Math 371
Quiz 11, May 1, 2014

Name: Solution

Derive an approximation formula for $f'(x)$ which has form

$$f'(x) \approx \frac{1}{2h}\left(4f(x+h) - 3f(x) - f(x+2h)\right)$$

with error term $\frac{1}{3}h^2 f'''(\xi)$.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(\xi_1), \quad \xi_1 \in [x, x+h]$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{3!}f'''(\xi_2), \quad \xi_2 \in [x, x+2h]$$

$$\Rightarrow 4f(x+h) - 3f(x) - f(x+2h)$$

$$= 4f(x) + 4hf'(x) + 2h^2 f''(x) + \frac{4}{3!}h^3 f'''(\xi_1) - 3f(x)$$

$$- \left(f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{3!}h^3 f'''(\xi_2)\right)$$

$$= 2hf'(x) + \frac{4}{3!}h^3 f'''(\xi_1) - \frac{8}{3!}h^3 f'''(\xi_2)$$

$$= 2hf'(x) + \frac{2}{3}h^3\left(f'''(\xi_1) - 2f'''(\xi_2)\right)$$

$$= 2hf'(x) + \frac{2}{3}h^3\left(-f'''(\xi)\right) \quad (\text{By IVT})$$

$$\rightarrow f'(x) = \frac{1}{2h}\left(f(x+h) - 3f(x) - f(x+2h)\right) + \frac{1}{3}h^2 f'''(\xi).$$