Heat Equation

- The one-dimensional Heat equation (diffusion equation) can be written as,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

- The equation is a parabolic PDE.
- For the initial condition $u(x,0) = f(x)$ and boundary cond. $u(0,t) = u(1,t) = 0$, the exact solution is,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha k^2 t} \sin(kx)$$

, where $A_n = 2\int_0^1 f(x) \sin(kx)\,dx$ and $k = n\pi$.

Numerical Methods for the Diffusion Equation

Simple Explicit Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + O[\Delta t, (\Delta x)^2]$$

The schem is stable, when. $0 \leq r \leq \frac{1}{2}$ where $r = \frac{\alpha \Delta t}{(\Delta x)^2}$

The modified equation is,

$$u_t - \alpha u_{xx} = \left[-\frac{1}{2}\alpha^2 \Delta t + \frac{\alpha (\Delta x)^2}{12}\right] u_{xxxx}$$
$$+ \left[\frac{1}{3}\alpha^3 (\Delta t)^2 - \frac{1}{12}\alpha^2 \Delta t (\Delta x)^2 + \frac{1}{360}\alpha (\Delta x)^4\right] u_{xxxxxx}$$
$$+ \cdots$$

- Note that when $r=\frac{1}{6}$, the truncation error becomes $O[(\Delta t)^2, (\Delta x)^4]$.
- Also note that there are no odd derivatives, which signifies that there are no dispersive errors.

  If we look at the amplication factor, $G = 1+2r(\cos\beta - 1)$ which has no imaginary part; since $\phi = \tan^{-1}\left[\frac{Img}{Real}\right]$

  and $\phi = \tan^{-1}\left[\frac{0}{Real}\right] = 0$ and hence there are no dispersive errors since there is no phase shift.
- At larger values of r, (closer to $\frac{1}{2}$), the scheme is highly dissipative.

Crank-Nidson Method (Implicit Scheme)

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha \frac{\delta_x^2 u_j^n + \delta_x^2 u_j^{n+1}}{2(\Delta x)^2}$$

There are a number of different ways to derive the Crank-Nicol. scheme. The best way is to integrate in time both sides of the equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$u(x, t+\Delta t) - u(x,t) = \alpha \int_t^{t+\Delta t} \frac{\partial^2 u}{\partial x^2}(x, s)\, ds.$$

Then using the trapezoid rule to evaluate the integral on the right hand side,

$$u(x, t+\Delta t) - u(x,t) = \frac{1}{2}\Delta t\,\alpha\left(u_{xx}(x,t) + u_{xx}(x, t+\Delta t)\right) - \frac{1}{12}(\Delta t)^3 u_{xxtt}(x, \theta_t)$$

If we replace the spatial derivatives with difference quotients, we get the expression,

$$u(x,t+\Delta t) - u(x,t) = \frac{1}{2}(\Delta t)\alpha\left[\frac{u(x+\Delta x, t+\Delta t) - 2u(x,t+\Delta t) + u(x-\Delta x, t+\Delta t)}{h^2}\right]$$

$$+ \frac{1}{2}(\Delta t)\alpha\left[\frac{u(x+\Delta x, t+\Delta t) - 2u(x,t+\Delta t) + u(x-\Delta x, t+\Delta t)}{h^2}\right]$$

$$+ \frac{1}{24}(\Delta x)^2(\Delta t)\alpha\left[\frac{\partial^4 u}{\partial x^4}(\eta_1, t+\Delta t) + \frac{\partial^4 u}{\partial x^4}(\eta_0, t)\right]$$

$$- \frac{1}{12}(\Delta t)^3 u_{xxtt}$$

In numerical form,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \alpha\,\frac{1}{2}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right]$$

- The algorithm is unconditionally stable and is $O[(\Delta t)^2, (\Delta x)^2]$
- The amplification factor is

$$G = \frac{1 - r(1-\cos\beta)}{1 + r(1-\cos\beta)}$$

The scheme can be written as in matrix - vector form,

$$(I + rK)u_j^{n+1} = (I - rK)u_j^n$$

where
$$K = \begin{bmatrix} +2 & -1 & 0 & \cdots & & & \\ -1 & 2 & -1 & \cdots & & & \\ 0 & -1 & 2 & -1 & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix}.$$

Keller Box method

- Its an implicit second order in time and space method.
- The scheme is derived in the following manner.
- For the 1D-heat equation, define $v = \frac{\partial u}{\partial x}$, so that the 2nd. order equation can be written as

$$\frac{\partial u}{\partial x} = v$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial v}{\partial x}$$

- A finite-difference scheme can then be written as,

$$u_{j-\frac{1}{2}}^{n+1} = \frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x_j}$$

$$\frac{u_{j-\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n}}{\Delta t_{n+1}} = \frac{\alpha\left(v_j^{n+1/2} - v_{j-1}^{n+\frac{1}{2}}\right)}{\Delta x_j}$$

Here the $\frac{1}{2}$ are defined as averages,

$$u_{j-\frac{1}{2}}^{n+1} = \frac{u_j^{n+1} + u_{j-1}^{n+1}}{2}, \quad v_j^{n+\frac{1}{2}} = \frac{v_j^n + v_j^{n+1}}{2}$$

If we substitute these equations into the FDEs, then the new difference equation can be written as

$$\frac{u_j^{n+1} - u_{j-1}^{n+1}}{\Delta x_j} = \frac{v_j^{n+1} + v_{j-1}^{n+1}}{2}$$

$$\frac{u_j^{n+1} + u_{j-1}^{n+1}}{\Delta t} = \alpha \frac{v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x_j} + \frac{u_j^{n} + u_{j-1}^{n}}{\Delta t} + \alpha \frac{v_j^{n} - v_{j-1}^{n}}{\Delta x_j}$$

Then the Keller box method can be written in matrix-vector form as,

$$[B] F_{j-1}^{n+1} + [D] F_j^{n+1} + [A] F_{j+1}^{n+1} = C$$

where $F = \begin{bmatrix} u \\ v \end{bmatrix}$ and B, D, A are 2×2 matrices.

- The system of equations is a block tridiagonal system.

Schemes for Two-Dimensional Heat Equation

The two-dimensional heat equation can be written as,

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

If we apply the simple explicit scheme used for the 1D equation, then the numerical scheme may be written as,

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \alpha \left[\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2} \right].$$

The stability of this scheme is, $\alpha (\Delta t) \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right] \leq \frac{1}{2}$

If $\Delta x = \Delta y$, then $\alpha(\Delta t)\left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta x)^2}\right] \leq \frac{1}{2}$

$$\frac{2\alpha(\Delta t)}{(\Delta x)^2} \leq \frac{1}{2}$$

$$r = \frac{\alpha(\Delta t)}{(\Delta x)^2} \leq \frac{1}{4},$$

Hence the stability limit reduces from, $\frac{1}{2}$ to $\frac{1}{4}$. This in turn reduces the (Δt) and renders the scheme impractical.

Crank–Nicolson for 2D Heat Eqn

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \frac{\alpha}{2}\left(\hat{\delta}_x^2 + \hat{\delta}_y^2\right)\left(u_{ij}^{n+1} + u_{ij}^{n}\right)$$

where $\hat{\delta}_x^2 u_{ij}^n = \dfrac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{(\Delta x)^2}$

$$\hat{\delta}_y^2 u_{ij}^n = \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{(\Delta y)^2}$$

- The scheme is unconditionally stable in 2D as well.
- However in 2D, the scheme results into a penta-diagonal system of equations.

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \frac{\alpha}{2}\hat{\delta}_x^2 u_{ij}^{n+1} + \frac{\alpha}{2}\hat{\delta}_x^2 u_{ij}^{n} + \frac{\alpha}{2}\hat{\delta}_y^2 u_{ij}^{n+1} + \frac{\alpha}{2}\hat{\delta}_y^2 u_{ij}^{n}$$

$$= \frac{\alpha}{2}\left[\frac{u_{i+1,j}^{n+1} - 2u_{ij}^{n+1} + u_{i-1,j}^{n+1}}{(\Delta x)^2}\right] + \frac{\alpha}{2}\left[\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2}\right]$$

$$+\frac{\alpha}{2}\left[\frac{u_{i,j+1}^{n+1}-2u_{ij}^{n+1}+u_{i,j-1}^{n+1}}{(\Delta y)^2}\right]+\frac{\alpha}{2}\left[\frac{u_{i,j+1}^{n}-2u_{ij}^{n}+u_{i,j-1}^{n}}{(\Delta y)^2}\right]$$ 51

Collect therterms,

$$-\frac{\alpha(\Delta t)}{2(\Delta y)^2}u_{i,j-1}^{n+1}-\frac{\alpha(\Delta t)}{2(\Delta x)^2}u_{i-1,j}^{n+1}+\left[1+\frac{\alpha(\Delta t)}{(\Delta x)^2}+\frac{\alpha(\Delta t)}{(\Delta y)^2}\right]u_{ij}^{n+1}$$

$$-\frac{\alpha(\Delta t)}{2(\Delta x)^2}u_{i+1,j}^{n+1}-\frac{\alpha(\Delta t)}{2(\Delta y)^2}u_{i,j+1}^{n+1}=\Big[1+\frac{\alpha(\Delta t)}{2}\Big(\frac{u_{i+1,j}^{n}-2u_{ij}^{n}+u_{i-1,j}^{n}}{(\Delta x)^2}$$

$$+\frac{\alpha(\Delta t)}{2}\Big(\frac{u_{i,j+1}^{n}-2u_{ij}^{n}+u_{i-1,j}^{n}}{(\Delta y)^2}$$

- If we define, $r_x=\dfrac{\alpha(\Delta t)}{(\Delta x)^2}$ and $r_y=\dfrac{\alpha(\Delta t)}{(\Delta y)^2}$, then

$$-\frac{1}{2}r_y u_{i,j-1}^{n+1}-\frac{1}{2}r_x u_{i-1,j}^{n+1}+[1+r_x+r_y]u_{ij}^{n+1}-\frac{1}{2}r_x u_{i+1,j}^{n+1}-\frac{1}{2}r_y u_{i,j}^{n}$$

$$=\left[1+\frac{1}{2}r_x\left(u_{i+1,j}^{n}-2u_{ij}^{n}+u_{i-1,j}^{n}\right)+\frac{1}{2}r_y\left(u_{j+1}^{n}-2u_{ij}^{n}+u_{i,j-1}^{n}\right)\right]$$

- If we then define,

$$a=-\frac{1}{2}r_y$$

$$b=-\frac{1}{2}r_x$$

$$c=1+r_x+r_y$$

d = Right-Hand-Side.

then the Crank-Nicolson scheme can be written as,

$$au_{ij-1}^{n+1}+bu_{i-1,j}^{n+1}+cu_{ij}^{n+1}+bu_{i+1,j}^{n+1}+au_{i,j+1}^{n+1}=d_{ij}^{n}$$

In matrix-vector form, it can be written as,

$$
\begin{bmatrix}
c & b & 0 & 0 & a & 0 & - & - & - & \cdots & 0 \\
b & c & b & 0 & 0 & a & - & - & - & \cdots & 0 \\
 & \ddots & \ddots & \ddots & & & \ddots & & & & \\
 & & \ddots & \ddots & \ddots & & & \ddots & & & \\
 & & & & & a & 0 & 0 & b & c & b \\
 & & & & & & a & 0 & 0 & b & c
\end{bmatrix}
\begin{bmatrix}
u_{i,2}^{n+1} \\ u_{i+1,2}^{n+1} \\ \vdots \\ u_{imax,2}^{n+1} \\ u_{i,3}^{n+1} \\ \vdots \\ u_{imax,3}^{n+1} \\ \vdots \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
d_{i,2}^{n} \\ d_{i+1,2}^{n} \\ \vdots \\ d_{imax,2}^{n} \\ d_{i,3}^{n} \\ \vdots \\ \vdots \\ \vdots \\ \vdots
\end{bmatrix}
$$

Since the system is pentadiagonal, either a special solver can be used or an iterative system as seen before can be employed as well.

ADI Methods

- In order to avoid solving a pentadiagonal system of equations, the alternating-direction implicit (ADI) scheme was introduced by Peaceman and Rachford in 1955.
- The scheme can be written in two steps,

Step 1
$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^{n}}{\Delta t/2} = \alpha\left(\hat{\delta}_x^2 u_{ij}^{n+\frac{1}{2}} + \hat{\delta}_y^2 u_{ij}^{n}\right)$$

Step 2
$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\hat{\delta}_x^2 u_{ij}^{n+\frac{1}{2}} + \hat{\delta}_y^2 u_{ij}^{n+1}\right)$$

- Each step results to a tridiagonal system of equations.
- The scheme is second-order accurate in time and space where $O[(\Delta t)^2, (\Delta x)^2, (\Delta y)^2]$
- The amplification factor, $G = \dfrac{[1 - r_x(1-\cos\beta_x)][1 - r_y(1-\cos\beta_y)]}{[1 + r_x(1-\cos\beta_x)][1 + r_y(1-\cos\beta_y)]}$

  where $\beta_x = k_m \Delta x$ and $\beta_y = k_m \Delta y$.

- In 2D, the method is unconditionally stable, however, in 3D its conditionally stable and the scheme is only first-order accurate in time, and second-order in space.

Approximate Factorization

- In order to recover the lost order in the temporal direction for the 3D ADI, in 1964 Douglas and Gunn, developed a general method for deriving ADI schemes that are unconditionally stable and second-order in time and space.
- First define $\Delta u_{ij} = u_{ij}^{n+1} - u_{ij}^{n}$,
- Then substitute this into the Crank–Nicolson scheme,

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \frac{\alpha}{2}\left(\hat{\delta}_x^2 + \hat{\delta}_y^2\right)\left(u_{ij}^{n+1} + u_{ij}^{n}\right)$$

$$\Delta u_{ij} = \frac{\alpha(\Delta t)}{2}\left(\hat{\delta}_x^2 + \hat{\delta}_y^2\right)\left(u_{ij}^{n+1} - u_{ij}^n + u_{ij}^n + u_{ij}^n\right)$$

$$= \frac{\alpha(\Delta t)}{2}\left[\frac{\hat{\delta}_x^2 \Delta u_{ij}}{(\Delta x)^2} + \frac{\hat{\delta}_y^2 \Delta u_{ij}}{(\Delta y)^2} + \frac{2\hat{\delta}_x^2 u_{ij}^n}{(\Delta x)^2} + \frac{2\hat{\delta}_y^2 u_{ij}^n}{(\Delta y)^2}\right]$$

Rearrange the equation and place Δu_{ij} on the left-hand-side

$$\left[1 - \frac{\alpha(\Delta t)}{2(\Delta x)^2}\hat{\delta}_x^2 - \frac{\alpha(\Delta t)}{2(\Delta y)^2}\hat{\delta}_y^2\right]\Delta u_{ij} = \frac{\alpha(\Delta t)}{(\Delta x)^2}\hat{\delta}_x^2 u_{ij}^n + \frac{\alpha(\Delta t)}{(\Delta y)^2}\hat{\delta}_y^2 u_{ij}$$

$$\left[1 - \tfrac{1}{2}r_x\hat{\delta}_x^2 - \tfrac{1}{2}r_y\hat{\delta}_y^2\right]\Delta u_{ij} = \left(r_x\hat{\delta}_x^2 + r_y\hat{\delta}_y^2\right)u_{ij}^n$$

If we the term on the left as a product of two quantities, then

$$\left[1 - \tfrac{1}{2}r_x\hat{\delta}_x^2 - \tfrac{1}{2}r_y\hat{\delta}_y^2\right] \approx \left(1 - \tfrac{1}{2}r_x\hat{\delta}_x^2\right)\left(1 - \tfrac{1}{2}r_y\hat{\delta}_y^2\right)$$

If we reexpand $\left(1 - \tfrac{1}{2}r_x\hat{\delta}_x^2\right)\left(1 - \tfrac{1}{2}r_y\hat{\delta}_y^2\right)$

$$= 1 - \tfrac{1}{2}r_x\hat{\delta}_x^2 - \tfrac{1}{2}r_y\hat{\delta}_y^2 + \tfrac{1}{4}r_x r_y\hat{\delta}_x^2\hat{\delta}_y^2$$

The last term is then an extra term that gets added to the truncation error.

$$\tfrac{1}{4}r_x r_y\hat{\delta}_x^2\hat{\delta}_y^2 = \frac{1}{4}\frac{\alpha(\Delta t)}{(\Delta x)^2}\frac{\alpha(\Delta t)}{(\Delta y)^2}\hat{\delta}_x^2\hat{\delta}_y^2$$

$$= \frac{1}{4}\frac{\alpha(\Delta t)^2}{(\Delta x)^2(\Delta y)^2}\hat{\delta}_x^2\hat{\delta}_y^2$$

- If we assume $\Delta x = \Delta y$, then the additional term is,

$$\frac{1}{4} \frac{\alpha (\Delta t)^2}{(\Delta x)^4} \hat{\delta}_x^2 \hat{\delta}_y^2$$

- If we define $\Delta u_{ij}^* = \left(1 - \frac{1}{2} r_y \hat{\delta}_y^2\right) \Delta u_{ij}$, then the scheme can be solved in two steps.

Step 1 : $\left(1 - \frac{1}{2} r_x \hat{\delta}_x^2\right) \Delta u_{ij}^* = \left(r_x \hat{\delta}_x^2 + r_y \hat{\delta}_y^2\right) u_{ij}^n$

Step 2 : $\left(1 - \frac{1}{2} r_y \hat{\delta}_y^2\right) \Delta u_{ij} = \Delta u_{ij}^*$

- And in 3D,

Step 1 : $\left(1 - \frac{1}{2} r_x \hat{\delta}_x^2\right) \Delta u^* = \left(r_x \hat{\delta}_x^2 + r_y \hat{\delta}_y^2 + r_z \hat{\delta}_z^2\right) u$

Step 2 : $\left(1 - \frac{1}{2} r_y \hat{\delta}_y^2\right) \Delta u^{**} = \Delta u^*$

Step 3 : $\left(1 - \frac{1}{2} r_z \hat{\delta}_z^2\right) \Delta u = \Delta u^{**}$

- Other possible schemes to solve the 2D Heat Equation are
  1) Splitting Methods
  2) ADE (Alternating - Direction Explicit)
  3) Hop scotch method.

Laplace Eqn.

- Laplace eqn can be written as,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$, which is an elliptic PDE

- It can represent the steady-state temperature distribution in a solid as well as the incompressible, irrotational fluid flo

- A basic finite-difference representation using a five point formula may be written as,

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\Delta y)^2} = 0$$

which has TE of $O[(\Delta x)^2, (\Delta y)^2]$.

- The modified equation is $u_{xx} + u_{yy} = -\frac{1}{12}\left[u_{xxxx}(\Delta x)^2 + u_{yyyy}(\Delta y)^2\right] + \cdots$

- As mentioned before, the equation above could be written as in matrix-vector form where the coeffs of the equation form the matrix (~~tri~~ penta-diagonal) and the vector is u. Then a direct or iterative solver can be used to solve for u.
- Instead of this form, the difference equation could be written in residual form.

- Let L be a difference operator (5-point difference representation, as mentioned in the previous page).

Thus the equation $\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\Delta y)^2} =$

can be written as,

$$L u_{ij} = 0$$

- Then define u_{ij} as $\tilde{u}_{ij} + \Delta u_{ij}$, where $\tilde{u}_{ij}$ is some solution

- We can then substitute $u_{ij} = \tilde{u}_{ij} + \Delta u_{ij}$ into $L u_{ij} = 0$ and obtain,

$$L u_{ij} = L\tilde{u}_{ij} + L\Delta u_{ij} = 0$$

Then define $L\tilde{u}_{ij} = R_{ij}$ as the residual, then.

$$L\Delta u_{ij} = -L\hat{u}_{ij} = -R_{ij}$$

- Methods for Solving the Laplace Eqns. (as mentioned before).
  - Direct solvers
  - Iterative
    - Jacobi
    - GS
    - SOR
    - SSOR
    - CG.
- Other approaches will be discussed later.

Burger's Equation

- The nonlinear Burgers equation is a parabolic PDE which can serve as a model equation for the boundary-layer equations as well as the Navier–Stokes equations.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}$$

- For a particular boundary and initial condition,

$$u(0,t) = u_0$$
$$u(L,t) = 0$$

the exact solution is $u = u_0 \bar{u}\left[\frac{1-\exp[\bar{u}\,Re_L(x/L - 1)]}{1-\exp[\bar{u}\,Re_L(x/L-1)]}\right]$

where $Re_L = \frac{u_0 L}{\mu}$ and

$\bar{u}$ is the solution to $\frac{\bar{u}-1}{\bar{u}+1} = \exp(-\bar{u}\,Re_L)$.

Numerical Methods to solve Burger's Eqn.

- FTCS Method (Forwar Time, Centered Space).

If we solve the linearized Burger's equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2},$$

then the scheme can be written as,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \mu\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

- The scheme is first-order in time, and second-order in space, where the TE $= O[(\Delta t), (\Delta x)^2]$.

- The modified eqn can be written as,

$$u_t + c u_x = \left(\mu - \frac{c^2 \Delta t}{2}\right) u_{xx} + \frac{c(\Delta x)^2}{3}\left(3r - \nu^2 - \frac{1}{2}\right) u_{xxx} + \cdots$$

Where $r = \frac{\mu(\Delta t)}{(\Delta x)^2}$ and $\nu = \frac{c(\Delta t)}{(\Delta x)}$

- If $r = \frac{1}{2}$, $\nu = 1$, then the coeffients on the right-hand-side are, zero and the equation is $u_t + c u_x = 0$ and hence the μu_{xx} is term is inarderntly eliminated.

- If we look at the first-term on the right-hand-side,

$\left(\mu - \frac{c^2 \Delta t}{2}\right) \geq 0$ since you want to model a dissipative phenomena.

Hence $\mu \geq \frac{c^2 \Delta t}{2}$

If we multiply both sides by $\frac{(\Delta t)}{(\Delta x)^2}$, then we have

$$\mu \frac{(\Delta t)}{(\Delta x)^2} \geq \frac{c^2 (\Delta t)^2}{2(\Delta x)^2}$$

$$r \geq \frac{1}{2}\nu^2$$

- From a Fourier stability analysis, the FTCS method has an amplification factor, $G = 1 + 2r(\cos\beta - 1) - i\nu \sin\beta$
- By examining the modulus of the amplification factor,

$\nu^2 \leq 2r$, and $r \leq \frac{1}{2}$

- In addition, we can also view these stability criteria by examining them in the context of the "mesh Reynolds number"

$$Re_{\Delta x} = \frac{c\Delta x}{\mu}.$$

$$= \frac{c\,\Delta x}{\mu} \cdot \frac{\Delta x}{\Delta x} \cdot \frac{\Delta t}{\Delta t} = \frac{c\Delta t}{\Delta x} \cdot \frac{(\Delta x)^2}{\mu\,\Delta t} = \nu \cdot \frac{1}{r}$$

$$= \frac{\nu}{r}$$

So, $\nu^2 \leq 2r$ and $r \leq \frac{1}{2}$

$\frac{\nu}{r} \leq \frac{2}{\nu}$ $\qquad 2\nu \leq \frac{\nu}{r}$

$Re_{\Delta x} \leq \frac{2}{\nu}$ $\qquad 2\nu \leq Re_{\Delta x}$

Hence

$$2\nu \leq Re_{\Delta x} \leq \frac{2}{\nu}$$

- Solutions to the Burger's viscous equation through the finite-difference methods have a tendency to produce wiggles or oscillations, especacially in the vicinity of high gradients.
- For the FTCS method, this occurs for mesh Reynolds number in the range

$$2 \leq Re_{\Delta x} \leq \frac{2}{\nu}$$

For values greater than $\frac{2}{\nu}$, the oscillations will grow and eventually the solution will blow-up.

- Its easy to see why this would be the case, since for values of $Re_{\Delta x} > \frac{2}{\nu}$, the scheme is unstable.

- However for values below this, the solution will still produce wiggles but the solution would be stable. To better understand this behavior, let us look at the numerical scheme but reorder the terms.

Frome page 58,

$$u_j^{n+1} = u_j^n - 2\mu \frac{(\Delta t)}{(\Delta x)^2} u_j^n - \frac{1}{2} c \frac{\Delta t}{\Delta x} u_{j+1}^n + \mu \frac{\Delta t}{(\Delta x)^2} u_{j+1}^n$$

$$+ \frac{1}{2} c \frac{\Delta t}{\Delta x} u_{j-1}^n + \mu \frac{(\Delta t)}{(\Delta x)^2} u_{j-1}^n$$

$$= (1-2r) u_j^n + \left(-\frac{1}{2}\nu + r\right) u_{j+1}^n + \left(\frac{1}{2}\nu + r\right) u_{j-1}^n$$

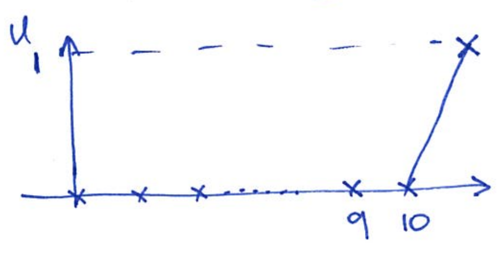
$$u_j^{n+1} = \frac{r}{2}\left(2 - Re_{\Delta x}\right) u_{j+1}^n + (1-2r) u_j^n + \frac{r}{2}\left(2 + Re_{\Delta x}\right) u_{j-1}^n$$

- Suppose we are trying to solve the Burger's viscous equation with initial conditions,

$$u(x,0) = 0 \qquad 0 \leq x \leq 1$$

and boundary conditions, $u(0,t) = 0$ and $u(1,t) =$

- If we use a 11-point mesh, then we would obtain the following values for u at $t=1$,



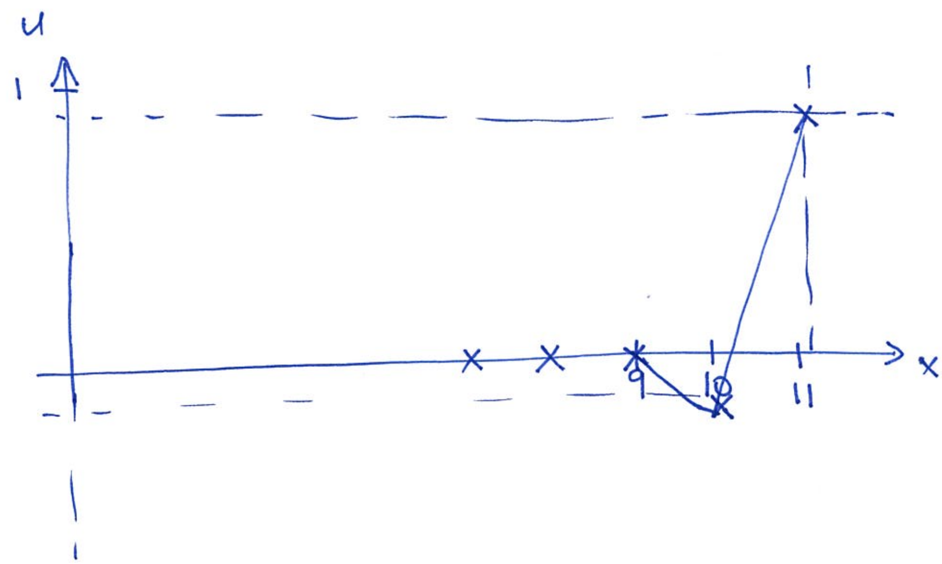
- Then at time level $(n+1)$, are the values of u are zero except at $j=10$,

$$u_{10}^{n+1} = \frac{r}{2}(2 - Re_{\Delta x})\, u_{11}^{n+1} + (1-2r)\, u_{10}^{n} + \frac{r}{2}(2 + Re_{\Delta x})u$$

$$= \frac{r}{2}(2 - Re_{\Delta x})(1) + (1-2r)(0) + \frac{r}{2}(2 + Re_{\Delta x})(0)$$

$$= \frac{r}{2}(2 - Re_{\Delta x})$$

If $Re_{\Delta x} > 2$, then $u_{10}^{n+1} < 0$ and will produce a kink in the solution for u, as shown below,

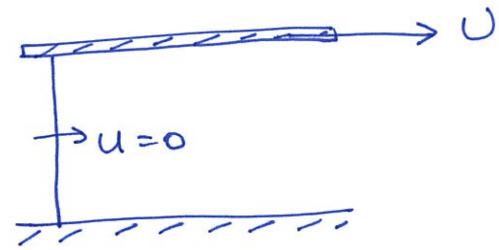


- At the next time step, the oscillation propagates inwards, and produces

- At subsequent time-levels, the ~~th~~ oscillations will continue to propagate but the solution will remain stable and the oscillations will be bounded.

- If $Re_{\Delta x} > 2$, the coef for u_{j+1}^n becomes negative as seen in the previous page. This behavior is unphysical. Let us compare this to Couette flow with similar boundary conditions;

If



, then the movement of the plate would cause the fluid below it to be convected at a speed below U and not in the opposite direction as suggested by the FTCS method.

- The oscillations can be removed by replacing the second-order central difference term for the convective flux with a first-order upwind term, but the scheme is rendered too dissipative.

Leap Frog

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \mu\,\frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n}{(\Delta x)^2}$$

The TE is of $O\left[(\Delta t/\Delta x)^2,\ (\Delta t)^2,\ (\Delta x)^2\right]$

The modified equation is $u_t + c u_x = \mu(1-\nu^2)u_{xx} + \dots$

The stability condition, $\nu \leq 1$

Lax-Wendroff method

We can write the Burgers equation in general form as,

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \quad \text{where} \quad F = cu \quad \text{or} \quad F = \frac{u^2}{2}.$$

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \quad , \text{where } A = \frac{\partial F}{\partial u}$$

$$\frac{\partial u}{\partial t} + \underset{\substack{\downarrow \\ A = c}}{c}\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} + \underset{\substack{\downarrow \\ A = u}}{u}\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}$$

Then the two-step Lax-Wendroff scheme can be written as,

step 1 : $$u_j^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n\right) - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right) + r\left[\left(u_{j-\frac{3}{2}}^n - 2u_{j-\frac{1}{2}}^n + u_{j+\frac{1}{2}}^n\right) + \left(u_{j+\frac{3}{2}}^n - 2u_{j+\frac{1}{2}}^n + u_{j-\frac{1}{2}}^n\right)\right]$$

step 2 : $$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) + r\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

The method is $O\left[\Delta t, (\Delta x)^2\right]$ with stability cond.

$$\frac{\Delta t}{(\Delta x)^2}\left(A^2 \Delta t + 2\mu\right) \le 1$$

MacCormack Method

Predictor : $u_j^{\overline{n+1}} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right) + r\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$

Corrector : $$u_j^{n+1} = \frac{1}{2}\left[u_j^n + u_j^{\overline{n+1}} - \frac{\Delta t}{\Delta x}\left(F_j^{\overline{n+1}} - F_{j-1}^{\overline{n+1}}\right) + r\left(u_{j+1}^{\overline{n+1}} - 2u_j^{\overline{n+1}} + u_{j-1}^{\overline{n+1}}\right)\right]$$

- The scheme is second-order in both time and space
- It is not possible to obtain a stability criterion but the following criterion was derived as a guide to ensure stability,

$$\Delta t \leq \frac{(\Delta x)^2}{|A|\Delta x + 2\mu}$$

Roe Method

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left[\left(F_{j+1}^n - F_{j-1}^n\right) - \left|\bar{u}_{j+\frac{1}{2}}^n\right|\left(u_{j+1}^n - u_j^n\right) + \left|\bar{u}_{j-\frac{1}{2}}^n\right|\left(u_j^n - u_{j-1}^n\right)\right]$$
$$+ r\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

where $F = \frac{u^2}{2}$ and $\bar{u}_{j+\frac{1}{2}}^n = \frac{u_j^n + u_{j+1}^n}{2}$

- The scheme is $O[\Delta t, (\Delta x)^2]$.