Chapter 9 - High Order Portfolios

Exercises

Exercise 9.1 *Show why* $\boldsymbol{\Sigma}\boldsymbol{x} = \boldsymbol{b}/\boldsymbol{x}$ *can be equivalently solved as* $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{b}/\boldsymbol{x}$*, where* $\boldsymbol{C}$ *is the correlation matrix defined as* $\boldsymbol{C} = \boldsymbol{D}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{D}^{-1/2}$ *with* $\boldsymbol{D}$ *a diagonal matrix containing* $\mathsf{diag}(\boldsymbol{\Sigma})$ *along the main diagonal. Would it be possible to use instead* $\boldsymbol{C} = \boldsymbol{M}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{M}^{-1/2}$*, where* $\boldsymbol{M}$ *is not necessaryly a diagonal matrix?*

Exercise 9.2 *If the covariance matrix is diagonal* $\boldsymbol{\Sigma} = \boldsymbol{D}$*, then the system of nonlinear equations* $\boldsymbol{\Sigma}\boldsymbol{x} = \boldsymbol{b}/\boldsymbol{x}$ *has the closed-form solution* $\boldsymbol{x} = \sqrt{\boldsymbol{b}/\mathsf{diag}(\boldsymbol{D})}$*. Explore whether a closed-form solution can be obtained for the rank-one plus diagonal case* $\boldsymbol{\Sigma} = \boldsymbol{u}\boldsymbol{u}^\mathsf{T} + \boldsymbol{D}$*.*

Exercise 9.3 *The solution to the formulation*

$$\begin{array}{ll} \underset{\boldsymbol{x}\geq\boldsymbol{0}}{\mathsf{maximize}} & \boldsymbol{b}^\mathsf{T}\log(\boldsymbol{x}) \\ \mathsf{subjectto} & \sqrt{\boldsymbol{x}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{x}} \leq \sigma_0 \end{array}$$

is

$$\lambda\boldsymbol{\Sigma}\boldsymbol{x} = \boldsymbol{b}/\boldsymbol{x} \times \sqrt{\boldsymbol{x}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{x}}.$$

Can you solve for λ *and rewrite the solution in a more compact way without* λ*?*

Exercise 9.4 *Newton's method requires computing the direction* $\boldsymbol{d} = \mathsf{H}^{-1}\nabla f$ *or, equivalently, solving the system of linear equations* $\mathsf{H}\boldsymbol{d} = \nabla f$ *for* $\boldsymbol{d}$*. Explore whether a more efficient solution is possible exploiting the structure of the gradient and Hessian:*

$$\begin{aligned} \nabla f &= \boldsymbol{\Sigma}\boldsymbol{x} - \boldsymbol{b}/\boldsymbol{x} \\ \mathsf{H} &= \boldsymbol{\Sigma} + \mathsf{Diag}(\boldsymbol{b}/\boldsymbol{x}^2). \end{aligned}$$

Exercise 9.5 *The MM algorithm requires the computation of the largest eigenvalue* λ_{max} *of matrix* $\boldsymbol{\Sigma}$*, which can be obtained from the eigenvalue decomposition of the matrix. A more efficient alternative is the power iteration method. Program both methods and compare the computational complexity.*

Exercise 9.6 *Consider the vanilla convex formulation*

$$\underset{\boldsymbol{x}\geq\mathbf{0}}{\mathsf{minimize}} \quad \tfrac{1}{2}\boldsymbol{x}^\mathsf{T}\boldsymbol{\Sigma}\boldsymbol{x} - \boldsymbol{b}^\mathsf{T}\log(\boldsymbol{x}).$$

Implement the cyclical coordinate descent method and the parallel SCA method in a high-level programming language (e.g., R, Python, Julia, or Matlab) and compare the converge vs CPU time for these two methods. Then, re-implement these two methods in a low-level programming language (e.g., C, C++, C#, or Rust) and compare the convergence again. Comment on the difference observed.