Cubics, Chaos and Newton's Method
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Cubics, chaos and Newton's method

THOMAS DENCE

It is common today for students in elementary calculus to possess a powerful graphics calculator, such as the TI-82. These machines have had a dramatic impact on how teachers present the subject. The teacher population is clearly divided when it comes to the emphasis on the use of the calculators. There are instances, though, where it is difficult to argue with the calculator's effectiveness as a teaching tool in calculus. One such case is in using Newton's method to find the roots of an equation, for once the algorithm is presented, it is merely a matter of performing the numerical calculations for the sequence of iterates, and this is an operation that the TI-82 does very well.

To illustrate, suppose we desire the intersection of e^x and $2 + \cos x$ in the first quadrant. On the TI-82, enter the function $y_1 = e^x - (2 + \cos(x))$ and its derivative $y_2 = e^x + \sin(x)$. Observing that the intersection is close to 1, we enter the following key strokes:

1 ENTER
ANS − y_1 (ANS)/y_2 (ANS) ENTER
ENTER
ENTER
ENTER

By the time the fifth ENTER has been recorded, a value of .9488147556 has been reached for the x-value of the intersection point. The accuracy of this result, and the ease and speed with which it was reached, is hard to beat.

One should not disregard the importance of 'exploring' and searching for patterns in mathematics. My calculus class recently enjoyed the following discovery when experimenting with Newton's method on cubic polynomials.

Consider, initially, the cubic $f(x) = (x - r_1)(x - r_2)(x - r_3)$ with three real roots $r_1 < r_2 < r_3$, and let $c_1 < c_2$ be the two critical points (where the derivative vanishes, $f'(c_i) = 0$). Let us say that we wish to solve for the largest root r_3. Using Newton's method, which is repeated application of $N(x) = x - f(x)/f'(x)$ with any initial estimate x_1, where $x_1 > c_2$, Figure 1, will produce a sequence $\{x_n\}$ of iterates that converges to r_3.

In an article [1] in an earlier issue of this journal, the author showed that an initial estimate x_1 equal to the midpoint m_2 of the interval $[r_2, r_3]$

$$x_1 = m_2 = \frac{r_2 + r_3}{2}$$

produces a sequence $\{x_n\}$ that converges (after one iterate) to the smallest root r_1. Incidentally, a similar result occurs if $x_1 = m_1 = (r_1 + r_2)/2$ with $\{x_n\}$ converging to r_3 (Figure 2).

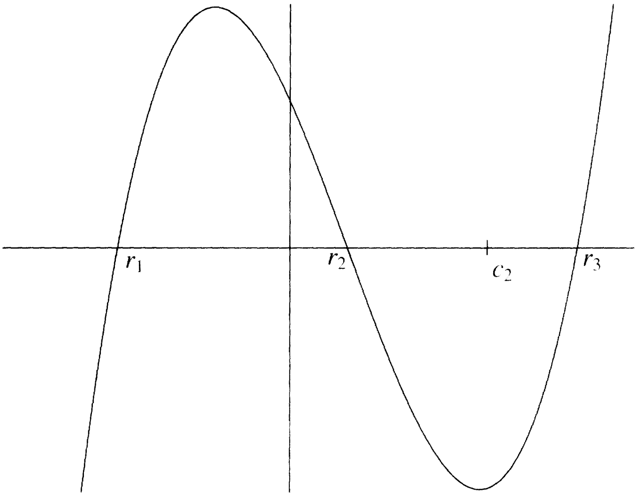

FIGURE 1

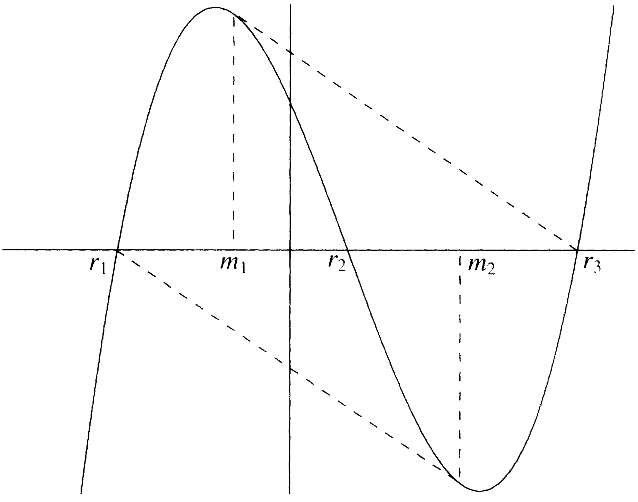

FIGURE 2

Because of the symmetry that occurs with $\{x_n\}$, depending on whether $x_1 < r_2$ or $x_1 > r_2$, let us assume the latter. It follows that $x_1 = m_2$ must be less than the critical point c_2, otherwise $\{x_n\}$ couldn't converge to r_1. Furthermore, any initial estimate x_1 chosen between m_2 and c_2 will yield a sequence $\{x_n\}$ that converges to r_1, as will any x_1 relatively close to, but less than, m_2. How close x_1 has to be to m_2 is determined by whether the second Newton approximation $x_2 = N(x_1)$ is less than the critical value c_1 (Figure 3). As a result, we have an open interval $\mathbf{I}_1 = (a_1, b_1)$ of length L_1, with

$b_1 = c_2$, such that the sequence $\{x_n\}$ of Newton approximates converges to r_1 for any initial estimate $x_1 \in \mathbf{I}_1$. The value of a_1 is the number that Newton's method maps to c_1 after one iteration, $N(a_1) = c_1$.

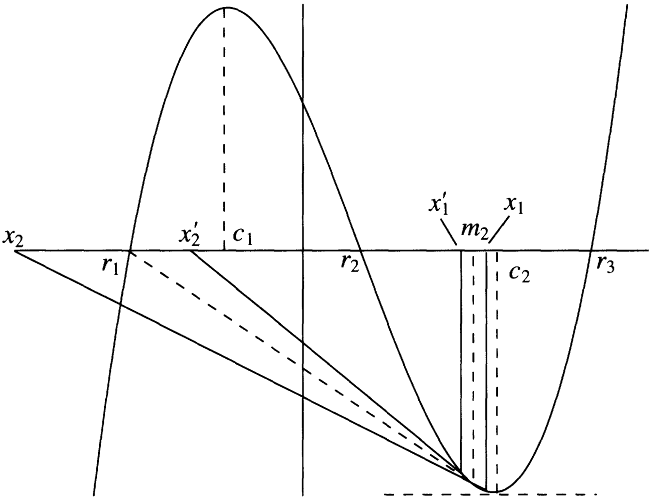

FIGURE 3

To illustrate what has been presented so far, the polynomial $p(x) = x^3 - 2x^2 - 11x + 12 = (x + 3)(x - 1)(x - 4)$ has three roots $r_1 = -3, r_2 = 1, r_3 = 4$ and two critical values (solving $p'(x) = 0$) c_1 and c_2 given by

$$c_1 = \frac{4 - \sqrt{148}}{6} \approx -1.360920843$$

$$c_1 = \frac{4 + \sqrt{148}}{6} \approx 2.694254177.$$

The interval $\mathbf{I}_1$ is located at (a_1, b_1) with $b_1 \approx 2.694254177$ and a_1 is the solution to $N(x) = c_1$, or

$$x - \frac{x^3 - 2x^2 - 11x + 12}{3x^2 - 4x - 11} \approx -1.360920843.$$

Once again the TI-82 is very good at solving this kind of equation, and we find $a_1 \approx 2.40993$.

A second interval $\mathbf{I}_2 = (a_2, b_2)$ of length $L_2 < L_1$ then exists adjacent to $\mathbf{I}_1$, with $b_2 = a_1$, and for which the Newton approximates $\{x_n\}$ converge to r_3 for all $x_1 \in \mathbf{I}_2$. These are precisely the numbers that map under N to this corresponding interval containing m_1 (Figure 4).

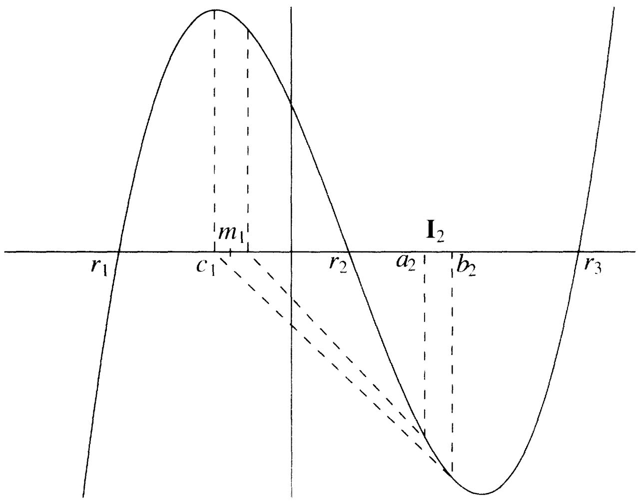

FIGURE 4

The pattern is now starting to unfold. Intervals $\mathbf{I}_n = (a_n, b_n)$, with $b_n = a_{n-1}$ and $L_n < L_{n-1}$ now exist along the x-axis with the property that any initial $x_1 \in \mathbf{I}_n$ will yield Newton iterates $\{x_n\}$ that converge alternately to r_1 or r_3! The interval lengths are restricted in that their sums converge, for $\{a_n\}$ converges to the largest of the nontrivial x values left fixed by the function $N^2 = N \circ N$. The equation $N^2(x) = x$ simplifies to a 9th degree polynomial with the three obvious roots r_1, r_2, r_3, and two other real roots $r_4 < r_5$, situated on alternate sides of r_2 (Figure 5). These two roots satisfy $r_4 = N(r_5)$ and $r_5 = N(r_4)$, and are special, for if $x_1 \in (r_4, r_5)$, the iterates

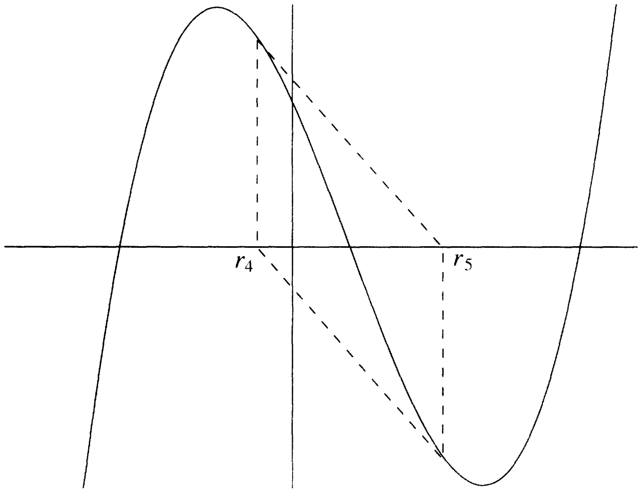

FIGURE 5

$\{x_n\}$ now converge to the middle root r_2. Otherwise, initial estimates x_1 with $x_1 < r_4$ or $x_1 > r_5$ produce iterates $\{x_n\}$ that, if they converge, do so to the other two roots r_1 and r_3.

Because the intervals $\mathbf{I}_n = (a_n, b_n)$ are shrinking in size with $L_n \to 0$, and $a_n \to r_5$, any x_1 chosen from $\mathbf{I}_n$ for a large n will present problems in determining exactly which root $\{x_n\}$ converges to. This depends greatly on the precision used for the computations, but it is also the nature of the beast. Minute differences in x_1 can cause dramatically different results. This sensitivity is characteristic of chaos, and the reader is referred to either [2] or [3] for further discussion on the topics of chaos and dynamical systems.

It is rather natural, yet numerically quite taxing, to measure each interval length L_n, because the points a_n (known as bifurcation points [4]) are difficult to represent explicitly, and good approximations are sought.

These concepts and results become clearer, perhaps, by looking at our previous example with $p(x) = x^3 - 2x^2 - 11x + 12 = (x+3)(x-1)(x-4)$. Newton's method yields the function

$$N(x) = x - \frac{x^3 - 2x^2 - 11x + 12}{3x^2 - 4x - 11}$$

and the two points fixed by N^2 are the nontrivial real roots to $N^2(x) = x$, or the rational function $g(x)/h(x) = 0$, where $g(x)$ is the polynomial

$$20x^9 - 120x^8 - 161x^7 + 1808x^6 - 129x^5 - 9380x^4 - 211x^3 + 26736x^2 - 2879x - 15684 = 0.$$

In addition to the three obvious roots of -3, 1, 4, this polynomial has 4 complex roots, and two nontrivial real roots where $r_4 \approx -0.7829393801$, $r_5 \approx 2.352836324$. The following table lists some specific points x_1 that are good approximations to bifurcation points along with the corresponding root that $\{x_n\}$ converges to. Note that all values of x_1 are greater than r_5, except the last one. Here $x_1 < r_5$ and consequently $\lim_{n\to\infty} x_n = 1 = r_2$.

TABLE 1 Some specific calculations

Initial point x_1	$\lim \{x_n\}$
2.69425417	−3
2.40993588	4
2.36055069	−3
2.35430681	4
2.35303997	−3
2.35287527	4
2.35284172	−3
2.35283735	4
2.352836327	−3
2.35 2836323	1

It is interesting to note in closing that if a cubic possesses only two distinct real roots, then Newton's method applied to an initial estimate of the mean of the two roots will yield a sequence that converges after just one term again to the double root. For if $p(x) = (x-a)(x-b)^2$, $x_1 = (a+b)/2$, then $p'(x) = (x-b)(3x-2a-b)$ and

$$N(x_1) = x_1 - \frac{(x_1 - a)(x_1 - b)}{(3x_1 - 2a - b)} = b.$$

Higher degree polynomials appear not to present the analogous result with initial estimates of the mean of adjacent roots yielding iterates $\{x_n\}$ that converge immediately to other roots, although there are instances where this does happen. For instance, if $p(x) = x(x-2)(x-3)(x+1)(x+3)$ then an initial estimate of either $x_1 = 1$ or $x_1 = -0.5$ yields a sequence $\{x_n\}$ that converges immediately to one of the roots. The other two midpoints of -2, 2.5 yield sequences that converge to the roots -1, 3, respectively, but not immediately. The underlying analysis accounting for this behaviour is still open for discovery.

References

1. Ray Dunnett, Newton-Raphson and the cubic, *Math. Gaz.* **78** (November 1994) pp. 347-348.
2. C. H. Edwards and D. Penney, *Calculus with analytic geometry* 4th edition, Prentice-Hall (1994) pp. 189-192.
3. Robert Devaney, *Chaos, fractals, and dynamics: computer experiments in mathematics*, Addison-Wesley, (1990).
4. D. Ruelle, *Elements of differentiable dynamics and bifurcation theory*, Academic Press (1989) p. 71.

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So that explains it

In the spring of 1954, and almost 17 years old, I acquired sufficient highers to guarantee admission to university. That one of these passes was in mathematics is a perpetual source of wonder to me, matched only by my surprise the day the Secretary of State asked me to conduct a national investigation into this very subject in England in 1988. This confirmed my father's fairly low opinion of both the English and myself.

Noticed by Peter Reynolds on page 37 in *Sunset on the Clyde*, by Duncan Graham.

An end to split shifts?

Police have doubled the number of officers on the beat to five.

From the Harrow *Independent* via the *Daily Telegraph*, 8 May 1997. This second-hand gleaning was spotted by Robert Pargeter.