Lax-Friedrichs

$$\frac{u_j^{n+1} - \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)}{\Delta t} + \frac{A}{2}\,\frac{u_{j+1}^n - u_{j-1}^n}{\Delta x} = 0$$

Substitute $u_j^n = e^{at} e^{ik_m x}$

$$\frac{e^{a(t+\Delta t)} e^{ik_m x} - \frac{1}{2}\left(e^{at} e^{ik_m(x+\Delta x)} + e^{at} e^{ik_m(x-\Delta x)}\right)}{\Delta t}$$

$$+ \frac{A}{2\Delta x}\left(e^{at} e^{ik_m(x+\Delta x)} - e^{at} e^{ik_m(x-\Delta x)}\right) = 0$$

Divide by $e^{at} e^{ik_m x}$

$$e^{a\Delta t} = \frac{1}{2}\left(e^{ik_m\Delta x} + e^{-ik_m\Delta x}\right) - \frac{1}{2}\,\frac{A\Delta t}{\Delta x}\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right)$$

$$G = \cos(k_m\Delta x) - i\nu\sin(k_m\Delta x)$$

$$|G| = \sqrt{\cos^2(k_m\Delta x) + (-\nu\sin(k_m\Delta x))^2} \leq 1$$

$$\cos^2(\beta) + \nu^2\sin^2(\beta) \leq 1$$

$$1 + (\nu^2 - 1)\sin^2(\beta) \leq 1$$

$$(\nu^2 - 1)\sin^2(\beta) \leq 0$$

$$\nu^2 \leq 1$$

$$\nu \leq 1$$

$$A\frac{\Delta t}{\Delta x} \leq 1$$

Lax - Wendroff

$$u_j^{n+1} = u_j^n - \frac{A\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{A\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$e^{a(t+\Delta t)} e^{ik_m x} = e^{at} e^{ik_m x} - \frac{1}{2}\nu\left(e^{at} e^{ik_m(x+\Delta x)} - e^{at} e^{ik_m(x-\Delta x)}\right) + \frac{1}{2}\nu^2 e^{at}\left(e^{ik_m(x+\Delta x)} - 2e^{ik_m x} + e^{ik_m(x-\Delta x)}\right)$$

$$e^{a\Delta t} = 1 - \frac{1}{2}\nu\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right) + \frac{1}{2}\nu^2\left(e^{ik_m\Delta x} - 2 + e^{-ik_m\Delta x}\right)$$

$$G = 1 - i\nu\sin(k_m\Delta x) - \nu^2 + \nu^2\cos(k_m\Delta x)$$

$$|G| = \sqrt{\left(1-\nu^2+\nu^2\cos\beta\right)^2 + \left(-\nu\sin\beta\right)^2} \leq 1.$$

$$= \left[1-\nu^2(1-\cos\beta)\right]^2 + \nu^2\sin^2\beta$$

$$= \left[1-2\nu^2\sin^2\left(\frac{\beta}{2}\right)\right]^2 + \nu^2\sin^2\beta$$

$$= 1 + 4\nu^4\sin^4\left(\frac{\beta}{2}\right) - 4\nu^2\sin^2\left(\frac{\beta}{2}\right) + \nu^2\sin^2\beta$$

$$= 1 + 4\nu^4\sin^4\left(\frac{\beta}{2}\right) - 4\nu^2\sin^4\left(\frac{\beta}{2}\right)$$

$$= 1 - (1-\nu^2)4\nu^2\sin^4\left(\frac{\beta}{2}\right) \leq 1$$

$$1-\nu^2 \geq 0$$

$$\nu^2 \leq 1$$

$$\nu \leq 1$$

$$\frac{A\Delta t}{\Delta x} \leq 1$$

Upwind Consistency

Uses Forward-Time, Backword-Space finite differences

$$u(x,t+\Delta t) = u(x,t) + \frac{\partial u(x,t)}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial t^2}(\Delta t)^2 + \frac{1}{6}\frac{\partial^3 u(x,t)}{\partial t^3}(\Delta t + \ldots$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial t^2}\Delta t + \ldots$$

$$u(x-\Delta x,t) = u(x,t) - \frac{\partial u(x,t)}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x}(\Delta x)^2 + \ldots$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{u(x,t) - u(x-\Delta x,t)}{\Delta x} - \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x + \ldots$$

$$TE = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial t^2}\Delta t + \frac{1}{6}\frac{\partial^3 u(x,t)}{\partial t^3}(\Delta t)^2$$

$$- \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x + \frac{1}{6}\frac{\partial^3 u(x,t)}{\partial x^3}(\Delta x)^2 + \ldots \; O((\Delta t)^3, (\Delta x)^3)$$

The truncation error obviously goes to 0 as Δt and Δx goes to 0.