Governing Equations

The Navier-Stokes equation written in vector divergence form is written as follows:

$$\frac{\partial W}{\partial t} + \frac{\partial (F - F_v)}{\partial x} + \frac{\partial (G - G_v)}{\partial y} + \frac{\partial (H - H_v)}{\partial z} = 0,$$

where the conserved variables are:

$$W = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_t \end{bmatrix} = \begin{bmatrix} \text{density} \\ x\text{-momentum} \\ y\text{-momentum} \\ z\text{-momentum} \\ \text{total energy per unit volume} \end{bmatrix}.$$

The flux vector are:

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (E_t + p) u \end{bmatrix} \qquad F_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - \dot{q}_x \end{bmatrix}$$

$$G = \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v w \\ (E_t + p) v \end{bmatrix} \qquad G_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx} + v\tau_{yy} + w\tau_{yz} - \dot{q}_y \end{bmatrix}$$

$$H = \begin{bmatrix} \rho w \\ \rho w u \\ \rho w v \\ \rho w^2 + p \\ (E_t + p) w \end{bmatrix} \qquad H_v = \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx} + v\tau_{zy} + w\tau_{zz} - \dot{q}_z \end{bmatrix}$$

The equation of state is written as,

$$p = (\gamma - 1)\left[E_t - \frac{\rho}{2}(u^2 + v^2 + w^2)\right]$$

and the shear stress can be written as,

$$\tau_{x_i x_j} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij}$$

where
1. $\langle u, v, w \rangle = \langle u_1, u_2, u_3 \rangle$
2. $\langle x, y, z \rangle = \langle x_1, x_2, x_3 \rangle$
3. using Einstien's notation, repeated indices signify the terms should be expanded and hence $\frac{\partial u_k}{\partial x_k} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$
4. $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\dot{q}_{x_i} = k \frac{\partial T}{\partial x_i}$$

μ = molecular viscosity (Sutherland's formula)

$$= (T)^{3/2}\left[\frac{1 + \tilde{c}/T_\infty}{T + \tilde{c}/T_\infty}\right], \text{ where } \tilde{c} = 110.4^\circ K$$

Divergence Form

- The classical form of the equations are typically not given in conservation form.
  For an example, in non-conservative form, the governing equations for momentum in the x-direction for the two-dimensional equations are,

$$\frac{\partial}{\partial t}(\rho u) + \rho \vec{V} \cdot \nabla u = -\frac{\partial p}{\partial x}$$

Assume steady flow,

$$\rho \vec{V} \cdot \nabla u = -\frac{\partial p}{\partial x}$$

$$\rho \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = -\frac{\partial p}{\partial x}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}$$

- To derive the conservative form,

$$\frac{\partial (\rho u^2)}{\partial x} = \rho u \frac{\partial u}{\partial x} + u \frac{\partial (\rho u)}{\partial x}$$

$$\rho u \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\rho u^2) - u \frac{\partial}{\partial x}(\rho u)$$

similarly, $$\rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\rho u v) - u \frac{\partial}{\partial y}(\rho v)$$

Then substitute these equations into the non-conservative form

$$\frac{\partial}{\partial x}(\rho u^2) - u\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho u v) - u\frac{\partial}{\partial y}(\rho v) = -\frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v) - u\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\right] = 0$$

since $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$ from continuity,

$\frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho u v) = 0$ is the conservation or divergence form of the two-dimensional x-momentum

- If discontinuities exist, then the conservation form must be used.
- For example across a shockwave the density and velocity jump in value, however the product of them ~~are~~ is a constant. Hence to ensure mass is constant, the divergence form is better numerically.

- The Navier-Stokes equations can be simplified either to reduce the computational cost or through the application of assumptions based on the given problem.

- The following are in the order of decreasing computational cost the various approaches to solving the governing equations.

Direct Numerical Simulation (DNS) — Full NS equations

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Large Eddy Simulation (LES)
- use low-pass filtering to eliminate small scales of the solution

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Reynolds Averaged Navier-Stokes. (RANS)
- time averaged equations of motion of fluid-flow.
- A combination of RANS and LES is called Detached Eddy Simulation (DES).

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Thin Layer NS equations
- restrict viscous effects to gradients normal to the surface

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Boundary Layer Equations
- introduce Prandtl boundary layer assumption
  - pressure is constant across layer
  - only use leading viscous term.

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Euler Equation

- assume no viscous effects.

$\downarrow$

Full Potential equations.

- assume irrotational flow.

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Transonic Small Disturbance Equation

- small disturbance approximation.

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Laplace Equation

- incompressible flow.

Full Potential Equation

- Before we derive the full potential equations, let us combine the equations in a special form called the gas dynamics equation.

- x, y, z will be defined by x_i for $i = 1, 2, 3$

- We start with the assumption that the flow is isentropic, then $\left.\frac{\partial p}{\partial \rho}\right|_s = a^2$ where $a = \sqrt{\gamma R T}$