Then the complete differential form can be written as,

$$\frac{\partial W}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}(FA) = Q .$$

where $W = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$, $F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}$, $Q = \begin{bmatrix} 0 \\ \frac{p}{A}\frac{dA}{dx} \\ 0 \end{bmatrix}$

$$p = (\gamma - 1)\rho\left(\frac{e}{\rho} - \frac{u^2}{2}\right) \quad \text{and} \quad A = A(x)$$

where A is the cross-sectional area as a function of x.

In integral form, we can write the equation as,

$$\frac{\partial W}{\partial t} + \frac{1}{V}\int_{cs} F\, dA = \frac{1}{V}\int_{cv} Q\, dV$$

Boundary Conditions for Euler and Navier-Stokes Equations

- To solve for values along inlet and outlet boundaries as well as the farfied, the method of Characteristics can be employed.

- It can be employed for solving hyperbolic sets of equations in two coordinates systems but not three.

– We start from the non-conservation form of the governing equations in the x and t coordinates.

From, $\dfrac{\partial W}{\partial t} + \dfrac{\partial F}{\partial x} = 0$

we have in non-conservation form,

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial W}\frac{\partial W}{\partial x} = 0$$

or $\dfrac{\partial W}{\partial t} + A\dfrac{\partial W}{\partial x} = 0$ where $A = \dfrac{\partial F}{\partial W}$

Then diagonialize A through a similarity transformation,

$$SAS^{-1} = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$$

where S is the similarity matrix,

Then from the non-conservation form of the equation,

$$S\frac{\partial W}{\partial t} + SA\frac{\partial W}{\partial x} = 0$$

since $SA = \Lambda S$

then $S\dfrac{\partial W}{\partial t} + \Lambda S\dfrac{\partial W}{\partial x} = 0$

If we introduce, $\frac{\partial}{\partial \xi_k} = \frac{\partial}{\partial t} + \lambda_k \frac{\partial}{\partial x}$

then the equation from the previous page can be written as a set of N ordinary differential equation where,

$$\sum_{n=1}^{N} S_{kn} \frac{\partial w_n}{\partial \xi_k} = 0 \quad , \text{ where } \quad k = 1, \dots, N$$

where $\{S_{kn}\} = S$

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

We can now write the characteristic relations for the one-dimensional Euler equations.

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} F = 0$$

where $w = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$, $F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p) u \end{bmatrix}$

Next we transform the equations to non-conservatio form,

- However since we want to use the final set of equations to satisfy boundary conditions, then it would be easer to transform the vector w to one which is only dependent on ρ, u, and p since these are the quantities that we want to specify or update ~~the the~~ at the boundaries,

so from

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x} = 0$$

$$\frac{\partial W}{\partial t}\frac{\partial V}{\partial V} + \frac{\partial F}{\partial x}\frac{\partial V}{\partial V} = 0 \qquad \text{where } V = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}$$

$$\frac{\partial W}{\partial V}\frac{\partial V}{\partial t} + \frac{\partial F}{\partial W}\frac{\partial W}{\partial x}\frac{\partial V}{\partial V} = 0$$

$$\frac{\partial W}{\partial V}\frac{\partial V}{\partial t} + A\frac{\partial W}{\partial V}\frac{\partial V}{\partial x} = 0$$

Define $T = \dfrac{\partial W}{\partial V}$

$$T\frac{\partial V}{\partial t} + AT\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + T^{-1}AT\frac{\partial V}{\partial x} = 0$$

Define $\hat{A} = T^{-1}AT$

$$\frac{\partial V}{\partial t} + \hat{A}\frac{\partial V}{\partial x} = 0$$

where $\hat{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix}$

- Now we can perform the similarity transformation,

$$S \frac{\partial V}{\partial t} + S\hat{A} \frac{\partial V}{\partial x} = 0$$

$$S\hat{A} = \Lambda S$$

$$S\hat{A}S^{-1} = \Lambda = \begin{bmatrix} u & & 0 \\ & u+c & \\ 0 & & u-c \end{bmatrix}$$

where $S = \begin{bmatrix} 1 & 0 & -1/c^2 \\ 0 & \rho c & 1 \\ 0 & -\rho c & 1 \end{bmatrix}$

$$S^{-1} = \begin{bmatrix} 1 & \frac{1}{2c^2} & \frac{1}{2c^2} \\ 0 & \frac{1}{2\rho c} & -\frac{1}{2\rho c} \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

then $S \frac{\partial V}{\partial t} + \Lambda S \frac{\partial V}{\partial x} = 0$

using the equation $\sum_{n=1}^{N} S_{kn} \frac{\partial W_n}{\partial \xi_k} = 0$

then,

$$k=1, \quad \sum_{n=1}^{N} S_{1n} \frac{\partial V_n}{\partial \xi_1} = 0$$

$$S_{11}\frac{\partial V_1}{\partial \xi_1} + S_{12}\frac{\partial V_2}{\partial \xi_1} + S_{13}\frac{\partial V_3}{\partial \xi_1} = 0$$

$$1\frac{\partial \rho}{\partial \xi_1} + 0 \cdot \frac{\partial u}{\partial \xi_1} - \frac{1}{c^2}\frac{\partial p}{\partial \xi_1} = 0$$

$$\boxed{\frac{\partial \rho}{\partial \xi_1} - \frac{1}{c^2}\frac{\partial p}{\partial \xi_1} = 0}$$

$$k=2, \quad S_{21}\frac{\partial V_1}{\partial \xi_2} + S_{22}\frac{\partial V_2}{\partial \xi_2} + S_{23}\frac{\partial V_3}{\partial \xi_2} = 0$$

$$0 \cdot \frac{\partial \rho}{\partial \xi_2} + \rho c \frac{\partial u}{\partial \xi_2} + 1 \cdot \frac{\partial p}{\partial \xi_2} = 0$$

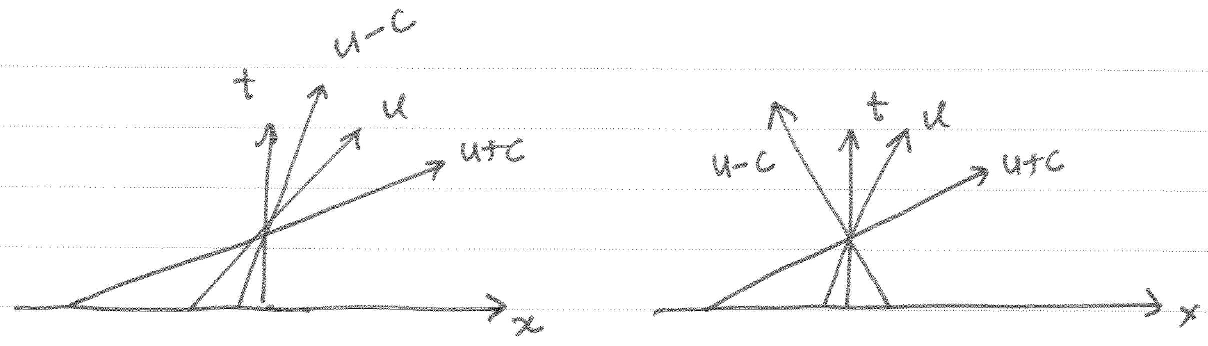
$$\boxed{\rho c \frac{\partial u}{\partial \xi_2} + \frac{\partial p}{\partial \xi_2} = 0}$$

$$k=3, \quad S_{31}\frac{\partial V_1}{\partial \xi_3} + S_{32}\frac{\partial V_2}{\partial \xi_3} + S_{33}\frac{\partial V_3}{\partial \xi_3} = 0$$

$$0 \cdot \frac{\partial \rho}{\partial \xi_3} - \rho c \frac{\partial u}{\partial \xi_3} + 1 \cdot \frac{\partial p}{\partial \xi_3} = 0$$

$$\boxed{-\rho c \frac{\partial u}{\partial \xi_3} + \frac{\partial p}{\partial \xi_3} = 0}$$

Judging by the eigenvalues of $\hat{A}$, where $\lambda_1 = u$, $\lambda_2 = u+c$, and $\lambda_3 = u-c$, then λ_1 and λ_2 are right running characteristics while λ_3 is left running if $M<1$ and all three are right runing if $M>1$



M > 1 M < 1

– Entrance (Inlet) and Exit Boundary conditions,

Based on the above characteristics of the 1D Euler equations the following is a summary of the boundary values / equations that need to be specified / solved.

	Quanties to be Specified	Characteristics that need to be solved
Inlet		
subsonic	P_t, T_t	1; u – c
supersonic	all	0
Exit		
subsonic	P_e (exit pressure)	2; u, u+c
supersonic	none	3, all

- subsonic inlet boundary
  - specify, p_t or/and T_t. Then static pressure and temperature can be evaluated fom isentropic relations,

$$p = p_t\left[1 - \frac{\gamma-1}{\gamma+1}\frac{u^2}{a_*^2}\right]^{\gamma/\gamma-1} = p(u)$$

$$T = T_t\left[1 - \frac{\gamma-1}{\gamma+1}\frac{u^2}{a_*^2}\right] = T(u)$$

$$\text{where } a_*^2 = 2\gamma\frac{\gamma-1}{\gamma+1}c_v T_t$$

To solve the left running characteristic, we need to solve from page 9 equation 3,

$$-\rho c\frac{\partial u}{\partial \xi_3} + \frac{\partial p}{\partial \xi_3} = 0$$

$$\text{since } \frac{\partial}{\partial \xi_k} = \frac{\partial}{\partial t} + \lambda_k\frac{\partial}{\partial x}$$

$$\text{then } \frac{\partial p}{\partial t} + (u-c)\frac{\partial p}{\partial x} - \rho c\left[\frac{\partial u}{\partial t} + (u-c)\frac{\partial u}{\partial x}\right] =$$

$$\frac{\partial p}{\partial t} - \rho c\frac{\partial u}{\partial t} = -(u-c)\left[\frac{\partial p}{\partial x} - \rho c\frac{\partial u}{\partial x}\right]$$

From the isentropic relations $\delta p = \frac{\partial p}{\partial u}\delta u$

Then using a first-order explicit finite-difference scheme, we have

$$\delta_t p - \rho c \, \delta_t u = -(u-c)\frac{\Delta t}{\cancel{\Delta x}}\left[\delta_x p - \rho c \, \delta_x u\right].$$

since $\delta_t p = \frac{\partial p}{\partial u}\delta_t u$

then $$\left(\frac{\partial p}{\partial u} - \rho c\right)\delta_t u = -(u-c)\frac{\cancel{\Delta t}}{\cancel{\Delta x}}\left[\delta_x p - \rho c \, \delta_x u\right]$$

Correction

$$\delta u = \frac{-(u-c)\Delta t\left[\delta_x p - \rho c \, \delta_x u\right]}{\frac{\partial p}{\partial u} - \rho c}$$

Since $\delta_t u = \frac{\delta u}{\Delta t}$

$$\boxed{\delta_t u = \frac{-(u-c)\frac{\cancel{\Delta t}}{\cancel{\Delta x}}\left[\delta_x p - \rho c \, \delta_x u\right]}{\frac{\partial p}{\partial u} - \rho c}}$$

Once $\delta_t u$ is computed then

Replace this with δu

$$u^{n+1} = u^n + \delta_t u \quad @ \text{ the boundary.}$$

$$p^{n+1} = p(u^{n+1})$$

$$T^{n+1} = T(u^{n+1})$$

$$\rho^{n+1} = \frac{p^{n+1}}{R T^{n+1}}$$

$$e^{n+1} = \rho^{n+1}\left[c_v T^{n+1} + \frac{1}{2}(u^{n+1})^2\right]$$

- subsonic or supersonic exit boundary, specify p_e if subsonic or none if supersonic,

Then from page 9 we must solve

$$\frac{\partial \rho}{\partial t} - \frac{1}{c^2}\frac{\partial p}{\partial t} = -u\left(\frac{\partial \rho}{\partial x} - \frac{1}{c^2}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial t} + \rho c\frac{\partial u}{\partial t} = -(u+c)\left(\frac{\partial p}{\partial x} + \rho c\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial p}{\partial t} - \rho c\frac{\partial u}{\partial t} = -(u-c)\left(\frac{\partial p}{\partial x} - \rho c\frac{\partial u}{\partial x}\right)$$

If subsonic you would only need to solve the first two equations while all three must be solved for supersonic

using first-order explicit discretizations,

$$\delta_t \rho - \frac{1}{c^2}\delta_t p = -u\cancel{\frac{\Delta t}{\Delta x}}\left[\delta_x \rho - \frac{1}{c^2}\delta_x p\right] = R_1$$

$$\delta_t p + \rho c\,\delta_t u = -(u+c)\cancel{\frac{\Delta t}{\Delta x}}\left[\delta_x p + \rho c\,\delta_x u\right] = R_2$$

$$\delta_t p - \rho c\,\delta_t u = -(u-c)\cancel{\frac{\Delta t}{\Delta x}}\left[\delta_x p - \rho c\,\delta_x u\right] = R_3$$

$$\delta p = \begin{cases} \dfrac{R_2 + R_3}{2} & \text{if } M_e > 1 \\ 0 & \text{if } M < 1 \text{ since } p_e \text{ is specifie} \end{cases}$$

then $\delta \rho = R_1 + \frac{1}{c^2}\delta_t p$ from the first eqn.

$\delta u = \dfrac{R_2 - \delta_t p}{\rho c}$ from the second eqn.

Then the exit values can be computed by,

$$\rho_e^{n+1} = \rho_e^n + d\rho$$

$$u_e^{n+1} = u_e^n + du$$

$$p_e^{n+1} = p_e^n + dp \quad \text{if } M_e > 1$$

$$T_e^{n+1},\ e^{n+1},\ \ldots \text{etc.}$$

2D/~~3D~~ Euler Equations.

In 2D, $\dfrac{\partial W}{\partial t} + \dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} = 0$

where $W = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}$ $F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{bmatrix}$, $G = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{bmatrix}$

The far-field boundary conditions are based on the method of characteristics while slip boundary condition, $\overline{V} \cdot \overline{n} = 0$, is employed along the wall.