**Assigned:** Wednesday, February 11, 2014
**Due:** Wednesday, February 18, 2014

# MTH 371: Homework 3
# Taylor Series

GENERAL HOMEWORK GUIDELINES:

- On the very first page of your homework, provide your name, date, and homework number.
- Homework will be graded in part on neatness, organization, and completeness of solutions. Multiple pages MUST BE STAPLED.
- Attach all Scilab code, output, and plots to the page *immediately following* each problem.
- Clearly label all plots (title, $x$-axis, $y$-axis, legend). Use the "subplot" when needed

1. Compute by hand the first 5 terms in the Taylor series (constant, linear, quadratic, cubic, and quartic terms) for the following functions.
   (a) $f(x) = 3\tan(x)$, about the point $x = \pi/4$.
   (b) $f(x) = e^{\cos(x)}$, about the point $x = 0$.

2. Using the both parts (a) and (b) from problem 1, make a single Scilab plot which contains all of the following.
   (a) a graph of $f(x)$ versus $x$ for $x \in (-3, 3)$,
   (b) a graph of $P_2(x)$.
   (c) a graph of $P_4(x)$.
   (d) a title, $x$-axis label, $y$-axis label, and a legend.

   What role does the center of the Taylor series play with these graphs?

3. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos(x)$ about $x_0 = 0$.
   (a) Using Taylor's theorem from class, find an upper bound for $|f(0.5) - P_2(0.5)|$. Compare this bound to the true error $|f(0.5) - P_2(0.5)|$.
   (b) Find a bound for the error $|f(x) - P_2(x)|$ where $P_2$ approximates $f$ on interval $[0, 1]$.
   (c) Plot $f$ and $P_2$ on $[0, 1]$ in Scilab. Approximately, where does the maximum error occur?
   (d) Approximate $\int_0^1 f(x)\, dx$ by $\int_0^1 P_2(x)\, dx$.
   (e) Find an upper bound for the error in (d) by using $\int_0^1 |R_2(x)|\, dx$. Compare this bound to the true error.

4. Give the Taylor series for $f(x) = x^3 - 2x^2 + 4x - 1$ using center $x = 2$. Discuss how this series compares to the original function.

5. (a) Derive the Maclaurin series for $f(x) = \cos(x)$. What is it's radius of convergence and why?
   (b) How many terms are needed in the series from part (a) to compute $\cos(x)$ for $|x| < 0.5$ accurate to 12 decimal places? Verify your results using Scilab.

6. An alternating series is of the form

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} b_k = b_1 - b_2 + b_3 - b_4 + \ldots, \qquad b_k > 0$$

If both $b_{k+1} \leq b_k$ for all $k$ and also $\lim_{k\to\infty} b_k = 0$, then the series is convergent. In addition, for the $n$th partial sum $S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$, we have the following error formula.

$$|S - S_n| \leq b_{n+1}$$

Use this result to answer the following.

(a) If you use the Maclaurin series for $\sin(x)$ to approximate $\sin(1)$ to within error $0.5 \times 10^{-6}$, how many terms in the series are needed? Use Scilab to check your answer.

(b) Derive a series for the natural logarithm $\ln(x)$ with center $x = -1$. How many terms of this series are needed to compute an approximation to $\ln(2)$ within error $0.5 \times 10^{-6}$?