Ramanujan and Pi

Some 75 years ago an Indian mathematical genius developed ways of calculating pi with extraordinary efficiency. His approach is now incorporated in computer algorithms yielding millions of digits of pi

by Jonathan M. Borwein and Peter B. Borwein

Pi, the ratio of any circle's circumference to its diameter, was computed in 1987 to an unprecedented level of accuracy: more than 100 million decimal places. Last year also marked the centenary of the birth of Srinivasa Ramanujan, an enigmatic Indian mathematical genius who spent much of his short life in isolation and poor health. The two events are in fact closely linked, because the basic approach underlying the most recent computations of pi was anticipated by Ramanujan, although its implementation had to await the formulation of efficient algorithms (by various workers including us), modern supercomputers and new ways to multiply numbers.

Aside from providing an arena in which to set records of a kind, the quest to calculate the number to millions of decimal places may seem rather pointless. Thirty-nine places of pi suffice for computing the circumference of a circle girdling the known universe with an error no greater than the radius of a hydrogen atom. It is hard to imagine physical situations requiring more digits. Why are mathematicians and computer scientists not satisfied with, say, the first 50 digits of pi?

Several answers can be given. One is that the calculation of pi has become something of a benchmark computation: it serves as a measure of the sophistication and reliability of the computers that carry it out. In addition, the pursuit of ever more accurate values of pi leads mathematicians to intriguing and unexpected niches of number theory. Another and more ingenuous motivation is simply "because it's there." In fact, pi has been a fixture of mathematical culture for more than two and a half millenniums.

Furthermore, there is always the chance that such computations will shed light on some of the riddles surrounding pi, a universal constant that is not particularly well understood, in spite of its relatively elementary nature. For example, although it has been proved that pi cannot ever be exactly evaluated by subjecting positive integers to any combination of adding, subtracting, multiplying, dividing or extracting roots, no one has succeeded in proving that the digits of pi follow a random distribution (such that each number from 0 to 9 appears with equal frequency). It is possible, albeit highly unlikely, that after a while all the remaining digits of pi are 0's and 1's or exhibit some other regularity. Moreover, pi turns up in all kinds of unexpected places that have nothing to do with circles. If a number is picked at random from the set of integers, for instance, the probability that it will have no repeated prime divisors is six divided by the square of pi. No different from other eminent mathematicians, Ramanujan was prey to the fascinations of the number.

The ingredients of the recent approaches to calculating pi are among the mathematical treasures unearthed by renewed interest in Ramanujan's work. Much of what he did, however, is still inaccessible to investigators. The body of his work is contained in his "Notebooks," which are personal records written in his own nomenclature. To make matters more frustrating for mathematicians who have studied the "Notebooks," Ramanujan generally did not include formal proofs for his theorems. The task of deciphering and editing the "Notebooks" is only now nearing completion, by Bruce C. Berndt of the University of Illinois at Urbana-Champaign.

To our knowledge no mathematical redaction of this scope or difficulty has ever been attempted. The effort is certainly worthwhile. Ramanujan's legacy in the "Notebooks" promises not only to enrich pure mathematics but also to find application in various fields of mathematical physics. Rodney J. Baxter of the Australian National University, for example, acknowledges that Ramanujan's findings helped him to solve such problems in statistical mechanics as the so-called hard-hexagon model, which considers the behavior of a system of interacting particles laid out on a honeycomblike grid. Similarly, Carlos J. Moreno of the City University of New York and Freeman J. Dyson of the Institute for Advanced Study have pointed out that Ramanujan's work is beginning to be applied by physicists in superstring theory.

Ramanujan's stature as a mathematician is all the more astonishing when one considers his limited formal education. He was born on December 22, 1887, into a somewhat impoverished family of the Brahmin caste in the town of Erode in southern India and grew up in Kumbakonam, where his father was an accountant to a clothier. His mathematical precocity was recognized early, and at the age of seven he was given a scholarship to the Kumbakonam Town High School. He is said to have recited mathematical formulas to his schoolmates—including the value of pi to many places.

When he was 12, Ramanujan mastered the contents of S. L. Loney's rather comprehensive *Plane Trigonometry,* including its discussion of the sum and products of infinite sequences, which later were to figure prominently in his work. (An infinite sequence is an unending string of terms, often generated by a simple formula. In this context the interesting sequences are those whose terms can be added or multiplied to yield an identifiable, finite value. If the terms are added, the resulting expression is called a series; if they are multiplied, it is called a product.) Three years later he borrowed the *Synopsis of Elementary Results in Pure Mathematics,* a listing of some 6,000 theorems (most of them given without proof) compiled by G. S. Carr, a tutor at the University of Cambridge. Those two books were the basis of Ramanujan's mathematical training.

In 1903 Ramanujan was admitted to a local government college. Yet total absorption in his own mathematical diversions at the expense of everything else caused him to fail his examinations, a pattern repeated four years later at another college in Madras. Ramanujan did set his avocation aside—if only temporarily—to look for a job after his marriage in 1909. Fortunately in 1910 R. Ramachandra Rao, a well-to-do patron of mathematics, gave him a monthly stipend largely on the strength of favorable recommendations from various sympathetic Indian mathematicians and the findings he already had jotted down in the "Notebooks."

In 1912, wanting more conventional work, he took a clerical position in the Madras Port Trust, where the chairman was a British engineer, Sir Francis Spring, and the manager was V. Ramaswami Aiyar, the founder of the Indian Mathematical Society. They encouraged Ramanujan to communicate his results to three prominent British mathematicians. Two apparently did not respond; the one who did was G. H. Hardy of Cambridge, now regarded as the foremost British mathematician of the period.

Hardy, accustomed to receiving crank mail, was inclined to disregard Ramanujan's letter at first glance the day it arrived, January 16, 1913. But after dinner that night Hardy and a close colleague, John E. Littlewood, sat down to puzzle through a list of 120 formulas and theorems Ramanujan had appended to his letter. Some hours later they had reached a verdict: they were seeing the work of a genius and not a crackpot. (According to his own "pure-talent scale" of mathematicians, Hardy was later to rate Ramanujan a 100, Littlewood a 30 and himself a 25. The German mathematician David Hilbert, the most influential figure of the time, merited only an 80.) Hardy described the revelation and its consequences as the one romantic incident in his life. He wrote that some of Ramanujan's formulas defeated him completely, and yet "they must be true, because if they were not true, no one would have had the imagination to invent them."

Hardy immediately invited Ramanujan to come to Cambridge. In spite of his mother's strong objections as well as his own reservations, Ramanujan set out for England in March of 1914. During the next five years Hardy and Ramanujan worked together at Trinity College. The blend of Hardy's technical expertise and Ramanujan's raw brilliance produced an unequaled collaboration. They published a series of seminal papers on the properties of various arithmetic functions, laying the groundwork for the answer to such questions as: How many prime divisors is a given number likely to have? How many ways can one express a number as a sum of smaller positive integers?

In 1917 Ramanujan was made a Fellow of the Royal Society of London and a Fellow of Trinity College—the first Indian to be awarded either honor. Yet as his prominence grew his health deteriorated sharply, a decline perhaps accelerated by the difficulty of maintaining a strict vegetarian diet in war-rationed England. Although Ramanujan was in and out of sanatoriums, he continued to pour forth new results. In 1919, when peace made travel abroad safe again, Ramanujan returned to India. Already an icon for young Indian intellectuals, the 32-year-old Ramanujan died on April 26, 1920, of what was then diagnosed as tuberculosis but now is thought to have been a severe vitamin deficiency. True to mathematics until the end, Ramanujan did not slow down during his last, pain-racked months, producing the re-

SRINIVASA RAMANUJAN, born in 1887 in India, managed in spite of limited formal education to reconstruct almost single-handedly much of the edifice of number theory and to go on to derive original theorems and formulas. Like many illustrious mathematicians before him, Ramanujan was fascinated by pi: the ratio of any circle's circumference to its diameter. Based on his investigation of modular equations (*see box on page 114*), he formulated exact expressions for pi and derived from them approximate values. As a result of the work of various investigators (including the authors), Ramanujan's methods are now better understood and have been implemented as algorithms.

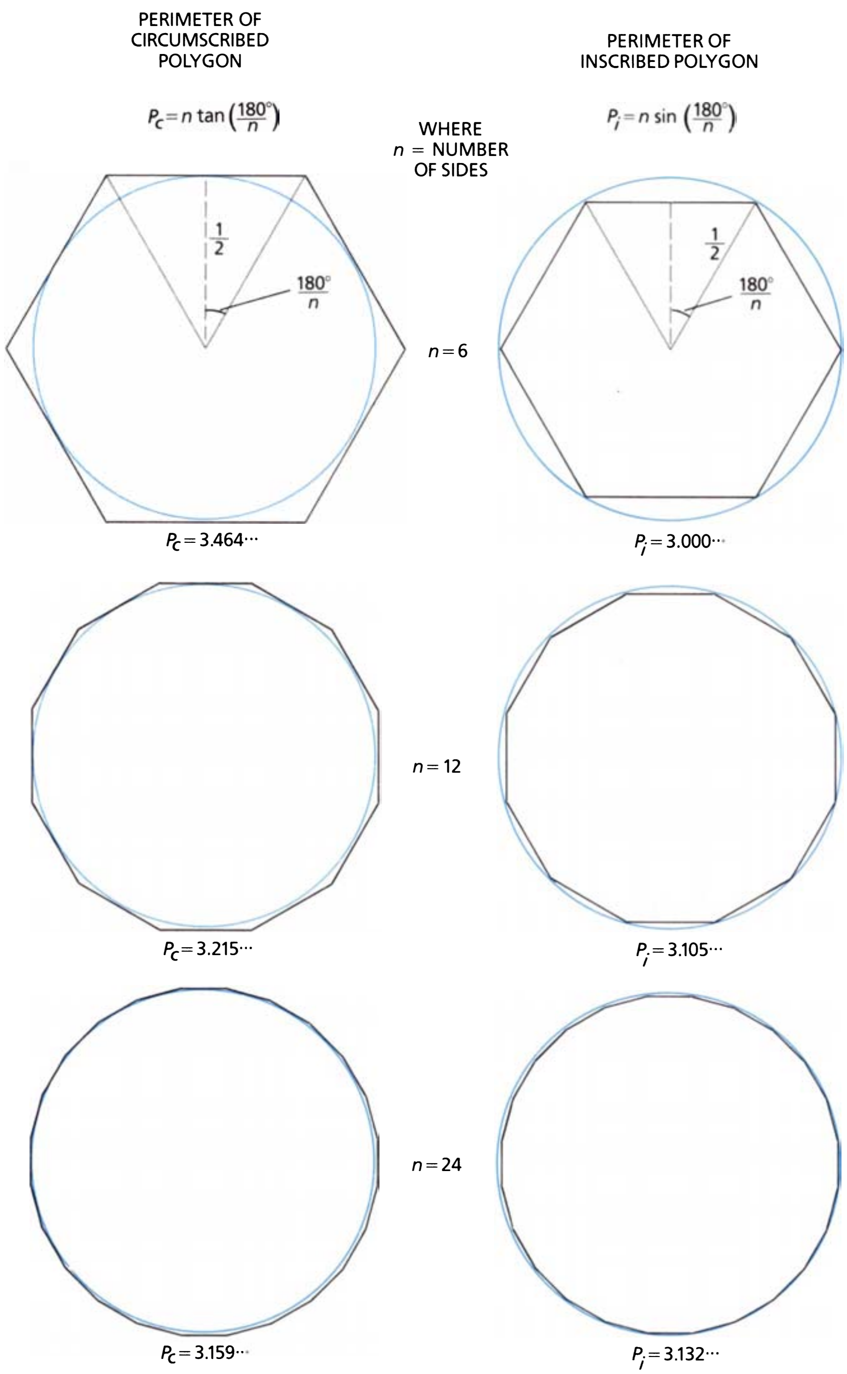
markable work recorded in his so-called "Lost Notebook."

Ramanujan's work on pi grew in large part out of his investigation of modular equations, perhaps the most thoroughly treated subject in the "Notebooks." Roughly speaking, a modular equation is an algebraic relation between a function expressed in terms of a variable x—in mathematical notation, $f(x)$—and the same function expressed in terms of x raised to an integral power, for example $f(x^2)$, $f(x^3)$ or $f(x^4)$. The "order" of the modular equation is given by the integral power. The simplest modular equation is the second-order one: $f(x) = 2\sqrt{f(x^2)}/[1 + f(x^2)]$. Of course, not every function will satisfy a modular equation, but there is a class of functions, called modular functions, that do. These functions have various surprising symmetries that give them a special place in mathematics.

Ramanujan was unparalleled in his ability to come up with solutions to modular equations that also satisfy other conditions. Such solutions are called singular values. It turns out that solving for singular values in certain cases yields numbers whose natural logarithms coincide with pi (times a constant) to a surprising number of places [*see box on page 114*]. Applying this general approach with extraordinary virtuosity, Ramanujan produced many remarkable infinite series as well as single-term approximations for pi. Some of them are given in Ramanujan's one formal paper on the subject, *Modular Equations and Approximations to* π, published in 1914.

Ramanujan's attempts to approximate pi are part of a venerable tradition. The earliest Indo-European civilizations were aware that the area of a circle is proportional to the square of its radius and that the circumference of a circle is directly proportional to its diameter. Less clear, however, is when it was first realized that the ratio of any circle's circumference to its diameter and the ratio of any circle's area to the square of its radius are in fact the same constant, which today is designated by the symbol π. (The symbol, which gives the constant its name, is a latecomer in the history of mathematics, having been introduced in 1706 by the English mathematical writer William Jones and popularized by the Swiss mathematician Leonhard Euler in the 18th century.)

Archimedes of Syracuse, the greatest mathematician of antiquity, rigorously established the equivalence of the two ratios in his treatise *Measurement of a Circle.* He also calculated a value for pi based on mathematical principles rather than on direct measurement of a circle's circumference, area and diameter. What Archimedes did was to inscribe and circumscribe regular polygons (polygons whose sides are all the same length) on a circle assumed to have a diameter of one unit and to consider



ARCHIMEDES' METHOD for estimating pi relied on inscribed and circumscribed regular polygons (polygons with sides of equal length) on a circle having a diameter of one unit (or a radius of half a unit). The perimeters of the inscribed and circumscribed polygons served respectively as lower and upper bounds for the value of pi. The sine and tangent functions can be used to calculate the polygons' perimeters, as is shown here, but Archimedes had to develop equivalent relations based on geometric constructions. Using 96-sided polygons, he determined that pi is greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$.

the polygons' respective perimeters as lower and upper bounds for possible values of the circumference of the circle, which is numerically equal to pi [*see illustration on opposite page*].

This method of approaching a value for pi was not novel: inscribing polygons of ever more sides in a circle had been proposed earlier by Antiphon, and Antiphon's contemporary, Bryson of Heraclea, had added circumscribed polygons to the procedure. What was novel was Archimedes' correct determination of the effect of doubling the number of sides on both the circumscribed and the inscribed polygons. He thereby developed a procedure that, when repeated enough times, enables one in principle to calculate pi to any number of digits. (It should be pointed out that the perimeter of a regular polygon can be readily calculated by means of simple trigonometric functions: the sine, cosine and tangent functions. But in Archimedes' time, the third century B.C., such functions were only partly understood. Archimedes therefore had to rely mainly on geometric constructions, which made the calculations considerably more demanding than they might appear today.)

Archimedes began with inscribed and circumscribed hexagons, which yield the inequality $3 < \pi < 2\sqrt{3}$. By doubling the number of sides four times, to 96, he narrowed the range of pi to between $3\frac{10}{71}$ and $3\frac{1}{7}$, obtaining the estimate $\pi \approx 3.14$. There is some evidence that the extant text of *Measurement of a Circle* is only a fragment of a larger work in which Archimedes described how, starting with decagons and doubling them six times, he got a five-digit estimate: $\pi \approx 3.1416$.

Archimedes' method is conceptually simple, but in the absence of a ready way to calculate trigonometric functions it requires the extraction of roots, which is rather time-consuming when done by hand. Moreover, the estimates converge slowly to pi: their error decreases by about a factor of four per iteration. Nevertheless, all European attempts to calculate pi before the mid-17th century relied in one way or another on the method. The 16th-century Dutch mathematician Ludolph van Ceulen dedicated much of his career to a computation of pi. Near the end of his life he obtained a 32-digit estimate by calculating the perimeter of inscribed and circumscribed polygons having 2^{62} (some 10^{18}) sides. His value for pi, called the Ludolphian number in parts of Europe, is said to have served as his epitaph.

The development of calculus, largely by Isaac Newton and Gottfried Wilhelm Leibniz, made it possible to calculate pi much more expeditiously. Calculus provides efficient techniques for computing a function's derivative (the rate of change in the function's value as its variables change) and its integral (the sum of the function's values over a range of variables). Applying the techniques, one can demonstrate that inverse trigonometric functions are given by integrals of quadratic functions that describe the curve of a circle. (The inverse of a trigonometric function gives the angle that corresponds to a particular value of the function. For example, the inverse tangent of 1 is 45 degrees or, equivalently, $\pi/4$ radians.)

(The underlying connection between trigonometric functions and algebraic expressions can be appreciated by considering a circle that has a radius of one unit and its center at the origin of a Cartesian x-y plane. The equation for the circle—whose area is numerically equal to pi—is $x^2 + y^2 = 1$, which is a restatement of the Pythagorean theorem for a right triangle with a hypotenuse equal to 1. Moreover, the sine and cosine of the angle between the positive x axis and any point on the circle are equal respectively to the point's coordinates, y and x; the angle's tangent is simply y/x.)

Of more importance for the purposes of calculating pi, however, is the fact that an inverse trigonometric function can be "expanded" as a series, the terms of which are computable from the derivatives of the function. Newton himself calculated pi to 15 places by adding the first few terms of a series that can be derived

WALLIS' PRODUCT (1665)

$$\frac{\pi}{2} = \frac{2\times 2}{1\times 3}\times\frac{4\times 4}{3\times 5}\times\frac{6\times 6}{5\times 7}\times\frac{8\times 8}{7\times 9}\times\cdots = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}$$

GREGORY'S SERIES (1671)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

MACHIN'S FORMULA (1706)

$$\frac{\pi}{4} = 4\ \text{arctan}\ (1/5) - \text{arctan}\ (1/239), \qquad \text{where arctan } X = X - \frac{X^3}{3} + \frac{X^5}{5} - \frac{X^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{X^{(2n+1)}}{2n+1}$$

RAMANUJAN (1914)

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9{,}801}\sum_{n=0}^{\infty}\frac{(4n)![1{,}103 + 26{,}390n]}{(n!)^4\, 396^{4n}}, \quad \text{where } n! = n \times (n-1) \times (n-2) \times \cdots \times 1 \text{ and } 0! = 1$$

BORWEIN AND BORWEIN (1987)

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)![212{,}175{,}710{,}912\sqrt{61} + 1{,}657{,}145{,}277{,}365 + n(13{,}773{,}980{,}892{,}672\sqrt{61} + 107{,}578{,}229{,}802{,}750)]}{(n)!^3(3n)![5{,}280(236{,}674 + 30{,}303\sqrt{61})]^{(3n+3/2)}}$$

TERMS OF MATHEMATICAL SEQUENCES can be summed or multiplied to yield values for pi (divided by a constant) or its reciprocal. The first two sequences, discovered respectively by the mathematicians John Wallis and James Gregory, are probably among the best-known, but they are practically useless for computational purposes. Not even 100 years of computing on a supercomputer programmed to add or multiply the terms of either sequence would yield 100 digits of pi. The formula discovered by John Machin made the calculation of pi feasible, since calculus allows the inverse tangent (arc tangent) of a number, *x*, to be expressed in terms of a sequence whose sum converges more rapidly to the value of the arc tangent the smaller *x* is. Virtually all calculations for pi from the beginning of the 18th century until the early 1970's have relied on variations of Machin's formula. The sum of Ramanujan's sequence converges to the true value of $1/\pi$ much faster: each successive term in the sequence adds roughly eight more correct digits. The last sequence, formulated by the authors, adds about 25 digits per term; the first term (for which *n* is 0) yields a number that agrees with pi to 24 digits.

as an expression for the inverse of the sine function. He later confessed to a colleague: "I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time."

In 1674 Leibniz derived the formula $1 - 1/3 + 1/5 - 1/7 \ldots = \pi/4$, which is the inverse tangent of 1. (The general inverse-tangent series was originally discovered in 1671 by the Scottish mathematician James Gregory. Indeed, similar expressions appear to have been developed independently several centuries earlier in India.) The error of the approximation, defined as the difference between the sum of n terms and the exact value of $\pi/4$, is roughly equal to the $n+1$th term in the series. Since the denominator of each successive term increases by only 2, one must add approximately 50 terms to get two-digit accuracy, 500 terms for three-digit accuracy and so on. Summing the terms of the series to calculate a value for pi more than a few digits long is clearly prohibitive.

An observation made by John Machin, however, made it practicable to calculate pi by means of a series expansion for the inverse-tangent function. He noted that pi divided by 4 is equal to 4 times the inverse tangent of 1/5 minus the inverse tangent of 1/239. Because the inverse-tangent series for a given value converges more quickly the smaller the value is, Machin's formula greatly simplified the calculation. Coupling his formula with the series expansion for the inverse tangent, Machin computed 100 digits of pi in 1706. Indeed, his technique proved to be so powerful that all extended calculations of pi from the beginning of the 18th century until recently relied on variants of the method.

Two 19th-century calculations deserve special mention. In 1844 Johann Dase computed 205 digits of pi in a matter of months by calculating the values of three inverse tangents in a Machin-like formula. Dase was a calculating prodigy who could multiply 100-digit numbers entirely in his head—a feat that took him roughly eight hours. (He was perhaps the closest precursor of the modern supercomputer, at least in terms of memory capacity.) In 1853 William Shanks outdid Dase by publishing his computation of pi to 607 places, although the digits that followed the 527th place were wrong. Shank's task took years and was a rather routine, albeit laborious, application of Machin's formula. (In what must itself be some kind of record, 92 years passed before Shank's error was detected, in a comparison between his value and a 530-place approximation produced by D. F. Ferguson with the aid of a mechanical calculator.)

The advent of the digital computer saw a renewal of efforts to calculate ever more digits of pi, since the machine was ideally suited for lengthy, repetitive "number crunching." ENIAC, one of the first digital computers, was applied to the task in June, 1949, by John von Neumann and his colleagues. ENIAC produced 2,037 digits in 70 hours. In 1957 G. E. Felton attempted to compute 10,000 digits of pi, but owing to a machine error only the first 7,480 digits were correct. The 10,000-digit goal was reached by F. Genuys the following year on an IBM 704 computer. In 1961 Daniel Shanks and John W. Wrench, Jr., calculated 100,000 digits of pi in less than nine hours on an IBM 7090. The million-digit mark was passed in 1973 by Jean Guilloud and M. Bouyer, a feat that took just under a day of computation on a CDC 7600. (The computations done by Shanks and Wrench and by Guilloud and Bouyer were in fact carried out twice using different inverse-tangent identities for pi. Given the history of both human and machine error in these calculations, it is only after such verification that modern "digit hunters" consider a record officially set.)

Although an increase in the speed of computers was a major reason ever more accurate calculations for pi could be performed, it soon became clear that there were inescapable limits. Doubling the number of digits lengthens computing time by at least a factor of four, if one applies the traditional methods of performing arithmetic in computers. Hence even allowing for a hundredfold increase in computational speed, Guilloud and Bouyer's program would have required at least a quarter century to produce a billion-digit value for pi. From the perspective of the early 1970's such a computation did not seem realistically practicable.

Yet the task is now feasible, thanks

MODULAR FUNCTIONS AND APPROXIMATIONS TO PI

A modular function is a function, $\lambda(q)$, that can be related through an algebraic expression called a modular equation to the same function expressed in terms of the same variable, q, raised to an integral power: $\lambda(q^p)$. The integral power, p, determines the "order" of the modular equation. An example of a modular function is

$$\lambda(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8 .$$

Its associated seventh-order modular equation, which relates $\lambda(q)$ to $\lambda(q^7)$, is given by

$$\sqrt[8]{\lambda(q)\lambda(q^7)} + \sqrt[8]{[1-\lambda(q)][1-\lambda(q^7)]} = 1 .$$

Singular values are solutions of modular equations that must also satisfy additional conditions. One class of singular values corresponds to computing a sequence of values, k_p, where

$$k_p = \sqrt{\lambda(e^{-\pi\sqrt{p}})}$$

and p takes integer values. These values have the curious property that the logarithmic expression

$$\frac{-2}{\sqrt{p}} \log\left(\frac{k_p}{4}\right)$$

coincides with many of the first digits of pi. The number of digits the expression has in common with pi increases with larger values of p.

Ramanujan was unparalleled in his ability to calculate these singular values. One of his most famous is the value when p equals 210, which was included in his original letter to G. H. Hardy. It is

$$k_{210} = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7})(\sqrt{10}-3)^2(\sqrt{15}-\sqrt{14})(4-\sqrt{15})^2(6-\sqrt{35}) .$$

This number, when plugged into the logarithmic expression, agrees with pi through the first 20 decimal places. In comparison, k_{240} yields a number that agrees with pi through more than one million digits.

Applying this general approach, Ramanujan constructed a number of remarkable series for pi, including the one shown in the illustration on the preceding page. The general approach also underlies the two-step, iterative algorithms in the top illustration on the opposite page. In each iteration the first step (calculating y_n) corresponds to computing one of a sequence of singular values by solving a modular equation of the appropriate order; the second step (calculating α_n) is tantamount to taking the logarithm of the singular value.

not only to faster computers but also to new, efficient methods for multiplying large numbers in computers. A third development was also crucial: the advent of iterative algorithms that quickly converge to pi. (An iterative algorithm can be expressed as a computer program that repeatedly performs the same arithmetic operations, taking the output of one cycle as the input for the next.) These algorithms, some of which we constructed, were in many respects anticipated by Ramanujan, although he knew nothing of computer programming. Indeed, computers not only have made it possible to apply Ramanujan's work but also have helped to unravel it. Sophisticated algebraic-manipulation software has allowed further exploration of the road Ramanujan traveled alone and unaided 75 years ago.

One of the interesting lessons of theoretical computer science is that many familiar algorithms, such as the way children are taught to multiply in grade school, are far from optimal. Computer scientists gauge the efficiency of an algorithm by determining its bit complexity: the number of times individual digits are added or multiplied in carrying out an algorithm. By this measure, adding two n-digit numbers in the normal way has a bit complexity that increases in step with n; multiplying two n-digit numbers in the normal way has a bit complexity that increases as n^2. By traditional methods, multiplication is much "harder" than addition in that it is much more time-consuming.

Yet, as was shown in 1971 by A. Schönhage and V. Strassen, the multiplication of two numbers can in theory have a bit complexity only a little greater than addition. One way to achieve this potential reduction in bit complexity is to implement so-called fast Fourier transforms (FFT's). FFT-based multiplication of two large numbers allows the intermediary computations among individual digits to be carefully orchestrated so that redundancy is avoided. Because division and root extraction can be reduced to a sequence of multiplications, they too can have a bit complexity just slightly greater than that of addition. The result is a tremendous saving in bit complexity and hence in computation time. For this reason all recent efforts to calculate pi rely on some variation of the FFT technique for multiplication.

Yet for hundreds of millions of dig-

(a) Let $y_0 = \frac{1}{\sqrt{2}}$ $\quad \alpha_0 = \frac{1}{2}$

and

$$y_{n+1} = \frac{1 - \sqrt{1-y_n^2}}{1 + \sqrt{1-y_n^2}} \qquad \alpha_{n+1} = [(1+y_{n+1})^2\alpha_n] - 2^{n+1}y_{n+1}$$

(b) Let $y_0 = \sqrt{2}-1$ $\quad \alpha_0 = 6 - 4\sqrt{2}$

and

$$y_{n+1} = \frac{1 - \sqrt[4]{1-y_n^4}}{1 + \sqrt[4]{1-y_n^4}}$$

$$\alpha_{n+1} = [(1+y_{n+1})^4\alpha_n] - 2^{2n+3}y_{n+1}(1+y_{n+1}+y_{n+1}^2)$$

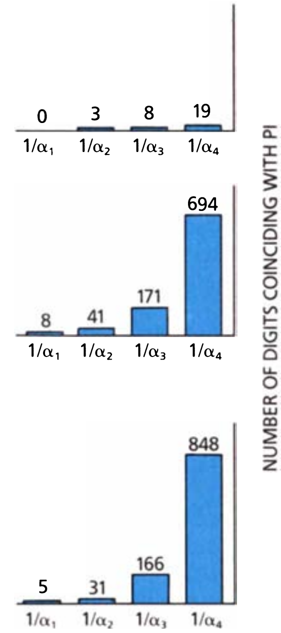
(c) Let $S_0 = 5(\sqrt{5}-2)$ $\quad \alpha_0 = \frac{1}{2}$

and

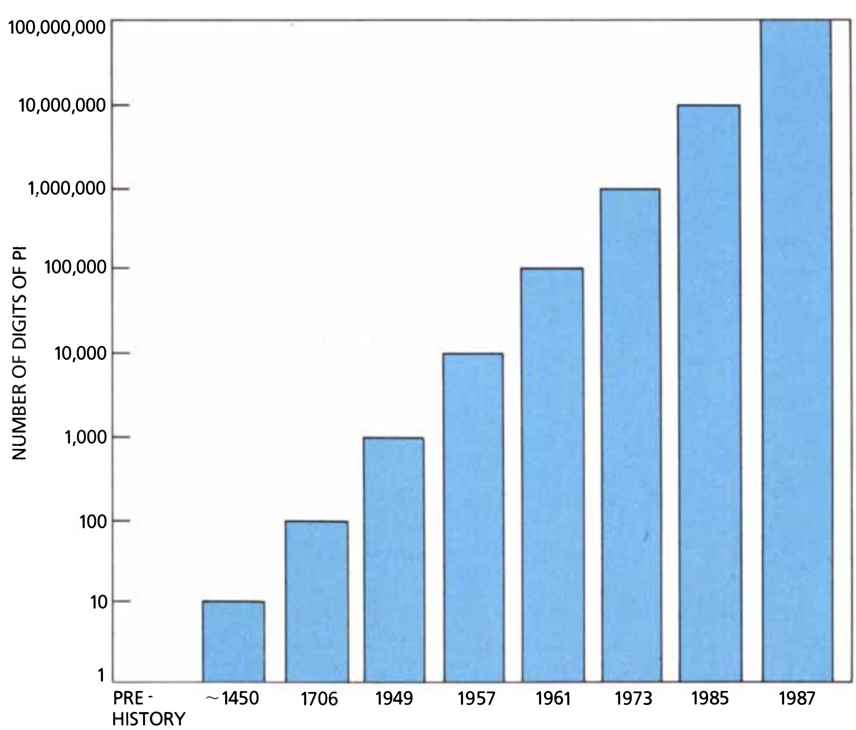
$$S_{n+1} = \frac{25}{S_n(Z + X/Z + 1)^2}, \text{ where } X = \frac{5}{S_n} - 1, \quad Y = (X-1)^2+7$$

$$\text{and } Z = \sqrt[5]{\frac{X(Y+\sqrt{Y^2-4X^3})}{2}}$$

$$\alpha_{n+1} = S_n^2\alpha_n - 5^n\left[\frac{S_n^2-5}{2} + \sqrt{S_n(S_n^2 - 2S_n + 5)}\right]$$



ITERATIVE ALGORITHMS that yield extremely accurate values of pi were developed by the authors. (An iterative algorithm is a sequence of operations repeated in such a way that the ouput of one cycle is taken as the input for the next.) Algorithm *a* converges to $1/\pi$ quadratically: the number of correct digits given by α_n more than doubles each time n is increased by 1. Algorithm *b* converges quartically and algorithm *c* converges quintically, so that the number of coinciding digits given by each iteration increases respectively by more than a factor of four and by more than a factor of five. Algorithm *b* is possibly the most efficient known algorithm for calculating pi; it was run on supercomputers in the last three record-setting calculations. As the authors worked on the algorithms it became clear to them that Ramanujan had pursued similar methods in coming up with his approximations for pi. In fact, the computation of s_n in algorithm *c* rests on a remarkable fifth-order modular equation discovered by Ramanujan.



NUMBER OF KNOWN DIGITS of pi has increased by two orders of magnitude (factors of 10) in the past decade as a result of the development of iterative algorithms that can be run on supercomputers equipped with new, efficient methods of multiplication.

its of pi to be calculated practically a beautiful formula known a century and a half earlier to Carl Friedrich Gauss had to be rediscovered. In the mid-1970's Richard P. Brent and Eugene Salamin independently noted that the formula produced an algorithm for pi that converged quadratically, that is, the number of digits doubled with each iteration. Between 1983 and the present Yasumasa Kanada and his colleagues at the University of Tokyo have employed this algorithm to set several world records for the number of digits of pi.

We wondered what underlies the remarkably fast convergence to pi of the Gauss-Brent-Salamin algorithm, and in studying it we developed general techniques for the construction of similar algorithms that rapidly converge to pi as well as to other quantities. Building on a theory outlined by the German mathematician Karl Gustav Jacob Jacobi in 1829, we realized we could in principle arrive at a value for pi by evaluating integrals of a class called elliptic integrals, which can serve to calculate the perimeter of an ellipse. (A circle, the geometric setting of previous efforts to approximate pi, is simply an ellipse with axes of equal length.)

Elliptic integrals cannot generally be evaluated as integrals, but they can be easily approximated through iterative procedures that rely on modular equations. We found that the Gauss-Brent-Salamin algorithm is actually a specific case of our more general technique relying on a second-order modular equation. Quicker convergence to the value of the integral, and thus a faster algorithm for pi, is possible if higher-order modular equations are used, and so we have also constructed various algorithms based on modular equations of third, fourth and higher orders.

In January, 1986, David H. Bailey of the National Aeronautics and Space Administration's Ames Research Center produced 29,360,000 decimal places of pi by iterating one of our algorithms 12 times on a Cray-2 supercomputer. Because the algorithm is based on a fourth-order modular equation, it converges on pi quartically, more than quadrupling the number of digits with each iteration. A year later Kanada and his colleagues carried out one more iteration to attain 134,217,000 places on an NEC SX-2 supercomputer and thereby verified a similar computation they had done earlier using the Gauss-Brent-Salamin algorithm. (Iterating our algorithm twice more—a feat entirely feasible if one could somehow monopolize a supercomputer for a few weeks—would yield more than two billion digits of pi.)

HOW TO GET TWO BILLION DIGITS OF PI WITH A CALCULATOR*

Let

$y_0 = \sqrt{2} - 1$	$\alpha_0 = 6 - 4\sqrt{2}$
$y_1 = [1-\sqrt[4]{1-y_0^4}]/[1+\sqrt[4]{1-y_0^4}]$	$\alpha_1 = (1+y_1)^4\,\alpha_0 - 2^3 y_1(1+y_1+y_1^2)$
$y_2 = [1-\sqrt[4]{1-y_1^4}]/[1+\sqrt[4]{1-y_1^4}]$	$\alpha_2 = (1+y_2)^4\,\alpha_1 - 2^5 y_2(1+y_2+y_2^2)$
$y_3 = [1-\sqrt[4]{1-y_2^4}]/[1+\sqrt[4]{1-y_2^4}]$	$\alpha_3 = (1+y_3)^4\,\alpha_2 - 2^7 y_3(1+y_3+y_3^2)$
$y_4 = [1-\sqrt[4]{1-y_3^4}]/[1+\sqrt[4]{1-y_3^4}]$	$\alpha_4 = (1+y_4)^4\,\alpha_3 - 2^9 y_4(1+y_4+y_4^2)$
$y_5 = [1-\sqrt[4]{1-y_4^4}]/[1+\sqrt[4]{1-y_4^4}]$	$\alpha_5 = (1+y_5)^4\,\alpha_4 - 2^{11} y_5(1+y_5+y_5^2)$
$y_6 = [1-\sqrt[4]{1-y_5^4}]/[1+\sqrt[4]{1-y_5^4}]$	$\alpha_6 = (1+y_6)^4\,\alpha_5 - 2^{13} y_6(1+y_6+y_6^2)$
$y_7 = [1-\sqrt[4]{1-y_6^4}]/[1+\sqrt[4]{1-y_6^4}]$	$\alpha_7 = (1+y_7)^4\,\alpha_6 - 2^{15} y_7(1+y_7+y_7^2)$
$y_8 = [1-\sqrt[4]{1-y_7^4}]/[1+\sqrt[4]{1-y_7^4}]$	$\alpha_8 = (1+y_8)^4\,\alpha_7 - 2^{17} y_8(1+y_8+y_8^2)$
$y_9 = [1-\sqrt[4]{1-y_8^4}]/[1+\sqrt[4]{1-y_8^4}]$	$\alpha_9 = (1+y_9)^4\,\alpha_8 - 2^{19} y_9(1+y_9+y_9^2)$
$y_{10} = [1-\sqrt[4]{1-y_9^4}]/[1+\sqrt[4]{1-y_9^4}]$	$\alpha_{10} = (1+y_{10})^4\,\alpha_9 - 2^{21} y_{10}(1+y_{10}+y_{10}^2)$
$y_{11} = [1-\sqrt[4]{1-y_{10}^4}]/[1+\sqrt[4]{1-y_{10}^4}]$	$\alpha_{11} = (1+y_{11})^4\,\alpha_{10} - 2^{23} y_{11}(1+y_{11}+y_{11}^2)$
$y_{12} = [1-\sqrt[4]{1-y_{11}^4}]/[1+\sqrt[4]{1-y_{11}^4}]$	$\alpha_{12} = (1+y_{12})^4\,\alpha_{11} - 2^{25} y_{12}(1+y_{12}+y_{12}^2)$
$y_{13} = [1-\sqrt[4]{1-y_{12}^4}]/[1+\sqrt[4]{1-y_{12}^4}]$	$\alpha_{13} = (1+y_{13})^4\,\alpha_{12} - 2^{27} y_{13}(1+y_{13}+y_{13}^2)$
$y_{14} = [1-\sqrt[4]{1-y_{13}^4}]/[1+\sqrt[4]{1-y_{13}^4}]$	$\alpha_{14} = (1+y_{14})^4\,\alpha_{13} - 2^{29} y_{14}(1+y_{14}+y_{14}^2)$
$y_{15} = [1-\sqrt[4]{1-y_{14}^4}]/[1+\sqrt[4]{1-y_{14}^4}]$	$\alpha_{15} = (1+y_{15})^4\,\alpha_{14} - 2^{31} y_{15}(1+y_{15}+y_{15}^2)$

$1/\alpha_{15}$ agress with π for more than two billion decimal digits

*Of course, the calculator needs to have a two-billion-digit display; on a pocket calculator the computation would not be very interesting after the second iteration.

EXPLICIT INSTRUCTIONS for executing algorithm *b* in the top illustration on the preceding page makes it possible in principle to compute the first two billion digits of pi in a matter of minutes. All one needs is a calculator that has two memory registers and the usual capacity to add, subtract, multiply, divide and extract roots. Unfortunately most calculators come with only an eight-digit display, which makes the computation moot.

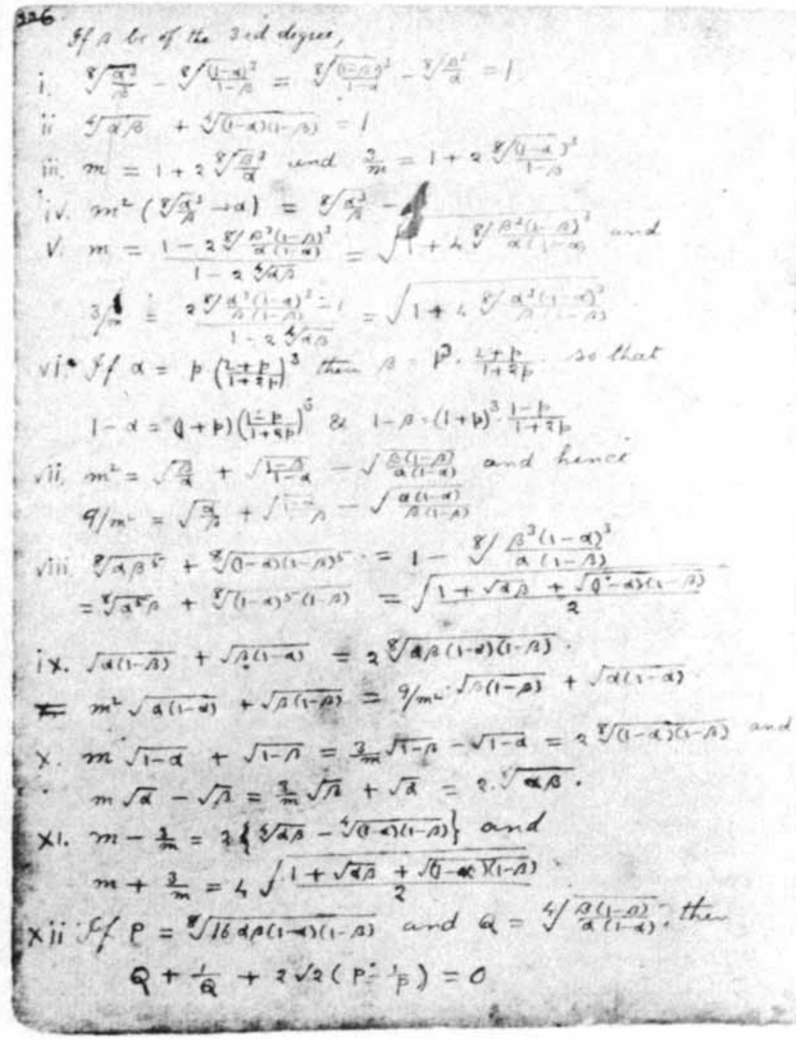
Iterative methods are best suited for calculating pi on a computer, and so it is not surprising that Ramanujan never bothered to pursue them. Yet the basic ingredients of the iterative algorithms for pi—modular equations in particular—are to be found in Ramanujan's work. Parts of his original derivation of infinite series and approximations for pi more than three-quarters of a century ago must have paralleled our own efforts to come up with algorithms for pi. Indeed, the formulas he lists in his paper on pi and in the "Notebooks" helped us greatly in the construction of some of our algorithms. For example, although we were able to prove that an 11th-order algorithm exists and knew its general formulation, it was not until we stumbled on Ramanujan's modular equations of the same order that we discovered its unexpectedly simple form.

Conversely, we were also able to derive all Ramanujan's series from the general formulas we had developed. The derivation of one, which converged to pi faster than any other series we knew at the time, came about with a little help from an unexpected source. We had justified all the quantities in the expression for the series except one: the coefficient 1,103, which appears in the numerator of the expression [*see illustration on page 113*]. We were convinced—as Ramanujan must have been—that 1,103 had to be correct. To prove it we had either to simplify a daunting equation containing variables raised to powers of several thousand or to delve considerably further into somewhat arcane number theory.

By coincidence R. William Gosper, Jr., of Symbolics, Inc., had decided in 1985 to exploit the same series of Ramanujan's for an extended-accuracy value for pi. When he carried out the calculation to more than 17 million digits (a record at the time), there was to his knowledge no proof that the sum of the series actually converged to pi. Of course, he knew that millions of digits of his value coincided with an earlier Gauss-Brent-Salamin calculation done by Kanada. Hence the possibility of error was vanishingly small.

As soon as Gosper had finished his calculation and verified it against Kanada's, however, we had what we needed to prove that 1,103 was the number needed to make the series true to within one part in $10^{10,000,000}$. In much the same way that a pair of integers differing by less than 1 must be equal, his result sufficed to specify the number: it is precisely 1,103. In effect, Gosper's computation became part of our proof. We knew that the series (and its associated algorithm) is so sensitive to slight inaccuracies that if Gosper had used any other value for the coefficient or, for that matter, if the computer had introduced a single-digit error during the calculation, he would have ended up with numerical nonsense instead of a value for pi.

Ramanujan-type algorithms for approximating pi can be shown to be very close to the best possible. If all the operations involved in the execution of the algorithms are totaled (assuming that the best techniques known for addition, multiplication and root extraction are applied), the bit complexity of computing *n* digits of pi is only marginally greater than that of multiplying two *n*-digit numbers. But multiplying two *n*-digit numbers by means of an FFT-based technique is only marginally more complicated than summing two *n*-digit numbers, which is the simplest of the arithmetic operations possible on a computer.



RAMANUJAN'S "NOTEBOOKS" were personal records in which he jotted down many of his formulas. The page shown contains various third-order modular equations—all in Ramanujan's nonstandard notation. Unfortunately Ramanujan did not bother to include formal proofs for the equations; others have had to compile, edit and prove them. The formulas in the "Notebooks" embody subtle relations among numbers and functions that can be applied in other fields of mathematics or even in theoretical physics.

Mathematics has probably not yet felt the full impact of Ramanujan's genius. There are many other wonderful formulas contained in the "Notebooks" that revolve around integrals, infinite series and continued fractions (a number plus a fraction, whose denominator can be expressed as a number plus a fraction, whose denominator can be expressed as a number plus a fraction, and so on). Unfortunately they are listed with little—if any—indication of the method by which Ramanujan proved them. Littlewood wrote: "If a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further."

The herculean task of editing the "Notebooks," initiated 60 years ago by the British analysts G. N. Watson and B. N. Wilson and now being completed by Bruce Berndt, requires providing a proof, a source or an occasional correction for each of many thousands of asserted theorems and identities. A single line in the "Notebooks" can easily elicit many pages of commentary. The task is made all the more difficult by the nonstandard mathematical notation in which the formulas are written. Hence a great deal of Ramanujan's work will not become accessible to the mathematical community until Berndt's project is finished.

Ramanujan's unique capacity for working intuitively with complicated formulas enabled him to plant seeds in a mathematical garden (to borrow a metaphor from Freeman Dyson) that is only now coming into bloom. Along with many other mathematicians, we look forward to seeing which of the seeds will germinate in future years and further beautify the garden.