$\downarrow$

Euler Equation
- assume no viscous effects.

$\downarrow$

Full Potential equations.
- assume irrotational flow.

$\downarrow$

Transonic Small Disturbance Equation
- small disturbance approximation.

$\downarrow$

Laplace Equation
- incompressible flow.

Full Potential Equation

- Before we derive the full potential equations, let us combine the equations in a special form called the gas dynamics equation.

- x, y, z will be defined by x_i for $i = 1, 2, 3$

- We start with the assumption that the flow is isentropic, then $\left.\frac{\partial p}{\partial \rho}\right|_s = a^2$ where $a = \sqrt{\gamma R T}$

- Then $\dfrac{\partial p}{\partial x_i} = \left.\dfrac{\partial p}{\partial \rho}\right|_s \dfrac{\partial \rho}{\partial x_i} = a^2 \dfrac{\partial \rho}{\partial x_i}$

- Multiply the x-momentum with u and the y-momentum with v, (Use the non conservative form)

$$\rho u^2 \frac{\partial u}{\partial x} + \rho u v \frac{\partial u}{\partial y} = -u \frac{\partial p}{\partial x}$$

$$u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} = -u \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} \quad \text{--- } x$$

$$uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y} = -\frac{v a^2}{\rho} \frac{d \rho}{\partial y} \quad \text{--- } y.$$

- From the continuity equation,

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = -\rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

- Now sum the x- and y-momentum equations,

$$u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y} = -\frac{u a^2}{\rho} \frac{\partial \rho}{\partial x} - \frac{v a^2}{\rho} \frac{\partial \rho}{\partial y}$$

$$= -\frac{a^2}{\rho} \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right)$$

- Substitute continuity into the above eqn.

$$u^2\frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v^2\frac{\partial v}{\partial y} = a^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$(u^2-a^2)\frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (v^2-a^2)\frac{\partial v}{\partial y} = 0$$

In three-dimensions, the equation can be written as,

$$(u^2-a^2)\frac{\partial u}{\partial x} + (v^2-a^2)\frac{\partial v}{\partial y} + (w^2-a^2)\frac{\partial w}{\partial z}$$
$$+ uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + vw\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + wu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0$$

The equation is known as the gas dynamics equation.

- Now to include the energy equation, we start from the material derivative of H, where $H = h + \frac{1}{2}(u^2+v^2)$.

Start from inviscid, adiabatic form of energy equation,

$$\frac{DH}{Dt} = 0.$$

Hence H must be a constant.

Then assume that the flow is both thermally and calorically perfect, $h = C_pT$ and,

$$H_0 = h + \frac{1}{2}(u^2+v^2)$$
$$C_pT_0 = C_pT + \frac{1}{2}(u^2+v^2).$$

Recalling that $a^2 = \gamma R T$, $R = C_p - C_v$, and $\gamma = \frac{C_p}{C_v}$

$$a^2 = \frac{C_p}{C_v}(C_p - C_v)T = \left(\frac{C_p - C_v}{C_v}\right) C_p T$$

$$a^2 = (\gamma - 1) C_p T$$

$$C_p T = \frac{1}{\gamma - 1} a^2$$

Then $a_0^2 = a^2 + \left(\frac{\gamma - 1}{2}\right)(u^2 + v^2)$.

Since a_0 is the freestream speed of sound, then a at any point can be computed from,

$$a^2 = a_0^2 - \frac{1}{2}(\gamma - 1)(u^2 + v^2)$$

and in 3D,

$$a^2 = a_0^2 - \frac{1}{2}(\gamma - 1)(u^2 + v^2 + w^2) .$$

- Now in the final step the gas dynamics equations can be written as the full potential equations with the assumption that the flow is irrotational,

For irrotational flow, $\nabla \times \vec{V} = 0$ and hence $\vec{V}$ can be defined as a gradient of a scalar quantity,

$$\vec{V} = \nabla \phi$$

$$u = \frac{\partial \phi}{\partial x} \quad , \quad v = \frac{\partial \phi}{\partial y} \quad , \quad w = \frac{\partial \phi}{\partial z}$$

From the gas dynamics equations, the full potential equation can be written as;

$$(\phi_x^2 - a^2)\phi_{xx} + (\phi_y^2 - a^2)\phi_{yy} + (\phi_z^2 - a^2)\phi_{zz}$$
$$+ 2\phi_x \phi_y \phi_{xy} + 2\phi_y \phi_z \phi_{yz} + 2\phi_z \phi_x \phi_{zx} = 0 .$$

The resulting equation is a single ~~pe~~ non-linear partial differential equation written in non-conservation form.

- Next we present special forms of the full-potential equations:

a) Small Disturbance form of the energy equation.

- The assumption is that the flowfield is only slightly disturbed by the body.

- From the energy equation in the form presented in previous pages

$$a^2 = a_0^2 - \frac{1}{2}(\gamma - 1)(u^2 + v^2)$$

In the farfield , $a_0^2 = a_\infty^2 + \frac{1}{2}(\gamma-1)U_\infty^2$

Then , $a^2 + \frac{1}{2}(\gamma-1)(u^2+v^2) = a_\infty^2 + \frac{1}{2}(\gamma-1)U_\infty^2$

Let $u = U_\infty + u'$ and $v = v'$

Then

$$a^2 + \frac{1}{2}(\gamma-1)\left[(U_\infty+u')^2 + v'^2\right] = a_\infty^2 + \frac{1}{2}(\gamma-1)$$

$$a^2 + \frac{1}{2}(\gamma-1)\left[U_\infty^2 + 2U_\infty u' + u'^2 + v'^2\right] = a_\infty^2 + \frac{1}{2}(\gamma-1)($$

$$a^2 = a_\infty^2 - \frac{1}{2}(\gamma-1)\left[2U_\infty u' + u'^2 + v'^2\right]$$

Assume $u' \lll U_\infty$ and $v' \lll U_\infty$

thus $\frac{u'}{U_\infty} < 1$ and $\left(\frac{u'}{U_\infty}\right)^2 \approx 0$

Then $u'^2 + v'^2$ can be neglected and hence

$$a^2 = a_\infty^2 - \frac{1}{2}(\gamma-1)\cdot 2U_\infty u'$$

$$a^2 = a_\infty^2 - (\gamma-1)U_\infty u'$$

This forms a linear relationship between the disturbance velocity and the speed of sound.

b) Small Disturbance form of the Full Potential Equation.

In 2D, $(\Phi_x^2 - a^2)\Phi_{xx} + 2\Phi_x\Phi_y\Phi_{xy} + (\Phi_y^2 - a^2)\Phi_{yy} = 0$

Now introduce in the potential function,

$$\Phi = U_\infty x + \phi(x,y)$$

Then $\Phi_x = u = U_\infty + \phi_x$

$\Phi_y = v = \phi_y$.

- Now assume that ϕ_x and ϕ_y are small compared to U_∞.

- Then

$$\Phi_x^2 - a^2 = (U_\infty + \phi_x)^2 - a_\infty^2 + \frac{1}{2}(\gamma-1)2U_\infty u'$$
$$= U_\infty^2 + 2U_\infty \phi_x + \phi_x^2 - a_\infty^2 + (\gamma-1)U_\infty \phi$$

where $u' = \phi_x$

Assume that $\phi_x^2 = u'^2 \approx 0$,

The equation is further simplified to,

$$\Phi_x^2 - a^2 \approx U_\infty^2 - a_\infty^2 + [2 + \gamma - 1] U_\infty \phi_x$$
$$\approx U_\infty^2 - a_\infty^2 + (\gamma+1) U_\infty \phi_x$$

Dividing by a_∞^2,

$$\frac{\Phi_x^2 - a^2}{a_\infty^2} \approx \frac{U_\infty^2}{a_\infty^2} - 1 + (\gamma+1)\frac{U_\infty \phi_x}{a_\infty^2} \cdot \frac{U_\infty}{U_\infty}.$$

$$\approx M_\infty^2 - 1 + (\gamma+1) M_\infty^2 \left(\frac{\phi_x}{U_\infty}\right)$$

- Divide the 2D form of the full potential equations with a_∞^2

$$\left(\frac{\Phi_x^2-a^2}{a_\infty^2}\right)\Phi_{xx} + \frac{2\Phi_x\Phi_y}{a_\infty^2}\Phi_{xy} + \left(\frac{\Phi_y^2-a^2}{a_\infty^2}\right)\Phi_{yy}$$

- since $\Phi_x = U_\infty + \phi_x$ and $\Phi_y = \phi_y$
- then $\Phi_{xx} = \phi_{xx}$ and $\Phi_{yy} = \phi_{yy}$ and $\Phi_{xy} = \phi_{xy}$.
- similar to the simplification of $\left(\frac{\Phi_x-a^2}{a_\infty^2}\right)$,

$$\left(\frac{\Phi_y^2-a^2}{a_\infty^2}\right) \approx -1 + (\gamma-1)M_\infty^2\left(\frac{\phi_y}{U_\infty}\right)$$

- Substitute all the above terms and equations and neglect higher order terms, and we arrive at

$$\left[M_\infty^2 - 1 + (\gamma+1)M_\infty^2\frac{\phi_x}{U_\infty}\right]\phi_{xx} + 2M_\infty^2\left(1+\frac{\phi_x}{U_\infty}\right)\frac{\phi_y}{U_\infty}\phi_{xy}$$

$$+ \left[-1 + (\gamma-1)M_\infty^2\frac{\phi_y}{U_\infty}\right]\phi_{yy} = 0$$

c) Transonic Small Disturbance Equation.

- Transonic flows contain regions of subsonic and supersonic velocities
- In transonic flows, flow properties vary rapidly in the streamwise direction.

- Therefore in transonic flows, $\frac{\partial}{\partial x} > \frac{\partial}{\partial y}$.

- Then by applying this Simplification to the final equation derived in the previous subsection, we get

$$\left[(M_\infty^2-1)+(\gamma+1)M_\infty^2\frac{\phi_x}{U_\infty}\right]\phi_{xx}+\cancelto{0}{2M_\infty^2\left(1+\frac{\phi_x}{U_\infty}\right)\frac{\phi_y}{U_\infty}\phi_{xy}}$$
$$+\left[-1+\cancelto{0}{(\gamma-1)M_\infty^2\frac{\phi_x}{U_\infty}}\right]\phi_{yy}=0$$

$$\left[(M_\infty^2-1)+(\gamma+1)M_\infty^2\frac{\phi_x}{U_\infty}\right]\phi_{xx}-\phi_{yy}=0$$

$$\left[(1-M_\infty^2)\mp(\gamma+1)M_\infty^2\frac{\phi_x}{U_\infty}\right]\phi_{xx}+\phi_{yy}=0$$

- The equation is still nonlinear because of the first term, $f(\phi_x\phi_{xx})$

- The sign of the coefficient of ϕ_{xx} can change depending on the type of flow,

d) Prandtl-Glauert Equation

- If the flow is entirely subsonic or supersonic, then all terms involving ~~of~~ products of small quantities can be removed,

$$(1-M_\infty^2)\,\phi_{xx} + \phi_{yy} = 0$$

- This is a linear equation.
- In 3D,

$$(1-M_\infty^2)\,\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

e) If the flow is incompressible, then $M_\infty \to 0$, and

$$\phi_{xx} + \phi_{yy} = 0.$$

- This is the Laplace equation.
- This equation is sometimes called potential equation as seen in undergraduate fluids courses.
- Hence the initial equation is deemed the 'Full Potential Equations'.

- Once ϕ is obtained from either the full potential equations or one of its simplified forms, the velocity components u and v, w can be obtained from $\vec{V} = \nabla\phi$.

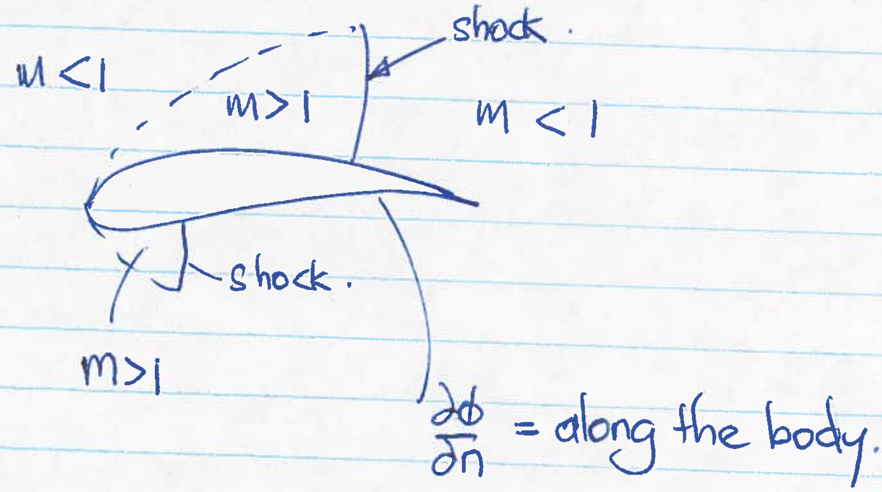
- We can then use the isentropic relations for a perfect gas to obtain

$$\rho \approx \rho_\infty\left[1 - M_\infty^2\left(\frac{u}{q_\infty} - 1\right)\right]$$

$$p \approx p_\infty\left[1 - \gamma M_\infty^2\left(\frac{u}{q_\infty} - 1\right)\right], \text{ where } q_\infty = U_\infty.$$

Murman and Cole Method for Transonic Small Disturbance Equation

The flow that we want to simulate is the flow around a two-dimensional airfoil. in transonic flow,



$$\left[(1-M_\infty^2) - (\gamma+1) M_\infty^2 \frac{\phi_x}{U_\infty}\right]\phi_{xx} + \phi_{yy} = 0$$

where
$$\phi_x = u'$$
$$\phi_y = v'$$
$$u = U_\infty + u'$$
$$v = v'$$

Let $A = \left[(1-M_\infty^2) - (\gamma+1) M_\infty^2 \frac{\phi_x}{U_\infty}\right]$

then

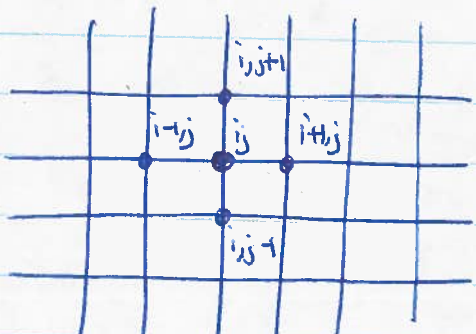
$$A\phi_{xx} + \phi_{yy} = 0$$

(i) subsonic region, $A>0$, use centered difference scheme for ϕ_{xx} and ϕ_{yy}

$$A_{ij}\,\delta_x^2\phi_{ij} + \delta_y^2\phi_{ij} = 0$$

where
$$\delta_x^2\phi_{ij} = \frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{(\Delta x)^2}$$
$$\delta_y^2\phi_{ij} = \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{(\Delta y)^2}$$

$$A_{ij} = (1-M_\infty^2) - (\gamma+1)\frac{M_\infty^2}{U_\infty}\,\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2(\Delta x)}$$



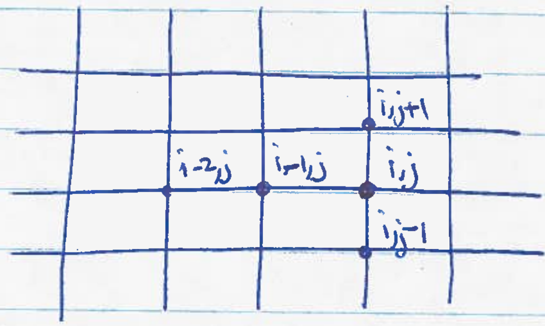
Stencil for $A>0$
because $M<1$

(ii) supersonic flow, $A<0$, use backward difference scheme for ϕ_{xx} and ϕ_x.

$$A_{i-1,j}\,\delta_x^2\phi_{ij} + \delta_y^2\phi_{ij} = 0$$

where
$$\delta_x^2\phi_{ij} = \frac{\phi_{ij} - 2\phi_{i-1,j} + \phi_{i-2,j}}{(\Delta x)^2}$$

$$A_{i-1,j} = (1-M_\infty^2) - (\gamma+1)\frac{M_\infty^2}{U_\infty}\,\frac{\phi_{ij}-\phi_{i-2,j}}{2\Delta x}$$



stencil for $A<0$
because $M>1$

(iii) sonic points ; If $A_{i+1,j} < 0$ and $A_{ij} > 0$ then omit ϕ_{xx} and the equation localy reduces to

$$\delta_y^2 \phi_{yy} = 0 .$$

(iv) Shock points ; If $A_{i+1} > 0$ and $A_{ij} < 0$ then include ϕ_{xx} twice.

$$A_{ij}\, \delta_x^2 \phi_{ij} + A_{i-1,j}\, \delta_x^2 \phi_{i-1,j} + \delta_y^2 \phi_{ij} = 0$$

- The four different scenarios above can be combined into a single difference equation using the following switch

$$\mu_{ij} = \begin{cases} 0 & \text{if } A_{ij} > 0 \\ 1 & \text{if } A_{ij} < 0 \end{cases}$$

- Note that the shock point has two derivatives in the x-direction and a Taylor expansion reveals that the scheme is not consistent with the PDE $A\phi_{xx} + \phi_{yy} = 0$

However the scheme is conservative and the jump conditions are satisfied in the limit of fine grids. The reason is that the notion of consistency is not relevant at a discontinuity but conservation is.

- The Murman-Cole method can then be written as,

$$(1-\mu_{ij})A_{ij}\,\delta_x^2\phi_{ij} + \mu_{i-1,j}A_{i-1,j}\,\delta_x^2\phi_{i-1,j} + \delta_y^2\phi_{ij} =$$

$$(1-\mu_{ij})A_{ij}\left(\frac{\phi_{i+1,j}-2\phi_{ij}+\phi_{i-1,j}}{(\Delta x)^2}\right) + \mu_{i-1,j}A_{i-1,j}\cdot$$

$$\cdot\left(\frac{\phi_{ij}-2\phi_{i-1,j}+\phi_{i-2,j}}{(\Delta x)^2}\right) + \frac{\phi_{i,j+1}-2\phi_{ij}+\phi_{i,j-1}}{(\Delta y)^2} = 0$$

$$\frac{(1-\mu_{ij})A_{ij}}{(\Delta x)^2}\phi_{i+1,j} + \left[\frac{\mu_{i-1,j}A_{i-1,j}}{(\Delta x)^2} - \frac{2(1-\mu_{ij})A_{ij}}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}\right]\phi_{ij} + \frac{1}{(\Delta y)^2}\phi_{i,j+1}$$

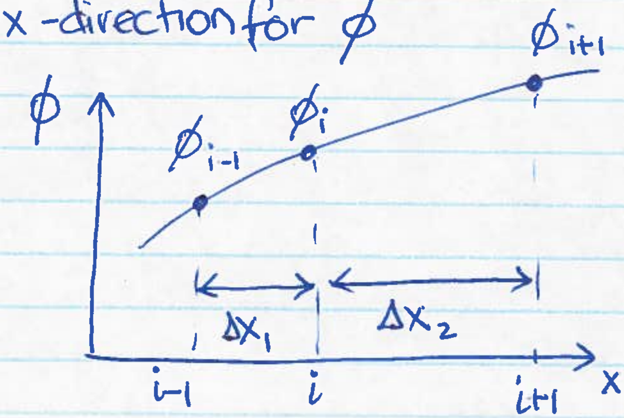
$$+\left[\frac{(1-\mu_{ij})A_{ij}}{(\Delta x)^2} - \frac{2\mu_{i-1,j}A_{i-1,j}}{(\Delta x)^2}\right]\phi_{i-1,j} + \left[\frac{\mu_{i-1,j}A_{i-1,j}}{(\Delta x)^2}\right]\phi_{i-2,j} + \frac{1}{(\Delta y)^2}\phi_{i,j-1}$$

$$= 0$$

The method can then be solved using Jacobi, GS, or SOR.

- To improve the accuracy of solving the TSD equations, a non-equally spaced mesh that concentrates points over the airfoil should be considered.

For an example, if the mesh is stretched in the x-direction for ϕ



Then $$\frac{\partial \phi}{\partial x} \approx \frac{\phi_{i+1} - \phi_{i-1}}{x_{i+1} - x_{i-1}}$$

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\dfrac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} - \dfrac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}}}{\frac{1}{2}(x_{i+1} - x_{i-1})}$$

If the finite-difference approximation is of second-order on an equally spaced mesh, it is also second-order on a stretched mesh if the stretching function is smooth, even though the leading TE term is not the same but they are still second-order.

In summary,

i) First-derivatives,

	Equally Spaced	Nonequal
$\delta_x^+ \phi_i =$	$\dfrac{\phi_{i+1} - \phi_i}{\Delta x}$	$\dfrac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i}$
$\delta_x^- \phi_i =$	$\dfrac{\phi_i - \phi_{i-1}}{\Delta x}$	$\dfrac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}}$
$\delta_x^0 \phi_i =$	$\dfrac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$	$\dfrac{\phi_{i+1} - \phi_{i-1}}{x_{i+1} - x_{i-1}}$

2) Second-derivatives,

$$\delta_x^2 \phi_i = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2} \qquad \frac{\dfrac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} - \dfrac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}}}{\frac{1}{2}(x_{i+1} - x_{i-1})}$$

$$\delta_x^2 \phi_{i-1} = \frac{\phi_i - 2\phi_{i-1} + \phi_{i-2}}{(\Delta x)^2} \qquad \frac{\dfrac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}} - \dfrac{\phi_{i-1} - \phi_{i-2}}{\cancel{\phi} x_{i-1} - x_{i-2}}}{\frac{1}{2}(x_i - x_{i-2})}$$

- The Murman — Cole method can be revisited with the above finite difference derivatives for nonequal grid spacing.

- To solve the equation, the same Jacobi, GS, SOR, or other schemes can be employed.

Murman-Cole Method for unequal Spaces

$$\rightarrow (1-\mu_{ij})A_{ij}\,\delta_x^2\phi_{ij} + \mu_{i-1,j}A_{i-1,j}\,\delta_x^2\phi_{i-1,j} + \delta_y^2\phi_{ij} = 0$$

$$(1-\mu_{ij})A_{ij}\left[\frac{\dfrac{\phi_{i+1,j}-\phi_{i,j}}{x_{i+1,j}-x_{ij}} - \dfrac{\phi_{i,j}-\phi_{i-1,j}}{x_{ij}-x_{i-1,j}}}{\frac{1}{2}(x_{i+1,j}-x_{i-1,j})}\right]$$

$$+\mu_{i-1,j}A_{i-1,j}\left[\frac{\dfrac{\phi_{ij}-\phi_{i-1,j}}{x_{ij}-x_{i-1,j}} - \dfrac{\phi_{i-1,j}-\phi_{i-2,j}}{x_{i-1,j}-x_{i-2,j}}}{\frac{1}{2}(x_{ij}-x_{i-2,j})}\right]$$

$$+\frac{\dfrac{\phi_{i,j+1}-\phi_{ij}}{y_{i,j+1}-y_{ij}} - \dfrac{\phi_{ij}-\phi_{i,j-1}}{y_{ij}-y_{i,j-1}}}{\frac{1}{2}(y_{i,j+1}-y_{i,j-1})} = 0$$

$$\rightarrow \frac{2}{(y_{ij}-y_{i,j-1})(y_{i,j+1}-y_{i,j-1})}\phi_{i,j-1} + \frac{2\mu_{i-1,j}A_{i-1,j}}{(x_{i-1,j}-x_{i-2,j})(x_{ij}-x_{i-2,j})}\phi_{i-2}$$

$$+\left[\frac{2(1-\mu_{ij})A_{ij}}{(x_{ij}-x_{i-1,j})(x_{i+1,j}-x_{i-1,j})} + \frac{-2\mu_{i-1,j}A_{i-1,j}}{(x_{ij}-x_{i-1,j})(x_{ij}-x_{i-2,j})}\right.$$

$$\left.+\frac{-2\mu_{i-1,j}A_{i-1,j}}{(x_{i-1,j}-x_{i-2,j})(x_{ij}-x_{i-2,j})}\right]\phi_{i-1,j}$$

$$+\left[\frac{-2(1-\mu_{ij})A_{ij}}{(x_{i+1,j}-x_{ij})(x_{i+1,j}-x_{i-1,j})} + \frac{-2(1-\mu_{ij})A_{ij}}{(x_{ij}-x_{i-1,j})(x_{i+1,j}-x_{i-1,j})}\right.$$

$$+\frac{2\mu_{i-1,j}A_{i-1,j}}{(x_{ij}-x_{i-1,j})(x_{ij}-x_{i-2,j})}+\frac{-2}{(y_{i,j+1}-y_{ij})(y_{i,j+1}-y_{i,j-1})}$$

$$+\frac{-2}{(y_{ij}-y_{i,j-1})(y_{i,j+1}-y_{i,j-1})}\Bigg]\phi_{ij}+\frac{2(1-\mu_{ij})A_{ij}\,\phi_{i+1,j}}{(x_{i+1,j}-x_{ij})(x_{i+1,j}-x_{i\cdots}}$$

$$+\frac{2}{(y_{i,j+1}-y_{ij})(y_{i,j+1}-y_{i,j-1})}\phi_{i,j+1}=0$$

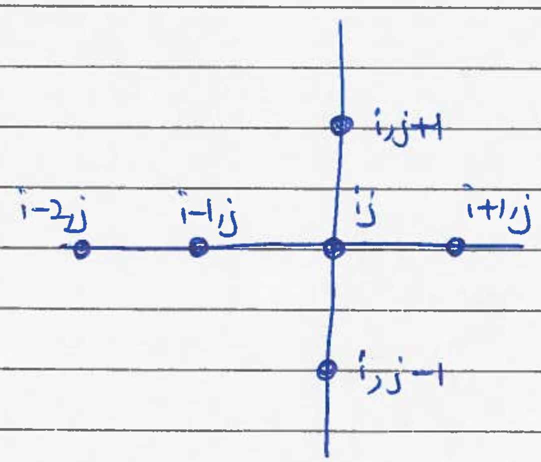
We can now simplify the equation to,

$$c_{ij}\phi_{i,j-1}+g_{ij}\phi_{i-2,j}+d_{ij}\phi_{i-1,j}+a_{ij}\phi_{ij}$$

$$+e_{ij}\phi_{i+1,j}+b_{ij}\phi_{i,j+1}=0$$

where a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, and g_{ij} are different for every point in the computational domain and are listed in the following page.

→ the stencil



$$a_{ij} = \frac{-2(1-\mu_{ij})A_{ij}}{(x_{i+1,j}-x_{i-1,j})(x_{i+1,j}-x_{ij})} + \frac{-2(1-\mu_{ij})A_{ij}}{(x_{i+1,j}-x_{i-1,j})(x_{ij}-x_{i-1,j})} \longrightarrow \phi_{ij}$$

$$+ \frac{-2}{(y_{i,j+1}-y_{i,j-1})(y_{i,j+1}-y_{ij})} + \frac{-2}{(y_{i,j+1}-y_{i,j-1})(y_{ij}-y_{i,j-1})}$$

$$+ \frac{2\mu_{i-1,j}A_{i-1,j}}{(x_{ij}-x_{i-1,j})(x_{ij}-x_{i-2,j})}$$

$$b_{ij} = \frac{2}{(y_{i,j+1}-y_{i,j-1})(y_{i,j+1}-y_{ij})} \longrightarrow \phi_{i,j+1}$$

$$c_{ij} = \frac{2}{(y_{i,j+1}-y_{i,j-1})(y_{i,j\cancel{+1}}-y_{i,j-1})} \longrightarrow \phi_{i,j-1}$$

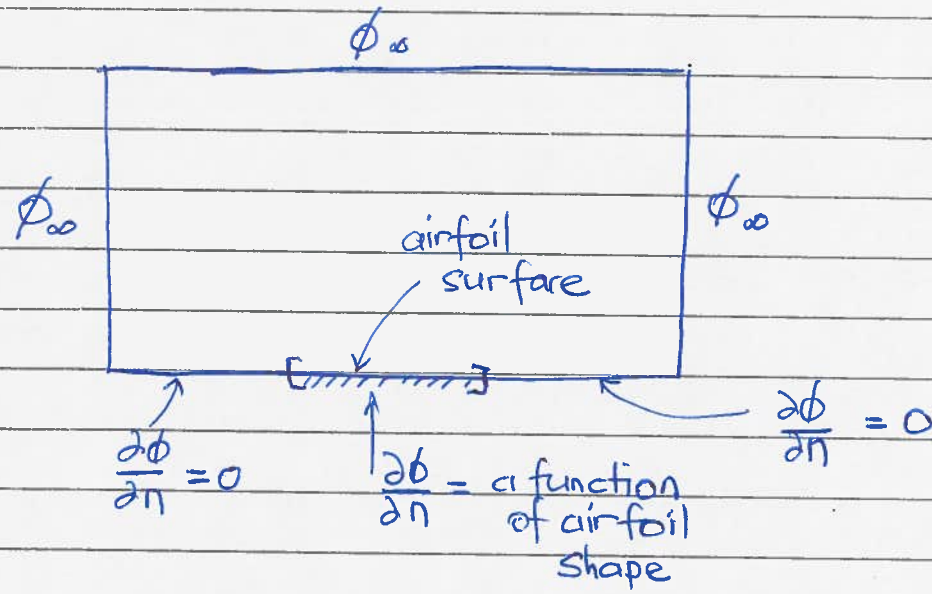
$$d_{ij} = \frac{+2(1-\mu_{ij})A_{ij}}{(x_{i+1,j}-x_{i-1,j})(x_{ij}-x_{i-1,j})} + \frac{-2\mu_{i-1,j}A_{i-1,j}}{(x_{ij}-x_{i-1,j})(x_{ij}-x_{i-2,j})}$$

$$+ \frac{-2\mu_{i-1,j}A_{i-1,j}}{(x_{ij}-x_{i-2,j})(x_{i-1,j}-x_{i-2,j})} \longrightarrow \phi_{i-1,j}.$$

$$e_{ij} = \frac{+2(1-\mu_{ij})A_{ij}}{(x_{i+1,j}-x_{i-1,j})(x_{i+1,j}-x_{ij})} \longrightarrow \phi_{i+1,j}$$

$$g_{ij} = \frac{+2\mu_{i-1,j}A_{i-1,j}}{(x_{i-1,j}-x_{i-2,j})(x_{ij}-x_{i-2,j})} \longrightarrow \phi_{i-2,j}.$$

boundary conditions.



→ ϕ_∞ can be set to 0.

Therefore all
$$\left.\begin{array}{l} \phi_{1,j} \ \forall\ j=1, jmax \\ \phi_{i,jmax} \ \forall\ i=1, imax \\ \phi_{imax,j} \ \forall\ j=1, jmax \end{array}\right\} = 0.$$

→ If airfoil is located along $j=1$, then airfoil is between X_{le} to X_{te} where le is leading edge and te is trailing edge

If $i = le$ corresponds to the index at the leading edge of the airfoil and $i = te$ is at the trailing edge, then

$$\left.\frac{\partial \phi}{\partial n}\right|_{j=1} = 0 \quad \forall\ i = 2, le-1 \text{ and } i = te+1, imax-1$$

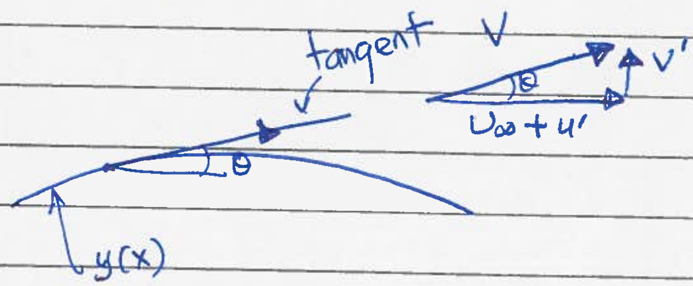
$$\left.\frac{\partial \phi}{\partial n}\right|_{j=1} = 0 \quad \Rightarrow \text{gradient of } \phi \text{ normal to } j=1 \text{ must be equal to zero}$$

$$\left.\frac{\Delta \phi}{\Delta n}\right|_{j=1} = 0$$

$$\left.\Delta \phi\right|_{j=1} = 0$$

$$-\phi_{i,2} + \phi_{i,1} = 0 \quad \forall \; i = 2, le-1 \text{ and } i = te+1, jmax$$

along the airfoil surface, we need to satisfy the tangency boundary condition that $\vec{u} \cdot \vec{n} = 0$



$$\vec{u} \cdot \vec{n} = 0$$

$$\begin{bmatrix} U_\infty + u' & v' \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$v' = 0$$

$$\tan\theta = \frac{v'}{U_\infty + u'} \approx \frac{v'}{U_\infty} \approx \frac{dy}{dx}$$

if $\theta \ll 1$

$$v' = \phi_y$$

so, $v' = U_\infty \dfrac{dy}{dx}$

$$\phi_n = \phi_y = U_\infty \frac{dy}{dx}$$

$$\frac{-\phi_{i,2} + \phi_{i,1}}{\Delta y} = U_\infty \frac{dy}{dx}$$

where $\dfrac{dy}{dx}$ is evaluated based on the shape of the airfoil.

- since the stencil requires $i-2,j$; $i-1,j$; $i+1,j$, $i,j+1$, and i,j then you may ~~set~~ along the boundaries, you may either convert the scheme to a first order method such as along the left boundary where $i-2,j$ are required or set values along $i=1$ and 2 for $\forall\ j=1,jmax$ as boundary values.

- pseudo-code to solve Murman-Cole equations.

a) Initialize $\phi=0\ \forall\ \Omega$ and $\partial\Omega$

b) satisfy boundary condition along lower wall.

$$\left.\frac{\partial\phi}{\partial n}\right|_{j=1} = 0 \quad \forall\ i=2, le-1 \text{ and } i=te+1, imax-1$$

$$\frac{\partial\phi}{\partial n} = V_\infty \frac{dy}{dx} \quad \forall\ i \in \text{(airfoil surface)}.$$

c) update $\phi_{ij}\ \forall\ i=3, imax-1\ ;\ j=2, jmax-1$.

- Solving the Murman-Cole equations in step c) above

a) point Jacobi,

$$c_{ij}^k \phi_{i,j-1}^k + g_{ij}^k \phi_{i-2,j} + d_{ij}^k \phi_{i-1,j}^k + a_{ij}^{k} \hat{\phi}_{ij}^{k+1} + e_{ij}^k \phi_{i+1,j}^k + b_{ij}^k \phi_{i,j+1} = 0$$

$$\hat{\phi}_{ij}^{k+1} = \frac{\left[-c_{ij}^k \phi_{i,j-1}^k - g_{ij}^k \phi_{i-2,j}^k - d_{ij}^k \phi_{i-1,j}^k - e_{ij}^k \phi_{i+1,j}^k - b_{ij}^k \phi_{i,j+1}\right]}{a_{ij}^k}$$

b) Gauss-Seidel

$$\phi_{ij}^{k+1} = \frac{\left[-c_{ij}^{k}\phi_{i,j-1}^{k} - \cdots\cdots\right]}{a_{ij}^{k}}.$$

to reduce the cost of updating $c_{ij}, \ldots a_{ij}$; these coefs can be computed once based on the previos iteration but not while sweeping through the domain during the current iteration.

c) SOR

As stated earlier.

d) Line implicit Gauss-Seidel.

$$c_{ij}^{k}\phi_{i,j-1}^{k+1} + a_{ij}^{k}\phi_{ij}^{k+1} + b_{ij}^{k}\phi_{i,j+1}^{k+1}$$

$$= -g_{ij}^{k}\phi_{i-2,j}^{k} - d_{ij}^{k}\phi_{i-1,j}^{k} - e_{ij}^{k}\phi_{i+1,j}^{k}$$

- This forms a set of equations along the current j line.
- since its tridiagonal, then a direct solver such as The Thomas Algorithm can be used.

- once ϕ converges, then the pressure can be updated using isentropic relations.

- $$\frac{p}{p_\infty} = \left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(1 - \frac{u^2+v^2}{V_\infty^2}\right)\right]^{\gamma/\gamma-1}$$

- The coef. of pressure can be computed from,

$$\cancel{\frac{p}{p_\infty}} - \quad C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{p/p_\infty - 1}{\frac{1}{2}\frac{\rho_\infty}{p_\infty} V_\infty^2}$$

$$= \frac{\frac{p}{p_\infty} - 1}{\frac{1}{2}\frac{\gamma}{a_\infty^2} V_\infty^2}$$

$$= \frac{\left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(1 - \frac{u^2+v^2}{V_\infty^2}\right)\right]^{\gamma/\gamma-1}}{\frac{1}{2}\gamma M_\infty^2}$$

$$= \frac{\left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(1 - \frac{(V_\infty+\phi_x)^2 + \phi_y^2}{V_\infty^2}\right)\right]^{\gamma/\gamma-1} - 1}{\frac{1}{2}\gamma M_\infty^2}$$

$$\approx \frac{\left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(1 - \frac{V_\infty^2}{V_\infty^2} - \frac{2V_\infty\phi_x}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1}{\frac{1}{2}\gamma M_\infty^2}$$

$$\approx \frac{\left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(-2\frac{\phi_x}{V_\infty}\right)\right]^{\gamma/\gamma-1} - 1}{\frac{1}{2}\gamma M_\infty^2}$$

From the binomial expression, $(1+\varepsilon)^n \approx 1 + n\varepsilon$

$$\left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(-2\frac{\phi_x}{V_\infty}\right)\right]^{\frac{\gamma}{\gamma-1}} \approx 1 - (\gamma-1) M_\infty^2 \frac{\phi_x}{V_\infty} \cdot \frac{\gamma}{\gamma-1}$$

$$\approx 1 - \gamma M_\infty^2 \frac{\phi_x}{U_\infty}$$

Therefore, $$C_p \approx \frac{1 - \gamma M_\infty^2 \phi_x / V_\infty - 1}{\frac{1}{2}\gamma M_\infty^2}$$

$$C_p \approx -\frac{2\phi_x}{U_\infty}$$

The loads on the airfoil may be computed by integrating the coef of pressure,

$$C_d = \int C_p \cos\theta \, dA$$

$$C_l = \int C_p \sin\theta \, dA$$