

You have two hours to complete this exam. Show all work, justify your solutions completely, simplify as much as possible. The only materials you should have on your desk are this exam and a pencil. If you have any questions, be sure to ask for clarification.

1. (12 points) (a) State the polynomial interpolation error theorem.

(b) Prove the polynomial interpolation error theorem.

2. (12 points) Let function  $g(x) = x^2(2 - x)$ .

(a) Complete the following statement. A sequence  $x_n$  converges to number  $L$  quadratically if

(b) Write down a fixed point iteration for finding the fixed points of  $g$ .

(c) Find the fixed points of  $g$  algebraically.

(d) To which of the fixed points found in (c) will the iteration in (b) converge quadratically? Why?

(e) Find a function  $f$  with roots the same as the fixed points of  $g$ .

(f) Write down Newton's method for finding the roots of function  $f$  from part (e).

(g) To which of the roots found of function  $f$  will the iteration in (f) converge quadratically? Why?

3. (12 points) Find the  $LU$  decomposition without partial pivoting for the following matrix. Show all steps!

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 5 & 1 & 1 \end{bmatrix}$$

4. (12 points) Prove that a polynomial interpolant of degree at most  $n$  through the  $(n + 1)$  points  $\{(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))\}$  must be unique.

5. (12 points) (a) A freshman calculus student was amazed to learn that the prime numbers have no predictable pattern. Mess with their head by using divided differences to produce a polynomial  $P$  for which  $P(n)$  gives the  $n$ th prime number for  $n = 1, 2, 3, 4, 5$ . (Recall, the first prime number is 2).

- (b) Assuming the existence of such a miraculous function  $f$  for which  $f(n)$  gives the  $n$ th prime number, provided the error formula for  $f(6) - P(6)$ .

6. (12 points) (a) List all 8 conditions for  $S(x)$  to be a natural cubic spline for a function  $f$  through points  $(-1, f(-1))$ ,  $(0, f(0))$ ,  $(1, f(1))$ .

- (b) Do there exist constants  $a, b, c, d$  such that the function

$$S(x) = \begin{cases} ax^3 + x^2 + cx, & -1 \leq x \leq 0 \\ bx^3 + x^2 + dx, & 0 \leq x \leq 1 \end{cases}$$

is a natural cubic spline for  $f(x) = |x|$  through points  $(-1, 1)$ ,  $(0, 0)$ ,  $(1, 1)$ .

7. (12 points) Recall that the error for the trapezoidal rule on one subinterval is given by

$$\int_a^b f(x) \, dx - T_1(a, b) = -\frac{(b-a)^3}{12} f''(\xi)$$

where  $T_1(a, b)$  denotes the trapezoidal rule you derived in part (a). Show that the error for the  $n$  step composite trapezoidal rule approximation is given by

$$\int_a^b f(x) \, dx - T_n(a, b) = -\frac{(b-a)}{12} f''(\eta) h^2$$

where  $T_n(a, b)$  denotes the  $n$  step composite trapezoidal rule and  $h$  is the size of each subinterval.

8. (12 points) (a) Find the needed constants  $A_0, A_1, x_0, x_1$  so that the quadrature rule of the form

$$\int_{-1}^1 f(x) \, dx \approx A_0 f(x_0) + A_1 f(x_1)$$

is exact for degree 3 polynomials. What is the name of this quadrature rule? (Hint: Assume  $A_0 = 1$ .)

- (b) Use part (a) to find constants  $B_0, B_1, t_0, t_1$  so that the quadrature rule of the form

$$\int_{-1}^1 t g(t) \, dt \approx B_0 g(t_0) + B_1 g(t_1)$$

which is exact for degree 3 polynomials.



9. (12 points) (a) Give the Taylor series expansion for  $f(x - h)$  and  $f(x + h)$  with center  $x$  up til error  $O(h^5)$ .

(b) Use part (a) to derive the central difference approximation formula for  $f'(x)$ . What is the order of the error?

(c) Use your results in part (b) along with Richardson Extrapolation to create an approximation of  $f'(x)$  which is  $O(h^4)$ .

10. (12 points) Consider the following initial value problem (IVP).

$$\begin{cases} y' = f(t, y), & a \leq t \leq b \\ y(a) = y_a. \end{cases}$$

(a) Give the Taylor series expansion for  $y(t+h)$  with center  $t$  up til error  $O(h^3)$ .

(b) Derive Euler's method for the above IVP. What is the local error and global error?

(c) Derive an order 2 Taylor method for the above IVP. What is the local error and global error?

(d) Write down the methods from (b) and (c) for the following IVP.

$$\begin{cases} y' = 2ty, & 0 \leq t \leq 1 \\ y(0) = 1. \end{cases}$$