

Chapter 11 - Risk Parity Portfolio

Exercises

Exercise 11.1 Show why $\Sigma \mathbf{x} = \mathbf{b}/\mathbf{x}$ can be equivalently solved as $\mathbf{C} \mathbf{x} = \mathbf{b}/\mathbf{x}$, where \mathbf{C} is the correlation matrix defined as $\mathbf{C} = \mathbf{D}^{-1/2} \Sigma \mathbf{D}^{-1/2}$ with \mathbf{D} a diagonal matrix containing $\text{diag}(\Sigma)$ along the main diagonal. Would it be possible to use instead $\mathbf{C} = \mathbf{M}^{-1/2} \Sigma \mathbf{M}^{-1/2}$, where \mathbf{M} is not necessarily a diagonal matrix?

Exercise 11.2 If the covariance matrix is diagonal $\Sigma = \mathbf{D}$, then the system of nonlinear equations $\Sigma \mathbf{x} = \mathbf{b}/\mathbf{x}$ has the closed-form solution $\mathbf{x} = \sqrt{\mathbf{b}/\text{diag}(\mathbf{D})}$. Explore whether a closed-form solution can be obtained for the rank-one plus diagonal case $\Sigma = \mathbf{u}\mathbf{u}^\top + \mathbf{D}$.

Exercise 11.3 The solution to the formulation

$$\begin{aligned} & \underset{\mathbf{x} \geq \mathbf{0}}{\text{maximize}} && \mathbf{b}^\top \log(\mathbf{x}) \\ & \text{subject to} && \sqrt{\mathbf{x}^\top \Sigma \mathbf{x}} \leq \sigma_0 \end{aligned}$$

is

$$\lambda \Sigma \mathbf{x} = \mathbf{b}/\mathbf{x} \times \sqrt{\mathbf{x}^\top \Sigma \mathbf{x}}.$$

Can you solve for λ and rewrite the solution in a more compact way without λ ?

Exercise 11.4 Newton's method requires computing the direction $\mathbf{d} = \mathbf{H}^{-1} \nabla f$ or, equivalently, solving the system of linear equations $\mathbf{H} \mathbf{d} = \nabla f$ for \mathbf{d} . Explore whether a more efficient solution is possible exploiting the structure of the gradient and Hessian:

$$\begin{aligned} \nabla f &= \Sigma \mathbf{x} - \mathbf{b}/\mathbf{x} \\ \mathbf{H} &= \Sigma + \text{Diag}(\mathbf{b}/\mathbf{x}^2). \end{aligned}$$

Exercise 11.5 The MM algorithm requires the computation of the largest eigenvalue λ_{\max} of matrix Σ , which can be obtained from the eigenvalue decomposition of the matrix. A more efficient alternative is the power iteration method. Program both methods and compare the computational complexity.

Exercise 11.6 Consider the vanilla convex formulation

$$\underset{\mathbf{x} \geq \mathbf{0}}{\text{minimize}} \quad \frac{1}{2} \mathbf{x}^\top \Sigma \mathbf{x} - \mathbf{b}^\top \log(\mathbf{x}).$$

Implement the cyclical coordinate descent method and the parallel SCA method in a high-level programming language (e.g., R, Python, Julia, or Matlab) and compare the converge vs CPU time for these two methods. Then, re-implement these two methods in a low-level programming language (e.g., C, C++, C#, or Rust) and compare the convergence again. Comment on the difference observed.