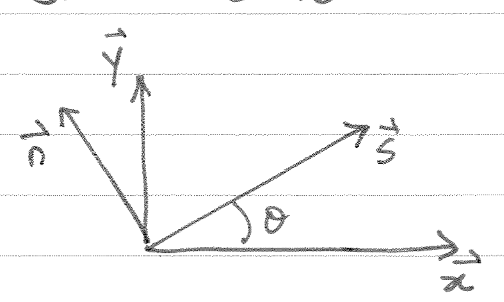
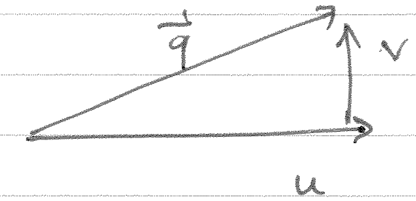
# Jameson's Rotated Difference Method

- The major difficulty in applying the Murman-Cole type dependent difference scheme is determining when and when not to change from central to backward difference approximation.
- In a two dimensional case, you could have a scenario where $\vec{q} = u\hat{i} + v\hat{j}$ where $|\vec{q}| > a$ and hence the flow is supersonic but each component is still subsonic, $|u| < a$ and $|v| < a$, which suggest central difference approximations would be used even if the flow is localy supersonic.

- Jameson recommended simply rotating the local coordinate system to the difection of the flow. Decisions made in the rotated coordinate system were then used to determine the difference approximations in the x-y coordinate system.

- Jameson's rotated coordinates are defined as,




$$\frac{\partial x}{\partial s} = \cos\theta = \frac{u}{|\vec{q}|}$$

$$\frac{\partial x}{\partial n} = -\sin\theta = \frac{v}{|\vec{q}|}$$

$$\frac{\partial y}{\partial s} = \sin\theta = \frac{-v}{|\vec{q}|}$$

$$\frac{\partial y}{\partial n} = \cos\theta = \frac{u}{|\vec{q}|}$$

- The full potential equation in 2D can be simplified to,
  $$(a^2 - q^2)\phi_{ss} + a^2 \phi_{nn} = 0$$

  from

  $$(a^2 - \phi_x^2)\phi_{xx} + -2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} = 0$$

- The Murman – Cole scheme can now be applied directly to the transformed equation as follows

  (i) The $\phi_{nn}$ term is central differenced.

  (ii) If
  $$q^2 \begin{cases} < a^2, & \text{then } \phi_{ss} \text{ is central differenced} \\ > a^2, & \text{then } \phi_{ss} \text{ is backward differenced.} \end{cases}$$

- In terms of the original coordinate system,

  $$\phi_{ss} = \frac{1}{q^2}(u^2\phi_{xx} + 2uv\phi_{xy} + v^2\phi_{yy})$$

  $$\phi_{nn} = \frac{1}{q^2}(v^2\phi_{xx} - 2uv\phi_{xy} + u^2\phi_{yy})$$

- Jameson's difference approximations to the full potential equation in the $x-y$ coordinate system, is as follows;

(i) $$\phi_{xx} \approx \delta_x^2 \phi_{ij} = \frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{\Delta x^2}$$

$$\phi_{xy} \approx \delta_{xy}^2 \phi_{ij} = \frac{\phi_{i+1,j+1} - \phi_{i-1,j+1} - \phi_{i+1,j-1} + \phi_{i-1,j-1}}{(2\Delta x)(2\Delta y)}$$

$$\phi_{yy} \approx \delta_y^2 \phi_{ij} = \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{(\Delta y)^2}$$

ii) If $q^2 < a^2$, the terms $\phi_{xx}$, $\phi_{xy}$, $\phi_{yy}$ appearing in $\phi_{ss}$, are central differenced as above

iii) If $q^2 > a^2$, then its supersonic,

$$\phi_{xx} \approx \delta_x^2 \phi_{i-1,j} = \frac{\phi_{i,j} - 2\phi_{i-1,j} + \phi_{i-2,j}}{(\Delta x)^2}$$

$$\phi_{xy} \approx \delta_{xy}^2 \phi_{i,j} \approx \frac{\phi_{ij} - \phi_{i-1,j} - \phi_{i,j-1} + \phi_{i-1,j-1}}{(\Delta x)(\Delta y)}$$

$$\phi_{yy} \approx \frac{\delta_y^2}{\cancel{\Delta y}} \phi_{i,j-1} \approx \frac{\phi_{i,j} - 2\phi_{i,j-1} + \phi_{i,j-2}}{(\Delta y)^2}$$

## Kutta Condition

- The solution to $\nabla^2 \phi = 0$ where the flow is assumed to be incompressible and irrotational can be written as a linear combination of elementary flows.

- Typically the potential $\phi$ can be written as,

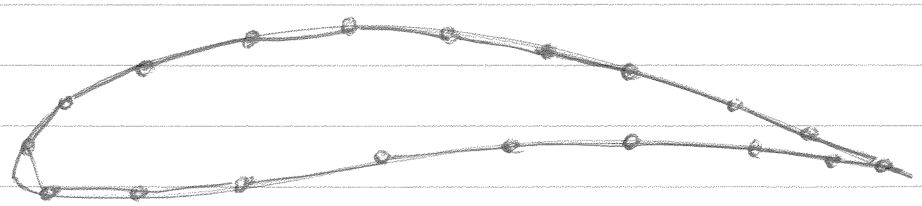
$$\phi(x,y) = \phi_\infty + \int_S \left[ \frac{q(s)}{2\pi} \ln r - \frac{\gamma(s)}{2\pi} \theta \right] ds$$

where $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Here $\frac{q(s)}{2\pi} \ln r$ represents a source with strength $q$

and $\frac{\gamma(s)}{2\pi} \theta$ represents a vortex with strength $\gamma$

- The approach is to break-up the surface into straight line segments, then assume that the source strength is constant over each line segment but has a different value for each panel and the vortex strength is constant and equal over each panel.



- The boundary conditions are the flow tangency boundary condition is $\vec{V} \cdot \vec{n} = 0$ and the Kutta condition.

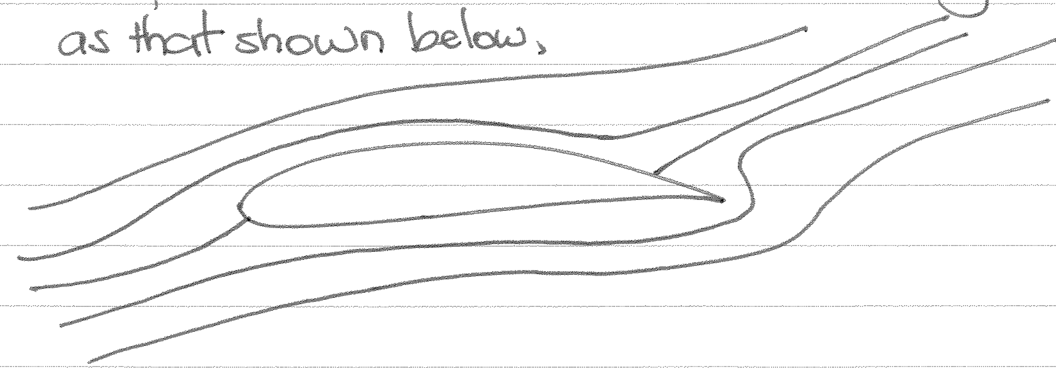
- The Kutta condition forces the rear stagnation point at the trailing edge. To ensure of this condition a circulation of sufficient strength is required.

Firstly, what is circulation? It is the line integral of the fluid velocity around a closed integral.

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s}$$

Circulation is also proportional to the strength of the vortex.

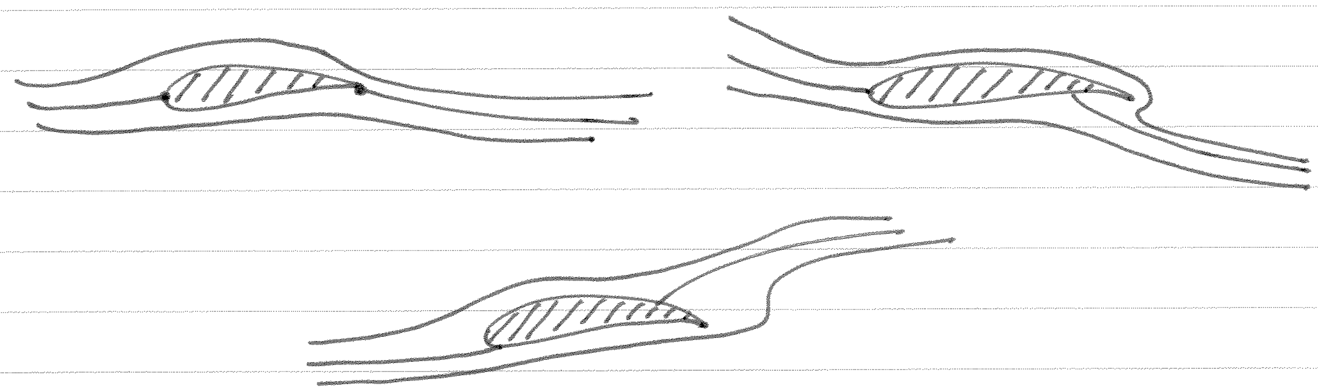
In a real fluid, as the viscous fluid begins to flow around an airfoil, imagine a rear stagnation point on the upper surface. Since this requires a very large local acceleration of the fluid to turn around the trailing edge as that shown below,



the air instead leaves the surface and produces a vortex just above the trailing edge. The vortex is formed as the stagnation point moves to the trailing edge and it is convected downstream. This is called the starting vortex and its strength is equal and opposite to the airfoil circulation.

- Without the Kutta condition an infinite number of potential flow solutions exist for a single airfoil at a fixed freestream pressure, velocity, and Mach number.

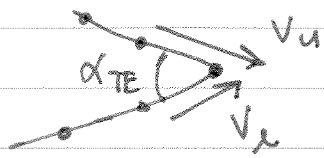
For an example,



All three cases above have different circulations, $\Gamma$ and from potential flow theory, the lift on an airfoil, $L = \rho V_\infty \Gamma$ and hence there would by different amounts of lift generated on the airfoils.

With the Kutta condition, the number of possible solutions reduces to one.

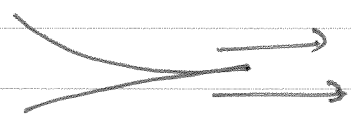
- However this introduces a new issue; in order to satisfy the Kutta condition, the trailing edge point must be the stagnation point, however in a real airfoil with a finite non-zero trailing edge angle, the velocities must be tangent to the surface,



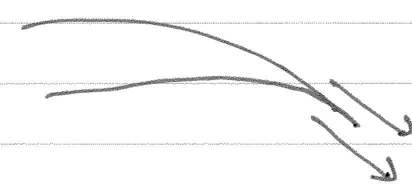

$\alpha_{TE}$ is the trailing edge angle.

This suggest two different velocities at the trailing edge, therefore the only possible solution is for $V_u = V_\ell = 0$. This introduces a singularity into the problem.

If the trailing edge angle is zero, then you have a cusped airfoil



or



In this case $V_u = V_\ell = 0$ is not necessary, since $V_u = V_\ell$ is a possible solution.

- When using the kutta condition to solve the potential flow, if the trailing edge angle is a finite non-zero number then you must ensure that the last panals are small and of equal length. Otherwise the error would affect the accuracy of your calculation and you would have an inconsistent approximation.