Pre-Calculus Notes

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Fun Stuff

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Course Introduction

1. Syllabus highlights
   (a) Grades:
      i. Know the expectation / what you are getting into.
      ii. 15perc A (excellent), 35perc B (good), 35perc C (satisfactory),10perc D (passing), some F (failing)

    iii. Expect lower grades than you are used to. I was a student once upon a time. I know what it's like to give some effort in a class and still get an A/B. Night before study, good enough?
    iv. Turn in an exam / project. Did you do good work?
    v. Many will start off doing good / satisfactory work. Improve to something more. C is not the worst thing in existence. These letters say nothing of your capability.
  (b) What does good mean? Good means good. Good job! Excellent means you showed some flair.
  (c) Expect: More work, more expectation on good writing.
  (d) Math is a challenging subject. Not a natural thing to think or write in. It takes work and practice to be better. My goal is to train you to be better and give you ideas of where it can go.
  (e) Fact that you are here shows you are smart and capable. Your goal should be to improve.
  (f) Why do I do this? I do it out of respect for you. You are smart enough. I want you to gain something valuable here. I wouldn't do this job if I didn't think you were gaining something of value.

2. Outline of this class
  (a) Most important idea: Function
  (b) How to reverse a function? Inverse function. Key example: Exponential and logarithms.
  (c) Trigonometry.
    i. Triangle, unit circle, graphs.
    ii. Go deep with all connections. More than you could ever want.
    iii. Applications. Distance, waves, circular motion. Universal ideas here: sound, light, vibration, rotation of earth, atomic vibration, so much.
  (d) Vectors and high dimensional space. Trigonometry is the foundation.

3. Grand scheme of things. Where does this class sit inside all of mathematics.
  (a) Basics: Algebra, arithmetic.
  (b) First steps: Geometry, functions. (us now)
  (c) Calculus: Math of change / infinity.
  (d) Linear algebra: Math of vectors. Anything with finite representation. Invention of computers fueled this one. Gateway to real math / applications.
  (e) Applied math. Any application you want. Physics, finance, marketing, material science, CFD, sports.
  (f) Abstract math. Create your own world of ideas. Number theory, analysis, algebra, topology, more.

Chapter 4 Exponential and Logarithmic Functions

.1 4.0 Review: Functions and Inverse functions, Chapter 2 in text

1. Function (most important idea of this class)
  (a) Basic idea of function
    i. A function is a machine which takes an input and produces at most one output.
    ii. What is the domain / range of each? Key is that every input corresponds to at most one output.
    iii. Direction: input $\to$ output (card to bank account, bank account to card, are they the same association? Are they the same function?)

iv. Restriction: When will things go wrong? One input to two output.
v. Domain: the collection of input
vi. Range: the collection of output
vii. Restricted domain: the range is automatically determined. Apply to one above example.

(b) Mathematical setting

i. Numbers and variable (rather than any object)
ii. Formal definition and percision of math language: A function f from set X to Y, written $f : X \to Y$ is a relation which associates each element of X with *exactly* one element in Y.
iii. Notation: $y = f(x) = x^2, r = g(t)$, $f(-2) = 4$
iv. Domain: all possible x
v. Range: all possible y (outputs)
vi. Representations: Words (not common), formula, graph

(c) **Example:** Graph $f(x) = -2x + 1$ and $g(x) = x^2 - 1$, list domain and range.

(d) Remarks

i. How to tell if equation is of a function? Graph and vertical line test. Are each a function of x? $y = x^2, x = y^2$.
ii. To define a function, we need to
- Indicate the rule (formula)
- State the domain (important)

iii. If not specified, the domain is all the possible values
- Typical for even radicals, zero division
- All real numbers $(-\infty, \infty)$

(e) Interval notation

i. (a, b): all the numbers between a and b
ii. (,), exclude, [,] include
iii. (a, ∞) all the numbers that are bigger than a
iv. $(-\infty, b)$ all the numbers that are less than b

(f) **Student Example:** Find the domain of each. Write in interval notation:

$$f(x) = 2 - \sqrt{1 + 2x}, \quad g(x) = \sqrt{x - 3} - \frac{1}{\sqrt{x + 2}}, \quad h(x) = \frac{1}{2x^2 + 5x + 2}$$

(g) Other function ideas

i. Major types of functions: Constants, lines, quadratics, polynomials, rationals, roots, exponential, log, trig, list goest on.
ii. Symmetry: Even ($f(-x) = f(x)$ all x) / odd ($f(-x) = -f(x)$ all x) functions.
iii. Transformations: Shifts, stretches, reflections
iv. Combinations: Sum, diff, prod, quotient, composition

2. Inverse functions: Same association, the opposite direction. Reverse of a function.

(a) Big questions:
- How to tell if a function is invertible?
- If f is invertible, how to find it?
- What is the relationship between a function and its inverse? (Domain/range, graph, etc)

(b) New mapping, given the output, find the input. Output $\to$ input. Once you know one direction, and you know it is invertible, you should know both directions.

(c) **Student Examples:** $f(x) = -2x+1$ and $g(x) = x^2 - 1$. Given an output, which are invertible and why?

- One to one function: two distinct input cannot have the same output in original function f
- Horizontal line test on graph

(d) Inverse function definition and notation: For f one-to-one, the inverse of f, written f^{-1} is the association which maps outputs of f to corresponding inputs. That is,

$$y = f(x) \quad \Longleftrightarrow f^{-1}(y) = x$$

Note: There exists notational confusion.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

(e) How to find the inverse function f^{-1} of a given function f?

- Check if it is one-to-one (HLT)
- y is known, so find x. Solve the equation $y = f(x)$ for x.
- Write it in the standard way
- Put the domain if necessary

(f) **Example:** Find the inverse function of $f(x) = -2x+1$ (when is the output 1, -4, y?), graph f and f^{-1}.

- Relation between graphs of f and f^{-1}: reflection with respect to straight line $y = x$ (give examples)
- Repeat with $g(x) = x^2 - 1$, restricted domain $x > 0$. Why / where is the restriction needed?

(g) **Student Example:** A bit more challenging, find inverse of $f(x) = \frac{x+2}{2x-1}$, list domain and range.

(h) Fact

$$\text{range of } f = \text{domain of } f^{-1}$$
$$\text{domain of } f = \text{range of } f^{-1}$$

(i) Verifying inverse function of each other, refresh function composition: $(f \circ g)(x) = f(g(x))$.

$$f(g(x)) = x \quad \text{for any x in the domain of g}$$

or

$$g(f(x)) = x \quad \text{for any x in the domain of f}$$

Example: are they inverse function to each other: $f(x) = \frac{x+2}{2x-1}$, f^{-1} as above

.2 4.1 Exponential functions

1. Motivation: Compound interest example. Invest 1000 dollars for a year at 5 percent interest compounded monthly. How much do you earn?

   (a) Quick example

   (b) General formula and explanation of each variable

   $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

   (c) Applied problem to find the amount given principal, compounding period, and rate.

2. Basic: Review laws of exponents! Refresher examples.

(a) LoE: $a^0, a^1, a^m a^n, a^m/a^n, a^n b^n, (a/b)^n, a^{-n}$.

(b) **Student Examples**: Simplify $\dfrac{\sqrt[3]{ab}\cdot b^2}{a^3\cdot b^{1/2}}$; $(-27)^{2/3}(4)^{-5/2}$; $\left(\dfrac{2x^{2/3}}{y^{1/2}}\right)\left(\dfrac{3x^{-5/6}}{y^{1/3}}\right)$

(c) What exponent means: $2^3, 2^{-1}, 2^{1/2}, 2^{-4/3}$, good for any rational number, $2^\pi, 2^i$ needs calculus, but we have faith..

(d) Solving exponential equations

- **Student Examples**: Solve for x: $2^{-x} = 8$; $8^{2x} = \dfrac{1}{2^{2-x}}$; $3(3^x) + 9(3^{-x}) = 28$ (rewrite as same base and hidden quadratic)

3. Exponential function: $f(x) = a^x$

(a) Definition: why $a > 0$ and $a \neq 1$

(b) Graphs

- Concrete examples: $f(x) = 2^x, 5^x, (1/3)^x = 3^{-x}$
- Domain and range
- $a^0 = 1$ gives the same y-intercept
- Increasing ($a > 1$) / decreasing ($a < 1$)
- Shape: depends on the a
- Horizontal Asymptote always at the x-axis
- Note they are all one-to-one, so we should be able to invert them

4. Reading exponential function

- Comparing base
- General format: $b \cdot a^x$
- Identify graphs with points and shift

5. Intuition / examples:

- Exponential function grows fast (mark pen example)
- Application: Student loan interest calculation, mortgage payment calculator.

.3 4.2 The Natural exponential functions

1. Motivation: Need for a single, uniform base.

- Which one is bigger? (3^4 or 4^3)
- The idea of a uniform base(base is not unique 2^{3x}, 4^x)

2. The natural base e

(a) Rather than lots of bases a, we would like a uniform base with nice properties (the natural exponential). Called natural since it shows up in interesting way (instantaneous, large populations and reproduction, many times, many things, life isn't always discrete).

(b) Continuous compound interest:

- Invest \$1000 at 5% per year.

$$1000 + (0.05)1000 = 1050$$

- Same, twice a year, $\frac{5\%}{2}$ each time.

$$1000 + (0.025)1000 + (0.025)(1000 + (0.025)1000) = 1000(1 + 0.05/2)^2 = 1050.625$$

- Quarterly, $\frac{5\%}{4}$ each time.
$$1000(1 + 0.05/4)^4 = 1050.945$$
- Daily: 1051.267 (let students choose and guess here, per day second etc)
- This seems to approach a limit / max.
- Desmos: $(1 + \frac{0.05}{n})^{n/0.05}$.

(c) Fact: modify above desmos, sort of growth rate 1.
$$(1 + \frac{1}{n})^n \to e, \quad \text{when } n \to \infty$$
where $e \approx 2.72$, Euler's number. Can show e is irrational as important as π, if not more. Shows up in applications all the time.

(d) The natural exponential function f
$$f(x) = e^x$$

3. Law of continuous growth formula
$$q = q_0 e^{rt}$$
   - q_0: initial quantity
   - r: the growth rate
   - t: time
   - e: natural base

   (a) Note:
      i. $r > 0$: growth rate
      ii. $r < 0$: decay rate
      iii. r is better in terms of identifying the increasing and decreasing rate, no longer have cases with the base
      iv. "real" base: e^r

   (b) Continuous compound interest.

   (c) When to apply:
      i. grows/decays proportional to its current value
      ii. continuously (instantaneously) changing

   (d) Uniform base: transform $y = ae^{kt}$ to ab^t (still need logs to get here)

4. Applications
   - Continuous compound interest: $A = Pe^{rt}$
   - Population growth: $P = P_0 e^{rt}$
   - Radioactive decay (half life)
   - Anything that grow/decays at a percentage
   - How to understand continuous (not all the time, but can happen any time)
   - `https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&idim=state:06000:48000&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=state:06000:48000:12000&ifdim=country&hl=en_US&dl=en&ind=false`

.4 4.3-4.4 Logarithmic functions and log properties

1. Basics

   (a) Finding growth rate involves finding an input corresponding to a known output. The inverse of exponential function (all one-to-one here).

   (b) Graph $f(x) = a^x$ for $a > 1$ and $0 < a < 1$, automatically can draw f^{-1}. Name $f^{-1}(x) = \log_a(x)$.

   (c) Careful definition of logarithm (defined to be inverse).

   $$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

   (d) The log as a function:

      i. Domain, range

      ii. Special point (1,0)

      iii. Special bases

      iv. Function composition of a^x and $\log_a(x)$.

         A. The logarithmic function with natural base: $\ln x$

         B. The common logarithmic function: $y = \log x$.

   (e) **Examples:**

      i. Compute $\log(1/100), \log_4(2), \log_5(1/5), \log^3(1), \log_8(4), \log_9(\sqrt{3})$ (easier to look at exponential form)

      ii. Solve for x: $\log_3(x+4) = \log_3(1-x)$ (one-to-one), $e^{2\ln(x)} = 9$ (inverses and domain restriction).

      iii. Find the domain and range: $\ln(\ln x)$.

2. Applications:

   (a) Originally for hand calculation because of log properties below. (Napier, slide rule, revolution of calculation)

   (b) Astronomical distance `https://en.wikipedia.org/wiki/Astronomical_system_of_units`

   (c) The Benford's law (first digit law) `https://en.wikipedia.org/wiki/Benford%27s_law`

   (d) Logarithmic transformation in data science: `https://en.wikipedia.org/wiki/Data_transformation_(statistics)`

   (e) Nature: `https://en.wikipedia.org/wiki/Logarithmic_spiral`

   (f) Solve exponential equation: $23x = 10$, $e^{2x} - 3e^x + 2 = 0$

3. Log properties:

   (a) $\log_a(xy)$, $\log_a(x/y)$

   (b) $\log_a(x^p)$

   (c) $\log_a x = \frac{\log_b(x)}{\log_b(x)}$ change of base

   (d) $a^{\log_a x} = x, \quad \log_a a^x = x$ inverse relation

   (e) These are just the laws of exponents written in logarithmic form. Write a^{s+t}, a^{st}, a^{-s} and draw parallels.

      - Prod to sum: Let $\log_a(x) = s, \log_a(y) = t$, then $a^s = x, a^t = y$.
      - $xy = a^s a^t = a^{x+t}$, rewrite in log form
      - $\log_a(xy) = s + t = \log_a(x) + \log_a(y)$
      - Rest are same idea.

   (f) As mentioned before, make calculation easier (product to sum, power to product, etc).

4. Typical problems
   (a) Express $\log_a \frac{x^3\sqrt{y}}{z^2}$ in terms of $\log x$, $\log y$, $\log z$
      i. Split $\cdot$ and $/$
      ii. Bring down the power
   (b) Express as one logarithm, opposite direction: $\ln(x+1) - 2\ln(x-1) + 5\ln(x)$
   (c) Why are we doing this? Solving equations? Solve real life problem.
      i. The population of La Crosse 50000 in 2000, 55000 in 2010, what will it be in 2020 assuming continuous growth?
      ii. Which would you choose and why? Invest \$100 at 4% or \$500 at 3%? When do they equal? Depends on length of investment.
      iii. Google population of Florida, Cali, and Texas. Which is growing faster? Let them guess and explain why. Care about growth rate here, use log plot instead. Care about slope of this new line. Not a realistic fit globally though! Population of sad North Dakota

$$y = Pe^{rt},\ \ln(y) = \ln(P) + rt,\ z = c + rt$$

      `https://www.google.com/publicdata/directory` Possible project here, fit exponential, logistic growth, etc

5. Solving equations examples, these main ideas are all there is.
   (a) $8^{2x}(\frac{1}{4})^{x-2} = 4^{-x}$. Rewrite in same base.
   (b) $2^x = 3^{1-x}$, cannot rewrite in same base, use logarithm of any base. Many equivalent but different looking solutions. Nice bases to choose are 2,3.
   (c) $\log_3(-x) + \log_3(8-x) = 2$. Beware of domain changes. Always need to check solution. Only $x = -1$ works here.
   (d) Hidden quadratic: $e^{2x} - e^x - 2 = 0$

6. Transfer anything to base e: y = 2^x
   (a) Connection between continuous and discrete cases
   (b) Everything is continuous
   (c) One formula but restrict your x to be integer.
   (d) Groupwork handout, treat as take home quiz.
      i. Tips:
      ii. Remove the log
      iii. Check the domain

.5 4.5 Exponential and logarithmic equations

Group work take home quiz!

.6 4.6 Modeling with exponential functions

Already discussed all this, just remind and formalize. New idea is that continuous growth assumes a population (or quantity) grows at a rate proportional to its size. Calculusness.

1. Population growth:
   (a) "Scheduled" growth: $P(t) = P_0 b^t$ (doubling time a gives $P(t) = P_0 2^{t/a}$)
   (b) Continuous growth: $P(t) = P_0 e^{rt}$ where r is the relative growth rate (La Crosse example done)

2. Radioactive decay of mass: $m(t) = m_0 e^{-rt}$.
3. Newton's law of cooling: $T(t) = T_s + T_0 e^{-kt}$.
4. Examples from text: 11 (population growth), 20 (radioactive decay), 27 (newton's law of cooling)

Chapter 5 Trigonometric functions: Right triangle approach

- Does everybody know trigonometry? Definition discussion on day one, math of position, relate xy to angle and magnitude
- Trigonometry: greek words, trigonon(triangle) - metria (measurement)
- Why is it useful? GPS, astronomy, surveying, music theory, acoustics, electronics, position and waves, interpret any signal as sine/cosine (Fourier series/transform).
- Purpose of learning, physics, martial arts, example?

.1 5.1 Angle measures

1. Goal: Carefully define the idea of an angle (two units, applications too)
2. Definition of angle:
   (a) Geometry: one vertex, two rays
   (b) Notation: $\angle\alpha$, $\angle AOB$
   (c) Initial side: OA, terminal side: OB
   (d) Unit: degree (360° for a whole circle. Why? Calendar, time, and nice divisions of 360.)
   (e) Angles bigger than 360 degree: draw some examples
      - Show it by rotating arrow
      - Coterminal angles
      - Positive (counter-clockwise), negative (clockwise)
3. How to compare two angles? Put into standard position (put angle into coordinate systems).
   (a) How:
      - Vertex: origin (0,0)
      - Initial side: positive x-axis
      - 4 quadrants
   (b) Positive/negative angle: give an example of angle AOB and BOA
      - Positive: counterclockwise
      - Negative: clockwise
   (c) Notation: greek symbols without angle symbol
   (d) Draw a standard angle: give examples
      - Terminal side
      - Rotating arrow
   (e) Which quadrant the terminal side lies in?

   $$\theta = 60^\circ, \quad \theta = -135^\circ, \quad \theta = 1200^\circ, \quad -700^\circ$$

   (f) Adding and subtracting angle: transfer geometry into algebra
   (g) Coterminal angles α, β: $\alpha - \beta$ is a multiple of 360° (the senator of all angles)

(h) All the coterminal angles with θ: $\theta + 2n\pi$, where n can be any integer (including negative)

4. Angle terminology:
   (a) Right angle: $90°$, perpendicular
   (b) Acute angle: $0 < \theta < 90°$
   (c) Obtuse angle: $90° < \theta < 180°$
   (d) Complementary angles α, β: $\alpha + \beta = 90°$
   (e) Supplementary angles α, β: $\alpha + \beta = 180°$
   (f) **Example:** What's the complementary angle of $32°$? Supplementary?

5. Fractional angles:
   (a) Can use decimal representation: $90.5°$.
   (b) Can also use degrees/minutes/seconds: $90°30'0''$
   (c) Can convert easily, see text for more.

$$31°41'59'' = 31° + \left(41 + \frac{59}{60}\right)' = \left(31 + \frac{41}{60} + \frac{59}{360}\right)^{\circ} = 31.847...°$$

6. Radian measure:
   (a) When dealing with circles, new termonology for angle measure: radians.
   (b) Draw circle, center C, points A, B, radius r, central angle θ, arc(AB)
   (c) One *radian* is the measure of central angle θ after traveling one arc of length the radius. Draw!
   (d) How many radians per revolution? A bit more than 6.

$$360° = 2\pi \text{ radians } = 2\pi(\text{no label here, only label degrees})$$

   (e) Important angles to *not memorize* $0°, 30°, 45°, 60°, 90°, 180°, 360°$ convert to radians.
   (f) Let two people compete to draw angles in degree and radians
   (g) Relation between degree and radius
      i. $360° = 2\pi$ rad or $180° = \pi$ rad
      ii. degree to radians: $\theta° = \frac{\pi}{180}\theta$ radians
      iii. radians to degree: θ radians $= \frac{180}{\pi}\theta°$
      iv. **Example:** Convert each: $225°$, $\frac{11\pi}{6}$, $\pi°$, beware of labels!

7. Circumference and arc length formulas are easy in radians. Draw picture.
   - A circle of radius r has circumference 2π.
   - The length of arc s with central angle θ is θr.
   - Note, default angle unit is radians here! Degrees differ, especially with arc length. Need to convert.

8. The sector area formula. Again, no need to memorize. Instead, know the idea.
   (a) A circle of radius r has area πr^2.
   (b) The area of sector t with central angle θ is

$$\pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}\theta r^2$$

(c) **Example:** which one you would buy? A half 9" pizza or a slice from a 20" pizza

9. Angular and linear speed, allow change of these over time.
   (a) If angle changes, point P moves around the circle. These are connected, how?

$$s = \theta r \quad \Rightarrow \quad \frac{s}{t} = r\left(\frac{\theta}{t}\right), \quad \text{(linear speed} = r\cdot \text{(angular speed))}$$

   (b) A clock with a 3ft long minute hand and an 2ft long hour hand. What are the angular/linear speed for both hands?
   (c) Why do we care? Wind generators, clock gears, bike gears, body joint movement, astronomy etc
   (d) RMP: 55pmh on 28 inch diameter tires. What is the angular speed in RPMs? Unit conversions!

$$\frac{s}{t} = 55\frac{mi}{h} = 55\frac{mi}{h}\frac{1hr}{60min}\frac{5280ft}{1mi}\frac{12in}{1ft} = \frac{55\cdot 5280\cdot 12}{60}\frac{in}{min} = 14\left(\text{angular speed in}\frac{rads}{min}\right)$$

$$\text{angular speed} = \frac{55\cdot 5280\cdot 12}{60\cdot 14}\frac{rads}{min} = \frac{55\cdot 5280\cdot 12}{60\cdot 14}\frac{rads}{min}\frac{1rev}{2\pi rads} = 660.265RMP$$

   (e) Bigger tire = gas saver?
   (f) 7000 RMP, max of 120 mpg, what is the tire diameter?

.2 5.2 Trigonometry of right triangles

1. We have angles, now let's return to triangles. How are the 3 sides related to an interior angle? (draw a right triangle, identify the adjacent, opposite and hypotenuse sides)
   (a) Definition of all the six trig functions
      i. **Example:** If $\csc(\theta) = \frac{\sqrt{13}}{3}$ for θ acute, find $\tan(\theta)$. Hint by drawing triangle, need hypotenuse.
      ii. Pythagorean Theorem is needed (write down, why? picture proof, amazing)
   (b) Trigonometric values have nothing to do with the size of the triangle, show similar triangles
   (c) These 6 are functions of θ where
      i. Input is naturally in radians (reason to come)
      ii. Domain main 3 (for now acute angles, but will push ahead later)
      iii. Range main 3 (for acute angles)
      iv. Important interval: $0 < \sin(\theta) < 1$, $0 < \cos(\theta) < 1$
      v. Random triangles to motivate above
2. Identities you need to know
   (a) Pythagorean theorem
   (b) Reciprocal identities: $\sin(\theta) = 1/\csc(\theta)$... etc (6 in total)
   (c) The tangent and cotangent identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

   (d) The Pythagorean identities (super important), show first via general triangle then rest come from first by division.

$$\sin^2(\theta) + \cos^2(\theta) = 1, \quad \tan^2(\theta) + 1 = \sec^2(\theta), \quad \cot^2(\theta) + 1 = \csc^2(\theta)$$

3. How to use the identities

(a) Special right triangles 30°/60° (comes from equilateral triangle), 45° (comes from two sides the same). If other, use a calculator....for now.

(b) Nice Pythagorean triples (3,4,5),(5,12,13), (7, 24, 25), etc

(c) Knowing one trig function, can find all the trig values for acute θ, more angles later.

(d) Solving right triangles (note that if you just have the trig info, the triangle is not unique, similar triangles)

- **Example:** $b = 6$, $\sin(\theta) = \frac{3}{5}$, find a, c
- **Example:** $\tan(\theta) = \sqrt{3}/3$, $c = 5$, find everything else including complimentary angle.
- Similar triangles show up.

(e) Word problem fun: Section 5.2, 61. Highlight the overall strategy without being given a picture. Focus is turning description into math / triangles.

.3 5.3 Trigonometric functions of angles

1. We have angles bigger than acute angles. How to trig those?

   (a) Draw θ be a angle in standard position and $P(x, y)$ be any point on the terminal side of θ in the first quadrant.

   (b) $\theta = 0°$, $\theta = 90°$, move the terminal side approaching the x-axis, y-axis, what do each approach? So our ration definitions still make sense for these flattened triangles!

   (c) For other quadrants, (draw for quadrant 2), need to be ok with generalizing to triangles with negative sides. Always draw *reference triangle* with acute *reference angle* to keep us sane.

   - Examples: Compute each $\cos(240 degrees), \csc(495 degrees), \cot(16\pi/3), \sec(-13\pi/4)$

   (d) Complete trig function definitions: For angle θ in standard position and (x, y) a point on its terminal side, if the hypotenuse is $r = \sqrt{x^2 + y^2}$, then

   $$\sin\theta = \frac{y}{r}, \quad \text{etc, draw picture}$$

   i. Domain and range of each:

   ii. Trigonometric values can be negative, which quadrants does each occur (draw 4 quadrants, have them fill in signs of six trig fcns), also list where undefined on xy axis.

2. **Examples:**

   (a) If $P(3, -4)$ is on the terminal side of θ, find all the trig values of θ. Find the quadrant containing θ. Draw a reference triangle with reference angle θ_R to make discussion easier.

   (b) If $\sin\theta = 3/5$, find $\tan(\theta)$? 2 cases here, note the nice symmetry (reminds of even functions, hint to later.). If $\frac{\pi}{2} \leq \theta \leq \pi$, find all the trig values of θ. Again draw reference triangle.

3. Formula for area of a triangle: $A = \frac{1}{2}ab\sin(\theta)$ where θ is between a and b. Draw a picture. Why is this true?

4. **Example:** Excel spreadsheet challenge! Give them sine/cosine values for 0,1,2,...,,45 degrees. Randomly generate 6ish angles between -1000 and 1000 degrees, one for each trig function. How many can they get in 10 minutes?

   - Note: Need cofunction ideas here. Give example with cos(61) first.

.4 5.4 Inverse Trigonometric Functions and Right Triangles

1. Example: Find ALL θ such that $\sin(\theta) = -\frac{1}{2}$. Stumble across symmetry and periodicity.

2. Fill out all 4 quadrants with the nice angles (0, 30, 45, 60, 90) and reflections of these.

3. Motivation: $\sin(x) = \frac{1}{3}$
   (a) Our calculator can solve equations like this. What is really going on?
   (b) Draw the unit circle and show sticking point. Not a nice angle.

4. Inverse function review:
   (a) Inverse relationship, expression, domain and range.
   (b) Are all functions invertible? Need one-to-one. Reversing may not be unique.
   (c) Even though $f(x) = x^2$ is not invertible, we can still solve equations like $x^2 = 25$. Key is domain restriction to make invertible, then use special features of the function. This is key idea for inverse trig, and we already do this to solve equations.

5. Inverse sine:
   (a) Draw unit circle to show not one-to-one. Already know this because of symmetry / periodicity.
   (b) How to restrict the domain so take on each output in $[-1, 1]$ only once? Let them decide. Many ways, which is best?
   (c) Restricted sine: $y = \sin(x), \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. All outputs for sine $[-1, 1]$ are realized only once.
   (d) Draw unit circle for restricted sine.
   (e) Def: $y = \sin(x) \Leftrightarrow \arcsin(y) = x$
   (f) Notation: $\arcsin(x) = \sin^{-1}(x)$, but we like arcsin to avoid exponent confusion.
   (g) Domain restriction is confusing: $y = \arcsin(x)$ is the inverse function of RESTRICTED sine.

6. **Example:** Solve $\sin(x) = \frac{1}{3}$ via arcsin on $[0, \pi/2], [0, \pi]$, everywhere.
   (a) Draw unit circle, $x = \arcsin(1/3)$ gives solution on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
   (b) Get the second solution on the unit circle, then periodicity.
   (c) Issue: We can write down the solution, but what is $\arcsin(1/3)$?
   (d) This issue is analogous to logarithms. $2^x = 10$ gives $x = \log_2(10)$ which is not nice. Calculus gives logarithnmic tables and slide rules for this. Inverse trig has similar story.
   (e) Have them repeat for $\cos(x) = \frac{1}{3}$.

7. What about the rest? Have them fill in a table.
   (a) $\sin, \cos, \tan$
   (b) Invertible? Yes or no.
   (c) Restricted domain
   (d) Range
   (e) Part of unit circle
   (f) What about the buddy functions? $(\sec, \csc, \cot)$

8. Inverse relations:
   (a) $\sin(\arcsin(y)) = y, \quad -1 \leq y \leq 1$
   (b) $\arcsin(\sin(\theta)) = \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
   (c) Note the domain restriction subtlety.

(d) What about $\cos, \tan$?

(e) **Examples:**

i. $\sin(\arcsin(-1/2)) = -1/2$

ii. $\arcsin(\sin(\frac{5\pi}{3})) =?$ Cannot use the identity since not in restricted domain. Subtract 2π via periodicity and life is grand.

9. **Student Examples :**

(a) $\arccos(-1/2)$ (Call angle θ, then travel through the inverse relation. Unit circle and restricted domain and reference triangles.)

(b) $\tan(\arctan(12))$

(c) $\arcsin(\cos(\pi/6))$

(d) $\arcsin(\sin(5\pi/6))$

(e) $\cos(\text{arcsec}(2))$

(f) $\sin(\arccos(3/5))$

10. Verifying trigonometric identities

(a) $\cos(\arcsin(x)) = \sqrt{1-x^2}$ (Weird? Or not?)

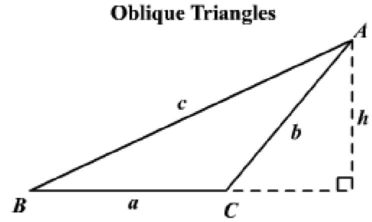
.5 5.5 Law of Sines

1. The primary goal of the next 2 sections is to extend trigonomety beyond right triangles.

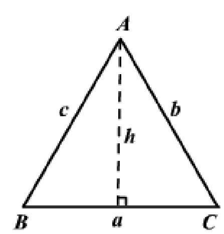
(a) An oblique triangle is a triangle with no right angles.

(b) Can still make use of right triangle trig. Use top picture.

Oblique Triangles



C is Obtuse



C is Acute

(c) How are A, B, C, a, b, c related? Create a right triangle.

$$\sin(B) = \frac{h}{c}, \quad \sin(C) = \cos(\frac{\pi}{2} - C) = \frac{h}{b}$$

$$h = c\sin(B) = b\sin(C) \quad \Rightarrow \quad \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

2. The law of sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

(a) Note, this is actually 3 equations.

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}, \quad \frac{\sin(A)}{a} == \frac{\sin(C)}{c}, \quad \frac{\sin(B)}{b} == \frac{\sin(C)}{c}$$

(b) Also, geometry gives a fourth: $A + B + C = \pi$.

3. How much information (a, b, c, A, B, C) is needed to solve a general triangle?
   (a) For right triangles, we need: (a) 2 sides or (b) 1 angle/1 side. Why does 2 angles not work? Note, we actually given 3 things here (right angle).
   (b) For general oblique triangle, need 3 things (not all angles). What are all the cases? Write all out. Which can we use law of sines on?
   - Case 1: ASA/AAS (LoSines)
   - Case 2: SSA (LoSines)
   - Case 3: SAS (LoCosines) Why can't we handle these?
   - Case 4: SSS (LoCosines) Why can't we handle these?
   - Case 5: AAA (hopeless for uniqueness by similar triangles)
4. **Examples:**
   (a) Solve the triangle with $A = 40°, B = 60°, a = 4$. What case is this? ASA. Can we use LoSines? Yes.
      i. $C = 80°, b = \frac{4\sin(60}{\sin(40)}, c = \frac{4\sin(80)}{\sin(40)}$
   (b) Solve the triangle with $A = 35°, a = 6, b = 8$. What case is this? SSA. Can we use LoSines? Yes, but...difficulty.
      i. Find B first. $\sin(B) = \frac{4\sin(35)}{3} \approx 0.76$. 2 solutions here! Does this make sense? Yes, swing the leg and see two possible triangles.
      ii. Case 1: $B < 90$, $C = 95.1, c = \frac{6\sin(95.1)}{\sin(35)}$
      iii. Case 2: $B > 90$, $C = 14.9, c = \frac{6\sin(14.9)}{\sin(35)}$
      iv. This is why SSA is in its own class. Does it always yeild 2 solutions? Think leg swinging....could be one unique solution ($\sin(B) = 1$, right triangle) or no solution ($\sin(B) > 1$, too short).
5. **Examples:** The real purpose? Applications. More in take home quiz.
   (a) pg 481, 36,
   (b) pg 481, 37, $\quad h = 175$ feet
   (c) pg 481, 40, $\quad$?? degrees.
   (d) Skip above applied problems, give take home quiz instead after covering Law of Cosines.

.6 5.6 The law of cosines

1. Two remaining cases leftover from Law of Sines:
   (a) SAS
   (b) SSS
2. The Law of Cosines (generalized Pythagorean theorem, all one formula by symmetry)
   (a) $a^2 = b^2 + c^2 - 2bc\cos(A)$
   (b) $b^2 = a^2 + c^2 - 2ac\cos(B)$

(c) $c^2 = b^2 + a^2 - 2ab\cos(C)$

Note: No ambiguous case possible for Law of Cosines.

3. **Example**: Solve the triangle with $c = 45°, A = 15, B = 10$. What case? Why can we not use Law of Sines at first? Use LoS as a second step and take car of multiple possibilities despite there being only a single solution.
4. Heron's formula
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$
where $s = \frac{1}{2}(a+b+c)$
   (a) Demonstrate on 3,4,5 right triangle and compare to usual area formula.
   (b) Project 5

Chapter 6: Trig functions and the unit circle approach

Chapter 5 is about geometry / triangles. Here we extend trig to further ideas (general functions, rotations, etc).

.1 6.1 The unit circle

1. Noted last time that the size of the triangle (similar triangles) yields the same trig values. Let's fix $r = 1$ to make life nicer. Draw in xy-plane, this draws the unit circle as θ varies.
   (a) Suppose θ is a real number and $P(x, y)$ is the point on the unit circle that corresponds to θ, then
   $$\sin(\theta) = y, \quad \cos(\theta) = x, \ldots$$
   Note the book uses t as a parameter for time.
   (b) Note (x, y) here, new coordinate system (polar coordinates). More later.
   (c) What is the equation of a circle? $x^2 + y^2 = 1$. Why? Distance formula / Pythagorean theorem, all the same. Note again $\sin^2(x) + \cos^2(x) = 1$.
2. How to tell if a point is on the unit circle?
   (a) Is $(1/2, 1/2)$ on the unit circle? No.
   (b) If $y = \frac{1}{\sqrt{2}}$, find the 2 points on the unit circle and corresponding angle θ.
3. Find the coordinates of the point with angle $\theta = 11\pi/6$.

.2 6.2 Trig functions of real numbers

1. Evaluating trig functions via the unit circle.
   (a) Compute $\sin(4\pi/3)$. Draw in the 3rd quadrant, want the y coordinate. Same as above example. Just need to find the point.
   (b) Draw the symmetry of the 4 quadrants with reference triangle. Note the sign of each trig just boils down to the sign of the (x, y) coordinate. This motivates some new identities.
2. Trig identities
   (a) Old: Defs, recriprocal, $\tan, \cot$ as $\sin, \cos$, pythagorean, cofunction.
   (b) New: Periodicity, even / odd (symmetry), details below
3. More examples to get fast at: $\cot(-11\pi)$, $\sec(-7\pi/6)$, $\cos(25\pi/2)$ $\csc(-11\pi/4)$.

4. Symmetry and periodicity of trig fcns. (know new formulas here)
   (a) Find $P(x, y)$ if $\theta = 0, \pi/4, \pi/3$, right triangle
   (b) Periodicity:
      i. $\sin(\theta), \cos(\theta)$: Modify last of above P as
      $$P(\theta + 2\pi), \quad P(\theta + 12\pi), \quad P(\theta - 2\pi)$$
      translates to
      $$\sin(\theta + 2\pi) = \sin(\theta), \quad \sin(\theta + 2n\pi) = \sin(\theta), \quad \text{for } n \text{ any integer, likewise for cos}$$
      ii. $\tan(\theta)$ repeats every π. Reason: $\tan(\theta + \pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin(x)}{-\cos(x)} = \tan(x)$.
      $$\tan(\theta + \pi) = \tan(\theta), \quad \tan(\theta + n\pi) = \tan(\theta), \quad \text{for } n \text{ any integer}$$
      iii. This translates to $\csc, \sec, \cot$.
      iv. Definition: Function f is *periodic* with period $k > 0$ if
      $$f(x + k) = f(x) \quad \text{for all } t \text{ in the domain}$$
   (c) Even / odd functions:
      i. Other symmetry? Back to θ in the first quadrant.
      $$P(-\theta) \quad \text{reminds of even and odd functions}$$
      translates to
      $$\sin(-\theta) = \sin(\theta), \quad \cos(-\theta) = \cos(\theta), \quad \text{also the rest.}$$
      ii. Definition: Function $f(x)$ is even if
      $$f(-x) = f(x) \quad \text{for all } x \text{ in the domain}$$
      Function $g(x)$ is odd if
      $$g(-x) = -g(x) \quad \text{for all } x \text{ in the domain}$$
      - Why even / odd for names? Power functions.
      - Useful property when solving equations.
      - Graphs are most important. Draw points to reason why.
      - Relate to reflections in function transformations (once reflected for even, twice for odd).

.3 6.3 Trig graphs / 6.4 More trig graphs

1. Distractions:
   - Spray paint oscillator: `http://www.youtube.com/watch?v=P-Umre5Np_0`
   - Pendulum wave: `http://www.youtube.com/watch?v=7_AiV12XBbI`, `https://www.desmos.com/calculator/xqaxvwxwtq`
   - Water speaker: `http://www.youtube.com/watch?v=uENITui5_jU`
   - Wave pool: `http://www.youtube.com/watch?v=cwNXNNifOAc`
   - Truth of traveling wave: `http://www.youtube.com/watch?v=-m_VDE-BSgc`
   - Resonance plate: `https://www.youtube.com/watch?v=wvJAgrUBF4w`
   - Voice resonance: `https://www.youtube.com/watch?v=Bs3uPbhIZxc`
   - Heart rate: `https://www.youtube.com/watch?v=GVm8pFDxUjU`

- Guitar strings: `https://www.youtube.com/watch?v=_014YFKqfvA`
- Bridge: `https://www.youtube.com/watch?v=j-zczJXSxnw`
- Sine waves and speed of sound. Time:10:15ish `https://www.ted.com/talks/clifford_stoll_on_everything`

2. Graphs of trig functions.
   (a) Draw the unit circle, and imagine how $\sin(\theta), \cos(\theta), \tan(\theta)$ changes. Draw $\sin(\theta)$ graph, have students imagine other two and sketch.
   (b) Graph of $\sin(\theta)$ (wave)
      i. Use desmos to find the graph (note the above).
      ii. Domain and range
      iii. Draw some special points (hit the 4 axis in unit circle)
      iv. Periodic function clear
      v. Connection to sign in quadrants
      vi. Connection to odd function (symmetry about the origin).
   (c) Graph of $\cos x$ via desmos, translate above
   (d) Graph of $\tan x$ via desmos, translate above, also asymptotes here
   (e) Graph of cot, sec and csc via desmos, no worries about knowing these for quizzes.
   (f) Show off desmos awesomeness and relate to project 1: `https://www.desmos.com/calculator/absl7xylcv`

3. **Examples:**
   (a) What is the advantage of graph? Global view of functions. Practical as well....but not more practical than unit circle.
   (b) Solve $\sin(t) = -\frac{\sqrt{2}}{2}$ in the interval $[0, 4\pi]$. Relate to both unit circle and graph. What about all t?
   (c) Where is $\sin(t) < -\frac{\sqrt{2}}{2}$?
   (d) If they are dying, do another for cosine or tangent. Same as previously just with symmetry and periodicity added.

4. Distraction tactics. Sine waves and speed of sound. Time:10:15ish
   - `https://www.ted.com/talks/clifford_stoll_on_everything`

5. Goal: Graph of $y = a\sin(bx + c) + d,\ y = a\cos(bx + c) + d$. What does each letter do? Motivation, to capture the idea of wave.
   - Drive with examples:
   - $\sin(x), 2\sin(x), -\frac{1}{2}\sin(x)$
   - $\sin(x), \sin(2x), \sin(\frac{1}{2}x)$
   - $\sin(x), \sin(x - \pi), \sin(x + \frac{\pi}{2})$
   - Combined: $y = 3\sin(2x - \frac{\pi}{3}) - 1$

   (a) a: amplitude
   (b) b: frequency, connected to period: $\frac{2\pi}{|b|}$
   (c) c: horizontal shift, shift we see is the phase shift: $-\frac{c}{b}$
   (d) d: vertical shift

(e) Speed of sound: `http://www.ted.com/talks/clifford_stoll_on_everything`
(f) Frequency: `http://www.youtube.com/watch?v=qNf9nzvnd1k`
(g) Cell phone oscilliscope

6. **Examples:** Find amplitude, period and phase shift. Graph each via function transformations.
   (a) $y = 3\cos(x + \frac{\pi}{6})$
   (b) $y = 9\cos(\frac{\pi}{4}x - \frac{\pi}{2})$
   (c) $y = 2\cos(x + 13\pi) - 2$ (think periodicity)
   (d) What if....negative leading coefficient, negative x coefficient?
7. A more efficient method: Keep track of change of five main points.
   - One complete cycle: $0 \le bx + c \le 2\pi$
   - Bisect above interval twice to get 5 total x values.
   - Compute corresponding y values.
   - Use periodicity to extend beyond basic interval.
   - A complete graph has details (coordinates of 5 points) for one period, then extends to $(-\infty, \infty)$.
8. Now, from a graph, can get a formula.
   (a) Boot up desmos, close eyes.
   (b) **Examples:** $y = 3\sin(2x + \pi/2)$, $y = -2\cos(2\pi x + 1/2)$
   (c) Controversy: is it sine or cosine, plus or minus? Either one $\sin(x - \pi/2) = \cos(x)$
9. What about the rest? Still in desmos, graph $y = \tan(x)$ and all the rest one by one. Graph of each and ask questions.
   (a) Domain, range
   (b) Special values
   (c) Period
   (d) Asymptotes
   (e) Odd, even

.4 6.5 Inverse trig functions and their graphs

1. Motivation: $\sin(x) = \frac{1}{3}$
   (a) Our calculator can solve equations like this. What is really going on?
   (b) Draw the unit circle and show sticking point. Not a nice angle.
2. Inverse function review: Everything from chapter 5, just hit on essentials.
   (a) Inverse relationship, expression, domain and range.
   (b) Are all functions invertible? Need one-to-one. Reversing may not be unique.
   (c) Even though $f(x) = x^2$ is not invertible, we can still solve equations like $x^2 = 25$. Key is domain restriction to make invertible, then use special features of the function. This is key idea for inverse trig, and we already do this to solve equations.
3. Inverse sine:
   (a) Draw graph to show not one-to-one. Already know this.

(b) How to restrict the domain so we pass the horizontal line test? Let them decide. Many ways, which is best?

(c) Restricted sine: $y = \sin(x), \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. All outputs for sine $[-1, 1]$ are realized.

(d) Draw graph of restricted sine, compare to unit circle.

(e) Def: $y = \sin(x) \Leftrightarrow \arcsin(y) = x$

(f) Notation: $\arcsin(x) = \sin^{-1}(x)$, but we like arcsin to avoid exponent confusion.

(g) Domain restriction is confusing: $y = \arcsin(x)$ is the inverse function of RESTRICTED sine.

4. **Example:** Solve $\sin(x) = \frac{1}{3}$ via arcsin.

   (a) Draw unit circle, $x = \arcsin(1/3)$ gives solution on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

   (b) Get the second solution on the unit circle, then periodicity.

   (c) Issue: We can write down the solution, but what is $\arcsin(1/3)$?

   (d) This issue is analogous to logarithms. $2^x = 10$ gives $x = \log_2(10)$ which is not nice. Calculus gives logarithnmic tables and slide rules for this. Inverse trig has similar story.

5. What about the rest? Have them fill in a table.

   (a) $\sin, \cos, \tan$

   (b) Invertible? Yes or no.

   (c) Restricted domain

   (d) Range

   (e) Graph of restricted trig

   (f) Part of unit circle

   (g) Graph of inverse trig

   (h) What about the buddy functions? $(\sec, \csc, \cot)$

6. Inverse relations:

   (a) $\sin(\arcsin(y)) = y, \quad -1 \le y \le 1$

   (b) $\arcsin(\sin(\theta)) = \theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

   (c) Note the domain restriction subtlety.

   (d) What about $\cos, \tan$?

   (e) **Examples:**

      i. $\sin(\arcsin(-1/2)) = -1/2$

      ii. $\arcsin(\sin(\frac{5\pi}{3})) =?$ Cannot use the identity since not in restricted domain. Subtract 2π via periodicity and life is grand.

7. **Student Examples :**

   (a) $\arccos(-1/2)$ (Call angle θ, then travel through the inverse relation. Unit circle and restricted domain.)

   (b) $\tan(\arctan(12))$

   (c) $\arcsin(\sin(\pi/6))$

   (d) $\arcsin(\sin(5\pi/6))$

   (e) $\cos(\operatorname{arcsec}(2))$

   (f) $\sin(\arccos(3/5))$

   (g) $\cos(2\arcsin(1/3))$

8. Verifying trigonometric identities
   (a) $\arcsin(-x) = -\arcsin(x)$ (Do we expect an odd function here?)
   (b) $\arccos(\frac{1}{x}) = \text{arcsec}(x)$
   (c) $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$
   (d) $\cos(\arcsin(4/5) + \arctan(3/4)) = 0$
   (e) $\cos(\arcsin(x)) = \sqrt{1-x^2}$ (Weird? Or not?)
   (f) Show it's not an identity: $\arctan x = \dfrac{\arcsin(x)}{\arccos(x)}$

.5 6.6 Modeling harmonic motion

Not on test, just show off idea.

1. The truth of traveling wave
   - Water tank: `http://www.youtube.com/watch?v=-m_VDE-BSgc`
2. Harmonic motion (pendulum)
   - Clock: `http://www.youtube.com/watch?v=3dtSnNfyENU`
   - Paint can: `http://www.youtube.com/watch?v=P-Umre5Np_0`
   - Spring mass: `http://www.youtube.com/watch?v=eeYRkW8V7Vg`
   - Foucault Pendulum: `https://www.youtube.com/watch?v=iqpV1236_Q0` `https://en.wikipedia.org/wiki/Foucault_pendulum`
3. Damped harmonic motion: `https://www.youtube.com/watch?v=sP1DzhT8Vzo`

$$y = ke^{-ct}\sin(\omega t), \quad y = ke^{-ct}\cos(\omega t)$$

   where c is the damping constant.

Chatper 7 Analytic Trigonometry

.1 7.1 Trigonometric identities

1. What identifies we already have?
   (a) Pythagorean
   (b) Reciprocal and basics
   (c) Even / odd ones
   (d) Unit circle ones like $\sin(\theta + \pi) = -\sin(\theta)$
   (e) Cofunction identites
   (f) Laws of sines and cosines
2. Basic simplifying:
   (a) $\dfrac{\cot(\theta)}{\csc(\theta) - \sin(\theta)}$
   (b) $\dfrac{\sin(t)}{1-\cos(t)} - \csc(t)$ (Need conjugate here.)
3. Verify/disprove an identity

(a) An identity is true if holds for all a, b. $(a+b)^2 = a^2 + 2ab + b^2$. Argue is true via geometry.

(b) An identity is not true: $(a+b)^2 = a^2 + b^2$. Show is false via contradiction ($a = b = 1$).

4. Verifying basic identity. Pick one side and work to the other.

   (a) $(\sin(x) + \cos(x))^2 = 1 + 2\sin(x)\cos(x)$

   (b) $\csc(x) - \sin(x) = \cos(x)\cot(x)$

   (c) $\sec(\theta) = \sin(\theta)(\tan(\theta) + \cot(\theta))$

   (d) $\dfrac{1}{\csc x - \cot x} = \csc x + \cot x$

   (e) $\dfrac{\cot\theta - \tan\theta}{\sin\theta + \cos\theta} = \csc\theta - \sec\theta$

   (f) $\dfrac{\tan x}{1 + \sec x} + \dfrac{1 + \sec x}{\tan x} = 2\csc x$

   (g) $\tan^2(2x) - \sin^2(2x) = \tan^2(2x)\sin^2(2x)$

   (h) $\dfrac{\csc(-t) - \sin(-t)}{\sin(-t)} = \cot^2 t$

   (i) $\ln(\sec\theta) = -\ln(\cos\theta)$

   (j) $\ln(\sec x + \tan x) = -\ln(\sec x - \tan x)$

   (k) $\dfrac{1 - \cos(x)}{\sin(x)} = \dfrac{\sin(x)}{1 + \cos(x)}$

5. **Examples:** Show the equation is not an identity.

   (a) $\sin(-t) = \sin(t)$

   (b) $\sin(x + y) = \sin(x) + \sin(y)$

   (c) $\cos(t + \pi) = \cos(t)$

   (d) $\log_2(x + y) = \log_2(x) + \log_2(y)$

   (e) $\sin(t - \pi) = \sin(t + \pi)$

   (f) $\sec^2(x) + \csc^2(x) = 1$

.2 7.2 Addition and subtraction formulas

1. How to compute exact trig values of more angles? $\cos(15°) =?$

2. Sum and difference formulas for cosine.

   (a) Subtraction formula for cosine

   $$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

   `http://www.maa.org/sites/default/files/321907348451.pdf.bannered.pdf`

   (b) Addition formula for cosine (do it yourself). Hint: $u + v = u - (-v)$

   $$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

   (c) **Examples**

   - $\cos(15°) =$
   - $\csc(\alpha) = 3/2$ and $\cot(\alpha) < 0$, find $\cos(\alpha - \frac{\pi}{4})$

3. Cofunction identities: Why 'co'sine? 'Co' is short for complimentary. $\sin(\theta) = \cos(\alpha)$ for complimentary angles.

(a) $\cos(\frac{\pi}{2} - u) = \sin u$
(b) Example for 3,4,5 right triangle.
(c) Proof by the right triangle
(d) Proof by function transformation of graph
(e) Proof by additional/subtraction formula (concrete version)
(f) Write the other 5 cofunction identities.

4. Addition and subtraction formulas for other trig functions. Cofunction identities are the gateway. Show each.

   (a) $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$
   (b) $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$ use oddness and subtraction to addition trick.
   (c) Have students try. Verify
   $$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$
   (d) Student conjecture $\tan(u-v)$.
   (e) Common misconceptions: Show false $\cos(u-v) = \cos u - \cos v$.
   (f) Write down all the formulas to summarize.
   (g) **Examples:**
      i. Find all the trig values for $\frac{7\pi}{12}$
      ii. If $\sin(\alpha) = 12/13$ and $\sec(\theta) = 4/5$ where α, β are in quadrant II and I, what is $\cos(\alpha+\beta)$, the quadrant containing $\alpha+\beta$?

5. **Examples:** Confirm or refute the following identites. If true, show it. If false, give a counterexample. Once done, solve the false equations.

   (a) $\sin(u+v)\cdot\sin(u-v) = \sin^2 u - \sin^2 v$ (True)
   (b) $\sin(u - \pi/2) = \cos(u)$ (False)
   (c) $\sin(4t)\cos(t) = \sin(t)\cos(4t)$ (False)
   (d) $\sin(\theta+\pi) = -\sin(\theta)$ (True)

6. **More examples:**

   (a) Reduction formula revisited $\sin(\pi/2 - \theta) = \cos\theta$
      i. Solve by algebra
      ii. Solve by unit circle
   $$\sin(\theta + \frac{3}{2}\pi)$$
   (b) Is this each an identity? If not, for which x does it work?
      - $\sin(x) = \cos(x - \frac{\pi}{2})$
      - $\tan(\pi - x) = \tan(x)$
      - $\sin(x+\pi) = \sin(x)$
      - $\tan(\frac{\pi}{2} - x) = \tan(x)$

7. Create your own identity $\sin(u+v+w) =?$

8. Combination of waves to get new waves. Why $\sin x + \cos x$ becomes another sine curve.
$$\sin(x) + \cos(x) = k\sin(x+\theta), \quad \sin(\theta) = \cos(\theta) = \frac{1}{k} = \frac{1}{\sqrt{2}}, \quad \theta = \frac{\pi}{4}$$

.3 7.3 Double, half, and product to sum formulas

1. Put the sum formulas to work. No new memorization here! Fill in the blank. Double angle formulas:
   (a) $\sin(2u) = 2\sin(u)\cos(v)$
   (b) $\cos(2u) = \cos^2(u) - \sin^2(u) = 1 - 2\sin^2(u) = 2\cos^2(u) - 1$ (three choice via the Pythagorean identity)
   (c) $\tan(2u) = \frac{2\tan(u)}{1-\tan^2(u)}$ (can also use sine and cosine values here)
2. How to use? **Examples:**
   (a) $\sin\alpha = 2/3$, and α is in quadrant II. Find all the trig value for 2α. Which quadrant is 2α?
   (b) Confirm or refute each. If true, show. If false, find when true.
      i. $\frac{\sin^2(2\theta)}{\sin^2(\theta)} = 4 - 4\sin^2(\theta)$ (True)
      ii. $\cos^4 x - \sin^4 x = \cos 2x$ (True)
      iii. $\cos(x) - \sin(2x) = 0$ (False)
      iv. $\frac{1+\sin 2x+\cos 2x}{1+\sin 2x-\cos 2x} = \cot x$ (True)
      v. $\sin x + \cos x = 1$ (False (again).) Solve by multiplying by $\frac{1}{\sqrt{2}}$ around then introducing sine addition identity with $\frac{\pi}{4}$. Past Method also works.
3. Multiple angle formulas. Can iterate the above idea to heart's content.
   (a) $\sin\alpha = 1/3$, what is $\cos 4\alpha$
   (b) $\sin\alpha = \frac{3}{5}$ and α is an acute angle. What is $\cos 6\alpha$?
4. Half angle formulas: Can turn double angle formulas around to go from doubling to halfing.
   (a) $\sin^2\frac{u}{2} = \frac{1-\cos u}{2}$
   (b) $\cos^2\frac{u}{2} = \frac{1+\cos u}{2}$
   (c) $\tan^2\frac{u}{2} = \frac{1-\cos u}{1+\cos u}$
   (d) **Examples:**
      i. What is all the trig values at $\frac{\pi}{8}$?
      ii. What are all the trigs for $\alpha/2$ if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$?
   (e) Need a formula without square at times. Need not fret which quadrant we lie in for tangent at least. Start with original formula, multiply by the conjugate on the top or the bottom. Note that $1 \pm \cos(x) \geq 0$ always and $\sin(u)$ and $\tan(\frac{u}{2})$ always agree in sign.

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$$
$$\tan\frac{u}{2} = \frac{\sin u}{1+\cos u}$$

5. Show that $\tan(\frac{u}{2})$ and $\sin u$ always have the same sign via graph transformation. Other ways here?
6. **Product to sum and sum to product** Read on own! Know they exist. Good enough.
   (a)

$$\begin{aligned}
\sin(u+v) &= \sin(u)\cos(v) + \cos(u)\sin(v)\\
\sin(u-v) &= \sin(u)\cos(v) - \cos(u)\sin(v)\\
\sin(u+v) + \sin(u-v) &= 2\sin(u)\cos(v)\\
\sin(u)\cos(v) &= \frac{1}{2}\left(\sin(u+v) + \sin(u-v)\right)
\end{aligned}$$

.4 7.4/5 Trigonometric equations

1. A tour of all the major cases:
   (a) Basic, nice angle: $\sin(\theta) = \dfrac{\sqrt{3}}{2}$
   (b) Basic, angry angle: $\sin(\theta) = -\dfrac{1}{3}$
   (c) Basic algebra: $\sqrt{2}\cos(\theta) - 1 = 0$
   (d) Factoring: $2\cos^2(\theta) - 7\cos(\theta) + 3 = 0$
   (e) Substitution: $\cos(5\theta) - 1 = 0$, find all solutions, find all from $[0, 2\pi]$
   (f) Identity needed: $2\sin^2(\theta) - \cos(\theta) = 1$
   (g) Square both sides, identity needed, check for extraneous solutions: $\sin(\theta) - 1 = \cos(\theta)$
   (h) Identity needed: $\cos(\theta)\cos(3\theta) - \sin(\theta)\sin(3\theta) = 0$
   (i) Identity needed: $\tan(\theta) + \cot(\theta) = 4\sin(2\theta)$
2. Once we reduce each to the basic version, there are the same steps.
   (a) Find location of reference angle θ_R. If not a special angle, will need calculator.
   (b) Draw the unit circle with all the possible solutions.
   (c) Identify the ones (positive/negative ones) on the graph.
   (d) Find the value of the two by geometry.
   (e) Add $2n\pi$.
3. Tips:
   (a) Reduce to $trig(angle) = number$ (case 1)
   (b) Use change of variable if necessary
   (c) Change to one trig if have multiple trig
   (d) Try to factor if the RHS is 0.
   (e) Your answer can be no solution.
   (f) If end up with an identity, the answer will be all the numbers in the original domain (check the domain).

Chapter 9 Vectorts in two and three dimensional space

.1 9.1 Vectors in two dimensions

1. The basics: Vector vs coordinates
   (a) Coordinates just give location. Where something is at. $P = (x, y) = (2, 3)$.
   (b) Vectors embody an action (displacement, force, velocity, etc). For intuition we will focus on displacement. Illustrate with marker example.
      i. Can draw the action in the plane. New notation $\vec{a}$. Note bold font used in the text.
      ii. Starting at a point A to another point B gives $\vec{a} = \vec{AB}$.
      iii. Notation: $\vec{a} = \langle 2, 3\rangle$. Starting point can be anywhere, terminal point can be anywhere. This action is independent of location.
      iv. What is we know the starting point $A = (1, 2)$ and $B = (-3, 4)$. What is $\vec{AB}$? Are $\vec{AB}$ and $\vec{BA}$ the same?

v. General component form of a vector:

$$\vec{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

2. Vector operations: Because vectors are actions, operations can be created to combine / modify vectors.
   (a) Give geometry first for each, then concrete numeric example in the plane.
   (b) To add vectors $\vec{a}$ and $\vec{b}$, perform one first, then the next. This yields the parallelogram law.
   (c) Scalar multiple. Perform vector action 2 times: $2\vec{a}$. Reverse the direction as $-\vec{a}$. No action as $0\vec{a}$.
   (d) Subtract vectors by adding a negative vector: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$. This yields the triangular law.
   (e) $\vec{0}$ is the absence of action, the only vector without orientation.
   (f) Properties of Vectors: (mimic real numbers since vector addition / scalar multiplication are made to be similar). Try to interpret each geometrically.
      i. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
      ii. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
      iii. $\vec{a} + \vec{0} = \vec{a}$
      iv. $\vec{a} + (-\vec{a}) = \vec{0}$
      v. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
      vi. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
      vii. $(cd)\vec{a} = c(d\vec{a})$
      viii. $1 \cdot \vec{a} = \vec{a}$

3. Unit and basis vectors:
   (a) A unit vector is of length 1, $|\vec{a}| = 1$. Then $\vec{a} = \langle \cos(\theta), \sin(\theta) \rangle$. The set of all unit vectors form the unit circle. Note the magnitude of latter is 1. Many a connection to trigonometry with all this, naturally. Example: Find the unit vector in the opposite direction of $\vec{a}$ with length 5. Perpendicular?
   (b) A set of basis vectors are sufficient to build the entire vector space.

4. Two main ways to describe vectors:
   (a) Initial and terminal point. Done above. Reformulate here with unit basis vectors (think of as the fundamental information of each direction). Not a lot of intuition here.
      i. $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath}$.
   (b) Magnitude and direction.
      i. Magnitude is vector length and we adopt absolute value notation. $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$. Note the book notation uses the same as absolute value here.
      ii. Note the unit basis vectors are of length one, thus the name unit.
      iii. If direction is angle θ in $[0, 2\pi]$, then can rewrite $\vec{a} = \|\vec{a}\| \cos(\theta)\vec{\imath} + \|veca\| \sin(\theta)\vec{\jmath}$.
      iv. Find the magnitued and direction of $\vec{a} = \langle -1, \sqrt{3} \rangle$

5. Thinking examples:
   (a) Find the vector in the direction of (1,2) with length 3.
   (b) Find the vector that is parallel to $y = 2x + 1$ with length 3.
   (c) Find the vector that is perpendicular to $y = 2x + 1$ with length 3. How are perpindicular vectors related?
   (d) Vector equation $\vec{r}$: find the vector $\vec{r}$ such that $\vec{r} + \vec{a} = \vec{b}$

(e) Add/subtract the following vectors in as many ways as possible to arrive at the zero vector (can add multiple times).

$$\vec{a} = \langle 1, 0\rangle, \quad \vec{b} = \langle 2, 2\rangle, \quad \vec{c} = \langle 1, 2\rangle, \quad \vec{d} = \langle 0, 2\rangle.$$

6. Applications:

   (a) Book problems 55 and 60.

7. End notes:

   (a) Key is that vectors are not the same as coordinates. Action vs location.
   (b) Notation is important. Otherwise confusion ensues.
   (c) Always think about things in three ways (algebra, geometry and unit vector). Even in high dimensional space, can still draw intuition from 2/3 dimensional geometry.

.2 9.2 The dot product

1. Given vectors $\vec{a} = \langle a_1, a_2\rangle$ and $\vec{b} = \langle b_1, b_2\rangle$, the dot product of $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

   Ie. sum the product of corresponding entries.

   (a) **Example:** $\langle -3, 1\rangle \cdot \langle 4, -1\rangle = -3(4) + 1(-1) = -13$
   (b) Properties of the dot product
      i. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
      ii. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
      iii. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
      iv. $m(\vec{a} \cdot \vec{b}) = (m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b})$
      v. $\vec{0} \cdot \vec{a} = 0$
      vi. $\vec{a} \cdot \vec{a} = 0$ only if $\vec{a} = \vec{0}$ All can be shown by definition. Show one or two. Note, can only take the dot product of two vectors.
   (c) **Examples:** These properties are easy to use.
      - If $\vec{a} = \langle 3, 1\rangle, \vec{b} = \langle 4, 1\rangle, \vec{c} = \langle 5, 9\rangle$, compute $2\vec{a} \cdot \left(3\vec{b} - 4\vec{c}\right)$
      - If $\|\vec{a}\| = \sqrt{13}, \vec{a} \cdot \vec{b} = 10, \|\vec{b}\| = \sqrt{17}$, compute $\|2\vec{a} - 3\vec{b}\|$.

2. This dot product you speak of....what is it good for? It comes from geometry! (rewrite the proof in your quiz)

   (a) Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos(\theta)$ (draw triangle)
   (b) Law of Cosines via vectors: $\vec{a}, \vec{b}, (\vec{a} - \vec{b})$, angle θ between $\vec{a}, \vec{b}$.

$$\|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos(\theta)$$

$$\|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos(\theta)$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

   In your face law of cosines! This is way easier to use when finding angle θ.

3. **Examples:**
   (a) Find the angle between two random vectors. Find the angle between vector and x-axis.
   (b) What if $\vec{a} \cdot \vec{b} = 0$ (vectors are orthogonal)? Find a orthogonal vector to $< 1, 2 >$. Infinitely many. Normal? Two here discluding zero vector. Note, zero vector is perpendicular to all.
   (c) What if $\vec{a} \cdot \vec{b} < 0$ or > 0, how close are these two vectors are? Use the pandora example, project.
   (d) Change geometry into algebra: if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$, show $|\vec{a}| = |\vec{b}|$. Geometry here, must be a rectangle for a right angle here.

4. **Examples:** Which are meaningful, which are hogwash?
   (a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ HW
   (b) $(\vec{a} \cdot \vec{b})\vec{c}$ M
   (c) $\|\vec{a}\|(\vec{b} \cdot \vec{c})$ M
   (d) $\vec{a} \cdot (\vec{b} + \vec{c})$ M
   (e) $\vec{a} \cdot \vec{b} + \vec{c}$ HW
   (f) $\|\vec{a}\| \cdot (\vec{b} \cdot \vec{c})$ HW

5. **Examples:** Vectors are swell, how to bring them into any geometric problem?
   (a) Find the angle $\angle ABC$ for points $A = (4, 3), B = (1, -1), C = (6, -4)$.

6. Component and projection of vector

$$proj_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}\vec{b}$$

$$comp_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

   Mention when $\vec{a} > 0$, $\vec{a} < 0$ and show via geometry. Distance from a point to a line. Why is this useful?
   (a) Projection example. Least squares?
   (b) Example: Assume my prius sits on a 15° incline. If the car is 2000lb, how much weight is being exerted on the wheels from rolling backwards?

$$\|\vec{r}\| = \text{comp}_{\vec{r}}(\vec{w}) = \|\vec{w}\| \cos(75°) = 2000 \cos(75°)$$

7. Work = force $\times$ distance = $F \times d$
   (a) Simple example: If I lift my 20 lb baby up 4 ft from the ground, I have done

$$W = F \times d = (20lb)(4ft) = 80 \text{ foot pounds}$$

   This simple case only happens when force is in the same direction as displacment.
   (b) If not in the same direction (pulling a wagon), vectors appear. Denote force ($\vec{F}$) and distance ($\vec{D}$) (displacement) as vectors. In which case vector projection appears. Draw distance vector $\vec{D}$ on the x axis and force vector $\vec{F}$ in quadrant 1. Then the effective magnitude of the force is projected as

$$\text{comp}_{\vec{D}}(\vec{F}) = \|\vec{F}\| \cos(\theta)$$

   and work is computed as

$$W = (force) \times (distance) = (\|\vec{F}\| \cos(\theta))\|\vec{D}\| = \|\vec{F}\|\|\vec{D}\| \cos(\theta) = \vec{F} \cdot \vec{D}$$

(c) Compute above with simple concrete numbers.

(d) Wagon example. If my kid weighs 20 pounds, and I pull with a force of 30 pounds on the 60° handle, how much work do I do to pull the wagon 100 feet?

$$\vec{W} = \vec{F} \cdot \vec{D} = \langle$$

8. Advantage: Why vector is better? Simpler point of view, easily extends to high dimensions.
   (a) Solving three dimensional object. $2, 3, 4$ rectangle. Angle on triangle across diagonals.
   (b) Solving vector equation
   (c) Geometry: $|\vec{r}| = 1$, $|\vec{r} - \vec{a}| = |\vec{r} - \vec{b}|$, $|\vec{r} - (0, 1)| = y + 1$ (parabola)

.3 9.3 Three dimensional coordinate geometry

1. Here we open up a new part of algebra. Prior to now, all functions have depended on only one variable, x or t, space or time. Never both or high dimensional space. New functions such as $f(x, t)$ are two dimensional for input and generate a surface in 3-space (as opposed to a curve in 2-space). Lucky for us, ideas of calculus generalize to 3-space (and n-space), though we need a new foundation to build on, vectors. The rest is left to calculus 3.

2. Three dimensional rectangular coordinate system
   (a) Right hand rule gives a consistent way to decide x, y, x axis directions. Octants now in place of quadrants.
   (b) Coordinate points (x, y, z).
   (c) Examples: Describe what each describes in 3-space.

   $$y = 0, \quad z = 2, \quad y = x, \quad x^2 + y^2 = 1, \quad y + z = 0, \quad x^2 + y^2 = z, \quad x^2 + y^2 = |z|, \quad y^2 + z^2 = 2$$

   A modification of intuition is required here.
   (d) Inequalities. Modify the above and see what they say.

3. Distance:
   (a) Distance formula idea, use Pythagoras twice. Motivate with simple three point example in first octant. Start with distance to opposite sides of a rectangular box. Then point to origin. Then point to point.
   (b) Formal distance formula for points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$.
   (c) Formula of a sphere: $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ Just the distance formula as in the two dimensional formula.
   (d) What is $x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$? A sphere if you can complete the square. Even if you can't I suppose. What if the stuff on the RHS is negative?
   (e) Middle point formula. Average two points to get the middle.

4. Projection:
   (a) Point (a, b, c) projected onto the xy-plane is $(a, b, 0)$. What about the other planes?
   (b) What are the projections of above shapes onto each plane? cylinder, pyramid, points, line?
   (c) Question: If all the three projections of an object are circle with radius 1, can we say the object is a sphere?
   (d) Consider the point $(-3, 1, -2)$, what's the distance to all the three plane and all the three axis?

.4 9.4 Vectors in three space

The beauty of vectors is that everything extends easily to 3 and higher dimensional space.

1. Component form of $\vec{a}$
2. $\vec{AB}$
3. $\|\vec{a}\|$
4. Basic operations: Add, subtract, scalar multiply
5. Unit basis vectors $\vec{i}, \vec{j}, \vec{k}$
6. Dot product
7. Angle between vectors
8. Direction angles α, β, γ with $\vec{i}, \vec{j}, \vec{k}$

   $$\cos(\alpha) = \frac{a_1}{\|\vec{a}\|}, \quad \cos(\beta) = \frac{a_2}{\|\vec{a}\|}, \quad \cos(\gamma) = \frac{a_3}{\|\vec{a}\|}$$

   giving

   $$\vec{a} = \langle \|\vec{a}\| \cos(\alpha), \|\vec{a}\| \cos(\beta), \|\vec{a}\| \cos(\gamma) \rangle$$

   and

   $$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$

9. Find the direction angles of vector $\langle 3, 4, 5 \rangle$.

.5 9.5 The cross product

1. Two vectors live in a plane in three dimensional space. What about the direction opposite (orthogonal) to that plane. Let $\vec{a}$ and $\vec{b}$ form the plane. Then opposite direction $\vec{c}$ is such that

   $$\vec{a} \cdot \vec{c} = 0, \quad \vec{b} \cdot \vec{c} = 0$$

   Then,

   $$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$$

   and

   $$b_1 c_1 + b_2 c_2 + b_3 c_3 = 0.$$

   Eliminating c_3 gives

   $$(a_1 b_3 - a_3 b_1) c_1 + (a_2 b_3 - a_3 b_2) c_2 = 0$$

   simplified as

   $$x c_1 + y c_2 = 0$$

   Choose $c_1 = y$ and $c_2 = -x$ giving

   $$c_1 = a_2 b_3 - a_3 b_2, \quad c_2 = a_3 b_1 - a_1 b_3$$

   resulting in

   $$c_3 = a_1 b_2 - a_2 b_1$$

   Assemble the new vector $\vec{c}$ resulting in our prize.

2. The cross product

   (a) Definition

   $$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

(b) Determinant algorithm to compute. Refresh two dimensional determinant first.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(c) Concrete example. Confirm that it actually worked.

3. Geometric meaning: a new vector

   (a) Direction: perpendicular to both with right hand rule

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

   (b) Theorem for length Length: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$. Comes from brute force length calculation with $|\vec{a} \times \vec{b}|^2$.

   (c) Useage:

      i. Find the unit vector that are perpendicular to both $\vec{i}$ and $\vec{j}$

      ii. What if $\vec{a} \times \vec{b} = 0$? Above formula with $\sin(\theta)$ ensures $\theta = 0$ and the angles must be parallel.

      iii. $|\vec{a} \times \vec{b}|$, the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$. This comes from drawing the picture, parallelogram law, and also

$$A = \|\vec{a}\|(\|\vec{b}\| \sin(\theta)) = \|\vec{a} \times \vec{b}\|$$

      What's the area of the triangle formed by P(1,4,6), Q(-2,5,-1) and R(1,-1,1): $5\sqrt{82}$

4. Algorithm

   (a) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

   (b) $m\vec{a} \times \vec{b} = \vec{a} \times m\vec{b} = m(\vec{a} \times \vec{b})$

   (c) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

   (d) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

   (e) $\vec{a}(\cdot\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot c$

   (f) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

5. Triple product gives a true 3 by 3 determinant.

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

   - Determinant
   - Volume of the parallelepiped. Volume is area of the base times the height. Use this with our dot product formula. Draw vectors b and c in the xy plane and vector a going verticalish.

$$V = Ah = \|\vec{b} \times \vec{c}\| |\|\vec{a}\|| \cos(\theta)|$$

   - If $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ we must be coplanar

Chapter 10 Sequence, series and probability

10.1 Infinite sequences and summation notation

1. Finite sequence: a list of numbers

2. Infinite sequences:
    (a) Direct formula: not unique (show)
        i. Arithmetic
        ii. Geometric
    (b) Recursive formula: (Fibonacci sequence)
    (c) Pattern or no pattern: read and say formula
3. Graphing a sequence (connection between sequences and functions)
4. Summation notation:
    (a) Why do we care? Monthly sale
    (b) Notation
    (c) Example: give a sale data, sum of the sale.
5. Infinite series:
    (a) Why do we care: calculas
    (b) things get weird

.1 10.2 Arithmetic sequences

1. Definition: common difference
2. Formula:
    (a) Recursive
    (b) Direct
3. Summation formula:
    (a) $S_n = \frac{n}{2}(a_1 +_n)$
    (b) $S_n = \frac{n}{2}(2a_1 + (n-1)d)$
4. Examples
    - Sum of a constant
    - Find the formula for $\sum_{i=1}^{n} n$
    - Find the summation of even integers less or equal than 91