Truncation error 2^{nd} order FD of Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+\Delta x, y) - 2u(x,y) + u(x-\Delta x, y)}{(\Delta x)^2}$$

$$u(x+\Delta x, y) = u(x,y) + \frac{\partial u}{\partial x}(x,y)\,\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,y)(\Delta x)^2$$

$$+ \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x,y)(\Delta x)^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(x,y)(\Delta x)^4 + \ldots$$

$$u(x-\Delta x, y) = u(x,y) - \frac{\partial u}{\partial x}(x,y)\,\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,y)(\Delta x)^2$$

$$- \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x,y)(\Delta x)^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(x,y)(\Delta x)^4 + \ldots$$

$$u(x+\Delta x, y) + u(x-\Delta x, y) = 2u(x,y) + \frac{\partial^2 u}{\partial x^2}(x,y)(\Delta x)^2$$

$$+ \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(x,y)(\Delta x)^4 + \text{EVEN TERMS}$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{u(x+\Delta x, y) + u(x-\Delta x, y) - 2u(x,y) - \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(x,y)(\Delta x)^4 - \ldots}{(\Delta x)^2}$$

$$TE = -\frac{2}{4!}\frac{\partial^4 u}{\partial x^4}(x,y)(\Delta x)^2 - \frac{2}{6!}\frac{\partial^6 u}{\partial x^6}(x,y)(\Delta x)^4 - \frac{2}{8!}\frac{\partial^8 u}{\partial x^8}(x,y)(\Delta x)^6 - \ldots$$

$$\boxed{TE \text{ is } O[(\Delta x)^2, (\Delta y)^2]}$$

The modified equation is

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\frac{\partial^4 u}{\partial x^4}(\Delta x)^2 - \frac{1}{360}\frac{\partial^6 u}{\partial x^6}(\Delta x)^4 - \text{EVEN TERMS}}$$