Calculus II and Euler also (with a Nod to Series Integral Remainder Bounds)

Richard Barshinger

A classical theorem of Maclaurin and Cauchy [**8**, p. 45] states that, if $f(x)$ is positive and decreases to zero, then

$$\gamma_f = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} f(k) - \int_1^n f(x)\, dx \right\}$$

exists. The constant γ_f is called an Euler constant, the original belonging to $f(n) = 1/n$ and having the value $\gamma = 0.57721566490\ldots$. [$\gamma$ arises, in applied mathematics, in the formulation of Bessel functions and the gamma function, among others.] Papers such as [**2**], [**3**], [**4**], and [**14**] have discussed these constants, as well as more generally considering rates of growth, for divergent series in particular.

This paper considers a suitable approach by which the computation of the canonical Euler constant (corresponding to $\lim_{n \to \infty}\{\sum_{k=1}^{n} 1/k - \ln n\}$, a classical indeterminacy of the form $\infty - \infty$) can find its way, in a pedagogically sound and useful fashion, into a first calculus course. Though a number of authors have used various intermediate to advanced techniques to explore some aspects of $\sum_{n=1}^{\infty} 1/n$ and γ, we use an elementary geometrical technique for accelerating the convergence of the above limit.

Evaluation of γ, Method O (brute force—almost!). Any student knows—or has to be taught—that most any computational device will claim that an infinite series whose terms decrease to zero will converge. Consequently, it is not sufficient simply to compute successive evaluations of $\sum_{k=1}^{n} 1/k - \ln n$. It is, however, clear that, for all n,

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} - \ln n.$$

It is easy to show that the two sequences are strictly monotonically increasing and monotonically decreasing, respectively (see, for example, [**13**], p. 669), and that both converge to γ. The following table of bounds can easily be classroom generated, when using, for example, the Sequences program in MicroCalc [**6**] and run on a 33 Mhz 486 microcomputer.

n			γ	
1800	0.576938	0.577493	0.577	(rounded)
2313	0.577000	0.577423	0.577	(truncated)
7557	0.577150	0.577282		
14778	0.577182	0.577249	0.5772	(rounded)

Computing about 15,000 terms of each of two sequences is, if nothing else, rather esthetically unattractive, however; so, alternatively, we might average the above pairs of values, which gives $\gamma = 0.577$ (truncated) as early as the 10th iterate. [This is based on $\sum_{k=1}^{n} 1/k - \ln\sqrt{n(n+1)}$ and involves the geometric mean between n and $n + 1$.]

Error Bounds for γ, Method 1. From a problem in [**12**, p. 344 & p. 621], based on Figure 1, it easily follows that $1/2 < \gamma < 1$. It is quite straightforward to improve substantially on these very crude bounds. Figure 2 shows that we may calculate a lower bound for γ by drawing secant lines from the endpoints of $f(x) = 1/x$ on the interval $[n, n+1]$ to the midpoint of the curve. The area above the secant lines and below the horizontal line $y = 1/n$ is smaller than the contribution to the value of γ by taking the area between the curve and $y = 1/n$. Similarly, an upper bound for γ may be obtained by calculating the area between $y = 1/n$ and the two intersecting lines that are drawn tangent to the curve, at $x = n$ and $x = n + 1$ respectively. Summing over n, we obtain

$$\frac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n(2n+1)} < \gamma < \sum_{n=1}^{\infty}\frac{1}{n(2n+1)}. \tag{1}$$

Papers such as [**1**], [**5**], [**9**], [**10**] & [**11**] have considered various ways to accelerate the convergence of a series. Let $f(n)$ be the generator of the terms of a convergent series and R_n be the remainder after the nth partial sum S_n. Assume f and $|f'|$ both decrease to zero (f is concave up). These inequalities, presented for archival purposes, then summarize the papers cited above.

$$\begin{aligned}
&\int_{n+1}^{\infty} f < \int_{n}^{\infty} f - \tfrac{1}{2}f(n) < \int_{n+1}^{\infty} f + \tfrac{1}{2}f(n+1) < \int_{n+1}^{\infty} f + \tfrac{3}{4}f(n+1) - \tfrac{1}{2}\int_{n+1}^{n+3/2} f \\
&\qquad < R_n < \int_{n+1/2}^{\infty} f + \tfrac{1}{4}f(n+1) - \tfrac{1}{2}\int_{n+1/2}^{n+1} f \\
&\qquad < \int_{n+1}^{\infty} f + \tfrac{1}{2}f(n+1) + \tfrac{1}{8}|f'(n+1)| \\
&\qquad < \int_{n+1/2}^{\infty} f < \int_{n}^{\infty} f - \tfrac{1}{2}f(n+1) - \tfrac{1}{4}|f'(n+1)| < \int_{n}^{\infty} f - \tfrac{1}{2}f(n+1) < \int_{n}^{\infty} f
\end{aligned} \tag{2}$$

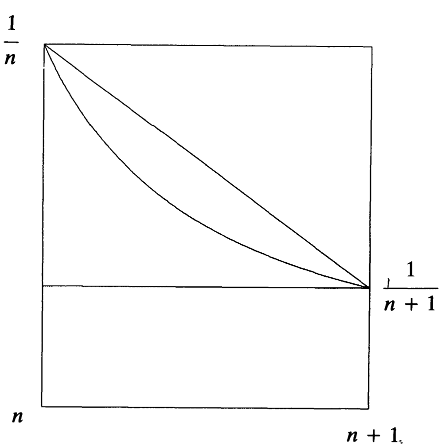

Figure 1

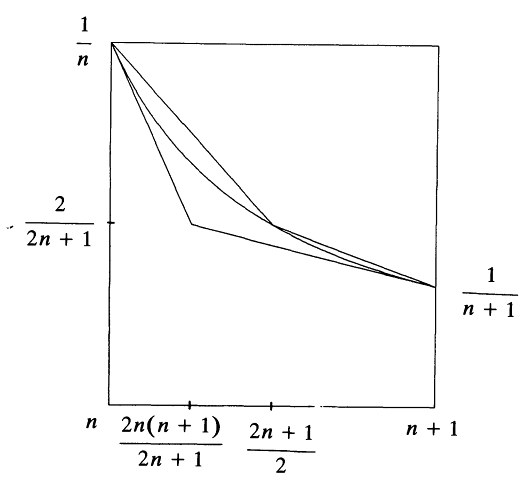

Figure 2

The weakest bounds are the basic elementary integral error bounds; the strongest bounds are newly computed by the author and are based on a suggestion in [**1**, p. 89].

Depending on the series in question, various bounds in (2) may give substantial, or only marginal, improvement over the use of other bounds also given in the above. For pedagogical purposes the author prefers the following inequality because of its clear improvement to the basic integral error bounds given in virtually all calculus texts. If $S = S_n + R_n$, papers [**9**] and [**10**] imply that

$$S_n + \int_{n+1}^{\infty} f + \tfrac{1}{2}f(n+1) < S < S_n + \int_{n}^{\infty} f - \tfrac{1}{2}f(n+1). \tag{3}$$

We adapt (3) to relationship (1) by letting $\underline{S}$, S and $\bar{S}$ be the sums of three series for which $\underline{S} < S < \bar{S}$. Also let $\underline{f}(n)$, $\bar{f}(n)$, $\underline{S}_n$ and $\bar{S}_n$ be the generators and partial sums for $\underline{S}$ and $\bar{S}$ respectively. Then

$$\underline{S}_n + \int_{n+1}^{\infty} \underline{f} + \tfrac{1}{2}\underline{f}(n+1) < S < \bar{S}_n + \int_{n}^{\infty} \bar{f} - \tfrac{1}{2}\bar{f}(n+1). \tag{4}$$

Applying (4) to (1) and summing the telescoping series, we obtain

$$\begin{aligned} \frac{1}{4} + \frac{1}{2}\left\{\sum_{k=1}^{n} \frac{1}{k(2k+1)} + \ln\frac{2n+3}{2n+2} + \frac{1}{(2n+2)(2n+3)}\right\} \\ < \gamma < \sum_{k=1}^{n} \frac{1}{k(2k+1)} + \ln\frac{2n+1}{2n} - \frac{1}{(2n+2)(2n+3)} \end{aligned} \tag{5}$$

This table easily follows.

n			
1	0.553238	0.688798	(all values are rounded)
2	0.555647	0.632667	
10	0.556824	0.614033	
49	0.556852	0.613709	
50	0.556853	0.613709	
77	0.556853	0.613707	
78	0.556853	0.613706	

After iteration 78 there is no further improvement; hence this method gives $0.55685 < \gamma < 0.61371$, with the lower bound conventionally truncated and the upper bound always rounded up.

Method 1 is less satisfactory than the almost 15,000 iterations necessary to generate four place rounded accuracy with Method 0. The following modification of the above technique, however, leads to bounds that converge to γ itself.

Convergence to γ, Method 1′. The difficulty with Method 1 is that the intervals (that generate those series giving lower and upper bounds for γ) are fixed at unit width, so that, on the interval $[n, n+1]$, the contributions to the bounds are poor when n is small. A way to remedy this is to replace the partial sums $\underline{S}_n$ and $\bar{S}_n$ in (4) by $\gamma_n \doteq \sum_{k=1}^{n} 1/k - \ln(n+1)$. From Figure 2, we have $\underline{S}_n < \gamma_n < \bar{S}_n$, $\underline{R}_n < \gamma_R < \bar{R}_n$ (where $\gamma_R = \gamma - \gamma_n$), and, with (4) above, easily see that

$$\begin{aligned} \underline{S}_n + \int_{n+1}^{\infty} \underline{f} + \tfrac{1}{2}\underline{f}(n+1) &< \gamma_n + \int_{n+1}^{\infty} \underline{f} + \tfrac{1}{2}\underline{f}(n+1) < \gamma_n + \underline{R}_n < \gamma_n + \gamma_R \\ &= \gamma < \gamma_n + \bar{R}_n < \gamma_n + \int_{n}^{\infty} \bar{f} - \tfrac{1}{2}\bar{f}(n+1) \\ &< \bar{S}_n + \int_{n}^{\infty} \bar{f} - \tfrac{1}{2}\bar{f}(n+1) \end{aligned} \tag{6}$$

Consequently, the bounds given by that part of (6) which is

$$\gamma_n + \int_{n+1}^{\infty} \underline{f} + \tfrac{1}{2}\underline{f}(n+1) < \gamma < \gamma_n + \int_{n}^{\infty} \bar{f} - \tfrac{1}{2}\bar{f}(n+1) \tag{7}$$

will converge to γ. By combining (1) with (7) we obtain

$$\frac{1}{4(n+1)} + \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) + \frac{1}{2}\ln\frac{2n+3}{2n+2} + \frac{1}{(4n+4)(2n+3)}$$
$$< \gamma < \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) + \ln\frac{2n+1}{2n} - \frac{1}{(2n+2)(2n+3)}.$$

From this we compute

n			γ	
1	0.568425	0.662318		
2	0.573701	0.600722		
9	0.576969	0.578069		
10	0.577014	0.577887	0.577	(truncated)
14	0.577111	0.577528		
15	0.577125	0.577483	0.577	(rounded)
35	0.577199	0.577257		
36	0.577200	0.577254	0.5772	(truncated)
107	0.577214	0.577220		
108	0.577214	0.577219	0.5772	(rounded)
355	0.577215	0.577216		
356	0.577216	0.577216	0.577216	(rounded)

In order to accelerate the convergence further, we could bisect the interval $[n, n+1]$ and repeat Method 1 for the subintervals $[n, n+1/2]$ and $[n+1/2, n+1]$, from which

$$\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{1}{4}\sum_{n=1}^{\infty}\frac{48n^2+44n+9}{n(2n+1)(4n+1)(4n+3)}$$
$$< \gamma < \sum_{n=1}^{\infty}\frac{8n+3}{n(4n+1)(4n+3)}$$

which converge, however, to values strictly lower/higher than γ. We then employ Method 1′, using the integral remainder bounds given by (4) and replacing the partial sums by γ, to get

$$\frac{1}{8(n+1)} + \gamma_n + \frac{1}{4}\ln\frac{(2n+3)(4n+5)(4n+7)}{32(n+1)^3}$$
$$+ \frac{48n^2+140n+101}{8(n+1)(2n+3)(4n+5)(4n+7)}$$
$$< \gamma < \gamma_n + \frac{1}{2}\ln\frac{(4n+1)(4n+3)}{16n^2} - \frac{8n+11}{2(n+1)(4n+5)(4n+7)}.$$

Convergence to a six-place (rounded) accurate value for γ is by the 183rd iterate. Informally, this appears to imply that the method is $O(h)$ in the step size (at least for this example).

Other Examples and Extensions. The techniques described in Methods 1 and 1′ above can be used to evaluate the Euler constant for a *convergent* series and, therefore, to accelerate finding the value of the series itself. For example, consider $\sum_{n=1}^{\infty} 1/n^2$, whose value is $\pi^2/6$. The computations for the Euler constant, given by Method 1′, converge to 0.644934 (rounded) by the 79th iteration. Adding a value of one thus gives an approximate value to the series. By contrast, calculations done in [5] and based on a slightly weaker form of (4) generate the same value by the 124th iteration. The above method thus converges faster, though at the expense of more analytical work.

Method 1 may also be used to accelerate the convergence of series that we do not normally associate with convergence tests for series of positive terms. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}/n = \ln 2$. We may rewrite this series in the form $\sum_{n=1}^{\infty}[1/(2n-1) - 1/2n] = \sum_{n=1}^{\infty} 1/[2n(2n-1)]$. [It is not by accident that we choose this example; its similarity to the series in (1) is clear. In fact, the bounds in (1) can be shown to be related to the alternating harmonic series and to have the exact values of $5/4 - \ln 2$ and $2 - 2\ln 2$, respectively.] When we apply (3) to the series in this form, we obtain

$$\sum_{k=1}^{n} \frac{1}{2k(2k-1)} + \frac{1}{2}\ln\frac{2n+2}{2n+1} + \frac{1}{(4n+4)(2n+1)}$$
$$< \ln 2 < \sum_{k=1}^{n} \frac{1}{2k(2k-1)} + \frac{1}{2}\ln\frac{2n}{2n-1} - \frac{1}{(4n+4)(2n+1)}.$$

Convergence to $\ln 2 = 0.693147$ (rounded) is by the 87th iterate. [If, instead, we use the strongest error bounds from (2) for a similar computation, we obtain the above value as early as the 25th iterate, a substantial improvement. In addition, see [7] for another approach to the evaluation of ln 2 by using various other series to estimate remainders.]

Conclusions. It is to be hoped that, with the ease and availability of machine computing, a somewhat more sophisticated approach to the evaluation of series may take place in the current calculus classroom. Error bounds such as those given in (3) or, indeed, any mix or match of bounds given in (2) are within the grasp of the student. In addition, the *sec/tan method* (Method 1′) for accelerating the convergence of the Euler constant perhaps merits some consideration because of its easily visualized geometry.

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Penn State-Scranton
120 Ridge View Drive
Dunmore, PA 18512

The Prisoner's Paradox Revisited

Awaiting the dawn sat three prisoners wary,
A trio of brigands named Tom, Dick and Mary.
Sunrise would signal the death knoll of two,
Just one would survive, the question was who.

Young Mary sat thinking and finally spoke.
To the jailer she said, "You may think this a joke"
But it seems that my odds of surviving 'til tea,
Are clearly enough just one out of three.

But one of my cohorts must certainly go,
Without question, that's something I already know.
Telling the name of one who is lost,
Can't possibly help me. What could it cost?"

The shriveled old jailer himself was no dummy,
He thought, "But why not?" and pointed to Tommy.
"Now it's just Dick and I" Mary chortled with glee,
"One in two are my chances, and not one in three!"

Imagine the jailer's chagrin, that old elf
She'd tricked him, or had she? Decide for yourself.

Richard E. Bedient
Department of Mathematics
and Computer Science
Hamilton College
Clinton, NY 13323