

# Neutron star mergers simulations

David Radice<sup>1,2</sup>



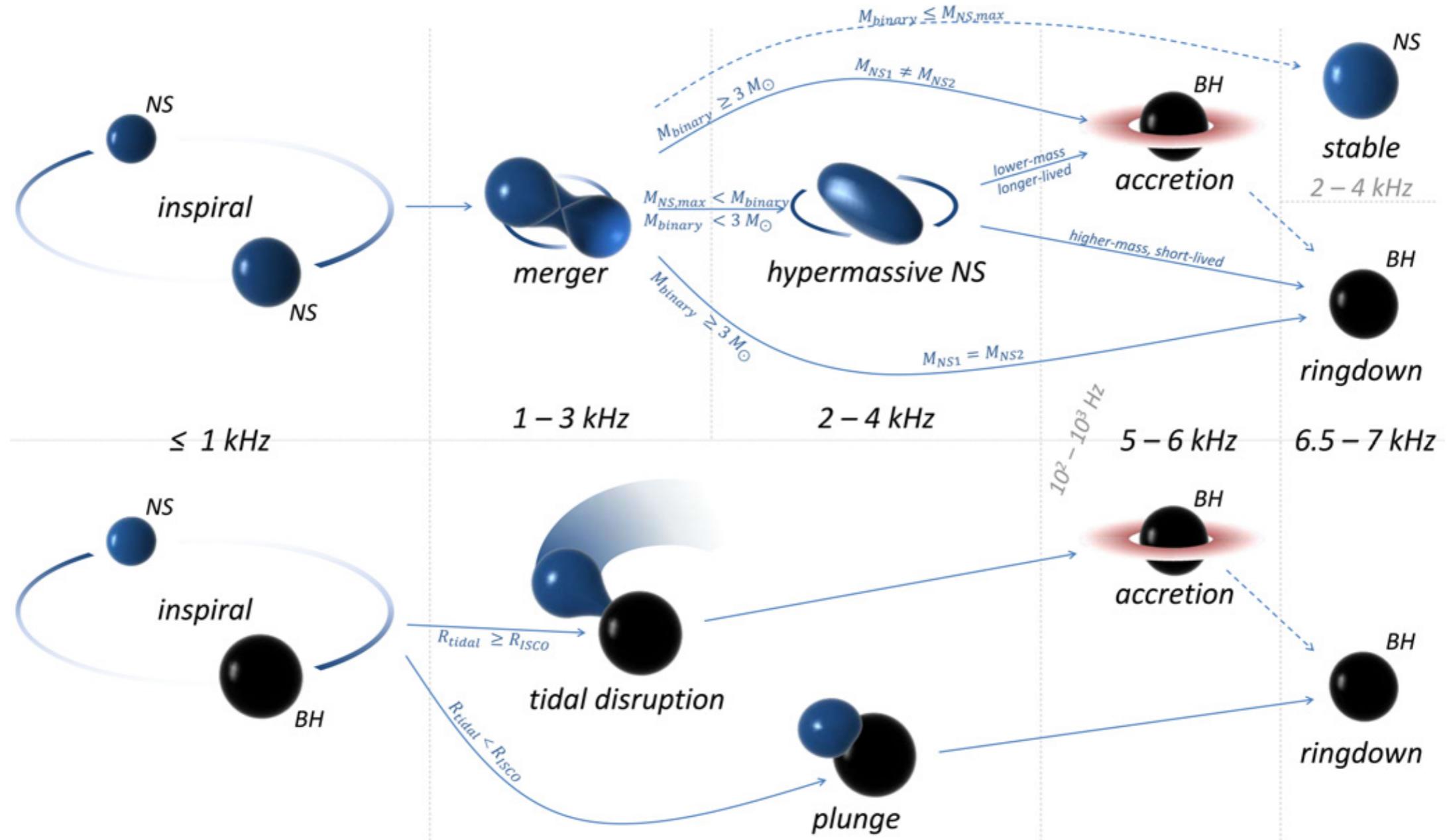
<sup>1</sup> Research Associate, Princeton University

<sup>2</sup> Taplin Member, Institute for Advanced Study

Neutron star mergers for non-experts:  
GW170817 in the multi-messenger astronomy and FRIB eras

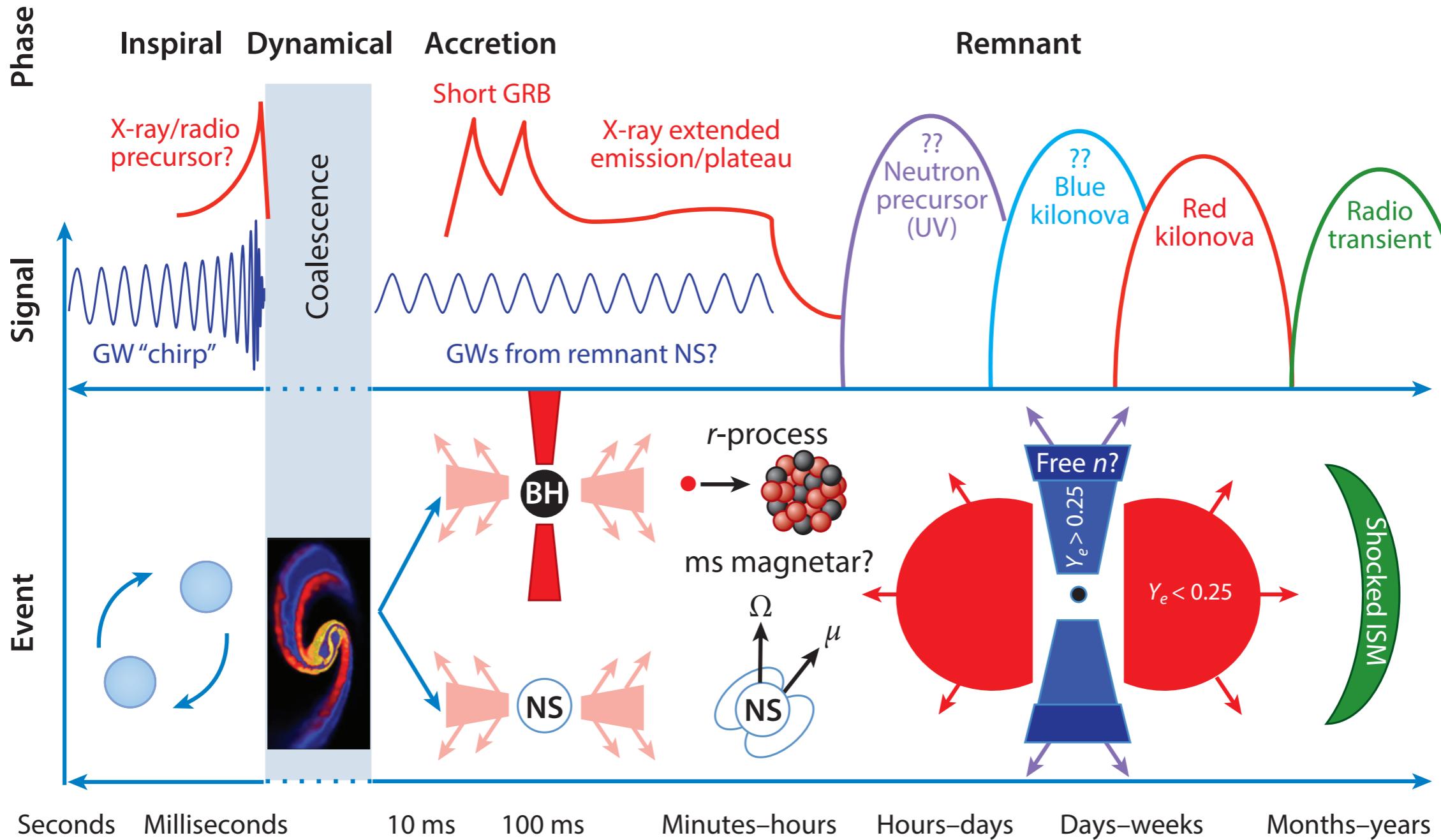
[https://github.com/dradice/JINA\\_MSU\\_School\\_2018.git](https://github.com/dradice/JINA_MSU_School_2018.git)

# Outcome of NS mergers



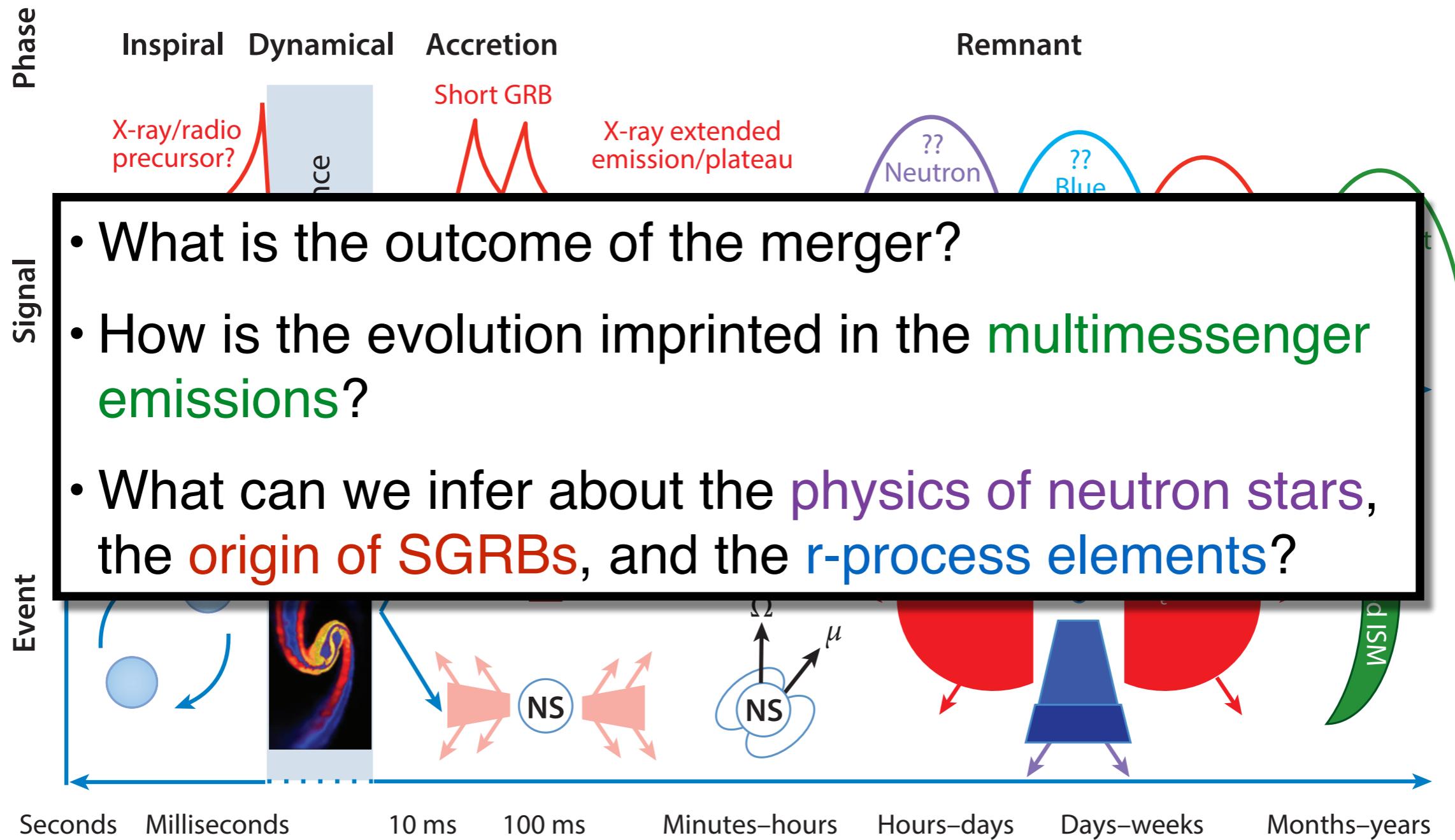
From Bartos, Brady, & Márka 2013

# Multimessenger emissions



From Fernández & Metzger 2016

# Multimessenger emissions



From Fernández & Metzger 2016

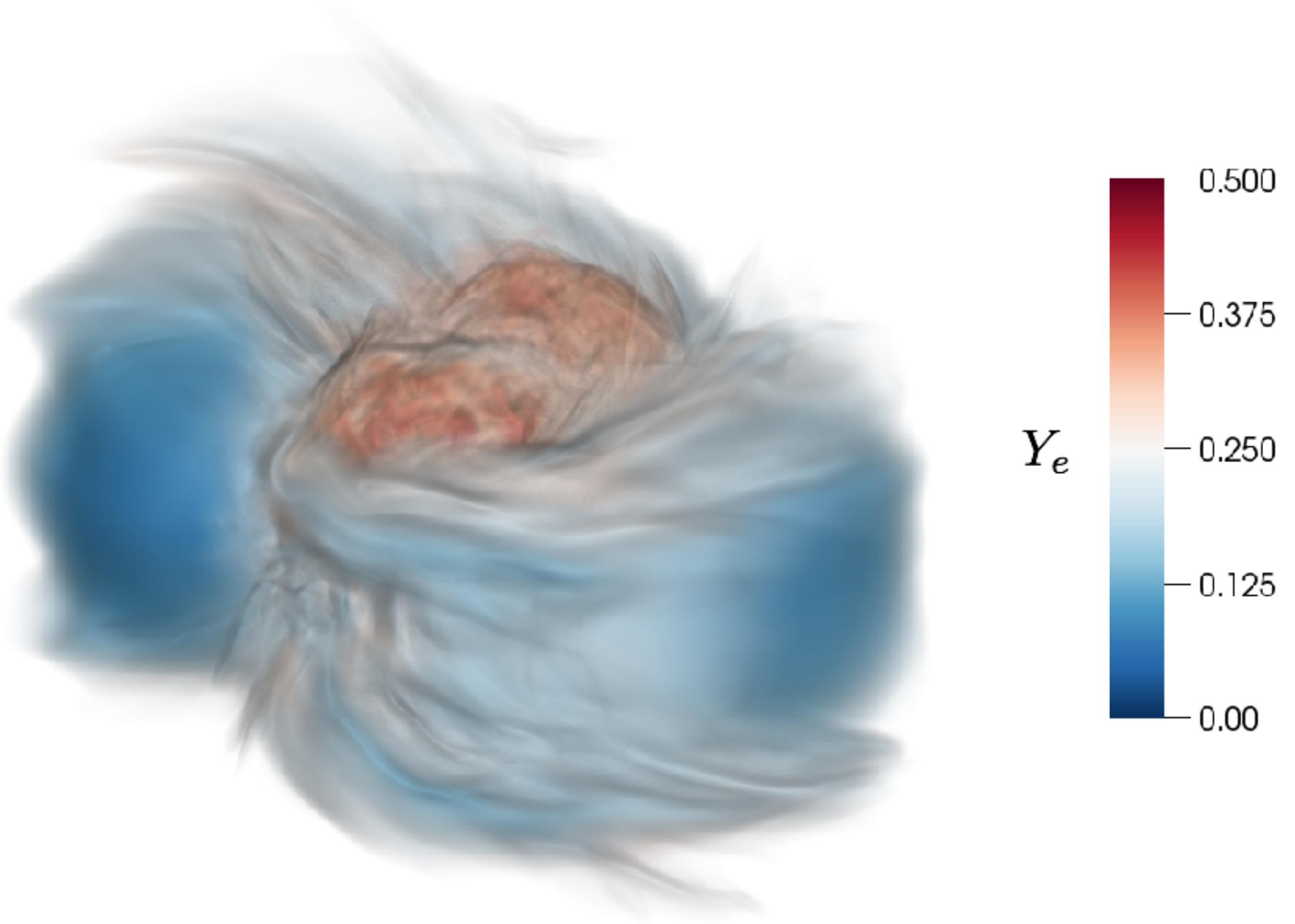
# A multi-physics problem

General relativity	Strong gravity
Nuclear EOS	Nuclear forces
Neutrino-matter interactions	Weak reactions
Relativistic MHD	Electromagnetic forces

## Multiple time and spatial scales

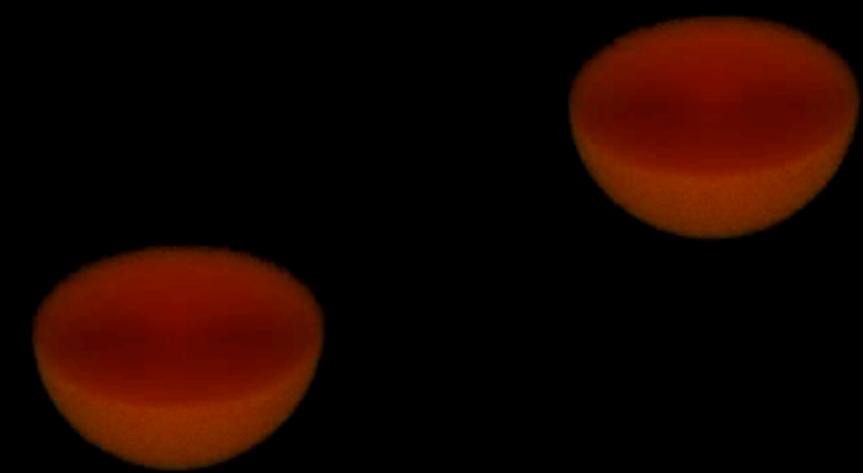
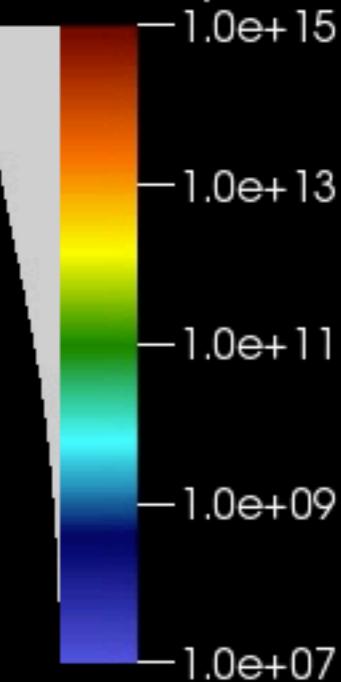
Dynamical time of a NS	$\sim 20 \mu\text{s}$
Postmerger GW emission	$\sim 10\text{-}20 \text{ ms}$
Viscous time	$\sim 0.01\text{-}1 \text{ sec}$
Neutrino cooling timescale	$\sim 2\text{-}3 \text{ sec}$
GRB launching timescale	$\sim 1.7 \text{ sec}$
Secular mass ejection	$\sim 5 \text{ sec}$

# Mass ejection



Volume

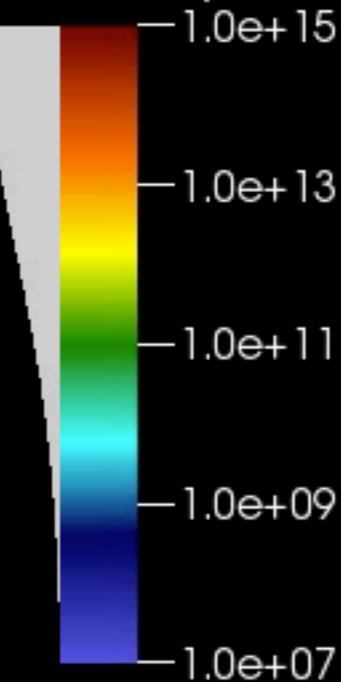
Var: density



Time = 0 ms

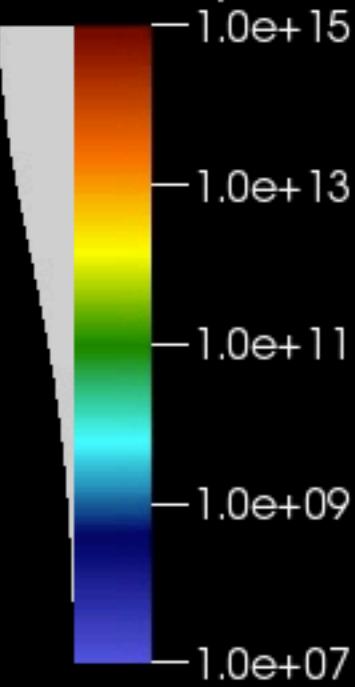
Volume

Var: density

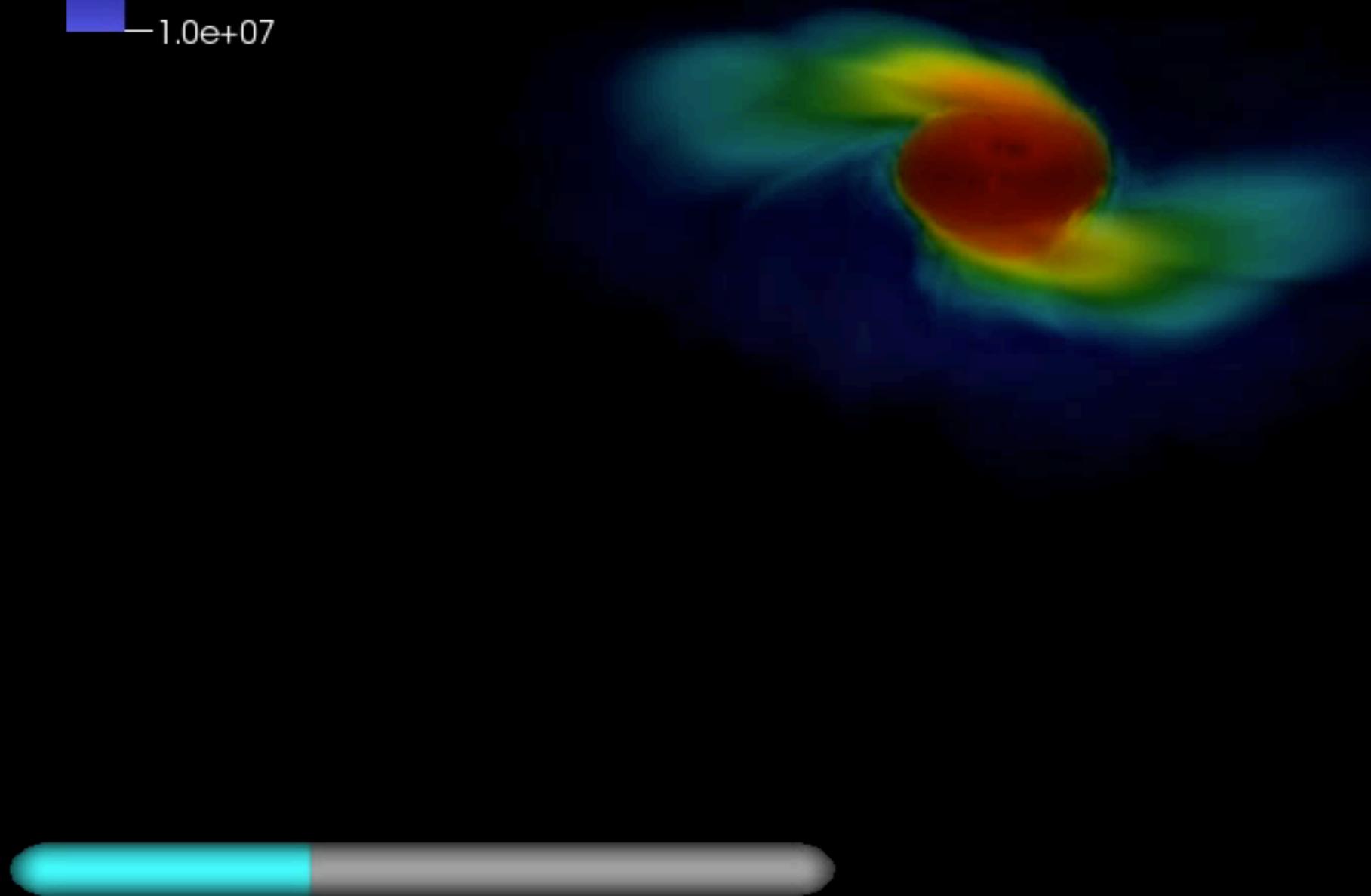


Time = 0 ms

Volume  
Var: density

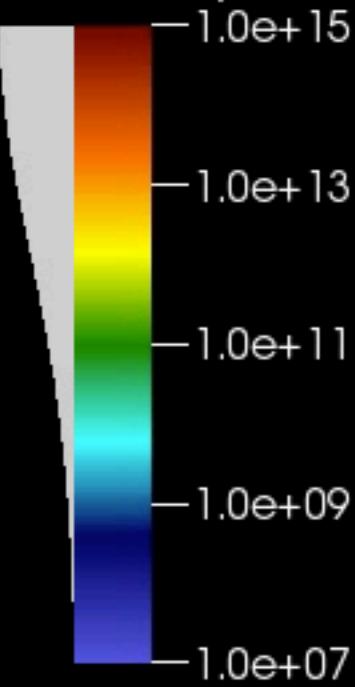


R

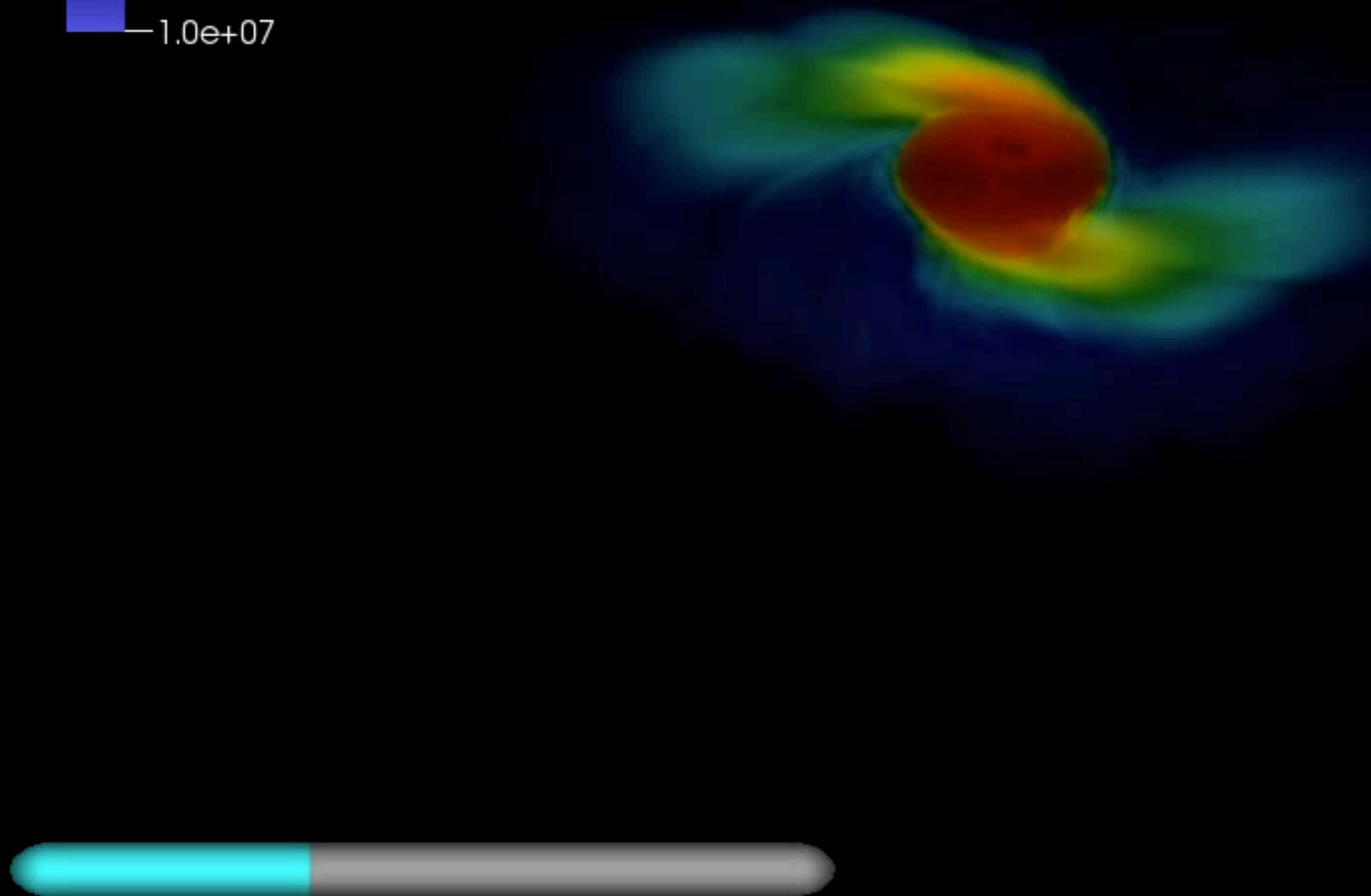


Time = 9.07331 ms

Volume  
Var: density

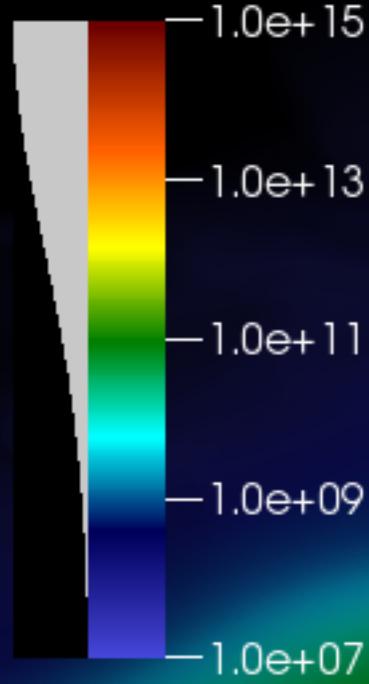


R



Time = 9.07331 ms

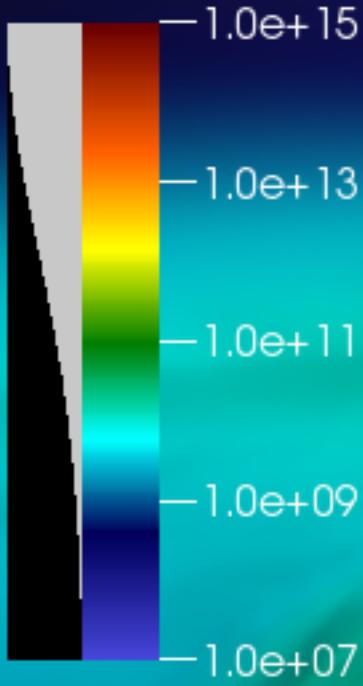
Volume  
Var: density



# Tidal tail

Time = 10.1196 ms

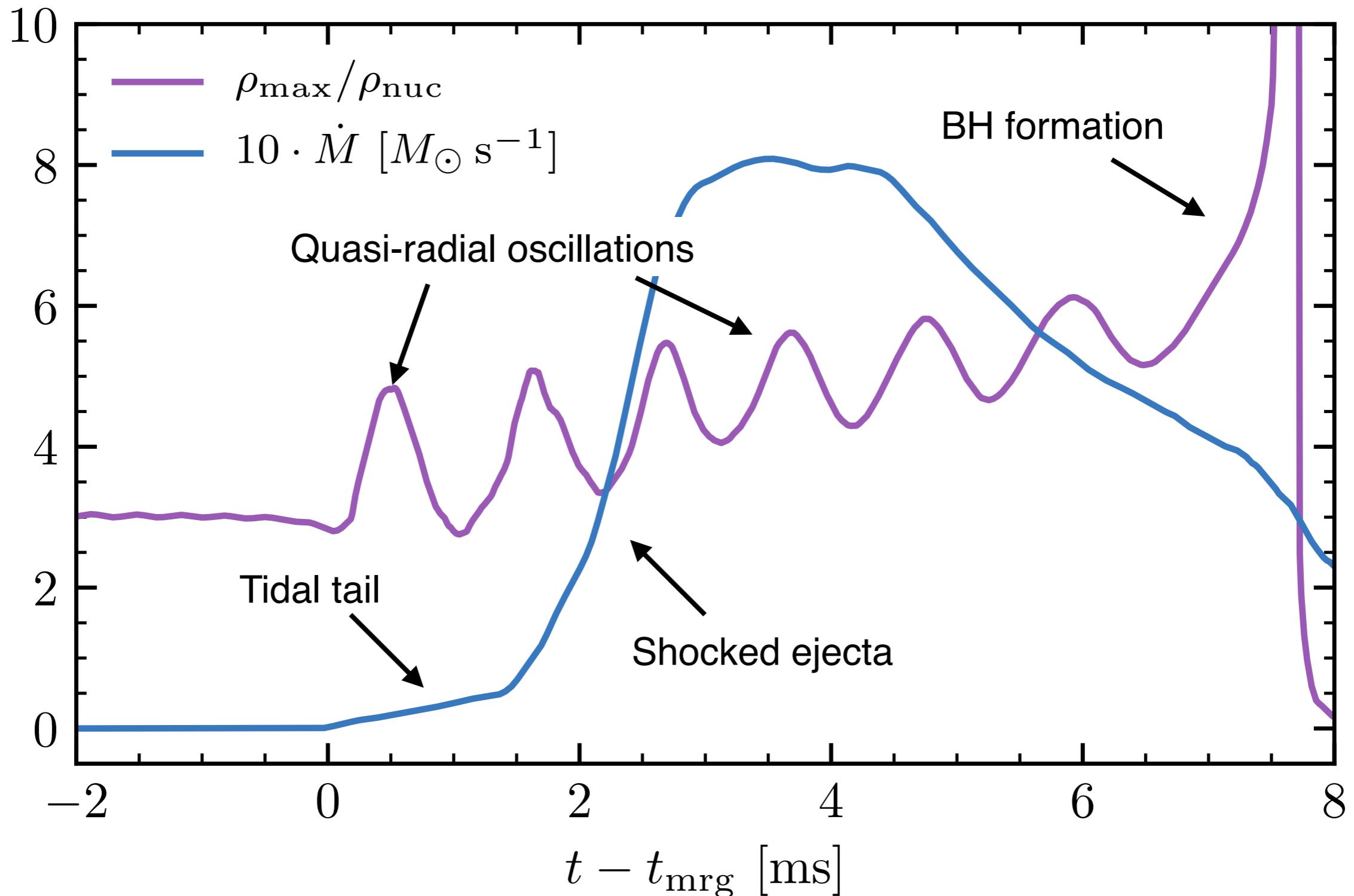
Volume  
Var: density



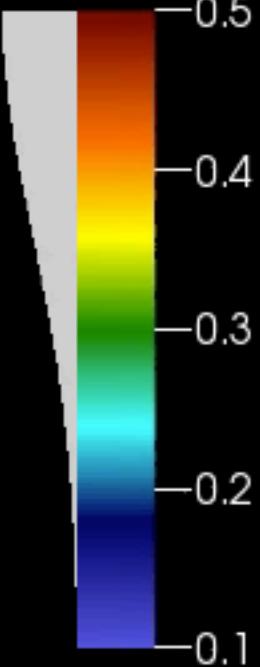
# Shocked ejecta

Time = 11.0003 ms

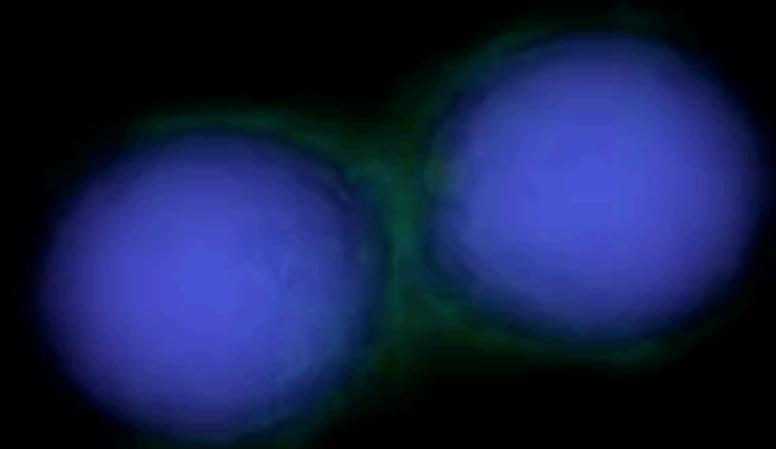
# Origin of the shocked ejecta



Volume  
Var: HYDROBASE-Y\_e

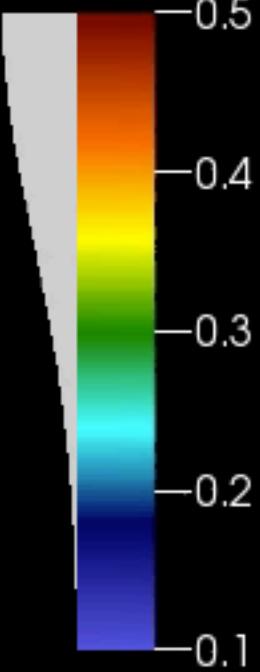


R

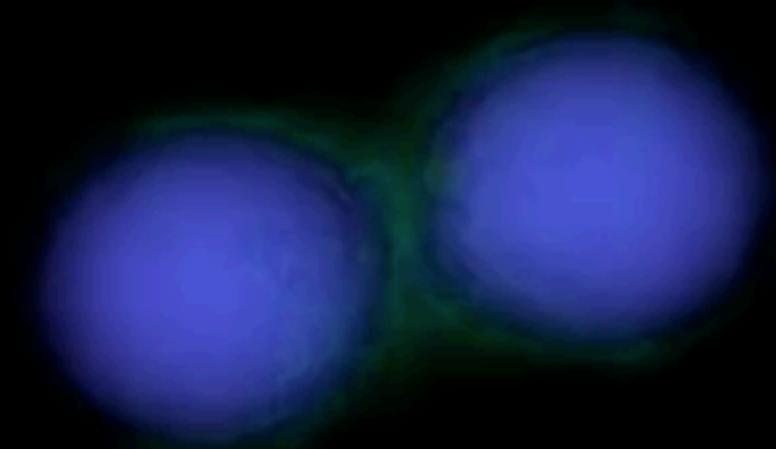


Time = 7.38869 ms

Volume  
Var: HYDROBASE-Y\_e

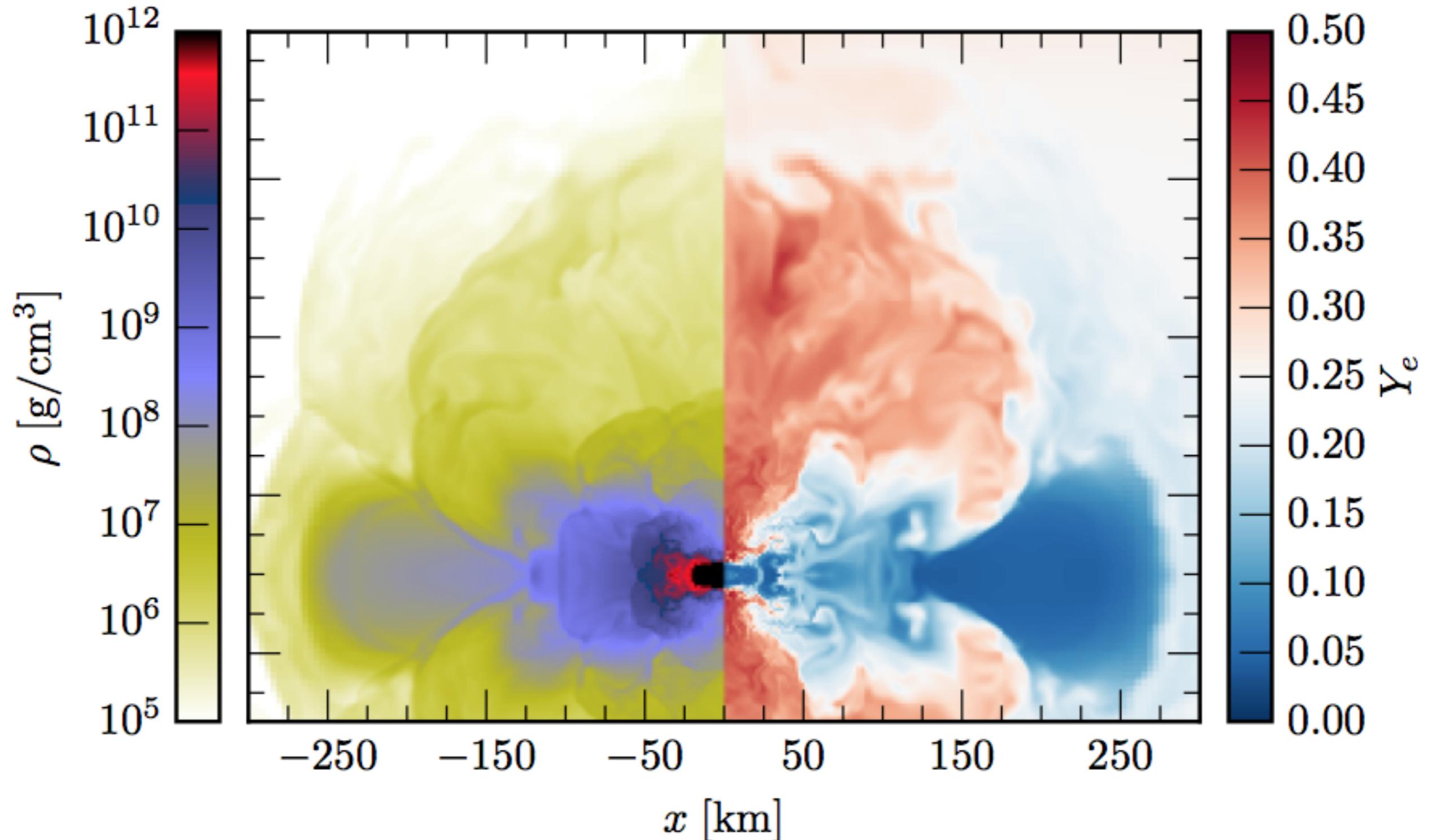


R



Time = 7.38869 ms

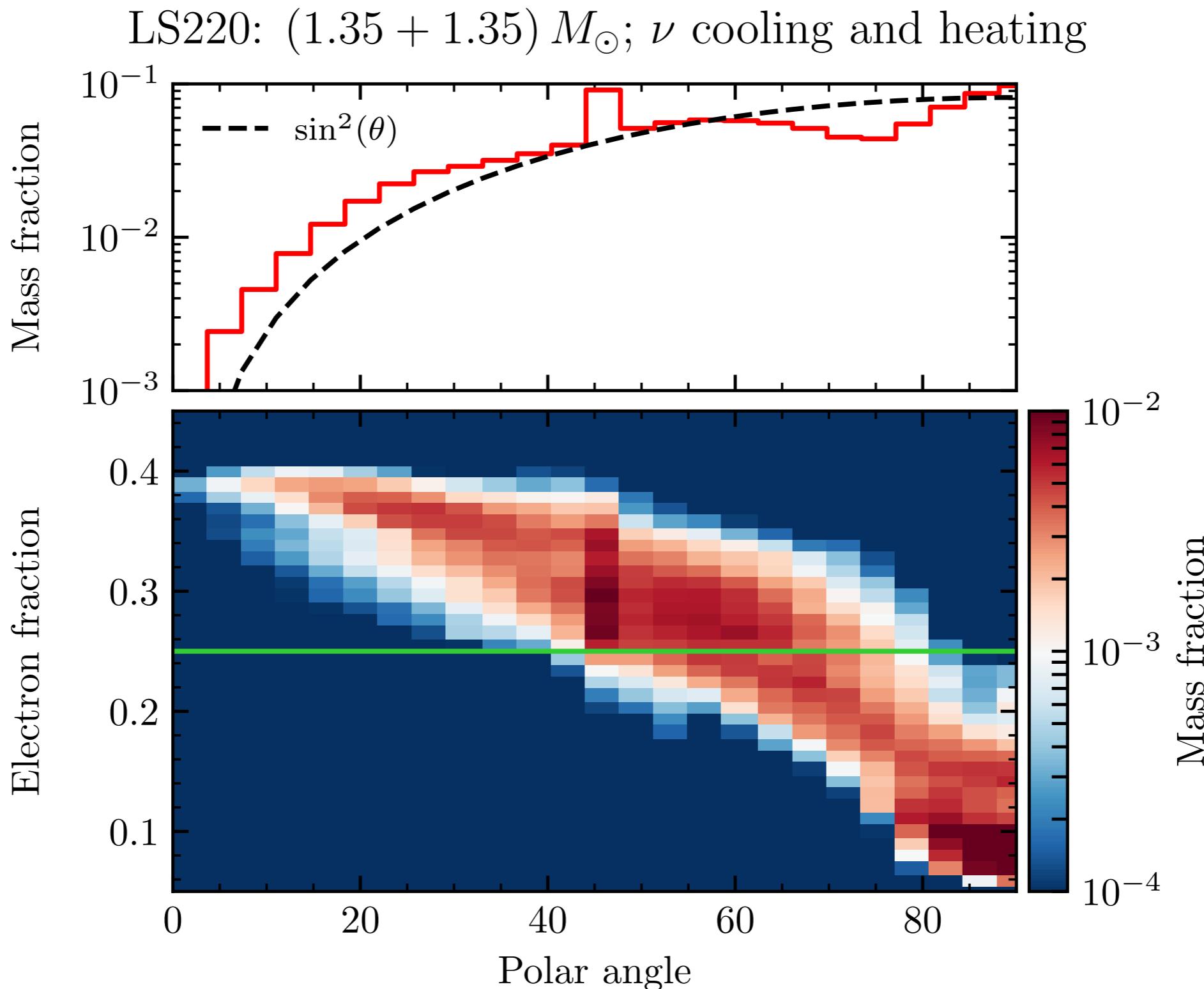
# Neutron-driven ejecta



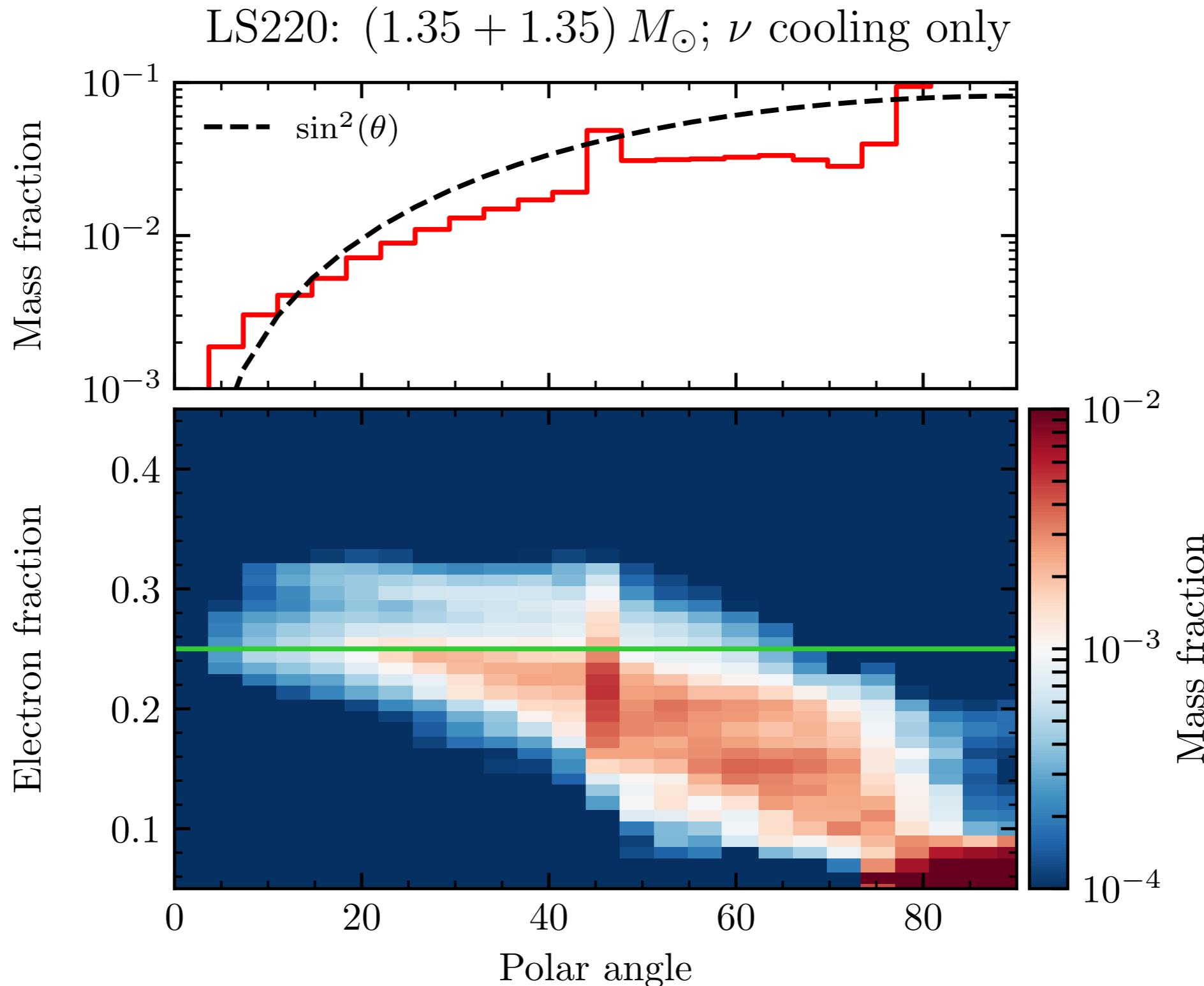
See also Wanajo+ 2014,  
Sekiguchi+ 2015, 2016, Foucart+ 2016

DR, Galeazzi+ MRAS 460:3255 (2016)

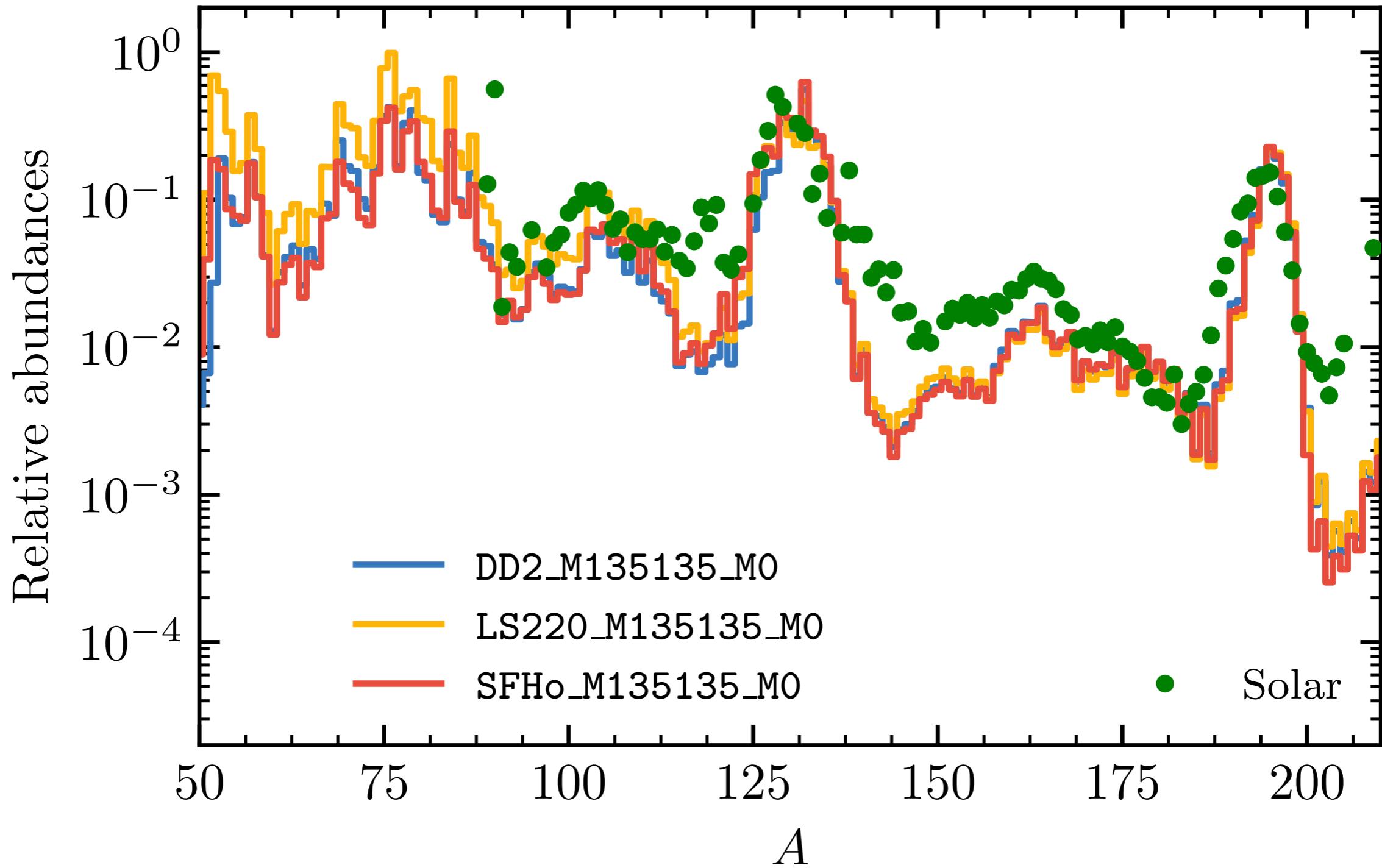
# Dynamical ejecta: composition



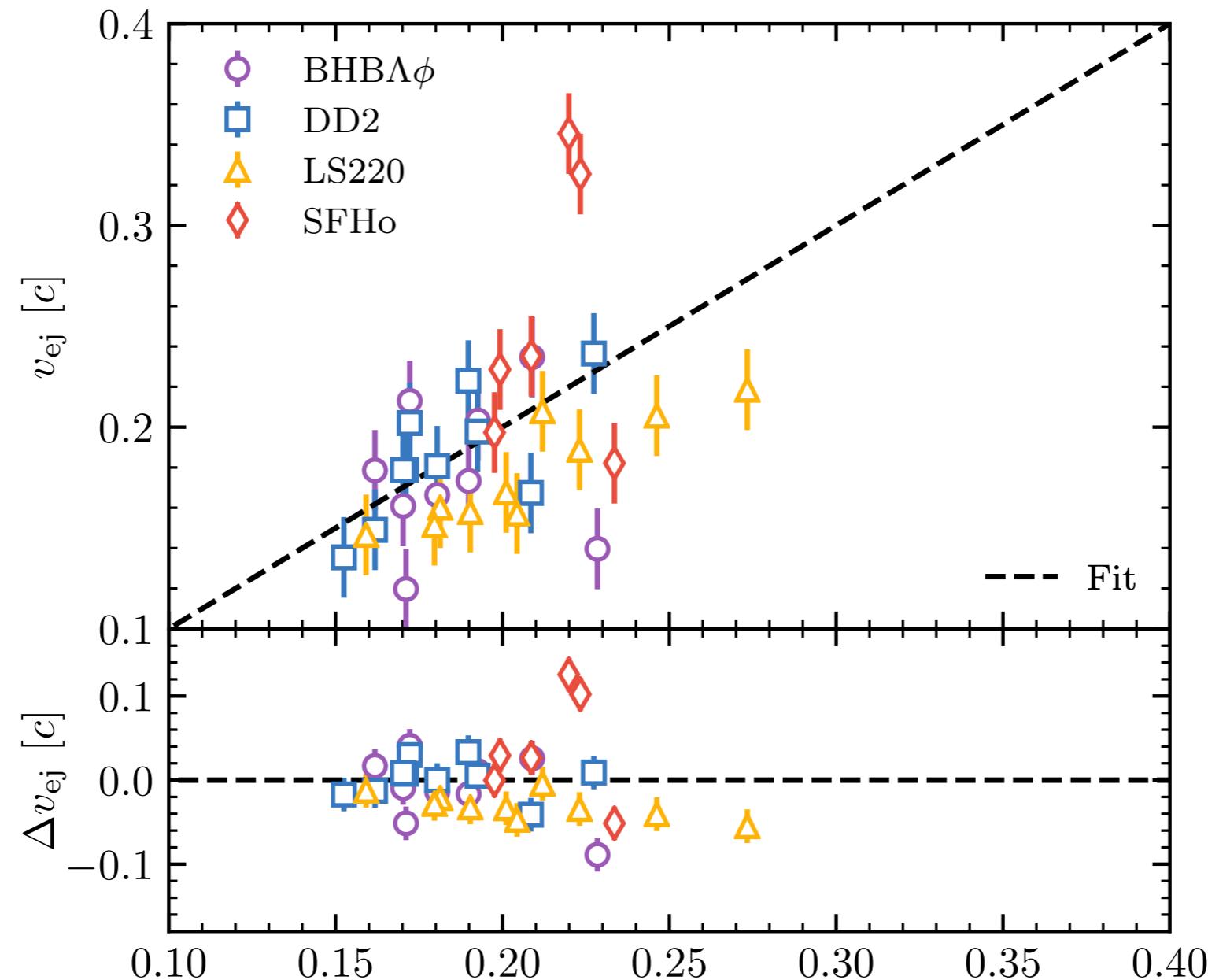
# Dynamical ejecta: composition



# Dynamical ejecta: yields



# Dynamical ejecta: rms velocity



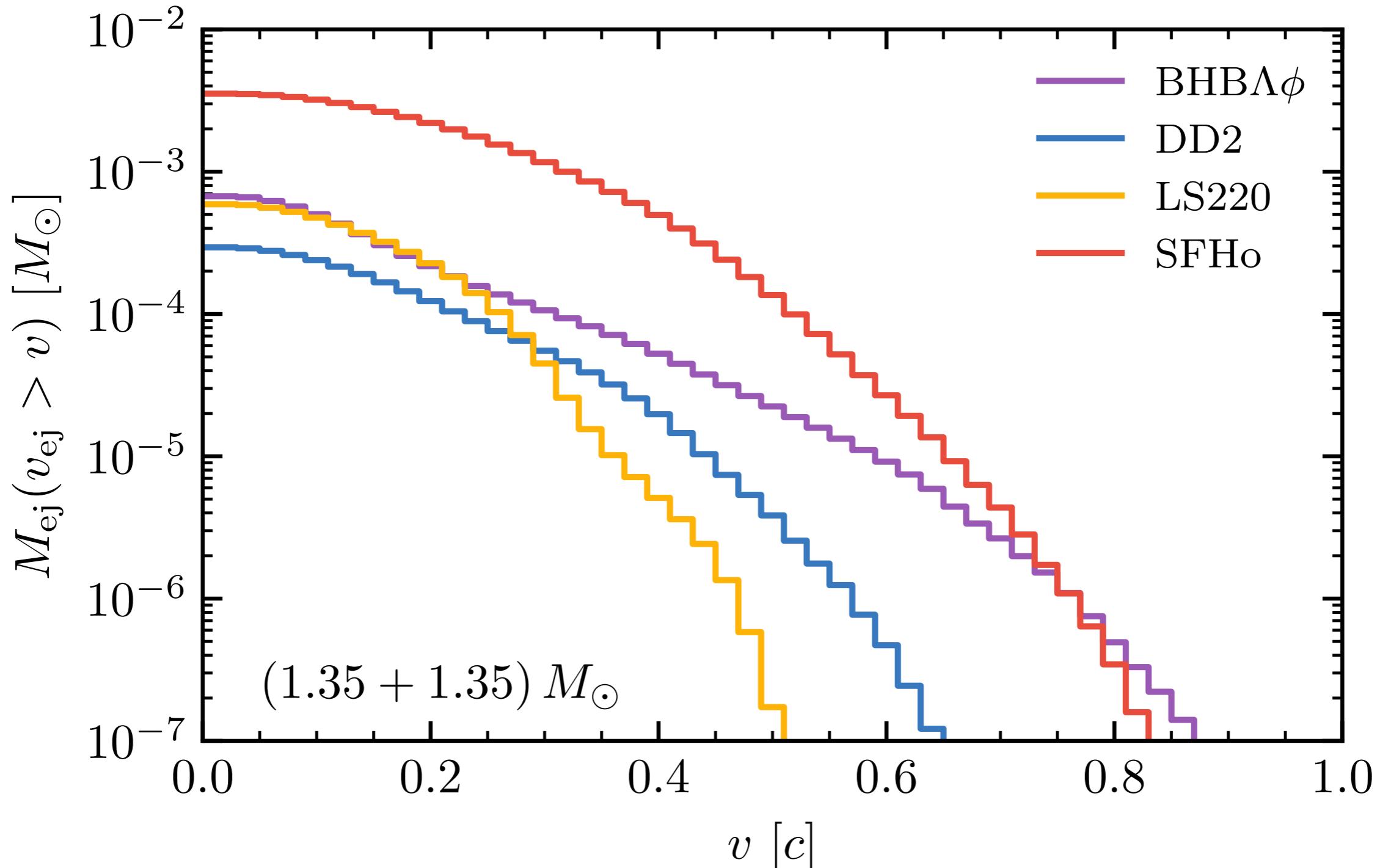
$$v_\rho = \left[ a \left( \frac{M_1}{M_2} \right) (1 + c C_1) \right] + (1 \leftrightarrow 2) + b.$$

Fitting formula from Dietrich+ CQG 34:105014 (2017)

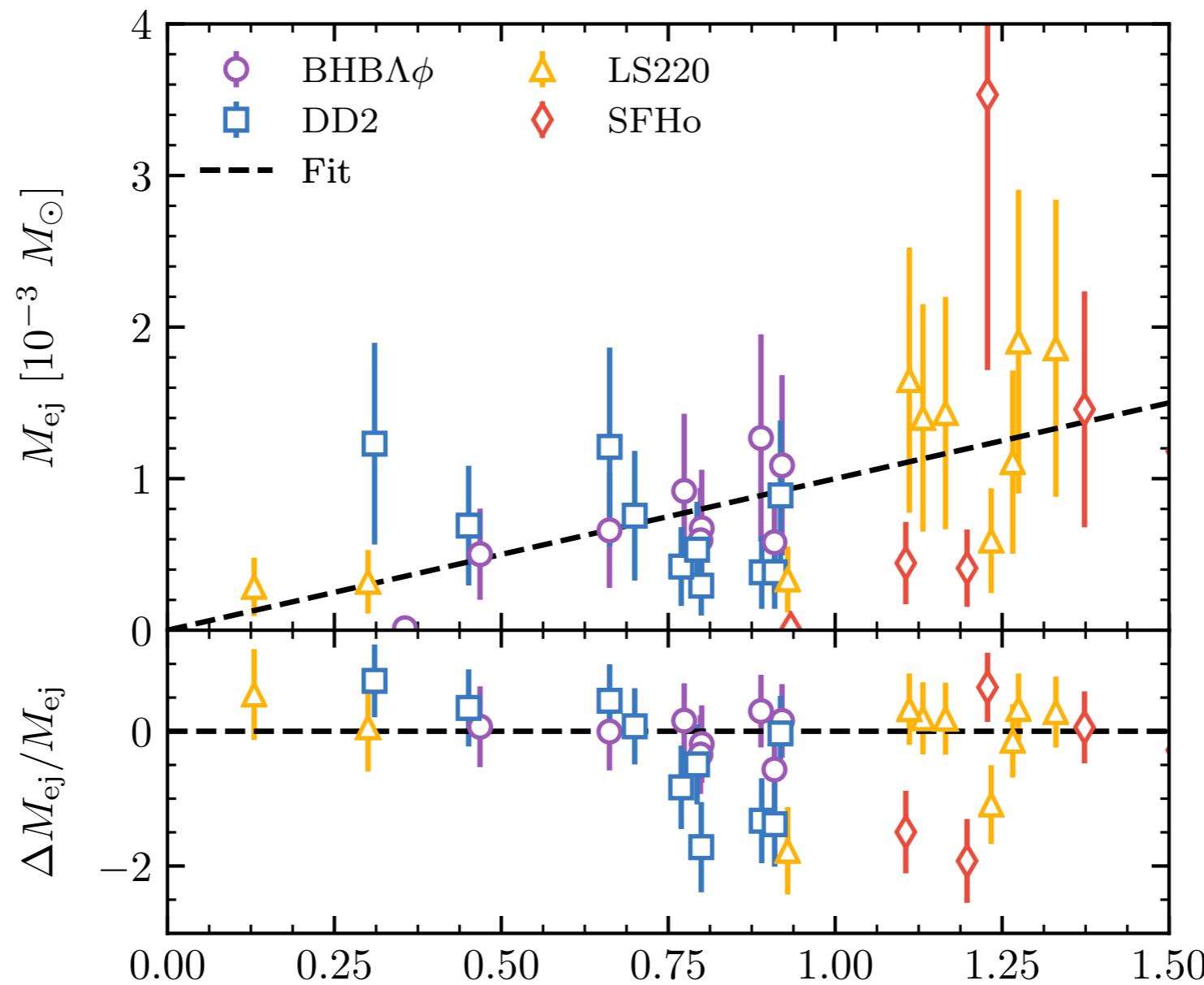
**Warning:** very different values for the fitting coefficients

DR et al. (2018), in prep

# High-velocity tail



# Dynamical ejecta: mass

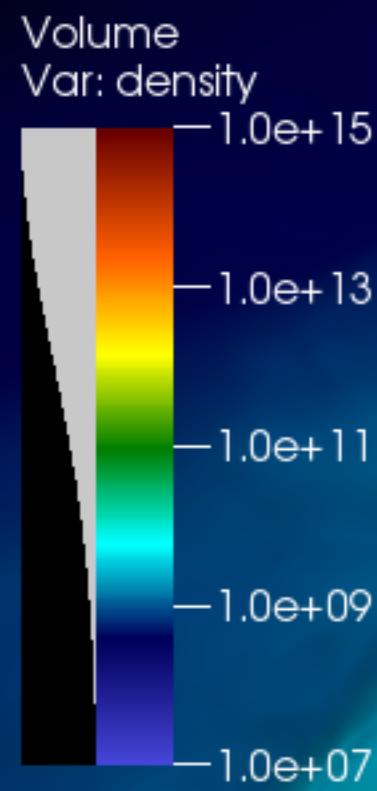


$$\frac{M_{\text{ej}}^{\text{fit}}}{10^{-3} M_{\odot}} = \left[ a \left( \frac{M_2}{M_1} \right)^{1/3} \left( \frac{1 - 2C_1}{C_1} \right) + b \left( \frac{M_2}{M_1} \right)^n + c \left( 1 - \frac{M_1}{M_1^*} \right) \right] M_1^* + (1 \leftrightarrow 2) + d.$$

Fitting formula from Dietrich+ CQG 34:105014 (2017)

**Warning:** very different values for the fitting coefficients

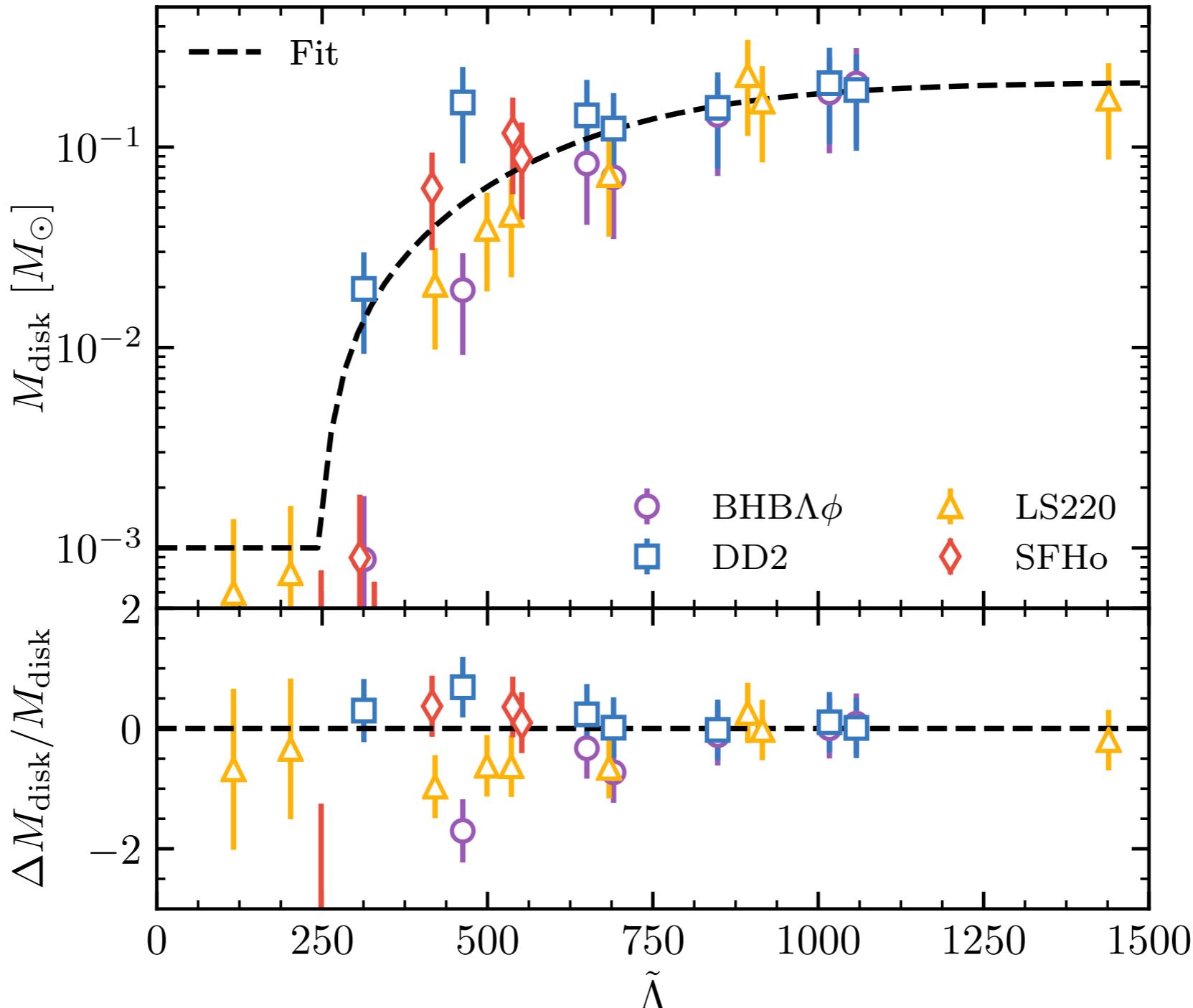
DR et al. (2018), in prep



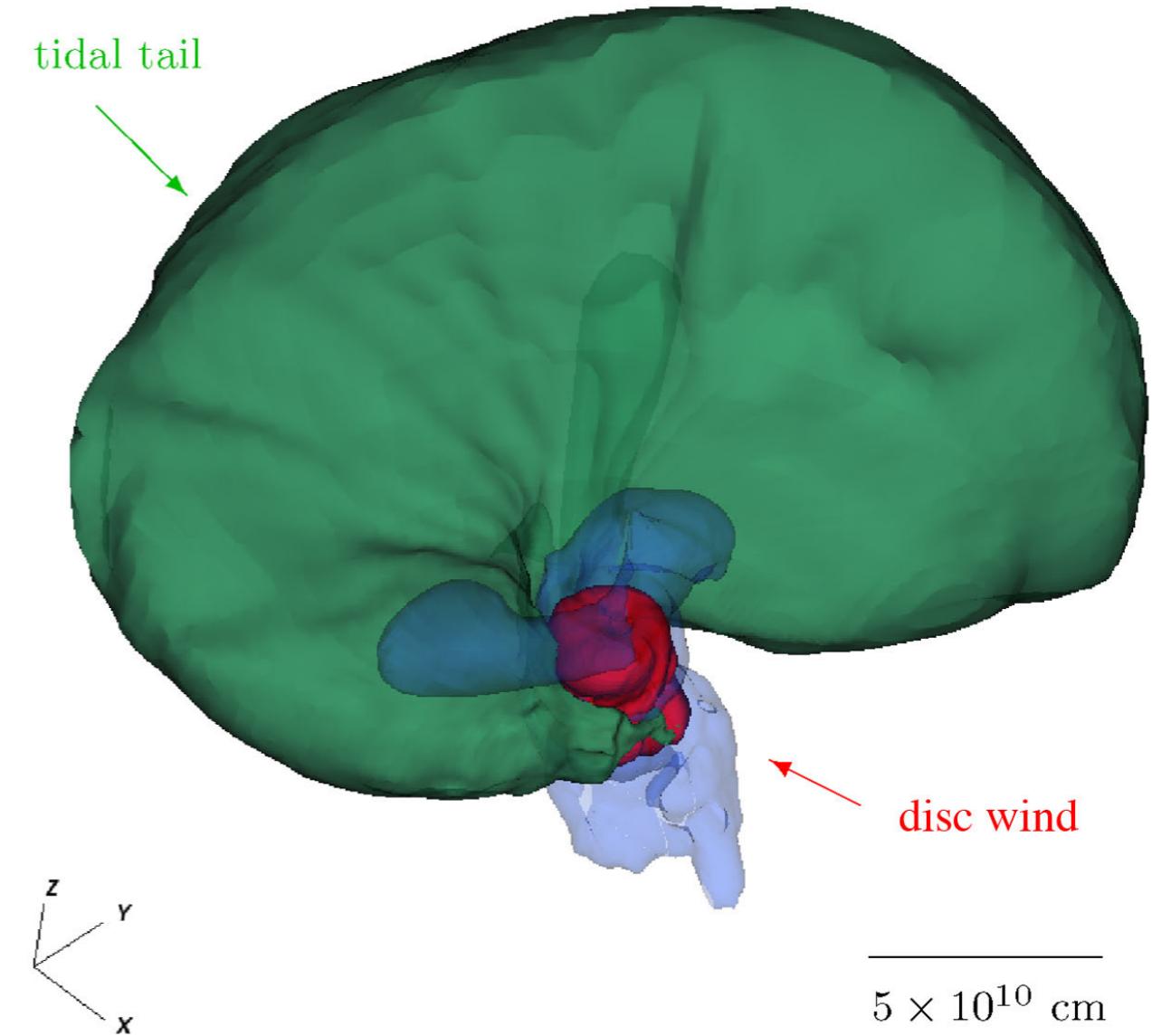
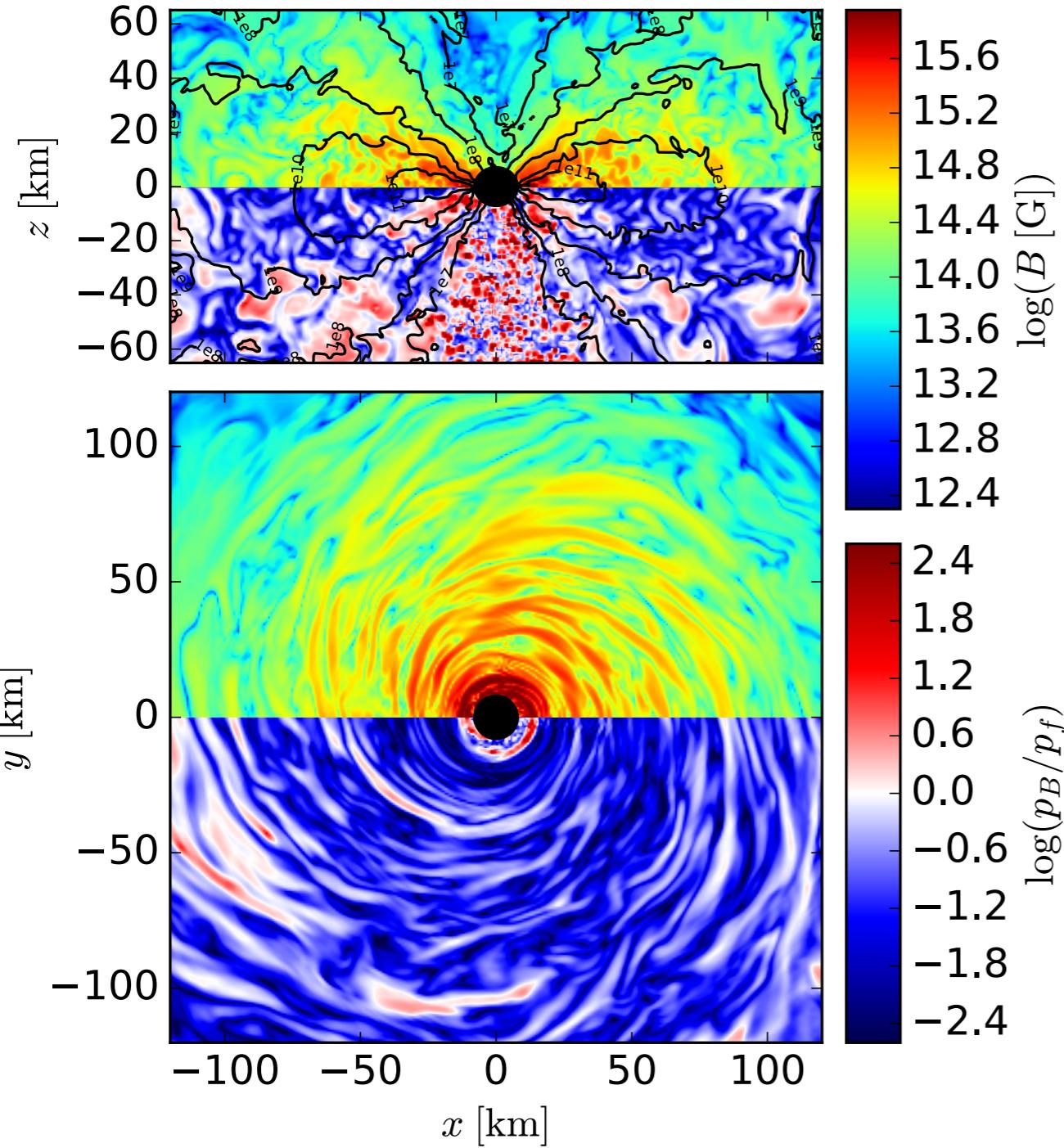
# Secular ejecta

Time = 23.6852 ms

# Typical disk masses



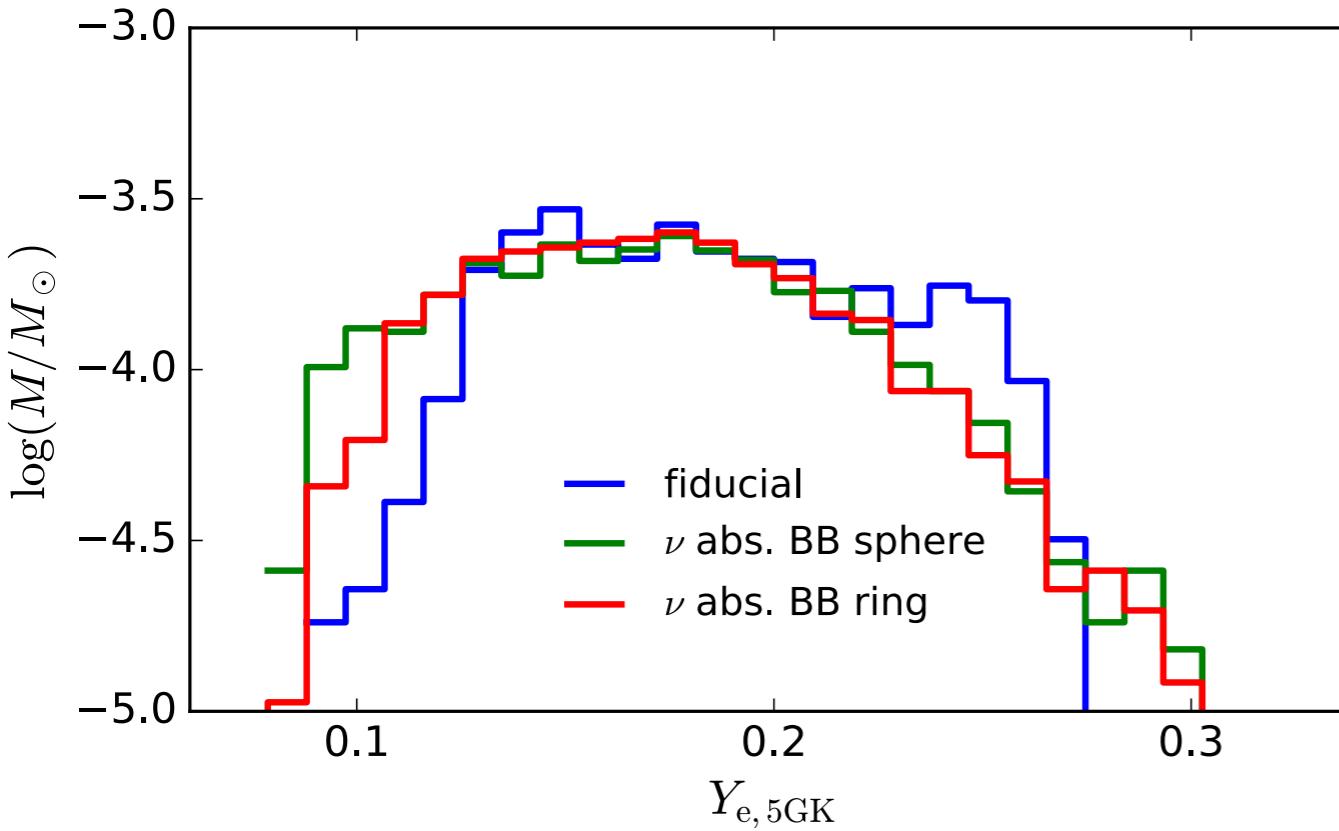
# Secular ejecta



From Siegel & Metzger, arXiv:1711.00868

From Fernandez+, MNRAS 449:390 (2015)

# Secular ejecta: properties

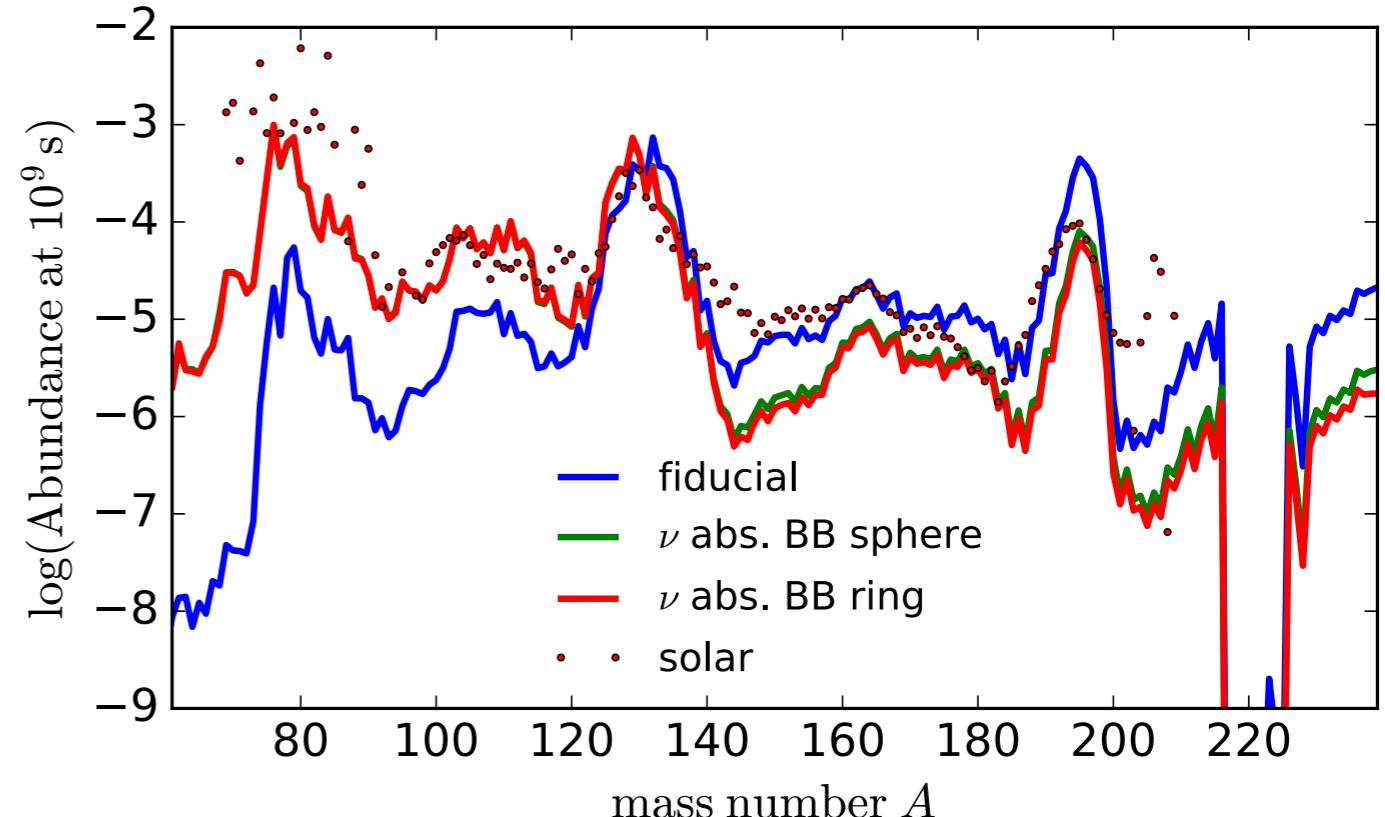


Mass ejection due to nuclear recombination. The characteristic energy gained by each baryon is:

$$\frac{1}{2} \left( \frac{v}{c} \right)^2 \sim \frac{8.8 \text{ MeV}}{931.5 \text{ MeV}}$$

This implies  $v \sim 0.1c$

Up to 50% of the remnant accretion disk is unbound!



# Summary

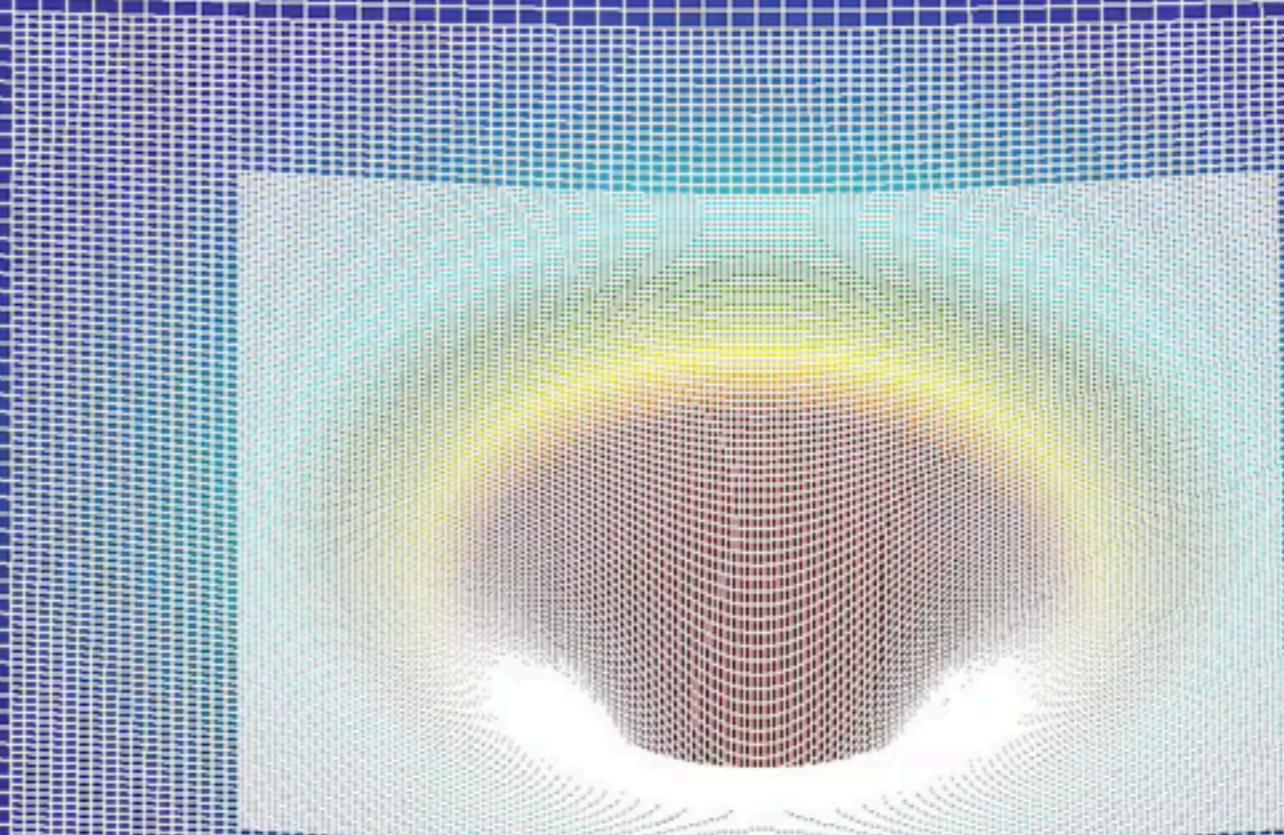
For moderate mass ratio NS-NS mergers:

- the dynamical ejecta determines the opacity of the outermost layer of the outflow;
- the disk wind dominates the nucleosynthetic yields.

The ejecta components from a NS-NS merger are:

- the **tidal tail**: neutron rich, fast;
- the **shocked ejecta**: moderate  $Y_e \sim 0.2$ , fast;
- the **neutrino-driven wind**: high  $Y_e > 0.25$ , fast;
- the **viscous-driven wind**: slower, wide  $Y_e$  distribution,  $v \sim 0.1c$

# How do the simulations work?



# Field equations

- Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

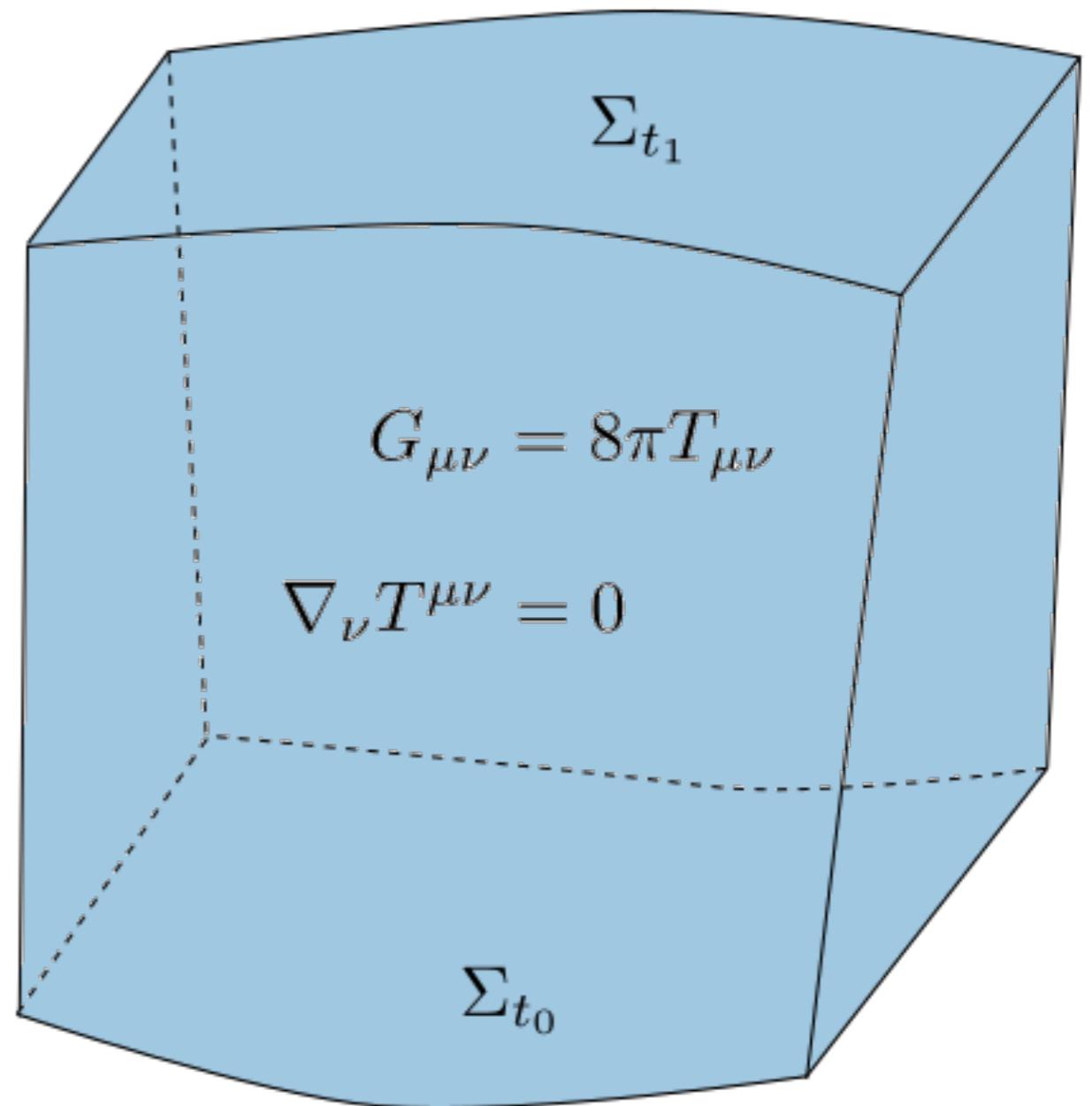
describes the geometry of spacetime

- Stress energy tensor

$$T_{\mu\nu}$$

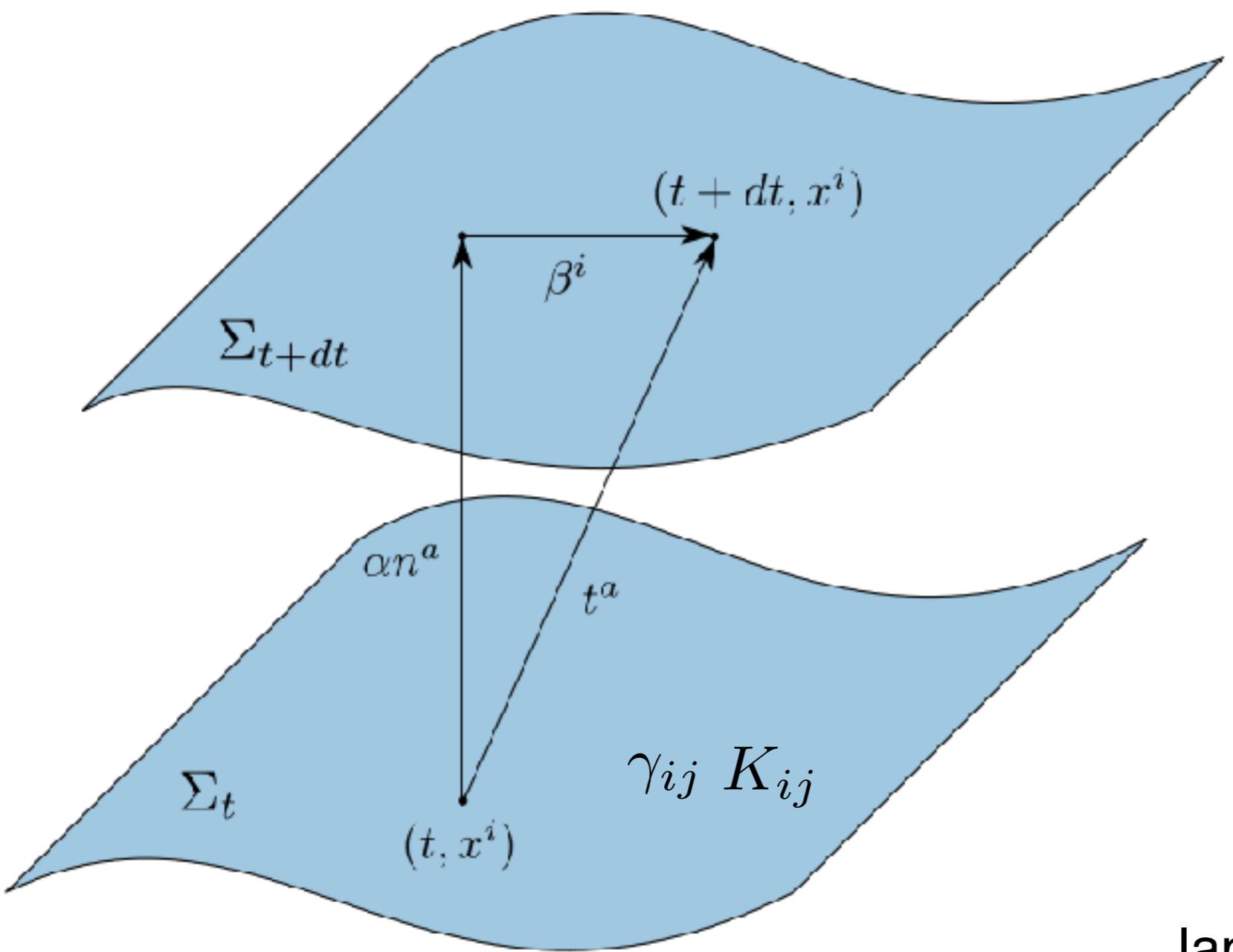
describes matter and radiation content of the spacetime

- Field equations and Bianchi identities connect geometry and flow



Note  $\mathbf{G} = \mathbf{c} = 1$  for us!

# 3+1 decomposition



$\gamma_{ij}$  spatial metric

$K_{ij}$  extrinsic curvature

$\vec{t}$  vector tangent to the constant  $x^i$  curves

$\vec{n}$  vector normal to the constant  $t$  hyper-surfaces

$$\vec{t} = \alpha \vec{n} + \vec{\beta}$$

lapse function      shift vector

Lapse and shift vector are **gauge degrees of freedom**

# Spatial metric

The spatial metric describes the intrinsic geometry of the spatial slices

$$d\ell^2 = \gamma_{ij} dx^i dx^j$$

As a four-dimensional object  $\gamma^\mu{}_\nu$  is used to project vectors tangentially to the spatial hypersurface

$$\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$$

The spatial metric is prescribed as initial data and then evolved

We use the symbol  $D_i$  for the **covariant derivative** associated with  $\gamma_{ij}$

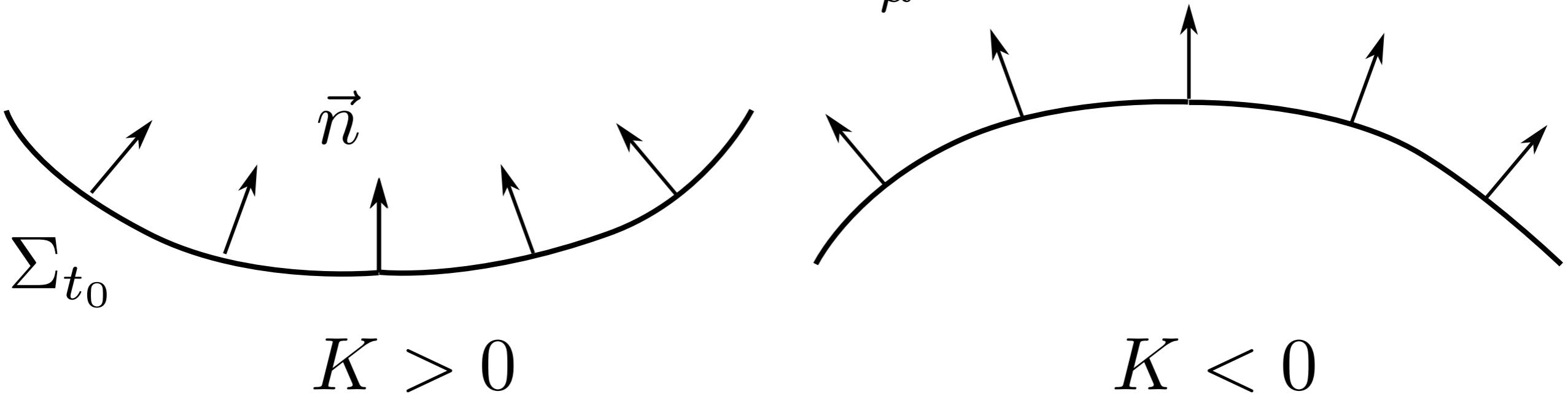
# Extrinsic curvature (I)

$$K_{\mu\nu} = -\gamma^\alpha_\mu \nabla_\alpha n_\nu$$

Tangent projection of the gradient of the normal vector

The trace of the extrinsic curvature is associated with the expansion of the world lines of the Eulerian observers:

$$K = -\nabla_\mu n^\mu$$



# Extrinsic curvature (II)

$$K_{ij} = -\frac{1}{2} \mathcal{L}_{\vec{n}} \gamma_{ij}$$

The extrinsic curvature is the Lie derivative of the metric along the normal to the spatial hypersurface

$K_{ij} = 0 \implies$  three-metric is **stationary!**

If  $\beta^i = 0$  then  $K_{ij} = -\frac{1}{2} \partial_t \gamma_{ij}$

The extrinsic curvature is also prescribed at the initial time and then evolved

# ADM equations

$$(\partial_t - \mathcal{L}_{\vec{\beta}})\gamma_{ik} = -2\alpha K_{ik}$$

$$(\partial_t - \mathcal{L}_{\vec{\beta}})K_{ik} = -D_i D_k \alpha + \alpha \left( {}^{(3)}R_{ik} - 2K_{ij}K^j{}_k + KK_{ik} \right) - 8\pi\alpha \left( S_{ik} - \frac{1}{2}\gamma_{ik}(S-E) \right)$$

$${}^{(3)}R + K^2 - K_{ik}K^{ik} = 16\pi E$$

$$D_k(K\gamma^k{}_i - K^k{}_i) = 8\pi j_i$$

$$S_{ik} = \gamma^\mu{}_i \gamma^\nu{}_k T_{\mu\nu} \quad S = S^i{}_i \quad j_i = -\gamma^\mu{}_i n^\nu T_{\mu\nu} \quad E = T^{\mu\nu} n_\mu n_\nu$$

- Solve constraint equations to create initial data
- Evolve metric and extrinsic curvature
- Note: numerically stable evolution requires additional modification to the equations: [BSSN](#), [Z4c](#)
- Another approach: [generalized harmonics](#)



Evolution equations



Constraints



Projections of the matter fields

# Modeling neutron stars

1. Assemble stress-energy tensor of matter and radiation

$$T_{\mu\nu} = T_{\mu\nu}^{(\text{HD})} + T_{\mu\nu}^{(\text{EM})} + T_{\mu\nu}^{(\nu)} + \dots$$

2. Solve for “conservation” of energy and momentum

$$\nabla_\nu T^{\mu\nu} = 0$$

3. Solve for conservation of baryon and lepton number

$$\nabla_\mu J^\mu = 0 \quad u^\mu \nabla_\mu Y_e = R$$

4. Add equation of state and other constitute relations

$$p = p(n, T, Y_e)$$

5. Equations for neutrino radiation and EM fields

....

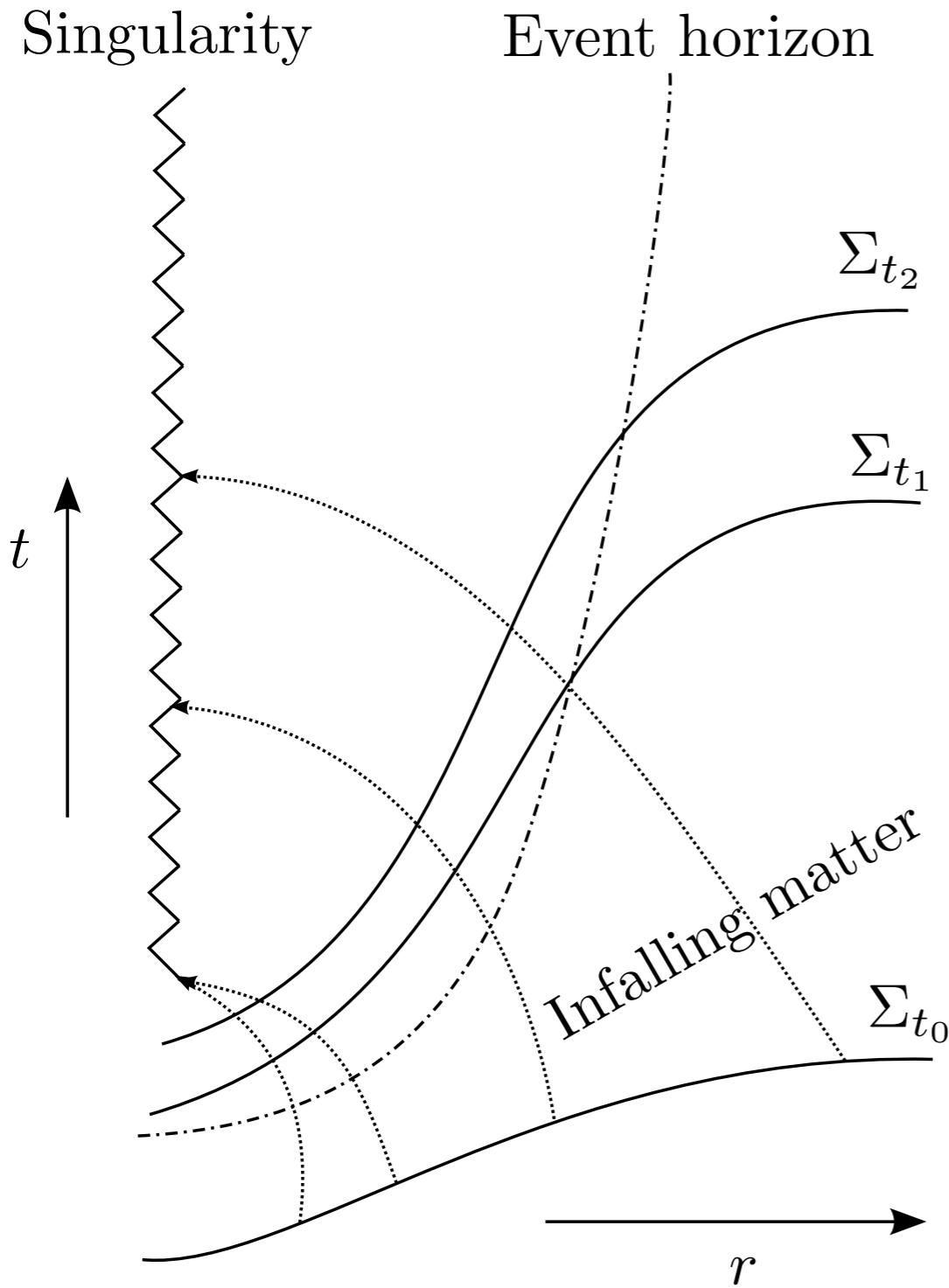
# Valencia formulation

$$\nabla_\nu T_\mu^\nu = 0 \rightarrow$$

$$\partial_t F_\mu^0 + \partial_i F_\mu^i = S_\mu$$

- **Projection** of  $\nabla_\nu T_\mu^\nu = 0$  on the space hypersurface
- Equations similar to the classical hydrodynamics equations
- **Discretize** with standard high-resolution shock-capturing (HRSC) schemes
- Note  $S_\mu$  is in general not zero! **Balance laws.**

# Slicing freedom



BH formation cartoon

Einstein equations do not tell us how to traverse the spacetime: we need to choose a slicing condition

Requirements:

- smoothness
- minimize grid distortion
- avoid singularities

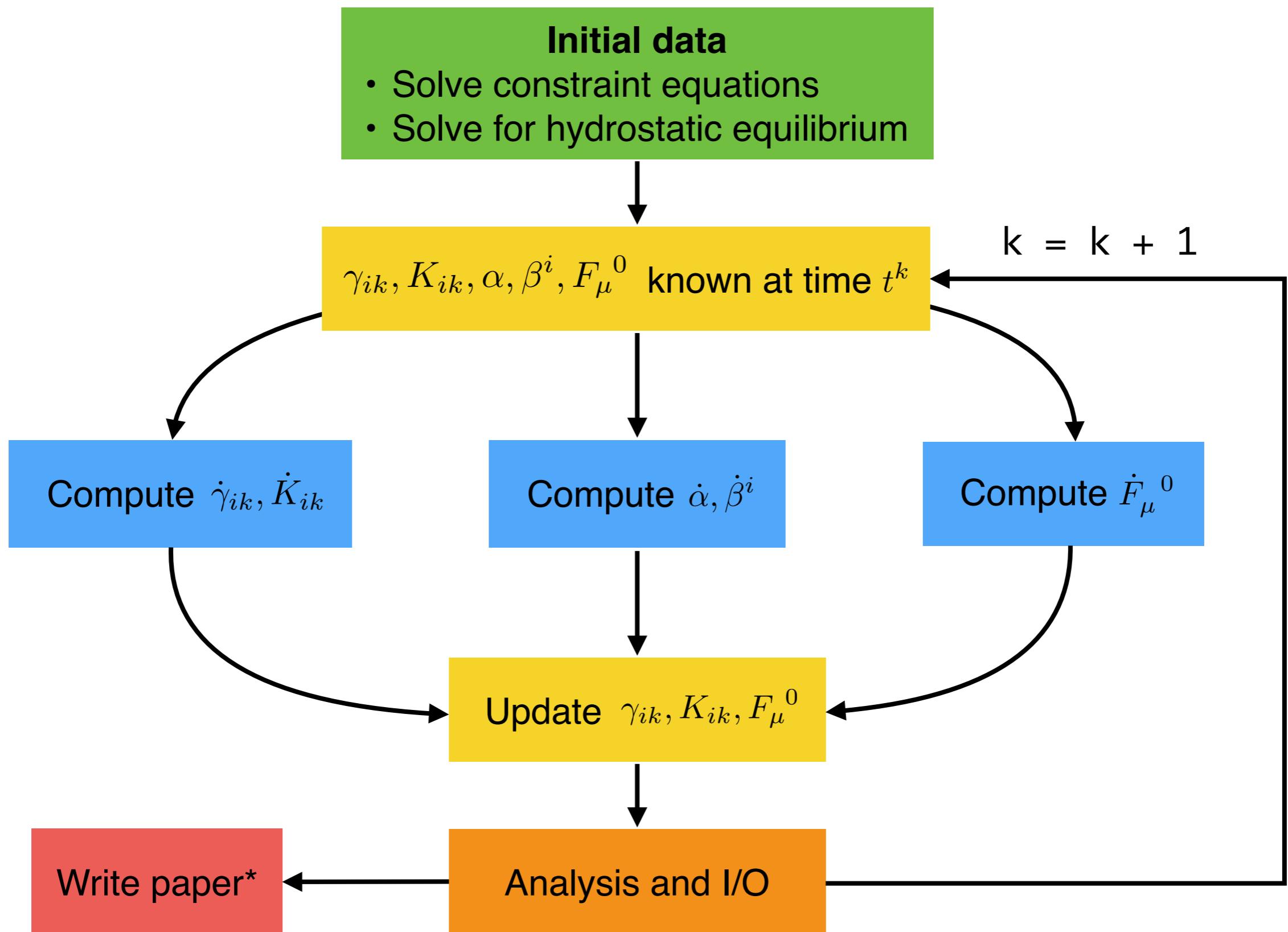
Two families that work in practice

- “puncture” gauge
- generalized harmonic

# Moving puncture gauge

$$(\partial_t + \beta^i \partial_i) \alpha = -2\alpha K$$

- No **caustics** can be formed ( $K = -\nabla_\mu n^\mu$ )
- **Singularity avoidance**: the lapse drops to zero before the foliation hits a singularity (diverging  $K$ )
- If there is a time symmetry, the slicing tries to adapt to it
- Advection with  $\beta^i$  allows black-holes to **move** through the grid



\* not implemented yet

[https://github.com/dradice/JINA\\_MSU\\_School\\_2018.git](https://github.com/dradice/JINA_MSU_School_2018.git)

The screenshot shows a GitHub repository page. At the top, the URL is https://github.com/dradice/JINA\_MSU\_School\_2018. The repository name is dradice / JINA\_MSU\_School\_2018. It has 4 commits, 1 branch, 0 releases, and 1 contributor. The 'Clone or download' button is highlighted with a red circle. Two specific files in the file list are also circled: 'data' and 'HomeWork.pdf'. The repository description at the bottom is 'Neutron Star Merger Simulations'.

No description, website, or topics provided. [Edit](#)

Add topics

4 commits 1 branch 0 releases 1 contributor

Branch: master [New pull request](#) [Create new file](#) [Upload files](#) [Find file](#) [Clone or download](#)

File	Description	Last Commit
dradice More instructions in the README		Latest commit a day ago
<a href="#">data</a>	First commit	7 days ago
<a href="#">HomeWork.pdf</a>	First commit	7 days ago
<a href="#">Lecture.pdf</a>	First commit	7 days ago
<a href="#">PyHRSC1D.py</a>	First commit	7 days ago
<a href="#">README.md</a>	More instructions in the README	a day ago
<a href="#">WorkBook.ipynb</a>	First commit	7 days ago
<a href="#">README.md</a>		

Neutron Star Merger Simulations

[https://github.com/dradice/JINA\\_MSU\\_School\\_2018.git](https://github.com/dradice/JINA_MSU_School_2018.git)

The screenshot shows a PDF document titled "HomeWork.pdf (page 1 of 4)". The title page contains the following text:

Relativistic Hydrodynamics: Fundamentals  
David Radice  
May 4, 2018

## Introduction

The goal of this exercise sheet is to become familiar with the equations of relativistic hydrodynamics and their basic properties, and to learn about the most important building blocks required to assemble the simulations codes used in numerical relativity. We will write a small python code that can solve the equations of special-relativistic hydrodynamics in 1D (1+1) starting from a toy code for non-relativistic gasdynamics. We will use units where  $c = 1$  throughout these notes.

## Basic equations

The special relativistic hydrodynamics equation describe the conservation of energy and momentum of a fluid:

$$\partial_\nu T^{\mu\nu} = 0, \quad (1)$$

where we have used Einstein's convention of sum over repeated indices. We consider a perfect fluid, i.e., we neglect viscosity and heat transport inside of the fluid so that the stress energy tensor reads

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p \eta^{\mu\nu}, \quad (2)$$

where  $\rho$  is the energy density of the fluid measured in a comoving frame,  $p$  is the pressure,  $u^\mu$  the fluid four-velocity and  $\eta = \text{diag}(-1, 1, 1, 1)$  is the metric of Minkowsky spacetime. We remind that the four-velocity can be written as

$$u^\mu = W(1, v^i), \quad (3)$$

Where

$$W = (1 - \eta_{ij} v^i v^j)^{-1/2} \quad (4)$$

is the Lorentz factor and  $v^i$  the spatial velocity.

[https://github.com/dradice/JINA\\_MSU\\_School\\_2018.git](https://github.com/dradice/JINA_MSU_School_2018.git)

```
3. Shell

[davide Downloads]$ git clone https://github.com/dradice/JINA_MSU_School_2018.git
Cloning into 'JINA_MSU_School_2018'...
remote: Counting objects: 20, done.
remote: Compressing objects: 100% (17/17), done.
remote: Total 20 (delta 7), reused 16 (delta 3), pack-reused 0
Unpacking objects: 100% (20/20), done.
[davide Downloads]$ cd JINA_MSU_School_2018/
[davide JINA_MSU_School_2018]$ jupyter-notebook-3.6
[I 17:57:47.580 NotebookApp] Serving notebooks from local directory: /Users/davide/t
mp/Downloads/JINA_MSU_School_2018
[I 17:57:47.580 NotebookApp] 0 active kernels
[I 17:57:47.580 NotebookApp] The Jupyter Notebook is running at:
[I 17:57:47.580 NotebookApp] http://localhost:8888/?token=2c1008017ea9594fa3c5a3d2ff
77a2e8fe251b11f7892278
[I 17:57:47.580 NotebookApp] Use Control-C to stop this server and shut down all ker
nels (twice to skip confirmation).
[C 17:57:47.584 NotebookApp]

Copy/paste this URL into your browser when you connect for the first time,
to login with a token:
http://localhost:8888/?token=2c1008017ea9594fa3c5a3d2ff77a2e8fe251b11f789227
8
[I 17:57:47.785 NotebookApp] Accepting one-time-token-authenticated connection from
::1
[W 17:58:28.514 NotebookApp] Notebook WorkBook.ipynb is not trusted
[I 17:58:29.209 NotebookApp] Kernel started: 3bb2c566-efc1-4a12-95fd-220453fcbcc8
```

[https://github.com/dradice/JINA\\_MSU\\_School\\_2018.git](https://github.com/dradice/JINA_MSU_School_2018.git)

The screenshot shows a Jupyter Notebook interface with the title "WorkBook". The notebook contains a section titled "Classical and relativistic hydrodynamics" with a description of its purpose and the packages used. Below this, two code snippets are shown in the "In [ ]:" cell.

```
In [ ]: from ipywidgets import interact, FloatSlider
import matplotlib.pyplot as plt
from math import fabs, sqrt
import numpy as np
import PyHRSC1D as hrsc
from scipy.optimize import bisect, brentq
```

```
In [ ]: import matplotlib as mpl
%matplotlib inline
```

## Classical and relativistic hydrodynamics

This notebook shows how to formulate and numerically solve the equations of classical and special-relativistic hydrodynamics in 1D, assuming a slab geometry.

We will use a custom python-3 library for the solution of balance laws in 1D, which is implemented using [numpy](#). This notebook also uses [matplotlib](#) and [ipywidgets](#) for the visualization and the root finding routines from [scipy](#). All of these packages are easily available through the [anaconda python distribution](#).

```
In [ ]: from ipywidgets import interact, FloatSlider
import matplotlib.pyplot as plt
from math import fabs, sqrt
import numpy as np
import PyHRSC1D as hrsc
from scipy.optimize import bisect, brentq
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```
In [ ]: import matplotlib as mpl
%matplotlib inline
```

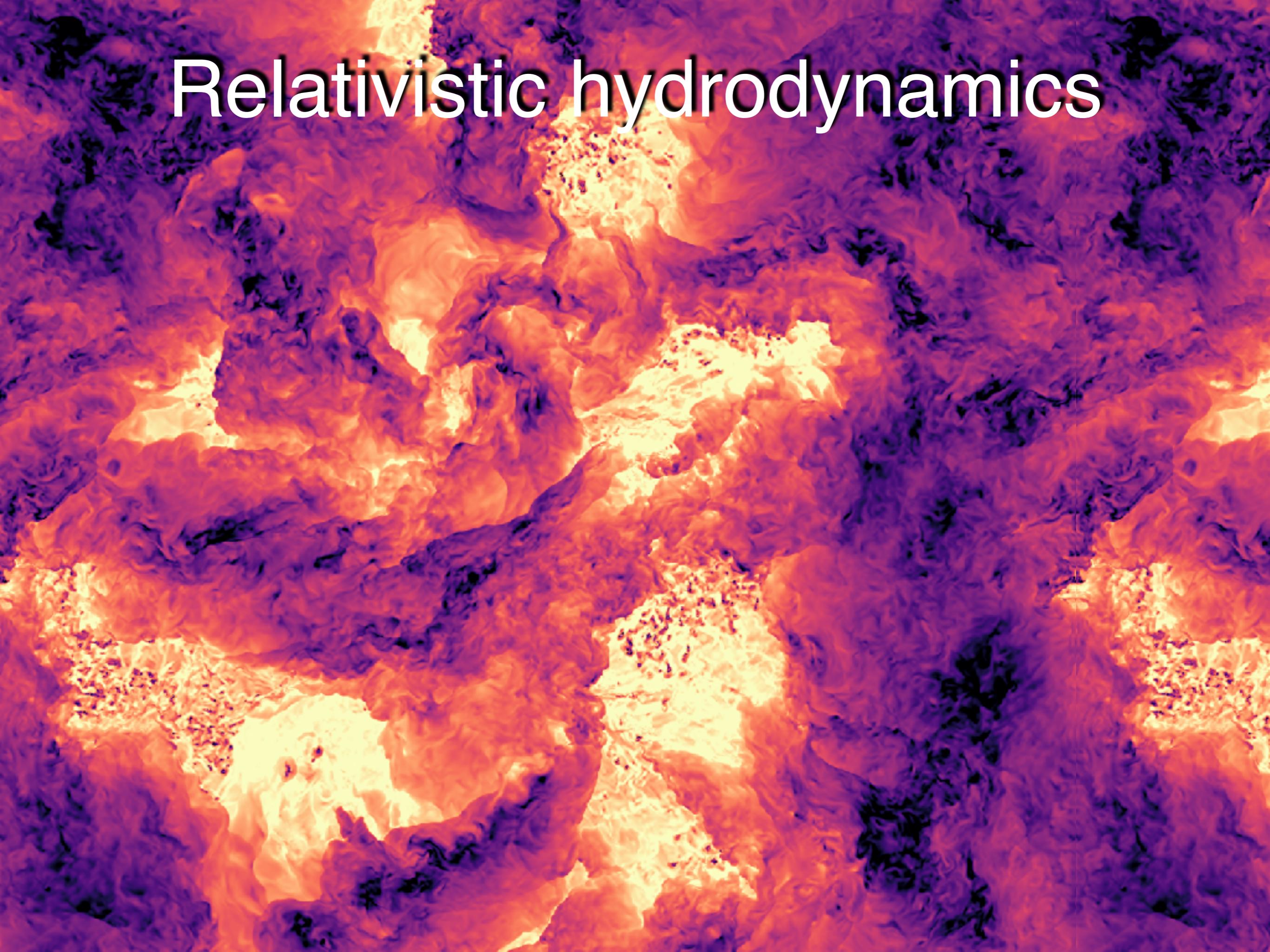
## Newtonian hydrodynamics

We start with the Newtonian hydrodynamics

### Basic equations

The classical equations of hydrodynamics describe the conservation of mass, momentum, and energy in a fluid. In one spatial dimension they are

# Relativistic hydrodynamics



# Field equations

- Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

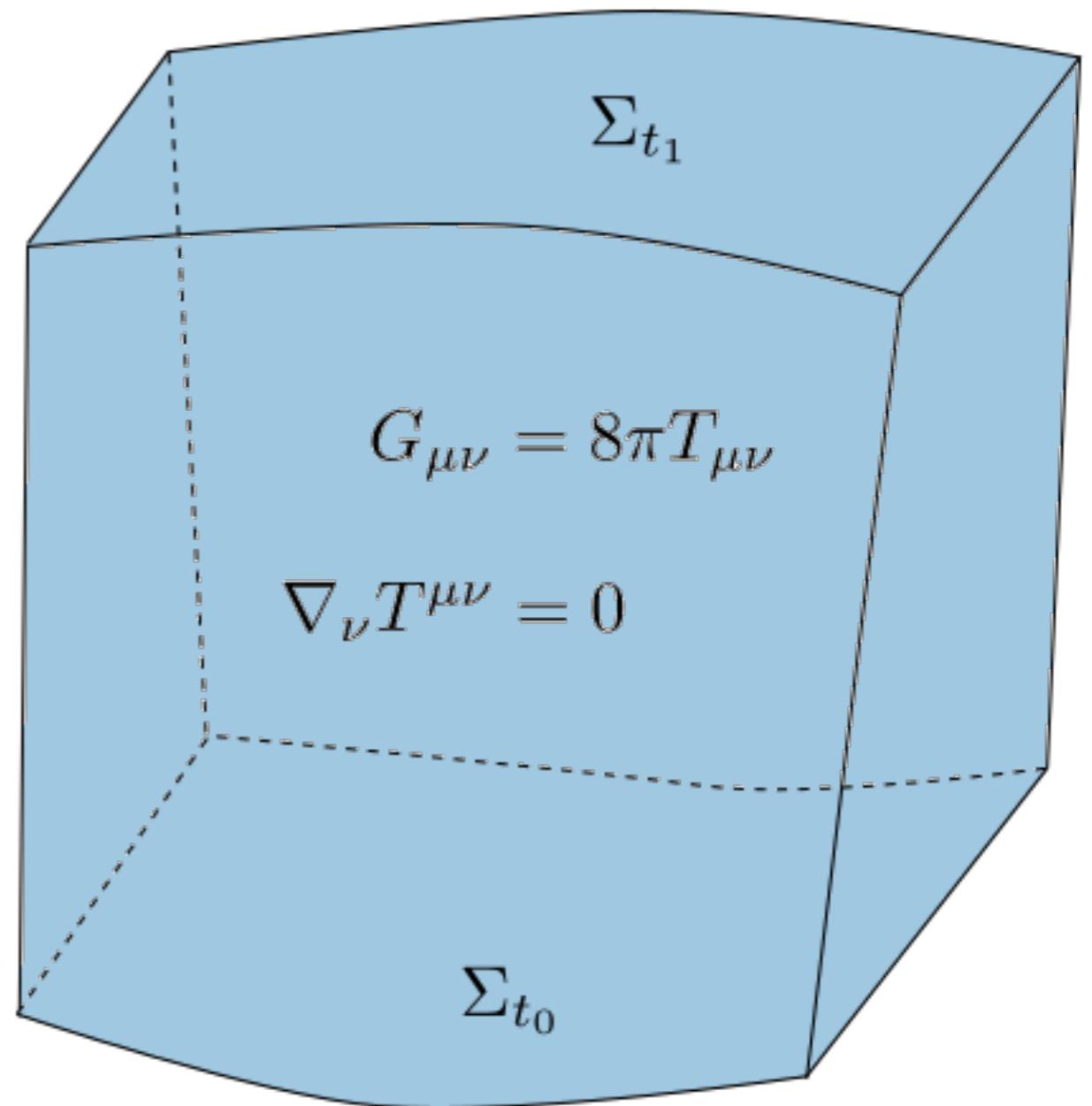
describes the geometry of spacetime

- Stress energy tensor

$$T_{\mu\nu}$$

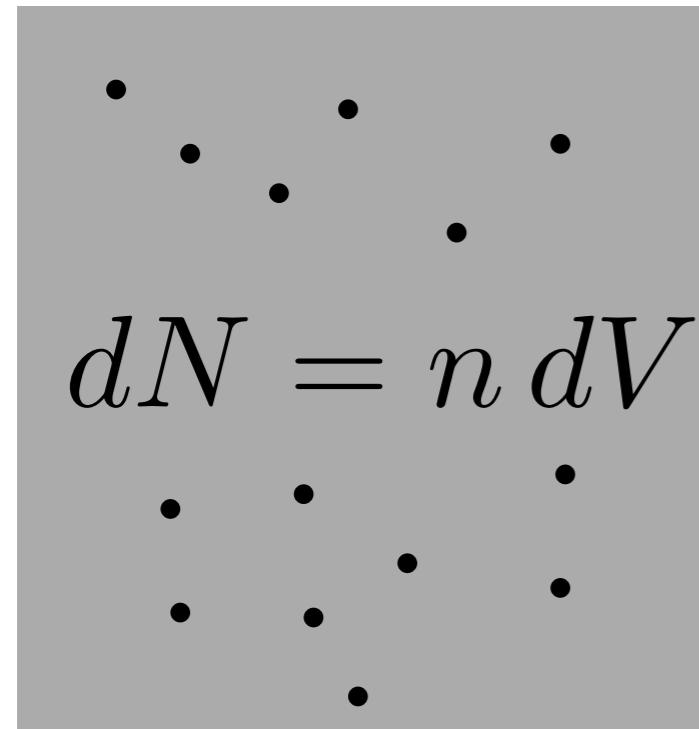
describes matter and radiation content of the spacetime

- Field equations and Bianchi identities connect geometry and flow



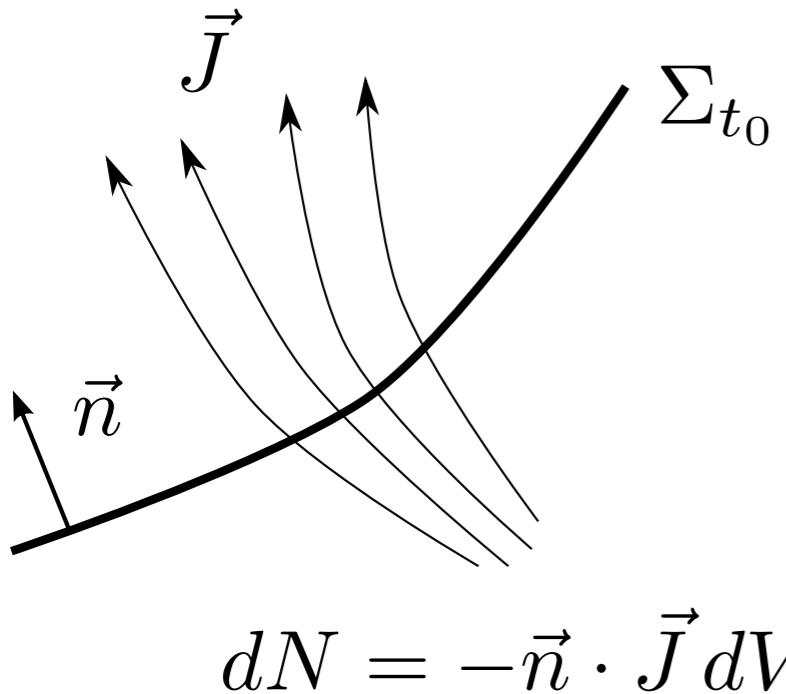
Note  $\mathbf{G = c = 1}$  for us!

# Relativistic hydrodynamics



## Classical picture

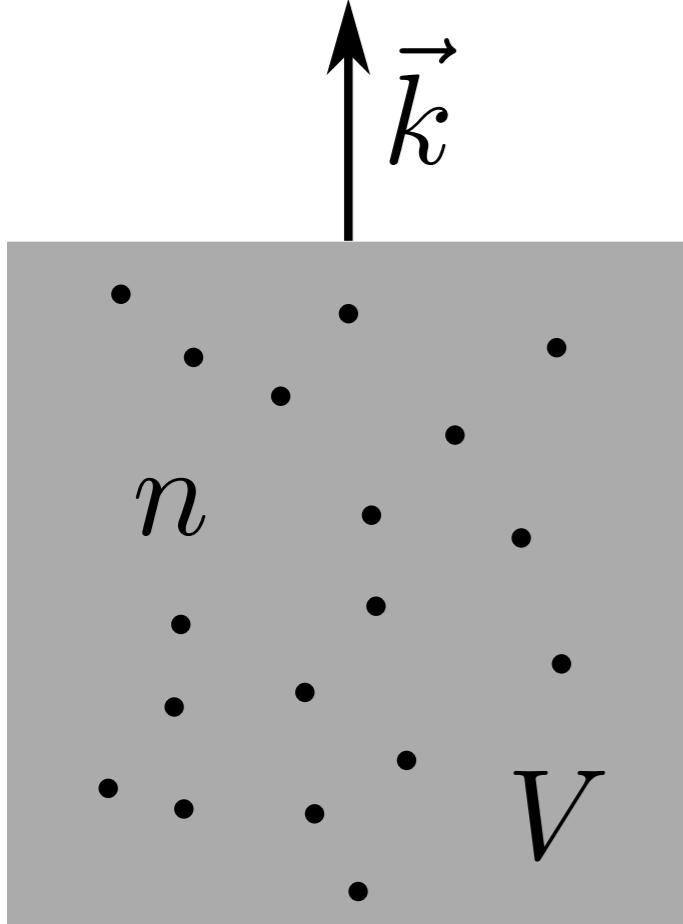
- Intrinsic variable: particle number density
- Equations describe the flux of matter through **space** as a function of **time**



## Relativistic picture

- Intrinsic variable: flux of particles through **spacetime**
- Equations describe the density as measured by any observer

# Classical continuity equation



$$\frac{d}{dt} \int_V n \, dV + \int_{\partial V} n \vec{v} \cdot \vec{k} \, dS = 0$$

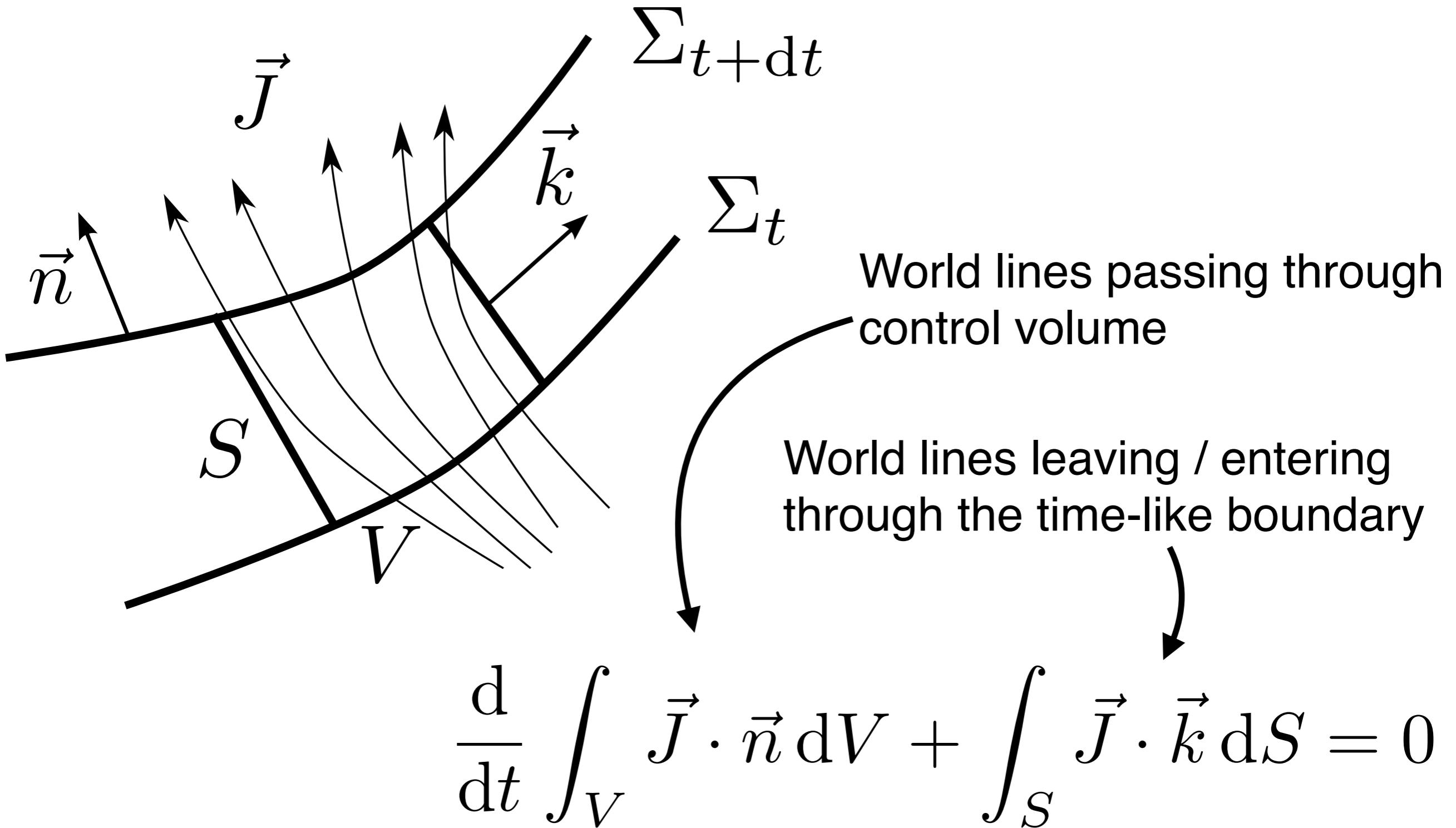
Number of  
particles in a  
control volume

Flux of particles across  
the boundaries of the  
control volume

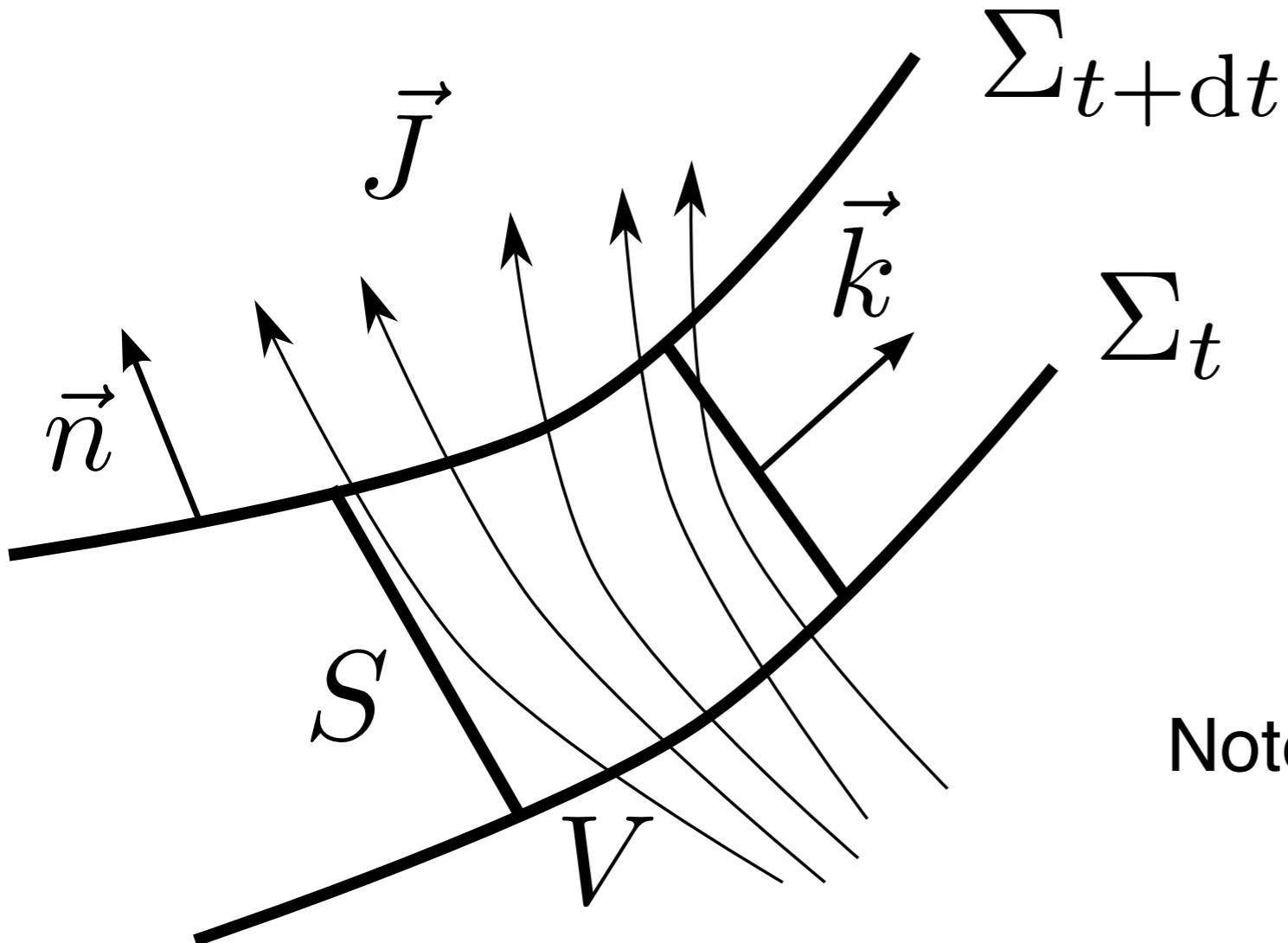
Differential form:

$$\partial_t n + \nabla_i (n v^i) = 0$$

# Relativistic continuity equation



# Relativistic continuity equation



4-velocity

Number current

$J^\mu = n u^\mu$

Note:  $-J^\mu u_\mu = n$

Differential form:

$$\nabla_\mu J^\mu = 0$$

# Stress-energy tensor

The number current of a fluid is such that

$$-J_\mu n^\mu = \text{“Number density measured by a fiducial observer”}$$

In analogy the stress-energy tensor of a fluid

$$T_{\mu\nu} n^\mu n^\nu = \text{“Energy density measured by a fiducial observer”}$$

$$-T_{\mu\nu} n^\mu (\partial_i)^\nu = \text{“Energy flux measured by a fiducial observer”}$$

$$T_{\mu\nu} (\partial_i)^\mu (\partial_j)^\nu = \Sigma_{ij} = \text{“Stress forces measured by a fiducial observer”}$$

Where we have chosen  $\{\vec{n}, \partial_i\}$  such that

$$n^\mu (\partial_i)_\mu = 0$$

# Modeling neutron stars

1. Assemble stress-energy tensor of matter and radiation

$$T_{\mu\nu} = T_{\mu\nu}^{(\text{HD})} + T_{\mu\nu}^{(\text{EM})} + T_{\mu\nu}^{(\nu)} + \dots$$

2. Solve for “conservation” of energy and momentum

$$\nabla_\nu T^{\mu\nu} = 0$$

3. Solve for conservation of baryon and lepton number

$$\nabla_\mu J^\mu = 0 \quad u^\mu \nabla_\mu Y_e = R$$

4. Add equation of state and other constitute relations

$$p = p(n, T, Y_e)$$

5. Equations for neutrino radiation and EM fields

....

# Perfect fluid

It is easy to write the stress energy tensor for NS matter in a locally-flat comoving frame  $\{\vec{u}, \partial_i\}$  assuming a **perfect fluid**

We impose the following condition

$$T_{\mu\nu} u^\mu u^\nu = \rho \quad \text{Energy density of the fluid}$$

$$T_{\mu\nu} u^\mu (\partial_i)^\nu = 0 \quad \text{No heat conduction}$$

$$T_{\mu\nu} (\partial_i)^\mu (\partial_j)^\nu = p \delta_{ij} \quad \text{Only isotropic pressure}$$

In this case the stress energy tensor takes the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \quad u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \xrightarrow{\text{General frame}}$$

$$\begin{aligned} T_{\mu\nu} &= \rho u^\mu u^\nu + p \perp^{\mu\nu} \\ \perp_{\mu\nu} &= g_{\mu\nu} + u_\mu u_\nu \end{aligned}$$

# “Conservation” laws (I)

The key part of the equations of motion is

$$\nabla_\nu T^{\mu\nu} = 0$$

what is really conserved?

# “Conservation” laws (II)

Multiply  $\nabla_\nu T^{\mu\nu} = 0$  by any smooth vector field  $p^\mu$  to get

$$\nabla_\mu (T^{\mu\nu} p_\nu) = -T^{\mu\nu} \nabla_\mu p_\nu = -\frac{1}{2} T^{\mu\nu} (\nabla_\mu p_\nu + \nabla_\nu p_\mu)$$



$$T^{\mu\nu} = T^{\nu\mu} = \frac{1}{2} (T^{\mu\nu} + T^{\nu\mu}) \quad = 0 \text{ for Killing vectors}$$

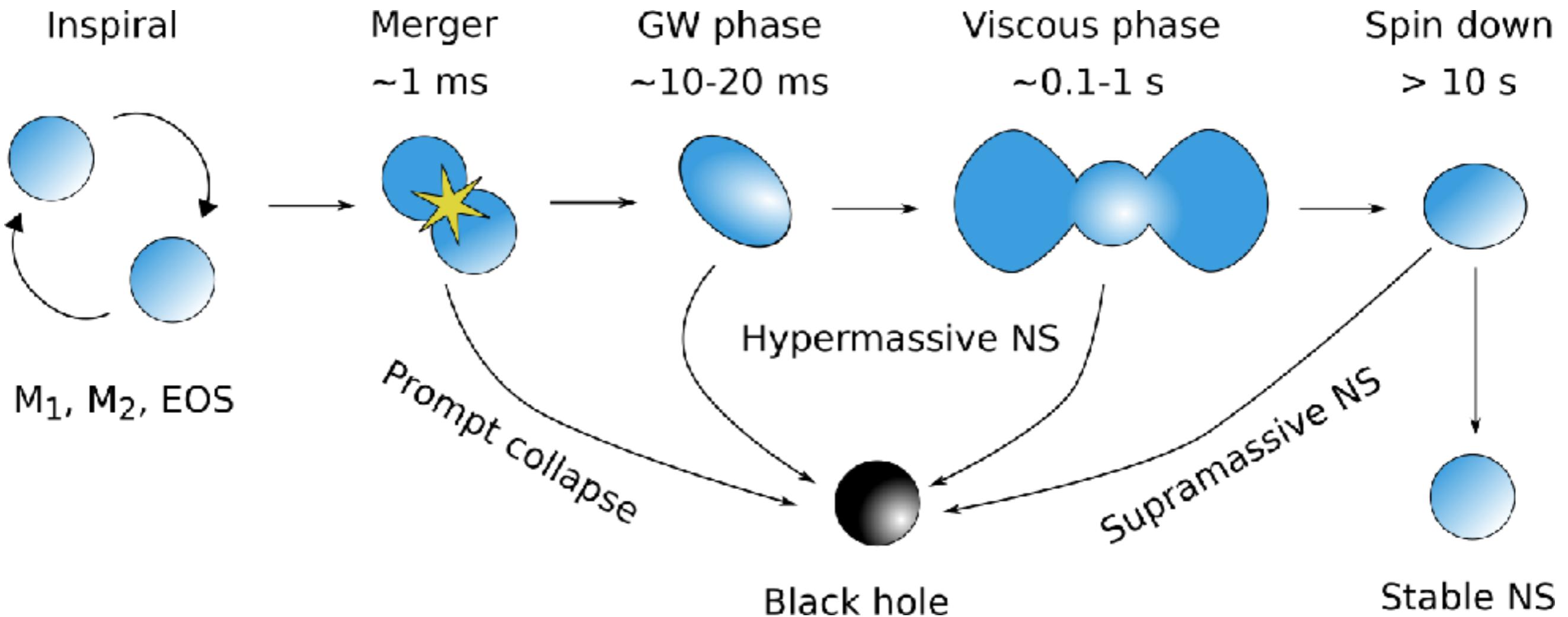
Killing vectors are direction of **symmetry** for the metric

$$0 = \nabla_\mu p_\nu + \nabla_\nu p_\mu \implies \mathcal{L}_{\vec{p}} g_{\mu\nu} = 0$$

Noether's theorem:

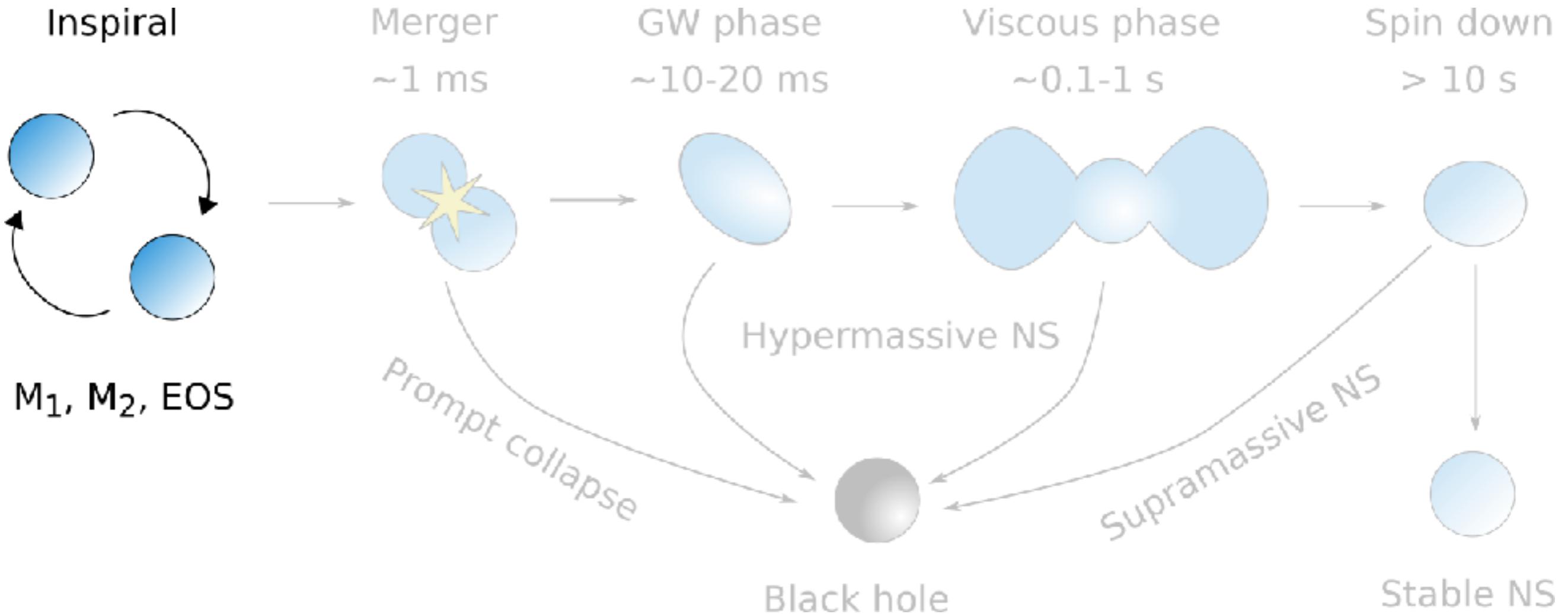
*Spacetime symmetries generate conserved charges*

$$p^\mu \text{ Killing vector} \implies C^\mu = T^{\mu\nu} p_\nu \text{ conserved}$$

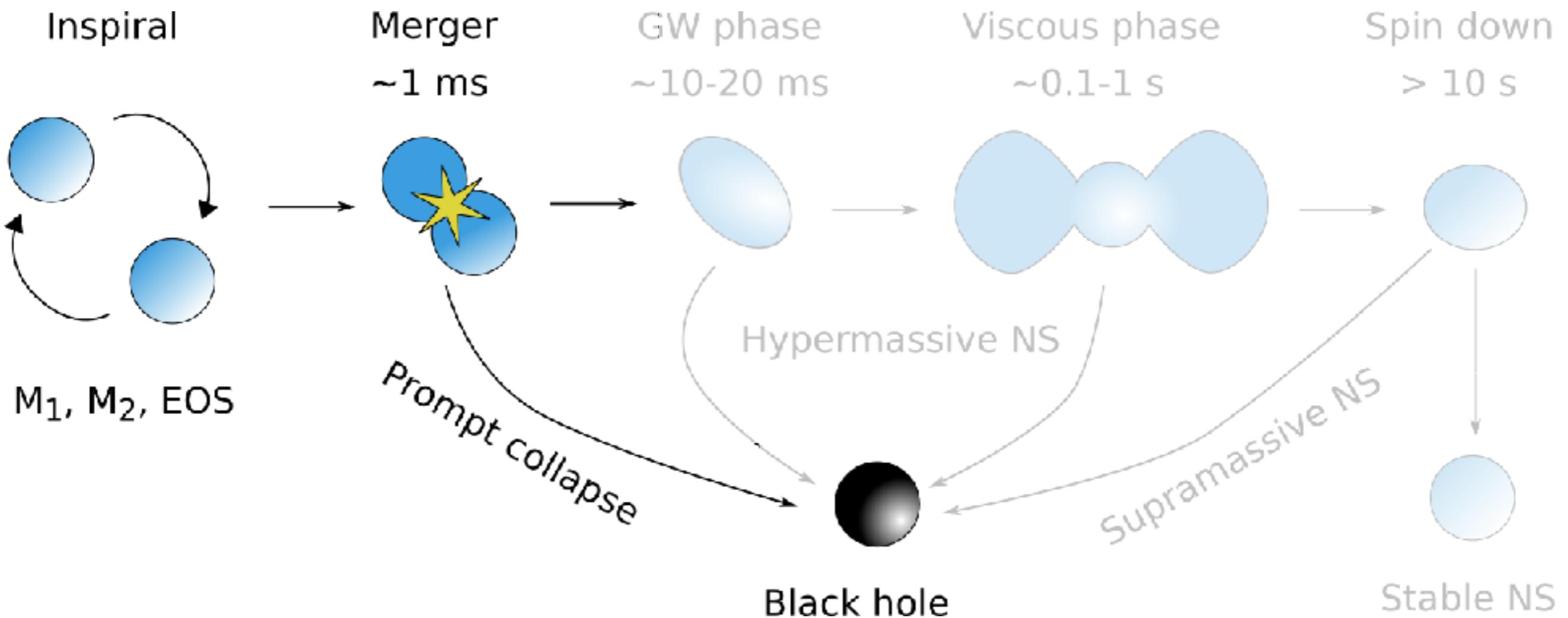


# Neutron star mergers phases and possible outcomes

The evolution and outcome of mergers depend on the **NS masses**, the **NS EOS**, and (possibly) **spins**.

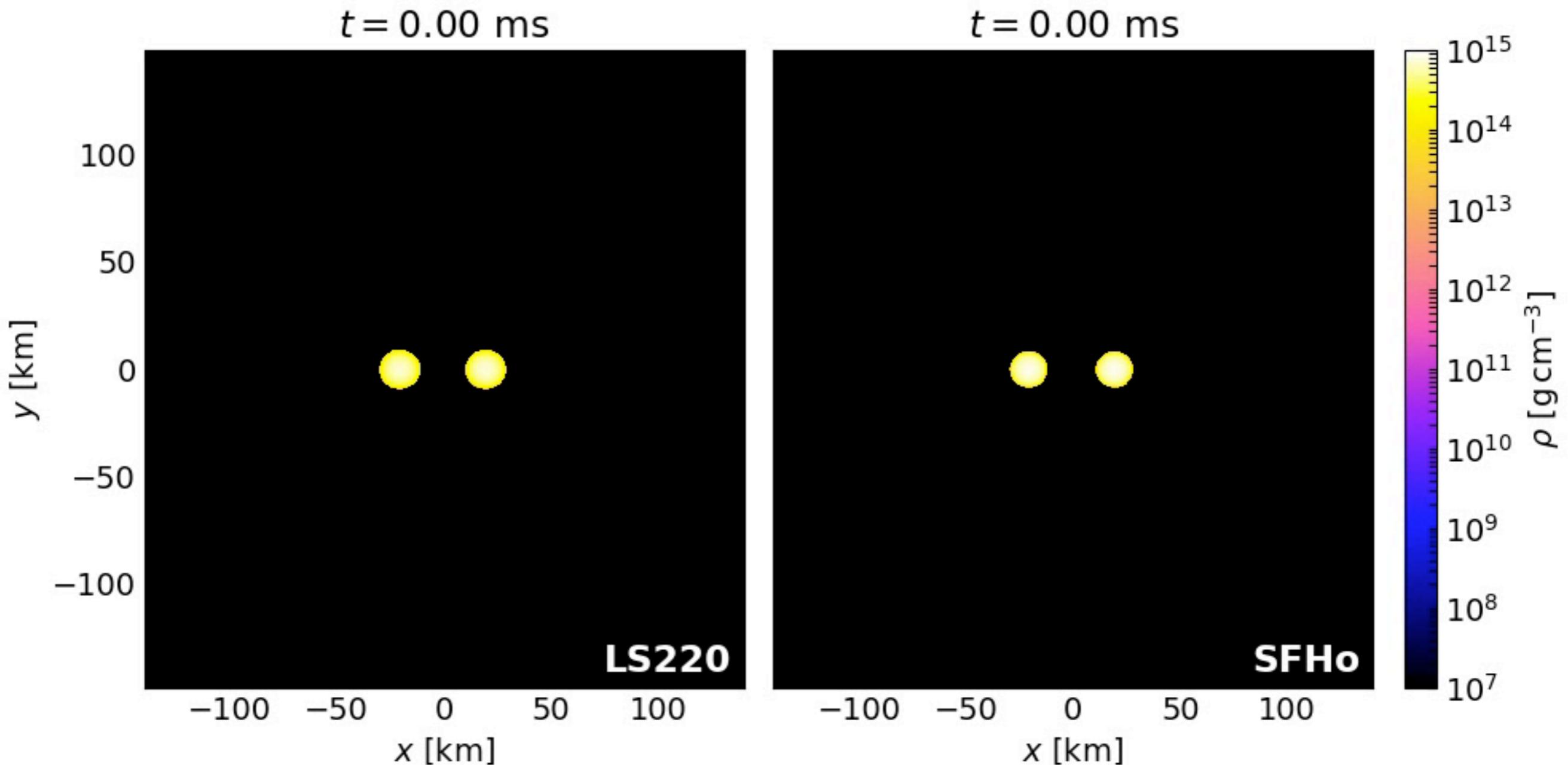


- The binary is in the LIGO/Virgo frequency band in the last few minutes of the inspiral ( $O(10^3)$  orbits).
- **Tidal effects** in this phase can be used to constrain the EOS of NSs
- Best understood phase of the binary evolution



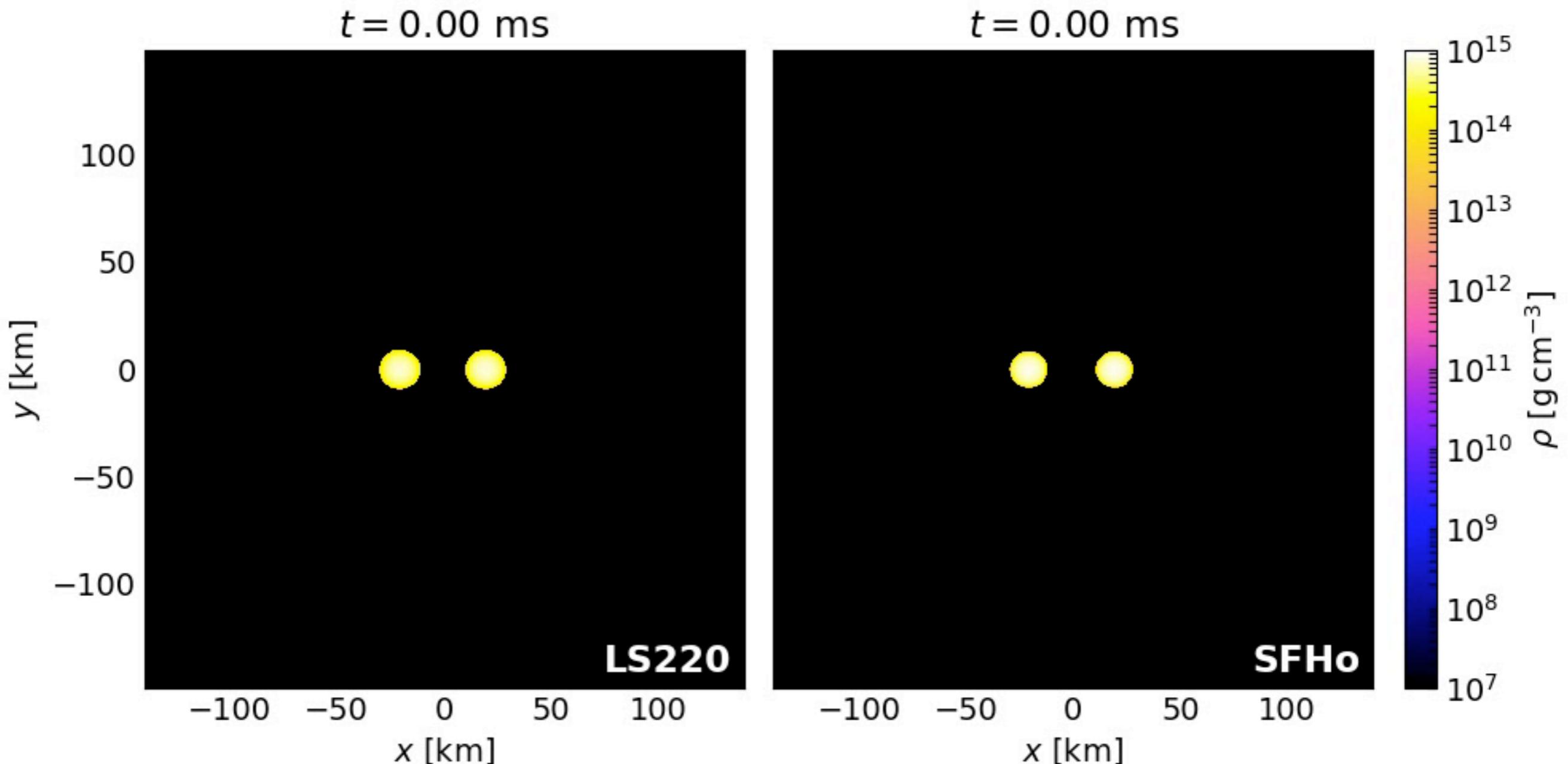
- Merger is the time when the GW signal reaches the maximum amplitude. Marks the end of the inspiral.
- If the total binary mass is larger than  $(1.3 - 1.6) M_{\text{max}}$  then a BH is formed **promptly** during merger.
- **Prompt BH collapse** leaves little debris: probably EM-quiet

# Prompt BH formation

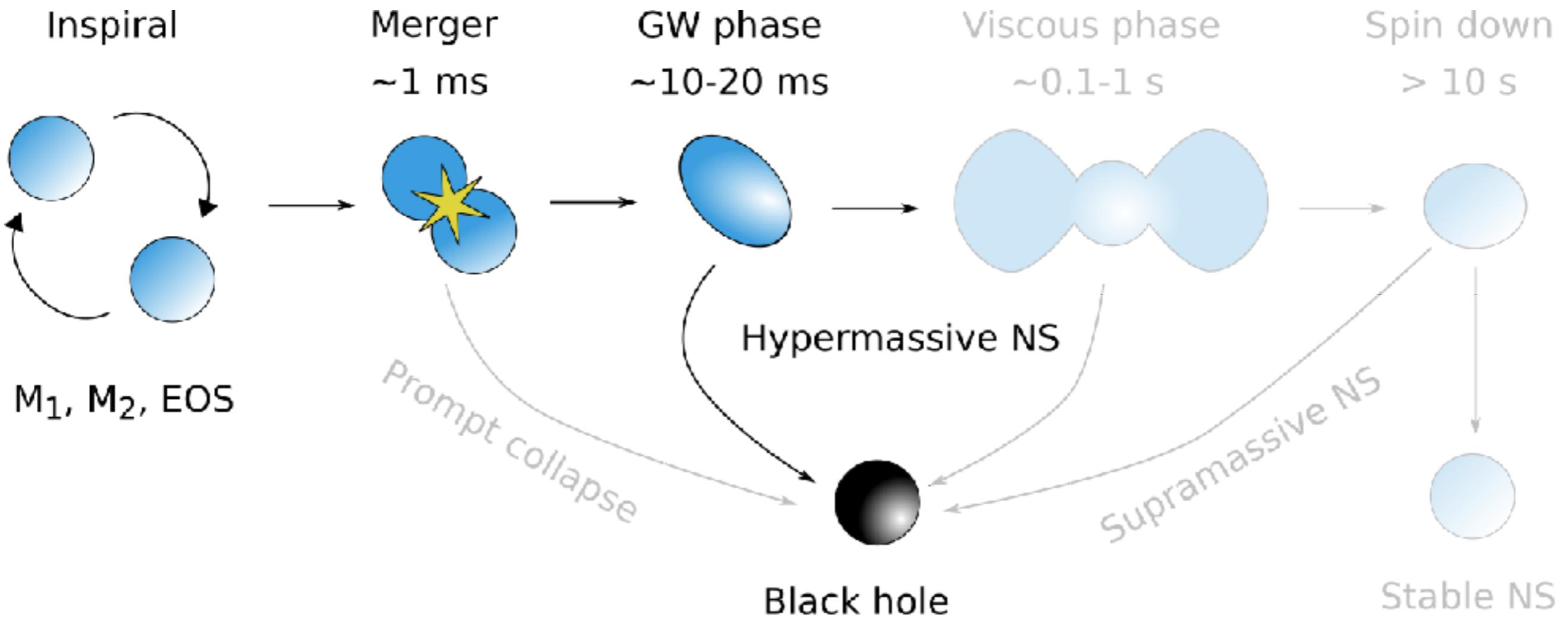


$(1.44 + 1.39) M_{\odot} - \text{B1913} + 13$

# Prompt BH formation

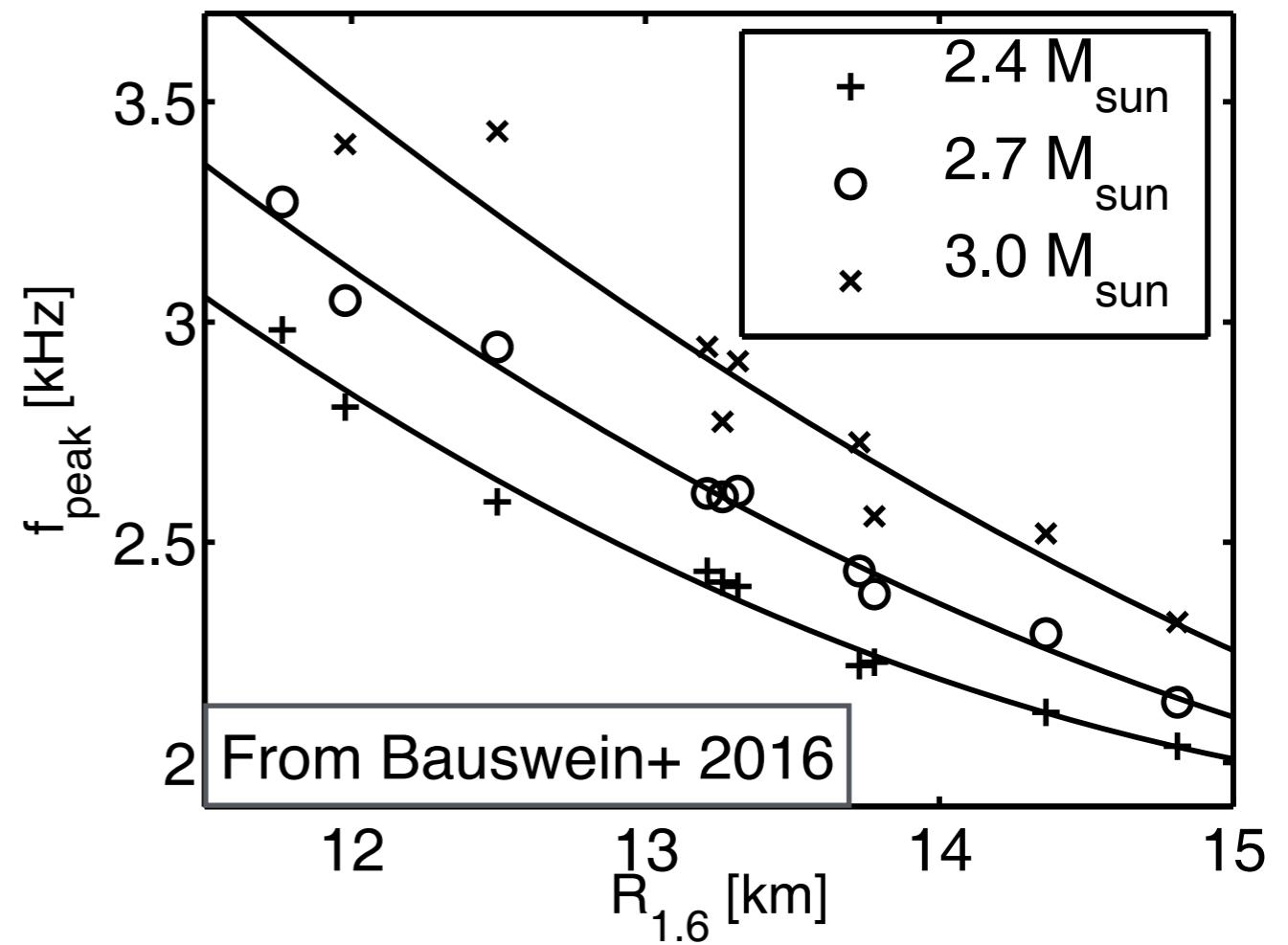
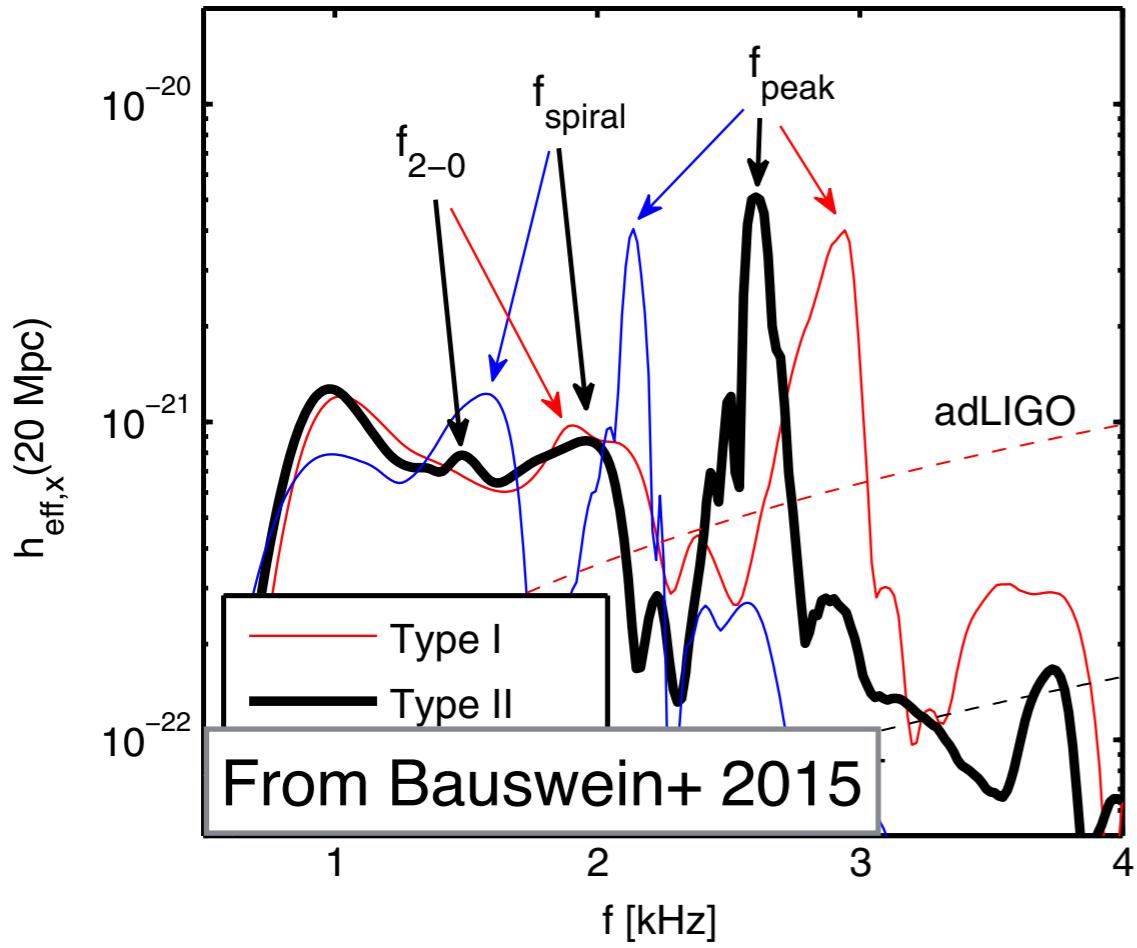


$(1.44 + 1.39) M_{\odot} - \text{B1913} + 13$



- The merger remnant is highly deformed and radiates GWs
- **Most luminous phase** of the evolution, but outside of LIGO band
- Postmerger GW could also constrain the NS EOS
- GW losses decay within  $\sim$ 10-20 ms
- The merger remnant might collapse because of the loss of angular momentum during this phase (**hypermassive NS**)

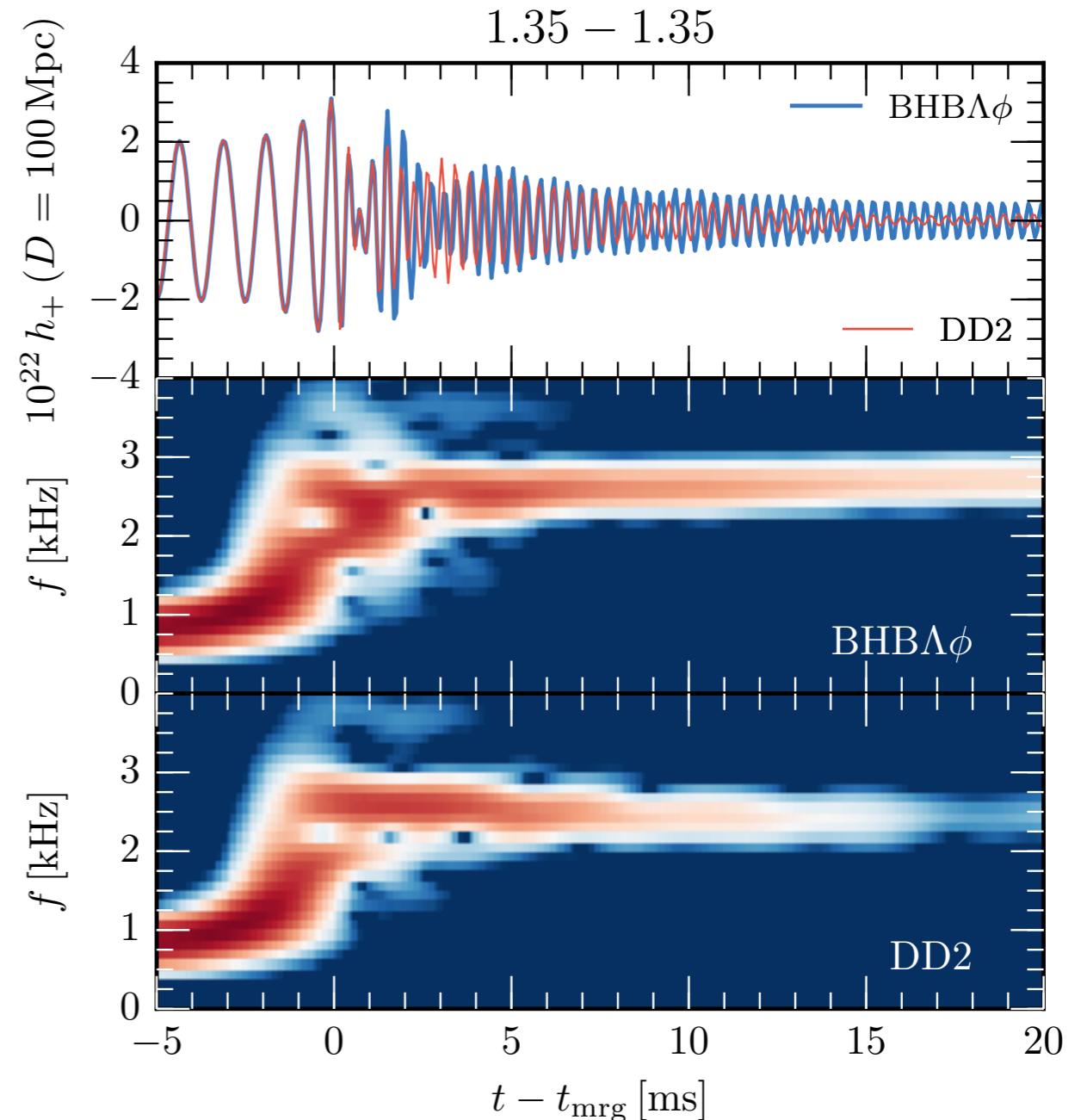
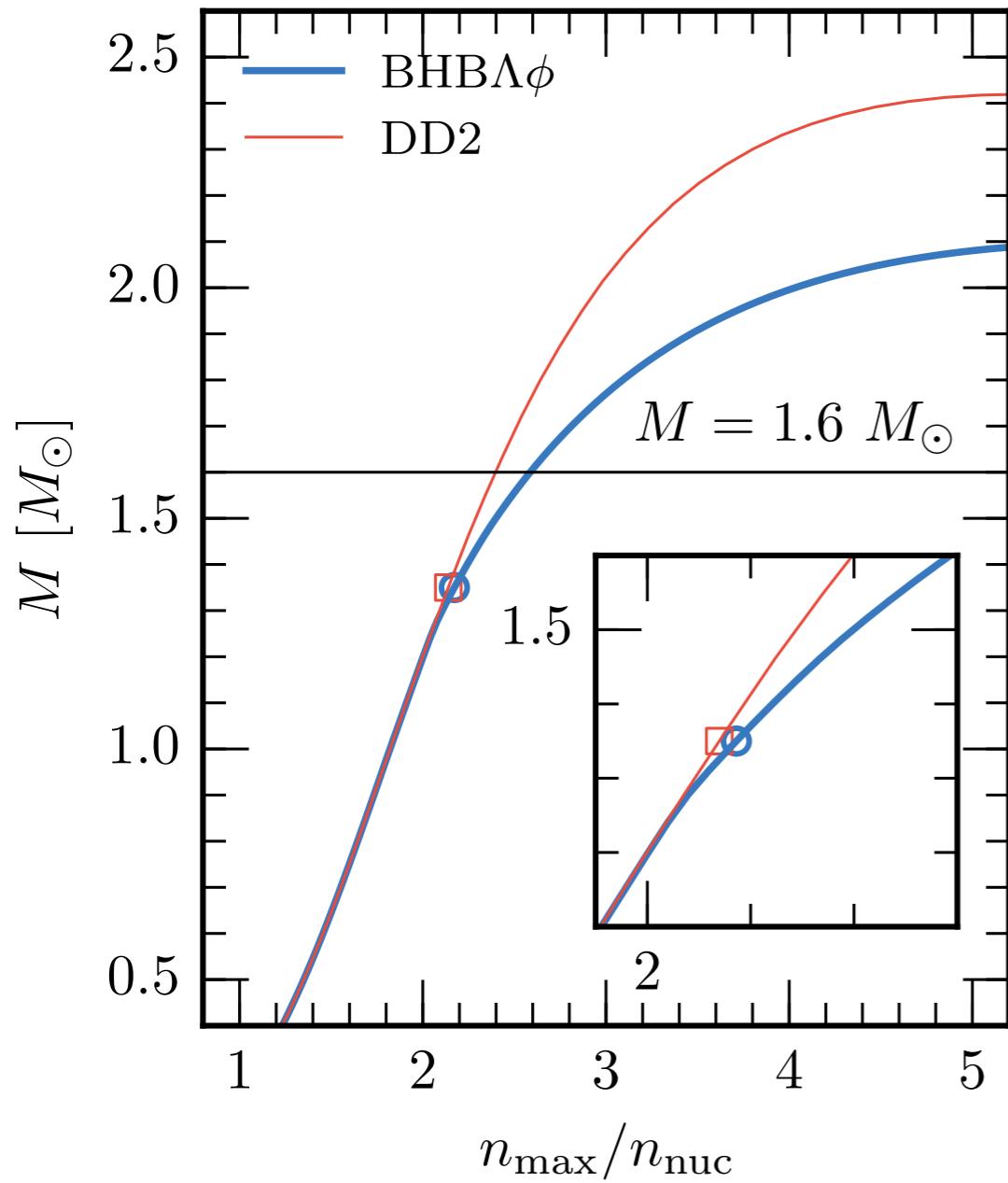
# Postmerger GWs signal (I)



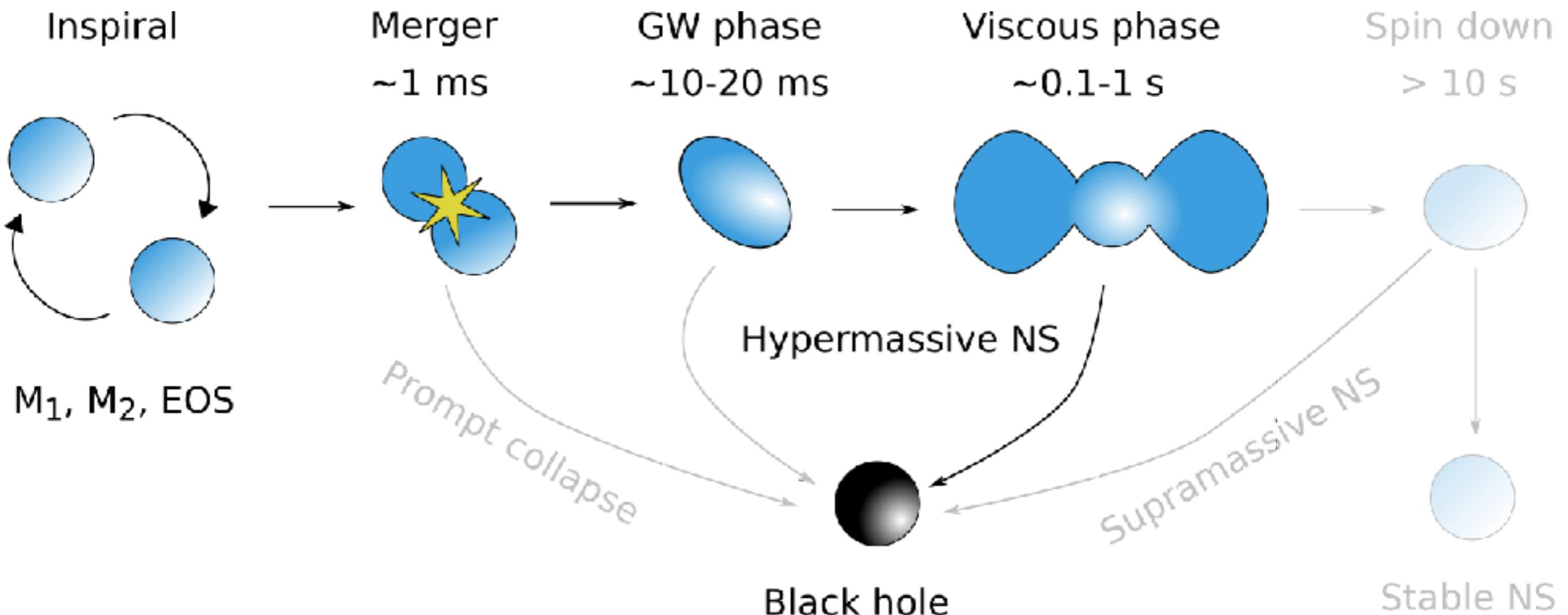
- Post-merger signal has a **characteristic peak frequency**
- $f_{\text{peak}}$  correlates with the NS radius
- **Small statistical uncertainty, systematics not understood yet**

See also Takami+ 2014; Rezzolla & Takami 2016; Dietrich+ 2016; Bose+ 2017

# Postmerger GW signal (II)

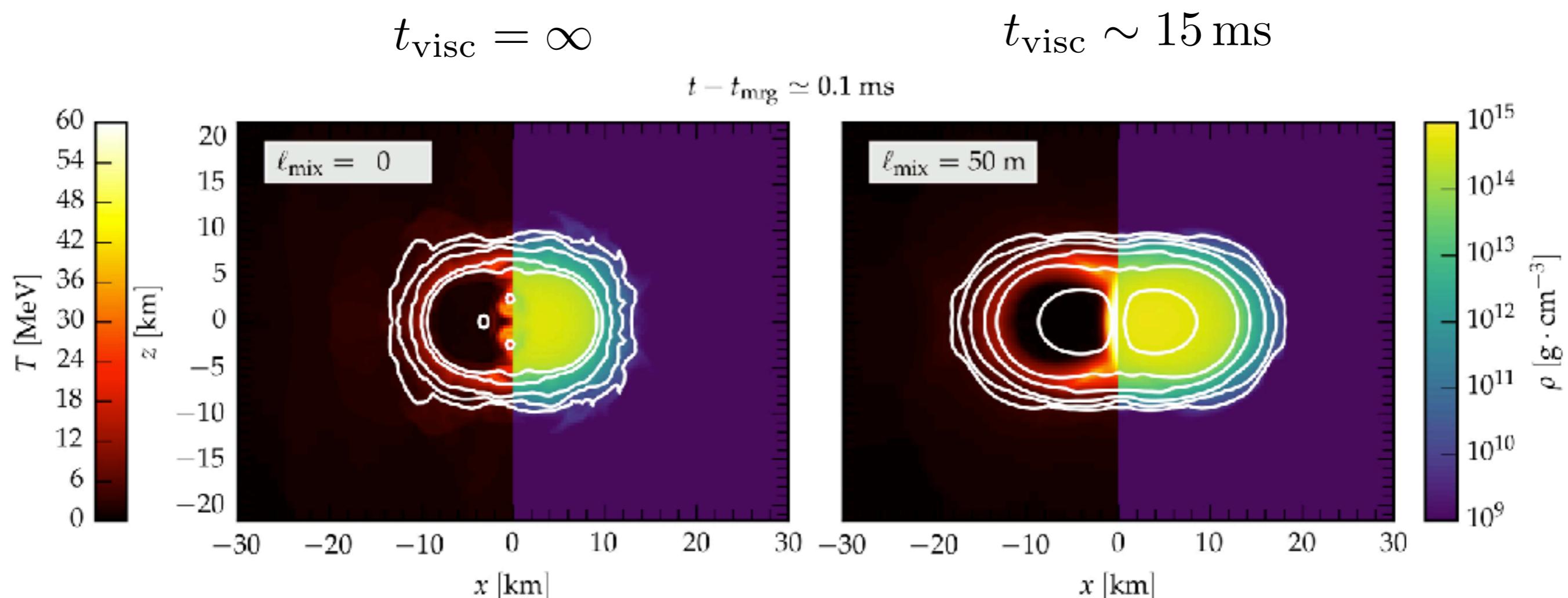


The GW amplitude could reveal the **outcome of the merger** and constraint the **properties of matter at extreme densities**

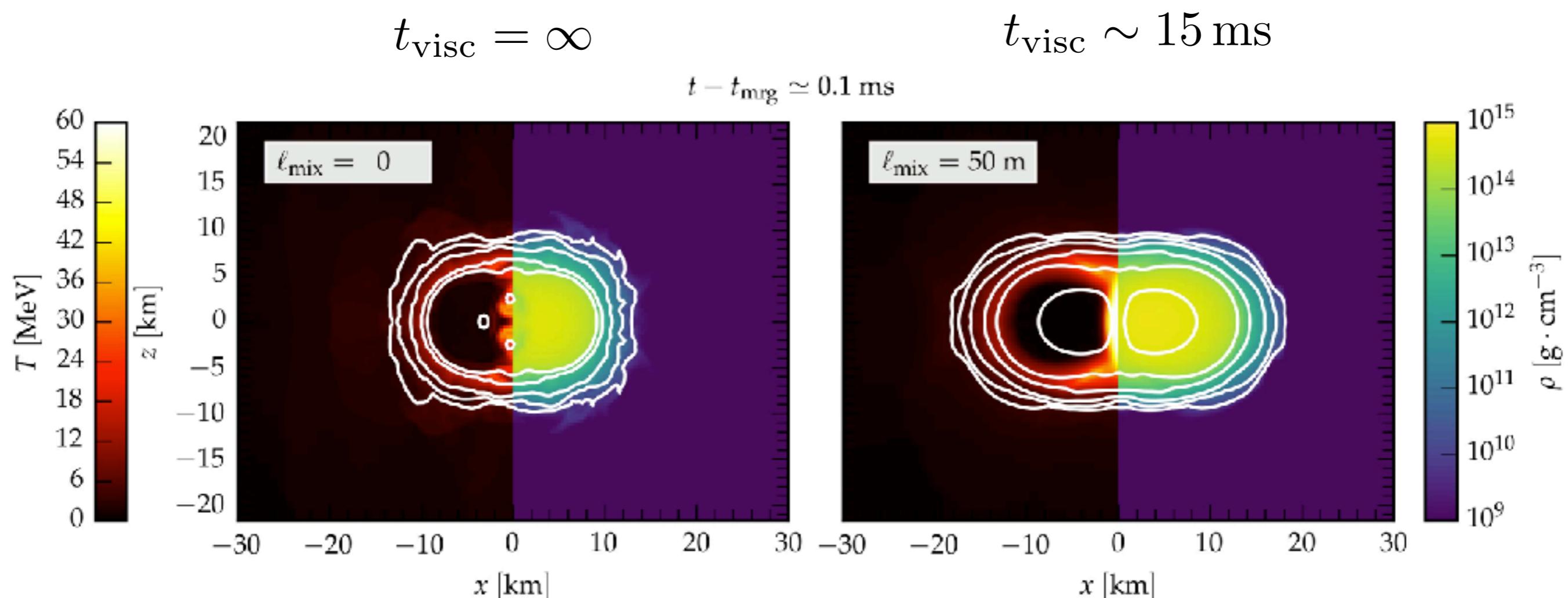


- After the first  $\sim 10\text{-}20 \text{ ms}$ , the remnant evolves because of the **effective viscosity** due to MHD turbulence, and **neutrino cooling**
- The remnant might still be hypermassive and undergo accretion-induced collapse during this phase

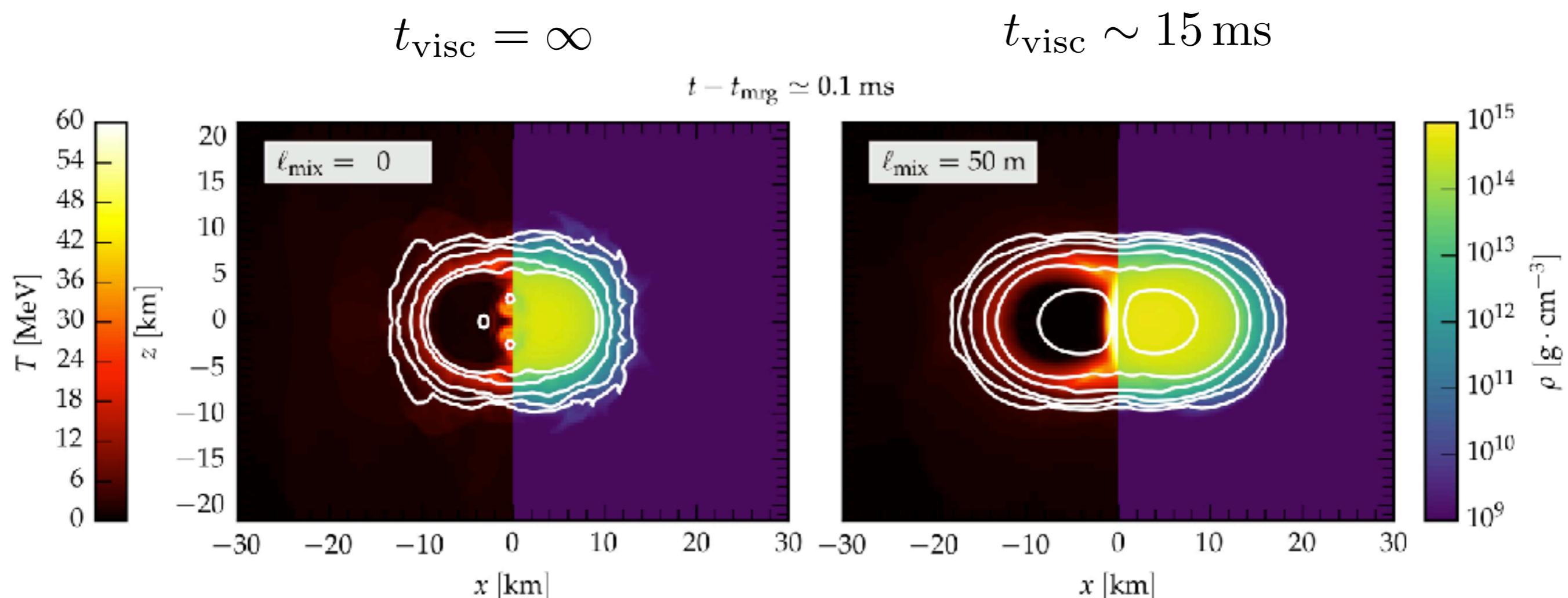
# Angular momentum transport



# Angular momentum transport

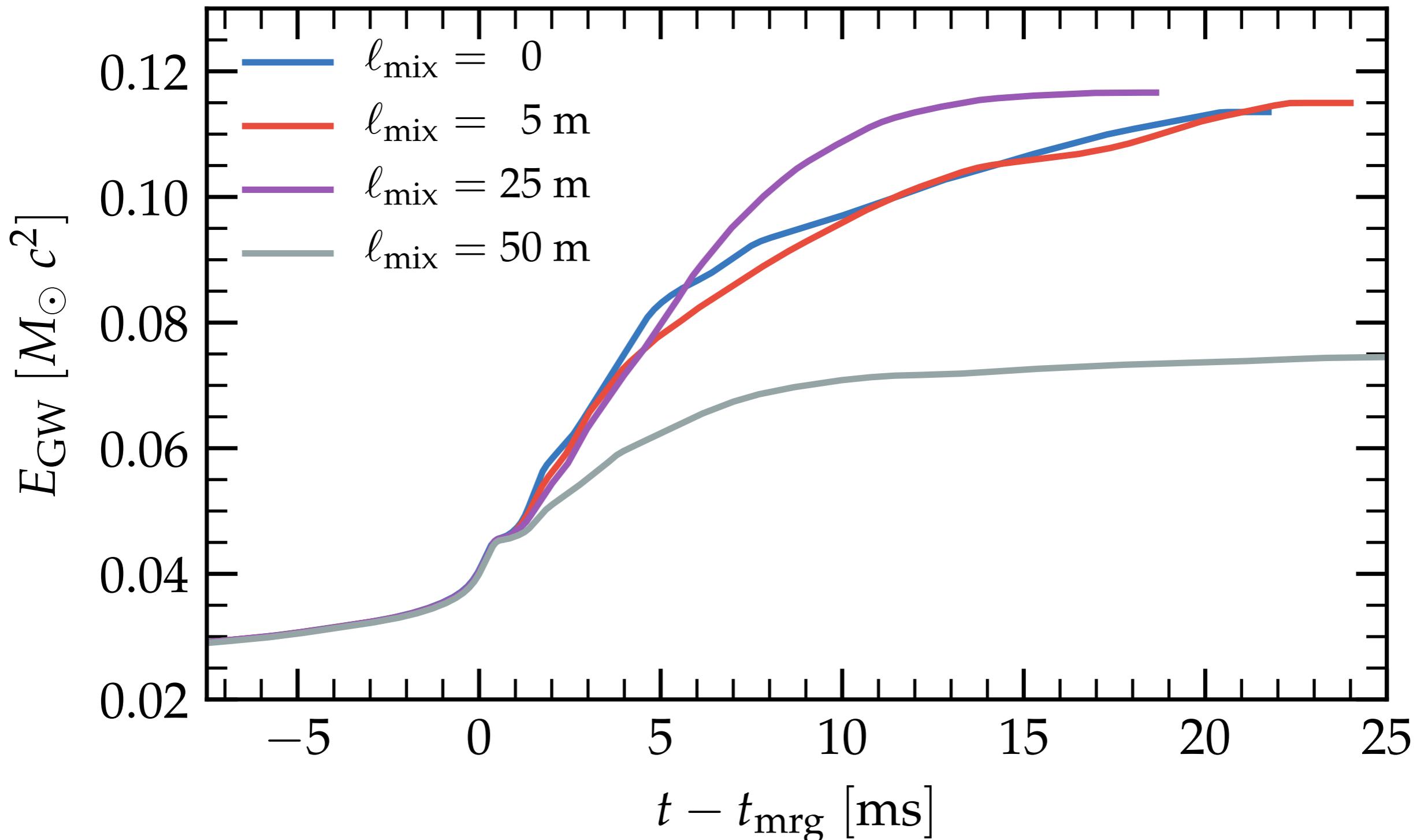


# Angular momentum transport

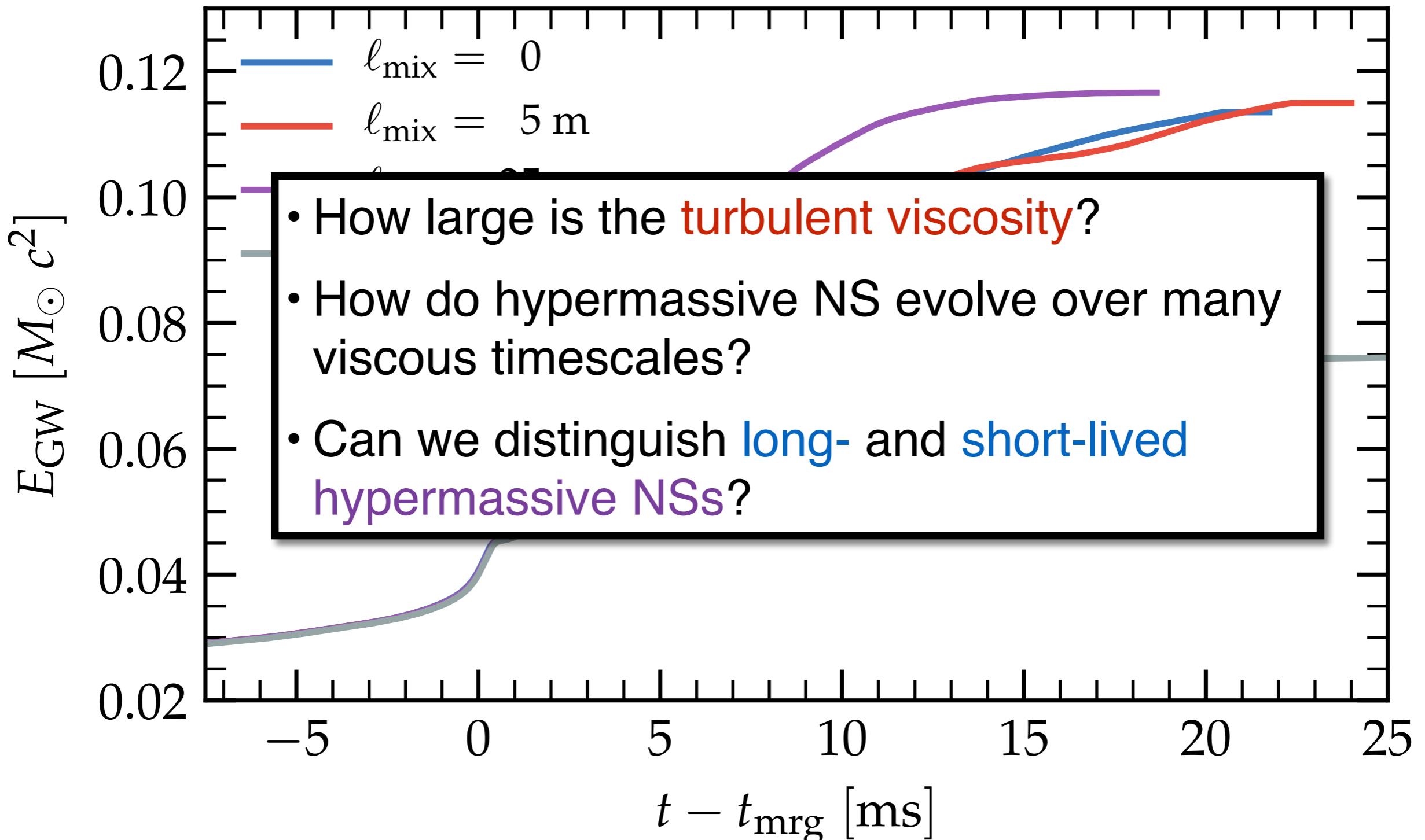


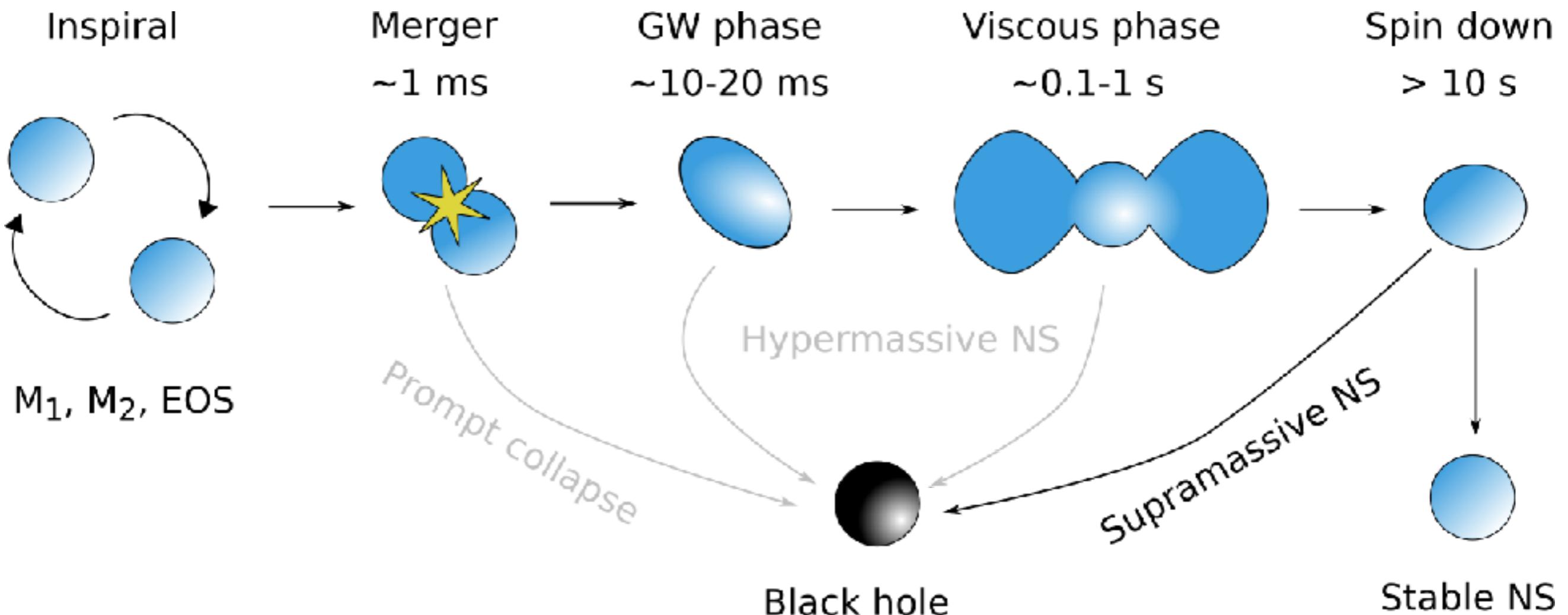
Delayed collapse!

# Gravitational waves



# Gravitational waves

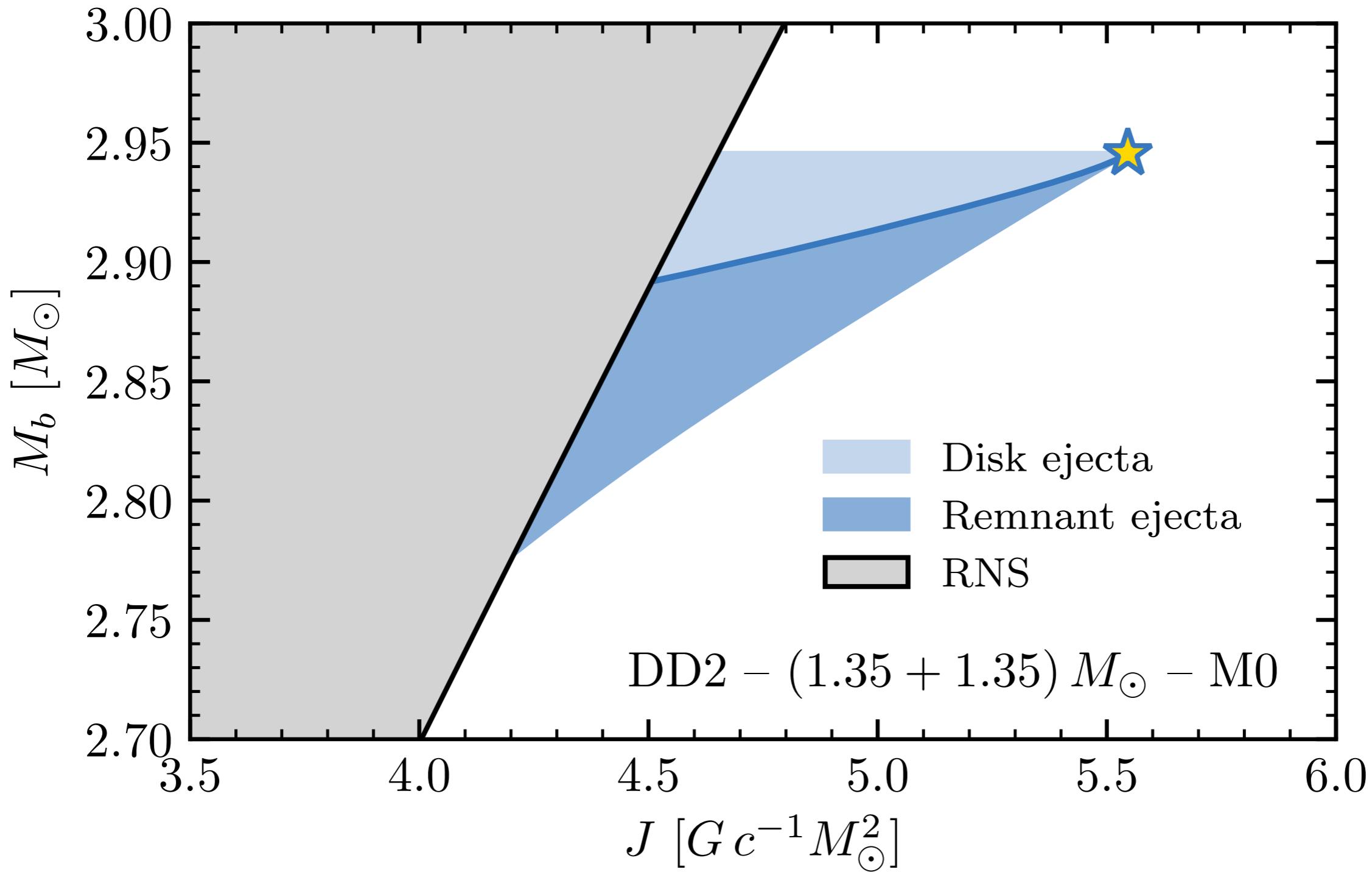




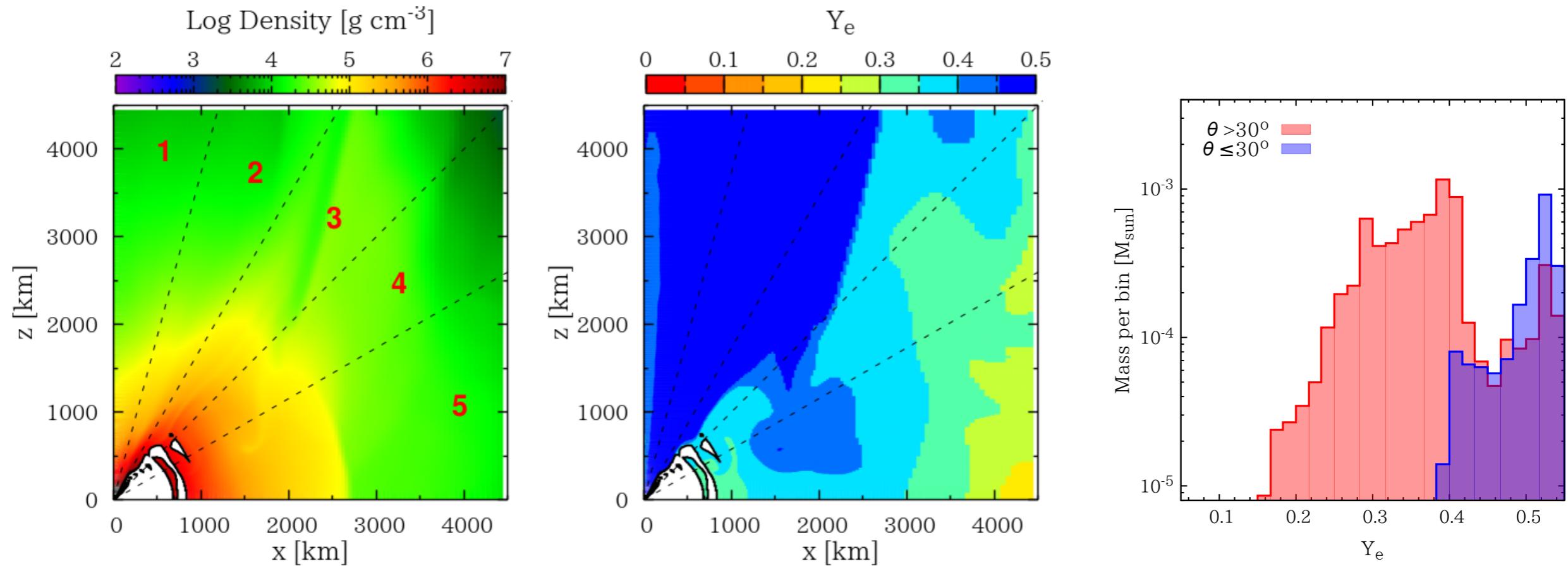
- If the binary mass is less than  $\sim 1.2 M_{\max}$  then it will be stable even after solid body rotation has been achieved
- Might collapse as it spins down due to magnetic torques (**supramassive remnants**) or might be stable forever
- How would a long-lived remnant look like in EM?

See e.g., Margalit & Metzger (2017) and DR, Perego, Bernuzzi, Zhang (2018)

# Viscous evolution (I)



# Viscous evolution (II)



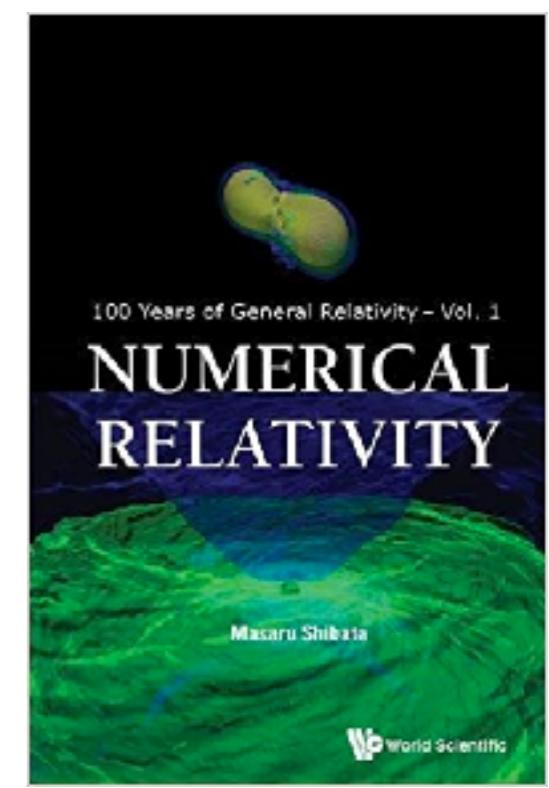
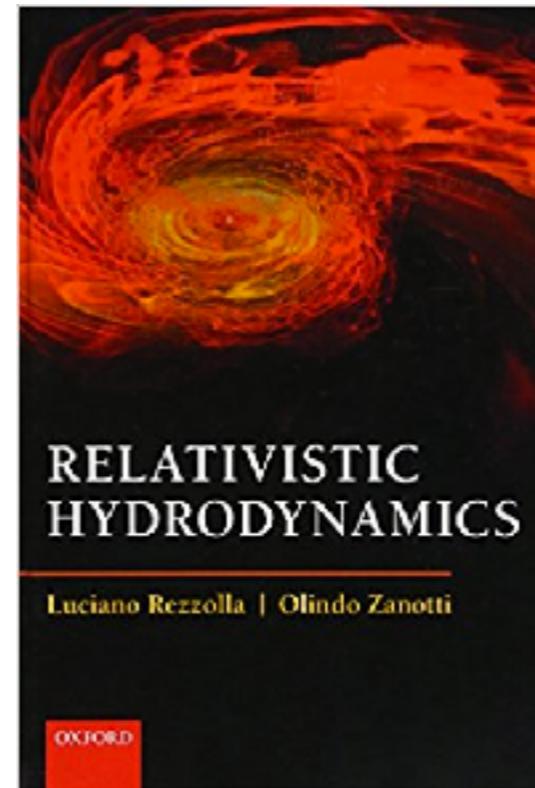
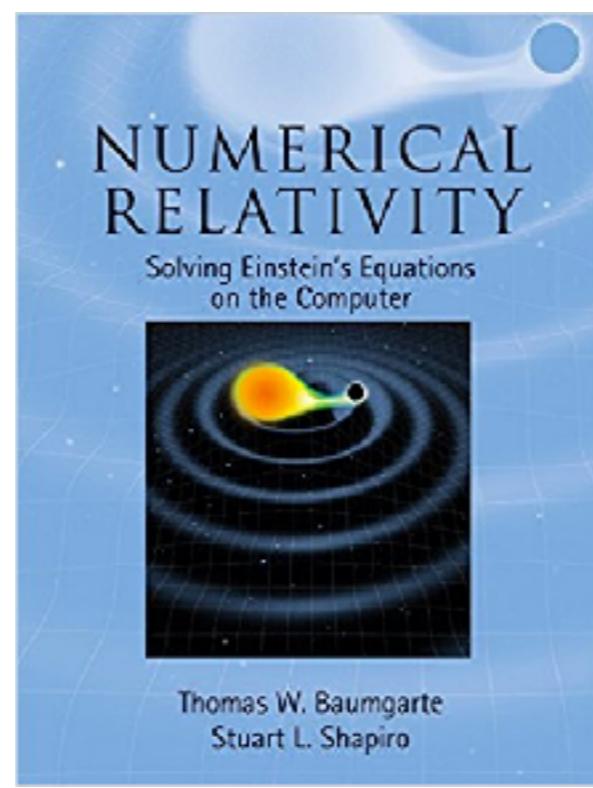
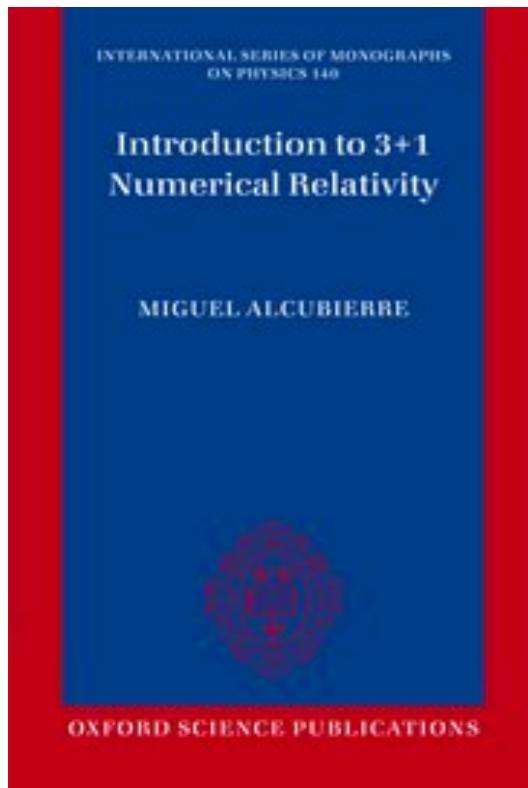
From Fujibayashi, Kiuchi+ (2017)

- Low-mass NS binaries exist\* and likely form stable remnants
- Viscous evolution of stable remnants NS is not trivial
- Smoking gun: a very bright kilonova with a blue component

# Summary

- The outcome of NS mergers depends on the **NS masses** and the (unknown) **NS EOS**
- The **long term evolution** of the remnants from NS mergers is not yet fully understood
- Future NS merger events might look very different from GW170817

# Further readings



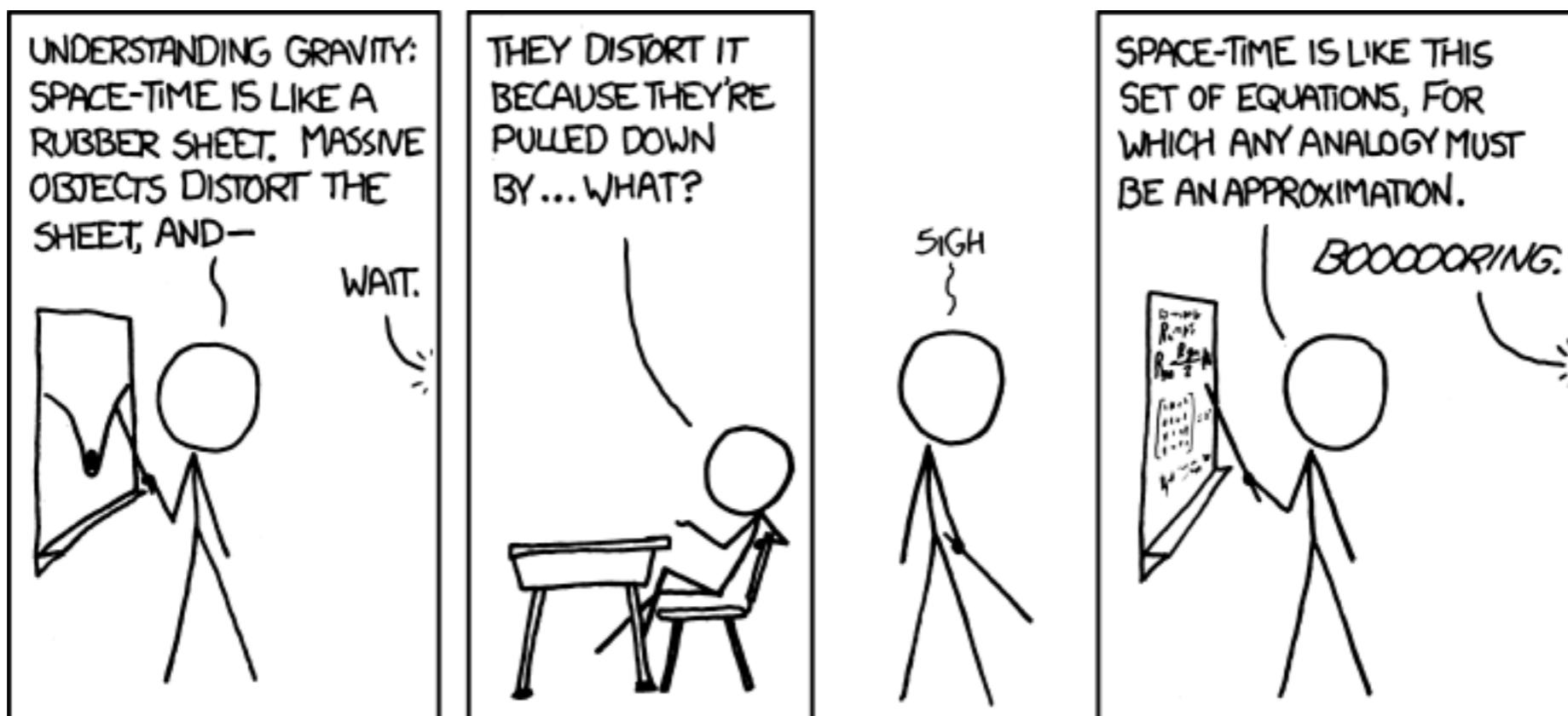
- Review articles on relativistic hydrodynamics  
Martí, Müller, Living Rev. Relativ. 6:7 (2003)  
Font, Living Rev. Relativ. 11:7 (2008)
- Review articles on binaries  
Shibata & Taniguchi, Living Rev. Relativ. 14:6 (2011) – NS+BH  
Baiotti & Rezzolla, Rep. Prog. Phys. 80 096901 (2017) – NS+NS

# Extra slides

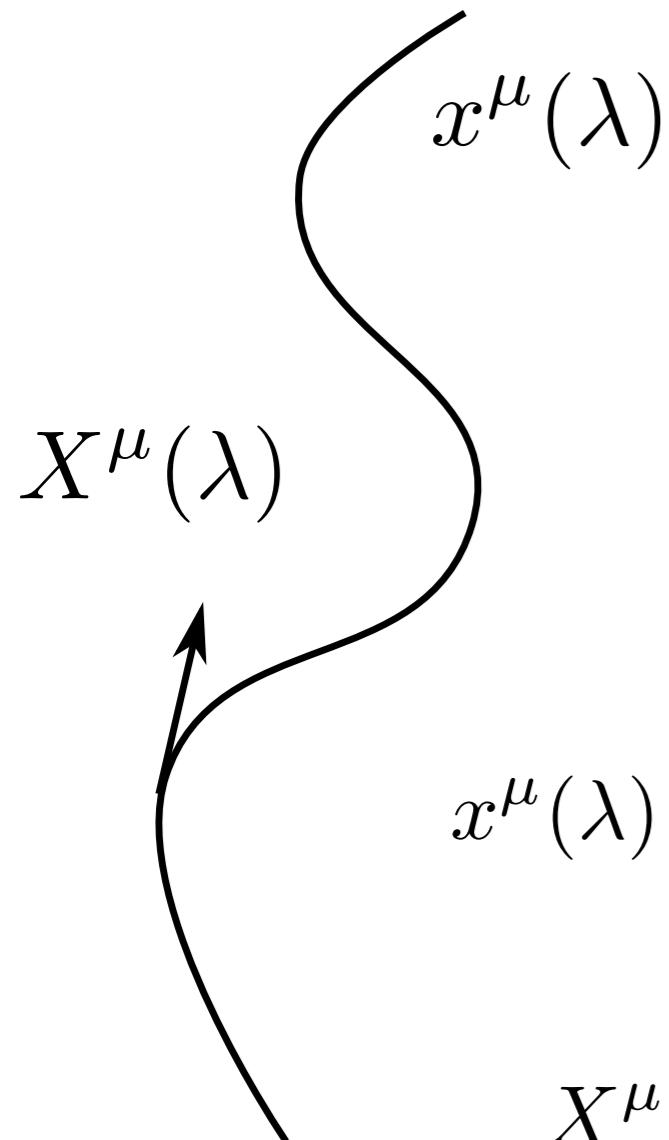
# General relativity

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad x^\mu = \{t, x, y, z\}$$

Gravity is encoded in the **geometry of spacetime**. The metric tensor defines the “distance” between **events** and the causal structure of the spacetime.



# Four-vectors



Vectors fields are generated as the tangents to families of curves

$$\frac{dx^\mu}{d\lambda} = X^\mu$$

Special case, coordinate basis vectors

$$x^\mu(\lambda) = \lambda \delta^\mu{}_0 \implies \vec{X} = \vec{\partial}_\mu$$

$$\delta^\mu{}_0 = \begin{cases} 1 & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu > 0 \quad \text{space-like vectors}$$

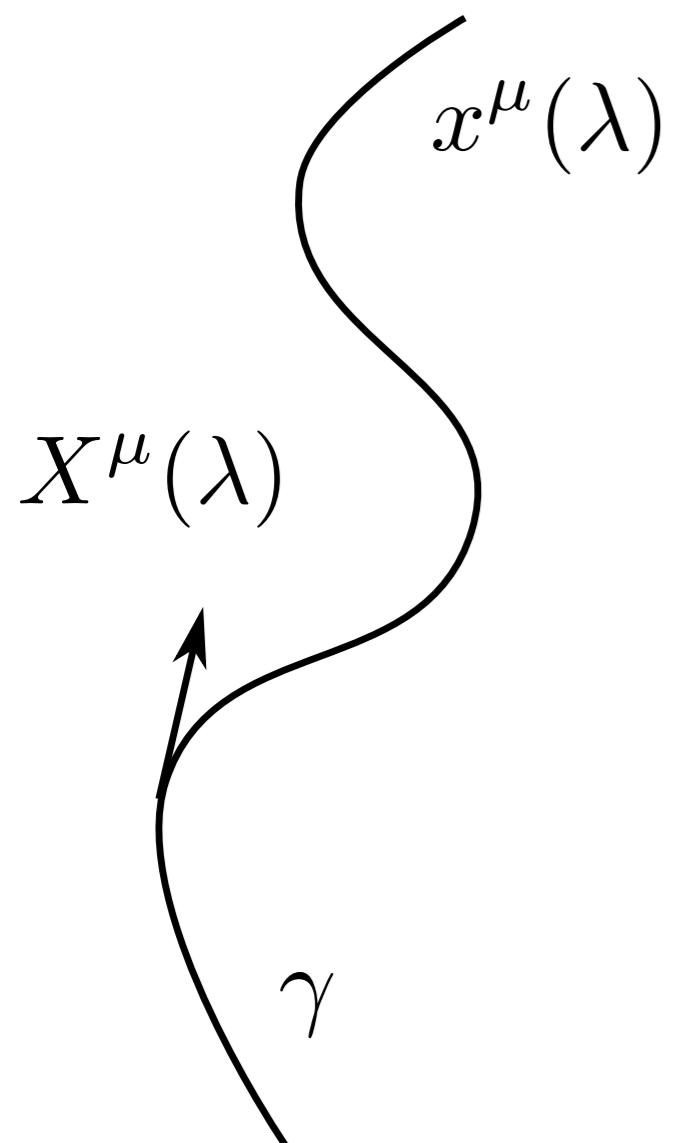
$$X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu < 0 \quad \text{time-like vectors}$$

$$X^\mu X_\mu = g_{\mu\nu} X^\mu X^\nu = 0 \quad \text{null vectors}$$

# Distances

Let  $\gamma$  be a curve with tangent vector  $X^\mu$

If  $X^\mu$  is always space-like



$$\ell = \int_{\gamma} \sqrt{g_{\mu\nu} X^\mu X^\nu} d\lambda$$

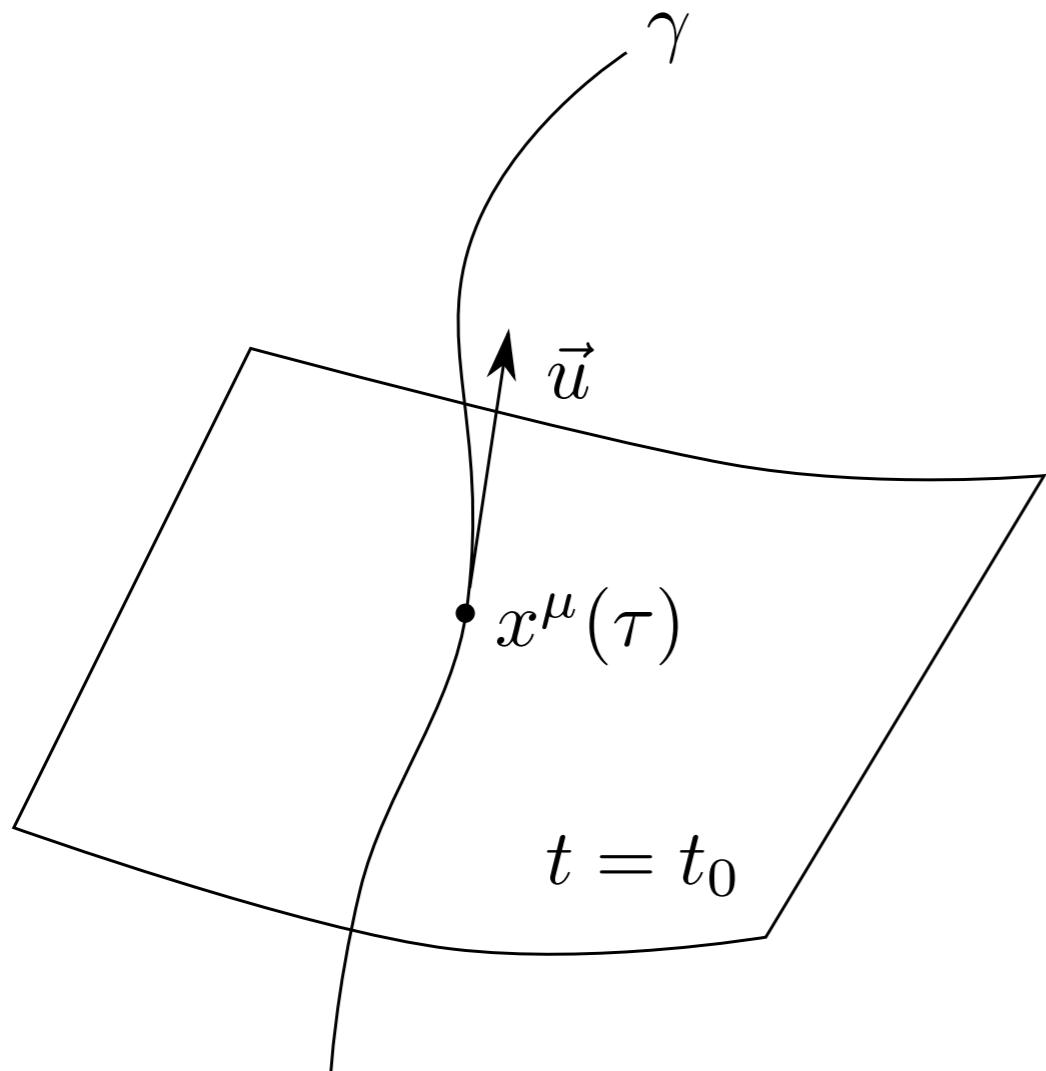
is the proper length of the along the curve  $\gamma$

If  $X^\mu$  is always time-like

$$\tau = \int_{\gamma} \sqrt{-g_{\mu\nu} X^\mu X^\nu} d\lambda$$

is the proper time experienced by an observer moving along the curve  $\gamma$

# Four-velocity



The rate of change of the position of a point in spacetime as a function of its proper time

$$\frac{dx^\mu}{d\tau} = u^\mu$$

Four-vectors also define the derivative of functions along the world line of a particle

$$u^\mu \nabla_\mu f = \frac{df}{d\tau}$$

Since  $\tau$  is the proper time:

$$g_{\mu\nu} u^\mu u^\nu = u^\mu u_\mu = -1$$

# Geodesic

In GR gravity is an “apparent force”. Test masses in a gravitational field do not feel any force and proceed in their motion on “straight lines” (**geodesics**).

Straight line: the tangent vector remains parallel to itself

$$0 = u^\mu \nabla_\mu u^\nu = u^\mu \partial_\mu u^\nu + \Gamma^\nu_{\mu\alpha} u^\mu u^\alpha$$

Remembering the definition of the 4-velocity we have

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

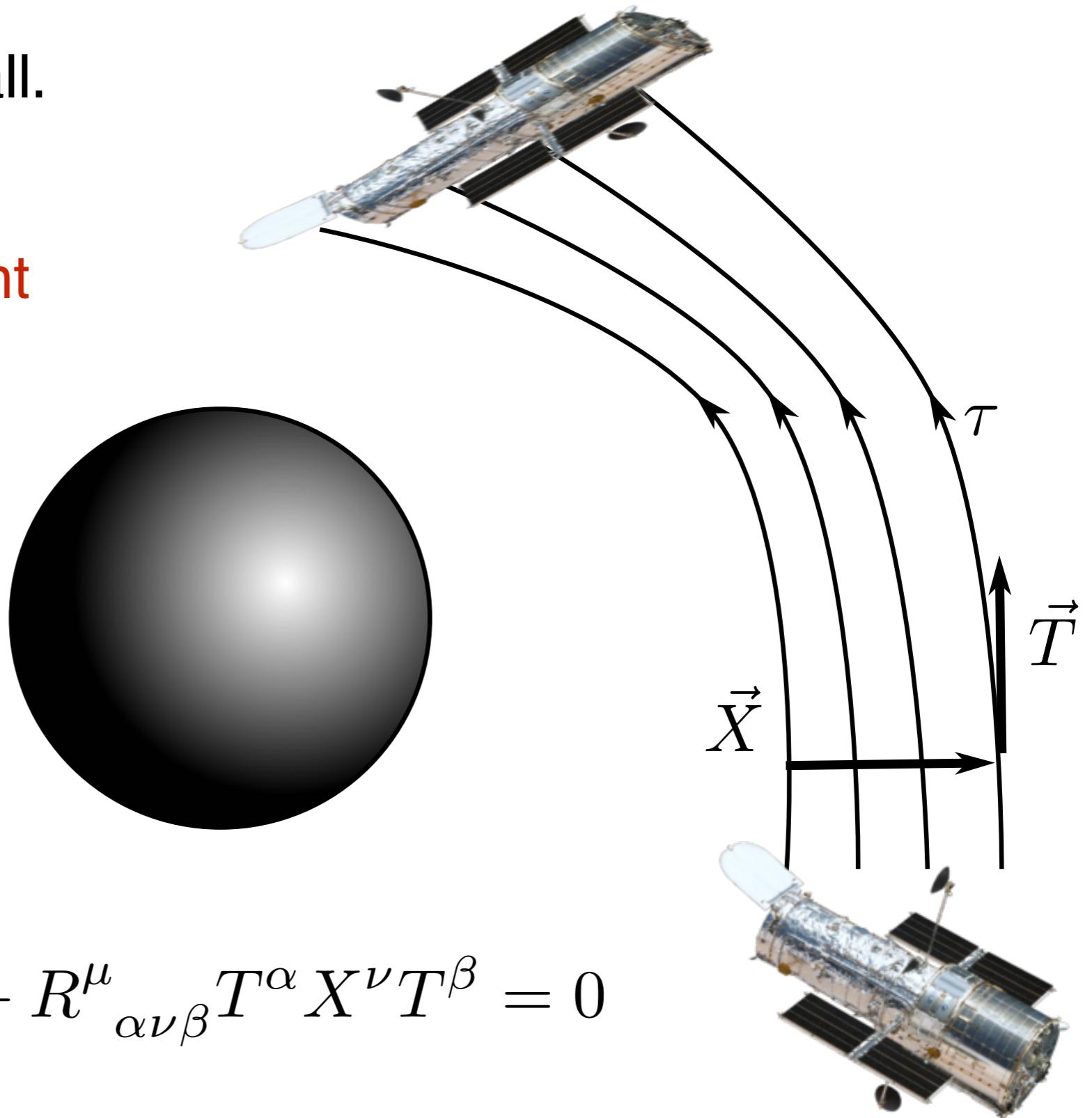
$\Gamma^\alpha_{\beta\gamma}$  are the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} (\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma})$$

# Geodesic deviation

Each particle is in free fall.

Problem: each particle  
wants to fall on a **different**  
**geodesic**: **tidal effects!**



$$\frac{D^2 X^\mu}{D\tau^2} + R^\mu{}_{\alpha\nu\beta} T^\alpha X^\nu T^\beta = 0$$

# Weak field limit

Consider the geodesic deviation equation in the case of slow motion

$$\frac{D^2 X^\mu}{D\tau^2} + R^\mu_{\alpha\nu\beta} T^\alpha X^\nu T^\beta = 0 \quad T^\alpha = \delta^\alpha_0$$

The spatial part of the relative acceleration reads

$$\frac{d^2 X^i}{dt^2} = -R^i_{0j0} X^j$$

Note that this is very similar to the Newtonian tidal field

$$\frac{d^2 X^i}{dt^2} = -\frac{\partial^2 \phi}{\partial x_i \partial x^j} X^j \quad \rightarrow \quad R^i_{0j0} \simeq \frac{\partial^2 \phi}{\partial x_i \partial x^j}$$

# Field equations

Our previous discussion suggest that Newton field equation could be generalized as

$$R_{00} = R^i_{0i0} \simeq \Delta\phi = 4\pi\rho \simeq 4\pi T_{00}$$

The Einstein field equations are only a couple of tweaks away

$$R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

Use 4D objects

This fixes the Newtonian limit

This term is needed to ensure  
conservation of energy

# Field equations

- Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

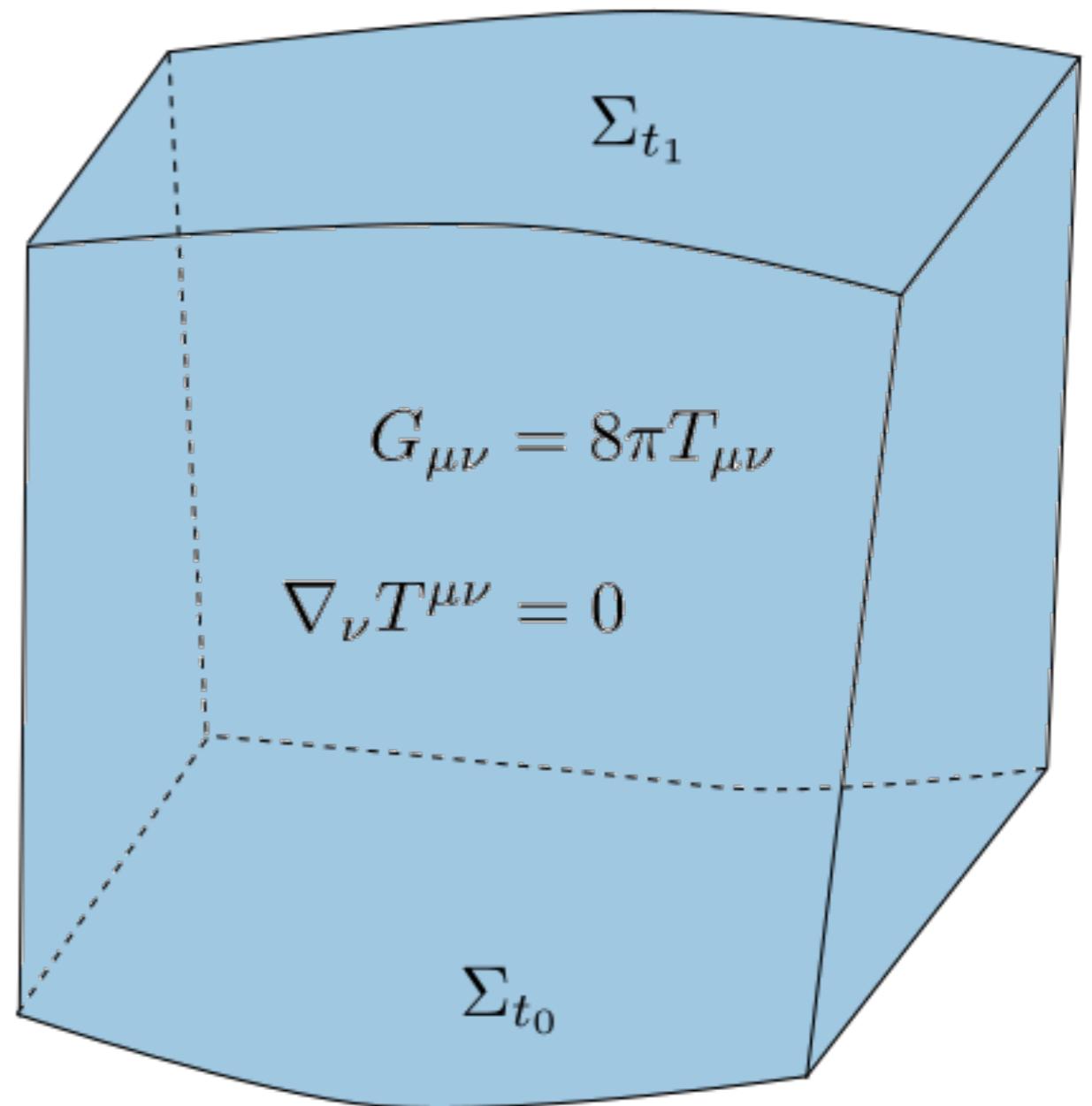
describes the geometry of spacetime

- Stress energy tensor

$$T_{\mu\nu}$$

describes matter and radiation content of the spacetime

- Field equations and Bianchi identities connect geometry and flow



Note  $\mathbf{G} = \mathbf{c} = 1$  for us!

# Vacuum spacetimes (I)

$$R_{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \dots = 0$$

Inverse of the metric

Ricci tensor      Wave operator acting on the spacetime metric       $T_{\mu\nu} = 0$

The diagram illustrates the vacuum Einstein field equation  $R_{\mu\nu} = 0$ . It features a horizontal brace under the term  $-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$ , which groups it with the right-hand side  $= 0$ . Three arrows point to different parts of the equation: one arrow points to the Ricci tensor  $R_{\mu\nu}$ , another points to the wave operator term, and a third points to the right-hand side  $T_{\mu\nu} = 0$ .

Non-linear wave equation: non-trivial curvature even in vacuum (field can source itself)

# Gravitational waves

Split the metric in a flat background part and a small perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \quad h_{\mu\nu} \ll 1$$

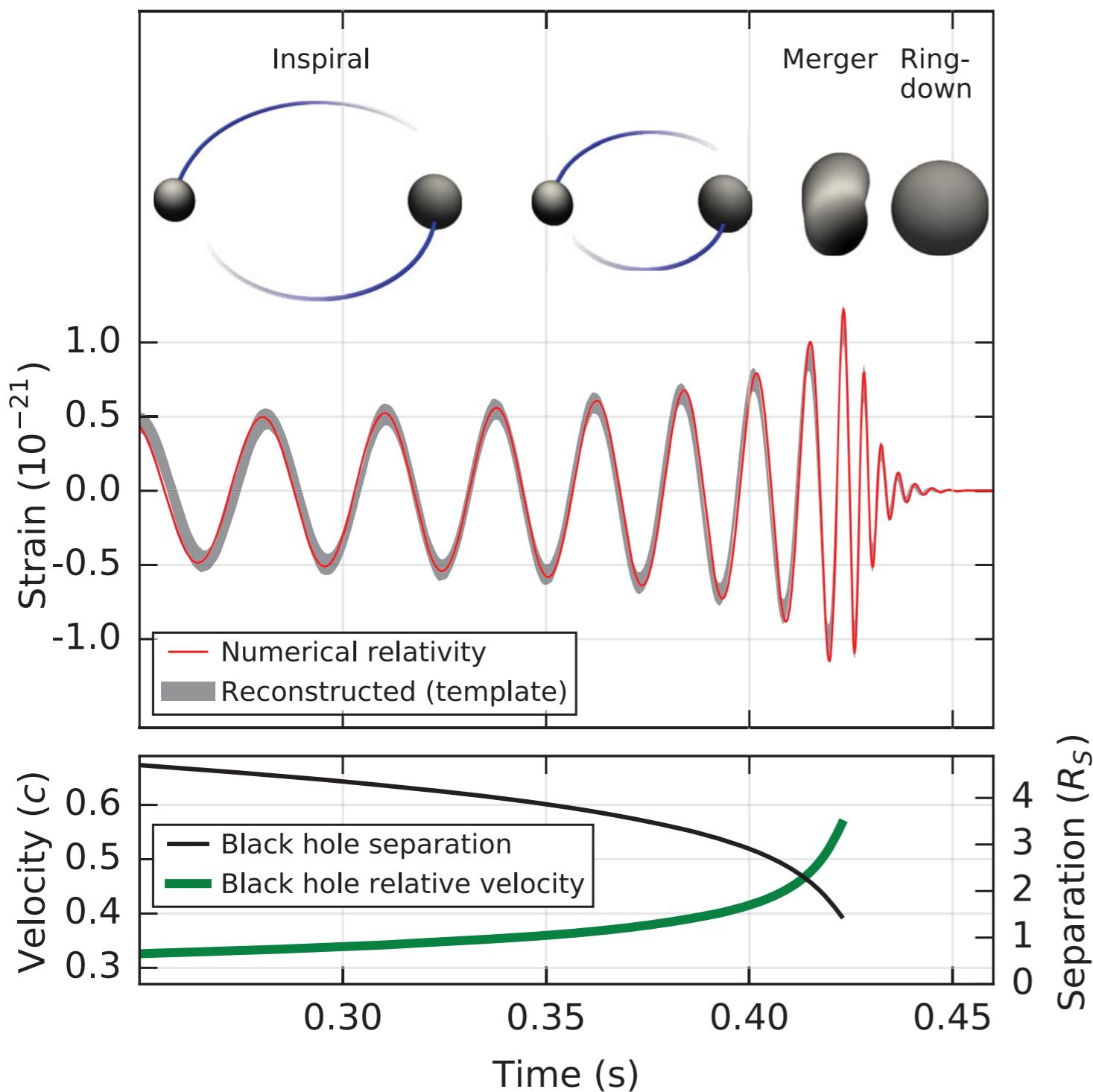
Linearize Einstein field equations

$$0 = R_{\mu\nu} \approx -\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \partial_\beta h_{\mu\nu} = 0$$

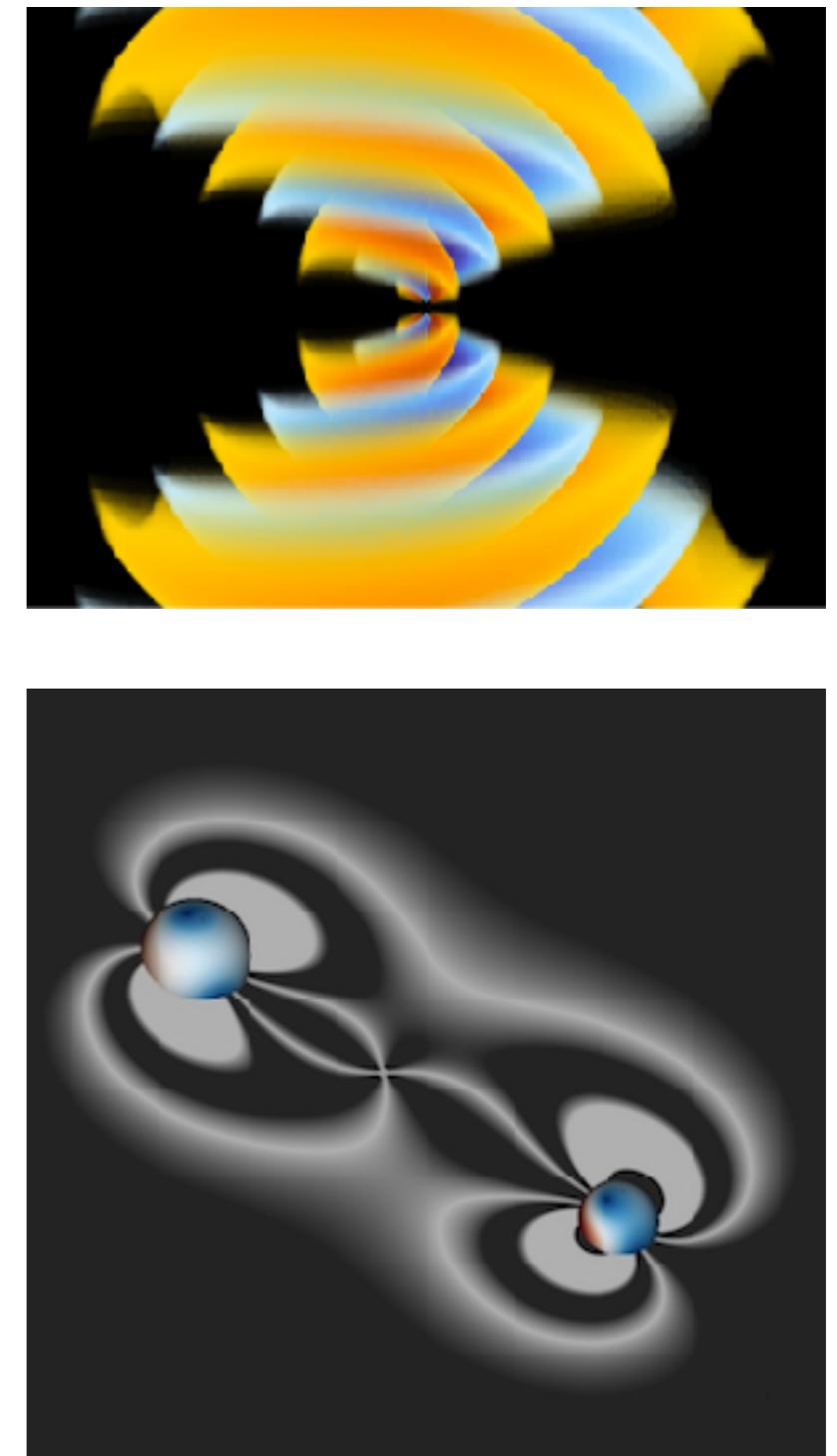
Perturbations satisfy the tensor wave equation: gravitational waves

$$\partial_t^2 h_{\mu\nu} - c^2 (\partial_x^2 + \partial_y^2 + \partial_z^2) h_{\mu\nu} = \square h_{\mu\nu} = 0$$

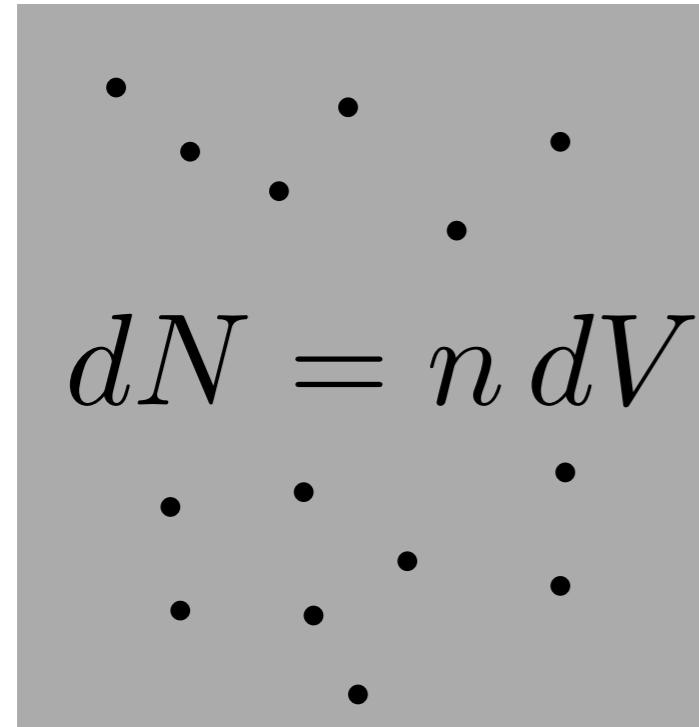
# Vacuum spacetimes (II)



From Abbott+ PRL 116, 061102 (2016)

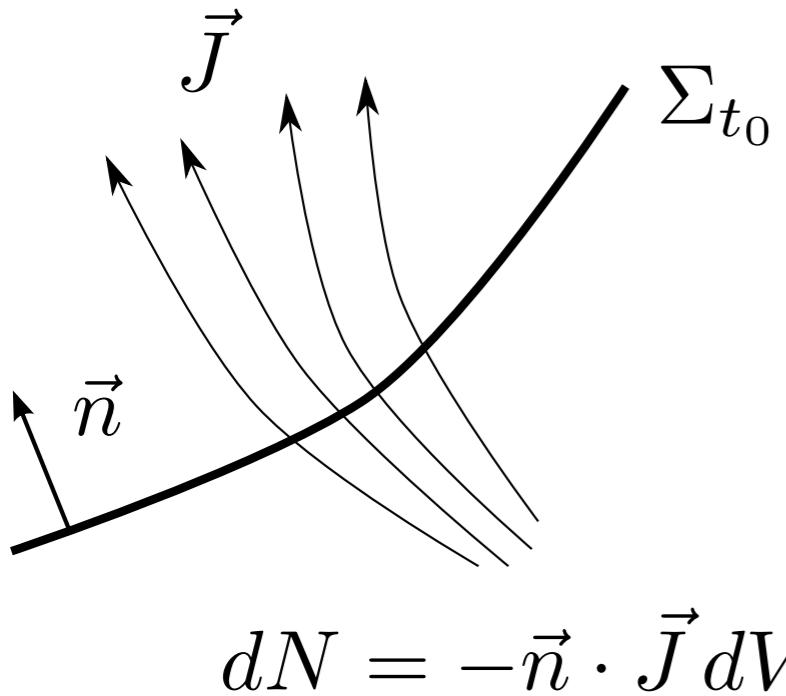


# Relativistic hydrodynamics



## Classical picture

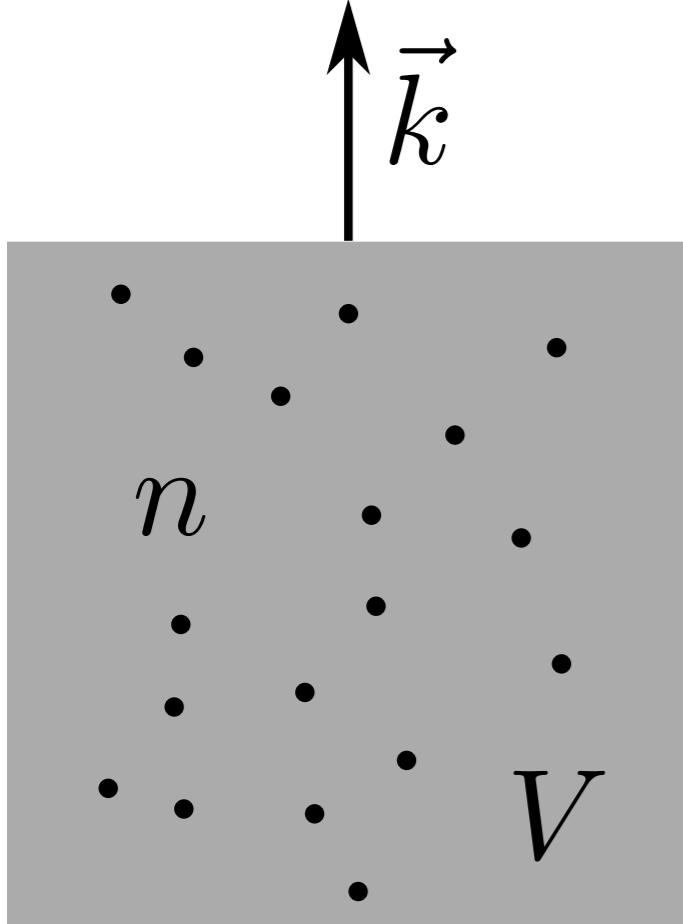
- Intrinsic variable: particle number density
- Equations describe the flux of matter through **space** as a function of **time**



## Relativistic picture

- Intrinsic variable: flux of particles through **spacetime**
- Equations describe the density as measured by any observer

# Classical continuity equation



$$\frac{d}{dt} \int_V n \, dV + \int_{\partial V} n \vec{v} \cdot \vec{k} \, dS = 0$$

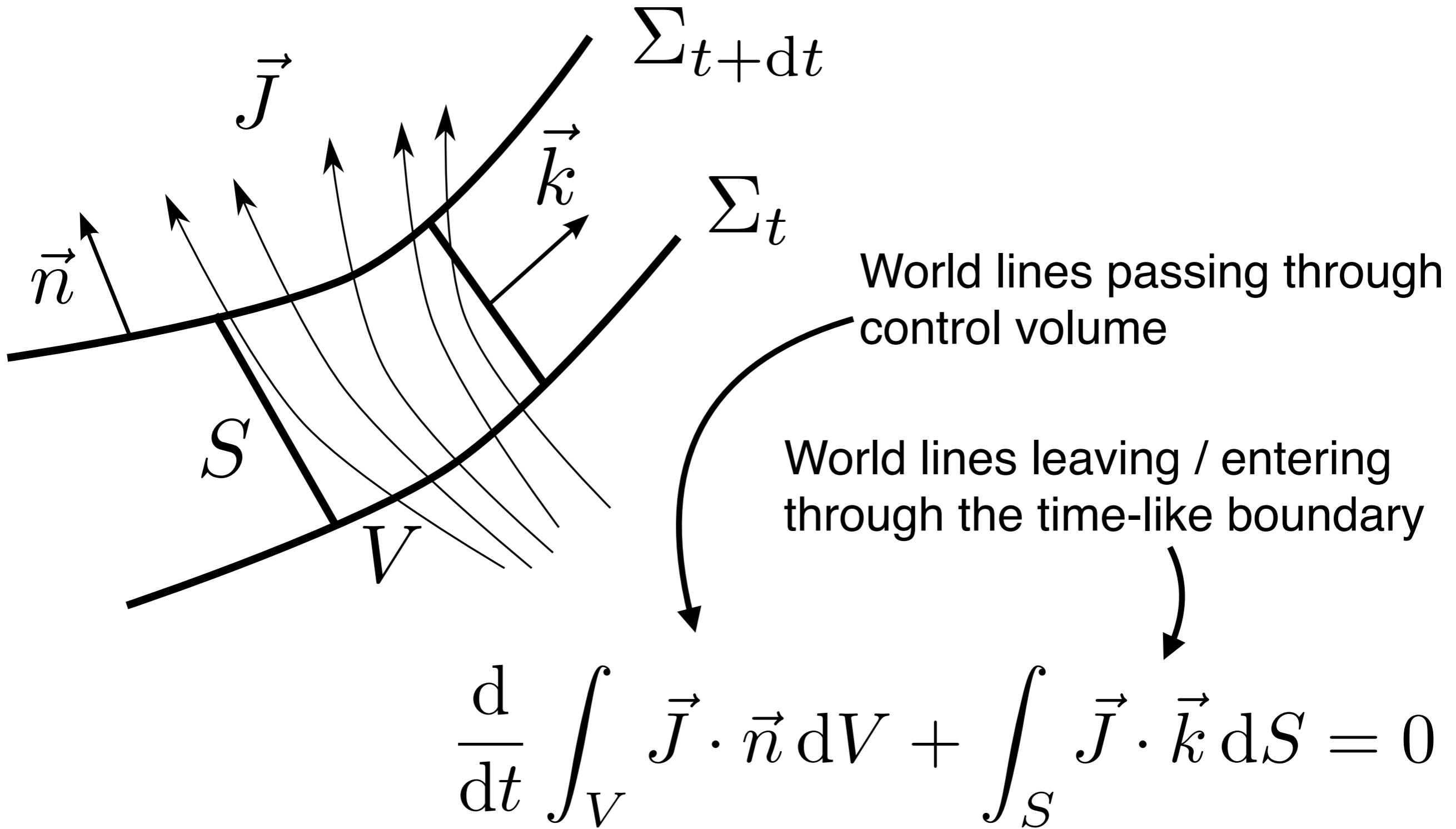
Number of  
particles in a  
control volume

Flux of particles across  
the boundaries of the  
control volume

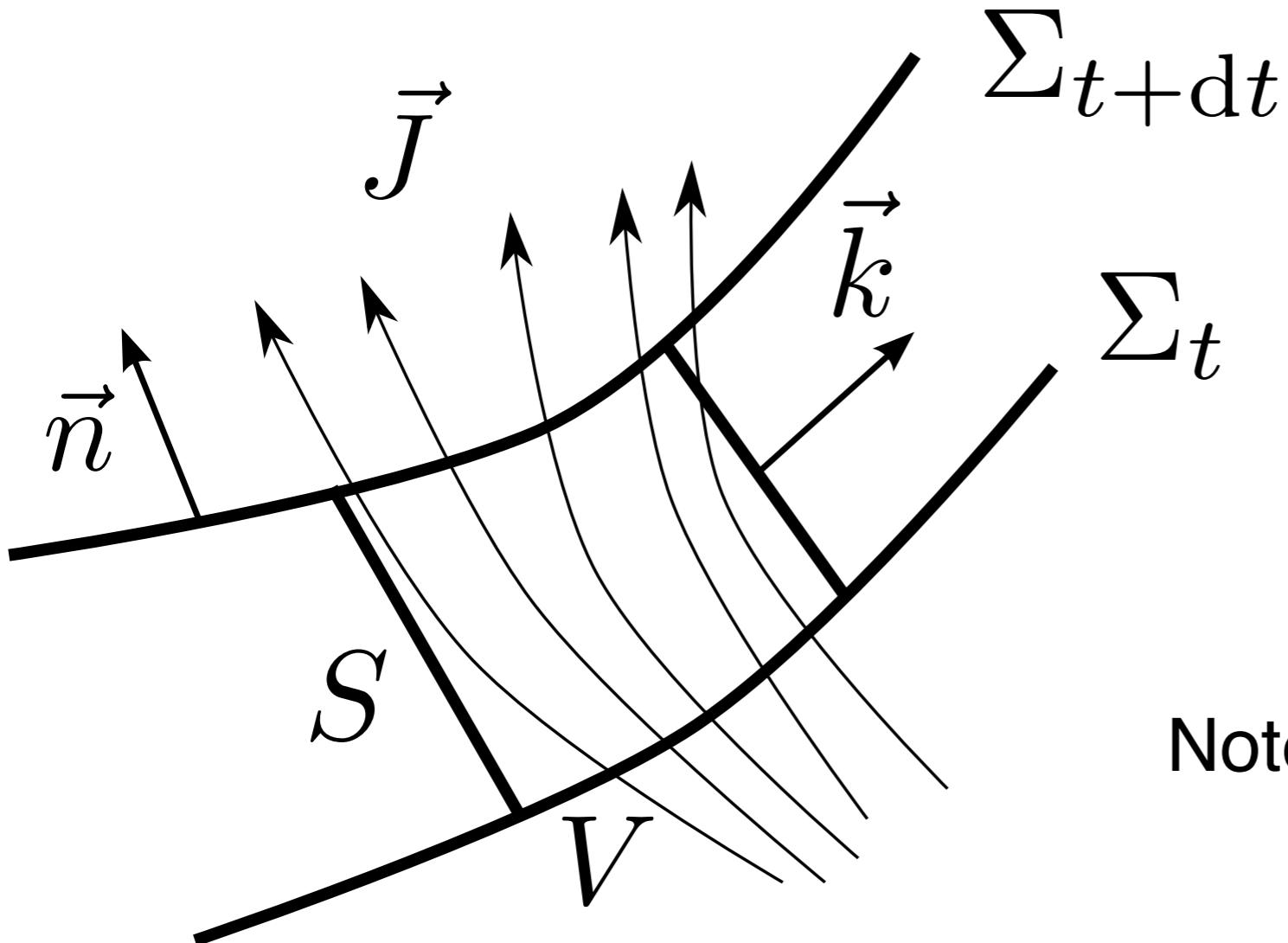
Differential form:

$$\partial_t n + \nabla_i (n v^i) = 0$$

# Relativistic continuity equation



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4-velocity

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Note:  $-J^\mu u_\mu = n$

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$$n^\mu (\partial_i)_\mu = 0$$

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$$T_{\mu\nu} (\partial_i)^\mu (\partial_j)^\nu = p \delta_{ij} \quad \text{Only isotropic pressure}$$

In this case the stress energy tensor takes the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \quad u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \xrightarrow{\text{General frame}}$$

$$\begin{aligned} T_{\mu\nu} &= \rho u^\mu u^\nu + p \perp^{\mu\nu} \\ \perp_{\mu\nu} &= g_{\mu\nu} + u_\mu u_\nu \end{aligned}$$

# “Conservation” laws (I)

The key part of the equations of motion is

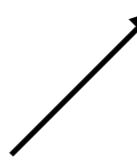
$$\nabla_\nu T^{\mu\nu} = 0$$

what is really conserved?

# “Conservation” laws (II)

Multiply  $\nabla_\nu T^{\mu\nu} = 0$  by any smooth vector field  $p^\mu$  to get

$$\nabla_\mu (T^{\mu\nu} p_\nu) = -T^{\mu\nu} \nabla_\mu p_\nu = -\frac{1}{2} T^{\mu\nu} (\nabla_\mu p_\nu + \nabla_\nu p_\mu)$$



$$T^{\mu\nu} = T^{\nu\mu} = \frac{1}{2} (T^{\mu\nu} + T^{\nu\mu}) \quad = 0 \text{ for Killing vectors}$$

Killing vectors are direction of **symmetry** for the metric

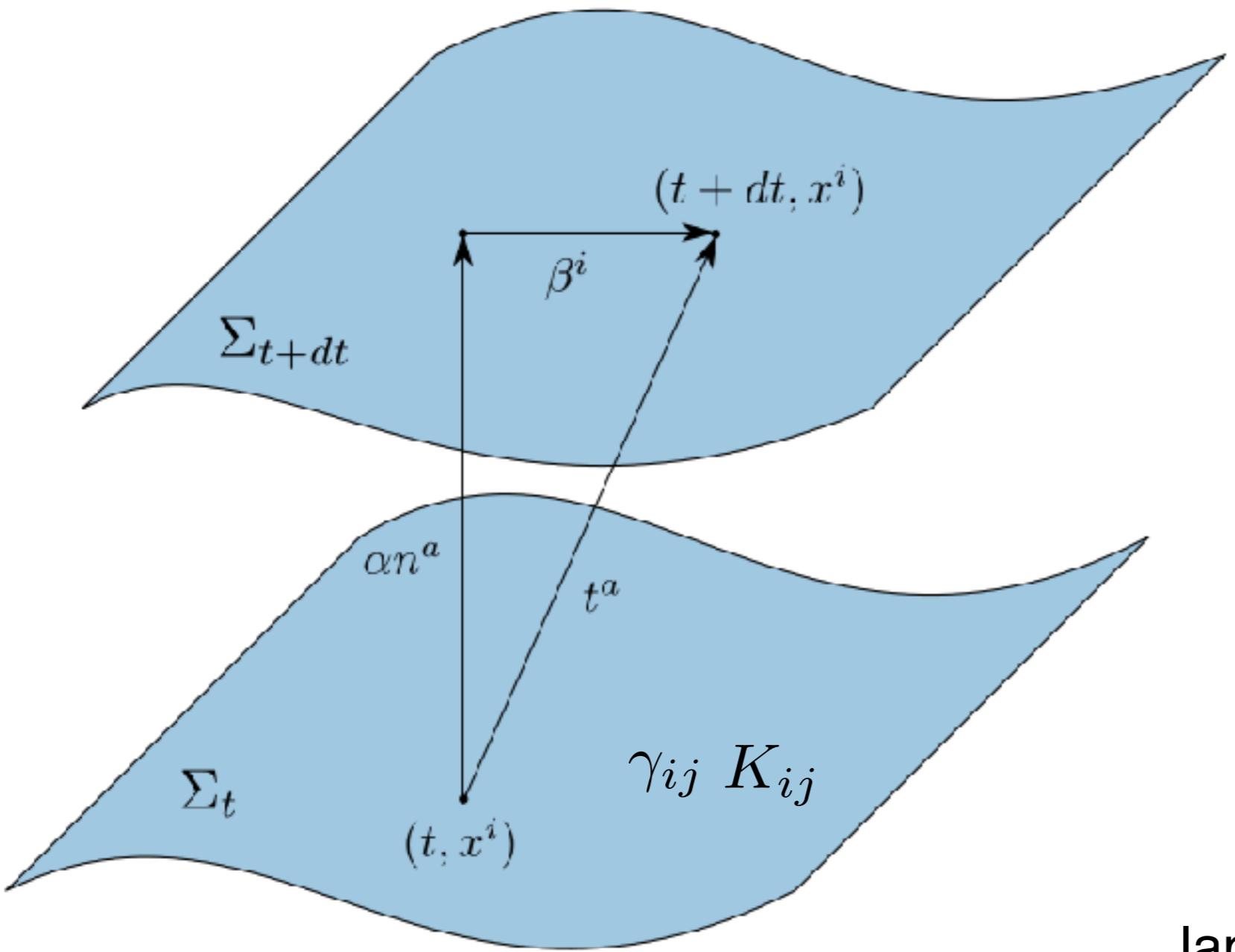
$$0 = \nabla_\mu p_\nu + \nabla_\nu p_\mu \implies \mathcal{L}_{\vec{p}} g_{\mu\nu} = 0$$

Noether's theorem:

*Spacetime symmetries generate conserved charges*

$$p^\mu \text{ Killing vector} \implies C^\mu = T^{\mu\nu} p_\nu \text{ conserved}$$

# 3+1 decomposition



$\gamma_{ij}$  spatial metric

$K_{ij}$  extrinsic curvature

$\vec{t}$  vector tangent to the constant  $x^i$  curves

$\vec{n}$  vector normal to the constant  $t$  hyper-surfaces

$$\vec{t} = \alpha \vec{n} + \vec{\beta}$$

lapse function

shift vector

Lapse and shift vector are **gauge degrees of freedom**

# Spatial metric

The spatial metric describes the intrinsic geometry of the spatial slices

$$d\ell^2 = \gamma_{ij} dx^i dx^j$$

As a four-dimensional object  $\gamma^\mu{}_\nu$  is used to project vectors tangentially to the spatial hypersurface

$$\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$$

The spatial metric is prescribed as initial data and then evolved

We use the symbol  $D_i$  for the **covariant derivative** associated with  $\gamma_{ij}$

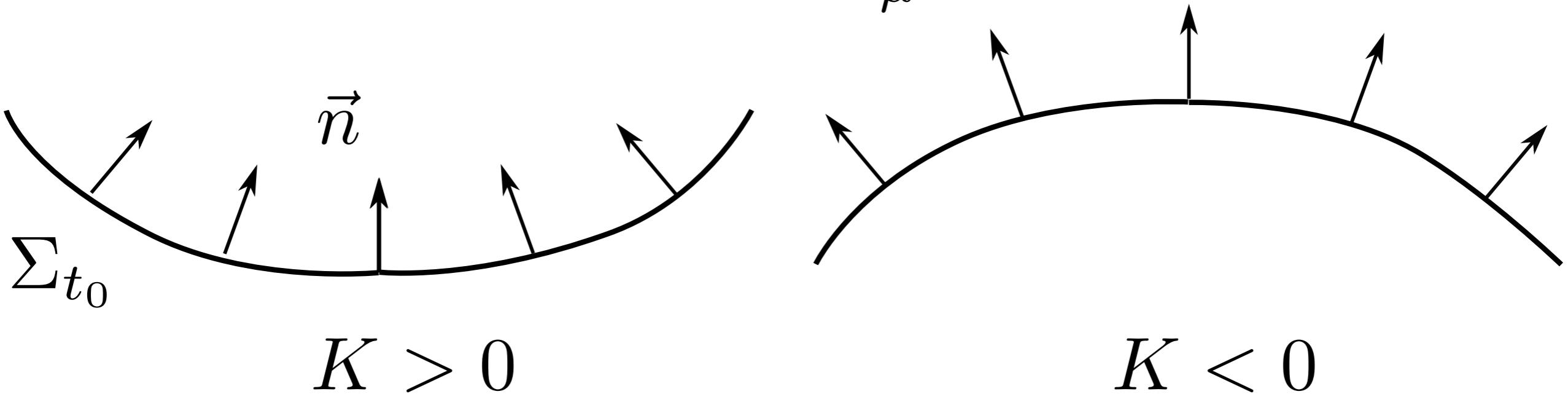
# Extrinsic curvature (I)

$$K_{\mu\nu} = -\gamma^\alpha_\mu \nabla_\alpha n_\nu$$

Tangent projection of the gradient of the normal vector

The trace of the extrinsic curvature is associated with the expansion of the world lines of the Eulerian observers:

$$K = -\nabla_\mu n^\mu$$



# Extrinsic curvature (II)

$$K_{ij} = -\frac{1}{2} \mathcal{L}_{\vec{n}} \gamma_{ij}$$

The extrinsic curvature is the Lie derivative of the metric along the normal to the spatial hypersurface

$K_{ij} = 0 \implies$  three-metric is **stationary!**

If  $\beta^i = 0$  then  $K_{ij} = -\frac{1}{2} \partial_t \gamma_{ij}$

The extrinsic curvature is also prescribed at the initial time and then evolved

# ADM equations

$$(\partial_t - \mathcal{L}_{\vec{\beta}})\gamma_{ik} = -2\alpha K_{ik}$$

$$(\partial_t - \mathcal{L}_{\vec{\beta}})K_{ik} = -D_i D_k \alpha + \alpha \left( {}^{(3)}R_{ik} - 2K_{ij}K^j{}_k + KK_{ik} \right) - 8\pi\alpha \left( S_{ik} - \frac{1}{2}\gamma_{ik}(S-E) \right)$$

$${}^{(3)}R + K^2 - K_{ik}K^{ik} = 16\pi E$$

$$D_k(K\gamma^k{}_i - K^k{}_i) = 8\pi j_i$$

$$S_{ik} = \gamma^\mu{}_i \gamma^\nu{}_k T_{\mu\nu} \quad S = S^i{}_i \quad j_i = -\gamma^\mu{}_i n^\nu T_{\mu\nu} \quad E = T^{\mu\nu} n_\mu n_\nu$$

- Solve constraint equations to create initial data
- Evolve metric and extrinsic curvature
- Note: numerically stable evolution requires additional modification to the equations: [BSSN](#), [Z4c](#)
- Another approach: [generalized harmonics](#)



Evolution equations

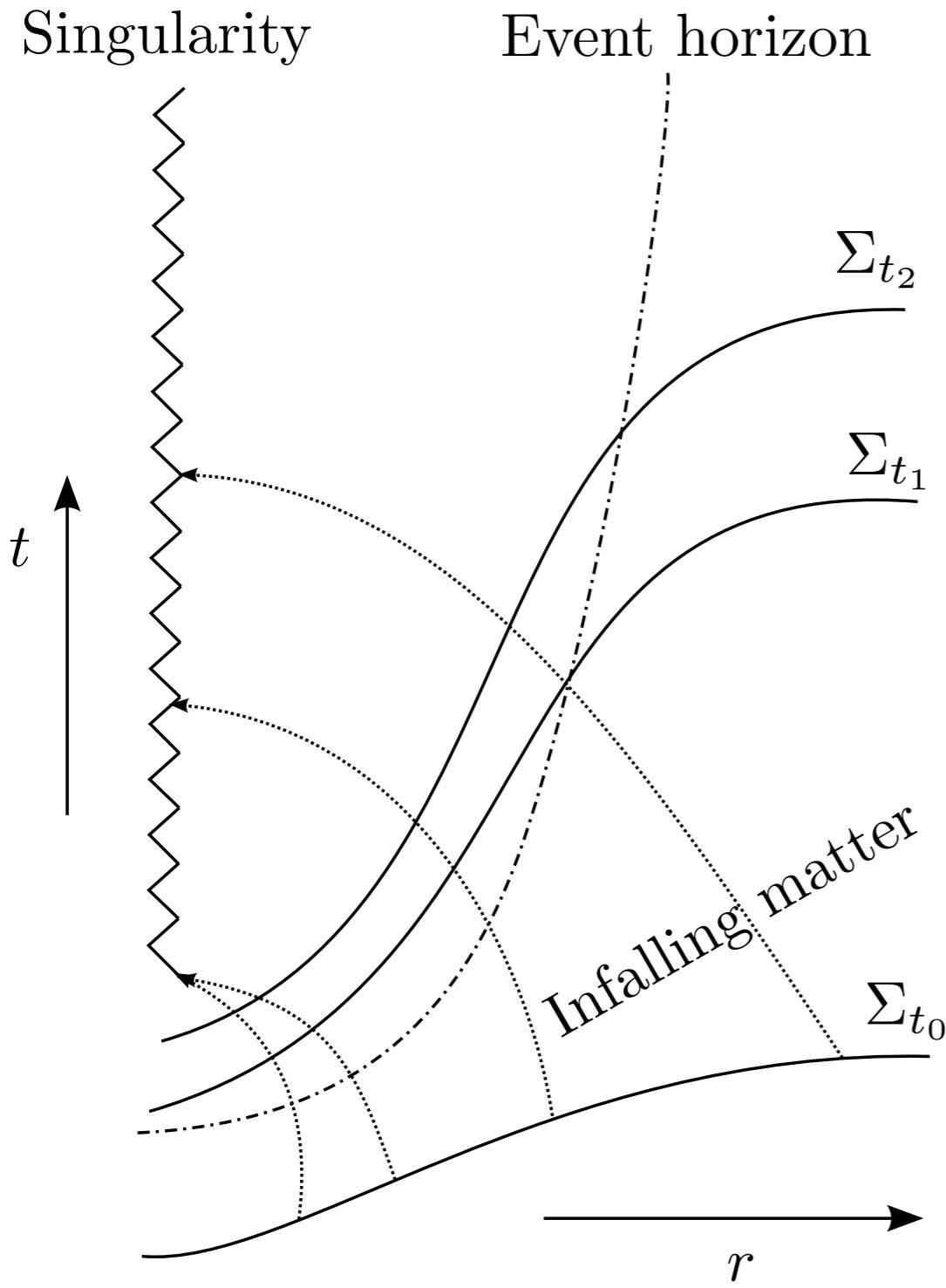


Constraints



Projections of the matter fields

# Slicing freedom



BH formation cartoon

Einstein equations do not tell us how to traverse the spacetime: we need to choose a slicing condition

Requirements:

- smoothness
- minimize grid distortion
- avoid singularities

Two families that work in practice

- “puncture” gauge
- generalized harmonic

# Moving puncture gauge

$$(\partial_t + \beta^i \partial_i) \alpha = -2\alpha K$$

- No **caustics** can be formed ( $K = -\nabla_\mu n^\mu$ )
- **Singularity avoidance**: the lapse drops to zero before the foliation hits a singularity (diverging  $K$ )
- If there is a time symmetry, the slicing tries to adapt to it
- Advection with  $\beta^i$  allows black-holes to **move** through the grid

# Valencia formulation

$$\nabla_\nu T_\mu^\nu = 0 \rightarrow$$

$$\partial_t F_\mu^0 + \partial_i F_\mu^i = S_\mu$$

- **Projection** of  $\nabla_\nu T_\mu^\nu = 0$  on the space hypersurface
- Equations similar to the classical hydrodynamics equations
- **Discretize** with standard high-resolution shock-capturing (HRSC) schemes
- Note  $S_\mu$  is in general not zero! **Balance laws.**