Applied Linear Algebra Notes, Fall 2021

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Course Introduction

.1 Data and Linear Algebra

1. Image pixel: LINK
2. Sports ranking: LINK
3. Word2Vec: LINK
4. Recommender system: LINK
5. Dimension reduction: LINK

Chapter 1: Linear Equations in Linear Algebra

.1 1.1 Systems of linear equations

1. Definition: A *linear equation* is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where x_i are unknown variables with a_i known constant coefficients and b known constant. Only powers of 1 per variable. No other products or quotients.

2. Fundamental problem of linear algebra:
   - Solve a system of linear equations (rich theory can completely study).
   - Key questions: Existence and uniqueness.

3. Familiar example, new ideas.

(a) Solve for x and y.

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

Linear equations, graphs are lines in 2d.

(b) Three perspectives of this class:

- Row picture (familiar)
- Column picture (new)
- Matrix representation (maybe new)

(c) Row picture:

- Graph in xy-plane. Solution is intersection of two lines. How to find? Substitute or elimination.
- In general, can see thee possibilities: Unique solution (lines differ in slope), infinite solutions (2 lines overlap), no solution (2 parallel non intersecting lines). No solution is called *inconsistent*. One or infinite many solutions called *consistent*.

(d) Column picture: Vector representation

- Remind of 2D vector geometry, scalar multiplication, vector addition, graph, and linear combination.
- Rewrite in vector form. How to think of this? What linear combination of column vectors $\vec{v}_1$ and $\vec{v}_2$ result in vector $\vec{b}$? Draw in the plane and sketch solution.
- Verify that solutions $x = 1, y = 2$ from before work.
- Again, three possibilities. What are the vector analogies regarding column vectors and RHS vector?
- Generalize: If we change the RHS vector, will we always have a solution? In this case yes since $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$. Change for parallel column vectors to see not always.

(e) Matrix representation:

- Rewrite as coefficient matrix times unknown vector equal a RHS vector.
- Notation: Note text uses bold face letters for vectors.

$$A, \quad \vec{x}, \quad \vec{b}$$

- Can also write short hand as an augmented matrix.
- Solve using the same elimination strategy as with linear equations. Think of this as a computational view. Next section covers this.
- Matrix A can be thought of as an operator on solution vector $\vec{x}$ with resulting vector $\vec{b}$. Studying this linear system equations to studying properties of matrix A.

4. Higher dimensions:

(a) 3 equations, 3 unknowns:

$$\begin{cases} x + 2y + 3z = 5 \\ 2x + 5y + 2z = 7 \\ 6x - 3y + z = -2 \end{cases}$$

Solution is $x = 0, y = 1, z = 1$.

(b) Row picture

- Ask graph of each linear equation. Graph in Geogebra 3d to see. Can anyone solve? Plot solution point as well.

- Again 3 cases here, but a bit richer. 1 solution, infinite solutions (plane or line of intersection), no solution (2 planes parallel but not the same).
- Solve by row reduction and backwards substitution. Goal is to replace system with equivalent, though simpler system. Summarize 3 elementary row operations (swap, scale, replace with row plus multiple of another). Why bother swap or scale? Take advantage of zeros and nice numbers. Computers care for high dimension to avoid roundoff error. Mention could eliminate all the way to Gauss Jordan form.

(c) Column picture: Linear combination of three vectors giving RHS vector. Use Geogebra 3d again. Again, think of three cases. Key is all three vectors are linearly independent.

(d) Matrix picture: Easy to write down? Now what?
- Can see columns of A are column vectors.
- What about row vectors? Will develop this.
- Augmented matrix. Algorithm in next section.

(e) Advantages / disadvantages of each picture: Combined they offer a complete theory.
- Row picture: Lots of info and intuition, cannot extend beyond 3d, will think in analogies.
- Column picture: Easy to extend, hard to solve, lots of info and intuition.
- Easy to adapt as algorithm, little intuition.

5. Homework: 3, 7, 13, 18, 19, 23, 25, 33, 34

.2 1.2 Row reduction and echelon form

1. 2 algorithms for solving linear systems of equations:
   - Gaussian elimination and backwards substitution (saw last time).
   - Gauss-Jordan elimination.
2. Example, 2×2: Solve the system using equation form.

$$\begin{cases} x - 2y = 1 \ (R_1) \\ 3x + 2y = 11 \ (R_2) \end{cases}$$

(a) Use the same forward reduction and back substitution idea as in last section.

$$R_2 \to -3R_1 + R_2$$

Check solution works. Recall 3 elementary row operations.

(b) Generalize: Use augmented matrix and aim towards a standard form.
- Row echelon form (GE)

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right]$$

- Reduced row echelon form (G-JE)

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

- Pivot entries correspond to locations of 1's in RREF. Pivot columns are columns which contain a pivot entry.
- Note, for any matrix REF is not unique but RREF is. Will prove the latter later.

(c) What if...

- No solution:
$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 8 \end{array}\right]$$
- Infinitely many solutions:
$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$
Here y is a free variable and all solutions are
$$\begin{cases} x = 1 + 2y \\ y \text{ free} \end{cases}$$
or written parametrically as
$$\begin{cases} x = 1 + 2t \\ y = t \end{cases}$$
for parameter t.

3. Example: Higher dimension, try on own:
$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$
REF and backwards sub vs RREF.

4. Homework: 1, 3, 5, 7, 11, 13, 15, 17, 21, 23, 33-34

.3 1.3 Vector equations

1. 3 view of linear algebra:
   - Equation (row picture)
   - Matrix
   - Vector (column picture): This section, this is where we get geometric reasoning with math rigor.
2. Definition: The vector space $\mathbb{R}^n$ consists of all column vectors $\vec{u}$ with n real valued components.
   - Notation: $\vec{u} = [u_1, u_2, \ldots, u_n]^T$, each entry is called a component.
   - Special case: $\vec{0}$.
3. Examples: Geometry of vectors, imagine displacement.
   - $vecu = [1, 2]^T \in \mathbb{R}^2$. Note not the same as (1,2). Vectors are location independent. Other examples in 4 quadrants. Sad zero vector.
   - $vecu = [-3, 1, 2]^T \in \mathbb{R}^3$
4. Definitions: Vector operations
   - Addition: $\vec{u} + \vec{v} = [u_1 + v_1, \ldots, u_n + v_n]^T$ in $\mathbb{R}^n$. Note need vectors of same length.
   - Scalar multiplication: $c\vec{u} = [cu_1, \ldots, cu_n]^T$ for scalar c.
   - Subtraction (triangular law): $\vec{u} - \vec{v}$
   - Bonus (dot product to compare direction, more later): $\vec{u} \cdot \vec{v}$
   - Bonus (norm or length, more later): $\|\vec{u}\|_n = \sqrt{u_1^2 + \cdots + u_n^2}$

5. Examples: $\vec{u} = [1, 2]^T, \vec{v} = [3, 1]^T$
   - $2\vec{u}, -\vec{u}, 4\vec{u}, 0\vec{u}, c\vec{u}$, set of all scalar multiples results in a line (rescaling gives name to scalar)
   - $\vec{u} + \vec{v}, \vec{v} + \vec{u}$ (Parallelogram law)
   - $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ (Triangular law)
6. Theorem (these mirror familiar algebraic properties, some proofs in HW): For all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and scalars
   (a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutative)
   (b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (Associative)
   (c) $\vec{u} + \vec{0} = \vec{u}$ (Identity)
   (d) $\vec{u} + (-\vec{u}) = \vec{0}$ for $-\vec{u} = (-1)\vec{u}$ (Inverse)
   (e) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ (Distribution)
   (f) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ (Distribution)
   (g) $c(d\vec{u}) = (cd)\vec{u}$ (Compatibility)
   (h) $1\vec{u} = \vec{u}$ (Identity)
7. Definition (the linear of linear algebra): Vector $\vec{y} \in \mathbb{R}^n$ is a linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if there exists scalars $c_1, \ldots, c_n$ (called weights) such that
   $$\vec{y} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$
8. Example (Vector equation): Show that $\vec{b} = [3, 1, -1]^T$ is a linear combination of vectors $\vec{a_1} = [2, 0, -1]^T$ and $\vec{a_2} = [-1, 1, 1]^T$.
   - This is equivalent to solving a linear system via GE.
   - Geogebra and geometric interpretation.
   - Is the same true for any $\vec{b}$? No, only if it lies in the plane generated by all linear combinations of $\vec{a_1}$ and $\vec{a_2}$. Consider a $\vec{b}$ which does not.
9. Definition: The collection of all linear combinations of $\vec{v_1}, \ldots, \vec{v_p} \in \mathbb{R}^n$ is called the Span$\{\vec{v_1}, \ldots, \vec{v_p}\}$ and is a subset of $\mathbb{R}^n$.
10. Homework: 1, 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 27

.4 1.4 The matrix equation $A\vec{x} = \vec{b}$

1. 3 views of linear algebra:
   - Row picture (lines and planes, done)
   - Column picture (vectors, done)
   - Matrix picture (now, idea is to capture linear combination as an operation)
2. Definition: For A a $m \times n$ matrix with columns $\vec{a_1}, \ldots, \vec{a_n}$ and $\vec{x} \in \mathbb{R}^n$, the produce $A\vec{x}$ is the linear combination of the columns of A with weights as entries in $\vec{x}$. That is,
   $$A\vec{x} = [\vec{a_1} \ldots \vec{a_n}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a_1} + \ldots x_n\vec{a_n}$$
   Note, the number of columns in A must match the number of entries of $\vec{x}$.

3. Example: Multiply a random $A_{2\times 3}$ matrix by a $\vec{x}_{3\times 1}$ vector.

   3 linear algebra POVs are here. For general $\vec{x}$, write

   - 2 equations (planes, geometry)
   - Linear combinations of 3 vectors (vectors, geometry)
   - Matrix equation $A\vec{x} = \vec{b}$ (operation on a vector, similar to idea of function). Important question is given A, can we solve $A\vec{x} = \vec{b}$ for any RHS vector $\vec{b}$.

   We will readily switch between these views to gain insight and perspective.

4. Example (entry-wise matrix multiplication): Multiply a random $A_{3\times 3}$ matrix by a $\vec{x}_{3\times 1}$ vector.

   - Linear combination of 3 row vectors. Important concept.
   - Dot product of rows and $\vec{x}$. This version is more convenient for hand calculation.

   Replace A with identity matrix $I_{3\times 3}$ and ask them to guess result.

5. Theorem (linearity of matrix multiplication): For matrix A $m \times n$, vectors $\vec{u}, \vec{v}$ $n \times 1$, and scalar c, we have

   (a) $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$ (distributive)

   (b) $A(c\vec{u}) = c(A\vec{u})$ (associative)

   Proof (of (a), $n = 3$ case, (b) in text): All we need is the corresponding result from vectors in previous section.

$$\begin{aligned} A(\vec{u} + \vec{v}) &= A \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \\ &= (u_1 + v_1)\vec{a_1} + (u_2 + v_2)\vec{a_2} + (u_3 + v_3)\vec{a_3} \\ &= (u_1\vec{a_1} + u_2\vec{a_2} + u_3\vec{a_3}) + (v_1\vec{a_1} + v_2\vec{a_2} + v_3\vec{a_3}) \\ &= A\vec{u} + A\vec{v} \end{aligned}$$

6. Theorem (big result for entire course, will grow this list): For A a $m \times n$ matrix, the following statements are either all true or all false.

   (a) For each $\vec{b} \in \mathbb{R}^n$, equation $A\vec{x} = \vec{b}$ has a solution.

   (b) Each $\vec{b} \in \mathbb{R}^n$ is a linear combination of the columns of A.

   (c) The columns of A span $\mathbb{R}^m$.

   (d) A has a pivot position in every row.

7. Homework: 5, 7, 9, 11, 13, 15, 17, 23, 29, 30

.5 1.5 Solution sets of linear equations

1. We want to characterize solutions to a linear system of equations $A\vec{x} = \vec{b}$ for A and $\vec{b}$ given and $\vec{x}$ unknown thru two perspectives:

   - Geometrically (picture, intuition)
   - Explicitly (formula, practical)

   Our approach will be to consider two related cases:

   - Homogeneous linear system: $A\vec{x} = \vec{0}$

- Nonhomogeneous linear system: $A\vec{x} = \vec{b}$

2. Homogeneous linear system: $A\vec{x} = \vec{0}$
   (a) For any A, $\vec{x} = \vec{0}$ is always a solution (called the trivial solution). We seek nontrivial solutions $\vec{x} \neq \vec{0}$. Will there always be a nontrivial solution? Only if the GE solution has at least one free variable.
   (b) Solve the homogeneous linear system:
   $$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \vec{x} = \vec{0}$$
   Solving by GE gives x_3 a free variable with
   $$\vec{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = x_3\vec{v} = span\{\vec{v}\}$$
   The set of these solutions are a line thru the origin parallel to $\vec{v}$.
   (c) Change above example so three rows are multiples of eachother giving 2 free variables.
   $$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 3 & -5 \\ 1 & 3 & -5 \end{bmatrix} \vec{x} = \vec{0}$$
   Solving by GE gives x_2, x_3 free variables with
   $$\vec{x} = \begin{bmatrix} -3x_2 + 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\vec{v_2} + x_3\vec{v_3} = span\{\vec{v_2}, \vec{v_3}\}$$
   generating a plane thru the origin. View in Geogebra.

3. Nonhomogenous linear system: $A\vec{x} = \vec{b}$
   (a) Example as from before:
   $$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$
   gives
   $$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
   Again x_3 is free and we have
   $$\vec{x} = \begin{bmatrix} -4x_3 - 5 \\ 3x_3 + 3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = \vec{p} + x_3\vec{v}$$
   for the same $\vec{v}$ as in the homogenous case. Graph same lines as before but first shifted by vector $\vec{p}$ away from the origin.
   (b) Solution to nonhomogenous equation is the same as the homogenous case but translated.
   (c) Theorem: For $A\vec{x} = \vec{b}$ consistent and $\vec{p}$ a particular solution, then the solution set of all $A\vec{x} = \vec{b}$ is all vectors of the form
   $$w = \vec{p} + \vec{v_h}$$
   where $\vec{v_h}$ is any solution to the homogeneous equation $A\vec{x} = \vec{0}$. (sketch the plane case in $\mathbb{R}^3$)

4. Homework: 1, 5, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31

.6 1.6 Applications of linear systems

1. Skip. Possible lab material.

.7 1.7 Linear independence

1. Here we rephrase homogeneous systems of linear equations as vector equations instead. So our example homogeneous linear system

$$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \vec{x} = \vec{0}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -8 \\ 9 \end{bmatrix} = \vec{0}$$

which brings us to an important definition for this course.

2. Definition: The set of vectors $\{\vec{v_1}, \dots, \vec{v_p}\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1\vec{v_1} + \cdots + x_p\vec{v_p} = \vec{0}$$

has only the trivial solution. If there are weights $x_1, \dots, x_p$ not all zero such that

$$x_1\vec{v_1} + \cdots + x_p\vec{v_p} = \vec{0}$$

then $\{\vec{v_1}, \dots, \vec{v_p}\}$ is linearly dependent.

3. Example: Previous work on

$$\begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \vec{x} = \vec{0}$$

gave solution set

$$\vec{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} = x_3\vec{v} = span\{\vec{v}\}$$

meaning that there are infinitely many solutions. Choosing $x_3 = 1$ gives $\vec{x} \neq 0$ so that

$$-4\vec{v_1} + 3\vec{v_2} + \vec{v_3} = \vec{0}$$

and so these three column vectors are linearly dependent. Alternatively,

$$\vec{v_3} = 4\vec{v_1} - 3\vec{v_2}$$

and there is redundant information in these columns. This points towards the following results.

4. Theorem: The columns of matrix A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

5. Theorem: The set of vectors $\{\vec{v_1}, \dots, \vec{v_p}\}$ is linearly dependent if one vector can be written as a linear combination of the others.

6. Intuition of linear dependence / independence:

   (a) One vector: Is the set of one vector linearly independent or dependent? Only if that vector is not the zero vector.

$$\vec{v_1} = [1, 2]^T$$

(b) Two vectors, $n = 2$: When are two vectors linearly dependent? If one is a scalar multiple of the other.

$$\vec{v_1} = [1,2]^T, \vec{v_2} = [5,10]^T, \text{ on the same line, same direction of information}$$

$$\vec{v_1} = [1,2]^T, \vec{v_2} = [1,10]^T, \text{ not on the same line, separate direction of information}$$

(c) Three vectors, $n = 2$: When are three vectors linearly dependent? Always. GE always yields a free variable. Graph example to show one vector as a linear combination of the other. Redundant information. This generalizes to the following result.

7. Theorem: The set $\{\vec{v_1}, \ldots, \vec{v_p}\}$ in $\mathbb{R}^n$ with $p > n$ is linearly dependent.

8. Note: With this section especially, we start to see the wide range of terminology in this course, much of it is a different perspective on the same root concept. Keeping this all straight is essential to avoid confusion.

9. Homework: 1, 3, 5, 7, 9, 15, 17, 21, 23, 25, 27, 31

.8 1.8 Introduction to the linear transformation

1. New perspective: Think of $A\vec{x} = \vec{b}$ as a matrix operation.
   (a) Similar to $f(x) = y$, function f acting on x to result in y.
   (b) Matrix A acts on vector $\vec{x}$ resulting in vector $\vec{y}$.

2. Def and terminology: A random 2×3 matrix times $\vec{x}$ giving $\vec{b}$.
   (a) Picture: Mapping of inputs to outputs
   (b) Inputs (domain) any vector in $\mathbb{R}^3$
   (c) Outputs (range) some vectors in $\mathbb{R}^2$ (codomain)
   (d) Linear transformation A mapping inputs to outputs
   (e) Notation: Matrix transformation $T(\vec{x}) = A\vec{x} = \vec{b}$ where $\vec{b}$ is the image of $\vec{x}$
   (f) Just as we try to understand a function for any input, we will try to understand a matrix transformation in general.

3. Example: Same 2×3 matrix as above. Define $T(\vec{x}) = A\vec{x}$.
   (a) Find the image of random vector $\vec{x}$.
   (b) For random vector $\vec{b}$, find input $\vec{x}$ if possible. Is it unique? If no, transformation is not invertible (reversible) as with function inverses.

4. Linear transformations: Defined and alternate forms.
   (a) Def: A transformation $T(\vec{x})$ is linear if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}), \quad \text{and} \quad T(c\vec{u}) = cT(\vec{u})$$

   for all vectors $\vec{u}, \vec{v}$ in the domain of T and all scalars c.
   (b) We have from before that all matrix transformations are linear transformations, but there are other linear transformations to be seen later on.
   (c) Theorem: If $T(\vec{x})$ is a linear transformation, then

$$T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v}), \quad \text{and} \quad T(\vec{0}) = T(\vec{0})$$

   for all vectors $\vec{u}, \vec{v}$ in the domain of T and all scalars c, d.

(d) Theorem: The superposition principle holds for any linear transformation $T(\vec{x})$. That is,

$$T(c_1\vec{u_1} + \cdots + c_p\vec{u_p}) = c_1T(\vec{u_1} + \cdots + T(c_p\vec{u_p})$$

(e) These two theorems are often more convenient.

5. Examples: Geometry of linear transformations. For vectors $\vec{u} = [3,1]^T$, $\vec{v} = [1,2]^T$ and $\vec{u}+\vec{v}$, what does transformation $T(\vec{x}) = A\vec{x}$ do? Use linearity for $\vec{u}+\vec{v}$. Draw the parallelogram to see effect.
   - Dilation
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
   - Contraction
$$A = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$
   - Reflection
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
   - Shear
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
   - 90 degree rotation
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
   - Projection
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
6. Homework: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 29, 31

.9 1.9 The matrix of a linear transformation

1. In the last section, we looked at a matrix transformation and saw geometry. Here we reverse. Given a geometric description, we will derive the needed linear transformation.
2. Unit basis in $\mathbb{R}^2$:
   - $\vec{e_1} = [1,0]^T, \vec{e_2} = [0,1]^T$, all other vectors in $\mathbb{R}^2$ are linear combinations of these two. Show example.
   - Amounts to geometric transformation of the unit square.
   - Using linearity, understanding $T(\vec{x}) = A\vec{x}$ action on these two unit basis will determine A. This is because for any $\vec{x} \in \mathbb{R}^2$,
$$\vec{x} = x_1\vec{e_1} + x_2\vec{e_2}$$
3. Example: Find linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that
$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$
   Since we have for any $\vec{x}$ that
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2,$$

then

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2,) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2) = [T(\vec{e}_1) + T(\vec{e}_2)]\,\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \\ 4 & 0 \end{bmatrix} \vec{x}$$

This holds for higher dimensional space as well.

4. Theorem: For linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, there exists a unique matrix A such that

$$T(\vec{x}) = A\vec{x} = [T(\vec{e_1}) \cdots T(\vec{e_n})]\,\vec{x}$$

for unit basis vectors $\vec{e_1}, \ldots, \vec{e_n}$.

Matrix A is called the standard matrix for the linear transformation T. Also see that any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is also a matrix transformation.

5. Example: Use the above theorem to find the linear transformation which
   - Projects $\vec{x}$ onto the main diagonal.
   - Rotates $\vec{x}$ 180 degrees about the origin.

6. Example: Use the above theorem to find the linear transformation $T(\vec{x})$ which rotates vector $\vec{x}$ by θ radians counter clockwise.
   - Draw $\vec{e_1}$ and $\vec{e_2}$ in the plane and resulting rotated vectors.
   - Use trig to find resulting vectors:

$$T(\vec{e_1}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad T(\vec{e_2}) = \begin{bmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{bmatrix} = \begin{bmatrix} \cos(\pi/2 - (-\theta)) \\ \sin(\pi/2 - (-\theta)) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

   - Theorem result says

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

7. Catalog of geometric transformations:
   - Thinking of what a transformation does to unit basis vectors $\vec{e_1}$ and $\vec{e_2}$ is equivalent to picturing its action on the unit square.
   - See text for list of common transformations.
   - Know these, do not memorize. Just think about what happens to $\vec{e_1}$ and $\vec{e_2}$

8. $A\vec{x} = \vec{b}$, existence and uniqueness rephrased in terms of linear transformations.

   (a) Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one (though maybe more) $\vec{x}$ in $\mathbb{R}^n$. This is existence. Draw picture to illustrate.

   (b) Definition: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one (though maybe none) $\vec{x}$ in $\mathbb{R}^n$. This is uniqueness. Draw picture to illustrate.

   (c) Return to textbook basic linear transformations. Which are onto? One-to-one? Both? Neither?

   (d) Example: Random 3×4 matrix A in REF. Is A onto? Yes, full set of pivots. One-to-one? No, free variable. So we can answer these questions via row reduction, but there is an easier way.

   (e) Theorem: Linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution.
      - If and only if means if statement P is true, then statement Q is also true. Further if Q is true, then P is also true.

- Here we prove this theorems in two steps. (1) Assume P is true, show Q is also true. (2) Assume P is false, then show Q also false (contrapositive of reverse direction).
- Proof of (1): Assume T is one-to-one. Then $T(\vec{x} = \vec{0}$ has only one solution. We know matrix transformations are such that $T(\vec{0}) = \vec{0}$. Then $\vec{x} = \vec{0}$.
- Proof of (2): Assume T is not one-to-one. Then for some $\vec{b}$ in $\mathbb{R}^m$ there are two vectors $\vec{u} \neq \vec{v}$ such that map to $\vec{b}$. But since T is linear

$$T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{b} - \vec{b} = \vec{0}$$

and hence $T(\vec{x}) = \vec{0}$ has a nontrivial solution.

(f) Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A. Then,

- T is onto if and only if the columns of A span $\mathbb{R}^m$.
- T is one-to-one if and only if the columns of A are linearly independent.

Why does this theorem make intuitive sense?

(g) Show T is a one-to-one linear transformation. Is T onto?

$$T(\vec{x}) = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$$

Show a matrix transformation with linearly indep columns, hence a one-to-one linear transformation. Two columns cannot span $\mathbb{R}^3$, so not onto.

9. Homework: 1, 3, 5, 7, 13, 15, 17, 23, 25, 27, 29, 30

.10 1.10 Linear models in business, science, and engineering

1. Possible lab material. Especially difference equations.

Chapter 2: Matrix algebra

.1 2.1 Matrix operations

1. Goal of this chapter: Treating A as an operator, we get a new view on $A\vec{x} = \vec{b}$.
   - Similar to $\frac{d}{dx}$ as an operator on $f(x)$
   - What are the properties of operator A?
   - How to reverse this operation (will call inverse)?
2. Basic matrix operations (easy): Arithmetic (addition and scalar multiplication)

   (a) Random 2×3 matrices A and B.

   (b) $2A$, entry-wise scalar multiplication

   (c) $A + B$, as with vectors, need dimensions to agree, entry-wise addition (and subtraction)

   (d) Theorem: For A, B, C matrices of the same dimension and scalars r, s,

   - $A + B = B + A$ (commutative)
   - $(A + B) + C = A + (B + C)$ (associative for addition)
   - $A + 0 = A$ (identity for addition)
   - $r(A + B) = rA + rB$ (scalar distribution)
   - $(r + s)A = rA + sA$ (matrix distribution)
   - $r(sA) = (rs)A$ (associative for mult)

3. Matrix multiplication:

   (a) Recall: $B\vec{x}$ as a linear combination of the column vectors of $n \times p$ matrix B

   $$B\vec{x} = x_1\vec{b}_1 + \cdots + x_p\vec{b}_p$$

   (b) Matrix composition: $A(B\vec{x})$ for A $m \times n$ and B $n \times p$.
      - Draw diagram: $\vec{x} \to B\vec{x} \to A(B\vec{x})$
      - One step arc on diagram: Think of AB as the new matrix operation for which $\vec{x} \to (AB)\vec{x}$.
      - Similar to function composition: $f(g(x)) = (f \circ g)(x)$
      - How to compute?

      $$B\vec{x} = x_1\vec{b}_1 + \cdots + x_p\vec{b}_p$$

      $$A(B\vec{x}) = A(x_1\vec{b}_1 + \cdots + x_p\vec{b}_p) = x_1A\vec{b}_1 + \cdots + x_pA\vec{b}_p = [A\vec{b}_1 \dots A\vec{b}_p]\vec{x}$$

      - What is the dimension of AB? $m \times p$
      - Definition: For A $m \times n$ and B $n \times p$, then

      $$AB = A[\vec{b}_1 \dots \vec{b}_p] = [A\vec{b}_1 \dots A\vec{b}_p]$$

      where matrix AB is $m \times p$.

   (c) Example: Random matrices A (2×3) and B (3×2).
      - AB column view (can get just a column this way):

      $$AB = A[\vec{b}_1\vec{b}_2] = [A\vec{b}_1A\vec{b}_2]$$

      - AB computational view (can get just an entry this way): Each row as row dot column
      - AB row view (can get just a row this way):

      $$AB = \begin{bmatrix} row_1(A) \\ row_2(A) \end{bmatrix} B = \begin{bmatrix} row_1(A)B \\ row_2(A)B \end{bmatrix}$$

      where this last step is done entry-wise.
      - Show $AB \neq BA$. Makes sense thinking of function composition.

4. Matrix multiplication in general:

   (a) Summary of matrix multiplication: For A $(m \times n)$, B $(n \times m)$, and $C = AB$ $(m \times p)$,
      - Column-wise in general

      $$C = AB = [A\vec{b}_1 \dots A\vec{b}_p]$$

      - Computational in general

      $$C = [c_{ij}], \quad c_{ij} = row_i(A) \cdot \vec{b}_j = \sum_{k=1}^{n} a_{ik}b_{kj}$$

      - Row-wise in general

      $$C = AB = \begin{bmatrix} row_1(A)B \\ \vdots \\ row_m(A)B \end{bmatrix}$$

   (b) Theorem (matrix multiplication properties): For A, B, C matrices of suitable dimension
      - $A(BC) = (AB)C$ (associative)
      - $A(B + C) = AB + AC$ (right distributive)
      - $(B + C)A = BA + CA$ (left distributive)

- $r(AB) = (rA)B = A(rB)$ (scalar commutative)
- $IA = A = AI$ (identity matrix multiplication, explain what I is)

Proofs in homework and book. These follow from vector properties shown previously.

(c) Warning: Matrix multiplication does not follow the intuition of scalar multiplication. In general

- $AB \neq BA$, not surprising since linear combos of cols of A need not equal linear combinations of cols of B.
- $AB = AC$ need not imply $B = C$.
- $AB = 0$ need not imply $A = 0$ or $B = 0$ for 0 the zero matrix.
- Construct you own examples for fun.

5. Powers of a matrix A

(a) Def: $A^k = A \cdot A \cdot \cdots \cdot A$, repeated multiplication k times

(b) Note, need a square matrix A $(n \times n)$.

(c) Think if repeating an operation over and over. Similar to repeat function composition.

(d) Will revisit this notion for important applications later.

6. Matrix transpose:

(a) Def: For $(m \times n)$ matrix A, the transpose of A written A^T is the $(n \times m)$ matrix whose columns are the rows of A

- Example: Random (2×3) matrix.
- Draw general picture of row and column vectors switching
- Entry-wise: $A_{m\times n} = [a_{ij}]$ gives $A^T_{n\times m} = [a_{ji}]$

(b) Theorem: Properties of matrix transpose. For matrices A and B of suitable dimensions and scalar r,

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = A^T B^T$ (only surprising result, shown in HW)

(c) Example: Random matrices and vectors $A_{3\times 2}, B_{2\times 2}, \vec{b}_3, \vec{c}_2$, find all possible products which are defined.

7. Homework: 1, 3, 5, 10, 11, 12, 15, 17, 19, 21, 23, 27, 33

.2 2.2 The inverse of a matrix

1. Reversing $A\vec{x} = \vec{b}$.

(a) Draw picture: Thinking of $A\vec{x} = \vec{b}$ as an operation $A : \vec{x} \to \vec{b}$, how to invert this process? Same idea as inverting a function. We need the operation to be one-to-one.

(b) Definition: Square matrix $A_{n\times n}$ is invertible if there exists matrix $A^{-1}_{n\times n}$ such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

for $I_{n\times n}$ the identity matrix. Note this only makes sense for square matrices.

(c) Connection: Think of as composition of linear operators.

$$\vec{x} = I\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x})$$

Draw picture. Similar to function inverses and composition, $(f \circ f^{-1})(x) = x$.

(d) Not all matrices A are invertible. If invertible, called non singular. If not invertible, called singular (alone and without a counterpart). Singular terminology may also refer to unusual. In face most square matrices randomly generated are invertible (non-singular), for (2×2) case, need both columns to be colinear which is less common than not. Singular may also reference troublesome. Last reason may referr to the determinant being zero resulting in zero division (singularity).

(e) Example: Show that

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$$

are inverses of eachother. Just need to check that $AB = BA = I$ to show $B = A^{-1}$.

2. Finding matrix inverses:

(a) For A a given matrix,

- How to check if A is invertible? For functions can check if $f(x)$ is one-to-one.
- How to compute A^{-1}? Method for functions as well, key is $(f^{-1} \circ f)(x) = x$, the inverse relation.

(b) General 2×2 case:

$$AB = A[\vec{b}_1 \vec{b}_2] = [A\vec{b}_1 A\vec{b}_2] = I$$

requires

$$A\vec{b_1} = \vec{e_1}, \quad A\vec{b_2} = \vec{e_2}.$$

These are two linear systems to solve. Likewise 3 linear systems for (3×3), and so on.

(c) Example: Find the inverse of

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}.$$

Previous example lets us know what to expect here.

- Solve two systems as separate augmented matrices.

$$A\vec{b_1} = \vec{e_1}, \quad A\vec{b_2} = \vec{e_2}$$

by using backwards substitution.
- Note redundancy and combine into a single augmented matrix

$$[A|I] \;\rightarrow\; [I|B] = [I|A^{-1}]$$

then use full Gauss-Jordan elimination.
- Elementary row operations are a key ingredient here. More shortly.
- Note: This approach of using Gaussian elimination extends to 3 or higher dimensions as well.

(d) Theorem: Can complete the (2×2) case in general. For any matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

A is invertible if $ad - bc \neq 0$ and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A is not invertible. In the (2×2) case, $ad - bc$ is called the determinant of A (note zero division singularity). Derive and verify on own.

(e) Validate for previous example.

(f) Above theorem generalizes to higher dimensions to a certain extent. Namely the idea of determinant generalizes via recursion. More later.

3. Using inverses to solve linear systems $A\vec{x} = \vec{b}$.

   (a) Theorem: If $A_{n\times n}$ is invertible, then for each $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution
   $$\vec{x} = A^{-1}\vec{b}.$$
   This isn't a practical method to solve (see previous example work), but it is important in reach (general, existence, uniqiness).

   Proof: Two steps:

   - Existence: Check that $\vec{x} = A^{-1}\vec{b}$ works. Key is inverse relation $AA^{-1} = I$.
   - Uniqiness: If $\vec{x}$ and $\vec{y}$ are two solutions, then $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$. Then we have $A\vec{x} = A\vec{y}$ and so $A^{-1}A\vec{x} = A^{-1}A\vec{y}$ implying $\vec{x} = \vec{y}$.

   (b) Example: Solving a linear system via inverse.
   $$\begin{cases} 3x_1 + 2x_2 = 3 \\ 7x_1 + 4x_2 = 2 \end{cases}$$
   Use the above calculation where $\vec{x} = A^{-1}\vec{b} = [-4, 11]^T$. Note the Gaussian elimination work as before was packaged into the inverse function calculation.

4. Properties of inverses

   (a) Theorem: For invertible matrices A and B of the same dimension,

      i. $(A^{-1})^{-1} = A$ (makes sense with respect to reversing an operator)

      ii. $(AB)^{-1} = B^{-1}A^{-1}$ (note the reverse of multiplication order, this is the reverse of operator composition)

      iii. $(A^T)^{-1} = (A^{-1})^T$ (note inverse of a symmetric matrix also symmetric)

   (b) Proofs of each, just need to check each works. Multiply to the identify.

      i. Need matrix C such that
      $$A^{-1}C = I, \quad CA^{-1} = I.$$
      By definition $C = A$ is what we have.

      ii. Compute $(AB)(B^{-1}A^{-1}) = \cdots = I$ and $(B^{-1}A^{-1})(AB) = \cdots = I$

      iii. This one relies on the reversing of multiplication for transpose.
      $$(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
      $$(A^{-1})^T(A^T) = (AA^{-1})^T = I^T = I$$

5. Elementary matrices and decomposing Gaussian elimination

   (a) Example: Think of three elementary row operations on matrix
   $$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

i. $R_1 \leftrightarrow R_3$: We seek matrix E_1 such that

$$E_1 A = E_1 \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 8 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Thinking bout the row picture for matrix multiplication,

$$\begin{bmatrix} row_1(E_1) \\ row_2(E_1) \\ row_3(E_1) \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} 4 & -3 & 8 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

So doing the same elem row operation on the identity matrix is the multiplier we need to swap rows 1 and 3.

ii. $R_1 \to 2R_1$. Thinking of the same row picture,

$$E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

iii. $R_3 \to R_3 + 2R_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(b) Elementary matrices:

- Definition: An elementary matrix is the matrix resulting from performing a single elementary row operation on the identity matrix I.
- So each elementary row operation can be performed as multiplication of an elementary matrix.
- Turns out all elementary matrices are invertible. The inverse can be found by construction (reversing the elementary row operation) and validating the inverse relation. Illustrate for above 3 examples.

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_1^{-1} = ?, \quad E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = ?, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad E_3^{-1} = ?.$$

(c) Example: Use Gaussian elimination to find the inverse of

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

Perform Gauss-Jordan elimination on

$$[A \mid I] \quad \to \quad \dots \quad \to \quad [I \mid A^{-1}]$$

resulting in

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

Easy to check this is correct: $AA^{-1} = I$.

(d) Thinking of elementary matricies, we must have

$$A^{-1} = E_5 E_4 E_3 E_2 E_1$$

and so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}.$$

Easy to check. This leads to a general result.

(e) Theorem: Square matrix A is invertible if and only if A is row equivalent to the identity matrix I.

6. Homework: 1, 5, 7, 9, 21, 25, 27, 29, 31, 35

.3 2.3 Characterizations of invertible matrices

1. Theorem: (Invertible matrix theorem)

   For A a square $n \times n$ matrix, the following statements are equivalent (either all true or all false).

   - A is an invertible matrix
   - A is row equivalent to the $n \times n$ identity matrix
   - A has n pivots positions
   - The equation $A\vec{x} = \vec{0}$ has only the trivial solution
   - The columns of A for a linearly independent set
   - The linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one
   - The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$
   - The columns of A span $\mathbb{R}^n$
   - The linear transformation $T(\vec{x}) = A\vec{x}$ is onto
   - There is a $n \times n$ matrix C such that $CA = I$
   - There is a $n \times n$ matrix D such that $AD = I$
   - A^T is an invertible matrix

2. Example: Problems 1-6 in the exercises. Decide if invertible or not.

   (a) Yes, LI columns

   (b) No, LD columns

   (c) Yes, 3 pivots after row reduction $(5, -7, -1)$. Don't need to do the row reduction here.

   (d) No, LD columns since zero vector included.

   (e) No after swapping rows 1 and 2 and doing row reduction, only 2 pivots

   (f) Do row reduction to see.

   (g) Note, 8 easy to see

3. Inverse of linear transformations:

   (a) Def: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that
   $$S(T(\vec{x})) = \vec{x}, \quad T(S(\vec{x})) = \vec{x}$$
   for all $\vec{x} \in \mathbb{R}^n$. S is called the inverse of T and we denote $S = T^{-1}$.

   (b) Theorem: For $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation with $T(\vec{x}) = A\vec{x}$, T is invertible if and only if A is an invertible matrix. In which case, $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

4. Homework: 1-7 odd, 11, 15-23 odd, 33

.4 2.4 Partitioned matrices

1. Idea: Generalize matrix multiplication to block multiplication
   - Certain problems naturally lead to symmetry / block structure of a matrix.
   - Can also block matrices to distribute computation to speed up compute time (parallel computing). High performance computing especially uses this approach.
2. Example:

$$A = \left[\begin{array}{cc|ccc} 2 & -3 & 1 & 0 & -4 \\ \hline 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{array}\right], \quad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ \hline -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{array}\right]$$

   (a) Note these are compatible for multiplication.
   (b) 2 approaches already: Row and columns

$$AB = A[\vec{b}_1 \ \vec{b}_2] = [A\vec{b}_1 \ A\vec{b}_2]$$

$$AB = \begin{bmatrix} row_1(A) \\ row_2(A) \\ row_3(A) \end{bmatrix} B = \begin{bmatrix} row_1(A)B \\ row_2(A)B \\ row_3(A)B \end{bmatrix}$$

   (c) New idea: Partition A and B into blocks.

$$AB = [A_1|A_2] \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = [A_1B_1 + A_2B_2]$$

   for A_1 (3×2), A_2 (3×3) and B_1 (2×2) and B_2 (3×2). Compare to the above 2 forms.
   (d) Note: We need submatrices to be compatible for multiplication.
   (e) Can partition further as

$$AB = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = [A_1B_1 + A_2B_2]$$

   for A_1 (1×2), A_2 (1×3), A_3 (2×2), A_4 (2×3) and B_1 (2×2) and B_2 (3×2). Compare to the above.
   (f) Think of other ways to partition:
      - A into 4 parts, B into 2
      - B into 3 parts
      - Repeat partitioning leads to the below theorem.
3. Theorem: For A $(m \times n)$ and B $(n \times p)$,

$$AB = [\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n] \begin{bmatrix} row_1(B) \\ row_2(B) \\ \vdots \\ row_n(B) \end{bmatrix}$$

4. Example: Inverses of partitioned matrices.
   (a) Find A^{-1} for A $(n \times n)$ where

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

   where A_{11} $(p \times p)$, A_{12} $(p \times q)$, A_{22} $(q \times q)$, and 0 $(q \times p)$.

(b) Find matrix B such that

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = I_{n\times n}$$

Multiplying, we have that

- $A_{11}B_{11} + A_{12}B_{21} = I_p$
- $A_{11}B_{12} + A_{12}B_{22} = 0$
- $A_{22}B_{21} = 0$
- $A_{22}B_{22} = I_q$

(c) The last bullet says $B_{22} = A_{22}^{-1}$ from the invertible matrix theorem.

(d) From the third bullet, $B_{21} = 0$ since A_{22} is invertible and multiplying by A_{22}^{-1}.

(e) The first bullet then gives $B_{11} = A_{11}^{-1}$.

(f) Finally, the second bullet gives

$$B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}.$$

(g) Finally,

$$A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

for B as derived.

(h) Note, this approach is especially nice if we can get down to (2×2) matrices where the inverse has a simple formula.

5. Homework: 1, 3, 7, 9, 11, 13, 15.

.5 2.5 Matrix factorizations

1. Lab content
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.6 2.6 The Leontief input-output model

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.7 2.7 Applications to computer graphics

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.8 2.8 Subspaces of $\mathbb{R}^n$

1. Here we generalize to the main theory of linear algebra as a way to delve deeper into $A\vec{x} = \vec{b}$.
2. Subspaces of $\mathbb{R}^n$.

   (a) Definition: A *subspace* of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ such that

   - The zero vector is in H
   - For each $\vec{u}, \vec{v} \in H$, we have $(\vec{u} + \vec{v}) \in H$
   - For each $\vec{u} \in H$, we have $(c\vec{u}) \in H$

This is says subspaces are *closed* under vector addition and scalar multiplication.

(b) Example: For $\vec{u}, \vec{v}, \vec{w}$ as

$$\vec{u} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix},$$

show $Span\{\vec{u}, \vec{v}, \vec{w}\}$ is a subspace of $\mathbb{R}^3$. Is

$$\vec{p} = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$$

in this subspace?

- Solution: For $Span\{\vec{u}, \vec{v}, \vec{w}\}$, need to check three properties. Note, $\vec{y} \in Span\{\vec{u}, \vec{v}, \vec{w}\}$ means

$$\vec{y} = x_1\vec{u} + x_2\vec{v} + x_3\vec{w}$$

  for some scalars x_1, x_2, x_3. Then the zero vector is there. Also, show addition and scalar multiplication are preserved.
- Note, spans will always be a subspace. Sometimes say subspace spanned by these vectors or subspace generate by these vectors.
- For $\vec{p}$, this amounts to solving a linear system via Gaussian elimination.

$$\left[\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  The system is consistent and so $\vec{p}$ is in the subspace.

(c) Example: $\{\vec{0}\}$ is a subspace of $\mathbb{R}^n$. Check three items. Called the zero subspace.

3. Column and null space of a matrix A. The two fundamental subspaces concerning $A\vec{x} = \vec{b}$.

(a) Definitions:

- The column space of matrix A, written $col(A)$, is the set of all linear combinations of the columns of A
- The null space of matrix A, written $nul(A)$, is the set of all vectors which solve the homogeneous system $A\vec{x} = \vec{0}$.
- Note the dimension of each relies on the number of rows and columns of A.

(b) Theorems:

- $col(A)$ is a subspace of $\mathbb{R}^n$. This holds since it is a span of vectors as discussed above.
- $nul(A)$ is a subspace of $\mathbb{R}^n$. This requires careful proof.

  Proof: Check the three items. $A\vec{0} = \vec{0}$. For $\vec{u}, \vec{v} \in nul(A)$, we have

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

$$A(c\vec{u}) = cA\vec{u} = \vec{0}$$

  which leverages linearity of a matrix transformation.

(c) Example: From previous example, $\vec{p}$ is in $col(A)$.

4. Basis for a subspace.

(a) Definition: A *basis* for a subspace H of $\mathbb{R}^n$ is linearly independent set H which spans H.

(b) Find a basis for the $nul(A)$ for above example.

(c) Find a basis for the $col(A)$ for the above example.

- Set of all columns would span $col(A)$, but they need not be linearly independent. In this case they aren't due to free variables.
- Eliminating to reduced row echelon form, we see how a column is a linear combination of the others.
- While the columns change thru row reduction, the system has the same solution and hence the linear dependence relation does not change.

(d) Theorem: The pivot columns of matrix A form a basis for $col(A)$.

- Note: These columns are the original columns of A, not the echelon form columns. Can see why echelon form wouldn't work with zeros in row entries.

(e) Columns of I form the standard unit basis for $\mathbb{R}^n$. Check that every system is consistent. Full set of pivots says the columns are a basis. Can now see where this terminology comes from.

(f) Find a basis for $col(A)$ and $nul(A)$. Note the difference in dimension of these spaces.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \end{bmatrix}$$

5. Homework: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

.9 2.9 Dimension and rank

1. Coordinate systems

(a) Given a basis $B = \{\vec{b_1}, \dots, \vec{b_p}\}$ of subspace H in $\mathbb{R}^n$, each element $\vec{x}$ of H can be written uniquely as a linear combination of basis elements of B. Reason: If there were two ways,

$$\vec{x} = c_1\vec{b_1} + \dots, \quad \vec{x} = d_1\vec{b_1} + \dots$$

then

$$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b_1}, \dots$$

Due to linear independence of the basis vectors, $c_1 - d_1 = 0$ gives $c_1 = d_1$ and likewise for the remaining weights.

(b) Definition: For basis $B = \{\vec{b_1}, \dots, \vec{b_p}\}$ of subspace H, each $\vec{x}$ in H can be expressed as the *coordinate vector* in $\mathbb{R}^p$ as

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

where $\vec{x} = c_1\vec{b_1} + \dots + c_p\vec{b_p}$.

(c) Example: For

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$$

show $B = \{\vec{v}_1, \vec{v}_2\}$ is a basis for the subspace H spanned by these vectors. Then show $\vec{x}$ is in H and find its coordinate vector. In the end

$$[\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Show picture of the book of the intuition of two dimensional subspace H and coordinates with respect to B.

(d) Note that H in the above example resembles $\mathbb{R}^2$ geometrically and with the coordinate vector. Further, $\vec{x} \to [\vec{x}]_B$ has a one-to-one correspondence since coordinate vectors are unique. Call this an *isomorphism* and say H is *isomorphic* to $\mathbb{R}^2$.

(e) Definition: The *dimension* of nonzero subspace H, denoted by $dim(H)$, is the number of vectors in any basis of H. The dimension of $\{\vec{0}\}$ is defined to be zero.

(f) Note: Can show if H has a basis with p vectors, then all possible basis for H must also have p vectors.

2. Return to linear system $A\vec{x} = \vec{b}$.

   (a) Definition: The *rank* of matrix A, denoted $rank(A)$, is the dimension of the column space of A.

   (b) Example: Number 12 in the book on screen. A has 5 total columns with 3 pivot columns. Then $rank(A) = 3$. Note, the dimension of the null space of A is the count of the remaining columns (free variables), 2 in this case. It is always the case that $rank(A) + dim(Nul(A)) = n$ where n is the total column number of A.

   (c) Theorem: For A matrix with n columns, we have that $rank(A) + dim(Nul(A)) = n$.

   (d) Terminology of *rank* is an important number which measures the "singularness" of a matrix. If full rank, nonsingular. If less than full rank, singular. Lower rank means "more" linear independence. Can also show the rank of A is the same as A^T. That is the number of LI columns matches the number of LI rows. Can see this from RREF form of a matrix.

   (e) Theorem: For H a p-dimensional subspace of $\mathbb{R}^n$, any linearly independent set of p elements in H is a basis for H. Likewise, any set of p elements which spans H is a basis for H.

3. Theorem: Last, we add to the Invertible Matrix Theorem

   (a) The columns of A form a basis for $\mathbb{R}^n$.

   (b) $Col(A) = \mathbb{R}^n$

   (c) $dim(Col(A)) = n$

   (d) $rank(A) = n$

   (e) $Nul(A) = \{\vec{0}\}$

   (f) $dim(Nul(A)) = 0$

4. Homework: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23

Chapter 3: Determinants

.1 3.1 Introduction to determinants

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.2 3.2 Properties of determinants

1. Homework:

.3 3.3 Cramer's rule, volume, and linear transformations

1. Homework:

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.1 4.1 Vector spaces and subspaces

1. Homework: 1-18, 23, 24

.2 4.2 Null spaces, column spaces, and linear transformations

1. Homework: 3-6, 17-26

.3 4.3 Linearly independent sets, bases

1. Homework: 21-25

.4 4.4 Coordinate systems

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.5 4.5 The dimension of a vector space

1. Homework:

.6 4.6 Rank

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.7 4.7 Change of basis

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.8 4.8 Applications to difference equations

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.9 4.9 Applications to Markov chains

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.1 5.1 Eigenvectors and eigenvalues

1. Homework:

.2 5.2 The characteristic equation

1. Homework: 25, 27

.3 5.3 Diagonalization

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.4 5.4 Eigenvectors and linear transformations

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.5 5.5 Complex eigenvalues

1. Homework:

.6 5.6 Discrete dynamical systems

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.7 5.7 Applications to differential equations

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.8 5.8 Iterative estimates to eigenvalues

1. Homework:

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.1 6.1 Inner product, length, and orthogonality

1. Homework:

.2 6.2 Orthogonal sets

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.3 6.3 Orthogonal projections

1. Homework: 19, 20

.4 6.4 The Gram-Schmidt process

1. Homework:

.5 6.5 Least-squares problems

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.6 6.6 Applications to linear models

1. Homework:

.7 6.7 Inner product spaces

1. Homework:

.8 6.8 Applications of inner product spaces

1. Homework:

Chapter 7: Symmetric matrices and quadratic forms

.1 7.1 Diagonalization of symmetric matrices

1. Homework:

.2 7.2 Quadratic forms

1. Homework:

.3 7.3 Constrained optimization

1. Homework:

.4 7.4 The singular value decomposition

1. Homework:

.5 7.5 Applications to image processing and statistics

1. Homework: