MECH 539: Computational Aerodynamics
Department of Mechanical Engineering, McGill University

Project #2: Solution to the Laplace Equation
Due 18th February, 2016

Consider the Laplace equation on a unit square with Dirichlet boundary conditions, $u(x,0) = u(0,y) = u(1,y) = 0$, and $u(x,1) = 1$. Discretize the second derivatives of u with respect to x and y with a second-order finite-difference spatial discretization. Write the following numerical codes to solve the linear system using the Jacobi, Gauss-Seidel, and Successive Over-Relaxation (SOR) on three grid sizes, 100×100, 200×200, and 400×400. [Note: You may use a single precision floating-point format.] Provide the following in a written report:

1. Derive the truncation error for the second-order finite-difference spatial discretization for the Laplace equation.
2. Derive the modified equation for the scheme and note whether the error is dissipative, dispersize, or a combination of the two. Explain why?
3. Demonstrate the solution of the Laplace Equation for the 400×400.
4. Convergence of the residual versus the number of iterations for all three methods on the same plot. Provide a plot for each grid size. Discuss the difference between the schemes. Compute the condition number of the matrix using the Forsythe-Moler method and discuss the results.
5. Convergence of the residual versus the CPU time for all three methods on the same plot. Discuss the difference between the schemes. Comment on the number of vectors and arrays that were necessary for each scheme and compare the algorithms in terms of memory usage.
6. Demonstrate that the order of accuracy of the scheme on a plot where the y-axis is the log of the error and the x-axis is Δx, where both $\Delta x = \Delta y$.
7. Effect of the relaxation parameter on the SOR. Try several different values and discuss your findings. Show plots of the convergence of the residual for various relaxation parameters. Is there an optimum relaxation parameter? Is the optimum the same for all grid sizes.

Bonus Questions (For 2 extra points)

1. Demonstrate that the amplification factor for the Leap-Frog method to the solution of the linear advection equation is $G = \pm\sqrt{1 - \nu^2 \sin^2 \beta} - i\nu \sin \beta$, where $\nu = c\frac{\Delta t}{\Delta x}$.
2. Demonstrate that the leading order term for the simple explicit scheme for the pure diffusion equation in one-dimension $u_t = \alpha u_{xx}$ is

$$\left[-\frac{1}{2}\alpha^2 \Delta t + \frac{\alpha (\Delta x)^2}{12}\right] u_{xxxx}$$

   and show that the stability of the scheme is $0 \le r \le \frac{1}{2}$, where $r = \alpha\frac{\Delta t}{(\Delta x)^2}$.