Calculus I Notes

Contents

Review

Prerequisite take home quiz assigned, refresh and keep track of what is important. Give short discussion / highlights below.

1. Day 1: What is a function? Answer in a way which explains to someone who doesn't know in the best way you can. Inspire via Feynman method: `https://www.youtube.com/watch?v=FrNqSLPaZLc`
2. Functions
   - Idea, def, domain/range, graph, vertical line test
   - What is a function good for? Why is one output so important?
3. Function graphs
   - Intercepts, odd/even function, function transformations, increasing/decreasing, asymptotes
4. Composite of functions, think of as combining multiple functions (first step, second, etc)
5. Inverse function (how to reverse a function? always possible?)
   - Horizontal line test, function composition with original, graph relations.
6. Simple functions (the logic of new concept, what is real world, appproximate real world, computers, etc)
   (a) Constant / linear / quadratic function
   (b) Polynomials (simple, computers inspire)
   (c) Rational functions
   (d) Root functions
   (e) Trigonometric functions (circular motion, everywhere)
   (f) Inverse trig functions
   (g) Exponential functions (growth / decay)
   (h) Logarithmic functions
7. Motivating examples: Graph, domain, range, compose.
   (a) $f(x) = -2x - 1$
   (b) $g(x) = x^2 + 3$, restrict to make invertible.
   (c) Piecewise combination of the two about $x = 0$. Domain, range invertible?
   (d) $h(x) = \frac{1}{x}$

Chapter 0

.1 0.0 Motivating Calculus

1. Where does calculus sit within mathematics? Evolution of ideas:

   (a) Develop math tools:
   - Arithmetic (combining numbers, quantify)
   - Algebra (equations and solving for unknowns, abstract)
   - Geometry (visualize, structure, intuition)
   - Functions (Machine to capture a process, polynomials, logarithms, trigonometry, graphs)
   - Calculus (Solve paradoxes of processes, change, area, limit, infinity)

   (b) Math fields (lots):
   - Linear algebra (data, matrices, high dimensional, discrete space)
   - Probability and statistics (chance, randomness, quantify uncertainty)
   - Differential equations (translation of world into calculus, modeling)
   - Analaysis (rigor, generalization, theory)
   - Much more (number theory, computational, hybrid, etc)

   (c) All the calculuses:
   - Calc 1: Main story of calculus, derivative connect to integral, limit is foundation, fundamental question of indeterminnant form
   - Calc 2: Full story of integration, generalize beyond functions, infinite series / power series big new idea
   - Calc 3: Extension to 3+ dimensional space, closer to the real world (eng, physics)

   (d) Calculus 1 contents:
   - Paradox of calculus (zero division and the tangent line, infinite accumulation and area under a curve)
   - Limit (solution to paradox, foundation of calculus)
   - Derivative (change, deep full story, applications)
   - Integral (area, accumulation)
   - Newton and Liebnitz connected last two via FTOC.

2. Two large application areas of calculus:

   (a) Optimization (will discuss soon)
   - `https://en.wikipedia.org/wiki/Mathematical_optimization`
   - `https://www.uwlax.edu/globalassets/offices-services/urc/jur-online/pdf/2016/meyers-jack-daniel.mth.pdf`

   (b) Differential equations (mentioned above)
   - `https://en.wikipedia.org/wiki/Differential_equation`
   - `https://en.wikipedia.org/wiki/List_of_named_differential_equations`

   (c) More as well

3. The big picture of calculus (intuition here, details for the rest of the semester)

   (a) Area under a curve: area of a circle.
   - Consider a hard problem (which we already know). What is the area of a circle with radius R. Pick $R = 3$ for now.

- Lots of ways to chop it up to try (vertical rectangles, triangles, circular rings). Let's try circular rings with thickness dr (change in r).
- Take one ring at location r. Unroll the ring. Approximate by a rectangle.

$$\text{Ring area } = 2\pi r\ dr$$

- Stack all these rectangles vertically in the plane (plot $y = 2\pi r$).
- The smaller dr, the closer we are. Looks to approach the area of a triangle.

$$\text{Triangle area } = \frac{1}{2}bh = \frac{1}{2}32\pi 3 = \pi 3^2$$

- For general radius R, we get an area of πR^2.

(b) Process: Hard problem $\Rightarrow$ sum of many small values $\Rightarrow$ area under a graph.

- A bit of a paradox here. Rectangles disappear, infinitely many.

(c) Area under a curve: velocity / distance.

- Suppose a car speeds up then comes to a stop.
- Assume we know the velocity everywhere. Plot a velocity function that makes sense.
- $d = r \cdot t$, so we can compute the distance over small time intervals to approximate. The smaller the dt, the better the approximation.
- These are rectangles under the curve for v which we are summing.

(d) Area under a curve: general problem.

- Of course math is about pushing conversation beyond a single problem. We generalize to create a more powerful theory.
- Example: $y = x^2$. Find the area under the curve on $[0, 3]$ or in general $[0, x]$. Denote this area $A(x)$ also known as the *integral of* x^2.
- If we change the area slightly, call it dA, can approximate as

$$dA \approx x^2 dx \quad \Rightarrow \quad \frac{dA}{dx} \approx x^2$$

  The smaller dx (and hence dA), the better the approximation.
- Derivative
$$\frac{dA}{dx} = f(x)$$
  connects the function to the area under the curve (integral)
- This idea is the fundamental theorem of calculus. More later on.

(e)

Chapter 2

.1 Introduction

1. Calculus and paradox

- Zeno paradox (Achilles and tortoise, tortoise always wins, infinite times when tortoise ahead) `https://en.wikipedia.org/wiki/Zeno%27s_paradoxes`
- 1=0.999999 (∞ as a process) `https://en.wikipedia.org/wiki/0.999...`

$$1 = 1 \cdot \frac{1}{3} = 1 \cdot (0.\overline{3}) = 1 \cdot (0.333...) = 0.999... = 0.\overline{9}$$

- $D = rt$ (inst veloc), newton quote `https://en.wikipedia.org/wiki/History_of_calculus`

$$D = rt \to r = \frac{D}{t}$$

  What is this as $t \to 0$?
- Achimedes and reductio ad absurdum: Practical solutions to be had: `https://en.wikipedia.org/wiki/The_Quadrature_of_the_Parabola`
- Used and criticised thru history, idea of limit formalized in 19th century, let to revolution in mathematical analysis.

2. Outline of chapter
   - Motivation: Tangent / velocity problem, paradox
   - Approach: Limit of a function, idea of solution
   - Techniques: Limit laws (structure), delta eps (rigor), infinity (more paradox)
   - Continuity: Big math idea applies to all functions
   - Derivative definition, develop deep in chapter 2

.2 2.1 The Tangent and Velocity Problems

1. Motivation: Playing the stock market
   - Calculus stock over time
   - When to buy and sell? How to tell what will happen next?
   - Average rate of change is easy (AROC) but gets weird as interval gets smaller.

$$\frac{\Delta S}{\Delta t}$$

   - Instantaneous rate of change makes sense with intuition, but not with calculation. 6/2 vs 6/0 vs 0/0.
   - Paradox of 0/0.
2. Motivation: Distance and velocity
   - My commute to work, plot velocity as I see on spedometer.
   - Can you draw distance? Δv vs Δd. Fast and slow Δd.
   - Using distance graph, how to get velocity? IROC at midpoint?
   - Connection: Average velocity.

$$d = rt \quad \to \quad r = \frac{d}{t}$$

   - Paradox of instantaneous velocity. 0/0.
3. AROC, IROC, and the difference quotient:
   - Graph general function $y = f(x)$ and label $x = a, b$.
   - Def of diff quotient.

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

   - Graph, secant line slope.
   - Connection to IROC. Can never get to IROC, our first paradox of calculus.

- Secant line trends to a tangent line.

4. Example: Try on your own.
   - $f(x) = x^2$, AROC over $[1, 2]$.
   - Try to approx IROC at $x = 2$. By hand, use calculator / computer.
   - Graph.
   - Compute AROC and draw secant line.
   - Use desmos.

5. Example: Alternate form of difference quotient.
   - a and b
   - a and $a + h$.
   - Graph to compare.
   - Second better for calculation.

.3 2.2 The Limit of a Function

1. Limit idea and notation Seems silly and weird and confusing.
   (a) Definition in words. For x near a, $f(x)$ is near L.
   $$\lim_{x \to a} f(x) = L$$
   (b) Important that L is finite here.
   (c) Reading notation: the limit of $f(x)$, as x approaches a, equals L.
   (d) Draw picture, careful language, how to read notation, idea only here, fuzzy and not careful.
   (e) Distinction between limit and f(a), may differ or same. Show can move f(a) in picture. Near does not mean equal.
   (f) Possible limit doesn't exist. Show picture.

2. Return to IROC:
   (a) Example from last section: IROC at $x = 2$ for $f(x) = x^2$
   (b) Limit of diff quotient, undefined at zero.
   (c) Plot diff quotient in desmos, show can remove zero division by factoring and simplifying, called removable discontinuity.
   (d) Limit def of IROC

3. Limit existence
   (a) Draw cases where exists, continuous, removable discontinuity
   (b) Draw cases where doesn't, jump discontinuity, asymptote (L must be finite), oscillatory case

4. Example: Piecewise function. Try on own.
   (a) Graph on own, and figure out limits everywhere in its domain. Where do limits not exist?
   $$f(x) = \begin{cases} 2 - x^2, & -1 \le x < 0 \\ 2 - x, & 0 < x \le 1 \\ 2x, & 1 < x < 2 \end{cases}$$

5. One sided limit.
   (a) Draw picture with jump disc.
   (b) Right and left side limit notation. Again, $f(a)$ doesn't matter.
$$\lim_{x \to a^{+/-}} f(x) = L$$
   (c) If they differ, regular limit doesn't exist. If same, regular limit is the same and agrees. Sometimes decomposing a limit into two sides is a good strategy.
$$\lim_{x \to a^{+}} f(x) = \lim_{x \to a^{-}} f(x) = L$$
   implies
$$\lim_{x \to a} f(x) = L$$
   and reverse as well.
6. Example: Previous problem. Explore one sided limits.
   (a) Graph on own, and figure out limits everywhere in its domain. Where do limits not exist?
$$f(x) = \begin{cases} 2 - x^2, & -1 \leq x < 0 \\ 2 - x, & 0 < x \leq 1 \\ 2x, & 1 < x < 2 \end{cases}$$
7. Infinite limits
   (a) Motivating examples: $f(x) = 1/x, 1/x^2$
   (b) Def of $\lim_{x \to a} = + - \infty$
   (c) Right / left limits can be one-sided, if agree get regular limit.
   (d) Have seen this before: VAs, bottom zero, top not
   (e) If limit is infty, still say limit DNE
   (f) Example: How to reason sign of infinity? Check in desmos.
$$f(x) = \frac{2 - x}{x + 1}, \quad g(x) = \frac{x^2 - 2x - 8}{x^2 - 5x + 6} \quad x \to 2$$

.4 2.3 Calculating Limits Using the Limit Laws

1. Current ways to calculate limit
   (a) graph (imprecise, unreliable)
   (b) calculator (impractical, not intuitive)
   (c) reasoning (fuzzy)
   (d) Need a precise approach for any function $f(x)$
2. Path of math
   (a) Precise foundation: Basic building block.
      - Soon will be $\delta - \epsilon$ def of limit, short version in next section
   (b) Build theory (skip to here for now): Prove more complicated, useful results.
      - Theorems, limit laws as base, combine these to handle very complex functions.

3. Limit laws (analytic / computational technique, practical)

   (a) Basics, for a, c constants.
   $$\lim_{x\to a} x = a, \quad \lim_{x\to a} c = c$$
   (b) Limit laws if *both limits exist (right, left agree and finite)* (SUBTLE) and c is a constant, then
      i. $f+g$
      ii. $f-g$
      iii. cf
      iv. $f\cdot g$
      v. $\frac{f}{g}$ if $\lim_{x\to a} g(x) \neq 0$
      vi. $f(x)^n$
      vii. $\sqrt[n]{f(x)}$
   (c) These laws match your reasoning, but need to be shown carefully using $\delta-\epsilon$ def of limit.
   (d) Why do we care about these laws? Practical.
      i. $\lim\limits_{x\to 2}\left(2x^2 - x + 2\right)$, reference corresponding limit law at each step.
      ii. Note need to simplify algebra first otherwise zero division: $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2-2x}$. Note $x \neq 2$ for the simplification steps and we don't care since limitness.
      iii. Check each in Desmos.
   (e) Return to IROC in previous section, $f(x) = x^2+1$ at $x=1$.
   (f) Powerful.
      i. Theorem: For $p(x)$ any polynomial and $r(x)$ any rational function, we can use direct substitution to evaluate limits.
      $$\lim_{x\to a} p(x) = p(a), \quad \lim_{x\to a} r(x) = r(a)$$
      provided a is in the domain of the rational function.

4. Challenge examples: Try on own first. Check in Desmos.

   (a) $\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x}$ (mult by conjugate)
   (b) $\lim_{x\to 0}(2x-1+|x|)$ (use def to remove abs val)
   (c) $\lim_{x\to 0} x\sin(1/x)$ (challenge, need squeeze theorem)

5. Squeeze theorem: The indirect attack.

   (a) Statement: if $f(x) \leq g(x) \leq h(x)$when x is near a (except at a) and
   $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$
   then
   $$\lim_{x\to a} g(x) = L$$
   (b) Draw picture of idea. Ask to draw on own first.
   (c) Hard to know when to use. Bounding sine is the key giveaway here and future.
   (d) Useful for proving other theorems in the future.

.5 2.4 The precise definition of a limit

1. Recall idea of limit:
   (a) Try to write down on own. Draw picture.
   (b) $\lim_{x\to a} f(x) = L$, near wording.
   (c) Note again $x \neq a$ and $f(x) \neq L$.
   (d) Issue: Fuzzy idea, lacks precision. How near is near?
2. Example: Motivation, try on own.
   (a) Design a circular plate. Boss cares about area. How off can the radius be?
   (b) Area 100 inches square +- 1 square inch. How off can the radius be?
   (c) Introduce function, use absolute value.
   $$A(r) = 100 \pm 1 \to |A(r) - 100| \leq 1$$
   (d) Draw graph of f and translate to L and a. Graph is parabola.
   (e) Boss comes back with +- 0.5 inch. Do in general once and for all. Update previous calculation and graphs.
3. $\delta - \epsilon$ definition of limit.
   (a) $\lim_{x\to a} f(x) = L$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that if
   $$|x - a| < \delta$$
   then
   $$|f(x) - L| < \epsilon$$
   (b) Draw picture. x window, f window.
   (c) Connect to previous example.
   (d) Key is no matter hos small ϵ is, can always find a δ.
4. Example: Prove limit of a random line.

.6 2.5 Continuity

1. Continuity before (draw one line without leaving the board)
   - Not good enough
   - Is $\sin(1/x)$ continuous at 0?
   - The Dirichlet function (so dense)
2. Def of continuous function: Function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.
   (a) Note: only one point here.
   (b) How about $sin(1/x)$ and Dirichlet function
   (c) Three things are involved here:
      i. limit exists (two sides)
      ii. function value definied
      iii. they are equal
3. Discontinuous at $x = a$ means one of them fails. What are the ways this can happen? We've already seen this.

(a) Removable discontinuity
(b) Jump discontinuity
(c) Infinite discontinuity
(d) High oscillation ($y = \sin(1/x)$ is not continuous, how weird from the grahp, wait, it makes sense, otherwise $f(0)$ must be defined)

4. One sided continuity can be considered (point to above jump discontinuity) graph, write down definition.

5. Continuity on an interval

   (a) Function f is continuous on an interval if it's continuous at every number in the interval
   (b) Continuous functions are continuous everywhere *in their domain.*
   (c) **Example:** Graph crazy piecewise function (removable, jump, infinite, not in domain).
      - Where is f discontinuous?
      - Where is f left / right discontinuous?
      - On what interval is f continuous.

6. Simple functions

   (a) Theorem: The following are continuous functions. (not surprising that they are familiar functions, each needs showing carefully)
      - Polynomials
      - Rational functions
      - Root functions
      - Trigonometric functions
      - Inverse trig functions
      - Exponential functions
      - Logarithmic functions

   (b) Theorem: if f and g are continuous at a and c is a constant, then the following functions are also continuous at a:
   $$f \pm g, cf, f \cdot g, f/g, \quad if \quad g(a) \neq 0$$
      - These are just the five limit laws!
      - **Example:** Where is the following function $f(x)$ continuous?
      $$\frac{\ln(x) + \arcsin(x)}{x^2 - 4}$$
      - Idea: Find the domain of all the simple functions and take the intersection, worry about zero division.

   (c) Function composition: If f is continuous at b and $\lim_{x\to a} g(x) = b$, then
   $$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$$
   (d) **Example:** Rewrite $f(x) = \left|\frac{x\sin(x)}{x^2+2}\right|$ as function composition ($f(x) = g(h(x))$). Why is f continuous everywhere?
   (e) Can now use continuity to compute limits! $\lim_{x\to\pi} f(x)$ for above.

7. Intermediate Value Theorem (recite every theorem that has a name, must be important if has a name!):
   If f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

8. Why is this useful?
   - Theory: Show that the function $x^3 + x^2 - 1 = 0$ has a root between $[0, 1]$
   - Practice: Can you find the root? Show by excel, Bisection method!
   - solve $x = \cos x$
9. Give the harder ones

.7 2.6 Limits at infinite: horizontal asymptotes

1. What do you know about horizontal asymptotes? (think about vertical asymptotes)?
   (a) Simple example: $f(x) = 1/x$. Draw comparison to vertical and horizontal asymptotes.
   (b) Def: Let f be a function defined on some interval (a, ∞), then
   $$\lim_{x\to\infty} f(x) = L$$
   means that the value of f(x) can be made arbitrarily close to L by taking x sufficiently large. Can also write: $f(x) \to L$ as $x \to \infty$.
   (c) Similar definition for $\lim_{x\to-\infty} f(x) = L$
   (d) Definition: the line $y = L$ is called a horizontal asymptote of the curve $f(x)$ if
   $$\lim_{x\to\infty} f(x) = L \quad or \qquad \lim_{x\to-\infty} f(x) = L$$
   (e) Questions:
   - How many horizontal asymptotes can a function have?
   - How many vertical asymptotes can a function have?
   - If $f(x)$ has a horizontal asymptote, then $f^{-1}(x)$ has a vertical asymptote. Vice versa?
   - Does $\sin(x)$ have a horizontal asymptote?
   - What can happen at infinity? Cases here?
     - Horizontal asymptote: limit is a finite number
     - Blow up: infinite limit, different cases (oblique, parabolique?)
     - Continued oscillation: no limit
2. Finding limit at infinity (finding horizontal asymptotes).
   (a) Note: limit laws mostly hold for limits at infinity.
   (b) Start from simple functions: the inverse function of functions with VAs now have HAs
   i. $\lim_{x\to-\infty} e^x = 0$
   ii. $\lim_{x\to\infty} \tan^{-1} x = \pi/2$.
   iii. $\lim_{x\to-\infty} \tan^{-1} x = -\pi/2$
   iv. For $r > 0$ is a rational number, then
   $$\lim_{x\to\infty} \frac{1}{x^r} = 0$$
   v. For $r > 0$ is a rational number such that x^r is defined for all x, then
   $$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$
3. Leading terms for "fractional functions"

(a) Example: try different variations within Desmos, add terms, roots

$$\lim_{x\to\infty} \frac{x^2+1}{3x^2-1}$$

- Rigorous computation
- Concept of leading terms (why does this work?)
- The concept only works for limit at ∞
- Infinite limits at infinity
- More leading terms

(b) Variations to consider:

$$\lim_{x\to\infty} \frac{x^4-x^2+1}{(x+1)^3(x-1)}; \qquad \lim_{x\to\infty} \frac{2x}{\sqrt{x^2-1}+\sqrt{3x^2+x}}; \qquad \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+1}}$$

(c) Now work out each carefully by dividing by enough x terms to use $\lim_{x\to\infty} \frac{1}{x^r} = 0$.

(d) Leading terms must happen at infinity, need to show via this careful computation.

4. Typical problems

(a) Leading terms other than polynomial, which grows more quickly?

$$e^x > \textit{polynomial} > \ln x$$

(b) Conjugate (again)

$$\lim_{x\to\infty} (\sqrt{x^2+1} - x)$$

(c) Composition function

$$\lim_{x\to\infty} \tan^{-1}(e^x)$$

(d) Squeeze theorem still works

$$\frac{\sin x}{x^2}$$

(e) Substitution (important idea, simple yet powerful)

$$\lim_{x\to 0^-} e^{\frac{1}{x}}$$

5. Indeterminate forms revisited: check all possible combinations of 0, c, ∞

- Case 1: mention that they are all equivalent, many things can happen with these, thus cannot determine result immediately.

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty\cdot 0, \quad 1^\infty, \quad 0^0, \quad \infty^0$$

- Transfer to case 1 (hopefully): $\infty - \infty$

$$x^2 - x^4$$

- The idea of comparing infinities using ratios instead differences makes things computatable.

$$\lim_{x\to 0} \frac{x}{x^2}$$

- Explain why $\infty - \infty$ will be indeterminate

$$1+2+3+4..., \quad 2+3+4+5...$$

- More techniques for these cases later.

.8 2.7-2.8 Derivatives and rates of change

1. Motivation: *instantaneous rates of change* (tangent line calculation). We return.
   (a) Idea in physics
   (b) By formula
   (c) **Definition:** (not a theorem!) The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope
   $$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
   provided that this limit exists.
   (d) Alternate formulation of difference quotient. Draw graph here. Same as above.
   $$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
   (e) By graph (show Desmos)
   (f) **Example:** What's the slope of the tangent line of $y = x^2$ at $(1, 1)$, what's the equation of the line? Graph these.
   (g) **Example:** Which formula is better? Do $y = x^2$ at $x = 1$ the second way.
   (h) **Example:** f(x) = 1/x at 3. Divide and conquer 2 different formulations.
   (i) **Example:** What is the slope of the tangent line of $\sqrt{x}$ at $(1, 1)$? Can use symmetry of inverse function here. (know what to expect here.)
   (j) Definition: The *derivative of a function* $f(x)$ *at* a is given by
   $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
   - What is the derivative of function $y = x^2$ at $(1, 1)$? Already did this.
   - Compare the derivative of $y = x^2$ at different place (0,1,-1,2,-2)
   - How about we generalize this to a function? Try and graph that function.

2. Definition: The *derivative function* of a $f(x)$ is
   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$
   - Derivative (function) of f means a function, this thing has a life of its own (domain, range, inc, dec), but should be tied closely to f somehow.
   - Find the derivative function first, then just plug in numbers to find a certain slope.
   - **Example:** the derivative function of x^2 (Draw it on the board and compare.)
   - **Example:** Find $f'(x)$ for $1/x$. When is the slope of the tangent line $-\frac{1}{4}$? Relate to graph. Happens twice.

3. The information from $f'(x)$:
   (a) Draw the changing rate (derivative) of water height function: Filling a glass, cone cup, then coke bottle. Helps to draw the height function also. `https://www.youtube.com/watch?v=APD91nEJgkA`, `https://teacher.desmos.com/waterline`
   (b) Give a M shaped function, draw the derivative, then draw the derivative of the derivative. (It's going to have corner and straight lines)
   (c) Benefits

- Increasing decreasing
- Concave and concave down.
- Finding the maximum and minimum

(d) Can we always find derivatives? Corner won't work, and it shouldn't. Separate ideas/theories handle this.

(e) Vertical asymptotes:
True or false: $f'(x)$ can't have fewer vertical asymptotes than $f(x)$ (bad to worse). Can we have more? Yes, vertical tangents.

(f) What about horizontal asymptotes? Always become HAs of $y = 0$ for f'.

(g) Can we always find derivatives?

4. Differentiability: When can the derivative be used?
   - A function f is differentiable at a if $f'(a)$ exists
   - A function is differentiable on an interval if it's differentiable at every number in the interval.
   - As mentioned above, a function is not always differentiable. When is it not differentiable? When limit fails
     - A discontinuity
     - A corner
     - A vertical tangent
     - **Example:** Show $f(x) = |x|$ is not differentiable at $x = 0$ as we expect. Compute $\lim_{h\to 0^-}$ and $\lim_{h\to 0^+}$
   - Theorem: If f is differentiable at c, then f is continuous at c.
   - Weierstrauss function

5. Differential notation

(a) Rate of change

i. Average rate of change, define $\triangle y$, $\triangle x$ and consider

$$\frac{\triangle y}{\triangle x}$$

ii. Instantaneous rate of change (same as derivative, application terminology)

$$\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}$$

(b) Differentiation notation

$$f'(x) = \frac{df}{dx}$$

- It contains the meaning of rate of change
- It's easy to see the variable (with respect to)
- Will be handy later when you have multi-variables.
- The idea of ratio
- Leibniz way vs Newton way

(c) The instantaneous rate of change with respect to time of

- displacement is the velocity
- velocity is the acceleration

(d) More notation: Distinguish between operator and function here.

$$\begin{aligned} f'(x) &= \frac{d}{dx}f = \frac{df}{dx} = \frac{df}{dx}(x) \\ f''(x) &= \frac{d^2f}{dx^2} = \frac{d}{dx}f' \\ f^{(n)}(x) &\quad \text{parenthesis needed here} \end{aligned}$$

(e) High order derivatives

Chapter 3 Differentiation rules

- Difference quotient is a pain

.1 Derivatives of polynomials and exponential functions

1. Derivatives of polynomials.

   (a) Theorem: Constant function derivative.

   $$\frac{d}{dx}c = 0 \quad \text{give proof}$$

   (b) Theorem: Straight line.

   $$\frac{d}{dx}(mx + b) = m \quad \text{give proof}$$

   (c) Power function: $f(x) = x^n$

   i. First order (done above)

   $$\frac{d}{dx}x = 1$$

   ii. Try second order, modify to cubic, pattern here, go for general.

   iii. Theorem (The power rule): if n is a positive integer, then

   $$\frac{d}{dx}(x^n) = nx^{n-1}$$

   Proof: Let $f(x) = x^n$, for n any positive integer. Compute

   $$f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a} = \lim_{x\to a}\frac{x^n - a^n}{x - a}$$

   Note, can factor here. Distribute to check.

   $$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots xa^{n-2} + a^{n-1})$$

   Then,

   $$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{(x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots xa^{n-2} + a^{n-1})}{x - a} \\ &= \lim_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots xa^{n-2} + a^{n-1}) \\ &= na^{n-1} \quad \text{QED} \end{aligned}$$

   iv. Theorem: The *general* power rule: (proof later, need more tools) if n is *any real number*, then

   $$\frac{d}{dx}(x^n) = nx^{n-1}$$

- **Example:** $\frac{d}{dx}x^5$
- **Example:** $\frac{d}{dx}\sqrt{x^3}$
- **Example:** $\frac{d}{dx}\frac{1}{x^5}$
- $n = 0$ case

(d) General polynomial functions? $\frac{d}{dx}(2x^2 - x + 5) =$? As with limits, have properties to handle things here. Since derivatives are actually limits, we already have all this.

2. Differentiation rules

(a) Theorem: For c any constant and f, g differentiable functions,

i. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$ (show this one?)

ii. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

(b) Explain by idea (vertical stretch, combine two functions, increase rate of change.

(c) These are just our limit laws!

(d) **Example:** General polynomial is now covered here. Illustrate for $\frac{d}{dx}(2x^2 - x + 5) =$? Where are the horizontal tangents?

(e) Revisit: $\frac{d}{dx}(mx + b)$

(f) **Example:** Try on own, show all steps: $\frac{d}{dx}\frac{x^{-1/2} - 2x^3 + 1}{x^2}$ (rewrite as sum of power functions).

(g) Caution, not all limit laws simply transfer.

$$(fg)' \neq f'g', \quad \frac{f}{g} \neq \frac{f'}{g'}$$

Illustrate for $f(x) = (1 - x)(x + 2)$

3. Exponential functions

$$\frac{d}{dx}e^x =?$$

(a) Sketch the graph of $f(x) = e^x$ and f. Look pretty similar.

(b) Detour to general exponential $f(x) = a^x, a > 0$.

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0}\frac{a^h - 1}{h} = a^x \cdot f'(0)$$

Then, $f'(x) = f'(0)a^x$. What is $f'(0) = 1$?

(c) Definition: Euler's number e is the number such that

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

It turns out that $e \approx 2.71828...$

(d) Theorem: $\frac{d}{dx}e^x = e^x$. (very useful property).

(e) **Examples:** $e^3, \quad e^x + x^e$

(f) Notes:

- e shows up in continuous compound interest (originally in history?) where another limit appeared. Compare to above. Same thing!

$$\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$$

- How to differentiate general exponentials a^x. Later!
- Keep differentiating e^x and it ALWAYS stays the same.
- Compare first derivative of any polynomial to e^x.

.2 3.2 The product and quotient rules

- Already noted that $(fg)' \neq f'g'$ and likewise $(f/g)' \neq f'/g'$, so what are they?

1. Theorem (product rule)L If both f and g are differentiable, then

$$\frac{d}{dx}(f \cdot g) = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)$$

or

$$(f \cdot g)' = f'g + g'f$$

   (a) Show the area proof: Can think of product as the area of a rectangle (dimensions $f \times g$). Let x change to $x + \Delta x$, then f, g change by $\Delta f, \Delta g$. Draw picture. So the change in the rectangle's area is

$$\Delta(f \cdot g) = (f + \Delta f)(g + \Delta g) - fg = f\Delta g + g\Delta f + \Delta f \Delta g$$
$$\frac{\Delta(f \cdot g)}{\Delta x} = f\frac{\Delta g}{\Delta x} + g\frac{\Delta f}{\Delta x} + \Delta f \frac{\Delta g}{\Delta x}$$

   Take $\Delta x \to 0$. Wild.

   (b) Show a rigorous proof: The power of adding zero.

$$\begin{aligned}(fg)' &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h} = f(x)g'(x) + f'(x)g(x)\end{aligned}$$

   Examples: $(x-1)(x+1)$ easier to distribute, x^2e^x, $xe^x + ex^e$

2. Theorem (quotient rule):

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

or

$$(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$$

   (a) Can prove via same trick as with power rule. See text.
   (b) Show the proof by finding $(1/g)'$ first via difference quotient, then apply product rule.
   (c) **Example:** $\frac{x^2-1}{x^3+6}$, find the second derivative of e^x/x.
   (d) Can now show carefully $(x^{-n})' = -nx^{-n-1}$ via the quotient rule.

3. Fancy differentiation with general constants (multivariable calculus).

   (a) **Example:** 2 ways, const mult and quotient rule.

$$\frac{d}{dx}[\frac{x}{a}]$$

   (b) **Example:** Try on own.

$$\frac{ax+b}{cx+d}$$

   (c) Notes:
   - Generalize: What is $(f \cdot g \cdot h)'$
   - Quotient rule can be derived / and replaced by product rule
   - Create own list of nice formulas. Will need to know these.

.3 3.3 Derivatives of Trigonometric functions

1. Derivative of $\sin x$:

$$\frac{d}{dx}\sin x = \cos x$$

   (a) Hint at the derivative of $\sin(x)$ by drawing the graph. Let them draw $\cos(x)$ and $\tan(x)$.

   (b) Limit definition of derivative. Need sum formula for sine ($\sin(u+v) = \sin(u)\sin(v) - \cos(u)\cos(v)$.

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h) - sin(x)}{h} = \cos x \lim_{h\to 0}\frac{\sin h}{h} + \sin x \lim_{h\to 0}\frac{\cos h - 1}{h}$$

   (c) Show the identity via the Squeeze Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

   - Draw the unit circle and show

$$\sin(\theta) < \theta \quad \text{(arc length greater than triangle height)}$$

     and

$$\theta < \tan(\theta) \quad \text{(triangle area bigger than sector area)}$$

     then combine as

$$\cos(\theta) < \frac{\sin(\theta)}{h} \leq \frac{1}{\cos(\theta)}$$

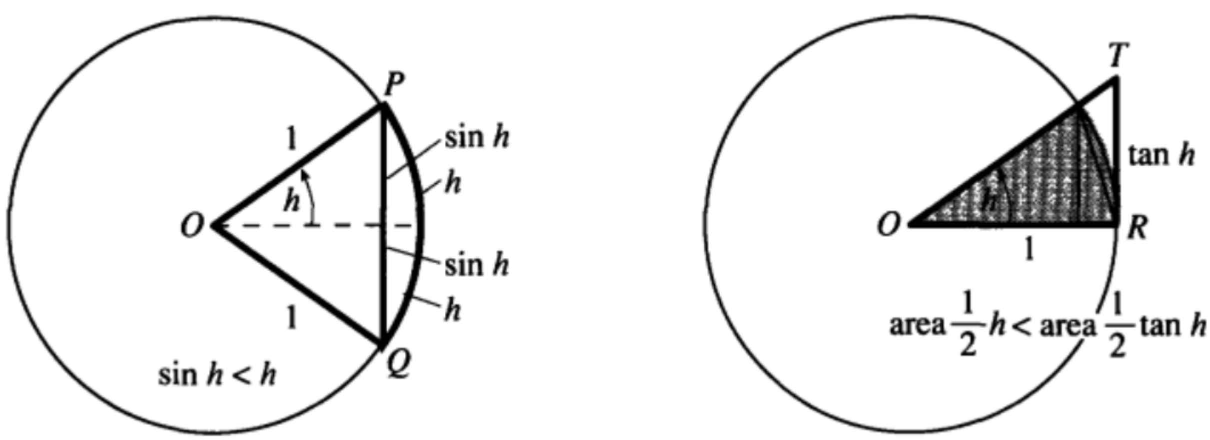

Fig. 2.11 Line shorter than arc: 2 sin *h* < 2*h*. Areas give *h* < tan *h*.

   (d) For $\lim_{h\to 0}\frac{\cos h - 1}{h}$, use previous result by multiplying by the conjugate and show the limit is zero

   (e) Returning, both limits exits and we can use limit laws.

$$\frac{d}{dx}\sin(x) = \cos x \lim_{h\to 0}\frac{\sin h}{h} + \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} = \cos(x)\cdot 1 + \sin(x)\cdot 0 = \cos(x)$$

   (f) Note: x must be in term of radians here, degrees differ in result by constant.

   (g) Important limits to take note of:

$$\frac{\sin(\theta)}{\theta} = 1 \quad \text{and} \quad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

2. Theorem: Derivative of all the trig functions ($\cos x$ is in homework 20, find others by yourself). Show these except cosine via quotient rule.

$$\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x), \quad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x), \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x), \quad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

3. **Examples:**

   (a)
   $$\frac{d}{dx}\frac{\sec x \sin x}{e^x + \tan x}$$

   (b) Find the second derivative of $\sec x$

   (c) Find the 99th derivative of $\sin x$

4. Above limit results can be useful in clever ways.

   (a)
   $$\lim_{\theta\to 0}\frac{\sin(7\theta)}{3\theta} = \lim_{\theta\to 0}\frac{\sin(7\theta)}{7\theta}\frac{7\theta}{3\theta} = \frac{7}{3}$$

   (b) Find
   $$\lim_{\theta\to 0}\frac{\sin(4x)}{\sin(6x)} = \frac{2}{3}$$

   (c) Mention limit law use and substitution ideas here.

   (d) Mention for future use.
   $$\lim_{\theta\to\infty}\frac{\sin(\theta)}{\theta} = 0$$

.4 3.4 The chain rule

How many differentiation rules do we have? List as many as they can say. What are we missing? Combining functions via function composition.

$$\frac{d}{dx}\sin(2x) = ?$$

1. Rate of change

   (a) A cheetah is 10x as fast as me. I am 2x as fast as my chicken. How much faster is the cheetah than my chicken? 20x as fast.

   (b) Example of temperature of La Crosse, temperature in the room, temperature in my storage case.

   (c) Explanation of chain idea: change in daytime light - changes - temperature - changes - growth of apple tree - changes - size of apple - changes - size of worm population

   (d) Draw classic function composition diagram, but write $y = f(g(x))$. Then, we have the chain rule. Rate of change in $f \circ g$ at x is the same as ROC of g at x times ROC of f at $g(x)$.

2. Theorem: For f and g differentiable, $f \circ g$ is also differentiable and

   $$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

   or equivalently in Leibniz notation.

   $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

(a) Proof idea:

$$\frac{d}{dx}f(g(x)) = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\frac{g(x+h)-g(x)}{h}$$

then change of variable and done.

(b) **Example:**

$$f(x) = x^2, \quad g(x) = 2x$$

(c) More intuition:

i. **Examples:** Carefully identify f and g. Challenge is identifying inside and outside functions.

$$(x^2+x+1)^{1000}, \quad \sin(1+\cos(x)), \quad \tan^3(4x)$$

ii. This is a versatile new technique. Quotient rule revisited:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx}f(x)(g(x))^{-1}$$

(d) Differential equation: population, mass-spring

3. General exponential functions

(a) $2^x = e^{\ln(2)x}$, now differentiate via the chain rule.

(b) Theorem:

$$\frac{d}{dx}a^x = a^x\ln(a)$$

(c) **Example:**

$$2^{3^{x^2}}$$

(d) Do speed review of inverse functions and logarithmic function

.5 3.5 Implicit differentiation

1. What remains?

(a) Inverse trig function, $\ln(x)$ (next section via chain rule), curves which aren't functions.

(b) Should be able to differentiate any curve via the same idea, just locally.

2. Implicit differentiation (chain rule in disguise)

(a) **Examples:** Find $\frac{dy}{dx}$ for $x^2+y^2=1$ (graph)

i. Idea: treat $y = y(x)$ and differentiate both sides of equation. Require chain rule for y terms.

ii. Is it useful? Just the same, but a bit more difficult to use. Find the equation of the tangent line of $x^2+y^2=1$ at $(\sqrt{2}/2, \sqrt{2}, 2)$

iii. Note:

- Always have multiple of y' for function y, actually the chain rule
- Your answer may involve both x and y
- All the other rules apply just the same
- Find the derivative by plugging in both x and y.

(b) **Examples:** Find $\frac{dy}{dx}$ for $x^3+y^3=6xy$ (Folium of Descartes)

i. Graph in desmos. Compare for following.

ii. Show (3,3) is on the graph. What is $\frac{dy}{dx}|_{x=3}$?

iii. Where is $\frac{dy}{dx} = 0$? Undefined?

(c) **Examples:** Find $\sin x \sin y + \sin y \sin x = 0$, draw the derivative in Desmos, nodal lines

- Youtube: amazing resonance effect
- `https://www.youtube.com/watch?v=wvJAgrUBF4w&noredirect=1`
- `https://www.youtube.com/watch?v=uENITui5_jU`

(d) **Examples:** Find $\frac{dy}{dx}$ for $\sin(x+y) = e^{xy}$

(e) **Examples:** Find the equation of the tangent line to $y \sin 2x = x \cos 2y$ at $(\pi/2, \pi/4)$

(f) Can show the power rule for rational powers:

$$y = x^n = x^{p/q}, \quad y^q = x^p, \quad qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

3. Derivatives of inverse trigonometric functions

(a) **Examples:** Find y' for $\sin y = x$: $y' = \frac{1}{\sqrt{1-x^2}}$

(b) Find all the inverse trig functions by yourself, we just did arcsin. Short review below (not invertible, domain restriction, Pythagorean identities, etc)

- $\arcsin(x), \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\arccos(x), \quad 0 \leq x \leq \pi$
- $\arctan(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
- $\mathrm{arccsc}(x), \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\mathrm{arcsec}(x), \quad 0 \leq x \leq \pi$
- $\mathrm{arccot}(x), \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) Add to your list of derivatives to know.

(d) **Examples:** Use two method to find the derivative ((students) chain rule, (me) draw triangle and simplify as algebraic expression).

$$y = \sin(\cos^{-1} x)$$

4. Derivative of general inverse functions

(a) Implicit differentiation, need not memorize this formula.

$$y = f^{-1}(x), \quad f(y) = x, \quad (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

(b) Illustrate by graph, reflection across main diagonal, how does the slope of the tangent change, slope is the reciprocal?

.6 3.6 Derivative of logarithmic functions

1. Historic motivational interestingness: `https://en.wikipedia.org/wiki/History_of_logarithms`
2. Theorem: (show via implicit differentiation of $x = e^y$)

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

and

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

(a) **Examples:** (with the Chain rule)

i. $\frac{d}{dx}(\ln(-x))$

ii. $\frac{d}{dx}(\log_5(x^2e^x))$ (2 ways, chain rule then log properties can be handy, need change of base formula to show match)

iii. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

3. Logarithmic differentiation: Logarithms are valued for their sweet properties.

   (a) Three nice property of $\log_a(x)$ (prod, quotient, powers), $\ln(x)$ is the nicest choice.

   (b) **Example:** $y = x^x$, then $y' =$? (no such rule)

   (c) **Example:** $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$, then $y' =$? (quotient, product, chain rule madness)

   (d) Summary of steps:

      i. Identify the situation (lots of multiplication, quotient, and powers)

      ii. Take log on both sides (if possible) and simplify using the log properties.

      iii. Differentiate implicitly with respect x

      iv. Solve for y'

      v. What if $y = f(x) < 0$ for some x? Use absolute value.

$$|y| = |f(x)|, \quad \ln(|y|) = \ln(|f(x)|), \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{f(x)}f'(x), \quad \frac{dy}{dx} = \dots$$

   (e) **Example:** Finally, the full power rule: $y = x^n$, n any real number, log differentiation.

4. More complicated cases:

   (a) **Examples:** Find y' for

$$y = x^{x^x}, \quad y = \sin x^{\cos x}, \quad y = \Big(\frac{(2x+3)^2\sqrt{2-x}}{(x-1)^3}\Big)^{x^2}$$

$$y^x = x^y$$

5. Important results to know:

   (a) Theorem:

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

   Reason: $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. So,

$$1 = f'(1) = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\ln((1+h)^{1/h})$$

   Then, because exponential functions are continuous,

$$\lim_{x\to 0}\ln(1+x)^{1/x} = 1 \quad \Rightarrow \quad e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0}(e^{\ln(1+x)^{1/x}}) = \lim_{x\to 0}(1+x)^{1/x}$$

   (b) Corollary: Take $n = \frac{1}{x}$ above,

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n \quad \text{(holy compound interest Batman!)}$$

   (c) Continuous compounded interest: PERT all ova the place.

$$\lim_{n\to\infty}P(1+\frac{r}{n})^{nt} = Pe^{rt}$$

.7 3.8 Exponential growth and decay

1. Things that *change* according to a rule (or rules) can be translated into a differential equation. See the many examples:
   - `https://en.wikipedia.org/wiki/Differential_equation`
   - `https://people.maths.ox.ac.uk/trefethen/pdectb.html`
   - Def: A differential equation is an equation involving derivatives where the unknown is a *function*. (analogous to algebraic equations)
2. Exponential growth and decay is a simple start.
   $$\frac{dy}{dt} = ky, \quad y(0) = y_0$$
   Here k is some constant ($k > 0$ growth, $k < 0$ decay) and y_0 is the initial size. $y = y(t)$ is an unknown function of time t need to find.
   (a) Read the DE as 'quantity y changes at a rate proportional to it's size'. What a language!
   (b) What things do you know which do this? Population, investment, radioactive decay, temperature of object, etc. Will see DEs in advanced related classes.
   (c) How to find solution $y(t)$? We already know a function whose derivative is (almost) the same. Guess $y(t) = Ce^{kt}$ for some constant C. Then,
   $$\frac{dy}{dt} = \frac{d}{dt}(Ce^{kt}) = ky$$
   What is C?
   $$y(0) = C = y_0$$
   Then, the solution is $y(t) = y_0 e^{kt}$. PERT formula!
   (d) Can show the ONLY solution to the above DE is $y(t) = y_0 e^{kt}$ (uniqueness).
3. **Examples:** (Optional)
   (a) Population growth, La Crosse had population 50000 in 2000 and 55000 in 2010, what will the population be in 2020? Here we *choose* the above model. What are the issues with this assumption?
   Log plots: Google populations of california, texas, florida. `https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&idim=state:06000:48000&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=state:06000:48000:12000&ifdim=country&hl=en_US&dl=en&ind=false`
   (b) Newton's Law of Cooling: Newton said the rate at which an object cools (or warms) is proportional to the difference in temp of the object and its surrounding temp. Language sound familiar?
   $$\frac{dT}{dt} = k(T - T_s), \quad T(0) = T_0$$
   where $T(t)$ is the unknown temperature function. Ideal coke temp 38°
   (c) Solution: We've already solved this type once and for all! Use the substitution $y(t) = T(t) - T_s$ and reduce to $\frac{dy}{dt} = ky$.
   (d) Coffee example if in high demand.

.8 3.9 Related rates

1. Motivation: A single phenomenon can give many different rates of change

   (a) A snowball is melting. What quantities are changing? Rates comparable? (Length vs area vs volume) `https://www.youtube.com/watch?v=LNEBZ8ekU18`

   (b) A ladder is sliding down a wall, driving a car, GPS, so many here.

   (c) Suppose a snow ball is melting at a rate $40cm^3/sec$, how fast is the diameter shrinking when the radius is 4cm?

   - How are these two quantities related?
   - Know: $V = \frac{4}{3}\pi r^3$, $d = 2r$
   - What is changing in time? $V = V(t), r = r(t), d = d(t)$
   - Differentiate equation (always implicit differentiation here)

2. Steps to solve related rates problem. Straightforward, but needs care to detail else will wander.

   (a) Make sure it's not an optimization problem (-est, most, least)

   (b) Assign all the variables

   (c) Express the rates in terms of derivative

   (d) Find the rate given and the rate you want to solve

   (e) Find a equation associating all the quantities in (3)

   (f) Use implicit differentiation to take derivatives of the equation (related rates)

   (g) Substitute in the missing information and solve

3. **Example:** A ladder 10 ft long is sliding against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6ft from the way?

   (a) Drawing picture is key, Pythagorean theorem is key.

   (b) What is the changing rate of θ?

4. **Example:** A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40ft from the pole?

5. Groupwork from text problems, pg 248, problems

   - 22 (pythagoras) Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.01$ meters/sec
   - 45 (trig right triangle) $\frac{10}{9}\pi$ km/min or 130 mph
   - 18 (similar right triangles) decreasing at a rate of 0.6 meters/sec
   - 42 (two pyth theorem comb to get rope length) $-\frac{10}{\sqrt{133}} \approx -0.87$ ft/sec

6. Global positioning system story of related rates. Student at MIT. `http://www.pcworld.com/article/2000276/a-brief-history-of-gps.html`

.9 3.10 Linear approximations and differentials

1. Motivation and idea:

   (a) Practical questions: What is $\sqrt{4.1}$, $\sin(46°)$?

   (b) Idea: Use the value of a function around a known $f(a)$ in a smart way.

(c) Think of

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When x is close to a they basically satisfies the relationship.

2. Linear approximation: the linear (tangent line) approximation of f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

Also known as the linearization of f at a. Compare to limit of difference quotient above.

(a) Idea: $f(x)$ is "locally" a line (around a), draw picture of f and L

(b) This is an approximation and may not be accurate at all

- depending on the original shape
- depending on how close your x is to a

3. **Examples:** Find the linearization of $\sqrt{x}$ at 4

(a) Use it to approximate $\sqrt{4.1}$, $\sqrt{4.5}$, $\sqrt{6}$ and compare to the real value.

(b) Find $\sin(44°)$, do the same thing.

(c) In physics, $\sin x \approx x$ when x is small. This is linearization.

4. Differentials:

$$dy = f'(x)dx$$

(a) What is this? Reminds of $\frac{dy}{dx} = f'(x)$. What if treat as a ration?

(b) Find the differential of x^2 at $x = 2$. Pick different dx and graph.

(c) Difference between dy and $\triangle y$. Actually, $dy = \delta L$.

(d) This is close to the original conceptualization of calculus.

5. **Example:** A sphere was measured and its radius was found to be 45 inches with a possible error of no more that 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \Delta V \approx dV = 4\pi r^2 dr$$

6. Can we replace $f(x)$ locally by a quadratic equation?

(a) Doable? (Yes, need first and second derivatives to match)

(b) More work? (Yes)

(c) Better accuracy? (Yes)

(d) Any polynomial? (Taylor polynomial, calculus 2)

(e) Why bother replacing functions with polynomials? (Biggest take-away of the section)

- Approximation of hard calculations
- Polynomials are nicer functions than anything else, so live in a better place.

(f) Can we use things other than polynomials? Sure thing (Fourier series) for periodic functions (light, sound, universe of waves).

.10 3.11 Hyperbolic functions

1. Motivation:

   (a) Think about a heavy flexible cable suspended between two points at the same height (the golden gate bridge, telephone cable). This is called a catenary. What is that curve? Not quite a parabola. `https://www.google.com/search?q=catenaries&espv=2&biw=1680&bih=921&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwj5552K2ejKAhVCFR4KHRoADp8QsAQIQw#tbm=isch&q=catenary&imgrc=ES8GEHgRx3OpXM%3A`

   $$\frac{e^x + e^{-x}}{2}$$

   (b) What's the derivative?

2. The family of hyperbolic functions, such parallels with regular trigonomety here.

   $$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

   (a) Regular division and such gives the rest. $\tanh(x) =, \ldots$.

   (b) Just as the points $(\cos(t), \sin(t))$ form a circle with a unit radius, the points $(\cosh(t), \sinh(t))$ form the right half of the equilateral hyperbola $x^2 - y^2 = 1$.

   (c) For some applications, this is the correct geometry (special relativity).

3. Hyperbolic identities:

   (a) Odd, even:

   $$\sinh(-x) = -\sinh(x), \quad \cosh(-x) = \cosh(x)$$

   (b) The "Pythagorean" identities:

   $$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 = sech^2 x$$

   (c) The sum formula:

   $$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sin(y)$$
   $$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sin(y)$$

   (d) Double angle formula:

   $$\sinh(2x) = 2\sinh x \cosh x$$

4. Derivatives of hyperbolic functions (show this)

   $$(\sinh x)' = \cosh x$$

5. Inversere hyperbolic function (show this, substitution, hidden quadratic)

   $$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

6. What you need to know:

   (a) Know they come from application. Be aware.

   (b) You don't have to memorize anything but the definition of the hyperbolic sine and cosine.

   (c) Feel free to check the book when you do the homework

   (d) I may test it as an exercise of derivatives.

Chapter 4 Applications of differentiation

.1 4.1 Maximum and minimum values

1. Motivation: Where do we see optimization
   -
   - What's the maximum/minimum speed (speeding ticket? Why do you care?
   - Buying stocks, when to take action?
   - Suppose you have a fence of 100 ft long. How to fence a rectangular region so that the area is maximum?
   - What's biggest rectangle inside a circle?
   - Who is the best March madness basketball team?
2. Extreme values definition: Let c be a number in the domain D of a function f. Then
   - $f(c)$ is the absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D
   - $f(c)$ is the absolute minimum value of f on D if $f(c) \geq f(x)$ for all x in D
3. Draw a graph typical to smooth nice functions already seen
4. True or False? (Keys: domain endpoints included/not, unbounded functions, discontinuities, singularities (corners), lots of cases here but not too many, draw pictures to summarize)
   - Domain D matters.
   - f can have more than 2 absolute values.
   - Every function must have an absolute value.
   - Absolute values occur where $f'(x) = 0$.
5. Lots of possibilities: Which have absolute max or mins? What are they?
   (a) $f(x) = 1/x$ (no domain listed implies all possible x.
   (b) $f(x) = x^2$
   (c) $f(x) = x^2$ on $[0, 2]$
   (d) $f(x) = x^2$ on $(0, 2)$
   (e) Keys: domain endpoints included/not, unbounded functions, discontinuities, singularities (corners), lots of cases here but not too many, draw pictures to summarize with endpoints, abs local min max/don't name, non differentiable, discontinuity.
6. **Extreme Value Theorem**: if f is continuous on a closed interval $[a, b]$, then f attains its extreme values on $[a, b]$
   (a) Check hypothesis before computing
   (b) If it fails, can you say no absolute values? Nope.
   (c) Require
       - Continuous
       - Closed interval (all endpoints included)
       - The max, min can be at a, b
       - Absolute max/min can occur multiple times
7. Local Extreme Values
   (a) **Definition:** a number $f(c)$ is a

- Local maximum value of f if $f(c) \geq f(x)$ when x is near c
- Local minimum value of f if $f(c) \leq f(x)$ when x is near c

(b) Distinctions:

- Difference between absolute extreme values and local extreme values (draw a picture to illustrate)
- Is an absolute max also local max?
- Why do we want these in applications? More important in some cases (stock market)

8. How to identify locations of local / absolute extrema? Already classified the cases.

   (a) **Definition:** A critical number of f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ DNE. (all cases here, see them? Stationary points, singular points, end points)

   (b) Fermat's little theorem: if f has a local max/min at c and if $f'(c)$ exists, then $f'(c) = 0$.

   (c) True or false?

   - If f'(c) = 0, then f has a local maximum or minimum at c
   - If f has a local maximum or minimum at c, then f'(c) = 0 Both are false

9. Finding critical numbers of

   (a) $f(x) = 2x - \tan x$ on $(-\pi, \pi)$

   (b) $g(x) = x^{1/3} - x^{-2/3}$

10. Method: Find the absolute extreme values of f: the closed interval method.

    (a) Check the extreme values theorem hypothesis

    (b) Find all the candidates: Critical numbers

    - Stationary points
    - Singular points
    - End points

    (c) Compare the f values at all the candidates.

    (d) Take the largest/smallest ones (may be more than one) and write the answer in the format: $f(x)$ has an absolute value ( ) at x = ( )

11. **Examples:** Find the absolute maximum and minimum values of each. Where is the absolute max/min and where does it occur?

    (a) $f(x) = x^3 + x^2 - x$ on $[-2, 2]$.

    (b) $f(x) = x^{\frac{2}{3}}$, no interval then add open / closed. Change to $x^{\frac{1}{3}}$

    (c) $f(x) = x + 2\cos(x)$ on $[-2\pi, \pi]$

.2 4.2 The mean value theorem

Write statements, then let them think about what it could mean.

1. **Rolle's Theorem:** Let f be a function that satisfies the following three hypotheses:

   - f is continuous on the closed interval $[a, b]$
   - f is differentiable on the open interval (a, b)
   - $f(a) = f(b)$

   Then there's a number c in (a, b) such that $f'(c) = 0$. Once they sort out, draw picture. Why is this useful?

- Proof:
- If f is a constant function, we are done.
- Assume $f(x) > f(a)$ for some x. Then by the etreme value theorem f has a max value which must be a local max. Then $f'(c) = 0$ by Fermat's theorem.
- Case where $f(x) < f(a)$ is similar.

2. **Example** Show that $x^3 + x - 1 = 0$ has only one real solution.
   (a) Using IVT on $[0, 1]$ to show existence.
   (b) What if had 2 zeros on $(0, 1)$, $f(a) = f(b) = 0$? Then Rolle's theorem says there is c in (a, b) such that $f'(0) = 0$. But, $f'(x) = 3x^2 + 1$. So, can only have 1 zero.

3. **Mean Value Theorem:** Let f be a function that satisfies
   - f is continuous on [a,b]
   - f is differentiable on (a,b)

   then there's a number c in (a,b) such that
   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
   (a) Draw picture to illustrate, drr...
   (b) Pro0f: Reduce the situation to Rolle's Theorem. Let $g(x)$ be the line through $(a, f(a)), (b, f(b))$.
   $$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$
   Define $s(x) = f(x) - g(x)$. Rolle's theorem gives $s'(c) = 0$ for some c in (a, b). Then, $f'(c) = \frac{f(b)-f(a)}{b-a}$.
   (c) Applications:
   - Average rate of change equals inst rat of change.
   - When you are driving, there'll always be a moment that your instantaneous velocity is the same as the average velocity.
   - Suppose you are driving from La Crosse to Madison: 150 mile, 1.5 hours. What should the speeding ticket be written for? 100mile/h

4. **Examples** Apply the Mean Value Theorem.
   (a) Show $(\cos A - \cos B) \leq (A - B)$.
   (b) If $f(0) = -3$ and $f'(x) \leq 5$ for all x, how large can $f(2)$ be?

5. **Theorem:** If $f'(x) = 0$ for all x in (a, b), then f is a constant on (a, b).
   (a) Why open interval here? Actual constant is not known. How to show this?
   (b) **Proof:** Assume $f'(x) = 0$ on (a, b). Need to show for any x_1, x_2 in (a, b), $f(x_1) = f(x_2)$. For $x_1 < x_2$, by the MVT
   $$0 = f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$
   for some c in (x_1, x_2). Done.

6. Antiderivative beginnings
   (a) Find $f(x)$ such that $f'(x) = 2x$. Lots here.
   (b) **Corollary:** If $f'(x) = g'(x)$ for all x in (a,b), then $f - g$ is a constant on (a,b)
   (c) That is: $f(x)$ is uniquely determined by $f'(x)$ subject to a change of constant. Terminology here is antiderivative.
   (d) **Proof:** Let $h(x) = f(x) - g(x)$, then $h'(x) = 0$. So $h(x) = C$ and $f(x) = g(x) + C$.

.3 4.3 How derivatives affect the shape of a graph

1. **Example:**
   (a) Graph $f(x)$, $f'(x)$ and $f''(x)$ for $f(x) = x^3 + x^2 - x$
   (b) How is f' related to f, f'' to f', f'' to f?

2. First derivative $f'(x)$
   (a) **Increasing/decreasing test**
   - If $f'(x) > 0$, then f is increasing ($a < b$ gives $f(a) < f(b)$)
   - If $f'(x) < 0$, then f is decreasing ($a < b$ gives $f(a) > f(b)$)

   (b) **The first derivative test** Suppose c is a critical number for f (possible local max/min). Let them fill in blank.
   - If $f'(x)$ changes from positive to negative at c, then .... f has a local max at c.
   - If $f'(x)$ changes from negative to positive at c, then .... f has a local min at c.
   - If $f'(x)$ does not change sign at c, then .... f has no local max or min at c. (called a saddle point)

3. **Examples:** Draw a number line and find the absolute and local maxs/mins for each. Plot the points.
   (a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
   (b) $h(x) = 1 + 2\sin(x)$ on $[0, 2\pi]$.
   (c) $g(x) = \sin(x)$
   (d) How to graph f? Have a pretty good picture. What else can we add for detail? Where are turning points? Zeros?

4. Second derivative $f''(x)$, take $\sin x$ as an example. Let them fill in blank.
   (a) **Concavity test**
   - If ... $f''(x) > 0$, then f is concave up
   - If ... $f''(x) < 0$, then f is concave down

   (b) Application: Filling a coke bottle with a constant volume of pouring. How does the height function change in concavity?
   (c) **The second derivative test:**
   - $f'(c) = 0$, $f''(c) > 0$, local min
   - $f'(c) = 0$, $f''(c) < 0$, local max

   (d) Used to find local max/min, easier than first derivative test.
   (e) If $f''(x) = 0$, it's inconclusive. Why? Think of a graph. This is an inflection point.
   (f) When can't the second derivative test be used? If f'' does not exist (corner)

5. **Examples:** Graph sketching
   (a) Give a random graph and decide where it's increasing, decreasing, concave up/down, local max/min, inflection points.
   (b) Give a random graph of f' and decide increasing, decreasing, concave up/down, local max/min, inflection points.
   (c) Sketch the graph of $3x^4 - 4x^3 - 12x^2 + 5$ with all detail and glory.
   (d) Sketch the graph of

$$f(x) = x^{2/3}(6-x)^{1/3} \quad \text{knowing} \quad f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}, \quad f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

.4 4.4 Indeterminate forms and L'Hospital Rule

1. We've already seen indeterminate forms for limits, but here we get a better way to handle them.
   (a) **Examples:** ...from the past...
      - $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = 1$ (0/0 IF, factor and cancel, hole in graph.)
      - $\lim_{x\to 1} \dfrac{\ln(x)}{x-1} = ?$ (0/0 IF, cannot factor)

   (b) Key: Indeterminate forms can be ANYTHING. Modify above example to show 2, 200, π, ∞, 0, etc.

   (c) Types of indeterminate form:
      - Type $0/0$
      - Type ∞/∞
      - Type $0 \cdot \infty$
      - Type $\infty - \infty$
      - Type $0^0, 1^\infty, \infty^0$

2. **Theorem: l'Hospital's Rule** If
   (a) f and g are differentiable in interval I containing $x = a$, and
   (b) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $0/0$ or ∞/∞

   then
   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
   if $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists or is $\pm\infty$

   (a) Why is this nice? $\lim_{x\to 1} \dfrac{x^2-1}{x-1} = 1, \lim_{x\to 1} \dfrac{\ln(x)}{x-1} = ?$ (0/0 IF, can use l'Hospital's Rule)

   (b) Why is LR true? **Proof** idea for 0/0 case by graph (slope over slope).
   $$\frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{m_f(x-a)}{m_g(x-a)} = \frac{m_f}{m_g} = \frac{f'(a)}{g'(a)} \quad \text{since } f(a) = g(a) = 0$$

   (c) **Proof** of ∞/∞ case: $\frac{f(x)}{g(x)} = \frac{1/g(x)}{1/f(x)}$ transfers to 0/0 IF.

   (d) Note: LR works for one sided limits also

3. **Examples:**
   (a) Can get crazy: Which grows faster, x^{1000} or e^x? How to tell? Look at the ratio: $\lim_{x\to\infty} \dfrac{x^{1000}}{e^x}$

   (b) Old limits are easier (?):
   $$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}, \quad \lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = 2/3$$

   (c) Beware of temptation:
      i. $\lim_{x\to 0} \cos x/x = 0$ What went wrong? What should I have had? ALWAYS check the hypothesis of LR first.

ii. $\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{x}$ =?, $\lim_{x\to\infty} \frac{\sqrt{9x+1}}{\sqrt{x+1}}$

iii. $\lim_{x\to\infty} \frac{x+\sin(x)}{x} = 1$ cannot apply right away.

4. Other indeterminate forms: Idea is to always transfer to $0/0$ or ∞/∞ form.
   (a) $0 \cdot \infty$ IF, for the second one way works but not other.
$$\lim_{x\to 0^+} x\ln(x),\ \lim_{x\to -\infty} xe^x$$
   (b) $\infty - \infty$ IF
$$\lim_{x\to 0} \cot x - 1/x$$
   (c) $0^0, \infty^0, 1^\infty$ IFs, cannot use algebra but logarithms are useful.
$$\lim_{x\to 0} x^x,\ \lim_{x\to\infty} (1+\frac{1}{x})^x$$
   Idea: $\lim_{x\to\infty} [f(x)]^{g(x)} = \lim_{x\to\infty} e^{\ln([f(x)]^{g(x)})} = e^{\lim_{x\to\infty} g(x)\ln(f(x))}$ assuming limit exists since e^x is continuous.
   (d) **Examples:**
$$\lim_{x\to 0^+} \frac{e^x - 1}{\cos x} = 0$$
$$\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2} = 1/3$$
$$\lim_{x\to \pi/2} \frac{1-\sin x}{1+\cos 2x} = 1/4$$
$$\lim_{t\to 0^+} (1+\sin 4t)^{\cot t} = e^4$$
   (e) Mention it's super important: indeterminant form is the most common case we want to work on. (derivative, def integral, etc)

.5 4.5 Summary of carve sketching

1. Guidelines for curve sketching (we've already covered this!):
   (a) Find the omain
   (b) Locate x and y intercepts
   (c) Does f have symmetry (even or odd)?
   (d) Asymptotes (horizontal, vertical, oblique)
   (e) Where is f increasing / decreasing?
   (f) Find local mins and maxes (critical pts and 1st or 2nd derivative test)
   (g) Concavity and points of inflection
   (h) Put all together to get a fantastic picture
2. **Examples:**
   (a) $f(x) = \frac{1+2x^2}{1-x^2}$ (horizontal and vertical asymptotes)
   (b) $g(x) = \frac{-3x^2+2}{x-1}$ (oblique asymptote, need long division)

   Read the section and finish the homework!

.6 4.7 Optimization problmes

This section is all about maximizing / minimizing some quantity subject to a given constraint. These problems are everywhere, some are just disguised (instructor scheduling problem, linear prog, genetic algorithm)

1. A farmer has 800 ft of fence and wants to fence off a rectangular field. How to arrange the fence so that it covers the biggest area? What if one side was my 60ft house? Almost the same.
2. Optimization problem strategy:
   (a) Make sure it's an optimization problem (-est, most, least)
   (b) Draw a picture to help
   (c) Find the variable y that you want to minimize/maximize, introduce other notation
   (d) Find the changing variable x
   (e) Write $y = f(x)$ as a function of x, eliminate other variables if needed
   (f) Identify an closed interval for x (why necessary? Extreme Value Theorem)
   (g) Find the extreme value of y
   (h) Answer the original question in words
3. **Examples:**
   (a) A cylindrical can is required to hold 1 liter of oil. Design the can to minimize the use of material.
   $$S = 2\pi r^2 + 2\pi rh, \text{ (eliminate } h\text{, can also use implicit diff)}$$
   (b) Find the point on the curve $y = 2x - 1$ closest to the point (3, 2).
   $$d = \sqrt{(x-2)^2 + (y-2)^2}, \text{ (can eliminate or implicit diff)}$$
   (c) What's the area of the biggest rectangle that can be inscribed inside a unit circle?
   $$A = 2xy = 2x\sqrt{1-x^2}$$
4. So many applications here, especially in business.

.7 4.8 Newton's method

1. **Motivating Example** Solve $x^3 - 3x + 1 = 0$
   - Pick place to start: $x_0 = 0$
   - Find the linearization at (x_0, y_0)
   - Follow linearization to get zero which approximates f's zero.
   - Show Desmos right away
   - Write down the formula
   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
   - Do the iteration by hand
   - Do $x_0 = 7$, 4.
2. Idea of Newton's method
   (a) Again, replace f by linearization (calculus), sweet move.

(b) Does it always work? What could go wrong? We need more than just a formula.

   i. Could be no root at all (IVT to check existence)
   ii. Could hit zero derivative, shoot to infinity (zero division)
   iii. Find wrong root.
   iv. Even though the formula stays the same, the result depends on the initial value x_1
   v. Slows down at roots with multiplicity.
   vi. Sometimes it just doesn't work (MVT, diverges to infinity)

$$x^{1/3} = 0, \quad x_1 = 1.$$

   vii. Under the right assumptions (continuous and differentiable around the root, choose x_1 close enough), can prove Newton's method is fast and effective.

3. Find the solution of $\cos x = x$ using the Newton's method

   (a) Draw a picture to see how many solutions are there
   (b) Find the iteration method
   (c) For which initial value does it fail
   (d) Assign the initial value
   (e) Compute the result
   (f) Mention fixed point methods if interested $x_n = \cos(x_{n-1})$

4. Mind-blowing awesomeness:

   (a) `http://octave-online.net/`
   (b) $\sqrt{2}$ via $x^2 = 2$.
   (c) π via $\sin(x) = 0$ fast, $\cos(x) = 1$ slow. Why? Multiplicity of root.
   (d) R pseudocode:

```
# newton's method
options(digits=16)

f <- function(x){x^2-2}
fp <- function(x){2*x}

x <- 1
for (i in c(1:10)){
  x <- x - f(x)/fp(x)
  print(x)
}
```

   (a) How to improve on Newton? Taylor series to higher order
   (b) Fractals and complex numbers

      - Zoomin: `https://www.youtube.com/watch?v=0jGaio87u3A`
      - Applications: `https://en.wikipedia.org/wiki/Fractal#Applications_in_technology`
      - Nature: `https://www.google.com/search?q=fractal+nature&espv=2&biw=1309&bih=781&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwjS677XsIbMAhUMMSYKHSwkBC0QsAQIGw`

.8 4.9 Anti-derivatives

1. Motivation: Goal is to reverse differentiation. Key here is lack of uniqueness.
   (a) Zombies touched on this situation already, leads to predator prey.
   (b) Find function $F(x)$ such that $F'(x) = 2x$
   (c) What about $F'(r) = 2\pi r$?, $F'(r) = 4\pi r^3$? Wha??
   (d) Conservation law.
   (e) Free fall $a(t) = 9.8m/s^2$, can get velocity and distance.
   (f) Free fall with drag $a(t) = 9.8 - kv$ (gravity const - drag).
   (g) Kepler's laws of planetary motion spurred Newton to work on Calculus and support theory of physics.
   (h) Any physical (or other) law.
2. **Def:** Function F is an antiderivative of f on interval I if $F'(x) = f(x)$ for all x on I. Comments
   (a) Note: An antiderivative is not unique. Why?
      - Graphically, the derivative only tells you the shape of the function subject to a vertical shift
      - By computation, differentiating a constant always gives zero.
      - Mean value theorem already gave us this as a theorem (lemma?). If F is an anti-derivative of f on an interval I, then the most general anti-derivative of f on I is

        $$F(x) + C$$

        were C is an arbitrary constant.
3. How to understand $F(x) + C$?
   (a) Give me three examples for function F you choose.
   (b) How to make it unique? Need more information. Find the anti-derivative of $f(x) = 2x$ that passes through point $(0, 1)$. Also, zombies already tackled this situation.
4. A new idea handles this lack of uniqueness.
   Definition: The set of all antiderivatives of a function is called the indefinite integral of f with respect to x, written

   $$\int f(x)\ dx = F(x) + C$$

   where C is an arbitrary constant. The process of computing integrals is integration. Here, we see differential notation and a stretched out S, reasons to come.
5. Properties of the indefinite integral come from differentiation. They are easily checked through that lens.
   (a) $\displaystyle\int cf(x)\ dx = c\int f(x)\ dx$
   (b) $\displaystyle\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$
6. bf Examples: Find the anti-derivatives of each. List all differentiation rules. Note each are easily checked, though integration is generally much harder than integration!
   (a) Integration rules: $k, x^n, \frac{1}{x}, \sin(x), \cos(x), \sec^2(x), \csc^2(x), \sec(x)\tan(x), \csc(x)\cot(x), e^x, e^{kx}, a^x, \frac{1}{1+x^2}$
   (b) $3x$
   (c) $x^2 + e^x$

(d) xe^x
(e) $2x\cos(x^2)$
(f) $\frac{2x}{1+x^2}$
(g) Find $F(x)$ such that $F''(x) = x^2$
(h) Find f such that $f'(x) = \frac{x^2 - x + 5x^3}{2x}$ and $f(1) = 2$.

Chapter 5

.1 5.1 Areas and distances

Motivation:

- The area of a parabola
- The area of a cylinder
- The volume of a half sphere and a cylinder with a cone removed.
- The area of two triangles between two parallel line
- The contradiction between two triangles.

1. Recall, when we motivated this class, we had 2 main questions:
   (a) Tangent problem (derivative)
   (b) Area problem (our change of focus, definite integral)
   (c) Surprisingly, these are deeply connected.
   (d) Example: Driving car at 60mph. Draw displacement and velocity. What is the connection back and forth?
      - $v'(t) = s(t)$
      - Accumulated area under s gives function values of v.
2. Find areas: Generally not so easy....
   (a) Rectangle, triangle, circle (talk Archimedes and limiting process)
   (b) Find the area under the curve $y = x^2 + 1$ on $[0, 2]$. Use a simple shape (rectangle) to approximate. Draw idea by hand.
   (c) Use Desmos to show Riemann sum, emphasize limiting procedure, hint at indeterminate form. `https://www.desmos.com/calculator/tgyr42ezjq`
3. Riemann Sum: Limiting procedure for area calculations.
   - Interval: $[a, b] = [0, 2]$
   - Function $f(x) = x^2 + 1 > 0$ on $[a, b]$
   - Find the area under curve $y = f(x)$ (between $f(x)$ and the x-axis) on $[a, b]$

   ‘

   (a) Finding the approximate area:
      i. Partition of [a,b]
      ii. Find the area of each rectangle
      iii. Take the sum
   (b) Details:

   i. Divide (a, b) into n sub-intervals
   ii. Name all the ending points x_0, x_1, ... , x_n
   iii. On each part, find the area of the rectangle
      - Width: $\Delta x = (b - a)/n$, uniform for simplicity.
      - Height: for the i_{th} rectangle, we have choices. 3 standard ways.
        - Left ending points: $h_i = f(x_{i-1})$
        - Right ending points: $h_i = f(x_i)$
        - Midpoints: $h_i = f\left(\frac{1}{2}(x_{i-1}) + f(x_i)\right)$
      - Area of each rectangle: $A_i = h_i \cdot \Delta x$
   iv. Take the sum:
   $$R_n = [h_1 \cdot \Delta x + h_2 \cdot \Delta x ... + h_n \cdot \Delta x]$$
   (c) For the true area, take $n \to \infty$
   $$A = \lim_{n\to\infty} R_n$$

4. Formal (make it look better)
   (a) Height $h_i = f(x_i^*)$
      - Left ending points: $x_i^* = x_{i-1}$
      - Right ending points: $x_i^* = x_i$
      - Midpoint: $x_i^* = (x_{i-1} + x_i)/2$
      - Does the answer depends on which sample we pick?
        - For approximation (finite number of rectangles), yes it does.
        - Doesn't matter much when $n \to \infty$ as long as $f(x)$ is continuous.
   (b) Compactify the summation formula of R_n: Sigma notation.
      - Review
      $$\sum_{i=1}^{5} i^2, \quad 3, \quad j, \quad \sum_{c=1}^{4} calculus$$
      - Summation formulas:
      $$\sum_{i=1}^{n} c = cn, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\text{(Gauss, telescoping via}(i+1)^2 - i^2 = 2i + 1)$$
      $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$
      - Riemann sum
      $$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$
      - Point to different notations and ask people the meaning of each term
      - Area = the limit of the Riemann sum:
      $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

5. Examples:
   (a) Find the approximate area under the graph of $f(x) = x^2 + 1$ on $[0, 2]$ using 4 left/right rectangles.
   (b) Find the approximate area under the graph of $\cos x$ from $(-\pi/2, \pi/2)$ using
      - 6 left rectangles
      - 6 right rectangles
   (c) Find the approximate area under the graph of $f(x) = x^2 + 1$ on $[0, 2]$ using n right rectangles. Take the limit to get true.

.2 5.2 The definite integral

1. Goal: Find the exact area of $f(x)$ on $[a, b]$.
   Idea: Take the limit of R_n as $n \to \infty$.

2. Definite integral of f from a to b

   $$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

   Provided the limit exists.

   - It's a limit of a sum
   - It's a number
   - Notation is complex. Keep reading it for what it is. Here it is clear why we use $\int$ to denote sum and dx in place of Δx. Analogy here with differentials.
   - Terminology:
     - Integrand: f(x)
     - Upper limit: b
     - Lower limit: a
     - Integration
   - Net area has physical meaning
     - Positive
     - Negative
     - Cancellation (sine curve)
     - Give random functions and determine whether its net area is positive or negative

3. **Examples:** We already computed one definite integral in previous section. Try this one on own via x_i^* (1) left end points, (2) right end points. Check your work via geometry. One triangle subtract another.

   $$\int_{-1}^{3} (-2x + 4)\ dx$$

   What if we had $\int_{-1}^{5}(-2x + 4)\ dx$? Can use symmetry to get zero.

4. **Examples:** Write the following expressions in terms of limit of a sum

   $$\int_0^1 \sin x\ dx, \quad \int_{-2}^{2} e^x \tan x\ dx$$

5. **Examples:** Write the limit as definite integrals

   - $$\lim_{n\to\infty} \sum_{i=1}^{n} x_i^* \sin x_i^* \frac{2}{n}, \quad \text{with lower limit } 0$$
   - $$\lim_{n\to\infty} \sum_{i=1}^{n} 3x_i^* \sqrt{x_i^* + 1}\frac{7}{n}, \quad \text{with upper limit } 4$$
   - $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin(x_i^*)^2}{\ln(x_i^* - 2)}\Delta x, \quad \text{on} \quad [1, 5]$$

6. Properties of the definite integral

   (a)
   $$\int_a^a f(x)dx = 0 \quad \text{no area}$$

   (b)
   $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{separate the interval}$$

   (c)
   $$\int_a^b f(x)dx = -\int_b^a f(x)dx \quad \text{reverse direction}$$

   (d)
   $$\int_a^b c\ dx = c(b-a), \quad \text{rectangular area is easy}$$

   (e)
   $$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx \quad \text{same as with summation}$$

   (f)
   $$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx \quad \text{same as with summation}$$

   (g)
   $$\int_a^b [cf(x)]dx = c\int_a^b f(x)dx, \quad \text{where c is a constant}$$

7. Comparison properties of the integral (draw pictures with each)

   (a) If $f(x) \geq g(x)$ on [a,b], then
   $$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$$

   (b) **Examples:**
   $$\int_0^1 x^2\ dx > 0, \qquad \int_0^1 x^2 + e^x\ dx > 1/2$$

   (c) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
   $$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a) \quad \text{Why? Try to decide}$$

   (d) **Example:**
   $$-4 \leq \int_{-2}^2 \sin x\ dx \leq 4 \quad \text{What is this one really though?}$$

8. If $\int_a^b f(x)\ dx$ exists, then f is integrable on $[a,b]$.

   - The limit doesn't always exist.
   - Riemann sum: why the limit exists much of the time.
     - Upper sum: overestimate
     - Lower sum: underestimate

- Squeeze theorem is in the works here.
- When will it fail: give an example (infinite discontinuity, unbounded interval, some of these cases actually work, Calc 2 and improper integrals)
- When will it converge: continuous on bounded intervals

- **Theorem:** If f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$. Draw picture of idea here.

9. Evaluating integrals

   (a) Reimann sum, then limit. Need a way to compute the sum via formulas.
   $$\int_0^4 (x + x^3/3)\ dx$$
   (b) Geometry
   $$\int_{-1}^1 \sqrt{1-x^2}\ dx$$
   (c) Either way, it's complicated and frustrating. Need something better.

.3 5.3 The fundamental theorem of Calculus

A better way to find the area under a curve (more general, easier, connects to differentiation.).

1. The Fundamental Theorem of Calculus (at last!)

   (a) Part 1: if f is continuous on [a,b], then the functor g defined by
   $$g(x) = \int_a^x f(t)\ dt, \quad a \le x \le b$$
   is continuous on [a,b] and differentiable on (a,b), and g'(x) = f(x)

   (b) Part 2: if f is continuous on [a,b], then
   $$\int_a^b f(x)\ dx = F(b) - F(a)$$
   where F is any anti-derivative of f.

   Part 1 connects definite integration and differentiation. Part 2 makes definite integrals easier (boils down to antiderivative problem).

2. What is part 1 about? A new object here called an accumulation function.
   $$g(x) = \int_a^x f(t)\ dt, \quad a \le x \le b$$
   Draw pictures to get idea.

3. **Example:** Part 2 is where the action is.

   (a)
   $$\int_0^2 (x^2+1)\ dx = \frac{8}{3} + 2 \quad \text{(same as before)}$$
   (b) Give many other examples. Anything antiderivative we can handle is fair game.

4. Well, that makes it easy. Why is it true? Think about distance problem. We had this connection already.

$$\int_a^b \text{velocity } dt = \quad \text{displacement during a, b}$$

5. **Proof of FTOC:** If f is continuous on [a,b], define

$$g(x) = \int_a^x f(t)\ dt$$

   (a) **Part 1:**

   - Assume $g(x)$ is differentiable (can show this), then we have access to difference quotient.

$$g'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ dt$$

   - Show that $g'(x) = f(x)$ via use of the Squeeze Theorem. If $m \leq f(x) \leq M$ on $[x, x+h]$, then

$$mh \leq \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ dt \leq Mh$$

   (b) **Part 2:**

   - From part 1, we have one antiderivative of f.

$$g(x) = \int_a^x f(t)\ dt, \qquad \text{where } g'(x) = f(x)$$

   - Then any antiderivative is $F(x) = g(x) + C$ (where C changes when a change).
   - Then,

$$F(b) - F(a) = g(b) - g(a) = g(b) - 0 = \int_a^b f(t)\ dt$$

6. A historic controversy: Issac Newton vs Gottried Leibniz

   - 1666: Newton start to work on calculus (manuscript)
   - 1674: Leibniz started to work on calculus
   - 1684: Leibniz published calculus
   - 1687: Newton's 1st publication about calculus
   - 1693: Newton's publication of fluxion
   - 1696: L'Hospital published his work and quote Leibniz's work
   - 1699: the controversy began (the royal society)
   - 1704: Newton's full work
   - 1711: the controversy broke out

7. **More examples:**

$$\int_3^6 \frac{3}{x+1}\ dx, \quad \int_0^{2\pi} \sin x\ dx$$

8. Part 1: This unassuming function has many applications in physics, chemistry and statistics. Let's have a closer look.

$$g(x) = \int_a^x f(t)\ dt$$

   Draw some function f. Where is g

(a) Increasing/decreasing

(b) Local maximum/minimum

(c) Concave up and down?

(d) Find g' where

$$g(x) = \int_a^x \ln t\ dt, \quad g(x) = \int_x^a \sin t\ dt, \quad g(x) = \int_1^{e^x} \ln t\ dt = xe^x, g(x) = \int_x^{x^2} e^{t^2}\ dt$$

.4 5.4 Indefinite integrals

1. We already discussed antiderivatives. Recall an indefinite integral is the collection of all antiderivatives. Ie

$$\int f(x)\ dx = F(x) + C$$

if $F'(x) = f(x)$

2. **Examples:**

(a) $\int (x+1)^2 dx$

(b) $\int (\frac{1}{x} - \frac{2}{x^2} + \frac{3}{x^2+1})dx$

(c) $\int \frac{t^6 - t\sqrt{t} + 2}{\sqrt[3]{t}} dx$

(d) $\int (e^x \tan x)' dx$

(e) True or false: $(\int x\ dx)' = \int (x)'\ dx$

(f) True or false: $\int \frac{\cos x}{\sin^2 x}\ dx = -\frac{1}{\sin x} + C$

3. **Examples:** Thanks to FTOC, finding definite integrals is just as easy. More ideas involved when it comes to area though. Even though cannot integrate directly, can still manage.

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{(some notation here)}$$

(a) $\int_{-1}^2 (x - 2|x|)\ dx$

(b) $\int_0^{\frac{3\pi}{2}} |\sin x|\ dx$

(c) $\int_{-\pi/2}^{\frac{3\pi}{2}} |\sin x|\ dx$

4. Net Change Theorem: If $r(t)$ is the rate of function $F(t)$, then

$$\int_a^b r(x)\ dx = F(b) - F(a)$$

(a) This is the same as the FTOC. The new idea is the interpretation.

(b) If I am driving at velocity $v(t)$, then $\int_0^1 0v(t)\ dt$ represents my displacement after 10mins.

(c) If $r(t)$ is the rate oil leaks from my car, then $\int_0^7 r(t)\ dt$ represents the amount of oil lost in a week.

(d) If $i(t), o(t)$ are the global birth,death rates in the US, then $\int_0^1 00b(t) - d(t)\ dt$ is the net population growth over 100 years.

(e) This is a primary way in which differential equations are derived (conservation laws).

.5 5.5 The substitution rule

1. The FTOC opens the door to many definite integrals we can now do. But, as we saw before, integration is much harder than differentiation. Here we start to grow our techniques by attempting to reverse more complicated differentiation rules.

   - Basic rules (constant, power rule, exponents, etc)
   - Addition, subtraction, constant multiple
   - Product, quotient (calculus II, integration by parts, let's be real, nobody wants to reverse the quotient rule....but can think of as product rule instead.)
   - Chain rule (now! substitution rule) How to reverse?

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

2. Hints at the idea of substitution:

   (a) Level 1: give direct examples

$$\int 2x\cos(x^2)\ dx, \quad \int 2\sin x\cos x\ dx$$

   (b) Level 2: modified by a constant

$$\int e^{2x}\ dx, \quad \int \frac{\tan^{-1} x}{1+x^2}\ dx, \quad \int \frac{x}{1+x^2}\ dx$$

   (c) Level 3: not so obvious, but doable.

$$\int x^5\sqrt{x^2+1}\ dx$$

3. **Chain Rule:** Formalize the idea. We rename things as u to help us see the structure. u is essentially a guess for the inside function.

$$\int f[g(x)]g'(x)dx = \int f(u)du$$

   (a) Note, the new integral is now in u only, not x. We update this variable as

$$u = g(x) \quad \rightarrow \quad du = g'(x)dx$$

   This is a differential version of $\frac{du}{dx} = g'(x)$. Squirrelly...

   (b) Hopefully we can integrate f.

   (c) After integrate, substitute back.

4. **Examples:**

   (a) Do all the easy ones again with this structure.

(b) Harder:

$$\int x^2 \cos(x^3+2)\ dx, \quad \int \tan x\ dx$$

(c) Harder yet:

$$\int x^5\sqrt{x^2+1}\ dx, \quad \int \sin^4 x \cos^3 x\ dx$$

Here we see the power of undoing such a simple rule. This opens the door to integrating many more functions. How about definite integrals?

5. **Theorem:** Substitution Rule for Definite Integrals
   If we assume

   - g' is continuous on $[a, b]$
   - f is continuous on the range of $u = g(x)$

   then we have

$$\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

6. **Examples:**

   (a) $\displaystyle\int_0^4 \sqrt{3x+4}\ dx = \frac{112}{9}$

   (b) $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}dx = \frac{1}{14}$

   (c) $\displaystyle\int_1^e \frac{\ln x}{x}dx = \frac{1}{2}$

   (d) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}dx = \frac{1}{2}$

   (e) $\displaystyle\int_{-1}^1 \sqrt{1-x^2}dx = \frac{\pi}{2}$ again via geometry. Can we actually compute? Yes, trig sub. Calc 2 will revisit...mwhuahahahahaha....

7. Integral of Symmetric Functions Suppose $f(x)$ is continuous on $(-a, a)$

   (a) If f is even, then

$$\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx$$

   (b) If f is odd, then

$$\int_{-a}^a f(x)\ dx = 0$$

8. More area intuitiveness. Show for any integrable f,

$$\int_a^b f(x+c)\ dx = \int_{a+c}^{b+c} f(x)\ dx$$

.6 6.1 Area between curves

1. Theorem: Condition

   - f and g are continuous
   - $f(x) \geq g(x)$ for all x in $[a, b]$

   Conclusion:

   $$A = \int_a^b [f(x) - g(x)]\ dx$$

   Example: find the are enclosed by y = x, y = 4-x, x = 0 and x=1
   *Find the area between $y = x^2$ and $y = 2 - x^2$

2. What if we don't have $f(x) > g(x)$: split the area
   Example: find the area bounded by $y = \sin x$, $y = \cos x$, x= 0 and x = $\pi/2$
   * Find the area between $y = x^3$ and $y = x$ *Find the area between $y = x - 2$ and $x = y^2$

3. Integrate with respect to y
   Find the area enclosed by $y = x$ and $y = x^2$ in both direction
   Find the area of a unit circle.

.7 6.2 Volumes

1. Idea:

   - Volume of a cylinder
   - Volume of a cone:
     * Partition in height
     * Cross sectional area: *Riemann sum
     * Take the limit to integral

     $$V = \int_a^b f(x)\ dx$$

     where the cross sectional area at position x is given by $f(x)$

2. Examples:

   - Volume of a pyramid with base sides l and height h
   - Volume of a Sphere
   - Find the volume of a circle rotated: donuts

3. Volume of revolution: `https://www.youtube.com/watch?v=M9-hAJ8IrmU`

   (a) Give sample pictures
   (b) Michael Jackson
   (c) Find the volume generated by rotating the region $y = \sqrt{x}$, $0 \leq x \leq 1$ with respect to the x axis
   (d) Find the volume generated by rotating the region $y = x^2$, $0 \leq x \leq 1$ with respect to the x axis
   (e) Find the volume generated by rotating the region $y = \sqrt{x}$, $0 \leq x \leq 1$ with respect to $y = -4$
   (f) What volume does the following expression represent?
   1.

   $$\pi \int_0^\pi \sin x\ dx$$

   2.

   $$\pi \int_{-1}^1 (1 - y^2)^2\ dy$$

4. The washer method: find the volume of the area between $y = x$ and $y = x^2$ rotated by x axis, y axis, $x = 4$, $y = 4$

$$\int_a^b \pi(R^2 - r^2)\, dx$$

5. Random shape with base and cross sectional area

   (a) The base of an object S is a circular disk with radius 1. Parallel cross sections perpendicular to the base are equilateral triangles

   (b) Bases of S is the triangular region with vertices (0,0), (1,0), and (0,1). Cross-sections perpendicular to the y axis are square

   (c) The basis of S is the region enclosed by the parabola $y = 1 - x^2$ and the x axis. The corrections perpendicular to the x-axis are isosceles triangles with height equal to the base.

.8 6.3 Volumes by cylindrical shells

* Find the volume of a rectangle rotated by y axis * Washer and toilet paper

`http://www.falconworkshop.co.uk/A2%20Washers.jpg`

`http://i00.i.aliimg.com/photo/v1/134273580/Kitchen_towel_tissue_paper.jpg`

1. The idea of cylindrical method:

   - Only works for volume by revolution
   - Formula

   $$\int_a^b 2\pi rh\, dr$$

   - Example
     - Donus: $(x-2)^2 + y^2 = 1$ rotated by x = 0
     - $y = x^2$ rotated by x = 0
     - $y = x$, $y = x^2$ rotated by $y = -2$
     - $y = x^3$, $y = 8$, $x = 0$ about x = 3
     - Cones:

.9 6.4 Work

1. Work: force times distance

2. Unit: ft-lb, Joule (m times N)

3. Formula: work done in moving the object from a to b

$$\int_a^b f(x)\, dx$$

   $f(x)$: force

4. Hooke's law: a force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to 15 cm. How much work is done in stretching the spring from 15cm to 18 cm?

5. A tank has the shape of an inverted circular cone with height 10 m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3)

6. A 10 ft chain weights 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it's level with the upper end.

.10 6.5 Average value of a function

1. Discrete average value
2. Average value via Riemann sum
3. Average value of a function formula)
4. The mean value theorem of integrals: if f is continuous on [a,b] then there exists a number c in [a,b] such that
$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\ dx$$
that is,
$$\int_a^b f(x)\ dx = f(c)(b-a)$$
5. Understanding from physics (average speed = dispacement / time)

Chapter 9 Differential Equations

.1 9.1 Modeling with differential equations

1. Motivation
   (a) Exponential growth
   (b) Logistic function
   $$y' = ky(1-y/R), \quad y = \frac{R}{1+e^{-kx}}$$
   (c) Physics
2. Differential equation
   (a) Definition: equations with derivatives
   (b) Order of differential equations
   (c) Definition of solution
3. Solving differential equations
   (a) Analytical
   (b) Direction field
   (c) Numerical: Euler's method
4. Initial value problem