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Classroom notes

Blending two major techniques in order to compute π

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Three major techniques are employed to calculate π. Namely, (i) the perimeter of polygons inscribed or circumscribed in a circle, (ii) calculus based methods using integral representations of inverse trigonometric functions, and (iii) modular identities derived from the transformation theory of elliptic integrals. This note presents a combination of the first two procedures, which allows the derivation of a family of series that may exhibit very fast convergence rates. The geometrical interpretation gives good insight into the acceleration method that is being implemented.

1. Introduction

Vièteś famous formula [1]:

$$\frac{2}{\pi_m} \approx \left(\frac{1}{2}\sqrt{2}\right)\left(\frac{1}{2}\sqrt{2+\sqrt{2}}\right)\cdots\underbrace{\left(\frac{1}{2}\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}\right)}_{m \text{ nested roots}} \tag{1}$$

as is well known, arises from the first major technique upon calculation of the perimeter of inscribed polygons of 2^{m+1} sides. Each successive product doubles the number of sides of the polygon starting with a square for $m=1$, and thus approaches the rectification of the circle.

On the other hand, the second major technique, based on calculus methods, uses the inverse trigonometric functions such as the inverse tangent Gregory series together with formulae where the angles involved are small in order to provide fast convergence. Two such cases are the inverse tangent series developed by Machin (1706) that involves the identity

$$\pi = 16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right) \tag{2}$$

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and Lehmer series that use the identity

$$\pi = 88\arctan\left(\frac{1}{28}\right) + 8\arctan\left(\frac{1}{443}\right) - 20\arctan\left(\frac{1}{1393}\right) - 40\arctan\left(\frac{1}{11018}\right) \tag{3}$$

1.1 *Geometric interpretation*

At first sight, these two major techniques have altogether different approaches. However, let us evoke a geometric interpretation of the second procedure that reveals a connection between them. To this end, consider the inverse sine function such as Newton did in order to calculate π

$$\arcsin(z) = \int_0^z \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}(2n+1)} z^{2n+1} \tag{4}$$

$$= z + \frac{z^3}{6} + \frac{3z^5}{40} + \cdots \tag{5}$$

This function represents the length of the arc subtended from 0 to z of a unit radius circle. The integral representation follows from the length formula $l = \int \sqrt{1+f'(x)^2}dx$ applied to the circle equation [2]. The series are obtained from term-to-term integration of the Taylor series expansion of the integrand. However, the order may be inverted obtaining first the Taylor series expansion of the circle equation $f(x) = \sqrt{1-x^2} = 1 - x^2/2 - x^4/8 \cdots$ and then calculating the length. This series approximates the circular arc with even order polynomials as depicted graphically in figure 1. The polynomials shown in the figure have been shifted in the y axis so that their end points are coincident at $(x, y) = \left(1/\sqrt{2}, \pm 1/\sqrt{2}\right)$; i.e. $y_{0\text{th order}} = 1/\sqrt{2}$, $y_{2\text{nd order}} = \left(2\sqrt{2}+1\right)/4 - x^2/2$, $y_{4\text{th order}} = (9+16\sqrt{2})/32 - x^2/2 - x^4/8$. These translations, of course, do not alter the length of the curves. The derivative squared of the circle equation series is $f'(x)^2 = x^2 + x^4 + x^6 + \cdots$;

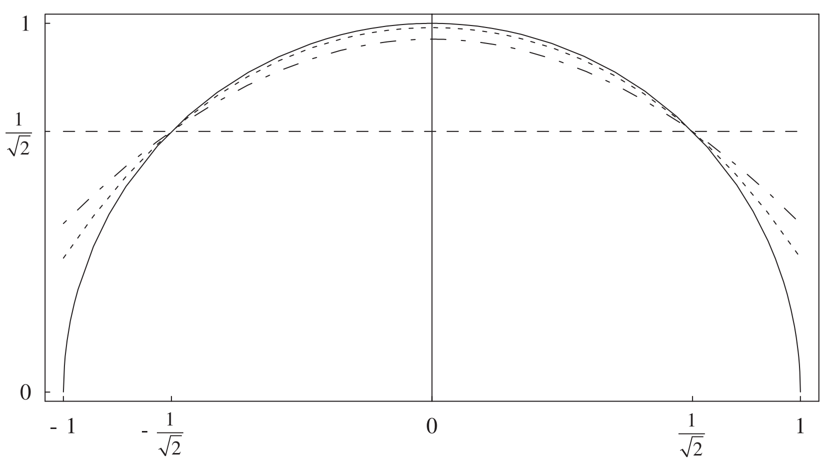

Figure 1. Approximation of the circular arc (——) by a straight line (– – –), a quadratic (- – - – - –) and a quartic (- - -) polynomial.

the length is then obtained from the term to term integration of the Taylor series expansion of the root's argument:

$$l = \int \sqrt{1 + x^2 + x^4 + \cdots}\, \mathrm{d}x = \int \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \cdots\right) \mathrm{d}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad (6)$$

which is of course identical to the inverse sine expansion. Thus, the inverse sine series may be viewed as the length of the $2n$th order polynomial approximation to the circular arc with fixed initial and final positions.

2. Polygon and calculus merged method

It is now plausible to link the two methods since the former evaluates the length of a straight line between two end points, which come closer to each other as the number of polygon sides is doubled. On the other hand, the calculus method approximates the length between two fixed points starting with a straight line and approximating with increasing order polynomial curves that approach the circular arc. The procedure is then to bring the end points closer and at the same time use a polynomial expansion for each segment. A pictorial representation of this method is then similar to taking a figure like a daisy flower and increasing the number of leaves while the shape of each leaf is trimmed closer to that of a circular arc.

In order to blend the two methods, consider the half angle identity written in terms of sine functions $\sin(\phi/2) = 1/2(2 - 2\sqrt{1 - \sin^2\phi})^{1/2}$. Successive application of this formula yields

$$z_m = \sin\left(\frac{\phi}{2^m}\right) = \frac{1}{2}\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + 2\sqrt{1 - \sin^2\phi}}}}_{(m-1)\text{ roots}}} \quad (7)$$

which may also be expressed as an iterative formula $z_m := (1/\sqrt{2})\sqrt{1 - \sqrt{1 - z_{m-1}^2}}$. If we now evaluate the inverse sine of this equation and substitute the series representation for the inverse sine function, we obtain:

$$\phi = 2^m \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}(2n+1)} \left[\frac{1}{2}\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + 2\sqrt{1 - \sin^2\phi}}}}_{(m-1)\text{ roots}}}\right]^{2n+1} \quad (8)$$

and for the particular case of $\phi = \pi/4$,

$$\pi_{m,n} = 2^{m+2} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}(2n+1)} \left[\frac{1}{2}\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{(m-1)\text{ roots}}}\right]^{2n+1} \quad (9)$$

This two parameter formula involves a polygon approximation method through the bisecting procedure and a calculus based method via the inverse sine series. A geometric representation of this poly-calculus algorithm is shown in figure 2, where the circle is being approximated by a rounded polygon; that is, a polygon with curved sides. The angle halving procedure, labelled with subindex m, brings closer the two end points of each segment. On the other hand, a polynomial curve of order

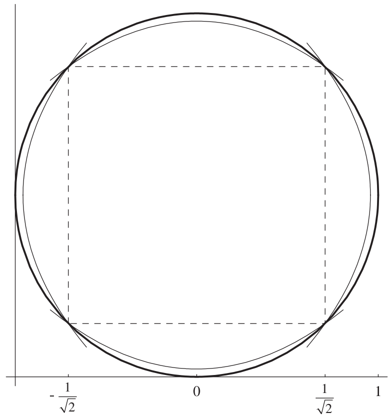

Figure 2. Geometric representation of the poly-calculus method. Example of approximation with four parabolic curves. Circle (———) with an inscribed square (- - - -) and a daisy like figure with four quadratic leaves (———), which circumscribe the square but are inscribed within the circle.

$2n$ rather than just a straight line approximates each segment. 2^{m+2} is the factor that takes into account the total number of segments. Viète's formula is then only the first term of the polynomial expansion, i.e. $n = 0$. Conversely, a variant of Newtonś type formula for π corresponds to the series in equation (8) with $m = 1$ and an initial angle of $\phi = \pi/3$.

Angle division is analogous to Archimedes 3×2^m-gons approximation method since the sine function successive angle bisection corresponds to doubling the number of polygon sides and is thus identical, for 2^m-gons, to Viète's formula. The equivalence of $(m-1)$ nested roots and the inverse of the infinite product with increasing nested roots may be obtained from the derivative of equation (7). For the particular case of $\phi = \pi/4$,

$$\sqrt{2 - \underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{(m-1)\text{ nested roots}}} = \frac{2}{\underbrace{\sqrt{2}\sqrt{2+\sqrt{2}}\cdots\sqrt{2 + \underbrace{\sqrt{2 + \cdots + \sqrt{2}}}_{(m-1)\text{ nested roots}}}}_{m\text{ products of nested roots}}} \tag{10}$$

and thus the inverse of Viète's product of nested roots may be written as a single term with $m-1$ nested roots subtracted from the first nested root.

2.1 *Convergence rate*

The above series converge to the value of π in the limit when either of the parameters tends to infinity, and, of course, when both of them do: $\lim_{m\to\infty}(\pi_{m,n}) = \lim_{n\to\infty}(\pi_{m,n}) = \lim_{m,n\to\infty}(\pi_{m,n}) = \pi_{\text{exact}}$. However, we should expect the convergence to be faster when both parameters are increased simultaneously. For an mth partitioned angle, the order of the nth term is given by

$$O(m, n) = 2^{m+2}\frac{(2n)!}{(n!)^2 2^{2n}(2n+1)}\left[\pi 2^{-(m+2)}\right]^{2n+1} \tag{11}$$

where the nested roots have been estimated to $\pi 2^{-(m+2)}$. Recalling Stirling's formula for large factorials $n! = \sqrt{2\pi n} n^n e^{-n}$, the above expression may be simplified to $O(m,n) = 2^{m+2}[\pi 2^{-(m+2)}]^{2n+1}/[\sqrt{\pi n}(2n+1)]$.

The approximation of π for the nth term is then correct up to the maximum value of the next term $O(m, n+1)$, namely

$$\pi_{\text{exact}} - \pi_{m,n} \approx \frac{2^{m+2}}{\sqrt{(n+1)\pi}(2n+3)}\left[\pi 2^{-(m+2)}\right]^{2n+1} \tag{12}$$

The dominant term for large m and large n is $O(m,n) = 2^{-2nm}$. Allow for the angle fraction m and the number of terms of the series n to be equal. The formula for π is then

$$\pi_{m,m} = 2^{m+2}\sum_{n=0}^{m}\frac{(2n)!}{(n!)^2 2^{4n+1}(2n+1)}\left[\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{(m-1)\text{ roots}}}\right]^{2n+1} \tag{13}$$

and the approximate order of convergence is $O(m,m) = 2^{-2m^2}$. Let us compare this series with the fast converging series developed by Ramanujan (1914) based on modular identities

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(26390n+1103)}{(n!)^4(4^{4n}99^{4n})} \tag{14}$$

and the even faster converging series derived by D.V. and G.V. Chudnovsky [3] (1989):

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!(545140134n+13591409)}{(n!)^4(3n)!(640320^3)^{n+(1/2)}} \tag{15}$$

These series based on the third major approximation technique described in the introduction are plotted in figure 3 for the first 13 terms together with the poly-calculus algorithm given by equation (9) for $m = 28 \Rightarrow \phi = \pi/536870912$. The angle division value in the blended poly-calculus method has been set so that it exhibits a slightly faster convergence rate. Although the terms in the poly-calculus method converge faster, the question of course arises regarding the speed of the calculations

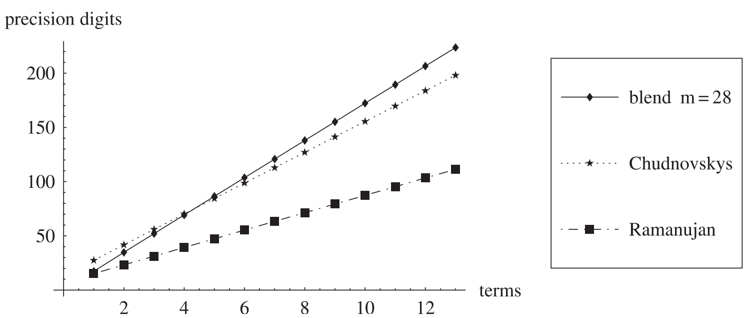

Figure 3. Comparison of convergence rates for the first thirteen terms using: the poly-calculus series with the inverse sine function (◆); Chudnovskys's series (⋆); Ramanujan's series (■).

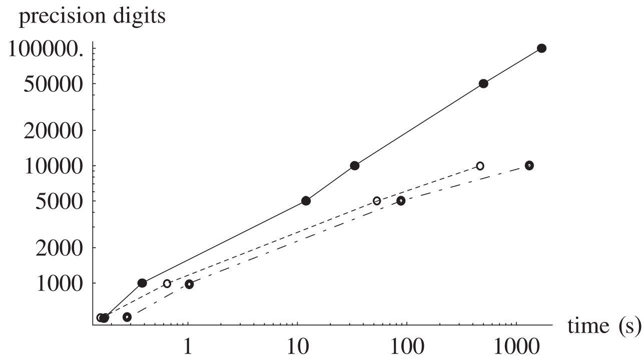

Figure 4. Time required versus precision using: the poly-calculus series derived from equation (13) (——); Chudnovskys's series (- - -); Ramanujan's series (- – – – – –).

in each case. In figure 4, we plot the calculation time versus the precision achieved using very simple non-optimized computer programs written in mathematica code and executed in an old personal computer. 10 000 digits precision is obtained in 1318 seconds using Ramanujan series, 467 seconds using Chudnovskys series and 33 seconds using the poly-calculus series. Precision of 100,000 digits with the poly-calculus method is obtained in 1694 seconds making $m = 410$ in equation (13). It is therefore reasonable to claim, as we stated in the introduction, that very fast converging series may be constructed by fusion of the rectification and calculus methods.

Recent iterative procedures invoking theta functions relationships and their calculation via the arithmetic geometric mean have extraordinarily fast converging rates [4]. The convergence of the poly-calculus series does not compete with these impressive AGM iterative methods.

3. Formulae for π with inverse trigonometric functions and succesive angle division

Let us attempt a more structured discussion of the various possibilities that may be proposed within this framework. First is the selection of the inverse trigonometric function. Second are the choices of the angle division factor and the initial angle.

3.1 *Inverse trigonometric function choice*

The usual option is the inverse tangent function. The half angle identity for the tangent function is $\tan(\theta/2) = (\sqrt{1 - \tan^2\theta} - 1)/\tan\theta$, which may be written as an iteration formula:

$$z_m := \frac{\sqrt{1 - z_{m-1}^2} - 1}{z_{m-1}} \tag{16}$$

Thus the inverse tangent series become a two parameter series

$$\pi_{m,n} = 2^{m+2} \arctan z_m = 2^{m+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z_m^{2n+1} \tag{17}$$

This particular result has been presented as an example of acceleration based on functional equations in order to achieve complexity reduction [5]. For $m = 2$

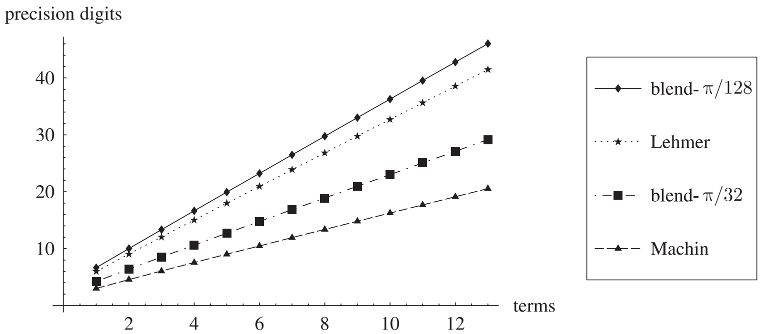

Figure 5. Comparison of convergence rates for the first thirteen terms of the series using: Machin's identity (▲); Lehmer's identity (⋆); and the blended Polygon–Calculus method. The inverse tangent function and angle bisection has been used with $m = 3$ (■) and $m = 6$ (◆).

and initial angle $z_0 = 1 = \tan(\pi/4)$, we obtain $z_2 = \tan(\pi/16) = (\sqrt{4 - 2\sqrt{2}} - 1)/(\sqrt{2} - 1) \approx 1/5$, which yields a series with very similar convergence to Machin series. This is so because the convergence rate is dominated by the largest angle involved in the invoked identities, e.g. 1/5 in Machin or 1/28 in Lehmer. In figure 5, the convergence for the first thirteen terms of these series is shown together with the poly-calculus inverse tangent series (18) that yield faster convergence rates than those exhibited by the Machin (1706) and Lehmer (1938) series [6]. The angle successive bisections for the tangent function yield rather awkward expressions that involve a large number of operations, which slow down the calculations. The sine bisection procedure yields faster calculation times because each iteration involves only the evaluation of one extra root. Although the inverse tangent or sine functions are the usual preferences, the reader may explore many other possibilities. For example, one may consider the *kin* trigonometric function defined as $\text{kin}\,\theta \equiv 1/\sqrt{2}(\cos\theta + \sin\theta)$. This function is of particular relevance in order to obtain a recipe for series representations of π in certain bases such as the BBP formula [7].

3.2 *Angle division procedure*

On the other hand, we may consider rather than bisection, trisection or qth order division of the angle. To this end, it is necessary to expand the trigonometric function in terms of the original angle $f_{\text{trig}}(q\theta) = g(f_{\text{trig}}(\theta))$ through successive angle bisection, which is straightforward. However, it is then necessary to find the inverse relationship $f_{\text{trig}}(\phi/q) = g^{-1}(f_{\text{trig}}(\phi))$, which usually amounts to solving a qth order polynomial and then choosing the appropriate root. Consider, for example, sine trisection $\sin(3\theta) = -4\sin^3\theta + 3\sin\theta$. From the three roots, the one that has a monotonic decreasing real value for the inverse sine function as the angle is trisected is

$$z_{3,m} = 2\text{Re}\left\{\frac{1}{4}\left(1 + i\sqrt{3}\right)\left(-z_{3,m-1} - i\sqrt{1 - z_{3,m-1}^2}\right)^{1/3}\right\} \tag{18}$$

and thus the calculation of π using the inverse sine function and angle trisection is

$$\phi = 3^m \arcsin z_{3,m} = 3^m \left[\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n}(2n+1)} z_{3,m}^{2n+1}\right] \tag{19}$$

There is finally the choice of initial angle. We need to evaluate the trigonometric function π/m_0, which follows a similar method to the angle division procedure. With the appropriate selection it is possible to reduce the polynomial order by one since we may choose the trigonometric function of the largest angle involved to be zero. Consider, for example, a decahedron with $q_0 = \pi/5$. The polynomial $16\sin^5 q_0 - 20\sin^3 q_0 + 5\sin q_0 = \sin(5q_0)$ is reduced to a fourth order polynomial and the sine of the initial angle readily calculated as $z_0 = \sin(\pi/5) = \sqrt{5-\sqrt{5}}/\left(2\sqrt{2}\right)$. The initial angle, as expected, has little impact in the calculation when a large number of angle divisions or series terms are evaluated.

4. Concluding remarks

Let us abridge the general poly-calculus formula as

$$\pi_{m,n}\left(f_{\text{trig}}, q, q_0\right) = q_0 q^m f_{\text{trig}}^{-1} z_{q,m} = q_0 q^m \left[\sum_{n=0}^{\infty} g_n\left(z_{q,m}\right)\right] \tag{20}$$

with $z_{q,m} := p_q\left(z_{q,m-1}\right)$; where $g_n\left(z_{q,m}\right)$ is the nth inverse trigonometric series term, q is the angle partition factor, q_0 is the initial angle and p_q is the function that relates the trigonometric function of the angle ϕ_{m-1} with ϕ_m where $\phi_m = \phi_{m-1}/q$. There are three degrees of freedom to select from in order to calculate this formula; namely, the choice of trigonometric function, the partition angle and the initial angle. These formulae allow the student or programmer to find his own particular approximation series. The series with fastest convergence rate for a given number of terms are given by $\pi_{m,m}$ in equation (13). The straightforward geometric interpretation as a daisy like figure or a polygon with rounded sides, that arises from the confluence of the polygon and calculus method, gives an attractive and clear insight of the procedure that is being invoked. These algorithms may prove useful for the generation of random numbers or encrypting processes with considerable speed.

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