Exam 1 Sample Problems
Math 371

1. Compute the rates of convergence of the following limits.
   (a) $\lim_{h\to 0} \frac{\sin(h) - h\cos(h)}{h}$
   (b) $\lim_{h\to 0} \frac{\sin(h)}{h}$
   (c) $\lim_{h\to 0} \frac{1 - e^h}{h}$
   (d) $\lim_{n\to\infty} \frac{n+5}{n^2}$
   (e) $\lim_{n\to\infty} \frac{2n^2}{n^2+1}$
   (f) $\lim_{n\to\infty} \sin(1/n)$
2. What empirical evidence usually suggests quadratic convergence?
3. Give the Taylor series expansion for function $f(x)$ about $x = a$.
4. State Taylor's theorem.
5. Compute by the hand the Taylor series expansion for
   (a) $f(x) = e^x$ about $x = 0$
   (b) $f(x) = e^{x^2} + e^{2x}$ about $x = 0$
   (c) $f(x) = \sin(x)$ about $x = 0$
   (d) $f(x) = \cos(x)$ about $x = 0$
   (e) $f(x) = \ln(x+1)$ about $x = 0$
   (f) $f(x) = \arctan(x)$ about $x = 0$
   (g) $f(x)x^4 - 3x^2 + 1 =$ about $x = 1$
   (h) $f(x) = \sqrt{x}$ about $x = 16$
6. What is the maximum absolue error posible when using the approximation
$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$
   for $-0.3 \le x \le 0.3$? For what x values is the approximation accurate to within 0.00005?
7. Give the degree 2 Taylor polynomial $P(x)$ for $f(x) = x^3$ about $x = 1$. Find the value of ξ in $[1, 3]$ such that $f(3) = P(3) + \frac{f''(\xi)}{6}(3-1)^3$.
8. Give absolute and relative errors in approximating $\frac{1}{3}$ with 0.33.
9. As an approximation of 1.23456, how many significant digits are there in $x = 1.237$?

10. Derive a formula for the following rootfinding methods. Give a complete explanation.
    (a) Bisection method
    (b) Secant method
    (c) Method of false position
    (d) Newtons method

11. Show that the Bisection method on the interval $[a, b]$ requires $n > \log_2((b-a)/TOL)$ to ensure absolute error less than some tolerance TOL.

12. Apply the result of the previous problem to show that $n = 14$ steps are required for the Bisection method to approximate the root of $f(x) = x^3 + 4x^2 - 10$ on $[1, 2]$ within absolute error less than 10^{-4}.

13. For the rootfinding problem $f(x) = e^x - 5x = 0$, write down two equivalent fixed point problems.

14. For the following fixed point problem $x = g(x) = \frac{1}{2}\left(x + \frac{3}{x}\right)$, write down an equivalent rootfinding problems.

15. State the Fixed Point Theorem from class, both the linear and quadratic convergence cases.

16. Let $g(x) = \frac{1}{10}(x^2 + x + 8)$.
    (a) Find the smallest positive fixed point of g.
    (b) Using the Fixed Point Theorem from class, show that starting with any $x_0 \in [0, 4]$, the sequence $x_n = g(x_{n-1})$ will converge to the smallest fixed point of g.

17. Suppose that $g(x)$ has a fixed point r in $[a, b]$ and that $|g'(x)| \le K < 1$ on $[a, b]$ for some constant K. Prove that starting with any $x_0 \in [a, b]$, the sequence $x_n = g(x_{n-1})$ converges to a fixed point of g.

18. Let $g(x) = \frac{x(2+x)}{-1+4x}$.
    (a) Find the two fixed points of g.
    (b) To which fixed point will convergence be quadratic?

19. Use the Fixed Point Theorem from class to show that Newton's method has quadratic rate of convergence for initial guess close enough to the root r provided $f(r) = 0$ and $f'(r) \neq 0$.

20. Consider the following system, $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

    (a) Perform regular Gaussian elimination to solve this system.
    (b) Find the LU decomposition of matrix A without pivoting, and use this decomposition to solve this system.
    (c) Perform Gaussian elimination with partial pivoting to solve this system.
    (d) Find the LU decomposition of matrix A with pivoting, and use this decomposition to solve this system.

21. Problem 1 on page 265 of the text.

22. Compute the number of floating-point operations (additions, subtractions, multiplications, and divisions) which are required for:
    (a) The product of a $(m \times n)$ matrix with a $(n \times p)$ matrix.
    (b) Forward substitution on $L\vec{y} = \vec{b}$ for L $(n \times n)$ resulting from LU decomposition.
    (c) Backwards substitution on $U\vec{y} = \vec{x}$ for U $(n \times n)$ resulting from LU decomposition.

23. What does it mean for a linear system to be ill-conditioned? What is the condition number of a matrix, and roughly, where does it come from?

24. Compute the least squares solution to the following overdetermined system.

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

25. Construct the Lagrange iterpolation polynomial for function $f(x) = 2^x$ using nodes $x_0 = 0$, $x_1 = 1$, and $x_2 = 3$. What is the absolute error at $x = 1$?

26. State the theorem giving the polynomial interpolation error.