Understanding the RANS equations

Revisit the mass-weighted Reynolds-Average conservation of momentum,

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{u}_i\tilde{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\overline{\tau_{ij}} - \overline{\rho u_i'' u_j''})$$

$$\text{where } \overline{\tau_{ij}} = \mu\left[\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right] + \mu\left[\left(\frac{\partial \overline{u_i''}}{\partial x_j} + \frac{\partial \overline{u_j''}}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \overline{u_k''}}{\partial x_k}\right]$$

$$\text{and } \overline{\tau_{ij}}^R = -\overline{\rho u_i'' u_j''}$$

However to represent the equation as closely as the original Navier-Stokes equations, the second part of $\overline{\tau_{ij}}$ can be moved into the definition of $\overline{\tau_{ij}}^R$, hence

$$\overline{\tau_{ij}} = \mu\left[\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial x_k}\right] = \left(\overline{\tau_{ij}}\right)^{lam}$$

where $\left(\overline{\tau_{ij}}\right)^{lam}$ is exactly the laminar stress gradient for the mean flow, and

$$\left(\overline{\tau_{ij}}\right)^R = \underbrace{-\overline{\rho u_i'' u_j''}}_{\substack{\text{apparent stress due to transport of momentum}\\ \text{by turbulent fluctuations}}} + \underbrace{\mu\left[\left(\frac{\partial \overline{u_i''}}{\partial x_j} + \frac{\partial \overline{u_j''}}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial \overline{u_k''}}{\partial x_k}\right]}_{\substack{\text{apparent stress due to turbulent deformations}\\ \text{due to fluctuations.}}}$$

apparent stress ~~gradients~~ due to transport of momentum by turbulent fluctuations

apparent stress due to turbulent deformations due to fluctuations.

- The second term in $(\overline{\tau_{ij}})^R$ is much smaller than the first.

We can perform a similar analysis on the Reynolds-averaged energy equation, where the original laminar term for the heat flux can be written as,

$$-(\nabla\cdot\vec{q})_{lam} = \frac{\partial}{\partial x_j}\left(k\frac{\partial\tilde{T}}{\partial x_j}\right) \qquad \text{from page 28}$$

and the apparent turbulent Reynolds heat flux can be written as,

$$-(\nabla\cdot\vec{q})_{turb} = \frac{\partial}{\partial x_j}\Big(-c_p\bar{\rho}\,\overline{T''u_j''} - c_p\,\overline{\rho' T'' u_j''} - \tilde{u}_j c_p\,\overline{\rho' T''}\Big)$$

The new set of equations have introduced new terms such as $(\overline{\tau_{ij}})^R$ and $-(\nabla\cdot\vec{q})_{turb}$ which introduces additional unknowns into the original Navier-Stokes equations.

New equations are required to complete the system of equations, also known as the closure problem and it has to be modelled through "turbulence models".

Parabolized and Thin-layer Navier Stokes.

These two forms of the Navier-Stokes equations are reduced forms of the NS or RANS equations by neglecting derivatives in a particular direction.

To derive them, first let us start from the complete differential-form in 3D,

$$\frac{\partial \vec{W}}{\partial t} + \frac{\partial \vec{F_1}}{\partial \xi} + \frac{\partial \vec{F_2}}{\partial \eta} + \frac{\partial \vec{F_3}}{\partial \zeta} = \frac{\partial \vec{F_{v_1}}}{\partial \xi} + \frac{\partial \vec{F_{v_2}}}{\partial \eta} + \frac{\partial \vec{F_{v_3}}}{\partial \zeta}$$

where $\vec{W} = \frac{1}{J}\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}$, $\vec{F_1} = \frac{1}{J}\begin{bmatrix} \rho V_1 \\ \rho V_1 u + \xi_x p \\ \rho V_1 v + \xi_y p \\ \rho V_1 w + \xi_z p \\ \rho V_1 H \end{bmatrix}$

$$\vec{F_{v_1}} = \frac{1}{J}\begin{bmatrix} 0 \\ \xi_x \tau_{xx} + \xi_y \tau_{xy} + \xi_z \tau_{xz} \\ \xi_x \tau_{yx} + \xi_y \tau_{yy} + \xi_z \tau_{yz} \\ \xi_x \tau_{zx} + \xi_y \tau_{zy} + \xi_z \tau_{zz} \\ \xi_x \Theta_x + \xi_y \Theta_y + \xi_z \Theta_z \end{bmatrix}$$

$$V_1 = \xi_x u + \xi_y v + \xi_z w$$

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\Theta_x = u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + k\frac{\partial T}{\partial x}$$

$$\frac{\partial u}{\partial x} = \xi_x \frac{\partial u}{\partial \xi} + \eta_x \frac{\partial u}{\partial \eta} + \zeta_x \frac{\partial u}{\partial \zeta}$$

$$\frac{\partial u}{\partial y} = \xi_y \frac{\partial u}{\partial \xi} + \eta_y \frac{\partial u}{\partial \eta} + \zeta_y \frac{\partial u}{\partial \zeta}$$

$$\vdots$$

etc.

Paracbolized Navier-Stokes.

The flow must meet the following three conditions,

1. steady
2. fluid moves only in one direction.
3. cross-flow components are negligible.

Then the following can be done.

1. derivatives of velocity components with respect to the streamwise component/direction are neglected.
2. derivatives of temperature with respect to the streamwise direction are neglected.
3. components of the viscous stress tensor an its work done in the streamwise component are dropped.

If ξ is the streamwise direction, then the Parabolized Navier–Stokes can be written as,

$$\frac{\partial W}{\partial t} + \frac{\partial \vec{F_1}}{\partial \xi} + \frac{\partial \vec{F_2}}{\partial \eta} + \frac{\partial \vec{F_3}}{\partial \zeta} = \frac{\partial \vec{F_{v_2}}}{\partial \eta} + \frac{\partial \vec{F_{v_3}}}{\partial \zeta}$$

$$\text{where } \frac{\partial u}{\partial x} \approx \eta_x \frac{\partial u}{\partial \eta} + \zeta_x \frac{\partial u}{\partial \zeta}$$

$$\frac{\partial u}{\partial y} \approx \eta_y \frac{\partial u}{\partial \eta} + \zeta_y \frac{\partial u}{\partial \zeta}$$

$$\vdots$$

etc.

Thin Layer Navies – Stokes

In very high Reynolds number flows, the viscous region is restricted to a very small region around the body.

It is then assumed that only gradients normal to the wall dominate the viscous terms, and hence ~~the~~ all derivaties for the viscous terms in the streamwise, $(\partial/\partial\xi)$ and cross flow directions, $(\partial/\partial\zeta)$ are dropped from the Navier–Stokes equations.

The resulting equations are called Thin Layer or Thin Shear Layer Navier-Stokes, and can be defined as,

$$\frac{\partial \vec{W}}{\partial t} + \frac{\partial \vec{F_1}}{\partial \xi} + \frac{\partial \vec{F_2}}{\partial \eta} + \frac{\partial \vec{F_3}}{\partial \zeta} = \frac{\partial \vec{F_{v_2}}}{\partial \eta}$$

$$\frac{\partial u}{\partial x} \approx \eta_x \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial y} \approx \eta_y \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial z} \approx \eta_z \frac{\partial u}{\partial \eta}.$$

$\vdots$

etc.

In the physical domain, the Thin-Layer Navier-Stokes equations can be written for incompressible fluid as, (Note: we are assuming incompressible flow, so that we can couple these equations to the solution of potential flow, which also assume incompressible flow).

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - \rho \frac{\partial}{\partial y} \overline{u''v''}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial y^2} - \rho \frac{\partial}{\partial y} \overline{(v'')^2}$$

$$\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial y^2} - \rho \frac{\partial}{\partial y} \overline{(v''w'')}$$

Boundary-Layer Equations

From the equations on page 34, we can further simplify the equations with the following two additional assumptions,

($\delta/L \ll 1$, thickness of boundary layer $\ll$ charac. length of flow)

1. There is noflow in the y-component and hence eliminate the y-component of momentum entirely,
2. Pressure, p is only assumed to be a function of x and z,

The equations for the momentum are then identical to the equation on 34 but with the y-component of momentum neglected.

In two-dimensions, the boundary-layer eqns can be written as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial y}\left(\overline{u''v''}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\dot{q}_w}{\rho C_p} + \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} - \frac{\partial}{\partial y}\left(\overline{T''v''}\right)$$

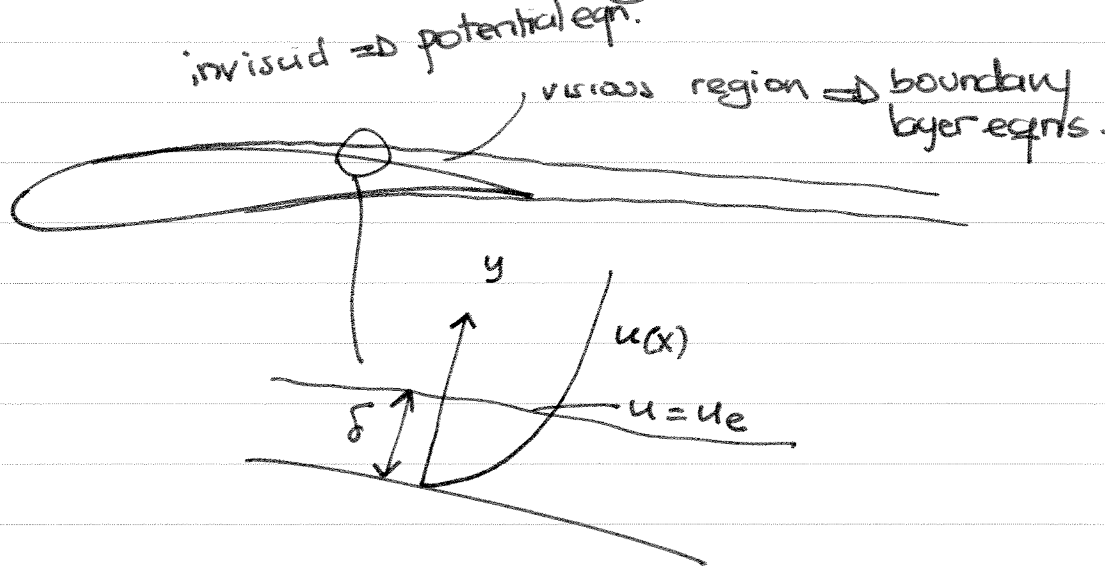
using the concept of eddy viscosity (explained later) the "apparent stress due to transport of momentum by turbulent fluctuations" or "Reynolds stress" can by written as

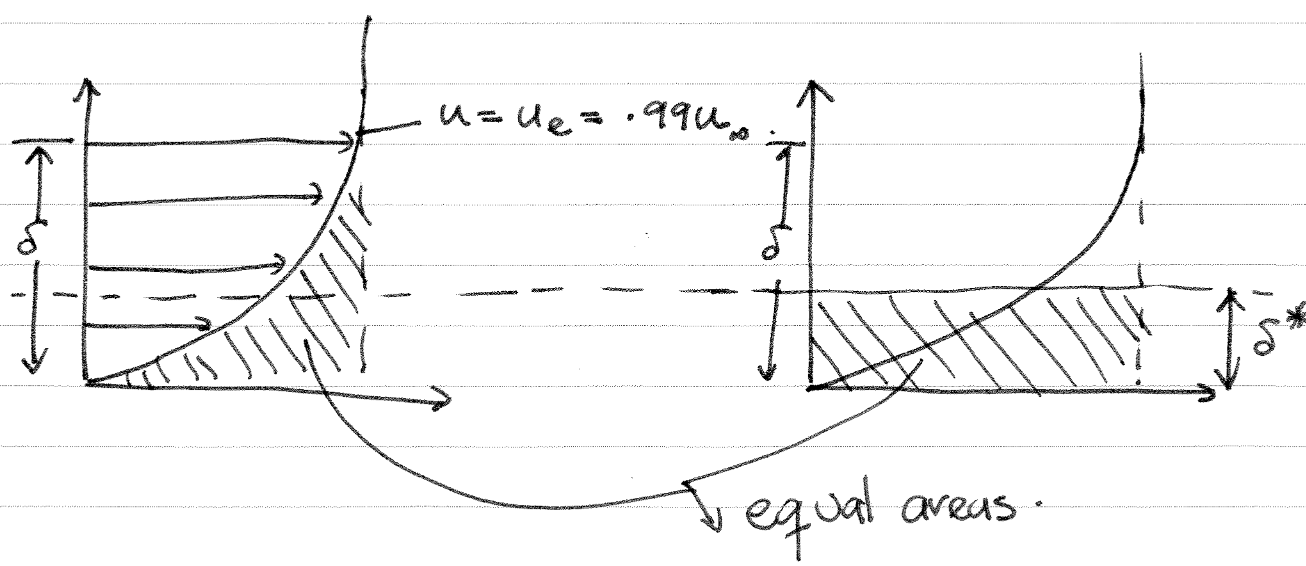
$$-\rho\overline{u''v''} = \rho\,\varepsilon_T\frac{\partial u}{\partial y} \quad \text{or} \quad \rho\mu_T\frac{\partial u}{\partial y}$$

The momentum equation can then be written as,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \underbrace{\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} + \frac{\mu_T}{\rho}\frac{\partial^2 u}{\partial y^2}}_{\frac{1}{\rho}(\mu+\mu_T)\frac{\partial^2 u}{\partial y^2}}$$

The boundary layer equations can then be coupled, together with the solution to the potential flow to provide a fast approach at modeling viscous effects.





δ^* = displacement thickness
- represents the amount that the thickness of the body must be increased so that the fictitious uniform inviscid flow has the same mass flow properties as the actual viscous flow.

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy \quad \text{or} \quad \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy.$$

There are two ways to couple the potential equations to the boundary layer equations.

Approach 1.

Developed by Le Balleur, and Carter and Wornom

1. Solve the potential flow and acquire u_{e_i} which is the inviscid velocity at every point on the airfoil.
2. Solve the boundary layer equations by first setting $-\frac{1}{\rho}\frac{dp}{dx} = u_{e_i}\frac{du_{e_i}}{dx}$,
3. Compute δ^* from the boundary layer solution.
4. Add δ^* to the airfoil geometry and recompute the potential eqns.

Steps 1 → 4 are repeated until convergence is achieved. A relaxation formula may be used to update the displacement thickness,

$$\delta^*(x) = \delta^{*o}(x)\left\{1 + \omega\left[\frac{u_{ev}(x)}{u_{e_i}(x)} - 1\right]\right\}$$

Approach 2

Developed by Veldman.

The external velocity, $u_e(x)$ and the displacement thickness, δ^* are treated as unknowns and solved simultaneously. As it iterates, the external velocity is updated via,

$$u_e(x) = u_e^{\circ}(x) + \delta u_e(x)$$

where $u_e^0(x)$ is the inviscid velocity and δu_e is the update calculated via,

$$\delta u_e = \frac{1}{\pi} \int_{x_a}^{x_b} \frac{d}{d\sigma}(u_e \delta^*) \frac{d\sigma}{x - \sigma}$$