become identica[l]

- Reynolds-averaging approach defines a time-averaged quantity $\bar{f}$ as

$$\bar{f} \equiv \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} f\, dt$$

  - $\Delta t$ must be larger than the period of the random fluctuations, but small with respect to the time constant for any variations in the flow field such as shedding of vortices, etc.

- We then replace the flow variables with their time averages plus fluctuations.

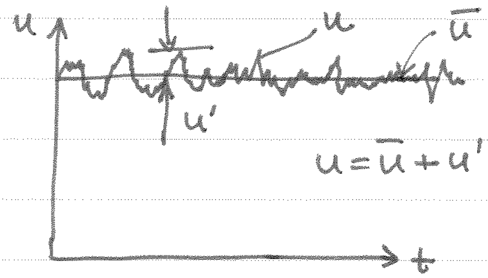
$$u_i = \bar{u}_i + u_i'$$
$$\rho = \bar{\rho} + \rho' \quad , \ldots \text{etc.}$$

Fluctuations in the fluid viscosity, $\mu$, thermal conductivity, $k$, and specific heats, $c_p$ and $c_v$, are usually smaller and are neglected!

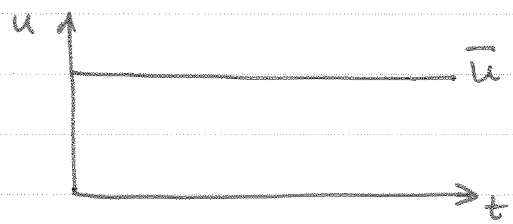
The following relations also hold.

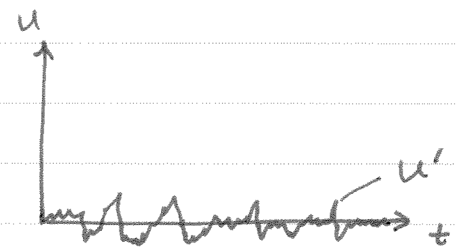
$$\overline{f'} = 0 \qquad \overline{\bar{f}g} = \bar{f}\bar{g} \qquad \overline{f'f'} \neq 0$$
$$\overline{\bar{f}g'} = 0 \qquad \overline{f+g} = \bar{f} + \bar{g}$$

let us understand these relations graphically,



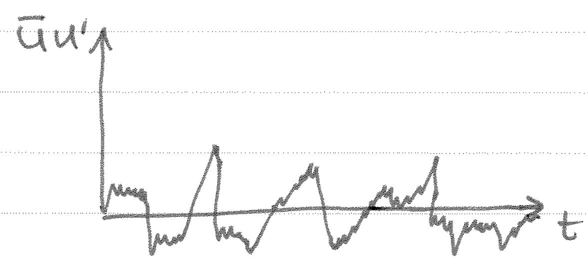

Decompose the plot into $\overline{u}$ and $u'$.




- ~~k~~ $\overline{u'} = 0$ . From the figure on the right its easy to see that averaging the fluctuation would result to 0.

- $\overline{u}u'$ would produce the following figure,



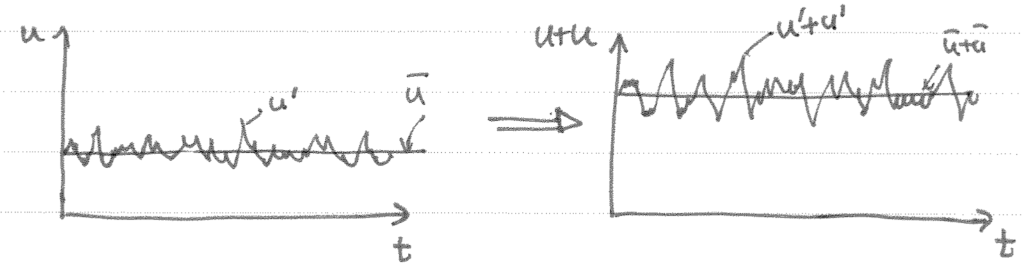

since $\overline{u}$ is a scalar value if the flow is steady then any scalar multiplied by $u'$ would only increase the magnitude of the signal.

hence an average of $\overline{\bar{u}u'}$ should be zero as easily seen in the figure.

– $\overline{\bar{u}u} = \overline{\bar{u}(\bar{u}+u')}$

$= \overline{\bar{u}\bar{u} + \bar{u}u'}$

To solve the above we need to understand that $\overline{f+g} = \bar{f} + \bar{g}$.
Let us look at what happens if $f = u$ and $g = u$, then. $u+u$ can be displayed as.




Its easily seen then that $\overline{u+u} = \bar{u} + \bar{u}$

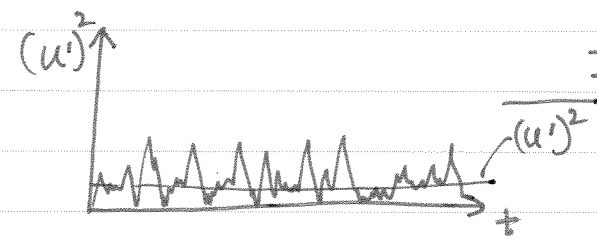
therefore $\overline{\bar{u}u} = \overline{\bar{u}\bar{u} + \bar{u}u'}$

$= \overline{\bar{u}\bar{u}} + \overline{\bar{u}u'}$

$= \bar{u}\bar{u}$

where $\overline{\bar{u}u'} = 0$ as previously stated.

– $\overline{f'g'}$ or $\overline{u'u'} = \overline{(u')^2}$




Its easily seen that the average $\overline{(u')^2} \neq 0$.

- Reynolds-averaging applied to the continuity equations

$$\frac{\partial}{\partial t}(\bar{\rho}+\rho') + \frac{\partial}{\partial x_i}\left[(\bar{\rho}+\rho')(\bar{u}_i+u_i')\right] = 0$$

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\bar{u}_i) + \frac{\partial}{\partial x_i}(\bar{\rho}u_i') + \frac{\partial}{\partial x_i}(\rho'\bar{u}_i)$$

$$+ \frac{\partial}{\partial x_i}(\rho' u_i') = 0$$

Then time-average the entire equation.

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \overline{\rho'}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\bar{\rho}\bar{u}_i}) + \frac{\partial}{\partial x_i}(\overline{\bar{\rho}u_i'}) + \frac{\partial}{\partial x_i}(\overline{\rho'\bar{u}_i})$$

$$+ \frac{\partial}{\partial x_i}(\overline{\rho' u_i'}) = 0$$

From the relations established ~~ong~~ in pages 19-21,

$$\overline{\rho'} = 0$$

$$\overline{\bar{\rho}\bar{u}_i} \neq 0$$

$$\overline{\bar{\rho}u_i'} = 0$$

$$\overline{\rho'\bar{u}_i} = 0$$

$$\overline{\rho' u_i'} \neq 0$$

Then the Reynold averaged form of the continuity equation can be written as,

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\bar{u}_i + \overline{\rho' u_i'}) = 0$$

- For compressible flows, a mass-weighted approach is more convienient.
  - First define mass-averaged variables as,

$$\tilde{f} = \frac{\overline{\rho f}}{\bar{\rho}}$$

Thus $\hat{u}_i = \frac{\overline{\rho u_i}}{\bar{\rho}}$ , $\hat{h} = \frac{\overline{\rho h}}{\bar{\rho}}$ , $\hat{T} = \frac{\overline{\rho T}}{\bar{\rho}}$ , $\hat{H} = \frac{\overline{\rho H}}{\bar{\rho}}$

As listed above, only velocity and thermal variable are mass-averaged! Pressure is not mass-averaged

We now define new fluctuating components,

$$u_i = \tilde{u}_i + u_i'' \quad , \ldots \text{ etc.}$$

Here we use $u_i''$ to distinguish the mass-average fluctuating component to the classical Reynolds averaging approach

Note that unlike previously shown, $\overline{u_i''} \neq 0$ since $\frac{\overline{\rho' u'}}{\bar{\rho}} = \overline{u_i''} \neq 0$ since $\overline{\rho' u'} \neq 0$

However $\overline{\rho f''} = 0$

because

$$\begin{aligned} \overline{\rho f} &= \overline{\rho (\hat{f} + f'')} \\ &= \overline{\rho \tilde{f}} + \overline{\rho f''} \\ &= \overline{\rho \frac{\overline{\rho f}}{\bar{\rho}}} + \overline{\rho f''} \\ &= \bar{\rho} \frac{\overline{\rho f}}{\bar{\rho}} + \overline{\rho f''} \end{aligned}$$

$$\overline{\rho f} = \overline{\rho f} + \overline{\rho f''}$$

hence $\overline{\rho f''} = 0$

and thus $\overline{\rho f} = \overline{\rho \tilde{f}} = \bar{\rho}\tilde{f}$

– From the continuity equations, we can now derive the mass-weighted averaged continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial}{\partial t}(\bar{\rho} + \rho') + \frac{\partial}{\partial x_i}\left[(\bar{\rho}+\rho')(\tilde{u}_i + u_i'')\right] = 0$$

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i}(\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_i}(\bar{\rho}u_i'') + \frac{\partial}{\partial x_i}(\rho'\tilde{u}_i)$$

$$+ \frac{\partial}{\partial x_i}(\rho' u_i'') = 0$$

Then time-averaged the enteire eqation,

$$\frac{\partial \bar{\rho}}{\partial t} + \underbrace{\cancel{\frac{\partial \overline{\rho'}}{\partial t}}}_{0} + \frac{\partial}{\partial x_i}(\overline{\bar{\rho}\tilde{u}_i}) + \frac{\partial}{\partial x_i}(\overline{\bar{\rho}u_i''}) + \underbrace{\cancel{\frac{\partial}{\partial x_i}(\overline{\rho'\tilde{u}_i})}}_{0}$$

$$+ \frac{\partial}{\partial x_i}(\overline{\rho' u_i''}) = 0$$

since $\frac{\partial}{\partial x_i}\left(\overline{\bar{\rho}u_i''} + \overline{\rho' u_i''}\right) = \frac{\partial}{\partial x_i}\left(\overline{\rho u_i''}\right)$

$= 0$

from $\overline{\rho u_i''} = 0$.

Then the mass-weighted continuity equation is

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_i) = 0$$

- Reynolds-averaging applied to the momentum equations.

From page (15), the x-momentum equation is,

$$\frac{\partial}{\partial t}(\rho u_1) + \frac{\partial}{\partial x_1}(\rho u_1 u_1) + \frac{\partial}{\partial x_2}(\rho u_2 u_1) + \frac{\partial}{\partial x_3}(\rho u_3 u_1)$$

$$+ \frac{\partial p}{\partial x_1} = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + \rho g_1$$

Neglect body forces and replace $[x_1, x_2, x_3]^T = [x, y, z]^T$ and $[u_1, u_2, u_3]^T$ with $[u, v, w]^T$

Then,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w) + \frac{\partial p}{\partial x}$$

$$= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

Substitute the flow properties with their Reynolds-average counterpart and time-average the equations

$$\frac{\partial}{\partial t}\left[\overline{(\bar{\rho}+\rho')(\bar{u}+u')}\right] + \frac{\partial}{\partial x}\left[\overline{(\bar{\rho}+\rho')(\bar{u}+u')^2}\right]$$

$$+ \frac{\partial}{\partial y}\left[\overline{(\bar{\rho}+\rho')(\bar{u}+u')(\bar{v}+v')}\right] + \frac{\partial}{\partial z}\left[\overline{(\bar{\rho}+\rho')(\bar{u}+u')(\bar{w}+w')}\right]$$

$$+ \frac{\partial}{\partial x}\left(\overline{\bar{p}+p'}\right) = \frac{\partial}{\partial x}\overline{\tau_{xx}} + \frac{\partial}{\partial y}\overline{\tau_{xy}} + \frac{\partial}{\partial z}\overline{\tau_{xz}}$$

$$\frac{\partial}{\partial t}\left[\bar{\rho}\bar{u} + \overline{\bar{\rho}u'} + \overline{\rho'\bar{u}} + \overline{\rho' u'}\right] + \frac{\partial}{\partial x}\left[\bar{\rho}\bar{u}\bar{u} + \overline{2\bar{\rho}\bar{u}u'} + \overline{\bar{\rho}(u')^2}\right.$$

$$\left. + \overline{\rho'\bar{u}\bar{u}} + \overline{2\rho'\bar{u}u'} + \overline{\rho'(u')^2}\right] + \frac{\partial}{\partial y}\left[\bar{\rho}\bar{u}\bar{v} + \overline{\bar{\rho}\bar{u}v'} + \overline{\bar{\rho}\bar{v}u'}\right.$$

$$\left. + \overline{\bar{\rho}u'v'} + \overline{\bar{u}\bar{v}\rho'} + \overline{\bar{u}\rho'v'} + \overline{\bar{v}\rho'u'} + \overline{\rho' u'v'}\right]$$

$$+ \frac{\partial}{\partial z}\left[\bar{\rho}\bar{u}\bar{w} + \overline{\bar{\rho}\bar{u}w'} + \overline{\bar{\rho}\bar{w}u'} + \overline{\bar{\rho}u'w'} + \overline{\bar{u}\bar{w}\rho'} + \overline{\bar{u}\rho'w'}\right.$$

$$\left. + \overline{\bar{w}\rho'u'} + \overline{\rho'u'w'}\right] + \frac{\partial\bar{p}}{\partial x} + \frac{\partial\overline{p'}}{\partial x} = \frac{\partial\overline{\tau_{xx}}}{\partial x}$$

$$+ \frac{\partial\overline{\tau_{xy}}}{\partial y} + \frac{\partial}{\partial z}\overline{\tau_{xz}}$$

From the above equation any terms that are $\overline{\bar{\rho}u_i'} = 0$ or $\overline{\bar{\rho}\bar{u}_i u_j'} = 0$.

The equation can then be further simplified once $\overline{\tau_{ij}}$ is expanded and simplified.

$$\text{From } \overline{\tau_{xx}} = \overline{2\mu S_{xx}} + \overline{\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

$$= \overline{\mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)} + \overline{\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

$$= 2\mu\overline{\frac{\partial u}{\partial x}} - \frac{2}{3}\overline{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}$$

$$= 2\mu\frac{\partial\bar{u}}{\partial x} - \frac{2}{3}\left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z}\right)$$

$$\text{since } \overline{\frac{\partial u}{\partial x}} = \overline{\frac{\partial}{\partial x}(\bar{u} + u')} = \frac{\partial\bar{u}}{\partial x} + \frac{\partial\overline{u'}}{\partial x} = \frac{\partial\bar{u}}{\partial x}$$

The remaining terms can be written as,

$$\frac{\partial}{\partial t}(\bar{\rho}\bar{u} + \overline{\rho' u'}) + \frac{\partial}{\partial x}(\bar{\rho}\bar{u}\bar{u} + \bar{u}\overline{\rho' u'}) + \frac{\partial}{\partial y}(\bar{\rho}\bar{u}\bar{v} + \bar{u}\overline{\rho' v'})$$

$$+ \frac{\partial}{\partial z}(\bar{\rho}\bar{u}\bar{w} + \bar{u}\overline{\rho' w'}) = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial \bar{u}}{\partial x} - \frac{2}{3}\frac{\partial \bar{u}_k}{\partial x_k}\right) - \bar{u}\overline{\rho' u'} - \bar{\rho}\overline{u' u'} - \overline{\rho' (u')^2}\right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) - \bar{v}\overline{\rho' u'} - \bar{\rho}\overline{u' v'} - \overline{\rho' u' v'}\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right) - \bar{w}\overline{\rho' u'} - \bar{\rho}\overline{u' w'} - \overline{\rho' u' w'}\right]$$

In tensor notation,

$$\frac{\partial}{\partial t}(\bar{\rho}\bar{u}_i + \overline{\rho' u_i'}) + \frac{\partial}{\partial x_j}(\bar{\rho}\bar{u}_i\bar{u}_j + \bar{u}_i\overline{\rho' u_j'})$$

$$= -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \bar{u}_j\overline{\rho' u_i'} - \bar{\rho}\overline{u_i' u_j'} - \overline{\rho' u_i' u_j'}\right)$$

$$\text{where } \bar{\tau}_{ij} = \mu\left[\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right]$$

using the mass-weighted Reynolds-averaging approach yields

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_i\tilde{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''}\right)$$

Note that the equation above is identical to the momentum equation on page 15 with the exception of the additional term $\tau_{ij}^R = -\overline{\rho u_i'' u_j''}$ which constitutes the Reynolds-stress tensor. It represents the transfer of momentum due to the turbulent fluctuations.

Note that $\bar{\rho}\,\overline{u_i'' u_j''}$ comes from the convective accelerati term on the left hand side and hence it represents the transfer of mamentum due to turbulence.

$$\frac{\partial}{\partial x_j}\left(-\bar{\rho}\,\overline{u_i'' u_j''}\right) = \frac{\partial}{\partial x_j}\bar{\tau}_{ij}^{R}$$

is the apparent stress gradients due to transport of momentum by turbulent fluctuations.

The Reynolds energy equation in mass-weighted variables becomes,

$$\frac{\partial}{\partial t}\left(\bar{\rho} c_p \tilde{T}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho} c_p \tilde{T} \tilde{u}_j\right) = \frac{\partial \bar{p}}{\partial t} + \tilde{u}_j \frac{\partial \bar{p}}{\partial x_j} + \overline{u_j'' \frac{\partial p}{\partial x_j}}$$

$$+ \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial \tilde{T}}{\partial x_j} + \kappa \overline{\frac{\partial T''}{\partial x_j}} - c_p \overline{\rho T'' u_j''}\right) + \overline{\tau_{ij} \frac{\partial u_i}{\partial x_j}}$$