Assigned: Monday, April 27, 2015
Due: Friday, May 8, 2015

MTH 371: Homework 12
Quadrature Rules, Numerical Differentiation, and Numerical Methods for ODEs

GENERAL HOMEWORK GUIDELINES:

- On the very first page of your homework, provide your name, date, and homework number.
- Homework will be graded in part on neatness, organization, and completeness of solutions. Multiple pages MUST BE STAPLED.
- Attach all Scilab code, output, and plots to the page immediately following each problem. Also, clearly indicate the problem they correspond to.

1. Below is a table of the zeros of the kth degree Legendre polynomial for $k = 2, 3, 4, 5$. These zeros give the nodes x_i for Gaussian quadrature rules on interval $[-1, 1]$. Using the method of undetermined coefficients, find the corresponding A_i values for each k via Scilab. Summarize your results in a table.

k	x_i
2	$\pm\sqrt{\frac{1}{3}}$
3	$0, \ \pm\sqrt{\frac{3}{5}}$
4	$\pm\sqrt{\frac{1}{7}(3-4\sqrt{0.3})}, \quad \pm\sqrt{\frac{1}{7}(3+4\sqrt{0.3})}$
5	$0, \quad \pm\sqrt{\frac{1}{9}\left(5-2\sqrt{\frac{10}{7}}\right)}, \quad \pm\sqrt{\frac{1}{9}\left(5+2\sqrt{\frac{10}{7}}\right)}$

2. With the transformation $t = \dfrac{2x-(a+b)}{b-a}$, a Gaussian quadrature rule of the form

$$\int_{-1}^{1} f(t)\ dt \approx \sum_{i=0}^{n} A_i f(t_i)$$

can be used over the interval $[a, b]$. Write a function `y = GaussianQuad(f,a,b,n)` which computes the the nth Gaussian quadrature rule

$$\int_{a}^{b} f(x)\ dx \approx \sum_{i=0}^{n} A_i f(x_i).$$

Use the x_i, A_i values found from problem 1 stored as a table as well as the above transformation. Assume input `f` is a defined Scilab function.

3. Use your code from problem 2 to compute the Gaussian quadrature approximations of the following functions for $n = 1, 2, 3, 4$.

   (a) $\int_0^1 \frac{1}{\sqrt{x}}\ dx$

   (b) $\int_0^1 \frac{\sin(x)}{x}\ dx$

4. Derive the following two formulas and establish formulas for the errors.

   (a) $f'(x) \approx \dfrac{1}{4h}\left(f(x+2h) - f(x-2h)\right)$

   (b) $f''(x) \approx \dfrac{1}{4h^2}\left(f(x+2h) - f(x) + f(x-2h)\right)$

   For each, determine the best choice of h to minimize the total error (combination of roundoff and theoretical error). Write a Scilab script to test your findings for derivatives of $f(x) = \sin(x)$ at $x = \pi/6$.

5. Consider the following initial value problem.

$$\begin{cases} y' = y + y^2, & t \in [1, 2.77] \\ y(1) = \frac{e}{16-e} \end{cases}$$

   (a) Verify that $y(t) = \frac{e^t}{16-e^t}$ is the exact solution.

   (b) Use Euler's method with $h = \frac{1}{100}$ to find an approximate solution. Compare the result with the exact solution by plotting them on the same graph and printing the 10 largest absolute errors.

   (c) Use a linear interpolant of your solution from (b) to approximate the solution at $t = \frac{\pi}{3}, \frac{\pi}{2}, \frac{e}{2}, e$. Compute the absolute error at these values.

   (d) Repeat parts (b) and (c) with

      i. a high order Taylor methods of global error $O(h^2)$,

      ii. a high order Taylor methods of global error $O(h^3)$,

      iii. and the midpoint method.

6. Consider the following initial value problem.

$$\begin{cases} y' = (2-t)y, & t \in [2, 5] \\ y(2) = 1 \end{cases}$$

   (a) Verify that $y(t) = \exp\left(-\dfrac{(t-2)^2}{2}\right)$ is the exact solution. (Here $\exp(x) = e^x$.)

   (b) Use a second order Runge-Kutta method with $h = \frac{1}{100}$ to find an approximate solution. Compare the result with the exact solution by plotting them on the same graph and printing the 10 largest absolute errors.

   (c) Repeat part (b) with a fourth order Runge-Kutta method.

   (d) Repeat part (b) by using the built in Scilab function `ode(y0,t0,t,f,'rk')`. Research this Scilab function and compare it to the previous two.

7. Suppose that a differential equation is solved numerically on an interval $[a, b]$ and that the local error is Ch^p for some constant C. Show that if all truncation errors have the same sign (the worst possible case), then the total truncation error is $(b-a)Ch^{p-1}$ where $h = \frac{(b-a)}{n}$.

8. (BONUS) In class, we discussed Richardson extrapolation as a way to improve an approximation. Apply this methodology to the composite trapezoidal rule below

$$\int_a^b f(x)\, dx = \frac{h}{2}\left(f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n\right) + Ch^2 + O(h^4)$$

where $h = (b-a)/n,\ x_i = a + ih,\ f_i = f(x_i)$, and C is a constant by completing the following steps. This is known as **Romberg integration** (Section 4.4 in the text).

   (a) Define $T_h = \frac{h}{2}\left(f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n\right)$. Find a linear combination of approximations T_h and $T_{h/2}$ which results in a method which is $O(h^4)$. Denote this new linear combination as $\phi_1(h)$. $\phi_1(h)$ should look familiar. What is it?

   (b) Assume the error for the above composite trapezoidal rule has only even powers of h, $h^2, h^4, h^6, \ldots$ Repeat part (a) twice to find methods, $\phi_2(h), \phi_3(h)$ which are $O(h^6)$ and $O(h^8)$ respectively.

   (c) To test your three new methods ϕ_1, ϕ_2, ϕ_3 for the following integrals. Compare these three results and their errors to the regular composite trapezoidal rule. Use $n = 5$ for each.

      i. $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx$

      ii. $\displaystyle\int_0^\pi \sin(x)\, dx$

      iii. $\displaystyle\int_0^1 \sqrt{x}\, dx$ Why are the results for this part worse than perhaps expected?

9. (BONUS) Solve the initial value problems from problems 5 and 6 exactly. Your final solution should match the given exact solution.

10. (BONUS) Use Simpson's rule to design an implicit method for a general initial value problem.

$$\begin{cases} y' = f(t,y), & t \in [a,b] \\ y(a) = y_a. \end{cases}$$

11. (BONUS) Design a third-order Runge Kutta method for a general initial value problem

$$\begin{cases} y' = f(t,y), & t \in [a,b] \\ y(a) = y_a. \end{cases}$$