

INFINITE SERIES FOR π

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1. INTRODUCTION

By forming antiderivatives of both sides of

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for } |x| < 1$$

and checking values at $x = 0$ to find the undetermined constant C , we get

$$(1) \quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \leq 1,$$

and letting $x = 1$ in this formula implies

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1},$$

so

$$(2) \quad \pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right) = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}.$$

This was discovered independently by Leibniz and James Gregory in the 1670s, and was known 300 years earlier by Madhava in India.¹ If we try to use (2) to compute π , we're faced with the reality that this series is *very* slowly convergent. For instance, the sum of the first 100 terms in (2) is approximately 3.15, which isn't even right in the second digit after the decimal point. The reason (2) is slowly convergent is that it comes from evaluating the infinite series for $\arctan t$ at an endpoint of its interval of convergence, and power series tend to converge less quickly near the edge of the interval of convergence.

We will use error estimates for alternating series to find π to several digits using both (2) and variations on (2) involving values of the arctangent function at points where there is more rapid convergence than at $x = 1$.

2. FIRST METHOD FOR ESTIMATING π

To compute π to 4 digits after the decimal point with (2), let's find an N that makes $|\pi - s_N| < 1/10^5$, where s_N is the N th partial sum of (2). Since (2) is an alternating series, the error estimate for alternating series tells us that $|\pi - s_N|$ is at most the $(N+1)$ -th term of (2) in absolute value: $|\pi - s_N| \leq 4/(2(N+1)+1) = 4/(2N+3)$. Therefore to make $|\pi - s_N| < 1/10^5$ we can choose N that makes $4/(2N+3) < 1/10^5$:

$$\frac{4}{2N+3} < \frac{1}{100,000} \iff 400,000 < 2N+3 \iff N > 199,998.5, \text{ so } N \geq 199,999.$$

The error bound says π lies in the interval

$$(s_{199,999} - 10^{-5}, s_{199,999} + 10^{-5}) = (3.141577\dots, 3.141597\dots),$$

¹Madhava's own writings on infinite series are lost, but later mathematicians in India wrote about it in 1500.

so $\pi = 3.1415\dots$

3. SECOND METHOD FOR ESTIMATING π

The series (1) has faster convergence the closer t is to 0, so by using a value of $\arctan t$ with $|t| < 1$ that involves π we will get formulas for π that converge more rapidly than (2).

For example, since $\tan(\pi/6) = \sin(\pi/6)/\cos(\pi/6) = \sin(30^\circ)/\cos(30^\circ) = (1/2)/(\sqrt{3}/2) = 1/\sqrt{3}$, we have $\arctan(1/\sqrt{3}) = \pi/6$: setting $t = 1/\sqrt{3}$ in (1), we have

$$\begin{aligned} \frac{\pi}{6} &= \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 9\sqrt{3}} - \frac{1}{7 \cdot 27\sqrt{3}} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+1}}{(2n+1)} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^n (1/\sqrt{3})}{(2n+1)} \\ &= \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}. \end{aligned}$$

Multiplying through by 6,

$$(3) \quad \pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \left(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 9} - \frac{1}{7 \cdot 27} + \dots \right).$$

The 3^n in the denominator makes this much more rapidly convergent than (2). This series, like (2), was known to Madhava in India long before calculus was developed in Europe, and Madhava used it to compute the first 10 digits of π after the decimal point.

The error in estimating π with (3) using N terms is at most the absolute value of the $(N+1)$ th term, which is $2\sqrt{3}/((2(N+1)+1)3^{N+1})$. That is less than $1/10^5$ for $N = 8$. Since

$$s_8 = 2\sqrt{3} \sum_{n=0}^8 \frac{(-1)^n}{(2n+1)3^n} = 3.141599773\dots,$$

we have

$$3.141589 < \pi < 3.141609\dots,$$

so $\pi = 3.141$ to 3 digits after the decimal point. We didn't get 4 digits of accuracy from this partial sum because its error interval allows for a 4th digit of 5 or 6, as seen above. The partial sum of (3) over $n \leq 9$ has an error bound that shows $\pi = 3.1415\dots$

The error bound is less than $1/10^9$ for $N = 16$, so

$$s_{16} = 2\sqrt{3} \sum_{n=0}^{16} \frac{(-1)^n}{(2n+1)3^n} = 3.14159265417\dots \implies \pi = 3.14159265\dots$$

4. THIRD METHOD FOR ESTIMATING π

While (3) is rapidly convergent, it has the defect of using the irrational number $\sqrt{3}$, so we need good estimates for $\sqrt{3}$ in order to get good estimates of π . Rapidly convergent formulas for π that don't involve any auxiliary irrational numbers can be found as sums of arctangent values, rather than single arctangent values. An example of such a formula is

$$(4) \quad \frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}.$$

Of course it's natural to ask where a formula like (4) comes from. Before addressing that, let's use the formula to estimate π with shorter partial sums than we used before. By (4) and the arctangent series (1),

$$(5) \quad \pi = 4 \left(\arctan \frac{1}{2} + \arctan \frac{1}{3} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1} \left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right).$$

The error in approximating π using the sum in (5) for $n \leq N$ is at most the magnitude of the $(N+1)$ th term, which is

$$(6) \quad \frac{4}{2(N+1)+1} \left(\frac{1}{2^{2(N+1)+1}} + \frac{1}{3^{2(N+1)+1}} \right).$$

The table below lists the error bound (6), the value of the corresponding partial sum in (5), and underlines the number of verifiably correct digits in π based on the error bound. (The estimate for $N = 6$ drops in accuracy because the error bound allows for a possible fourth digit equal to 5 or 6.)

N	Error bound (6)	N th partial sum of (5)
4	.00017960...	<u>3.141</u> 74119...
5	.00003775...	<u>3.1415</u> 6158...
6	.00000815...	<u>3.1415</u> 9934...
7	.00000179...	<u>3.1415</u> 9118...
8	.00000040...	<u>3.1415</u> 9298...
9	.00000009...	<u>3.1415</u> 9257...
10	.00000002...	<u>3.1415</u> 9267...

Let's return now to explaining how (4) can be found. It is based on the addition formula for the tangent function:

$$(7) \quad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

To make $x+y = \pi/4$, we want $\tan(x+y) = \tan(\pi/4) = 1$. Write $x = \arctan(1/a)$ and $y = \arctan(1/b)$ for unknown positive numbers a and b . Then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{1/a + 1/b}{1 - 1/(ab)} = \frac{a+b}{ab-1},$$

so

$$\frac{\pi}{4} = x+y \implies 1 = \frac{a+b}{ab-1} \implies ab = a+b+1.$$

The simplest solution to this last equation is $a = 2$ and $b = 3$, and that's how you can discover for yourself that $\pi/4 = \arctan(1/2) + \arctan(1/3)$.

(Strictly speaking, we didn't really show $\arctan(1/2) + \arctan(1/3)$ is $\pi/4$, but only that it has the same tangent value as $\pi/4$: if $\tan x = 1/2$ and $\tan y = 1/3$ then by (7)

$$\tan(x+y) = \frac{1/2 + 1/3}{1 - (1/2)(1/3)} = \frac{5/6}{5/6} = 1 = \tan\left(\frac{\pi}{4}\right),$$

but this doesn't force the two numbers to be equal since different numbers can have the same tangent value. After all, $\tan \theta$ is periodic. But two numbers in the interval $(-\pi/2, \pi/2)$ that have the same tangent value *do* have to be equal since $\tan \theta$ is increasing on $(-\pi/2, \pi/2)$. Let's check $\arctan(1/2) + \arctan(1/3)$ lies in $(-\pi/2, \pi/2)$: the alternating power series (1) implies $0 < \arctan x < x$ for $0 < x < 1$, so $\arctan(1/2) + \arctan(1/3)$ is positive and less than $1/2 + 1/3 < 1 < \pi/2$. Therefore

$$\tan\left(\arctan \frac{1}{2} + \arctan \frac{1}{3}\right) = 1 \implies \arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4},$$

and that completes the verification of (4).)

5. FOURTH METHOD FOR ESTIMATING π

An historically important formula for π is due to Machin (1706):

$$(8) \quad \frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

This equality can be *verified* in the same way we verified (4); the algebra is just more complicated. I don't know how Machin actually *found* this formula, however.

Using the series for $\arctan t$, (8) gives us the series formula

$$(9) \quad \pi = 4 \left(4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1} \left(\frac{4}{5^{2n+1}} - \frac{1}{239^{2n+1}} \right).$$

When computing π with the partial sum of (9) over $n \leq N$, the error is at most

$$(10) \quad \frac{4}{2(N+1)+1} \left(\frac{4}{5^{2(N+1)+1}} - \frac{1}{239^{2(N+1)+1}} \right),$$

which is less than $1/10^8$ when $N = 5$: the error bound is $.000000001008 \dots \approx 10^{-8}$. The partial sum of (10) for $n \leq 5$ with its error bound puts π in $(3.141592651 \dots, 3.141592653 \dots)$, so $\pi = 3.14159265 \dots$. The partial sum of (9) over $n \leq 10$ with its error bound verifiably gives π to 15 digits after the decimal point:

$$\pi = 3.141592653589793 \dots$$

To appreciate the successive improvements in the use of (2), (3), (5), and (9) **with their error terms** as methods of approximating π , they each rigorously tell us $\pi = 3.1415 \dots$ using partial sums over the following respective ranges: $n \leq 199,999$ for (2), $n \leq 9$ for (3), $n \leq 5$ for (5), and $n \leq 3$ for (9).

Machin used his formula (9) to compute 100 digits of π . Williams Shanks used (9) to compute π to 707 digits in 1853 (all calculations by hand!) and an assistant checked the first 440 digits were correct. When desk calculators were first developed in the mid-20th century it was discovered in 1944 that Shanks made an error in the 528th digit, so everything from this point on was wrong. A few years later a computer calculated π to 2000 digits.

Here are some more arctangent formulas for $\pi/4$ that have been used to compute π :

$$\begin{aligned} \frac{\pi}{4} &= 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}, \\ &= 5 \arctan \frac{1}{7} + 2 \arctan \frac{3}{79}, \\ &= 4 \arctan \frac{1}{5} - \arctan \frac{1}{70} + \arctan \frac{1}{99}, \\ &= 6 \arctan \frac{1}{8} + 2 \arctan \frac{1}{57} + \arctan \frac{1}{239}, \\ &= \arctan \frac{1}{2} + \arctan \frac{1}{5} + \arctan \frac{1}{8}. \end{aligned}$$