Partial Differential Equations (PDEs)

PDEs can be classified by two different methods; either a physical classification or a mathematical classification.

Physical Classification

There are two different ways to classify a PDE with this approach; the first is equilibrium problems; second, marching problems.

1) Equilibrium Problems :

- A solution of a given PDE is required within a closed domain and prescribed boundary conditions.
- Examples :
  a) steady-state temperature distribution.
  b) incompressible inviscid flows.
  c) Equilibrium Stress Distributions in Solids.

- Example 1 : The steady-state temperature distribution can be solved by the Laplace Equation.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad , \quad 0 \leq x \leq 1 \, , \quad 0 \leq y \leq 1$$

with boundary conditions

$$T(0,y) = 0 \; ; \quad T(1,y) = 0 \, , \quad T(x,0) = T_0 \, ,$$
$$T(x,1) = 0 .$$

This is a linear PDE and it can be solved via separation of variables. Thus the temperature, T can be written as the product of a function of x and a function of y, $\quad T(x,y) = X(x)\, Y(y)$

- In 1965, Weinberger showed that this is the only solution to the problem if the solution satisfies both the PDE and the boundary conditions.

- $T(x,y)$ is then substituted into the Laplace Equation, two ODE arise from it,

$$X'' + \alpha^2 X = 0 \qquad Y'' - \alpha^2 Y = 0$$
$$X(0) = X(1) = 0 \qquad Y(1) = 0$$

, where α is a by-product of the separation process.

- The general solution to the ODEs can then be written as

$$X(x) = A \sin(n\pi x)$$
$$Y(y) = C \sinh[n\pi(y-1)]$$

- Combining the two solutions produces

$$T(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh[n\pi(y-1)]$$

as the general solution to the PDE.

- From the boundary condition $T(x,0) = T_0$, the coefficients A_n can be determined to be,

$$A_n = \frac{2T_0}{n\pi} \frac{[(-1)^n - 1]}{\sinh(n\pi)}$$

2) Marching Problems :

- A solution of a given PDE is required on an open domain subject to both initial and boundary conditions.

- Example 1 : Determine the displacement $y(x,t)$ of a string of length l between $x=0$ and l if $y(x,0) = \sin \pi x/l$.

  - This problem can be modeled by the wave equation.

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$, where a is a positive constant.

The boundary conditions are $y(0,t) = y(l,t) =$

and initial conditions, $y(x,0) = \sin \frac{\pi x}{l}$,

$$y'(x,t)\Big|_{t=0} = 0 .$$

- The general solution to this problem is

$$y(x,t) = \sin\left(\pi \frac{x}{l}\right) \cos\left(a\pi \frac{t}{l}\right).$$

- Example 2 : Determine the time dependent temperature distribution in a 1D solid if the boundary conditions are $T(0,t) = 0$ and $T(1,t) = T_0$ and initial conditions are $T(x,0) = ($

- The solution to this problem is generally solved by setting

$$T(x,t) = u(x) + v(x,t).$$

, where the solution is the sum of two terms. The first, $u(x)$, is the solution acquired when t is very large. In other words, $u(x)$ is the steady state solution. The second, $v(x,t)$ is the transient solution that dies out at large t.

- Let os substitute $T(x,t)$ into the goverming equation.

Gorverning Equation : $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

, where α is the thermal diffosivity.

$$\frac{\partial}{\partial t}\left(u(x) + v(x,t)\right) = \alpha \frac{\partial^2}{\partial x^2}\left(u(x) + v(x,t)\right).$$

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial^2 v}{\partial x^2}$$

Subject to $u(0) = 0$ and $u(1) = T_0$.

To satisfy these boundary conditions, $\frac{\partial^2 u}{\partial x^2} = 0$

The solution to the steady problem is then $u(x) = T_0 x$.

- Then the equation $\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}$ must satisfy $v(0,t) = v(1,t) = 0$ and initial condition $T(x,0) = u(x) + v(x,0) = 0$

$$v(x,0) = -T_0 x .$$

Through the method of separation of variables we acquire the general solution that satisfies the PDE,

$$v(x,t) = e^{-\alpha n^2 \pi^2 t} \sin(n\pi x)$$

The complete general solution can then be written as

$$T(x,y) = u(x) + v(x,t)$$

$$= T_0 x + \sum_{n=1}^{\infty} \frac{2T_0 (-1)^n}{n\pi} e^{-n^2\pi^2 \alpha t} \cdot \sin(n\pi x$$

Mathematical Classification

- Characteristic lines or surfaces are directions in which informatio is propagated through the domain that is governed by partial differential equations
- Along these characteristic lines or surfaces, a particular properties' value or derivative may stay constant or conta a discontinuity.
- The classification of PDEs is then based on the type of characteristic line or surface.

- Let us consider a general 2nd. Order PDE,

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi = g.$$

where $a, b, \dots, f$ are either functions of x and y or just constants.

- The classification of a second-order PDE depends only on the second-derivatives, so the equation above may be written as

$$a\phi_{xx} + b\phi_{xy} + c\phi_{yy} = -(d\phi_x + e\phi_y + f\phi - g)$$
$$= H.$$

- The next step is to identify the characteristic lines or curves. Therefore what we really want to find is whether there are locations where the second derivatives are discontinuous? Since the characteristics are the locus of points along which the second derivatives may not be continuous. To accomplish this, the following approach is required.

- Define τ as a parameter that varies along a curve C in the plane x-y. Hence $x = x(\tau)$ and $y = y(\tau)$.

- Since x and y are functions of τ, then we can also write the following,

$$\phi_x = p(\tau), \quad \phi_y = q(\tau), \quad \phi_{xx} = u(\tau), \quad \phi_{xy} = v(\tau)$$
$$\phi_{yy} = w(\tau).$$

The equation for the second-order PDE can then be written as

$$a u(\tau) + b v(\tau) + c w(\tau) = H$$

To relate the first derivatives to the second,

$$\frac{d\phi_x}{d\tau} = \frac{dp}{d\tau} = \frac{\partial^2 \phi}{\partial x^2}\frac{dx}{d\tau} + \frac{\partial^2 \phi}{\partial x \partial y}\frac{dy}{d\tau}$$

$$= u\frac{dx}{d\tau} + v\frac{dy}{d\tau}$$

Similarly, $\quad \dfrac{d\phi_y}{d\tau} = \dfrac{dq}{d\tau} = v\dfrac{dx}{d\tau} + w\dfrac{dy}{d\tau}$

If we consider that $u, v,$ and w are the unknowns, then the three equations above can be written as a system of equations,

$$\begin{bmatrix} a & b & c \\ \frac{dx}{d\tau} & \frac{dy}{d\tau} & 0 \\ 0 & \frac{dx}{d\tau} & \frac{dy}{d\tau} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} H \\ \frac{dp}{d\tau} \\ \frac{dq}{d\tau} \end{bmatrix}$$

In order to locate values of $u, v,$ and w that might be discontinoous, we are actually looking for non-unique solutions to the system of equations above.

From the definition of a non-unique soln. to a system of equations, this is only true if the determinant is zero.

Hence $\det(A) = 0$

or

$$a\left(\frac{dy}{d\tau}\right)^2 - b\left(\frac{dx}{d\tau}\right)\left(\frac{dy}{d\tau}\right) + c\left(\frac{dx}{d\tau}\right)^2 = 0$$

$$a(dy)^2 + (-b\,dx\,dy) + c(dx)^2 = 0$$

If $h = \frac{dy}{dx}$, then $ah^2 - bh + c = 0$,

then

$$h = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since $h = \frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a}$, then the

solution represents the slope of the curve $y = y(x)$ which is called the characteristics of the PDE.

To repeat, along these curves the second derivatives are not uniquely determined by the values of ϕ or $\frac{d\phi}{dx}$

or $\frac{d\phi}{dy}$.

Hyperbolic

- If $b^2 - 4ac > 0$, then $\frac{dy}{dx}$ has two distinct real characteristic curves.

Parabolic

- If $b^2 - 4ac = 0$, then $\frac{dy}{dx}$ has only one distinct real characteristic curve

Elliptic:

- If $b^2 - 4ac < 0$, then no real characteristic exists.

- If a, b, and c are not constants, then the equation may change from region to region.

- Equations of PDEs can be transformed into a characteristic coordinate system by a coordinate transforonation. (canonical form)
- We desire to transform or map the coordinates from the (x,y) coordinate system to the (ξ,η) coordinate system.
- The new PDE in the transformed coordinate system can be represented by,

$$A\phi_{\xi\xi} + B\phi_{\xi\eta} + C\phi_{\eta\eta} + \cdots\cdots = g(\xi,\eta)$$

where $A = a\xi_x^2 + b\xi_x\xi_y + c\,\xi_y^2$, where $\xi_x = \frac{\partial\xi}{\partial x}$

$$B = 2a\xi_x\eta_x + b\xi_x\eta_y + b\xi_y\eta_x + 2c\xi_y\eta_y$$

$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

The discriminant b^2-4ac that was used to classify the equations is now

$$B^2-4AC = (b^2-4ac)(\xi_x\eta_y - \xi_y\eta_x)^2$$

Hyperbolic PDEs

- Two distinct real characteristics exists,

$$\frac{dy}{dx} = \lambda_1 \quad , \quad \frac{dy}{dx} = \lambda_2$$

- We then obtain, $y = \lambda_1 x + K_1$ and $y = \lambda_2 x + K_2$
- A hyperbolic PDE can be written in canonical form as

$$\phi_{\xi\eta} = f(\xi,\eta,\phi,\phi_\xi,\phi_\eta)$$

In the transformed cooridinate $\xi = y - \lambda_1 x$

$$\eta = y - \lambda_2 x$$

Example : Second-order wave Equation.

$$u_{tt} = c^2 u_{xx} \quad \text{on the interval } -\infty < x < \infty$$

with initial data

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

The transformation produces

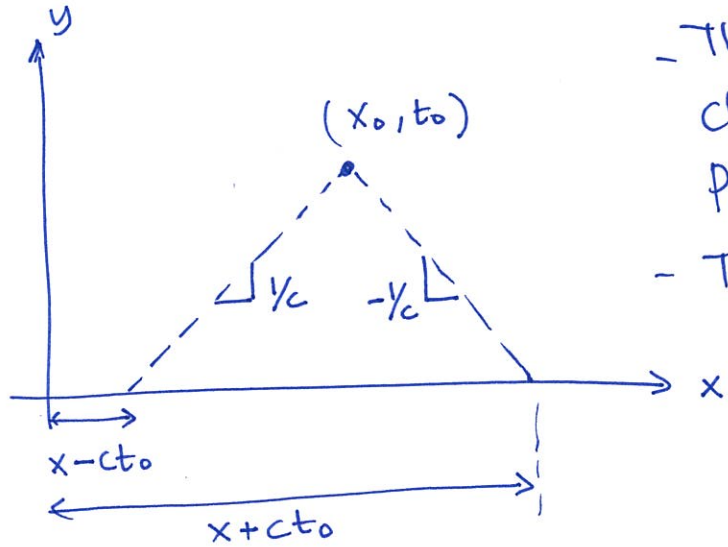
$$u_{\xi\eta} = 0 \,, \quad \text{where } \xi = x + ct$$
$$\eta = x - ct.$$

We integrate to obtain the solution

$$u(x,t) = F_1(\xi) + F_2(\eta) = \underbrace{F_1(x+ct) + F_2(x-ct)}_{\text{D'Alembert's solution.}}$$

Finally this results to,

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau$$



- There are two characteristics that pass through (x_0, y_0)
- The solution $u(x,t)$ at (x_0, t_0) depend only upon the initial data contain in

$$x_0 - ct_0 \leq x \leq x_0 + ct_0$$

The first term $\frac{f(x+ct)+f(x-ct)}{2}$ represents the initial data along the characteristics, while the second term $\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$ represents the effect of the information at $t=0$ on the closed interval.

- The interesting aspect of this observation is that the solution at (x_0, t_0) is only dependent on the information bounded by these characteristics.

- This would be equivalent observation to marching problems made earlier where once the initial conditions are specified, then the solution is marched outward in time.

- All hyperbolic problems have a similar nature due to the presents of two distinct real characteristics.
- Such PDEs describe time-dependent physical processes that are not evolving toward a steady state.

Parabolic Equations

- The parabolic case occurs when $b^2-4ac=0$

  hence $\frac{dy}{dx} = \frac{b}{2a}$

- In canonical form, $\phi_{\xi\xi} = g(\phi_\xi, \phi_\eta, \phi, \xi, \eta)$.

- Because only one characteristic exist, parabolic PDEs do not have the limited bounded regions that hyperbolic PDEs have. Instead, the solution at t depends entirely upon the complete phyisical solution from the previous time $t \leq t_1$.

- Example : Diffusion Equation.

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$

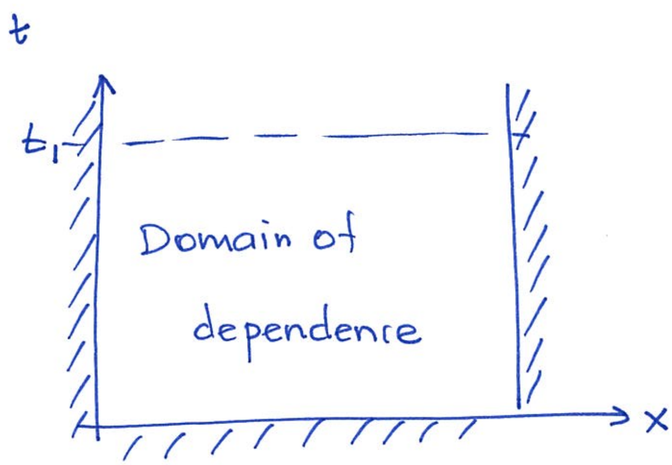
, where μ is the kinematic viscosity of the fluid.

The left-term is the local acceleration while the right-hand-side is the shear stress in the fluid, where

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial \tau}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right).$$

This equation is subject to

$$u(0,y) = 0$$
$$u(t,0) = U \quad t > 0$$
$$u(t,\infty) = 0$$



Through the use of a similarity transformation approach,

$$u = U\left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta\right)$$

where $\eta = \dfrac{y}{2\sqrt{\mu t}}$

- Parabolic PDEs descibe time-dependent processes that are evolving toward a steady state

Elliptic PDEs

- For elliptic PDEs, $b^2 - 4ac < 0$,
- Then $y - c_1 x + i c_2 x = k_1$

  $y - c_1 x - i c_2 x = k_2$
- In canonical form, $\phi_{\xi\xi} + \phi_{\eta\eta} = h(\xi, \eta, \phi, \phi_\xi, \phi_\eta)$.

- If we select ξ and η to be the real and imaginary parts of the complex conjugate function,

  $\xi = y - c_1 x$ and $\eta = c_2 x$.

Example : Solve the Laplace equation

$$\nabla^2 u = 0 \qquad 0 \leq r < 1 \quad , -\pi \leq \theta \leq \pi$$

subject to boundary conditions,

$$\frac{\partial u}{\partial r}(1, \theta) = f(\theta) \quad , \; -\pi \leq \theta \leq \pi .$$

Solution can be written as

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

where a_n and b_n depend on the boundary conditions on the disk.

- This is characteristic of elliptic problems, where the boundary conditions dictate the solution in the domain.
- Elliptic PDEs describe processes that have already reached a steady state.

Classification of systems of Equations

- Consider

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = 0$$

where A and B are functions of t, x, and y.

- The system is considered hyperbolic at (x,t) if the eigenvalues of A are real and distinct and can be written as $T^{-1}\lambda T$ where λ is a diagonal matrix containing the eigenvalues and T are the eigenvectors.
- The same can be said for matrix B for the y-direction
- Hence the system of equations can exhibit hyperbolic behavior in (x,t) but elliptic in (y,t) depending on the eigenvalues of A and B.

- Example 1:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

where $u = \begin{bmatrix} v \\ w \end{bmatrix}$ and $A = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}$

This equation is actually the second-order wave equation written as a system of two-first-order systems.

Let $v = \frac{\partial u}{\partial t}$ and $w = c\frac{\partial u}{\partial x}$

Then $\frac{\partial v}{\partial t} = c\frac{\partial w}{\partial x} \Longrightarrow \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$\frac{\partial w}{\partial t} = c\frac{\partial v}{\partial x} \Longrightarrow c\frac{\partial^2 u}{\partial t \partial x} = c\frac{\partial^2 u}{\partial x \partial t}$

The eigenvalues of A are

$$\det |A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & -c \\ -c & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - c^2 = 0$$

$$\lambda = \pm c$$

The characteristics are then $\left(\frac{dx}{dt}\right)_1 = c$

$\left(\frac{dx}{dt}\right)_2 = -c$

Thus the equation is hyperbolic,
If A has complex eigenvalues, then its elliptic.

Example 2 :

Consider a system with two independent variables,

$$a_1\frac{\partial u}{\partial x} + b_1\frac{\partial v}{\partial x} + c_1\frac{\partial u}{\partial y} + d_1\frac{\partial v}{\partial y} = f_1$$

$$a_2\frac{\partial u}{\partial x} + b_2\frac{\partial v}{\partial x} + c_2\frac{\partial u}{\partial y} + d_2\frac{\partial v}{\partial y} = f_2$$

The system can be written as,

$$A \frac{\partial w}{\partial x} + B \frac{\partial w}{\partial y} = F$$

where $w = \begin{bmatrix} u \\ v \end{bmatrix}$, $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$

Considering only solutions along curve C,

$$\frac{du}{d\tau} = \frac{\partial u}{\partial x}\frac{dx}{d\tau} + \frac{\partial u}{\partial y}\frac{dy}{d\tau}$$

$$\frac{dv}{d\tau} = \frac{\partial v}{\partial x}\frac{dx}{d\tau} + \frac{\partial v}{\partial y}\frac{dy}{d\tau}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ \frac{dx}{d\tau} & 0 & \frac{dy}{d\tau} & 0 \\ 0 & \frac{dx}{d\tau} & 0 & \frac{dy}{d\tau} \end{bmatrix} \begin{bmatrix} \frac{du}{dx} \\ \frac{dv}{dx} \\ \frac{du}{dy} \\ \frac{dv}{dy} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \frac{du}{d\tau} \\ \frac{dv}{d\tau} \end{bmatrix}$$

Similar to what was done before,

$$\left(-\frac{dy}{d\tau}\right)^2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \frac{dx}{d\tau}\frac{dy}{d\tau}\left(\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \right)$$

$$- \left(\frac{dx}{d\tau}\right)^2 \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$$

This can be simplified to,

$$\hat{A}\left(\frac{dy}{dx}\right)^2 - \hat{B}\left(\frac{dy}{dx}\right) + \hat{C} = 0$$

where $\hat{A} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and so forth.

Then the classification can be done by computing.

$$\hat{B}^2 - 4\hat{A}\hat{C}$$

If $\hat{B}^2 - 4\hat{A}\hat{C} > 0$, the equations are hyperbolic.

The rest follows.

Basics of Discretization Methods

Let us consider the heat equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Then use a forward-difference approximation to model $\frac{\partial u}{\partial t}$ and a second-order central-difference to approximate the spatial derivative; the heat equation can then be approximated by,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

We can then write the following,

$$\underbrace{\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} = \underbrace{\frac{u_{j\cancel{+1}}^{n+1} - u_j^n}{\Delta t} - \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j+1}^n\right)}_{\text{FDE}}$$

$$= \underbrace{\left[-\frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)\Delta t + \frac{\alpha}{12}\left(\frac{\partial^2 u}{\partial x^4}\right)(\Delta x)^2 + \cdots\right]}_{\text{Truncation Error (TE)}}$$

$$\parallel$$

$$O(\Delta t, \Delta x^2).$$

For the finite-difference representation to be correct or in other words, "Does it solve the same PDE?", then we have to check for the consistency and stability of the scheme.

Consistency

- A FDE is consistent if in the limit as the mesh is refined, the PDE is recovered. We have to show this by demonstrating that the truncation error vanishes under mesh refinement.

- From the previous page,

$$TE = -\frac{1}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)\Delta t + \underbrace{\frac{\alpha}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)(\Delta x^2)}_{\text{must vanish.}} + \cdots$$

but what about Δt?

- Hence, the scheme is only consistent if $\lim_{\Delta t, \Delta x \to 0} \frac{\Delta t}{\Delta x} \to 0$.

- In conclusion both Δt and Δx must approach zero for the truncation error to vanish.

- But what happens if they approach zero at the same rate; Let us examine, the DuFort – Frankel approach to solving the heat equation,

$$\underbrace{\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} = \underbrace{\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^{n} - u_j^{n+1} - u_j^{n-1} + u_{j-1}^{n} \right)}_{\text{FDE}}$$

$$= \underbrace{\frac{\alpha}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)(\Delta x)^2 - \alpha\left(\frac{\partial^2 u}{\partial t^2}\right)\left(\frac{\Delta t}{\Delta x}\right)^2 - \frac{1}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)(\Delta t)^2}_{\text{TE}}$$

If we examine what happens when $\lim_{\Delta t, \Delta x \to 0} \left(\frac{\Delta t}{\Delta x}\right) = ?$

The first term $\frac{\alpha}{12}\left(\frac{\partial^2 u}{\partial x^4}\right)(\Delta x)^2 \longrightarrow 0$ when $(\Delta x) \longrightarrow 0$

The third term $-\frac{1}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)(\Delta t)^2 \longrightarrow 0$ when $(\Delta t) \longrightarrow 0$

- What happens to the second term?

If $\Delta t \to 0$ faster than $\Delta x \to 0$, then $\alpha\left(\frac{\partial^2 u}{\partial t^2}\right)\left(\frac{\Delta t}{\Delta x}\right)^2 \to 0$

But if $\cancel{\left(\frac{\Delta t}{\Delta x}\right) \to 0}$ ~~at the~~ $(\Delta t) \to 0$ and $(\Delta x) \to 0$ at the same rate, then $\left(\frac{\Delta t}{\Delta x}\right) \to \beta$ and the scheme would be solving,

$$\frac{\partial u}{\partial t} + \boxed{\alpha\beta^2 \frac{\partial^2 u}{\partial t^2}} = \alpha \frac{\partial^2 u}{\partial x^2}$$ and not the heat eq

- The additional term comes from the truncation error and renders the Dufort – Frankel scheme inconsistent.

Stability.

- A scheme is stable if any error is not permitted to grow as the calculation proceeds.
- Let us consider the scheme below to approximate the heat equation,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

- The exact solution of this equation can be denoted by, D.
- If we use a computer with ~~e~~ a finite precision such as single or double precision, would yield the answer N.
- Let us denote A as the exact solution of the PDE.

The difference between D and N is then the ~~d~~round-off error while the difference between D and A is the discretization error,

Discretization error, $= D - A$

Round-off error, $= N - D$

- To address the stability of the scheme, we have to ensure that the two errors above damp as the solution progresses.
- To achieve this a Fourier or von Neumann analysis is used.

- For example let us define ε as $N - D$ (round off error)
- Then the scheme can be written as,

$$\frac{D_j^{n+1} + \varepsilon_j^{n+1} - D_j^n - \varepsilon_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(D_{j+1}^n + \varepsilon_{j+1}^n - 2D_j^n - 2\varepsilon_j^n + D_{j-1}^n + \varepsilon_{j-1}^n\right)$$

- since D is the exact solution to the FDE, then, it will satisfy the equation above without any error and hence, this results to,

$$\frac{\varepsilon_j^{n+1} - \varepsilon_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(\varepsilon_{j+1}^n - 2\varepsilon_j^n + \varepsilon_{j-1}^n\right)$$

- This demonstrates that the error equation must satisfy the same finite difference equation, however any error solution at the time-level, n must diminish in amplitude such that ε_j^{n+1} is smaller. This ensures that the errors are prevented from growing and this makes for a stable scheme.

- To proof this let us assume that at time-level, n, the error has a solution,

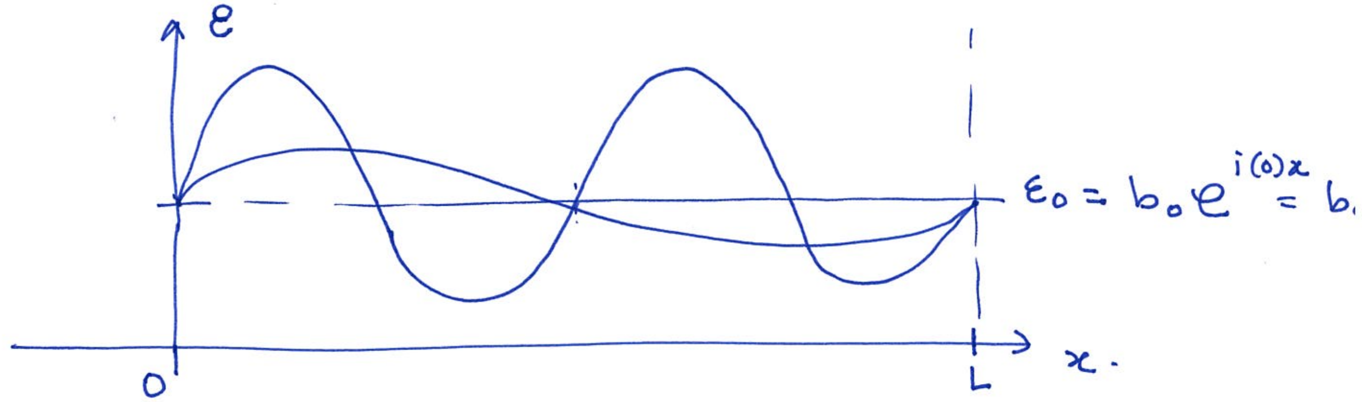
$$\varepsilon(x,t) = \sum_m b_m(t)\, e^{ik_m x},$$ where the term on the right-hand-side is a Fourier series; $b_m(t)$ are fourier coefficients, and k_m is the wave number

$$k_m = \frac{2\pi m}{2L} \quad ; \quad m = 0, 1, \ldots, M$$

where M is the total number of grid points.

The corresponding frequency is, $f_m = \frac{k_m}{2\pi} = \frac{m}{2L}$

- If $m = 0$, then $f_0 = 0$ which corresponds to the zeroth mode or the constant error throughout the domain.
- If $m = M$, then $f_M = \frac{M}{2L}$, which corresponds to the highest frequency,



- Therefore the error is a superposition of errors at all frequencies that are possible within the domain.

- Notice that $\varepsilon(x,t) = \sum_m b_m(t) e^{ik_m x}$, where $b_m(t)$ is only a function of t while $e^{ik_m x}$ is a function of x, therefore we can write $\varepsilon(x,t) = \sum_m (e^{at})_m e^{ik_m x}$.

- Each error can then be written as $\varepsilon_m = e^{at} e^{ik_m x}$

- If we substitute this equation into the equation on page 21, then we obtain,

$$\frac{e^{a(t+\Delta t)} e^{ik_m x} - e^{at} e^{ik_m x}}{\Delta t} = \frac{\alpha}{(\Delta x)^2} \left[e^{at} e^{ik_m (x+\Delta x)} - 2 e^{at} e^{ik_m x} + e^{at} e^{ik_m (x-\Delta x)} \right]$$

- Divide by $e^{at} e^{ik_m x}$,

$$\frac{e^{a\Delta t} - 1}{\Delta t} = \frac{\alpha}{(\Delta x)^2} \left[e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right]$$

$$e^{a\Delta t} - 1 = \frac{\alpha \Delta t}{(\Delta x)^2} \left[e^{ik_m \Delta x} + e^{-ik_m \Delta x} - 2 \right]$$

From the identity $\frac{e^{ik_m \Delta x} + e^{-ik_m \Delta x}}{2} = \cos(k_m \Delta x)$,

$$e^{a\Delta t} = \frac{\alpha \Delta t}{(\Delta x)^2} \left[2\cos(k_m \Delta x) - 2 \right] + 1$$

$$e^{a\Delta t} = 1 + 2r \left(\cos(k_m \Delta x) - 1 \right)$$

$$= 1 - 4r \sin^2 \frac{(k_m \Delta x)}{2}$$

- $e^{a\Delta t}$ reprents the growth of the error after Δt step, Hence we can write $\varepsilon_j^{n+1} = e^{a\Delta t} \varepsilon_j^n$.

- For the scheme to be stable $\left| e^{a\Delta t} \right| \leq 1$

This requires $\underbrace{\left| 1 - 4r\sin^2 \frac{(k_m \Delta x)}{2} \right|}_{\text{amplification factor}} \leq 1$

The two solutions to this equation are

$1 - 4r\sin^2 \beta/2 \geq -1$ or $1 - 4r\sin^2 \beta/2 \leq 1$

$-4r\sin^2 \frac{\beta}{2} \geq -2$ $\quad$ $-4r\sin^2 \frac{\beta}{2} \leq 0$

$r\sin^2 \frac{\beta}{2} \leq \frac{1}{2}$ $\quad$ $r\sin^2 \frac{\beta}{2} \geq 0$.

so $r \leq \frac{1}{2}$ and $r \geq 0$ for the scheme to be stable.

- Therefore $\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$

If α and Δx are chosen apriori, then Δt can be compute from $\Delta t \leq \frac{(\Delta x)^2}{2\alpha}$.

This equation bounds the value of Δt.

- For the implicit scheme,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right),$$

The equation for the growth of the error or amplification factor can be written as,

$$e^{a\Delta t} = \frac{1}{1+2r-2r\cos\beta} = \frac{1}{1+4r\sin^2\frac{\beta}{2}}$$

To ensure stability $|e^{a\Delta t}| \le 1$

$$\left|\frac{1}{1+4r\sin^2\frac{\beta}{2}}\right| \le 1.$$

From this as long as $r \ge 0$, the scheme is stable

so,
$$\frac{\alpha\,\Delta t}{(\Delta x)^2} \ge 0$$
$$\Delta t \ge 0$$

Hence there is no restriction on the value of Δt.

- Let us repeat this for the first-order wave equation.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$ where c is a constant.

If we use the Lax scheme to solve the equation,

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - c\frac{\Delta t}{\Delta x}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2}\right).$$

then $e^{a\Delta t} = \cos(k_m \Delta x) - i\nu \sin(k_m \Delta x)$.

where $\nu = \dfrac{c\,\Delta t}{\Delta x}$ which is also called the CFL or Courant number.

To ensure stability,

$$|G| \leq 1$$

$$|e^{a\Delta t}| \leq 1$$

$$|\cos(k_m \Delta x) - i\nu \sin(k_m \Delta x)| \leq 1$$

$$|\cos(k_m \Delta x) - i\nu \sin(k_m \Delta x)|^2 \leq 1^2$$

$$\cos^2(k_m \Delta x) + \nu^2 \sin^2(k_m \Delta x) \leq 1$$

Since $\cos^2 x + \sin^2 x = 1$, then the scheme is stable as long as $\nu^2 \leq 1$

or $|\nu| \leq 1$.

Stability Analysis for Systems of Equations

Let us consider, $\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$ where E and F are vectors and that

$$F = F(E).$$

Then we can write $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial E}\frac{\partial E}{\partial x}. = A\frac{\partial E}{\partial x}$ where

A is the Jacobian matrix,

Hence we can write, $\frac{\partial E}{\partial t} + A\frac{\partial E}{\partial x} = 0$

If we hold A constant while trying toadvance E, then we can apply our Fourier analysis, to produce,

$$E_j^{n+1} = \frac{\cancel{A}^n E_{j+1}^n + \cancel{A}^n E_{j-1}^n}{2} - \frac{\Delta t}{\Delta t}\left(\frac{A^n E_{j+1}^n - A^n E_{j-1}^n}{2}\right).$$

$$= \frac{1}{2}\left(I - \frac{\Delta t}{\Delta x}A^n\right)E_{j+1}^n + \frac{1}{2}\left(I + \frac{\Delta t}{\Delta x}A^n\right)E_{j-1}^n)$$

for the Lax approach.

Then from our Fourier analysis, we can write

$$\left[e^{a\Delta t}\right] = I\cos(k_m\Delta x) - i\left(\frac{\Delta t}{\Delta x}\right)A\sin(k_m\Delta x)$$

$\uparrow$

~~vector~~
matrix.

To ensure stability, $\underbrace{\left[e^{a\Delta t} \right]}_{\text{matrix}} \leq 1$.

To satisfy this requirement we have to ensure that the eigenvalue of matrix $\left[e^{a\Delta t} \right]$ is less than 1.

or $|\sigma_{max}| \leq 1$ where σ_{max} is the largest eigenvalue.

- This leads to the requirement that the largest eigenvalue of A, λ_{max} must be $\left| \lambda_{max} \frac{\Delta t}{\Delta x} \right| \leq 1$.