Leap-Frog Gain

$$\frac{u_j^{n+1} - u_j^{n-1}}{2} + A\frac{\Delta t}{\Delta x}\frac{1}{2}\left(u_{j+1}^{n} - u_{j-1}^{n}\right)$$

$$e^{a(t+\Delta t)} = e^{a(t-\Delta t)} - \nu\left(e^{i\beta} - e^{-i\beta}\right)e^{at}$$

$\nu = A\frac{\Delta t}{\Delta x}$

$\beta = k_m \Delta x$

$$e^{a(t+\Delta t)} = G^2 e^{a(t-\Delta t)}$$

$$e^{at} = G e^{a(t-\Delta t)}$$

$$G^2 = 1 - 2i\nu G\sin\beta$$

$$G^2 + 2i\nu\sin\beta\, G - 1 = 0$$

$$G = \frac{\pm\sqrt{4 - 4\nu^2\sin^2\beta} - i2\nu\sin\beta}{2}$$

$$= \pm\sqrt{1 - \nu^2\sin^2\beta} - i\nu\sin\beta$$

Simple Explicit Scheme Truncation Error

$$\frac{u_j^{n+1} - u_j}{\Delta t} = \alpha \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

$$u_j^{n+1} - u_j^n = \frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}(\Delta t)^2 + \dots$$

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t} + \underbrace{\left[\frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \dots \right]}_{\text{TE in time}}$$

$$FD = u_{j+1}^n - u_j^n = \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(\Delta x)^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(\Delta x)^4 + \dots$$

$$BD = u_j^n - u_{j-1}^n = \frac{\partial u}{\partial x}\Delta x - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(\Delta x)^3 - \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(\Delta x)^4 + \dots$$

$$FD - BD = u_{j+1}^n - 2u_j^n + u_{j-1}^n = \frac{\partial^2 u}{\partial x}(\Delta x)^2 + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}(\Delta x)^4 + \text{EVEN TERMS}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} - \underbrace{\left[\frac{1}{12}\frac{\partial^4 u}{\partial x^4}(\Delta x)^2 + \text{EVEN TERMS}\right]}_{\text{TE in space}}$$

Want $\frac{\partial^2 u}{\partial t^2} = u_{tt}$ in term of $\frac{\partial^4 u}{\partial x^4}$

$$u_t - \alpha u_{xx} = \frac{-1}{2}u_{tt}(\Delta t) + \alpha\frac{1}{12}u_{xxxx}(\Delta x)^4 - \dots + \dots$$

① $\frac{\partial}{\partial t}$

$$u_{tt} - \alpha u_{xxt} = -\frac{1}{2}u_{ttt}(\Delta t) + \alpha\frac{1}{12}u_{xxxxt}(\Delta x)^4 + \dots$$

② $\alpha\frac{\partial^2}{\partial x^2}$

$$\alpha u_{txx} - \alpha^2 u_{xxxx} = -\alpha\frac{1}{2}u_{ttxx}(\Delta t) + \alpha^2\frac{1}{12}u_{xxxxxx}(\Delta x)^4 + \dots$$

① + ②

$$u_{tt} = \alpha^2 u_{xxxx} + \Delta t(\dots) + (\Delta x)^4(\dots) + \dots$$

+ higher order

$$\text{TE in space and time} = \frac{-1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t + \frac{1}{12}\frac{\partial^4 u}{\partial x^4}\Delta x^4 = \left[-\frac{1}{2}\alpha^2\Delta t + \frac{1}{12}\alpha(\Delta x)^2\right]\frac{\partial^4 u}{\partial x^4}$$

Stability of Simple Explicit Scheme

$$u_j^{n+1} - u_j^n = \frac{\alpha \Delta t}{(\Delta x)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$e^{\Delta t} = 1 + \nu\left(e^{i\beta} - 2 + e^{-i\beta}\right) \qquad \nu = \frac{\alpha \Delta t}{(\Delta x)^2} \quad \beta = k_m \Delta x$$

$$= 1 - 2\nu + 2\nu\cos\beta$$

$$= 1 - 2\nu(1 - \cos\beta)$$

$$= 1 - 4\nu\sin^2\left(\frac{\beta}{2}\right)$$

$$1 - 4\nu\sin^2\frac{\beta}{2} \leq 1$$

$$-4\nu\sin^2\frac{\beta}{2} \leq 0$$

$$\nu \geq 0$$

$$1 - 4\nu\sin^2\frac{\beta}{2} \geq -1$$

$$\nu\sin^2\frac{\beta}{2} \leq \frac{1}{2}$$

$$\nu \leq \frac{1}{2}$$

$$0 \leq \nu \leq \frac{1}{2}$$