Differential Equations Notes

Contents

Fun Stuff

1. Course resource: `https://ocw.mit.edu/courses/mathematics/18-03-differential-equations-spring-` `index.htm`
2. Feynman Method: `https://www.youtube.com/watch?v=FrNqSLPaZLc`
3. Bad math writing: `https://lionacademytutors.com/wp-content/uploads/2016/10/sat-math-section.` `jpg`
4. Google AI experiments: `https://experiments.withgoogle.com/ai`
5. Babylonian tablet: `https://www.maa.org/press/periodicals/convergence/the-best-known-old-babyl`
6. Parabola in real world: `https://en.wikipedia.org/wiki/Parabola#Parabolas_in_the_physical_` `world`
7. Parabolic death ray: `https://www.youtube.com/watch?v=TtzRAjW6KO0`
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16. Polynomials and pi/taylor series: Matlab/machin https://en.wikipedia.org/wiki/Chronology_of_computation_of_%CF%80 https://en.wikipedia.org/wiki/Approximations_of_%CF%80#Machin-like formula https://en.wikipedia.org/wiki/William_Shanks

17. Deepfake: face https://www.youtube.com/watch?v=ohmajJTcpNk
dancing https://www.youtube.com/watch?v=PCBTZh41Ris

18. Pi digit calculations: https://en.wikipedia.org/wiki/Chronology_of_computation_of_%CF%80, poor shanks...https://en.wikipedia.org/wiki/William_Shanks

Course Introduction

1. Syllabus highlights
   (a) Grades:
      i. Know the expectation / what you are getting into.
      ii. 15perc A (excellent), 35perc B (good), 35perc C (satisfactory),10perc D (passing), some F (failing)
      iii. Expect lower grades than you are used to. I was a student once upon a time. I know what it's like to give some effort in a class and still get an A/B. Night before study, good enough?
      iv. Turn in an exam / project. Did you do good work?
      v. Many will start off doing good / satisfactory work. Improve to something more. C is not the worst thing in existence. These letters say nothing of your capability.
   (b) What does good mean? Good means good. Good job! Excellent means you showed some flair.
   (c) Expect: More work, more expectation on good writing.
   (d) Math is a challenging subject. Not a natural thing to think or write in. It takes work and practice to be better. My goal is to train you to be better and give you ideas of where it can go.
   (e) Fact that you are here shows you are smart and capable. Your goal should be to improve.
   (f) Why do I do this? I do it out of respect for you. You are smart enough. I want you to gain something valuable here. I wouldn't do this job if I didn't think you were gaining something of value.

2. Outline of this class
   (a) How to model real world with differential equations:
      - Population growth (cooling, investment, etc), harmonic motion (signals, spring, vibration), disease transfer (SIR model), more.
      - Modeling will not be a big focus of this class, but it can be a big focus depending on your area (math biology).
   (b) Main topics:
      - First order differential equations: Calculus 1,2

$$\frac{dy}{dt} = f(t, y), \quad \frac{dy}{dt} = ky, \quad \frac{dy}{dt} = f(y),$$

        Given a specific f, goal is to solve for unknown function y.
      - First order systems (many coupled equations): Vector calculus

$$\frac{d\vec{Y}}{dt} = f(t, \vec{Y})$$

- Linear systems: Linear algebra

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \quad \text{(akin to } \lambda\vec{x} = A\vec{x}\text{)}$$

- Second (and higher) order differential equations: Calculus and linear algebra

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t) \quad \text{(Newton's 2nd law: } F = ma\text{)}$$

- Laplace transforms: Extension of power series

(c) Main tools:

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(d) Big views:

- Analytic (solve via Calculus and integration)
- Qualitative (visualize DE to understand global behavior)
- Numerical (program to find approximate solutions)
- Theoretical (existence and uniqueness, minor in this class, big in this field)

3. Why do differential equations?

(a) Theories in many fields of science are translated into differential equations.

- Fliud dynamics, gravitational waves, optics, finance, etc
- List is long: `https://en.wikipedia.org/wiki/List_of_nonlinear_partial_differential_equations`
- Trouble is almost none can be solved, instead they are studied (theory).

(b) Millennium prize: `https://en.wikipedia.org/wiki/Millennium_Prize_Problems#Navier%E2%80%93Stokes_existence_and_smoothness`

(c) Youtubeness:

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Chapter 1: First-order differential equations

.1 1.1: Modeling via differential equations

1. Modeling via differential equations

(a) A mathematical model is a translation of some real world problem into math (algebra, calculus, linear algebra, etc). Making assumptions translates the problem into an equation. The more assumptions, the easier to solve, though the farther we travel from reality.

(b) Differential equations involve calculus and model change (derivatives).

2. Examples of differential equations. $y = y(t)$ is an unknown function that we need to find.

(a) $\dfrac{dy}{dt} = y$ has solution $y(t) = e^t$ giving exponential growth.

(b) $\dfrac{dy}{dt} = -y$ has solution $y(t) = e^{-t}$ giving exponential decay.

(c) $\dfrac{dy}{dt} = 2ty$ has solution $y(t) = e^{t^2}$ giving another exponential.

(d) $\dfrac{dy}{dt} = y^2$ has solution $y(t) = \frac{1}{1-t}$ giving geometric series summation.

(e) Note, the first three equations are linear in y while the last is nonlinear in y. These are two important cases for us. Can check each solution by plugging into the equation and ensuring equality holds.

(f) In general, all first order equations can be written as

$$\frac{dy}{dt} = f(t, y)$$

for some given function f. They all model how y changes.

(g) There are really infinite solutions to each. The first has another solution $y = 2e^t$, and in general $y(t) = Ce^t$ for any constant C is called the analytic solution to this equation. Sketch all these solutions given many values of C. This gives a global view of this differential equation (all solutions possible). Note $y(t) = 0$ is called the equilibrium solution since it never changes (constant).

(h) How do we know which one we want? Applications guide this.

3. Modeling with differential equations: basic models.

(a) Simple population growth

i. Assume a population $P(t)$ grows at a rate proportional to it's size. This translates to

$$\frac{dP}{dt} = kP \quad \to \quad P(t) = Ce^{kt} = P_0e^{kt}$$

where k is called the growth rate.

ii. If La Crosse had population 50000 in 2010 and 55000 now, what will the population be in 2030? $P(0) = 50000$ is called the initial condition.

iii. Note this model is exactly the same for bacteria growth, continuous compound interest, radioactive decay, etc. Also the same model discussed above.

(b) Newton's law of cooling

i. Assume an item cools (or heats) at a rater proportional to the difference between its temp ($T(t)$) and surrounding temp (constant T_s). Then,

$$\frac{dT}{dt} = k(T - T_s)$$

ii. A quick change of variable as $y = T - T_s$ turns this into the familiar

$$\frac{dy}{dt} = ky$$

resulting in exponential decay.

iii. Note $T(t) = T_s$ is now the equilibrium solution, and the name makes more sense.

(c) Logistic population growth

i. If you compare our La Crosse population growth model to census data, it is clear that the model is good at first but then wildly inaccurate. Making more assumptions can correct this but leads to a more challenging model.

ii. Assume the carrying capacity of La Crosse is $N = 80000$. Then revise the model as

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) = f(P)$$

iii. This is nonlinear and harder to solve (will see next time), though we can do a qualitative analysis by looking at $f(P)$ graphically. Plot f noting that it is a parabola with zeros (equilibrium) at $P = 0, N$. Note we can see decay and growth situations as well where f is positive or negative. Using this we can reason out solutions between the equilibrium. Two main cases, if above capacity we have decay to N, if below and positive we have growth to N, third case doesn't make sense in application.

iv. $P = N$ is called stable equilibrium while $P = 0$ is unstable. We can check for stability via $f'(P)$ ($f'(N) > 0$ is stable, $f'(0) < 0$ is unstable).

v. If we knew the general solution, we could be more precise here. Will find later.

(d) Predator prey model

i. Touch of high dimension. What if we have two quantities changing in a coupled way?

ii. If rabbits and foxes live together, assume

- Rabbit growth rate is 2
- Rabbit harvest rate is 1.2
- Fox death rate is 1
- Fox growth rate is 0.9

iii. The resulting model is then

$$\begin{cases} \frac{dR}{dt} &= 2R - 1.2F \cdot R \\ \frac{dF}{dt} &= -F + 0.9F \cdot R \end{cases}$$

iv. This gives a coupled nonlinear system and equilibrium solutions have new interest. For a linear system, we can borrow linear algebra to decouple the system. More on this later.

4. How to solve differential equations? Three ways.

(a) Analytic: Use calculus to find $y(t)$ explicitly. This gives the exact solution, but ignores global behavior of the DE. Most of the time this is impossible to do (integration is hard).

(b) Qualitative: We will use graphs to visualize all cases of curves (as with logistic population growth) to get a global view. Drawback here is cannot get actual solution values (impractical).

(c) Numerical: When analytic methods fail (most of the time), we can design computer algorithms to give us numeric answers. These are efficient and easy to use, but we don't get functions as result. Only numbers.

5. Homework: 1-3, 5, 11, 12, 16-18, 22, 23

.2 1.2: Analytic technique: separation of variables

1. Separable equations

(a) There are few nice cases of first order DEs which we can completely solve by hand.

(b) Separable equation: f can be separated as a product of a function of t and function of y.

$$\frac{dy}{dt} = f(t, y) = g(t)h(y)$$

(c) Once separated the solution can be found via direct integration. Our focus is then just on calculus.

2. Examples:

(a) $\dfrac{dy}{dt} = 2ty$ is separable. How to find y?

$$\begin{aligned} \frac{dy}{dt} &= 2ty \\ \int \frac{1}{y}\frac{dy}{dt}\, dt &= \int 2t\, dt \\ \int \frac{1}{y}\, dy &= \int 2t\, dt \\ \ln|y| &= t^2 + \hat{C} \\ |y| &= e^{t^2+\hat{C}} \\ y &= Ce^{t^2} \quad \text{(constant } C \text{ handles the sign)} \end{aligned}$$

Note the use of the chain rule. We will usually skip this step. Now we see where the family of solutions comes from (integration constant). Given an initial condition $y(0) = y_0$ gives

$$y = y_0 e^{t^2}.$$

If we had an initial value problem of $y(0) = y_0$, we also could have handled as a definite integral from t_0 to t giving

$$\int_{y_0}^{y} \frac{1}{y}\, dy = \int_{t_0}^{t} 2t\, dt \quad \Rightarrow \quad y = y_0 e^{t^2}.$$

(b) Logistic growth:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

Let's simplify things and take $k = 1$, $N = 1$.

$$\frac{dP}{dt} = P\,(1 - P) = f(P)$$

Again we are separable, but note the right hand side only depends on P. This is called an *autonomous* equation. We solve the same way.

$$\begin{aligned}
\frac{dP}{dt} &= P\,(1 - P) \\
\int \frac{1}{P(1-P)}\, dP &= \int\, dt \\
\int \frac{1}{P} + \frac{1}{1-P}\, dP &= \int\, dt \\
\ln|P| - \ln|1 - P| &= t + \hat{C} \\
\ln\left|\frac{P}{1-P}\right| &= t + \hat{C} \\
\frac{P}{1-P} &= e^{t+\hat{C}} \\
\frac{P}{1-P} &= Ce^{t} \\
P &= (1 - P)Ce^{t} \\
P &= \frac{Ce^{t}}{Ce^{t} + 1} = \frac{1}{1 + Ae^{-t}}
\end{aligned}$$

Note the use of partial fraction decomposition. For $A = 1$ this is the sigmoid function (logistic curve). For the original equation with rate k and carrying capacity N, we get

$$P = \frac{NP_0 e^{rt}}{N + P_0(e^{rt} - 1)}.$$

3. Difficulties with separation:

(a) Implicit solutions, and missing solutions.

$$\frac{dy}{dt} = \frac{y}{1 + y^2}$$

has solution

$$\ln|y| + \frac{y^2}{2} = t + C$$

and it is not possible to solve for y (known as an implicit solution). Also, $y = 0$ is an *equilibrium solution*, yet our implicit solution does not make sense for $y = 0$.

(b) Integration is not always possible.

4. Mixing problems: A 200 gal tub contains pure water. Sugar water is pumped in at 5 tablespoons sugar per gallon at a rate of 3 gals per minute. The take is mixed and outflow rate is the same. How much sugar is in the tank after 10 mins? What is the equilibrium solution? Setting up the equation is the challenge.
$$\frac{dS}{dt} = 5 \cdot 3 - \frac{S}{200} \cdot 3$$
This is a separable equation yet again.

5. Homework: 3, 4, 5-38 odd, 39, 41, 43

.3 1.3: Qualitative technique: slope fields

1. dfeild software: `https://math.rice.edu/~dfield/dfpp.html`

2. Finding exact solutions is usually out of reach in which case we need other options.
   - Visualize the solution (slope fields AKA direction fields): DE gives the slope of the tangent line of all solutions.
   - Compute solution numerically (no formula, just numbers, approximation and error involved)
   - Test if solution is stable (less information, but might be important information)

3. Slope fields: The geometry of $\frac{dy}{dt} = f(t, y)$.

   (a) Example (autonomous equation): $\frac{dy}{dt} = 2y, y(0) = 1$ giving $y = e^{2t}$.
   - Plot the true solution. Draw small tangent lines at $t = -1, 0, 1, 2$ to show off DE connection.
   - DE actually contains this information for all solutions and all times. Rest are just horizontal shifts of this curve (slope does not change in t for this case).

   (b) Example (autonomous equation): $\frac{dy}{dt} = 2 - y$.
   - Again $f(t, y)$ is independent of t and slopes stay the same horizontally. Plot slopes for $y = 0, 1, 2, 3, 4, 5$.
   - Note $y = 2$ is a stable equilibrium solution.
   - True solution ends up being $y = 2 + Ce^{-t}$. Can see this from the slope field.

   (c) Key ideas:

      i. Solution curves $y = 2 + Ce^{-t}$ are tangent to the direction arrows.

      ii. Plot arrows along curves $f(t, y) = \textit{constant}$ (known as *isoclines*). In our case $2 - y = s$ for slope s. In this case we only have horizontal isoclines since autonomous.

      iii. Note, curves can cross over isoclines, just not the zero isocline ($f(t, y) = 0$) known as the *nullcline*. The nullcline gives asymptotic behavior in this case.

   (d) Example (autonomous equation): Return to logistic equation: $\frac{dP}{dt} = P(1 - P)$.
   - Procedure: Plot nullcline first, then follow isoclines (horizontal again since autonomous).
   - Nullcline: $f(t, P) = P(1 - P) = 0$ for $P = 0, 1$.
   - Isoclines: $f(t, P) = P - P^2 = s$ will be positive for $0 < P < 1$ and negative otherwise. Helps to plot $z = P(1 - P)$ to see symmetry.
   - $y = 1$ is a stable equilibrium (solutions around $y = 1$ are attracted), while $y = 0$ is unstable (solutions around $y = 0$ are repelled).
   - Note again the sigmiod curve in between.

(e) Example (non-autonomous equation): $\dfrac{dy}{dt} = 1 + t - y$.

- Nullcline: $f(t, y) = 1 + t - y = 0$ gives $y = t + 1$.
- Isoclines: $f(t, y) = 1 + t - y = s$ gives $y = t + 1 + s$ (vertical shifts of $y = t + 1$). Consider $s = -2, -1, 1, 2$.
- What is the equilibrium in this case? Looks like $y = t$. Makes sense since this solves the DE and the slop of that curve is unchanging (always 1).
- True solution: $y = t + Ce^{-t}$ and we see that we asymptotically approach $y = t$.
- Note the lobster trap: once you pass an isocline, you cannot turn back forcing solutions off to infinity following the nullcline.

(f) Example (second special case): $\dfrac{dy}{dt} = f(t)$

- What do you think will happen with the direction field? Nullclines? Isoclines?
- Example: $\dfrac{dy}{dt} = 2t$ gives $y = t^2 + C$. All solutions are a vertical shift of eachother meaning slopes do not change along vertical lines (ie isoclines).

4. dfield: Demo software for each of the above examples.
   - Slopes are computed brute force.
   - Solution curves are computed via numerical method.
5. Key takeaways:
   - Solution curves are tangent to direction arrows: $y' = f(t, y)$
   - Plotting isoclines ($f(t, y) = s$) helps when doing by hand (special case is nullcline).
   - Solution curves for autonomous equations shift from left to right (horizintal lines have same slope).
   - Even if we cannot find the solution, we get global behavior as well as equilibrium (asymptotics). Not much calculus is used at all (no integration)!
   - This method is good for intuition, but not practical if you care about calculation (still will need analytic or numerical solutions).
6. Homework: 1-16, 20, 21

.4 1.4: Numerical technique: Euler's method

1. For most differential equations, analytic solutions are not possible. We need numerical methods.
   - We trade exact for approximate, though as close to the the exact solution as desired if we are careful.
   - The simplest idea harnesses tangent lines (just like direction fields) and is called Euler's method.
   - Drawback with this simple idea will be inefficiency, but this can be improved with more advanced methods.
2. Euler's method for approximating solutions to $\dfrac{dy}{dt} = f(t, y)$:

   (a) First step: $y_1 = y_0 + \Delta t f_0$

   - Notation: $y(0) = y_0$ (given initial value), $y_1 \approx y(t_1)$, $t_1 = \Delta t$ (some chosen time step), $f_0 = f(t_0, y_0)$ (slope of the solution curve at t_0)
   - Follow the tangent line to get the first approximation:

$$y_1 = y_0 + \Delta t f_0$$

- Example: $y' = 2y$ with $t_0 = 0, y(t_0) = 1$. Then the slope of the tangent is $f_0 = 2$ ad the tangent line is $y = y_0 + tf_0$ giving

$$y_1 = y_0 + \Delta t f_0 = 1 + 2\Delta t.$$

Graph with the true solution $y(t) = e^{2t}$. Note we cannot take Δt too big else we will stray far from the curve.

(b) Rearranging this is the same as approximating the derivative with a difference quotient.

$$\frac{dy}{dt} = f(t,y) \quad \Leftrightarrow \quad \frac{y_1 - y_0}{\Delta t} = f_0$$

(c) General nth time step: $y_{n+1} = y_n + \Delta t f_n$

- Following the second tangent line in above example gives $y_2 = y_1 + \Delta t f_1$. In general,

$$y_{n+1} = y_n + \Delta t f_n \qquad \left(\text{rearranged as } \frac{y_{n+1} - y_n}{\Delta t} = f_n\right)$$

- Example: Returning to $y' = 2y$, we get

$$\begin{aligned} y_{n+1} &= y_n + \Delta t f_n \\ &= y_n + \Delta t 2 y_n \\ &= (y_{n-1} + \Delta t 2 y_{n-1}) + \Delta t 2 (y_{n-1} + \Delta t 2 y_{n-1}) \\ &= (1 + 2\Delta t)^2 y_{n-1} \\ &= \vdots \\ &= (1 + 2\Delta t)^n y_0 \end{aligned}$$

If $y(0) = 1$, say we are truly after an approximation to $y(1)$. Then if we take n time steps ($\Delta t = \frac{1}{n}$)

$$y_{n+1} = \left(1 + \frac{2}{n}\right)^n \approx e^2 \qquad \left(\text{recall } (1 + \frac{1}{n})^n \to e\right)$$

and it looks like it is working!

- Taking smaller steps (Δt) results in more steps needed (larger n) to get to end time T. So we converge to the true solution.
- Draw on $y' = 2y$ example 2 steps to $T = 1$ vs 4 steps. Close approximation in the end, but how much better?

3. Euler's method error

(a) Define error at the nth step as the exact solution minus the approximate solution:

$$E_n = y(t_n) - y_n = y(n\Delta t) - y_n$$

To measure this error, we break the process into two steps:

- Local error (error from tangent line approximation)
- Global error (summation of all local errors)

(b) Local error: How good is a tangent line approximation? Need Taylor series to tell.

- Taylor series: ($a = 0$ gives Maclaurin series)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

with convergence for $|x - a| < R$ some radius of convergence.

- Maclaurin series you should know: $e^x, \sin(x), \cos(x), \dfrac{1}{1-x}$.
- For us,

$$y(t+\Delta t) = y(t) + \Delta t y'(t) + \frac{1}{2}\Delta t^2 y''(t) + \cdots$$

and we approximate

$$y(t+\Delta t) \approx y(t) + \Delta t y'(t) = y(t) + \Delta t f(t,y)$$

giving error

$$+\frac{1}{2}\Delta t^2 y''(t) + \cdots \leq \frac{1}{2}\Delta t^2 \max|y''|$$

via the mean value theorem (Taylor's theorem).
- So local error is of the order Δt^2 (constant times this value we control).

(c) Global error: Euler's method is a linear method

- Taking n steps to reach final destination $T = n\Delta t$ yields error accumulation of

$$\sum_{i=1}^{n} C\Delta t^2 = nC\Delta t^2 = TC\Delta t$$

So the global error is of order Δt.
- We say Euler's method is a linear method. As a result, if we take $2n$ steps instead of size $\frac{\Delta t}{2}$, our error divided by a factor of 2. Not much gain for twice the work.
- Example: Returning to $y' = 2y$,

$$(1+2\Delta t)^n = \left(1 + \frac{2T}{n}\right)^n \approx e^{2T} \quad \text{with error } \frac{C}{n}$$

- Script example to compare e, $\left(1+\frac{1}{n}\right)^n$, Taylor series for e, errors for $n = 1, 2, \cdots, 10$. Note the halving of error at each step for forward Euler, Taylor series is much faster.

4. Stability of numerical methods

(a) Instability:

- Issue: Sometimes local error can grow out of control, and accumulating bad local errors can result in instable global error.
- Solution: Take Δt small enough to control instability.

(b) Example: $\dfrac{dy}{dt} = -100y$ (rapid exponential decay)

- Start at $y(0) = 1$ and end up at $y(T) = e^{-100T}$, very small value.
- Euler's method:

$$y_{n+1} = y_n + \Delta t f_n = (1 - 100\Delta t)y_n \quad \Rightarrow \quad y_n = (1-100\Delta t)^n y_0$$

- What if we choose $\Delta t = 0.03$ which is reasonably small? Then $100\Delta t = 3$ and the above becomes

$$y_n = (-2)^n y_0 \Rightarrow y_n = 1, -2, 4, -8, \cdots$$

which grows exponentially. This is instability. Local error grows faster than the true solution.
- Need $|1 - 100\Delta t| \leq 1$ giving $\Delta t \leq \frac{2}{100}$.
- This is an example of a *stiff* equation (true solution is varying slowly (stiff), but nearby solutions vary rapidly) resulting in instability for Δt too large. For this example, not a huge

5. Backwards Euler and implicit methods

(a) Instead of going forward to y_{n+1} from y_n, t_n, f_n, we trace tangent lines backwards.

(b) Backward Euler:

$$\frac{y_{n+1}^B - y_n}{\Delta t} = f_{n+1} = f(t_{n+1}, y_{n+1}^B) \quad \Rightarrow \quad y_{n+1}^B = y_n + \Delta t f_{n+1}$$

Note, we need to solve for y_{n+1} at each step which can be hard for complex f.

(c) Example: $y' = -100y$ gives division by $(1 + 100\Delta t)$ instead of multiplying by $(1 - 100\Delta t)$.

$$(1 + 100\Delta t) y_{n+1}^B = y_n \quad \Rightarrow \quad y_n^B = \left(\frac{1}{1 + 100\Delta t}\right)^n y_0$$

which will always decay regardless of size of Δt. Implicit Euler is a good option for this linear equation.

(d) Implicit methods are more stable, but also more computationally expensive.

6. How to improve?

   (a) Conquer more terms in the Taylor series expansion (central difference).

   (b) Runge-Kutta (most often used in practice): Predict at $t + \Delta t$, correcting at $t + \Delta t$, then adjust stepsize Δt.

7. Programming assignment: In this section demo the results of first coding assignment in class. Subsequent assignment ideas:

   - Higher order (Runge-Kutta)
   - Systems of equations
   - Application project (SIR?)

8. Homework: 1-11, 13-14

.5 1.5: Existence and uniqueness of solutions

1. Here we show an important theoretical (but also practical) result. When can we be sure a solution exists? Do we know it is unique?

2. Past experience:

   (a) Quadratic equations: How many solutions does $ax^2 + bx + c = 0$ have? How do we know?

   (b) Linear systems: When does $A\vec{x} = \vec{b}$ have a unique solution? $\det(A) \neq 0$, full rank, columns linearly independent...

   (c) Others?

3. Existence and uniqueness for $\dfrac{dy}{dt} = f(t, y)$. Requirement is that we need f to be nice enough.

   (a) Existence: Starting at given $y(0)$ at time $t = 0$, do we have a solution?

   - Theorem: A solution exists if $f(t, y)$ is a continuous function for t near 0 and y near $y(0)$. (note the word near, near could be rather small)

   (b) Uniqueness: Is there more than one solution through the same $y(0)$?

   - Theorem: There cannot be two solutions with the same $y(0)$ if $\frac{\partial f}{\partial y}$ is also continuous.

   (c) Examples: Note each starts at $\frac{0}{0}$ which is bound to be trouble.

   - $\dfrac{dy}{dt} = \dfrac{y}{t}, y(0) = 0$ has infinitely many solutions $y = Ct$. Check they it work.

- $\dfrac{dy}{dt} = \dfrac{t}{y}, y(0) = 0$ has two solutions $y(t) = t, y(t) = -t$. Check that they work.
- Existence was not even ensured, though it is clear that f is not continuous at $t = 0$ for the first and $y(0) = 0$ for the second.

(d) Avoiding future bad behavior (finite time blow up).

- Fact: Continuity of f and $\frac{\partial f}{\partial y}$ at all points does not ensure that the solution will reach $t = \infty$.
- Example: $\dfrac{dy}{dt} = y^2, y(0) = 1$ has continuous f as well as $\frac{\partial f}{\partial y} = 2y$. Separation of variables gives the solution

$$y(t) = \frac{1}{1-t}$$

which blows up to ∞ at $t = 1$. To ensure controlled growth of the solution, we require more of f.

- Theorem (big one for this chapter, maybe course?): If

$$\left|\frac{\partial f}{\partial y}\right| \leq L$$

for all t, y, then there is a unique solution for all $y(0)$ reaching all t. (Essentially we need to control the growth of f in y)

- Let's prove this theorem. How do we show a solution exists when we have no formula for y or f? The answer is to construct a solution.
- Example: Show off idea with $y' = y$, $y(0) = 1$ (know the solution is e^t). Construct solution y via a sequence of functions y_k:

$$\begin{aligned}
&y_0' = 0, y(0) = 1\\
&y_1' = y_0 \Rightarrow y_1 = 1 + t\\
&y_2' = y_1 \Rightarrow y_2 = 1 + t + \frac{t^2}{2}\\
&y_3' = y_2 \Rightarrow y_3 = 1 + t + \frac{t^2}{2} + \frac{t^3}{6!}\\
&\vdots
\end{aligned}$$

generating the power series for solution e^t. Do need to show that series converges (ratio test). The proof of our theorem follows the sameish approach.

- Proof: We construct our sequence as

$$\text{(equation:)}\ \frac{dy_{n+1}}{dt} = f(t, y_n(t)), \quad \text{(solution:)}\ y_{n+1} = y_0 + \int_0^t f(s, y_n(s))\, dt$$

Then,

$$y_{n+1}(t) - y_n(t) = \int_0^t [f(s, y_n(s)) - f(s, y_{n-1}(s))\, ds$$

When $|\partial f/\partial y| \leq L$, the difference $|f(y_n) - f(y_{n-1})| \leq L|y_n - y_{n-1}|$. Then,

$$|y_2 - y_1| \leq \int_0^t L|y_1 - y_0|\, ds \leq Lt|y_1 - y_0|_{max}$$

and also

$$|y_3 - y_2| \leq \int_0^t L|y_2 - y_1|\, ds \leq \int_0^t L^2 t|y_1 - y_0|_{max}\, ds = \frac{L^2t^2}{2}|y_1 - y_0|_{max}$$

leading to

$$|y_n - y_{n-1}| \le \frac{L^n t^n}{n!} |y_1 - y_0|_{max}$$

which rapidly approaches zero. Then for n larger and N larger,

$$|y_N - y_n| \le |y_N - y_{N-1}| + |y_{N-1} - n_{N-2}| + \cdots + |y_{n+1} - y_n| \le C \frac{L^n t^n}{n!}$$

which also approaches zero. This is a Cauchy sequence which implies y_n converges to some limit $y(t)$. This implies y_{n+1} does likewise giving

$$y_{n+1} = y_0 + \int_0^t f(s, y_n(s))\, dt \quad \to \quad y(t) = y_1 + \int_0^t f(s, y(s)\, ds$$

solving $\frac{dy}{dt} = f(t, y)$.

4. Practical side of uniqueness theorem:

   - For f nice enough, if any two solutions are in the same place at the same time, they are the same function. Can use this fact to bound solutions.
   - Example: Logistic equation $\frac{dy}{dt} = y(1-y)$ has equilibrium solutions at $y = 0, 1$. Then if we know $y(0) = \frac{1}{2}$, we know this solution $0 < y(t) < 1$ for all t. Also, since $f(y_0) > 0$, we know $y(t)$ is always increasing. Then $y(t)$ will have to approach 1 as a horizontal asymptote. If not, say $y(t)$ had asymptote $L < 1$, since $f(L) > 0$, when y is close to L it must increase past L which hence cannot be an asymptote.

5. Homework: 1-8, 11-17

.6 1.6: Equilibria and the phase line

1. Return to autonomous equations $\dfrac{dy}{dt} = f(y)$.

   (a) Example: Logistic equation $\dfrac{dy}{dt} = y(1-y)$.

   - Already noted that direction fields of autonomous equations are constant on horizontal lines. Why? $f(t, y)$ does not depend on t.
   - A phase line compresses all this information into a single line. Draw vertical line next to the direction field. Already saw this in the previous sections.

   (b) Steps to draw a phase line

   - Find equilibrium solutions, label on number line. These are the only places f can change sign assuming continuous f.
   - Find the sign of f between equilibrium solutions. This gives increase / decrease of solution y.

2. Classification of equilibrium points: Three cases

   (a) Sink (stable): Approach y_0 from both above and below as $t \to \infty$. Test if $f'(y_0) < 0$ (draw graph of f to illustrate).

   (b) Source (unstable): Repel y_0 from both above and below (approach as $t \to -\infty$). Test if $f'(y_0) > 0$.

   (c) Node: Opposite behavior above and below (neither sink nor source). Test if $f'(y_0) = 0$ plus more information.

3. Example: Illustrate the above for

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)\left(\frac{P}{M} - 1\right)$$

where $N > M > 0$. What is this? Modified logistic growth with minimal sustainable threshold M. Choose nice numbers $N = 10, M = 2$ and illustrate phase line and classify equilibrium.

4. Homework: 1-36, 39, 43,

.7 1.7: Bifurcations

1. Here we study families of differential equations with general parameters.
   - Example: $\frac{dP}{dt} = kP$ where k is a parameter (any real number).
   - If a change in a parameter drastically changes the long term behavior of the solution, we call this a *bifurcation*.

2. Logistic equation with harvesting
   (a) Consider a logistic equation with harvesting rate $-h$

   $$\frac{dP}{dt} = 4P - P^2 - h = f(P)$$

   where $h > 0$ is our parameter.
   - Plot $f(P)$ for ...
   - $h = 0$ (no harvesting), two steady states
   - $h = 3$ (light harvesting), two steady states
   - $h = 4$ (critical harvesting), one steady state
   - $h = 5$ (over harvesting), no steady states
   - $h = 4$ is the *bifurcation value*. Find it via $f'(P) = 0$ and choosing the parameter such that $f = 0$ at that critical point.

3. Homework: 1-13, 15-21

.8 1.8: Linear equations

1. We end the chapter with one more useful analytic technique. Though there are many more techniques available, these two are the most widely useful:
   (a) Separable equations (already done)
   (b) Linear equations (now and next section)

2. First order linear equations:

   $$\frac{dy}{dt} = a(t)y + b(t)$$

   - If $b = 0$, this is homogeneous (or non-forced).
   - If $b \neq 0$, this is nonhomogeneous (or forced). Think of b as new input after starting time $t = 0$.

3. Meaning of linearity:
   (a) Homogeneous solution: If $y_h(t)$ solves the homogeneous equation $\frac{dy}{dt} = a(t)y$, then any constant times y_h also solves it.
   (b) Particular solution: $y_p(t)$ solves the nonhomogeneous equation $\frac{dy}{dt} = a(t)y + b(t)$.

(c) Complete solution: For linear equations, adding these two parts gives the complete solution

$$y = y_h + y_p$$

Easy to plug in and see why.

(d) Note: This only works because of the linear term (y^1). If we instead had y^2 adding y_h^2 to y_p^2 would not give $(y_h + y_p)^2$.

4. Method for solving linear equations:
   (a) Solve the homogeneous equation (separable equation) to get y_h.
   (b) Find a particular solution y_p. Two techniques:
      - Undetermined coefficients
      - Integrating factor (next section)
   (c) Add two solutions.

5. Method of undetermined coefficients:
   (a) Example: $\dfrac{dy}{dt} = -2y + e^t$.
      - The homogeneous equation is $y' = -2y$ with solution $y_h = Ce^{-2t}$.
      - For the particular solution, we need

        $$y' + 2y = e^t$$

        Then y better be some multiple of e^t. Let

        $$y_p = \alpha e^t$$

        and find the α which works.

        $$y_p' + 2y_p = \alpha e^t + 2\alpha e^t = 3\alpha e^t = e^t$$

        Then $\alpha = 1/3$ and $y_p = \frac{1}{3}e^t$.
      - The complete solution is then

        $$y = y_h + y_p = Ce^{-2t} + \frac{1}{3}e^t$$

   (b) This guessing really only works for nice functions for $b(t)$.
      - e^t needs αe^t
      - $\sin(t)$ (or $\cos(t)$) needs $\alpha \sin(t) + \beta \cos(t)$ (why both?)
   (c) Of course guessing isn't a good technique: Example

      $$y' = -2y + e^{-2t}$$

      - Why doesn't $y_p = \alpha e^{-2t}$ work? It solves the homogeneous equation.
      - What might work instead? $y_p = \alpha t e^{-2t}$ via the product rule.
      - The next section trades guessing for computation (integration).

6. Homework: 1-24, 33, 34

.9 1.9: Integrating factors for linear equations

1. This section uses a trick to integrate first order linear nonhomogeneous equations directly.
   - Advantage: Straightforward technique for finding analytic solutions.
   - Disadvantage: Integration is hard, even impossible much of the time.

2. Example: $\dfrac{dy}{dt} = -2y + e^t$ revisited.

   (a) Rearrange and note the similarity to the product rule.

   $$\frac{dy}{dt} + 2y = e^t$$

   We need to multiply the LHS by e^{2t} (called the *integrating factor*) in order to see the product rule exactly.

   $$\begin{aligned} e^{2t}\frac{dy}{dt} + 2e^{2t}y &= e^{2t}e^t \\ \frac{d}{dt}\left(e^{2t}y\right) &= e^{3t} \\ e^{2t}y &= \int e^{3t}\ dt \\ e^{2t}y &= \frac{1}{3}e^{3t} + C \\ y &= \frac{1}{3}e^t + Ce^{-2t} \end{aligned}$$

   as we found in the last section.

3. Method of integrating factor: General first order equation

   $$\frac{dy}{dt} = a(t)y + b(t) \quad \Rightarrow \quad \frac{dy}{dt} - a(t)y = b(t)$$

   (a) Rearrange, introduce exponential (integrating factor), then compact as product rule:

   $$\begin{aligned} \frac{dy}{dt} - a(t)y &= b(t) \\ e^{-\int a(t)\ dt}\frac{dy}{dt} - a(t)e^{-\int a(t)\ dt}y &= b(t)e^{-\int a(t)\ dt} \\ \frac{d}{dt}\left(e^{-\int a(t)\ dt}y\right) &= b(t)e^{-\int a(t)\ dt} \\ \frac{d}{dt}\left(\mu(t)y\right) &= b(t)\mu(t) \\ \mu(t)y &= \int b(t)\mu(t)\ dt \\ y &= \frac{1}{\mu(t)}\int b(t)\mu(t)\ dt \end{aligned}$$

   Note, check the differentiation of the integrating factor $\mu(t) = e^{-\int a(t)\ dt}$:

   $$\frac{d}{dt}e^{-\int a(t)\ dt} = e^{-\int a(t)\ dt}(-a(t))$$

   Better to execute this as a strategy rather than try to memorize the formula.

(b) Example: Solve $\dfrac{dy}{dt} = -\dfrac{y}{t} + 2, y(1) = 3$

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{y}{t} + 2 \\
\frac{dy}{dt} + \frac{y}{t} &= 2 \\
e^{\int 1/t \ dt}\frac{dy}{dt} + e^{\int 1/t \ dt}\frac{1}{t}y &= 2e^{\int 1/t \ dt} \\
e^{\ln|t|}\frac{dy}{dt} + e^{\ln|t|}\frac{1}{t}y &= 2e^{\ln|t|} \\
t\frac{dy}{dt} + t\frac{1}{t}y &= 2t \\
t\frac{dy}{dt} + y &= 2t \\
\frac{d}{dt}(ty) &= 2t \\
ty &= \int 2t \ dt \\
ty &= t^2 + C \\
y &= t + \frac{C}{t}
\end{aligned}$$

$y(1) = 3$ gives $C = 2$ and

$$y = t + \frac{1}{t}$$

(c) Trouble with this technique is two possibly challenging integrations:

- $\mu(t) = e^{-\int a(t) \ dt}$
- $\displaystyle\int b(t)\mu(t) \ dt$

4. Homework: 1-24

5. Chapter 1 review: 1-54

Chapter 2: First order systems

1. We completed first order systems, now there are two possible next steps.

   (a) Single function y, higher order. Second order equations (think displacement, velocity, acceleration), then third and higher.

   (b) Multiple functions and first order systems of equations.

   We will choose the latter, though it turns out these are connected. Higher order linear equations can be rewritten as linear systems (and vice versa). We will return to high order equations in Chapter 3.

2. Again, three main approaches: This chapter takes the first two.

   (a) Qualitative: Trade direction fields for phase portraits which compare the derivatives of two unknown functions.

   (b) Numerical: Bend Euler's method to systems.

   (c) Analytic: Only special cases can be handled here. Chapter 3 focuses here.

.1 2.1: Modeling via systems

1. Example: Predator-prey first order system

$$\begin{cases} \frac{dR}{dt} &= 2R - 1.2F \cdot R \\ \frac{dF}{dt} &= -F + 0.9F \cdot R \end{cases}$$

where R is rabbit population and F is fox population. This is a nonlinear system since there are interaction terms ($F \cdot R$).

   (a) Equilibrium solutions: Require both $\dfrac{dR}{dt} = \dfrac{dF}{dt} = 0$.

$$R(2 - 1.2F) = 0$$
$$F(-1 + 0.9R) = 0$$

   - If $R = 0$ and $F = 0$. So if no rabbits or fox, equilibrium.
   - If $R = \frac{1}{0.9} = \frac{10}{9}$, then need $F = \frac{2}{1.2} = \frac{20}{12} = \frac{5}{3}$. This is a nontrivial equilibrium where the system lies in balance.

   (b) Solution curves: Finding these explicitly is usually not possible and numerical methods are relied upon. Lucky for us this one twists into a separable equation.

$$\frac{dR/dt}{dF/dt} = \frac{dR}{dF} = \frac{R(2-1.2F)}{F(0.9R-1)} \quad \Rightarrow \quad \int \frac{0.9R-1}{R}\,dR = \int \frac{2-1.2F}{F}\,dF$$

   resulting in a family of solution curves

$$0.9R - \ln(R) = 2\ln(F) - 1.2F + C$$

   Plot a few in Desmos. Note we lose time (direction of the orbit). Euler's method brings back time in section 2.5.

   (c) Phase portrait: Plotting solution curves against R and F results in a phase portrait. Note our two equilibrium solutions.

   - $R = \frac{10}{9}, F = \frac{5}{3}$ is the center of the circle-like orbits. If start at that point, remain stationary.
   - $R = F = 0$ is also stationary. If only $F = 0$, $R \to \infty$. Also, if only $R = 0$, $F \to 0$ as expected with preyless foxes. To add direction to the phase portrait, just compute the tangents at any point on solution curves. If $R = F = 2$,

$$\frac{dR/dt}{dF/dt} = \frac{dR}{dF} = \frac{R(2-1.2F)}{F(0.9R-1)} = \frac{2(2-1.2\cdot 2)}{2(0.9\cdot 2-1)} = \frac{2(20-12\cdot 2)}{2(9\cdot 2-10)} = \frac{-8}{16} = \frac{-1}{2}$$

     decreasing rabbits yet increasing foxes as we see in pplane.

   (d) This system is the famous Lotka-Volterra equations dating back to 1925.

2. Harmonic motion (mass on a spring)

   (a) Notation:

   - $y(t)$ = displacement ($y = 0$ at rest)
   - $y'(t)$ = velocity (speed and direction)
   - $y''(t)$ = acceleration ($y'' > 0$ speeding up, $y'' < 0$ slowing down)
   - Assuming no damping of spring, how do these relate?

   (b) Video demos:

   - Mass on spring: `https://www.youtube.com/watch?v=eeYRkW8V7Vg`
   - Paint can solution: `https://www.youtube.com/watch?v=p9uhmjbZn-c`

- Known as a harmonic oscillator (simple harmonic motion).

(c) Model formulation:
   - Newton's second law: $F = ma$. For us,
   $$F = my''$$
   - Then how does the force of the spring F behave? Hook's law says
   $$F = -ky.$$
   Translation: Force is proportional to the stretching distance. More stretch, more force.
   - Model:
   $$F = ma \quad \Rightarrow \quad -ky = m\frac{d^2y}{dt^2} \quad \Rightarrow \quad m\frac{d^2y}{dt^2} + ky = 0$$
   This is a second order equation. The solution should involve sine and cosine terms. If we added damping (friction on the spring), we would also include a $b\frac{dy}{dt}$ term to slow velocity (section 2.3).

(d) Solution: Use method of undetermined coefficients on $\cos(\omega t)$ (or $\sin(\omega t)$) to get general solution as
$$y = c_1 \cos\left(\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\right)$$
which can be combined into a single cosine curve as
$$y = R\cos(\omega t - \alpha)$$
with amplitude, frequency, and phase shift all depending on initial conditions $y(0)$ and $y'(0)$. More complete story on all this later.

(e) Can translate to a first order system for an alternate view. Let our two variables be y and $v = \frac{dy}{dt}$. Then, our second order equation becomes
$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\frac{k}{m}y \end{cases}$$

(f) Example: Simplify this by letting $\frac{k}{m} = 1$.
   - Equilibrium solution: $y = v = 0$. No displacement or velocity.
   - General solution is
   $$y = c_1\cos(t) + c_2\sin(t)$$
   If a specific solution is $y(0) = 1$, $y'(0) = 0$, then
   $$y = \cos(t), \quad v = \sin(t)$$
   and this curve is the unit circle in the $y - v$ plane.
   - All solution curves are
   $$y = R\cos(t - \alpha), \quad v = -R\sin(t - \alpha)$$
   which are circles of radius R in the y, v plane.

(g) Second order equation vs first order system:
   - Second order equation is simpler and directly translates mechanical law. Also good for finding analytic solution.
   - System is more complex, but gives good qualitative analysis as well as better numeric solution.

3. Homework: 1-16, 19-24

.2 2.2: The geometry of systems

Jeff skip

1. Translating first order systems into the language of vectors makes it easier to see how phase portraits are generated.
   (a) Example: Predator prey
   $$\begin{cases} \frac{dR}{dt} & = 2R - 1.2F \cdot R \\ \frac{dF}{dt} & = -F + 0.9F \cdot R \end{cases}$$
   - Introduce vector $\vec{Y}$ and RHS vector field $\vec{F}(\vec{Y})$ as
   $$\vec{Y} = \langle R, F \rangle, \quad \vec{F}(\vec{Y}) = \langle 2R - 1.2F \cdot R, -F + 0.9F \cdot R \rangle$$
   - Then, at any point in the $R - F$ plane, we have a corresponding vector resulting from $\vec{F}$. If $R = F = 2$ as we saw last time,
   $$\vec{F} = \langle 4 - 4.8, -2 + 3.6 \rangle = \langle -0.8, 1.6 \rangle.$$
   - Plot all these vectors in the $R - F$ plane and we see our phase portrait.

   (b) Distinction: A direction field just gives direction while a phase portrait gives both magnitude and direction.
2. Homework: 1-18, 21

.3 2.3: The damped harmonic oscillator

Jeff skip

1. Here we accomplish two things:
   (a) Enhance the model for harmonic oscillation by adding damping (friction of the spring).
   (b) Introduce a general approach for second order constant coefficient equations:
   $$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$$
2. Damped harmonic oscillator:
   (a) We alter the precious model by adding damping in the form of $-b\dfrac{dy}{dt}$ for damping coefficient $b > 0$. Model: $F = ma$
   $$-ky - b\frac{dy}{dt} = m\frac{d^2y}{dt^2} \quad \Rightarrow \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$
   Generally we rewrite to mimic quadratic equations (will see why shortly).
   $$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0$$
   (b) Solution: It seems that the solution has to be an exponential $y = e^{st}$ for some number s. Plug it in and see.
   $$\begin{aligned} A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy &= 0 \\ As^2e^{st} + Bse^{st} + Ce^{st} &= 0 \\ (As^2 + Bs + C)e^{st} &= 0 \\ As^2 + Bs + C &= 0 \end{aligned}$$

This quadratic is called the *characteristic equation* and gives a way to find s.

$$s = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

with three cases:

- $B^2 > 4AC$ (two real roots, over damping)
- $B^2 = 4AC$ (repeated real roots, critical damping)
- $B^2 < 4AC$ (complex roots, under damping)
- Also include $B = 0$ (no damping)

(c) Note this translates into a first order system: Dividing by A yields

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\frac{C}{A}y - \frac{B}{A}v \end{cases}$$

and is of the form $\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y})$

(d) Example: $y'' + 3y' + 2y = 0$ has characteristic equation

$$s^2 + 3s + 2 = 0 \quad \Rightarrow \quad (s+2)(s+1) = 0 \quad \Rightarrow \quad s = -1, -2$$

Then the general solution is

$$y = c_1 e^{-t} + c_2 e^{-2t}.$$

This solution is stable and tends to zero. Can translate this solution to

$$\vec{Y} = c_1 \vec{Y}_1 + c_2 \vec{Y}_2$$

for

$$\vec{Y}_1 = \langle y_1, v_1 \rangle = \langle e^{-t}, -e^{-t} \rangle, \quad \vec{Y}_2 = \langle y_2, v_2 \rangle = \langle e^{-2t}, -2e^{-2t} \rangle$$

and view in phase portrait.

(e) Can imagine interesting cases here:

- $s_1, s_2 > 0$ will be unstable (tend to infinity).
- $s_1 < 0, s_2 > 0$ will be unstable ans stable?
- $s_1 = s_2$ will take thinking to find a second solution.
- s complex will involve Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

  resulting in sine and cosine terms (oscillation).

More on these later.

3. Homework: 1-10

.4 2.4: Additional analytic methods for special systems

Jeff skip

1. A general linear system is of the form

$$\begin{cases} \frac{dx}{dt} = Ax + By \\ \frac{dy}{dt} = Cx + Dy \end{cases}$$

For nice cases (decouple $B = C = 0$ or partially decoupled $B = 0$ or $C = 0$), we can draw on previous experience.

2. Example:

$$\begin{cases} \frac{dx}{dt} = 2x + 3y \\ \frac{dy}{dt} = -4y \end{cases}$$

(a) Solve for y first:

$$y = c_1 e^{-4t}$$

(b) Solve for x next:

$$\frac{dx}{dt} = 2x + 3c_1 e^{-4t}$$

This is a nonhomogeneous equation which we can solve to get

$$x = c_2 x_h + x_p = c_2 e^{2t} - \frac{1}{2} c_1 e^{-4t}$$

3. Homework: 1-13

.5 2.5: Euler's method for systems

Jeff skip

1. Euler's method for first order equations extends to systems in a straightforward way.

   (a) First order equation: $\frac{dy}{dt} = f(t, y)$

   $$y_{n+1} = y_n + \Delta t f_n$$

   Idea is to follow the tangent line by traveling stepsize Δt.

   (b) First order system:

   $$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases}$$

   Idea here is to follow the vector field by traveling $\Delta t \cdot \vec{F}(\vec{Y})$ where $\vec{F}$ is the vector at location (x, y).

   $$\begin{aligned} \langle x_1, y_1 \rangle &= \langle x_0, y_0 \rangle + \Delta t \vec{F}(x_0, y_0) \\ \langle x_2, y_2 \rangle &= \langle x_1, y_1 \rangle + \Delta t \vec{F}(x_1, y_1) \\ &\vdots \\ \langle x_{n+1}, y_{n+1} \rangle &= \langle x_n, y_n \rangle + \Delta t \vec{F}(x_n, y_n) \end{aligned}$$

   which simply gives componentwise

   $$x_{n+1} = x_n + \Delta t f_n, \quad y_{n+1} = y_n + \Delta t g_n.$$

2. Coding project assignment.

.6 2.6: Existence and uniqueness for systems

Jeff skip

1. Here we state an important theorem, no homework involved.

2. Theorem: Existence and uniqueness for systems.

   For $\vec{Y} = \langle x, y \rangle$ and $\vec{F} = \langle f(t, x, y), g(t, x, y) \rangle$, with $\vec{F}$ *continuously differentiable*, the system

   $$\frac{d\vec{Y}}{dt} = \vec{F}, \quad \vec{Y}(t_0) = \vec{Y}_0$$

   has a unique solution Y for t near t_0.

   Note, solutions may blow up in finite time as we saw in the one dimensional case.

3. Homework: None

.7 2.7: The SIR model of an epidemic

Jeff skip

1.

2. Coding assignment. Reproduce paper figures / tables.

.8 2.8: The Lorenz equations

Jeff skip

1.

2. Homework: None

Chapter 3: Linear systems

.1 3.1: Properties of linear systems and the linearity principle

1. Here we turn to analytic techniques for systems of equations, though we are tied to a special case with a wide range of applications: first order linear systems with constant coefficients:

   (a) 2×2 system:

   $$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

   (b) Vector form:

   $$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

   where

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   (c) Recall matrix multiplication definition.

   (d) Note, results will easily extend to $n \times n$ systems.

   (e) Linearity principle (AKA superposition principle): If $\vec{Y}_1$ and $\vec{Y}_2$ solve the linear system, so do any linear combination $c_1\vec{Y}_1 + c_2\vec{Y}_2$.

   Easy to show via derivative and matrix multiplication properties. Both are linear properties.

   $$\frac{d}{dt}(c_1\vec{Y}_1 + c_2\vec{Y}_2) = c_1\frac{\vec{Y}_1}{dt} + c_2\frac{\vec{Y}_2}{dt} = c_1A\vec{Y}_1 + c_2A\vec{Y}_2 = A(c_1\vec{Y}_1 + c_2\vec{Y}_2)$$

2. Recall high dimensional systems can be rewritten as linear systems. We will revisit the second order equations (harmonic oscillators) soon.

   (a) Second order equation with damping:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

   (b) First order linear system: Substitute $v = \frac{dy}{dt}$ and name $\vec{Y} = \langle y, v\rangle$ giving

$$\frac{dY}{dt} = \begin{bmatrix} v \\ -\frac{k}{m}y - \frac{b}{m}v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \vec{Y} = A\vec{Y}$$

3. For the general 2×2 system

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

   a key property is that matrix A is non-singular.

   (a) Recall, A is nonsingular if
      - $\det(A) \neq 0$, A^{-1} exists, only $\vec{0}$ in null(A), linearly independent columns, ...
      - Refresh: `https://en.wikipedia.org/wiki/Invertible_matrix#The_invertible_matrix_theorem`

   (b) Determinant of a 2×2 matrix A.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

      - Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

      - 3×3

   (c) Theorem: For A nonsingular, the system $\frac{d\vec{Y}}{dt} = A\vec{Y}$ only equilibrium point is the origin $\vec{0}$.

4. General solutions to linear systems:

   (a) Theorem: For $\vec{Y}_1, \vec{Y}_2$ solutions to the 2×2 linear system $\frac{d\vec{Y}}{dt} = A\vec{Y}$. If $\vec{Y}_1(0)$ and $\vec{Y}_2(0)$ are linearly independent, we can always find constants c_1 and c_2 such that

$$c_1\vec{Y}_1 + c_2\vec{Y}_2 \quad \text{(known as the general solution)}$$

   solves the initial value problem.

   (b) Definition: Vectors $\vec{x}$ and $\vec{y}$ are linearly independent if for any vector $\vec{a}$, there exists constants c_1 and c_2 such that

$$c_1\vec{x} + c_2\vec{y} = \vec{a}.$$

   Note it is equivalent to say matrix $A = [\vec{x}\vec{y}]$ is nonsingular. Why?

   (c) Example: Find the solution to the initial value problem

$$\frac{d\vec{Y}}{dt} = A\vec{A}, \quad A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}, \quad \vec{Y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   We already found two solutions last time.

$$x = c_1e^{2t} - \frac{1}{2}c_2e^{-4t}, \quad y = c_2e^{-4t}.$$

Then two vector solutions are

$$\vec{Y}_1 = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}, \quad \vec{Y}_2 = \begin{bmatrix} -\frac{1}{2}e^{-4t} \\ e^{-4t} \end{bmatrix}.$$

Then we need

$$c_1\vec{Y}_1(0) + c_2\vec{Y}_2(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \vec{Y}_1(0) & \vec{Y}_2(0) \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and we see why linear independence is again key. Result is $c_2 = 2$ and $c_1 = 2$ which leads to the solution to the IVP.

5. Homework: 5-19, 24, 30-32, 34-35

.2 3.2: Straight-line solutions

1. With this section we make an important connection to linear algebra (eigenvalues and eigenvectors).
2. Example: Same example as before.

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{Y} = A\vec{Y}$$

with general solution

$$\vec{Y} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = c_1 e^{2t}\vec{V}_1 + c_2 e^{-4t}\vec{V}_2$$

   (a) pplane: Note straight lines in phase portrait coincide with two vectors. Also, direction followed relates to the sign of the exponent (decay or growth).
   (b) Once we are on a straight line, we remain on it. Reason:

$$A\vec{V}_1 = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\vec{V}_1$$

3. $A\vec{V} = \lambda\vec{V}$ gives eigenvalues and eigenvectors (the characteristic direction of a matrix). Advantage of eigenness:
   (a) Matrix multiplication turns into scalar multiplication
   (b) Matrix diagonlization: If A has n linearly independent eigenvectors,

$$A = V\Lambda V^{-1} \quad \Rightarrow \quad A^k = V\Lambda^k V^{-1}$$

   (c) How to find λ and $\vec{V}$?
      - Eigenvalues: Need

$$A\vec{V} = \lambda\vec{V} \quad \Rightarrow \quad (A - \lambda I)\vec{V} = \vec{0}$$

        for $\vec{V} \neq \vec{0}$. This implies $(A - \lambda I)$ is singular and

$$\det(A - \lambda I) = 0 \quad \text{(called the characteristic equation)}$$

      - Eigenvectors: Once found λ, solve $(A - \lambda I)\vec{V} = \vec{0}$ via Gaussian elimination.
4. Example: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix}$$

(a) Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -5-\lambda & -2 \\ -1 & -4-\lambda \end{vmatrix} = (5+\lambda)(4+\lambda) - 2 = \lambda^2 + 9\lambda + 18 = (\lambda+6)(\lambda+3) = 0$$

Then $\lambda = -3, -6$.

(b) Eigenvectors:

- $\lambda = -3$

$$\begin{bmatrix} -5+3 & -2 \\ -1 & -4+3 \end{bmatrix} \vec{V} = \vec{0}$$

giving $\vec{V} = \langle 1, -1 \rangle$.

- $\lambda = -6$

$$\begin{bmatrix} -5+6 & -2 \\ -1 & -4+6 \end{bmatrix} \vec{V} = \vec{0}$$

giving $\vec{V} = \langle 2, 1 \rangle$.

5. Example: What can we expect from the linear system?

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -5 & -2 \\ -1 & -4 \end{bmatrix} \vec{Y} = A\vec{Y}$$

(a) Straight line solutions are along eigenvectors. Both solutions decay along these lines.

(b) General solution is

$$\vec{Y} = c_1 e^{\lambda_1 t} \vec{V}_1 + c_2 e^{\lambda_2 t} \vec{V}_2 = c_1 e^{-3t} \langle 1, -1 \rangle + c_2 e^{-6t} \langle 2, 1 \rangle$$

Key here is *linearly independent eigenvectors* (as we saw in last section). Equivalently, need *distinct eigenvalues.*

(c) All solutions should decay.

6. Theorem: If matrix A has *distinct real* eigenvalues λ_k with corresponding eigenvectors $\vec{V}_k$, then

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

has straight line solutions as

$$Y = c_k e^{\lambda_k t} \vec{V}_k$$

has general solution

$$Y = c_1 e^{\lambda_1 t} \vec{V}_1 + c_2 e^{\lambda_2 t} \vec{V}_2 + \cdots + c_n e^{\lambda_n t} \vec{V}_n$$

- Note dimension makes no difference here.
- Real distinct eigenvalues is important. We will look at complex and repeated eigenvalues later.

7. Homework: 1-25

.3 3.3: Phase portraits for linear systems with real eigenvalues

1. There are three main types of equilibrium points for 2×2 linear systems with real eigenvalues:

(a) Sinks: $\lambda_1 < \lambda_2 < 0$ implies decay to $\vec{0}$ from all directions.

(b) Sources: $0 < \lambda_1 < \lambda_2$ implies repelling from $\vec{0}$ to $\pm\infty$.

(c) Saddles: $\lambda_1 < 0 < \lambda_2$ decays to $\vec{0}$ along a single straight line solution but repels from $\vec{0}$ to $\pm\infty$ otherwise.

2. Stability:

   (a) Sinks are stable meaning solutions near eachother stay near eachother. Small changes to the initial value result in small changes in the solution final value.

   (b) Sources and saddles are unstable meaning solutions near eachother can end up far away. Small changes to the initial value can result in largely different final values.

3. Examples:

   (a) Sink:

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{Y}$$

   has general solution

   $$\vec{Y} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   (b) Source:

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \vec{Y}$$

   has general solution

   $$\vec{Y} =$$

   (c) Saddle:

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 5 & 4 \\ 9 & 0 \end{bmatrix} \vec{Y}$$

   has general solution

   $$\vec{Y} =$$

   (d) Plot phase portraits in pplane.

4. Homework: 1-16, 19-22, 27

.4 3.4: Complex eigenvalues

1. There are four main cases with 2D linear systems:

   (a) Two distinct real eigenvalues

   (b) Repeated real eigenvalues (next section)

   (c) Zero eigenvalue (next section)

   (d) Complex eigenvalues (now)

   - We will see that complex eigenvalues result in rotations / oscillation. Hence the harmonic oscillator will return.

2. Example:

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \vec{Y}, \quad \vec{Y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

   (a) Eigenvalues:

   $$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 + 1 = 0$$

   Then

   $$\lambda_1 = -2 + i, \quad \lambda_2 = -2 - i.$$

(b) Eigenvectors: We need to solve the systems $A\vec{Y_i} = \lambda_i \vec{V_i}$ and the result is

$$\vec{V_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \vec{V_2} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

(c) General solution: If we follow the mold from before, the result is

$$\vec{Y} = c_1 e^{\lambda_1 t}\vec{V_1} + c_2 e^{\lambda_2 t}\vec{V_2}$$

A few troubles here:

- How to make sense of $e^{(-2+i)t}$? Need Eulers formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

  Recall the power series formulation proof (`https://en.wikipedia.org/wiki/Euler%27s_formula`) and the use in the complex plane (unit circle). This formula is a big deal. Know it. Then,

$$e^{(-2+i)t} = e^{-2t}e^{it} = e^{-2t}(\cos(t) + i\sin(t))$$

  Likewise for λ_2.
- We need $\vec{Y}$ to be real valued, not complex valued. Simplify to get use there.

$$\begin{aligned} \vec{Y_1} &= e^{(-2+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{-2t}(\cos(t) + i\sin(t))\begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{-2t}\begin{bmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{bmatrix} \\ &= e^{-2t}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + ie^{-2t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \end{aligned}$$

  $\vec{Y_2}$ results in the same two vectors. Can show these vectors both solve the system are linearly independent via

$$\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1 \neq 0$$

Then the general solution is

$$\vec{Y} = c_1 e^{-2t}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

Note no more complex numbers.

(d) IVP solution: $\vec{Y}(0) = \langle 6, 2 \rangle$.

$$c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c1 \\ c2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

gives $c_1 = 6, c_2 = 2$ and

$$\vec{Y} = e^{-2t}\begin{bmatrix} 6\cos(t) + 2\sin(t) \\ -6\sin(t) + 2\cos(t) \end{bmatrix}$$

(e) Plot individual solutions to see oscillation:

$$x = e^{-2t}(\cos(t) + \sin(t)), \quad y = e^{-2t}(-\sin(t) + \cos(t))$$

Note e^{-2t} contributes decay, and the sine/cosine terms oscillation.

(f) Phase portrait: Note no straight line solutions, spiral sink in this case. Follow a single trajectory to see spiral.

3. Theorem: For $\vec{Y}$ a complex valued solution to $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$ where

$$\vec{Y} = \vec{Y}_{re} + i\vec{Y}_{im},$$

then both $\vec{Y}_{re}$ and $\vec{Y}_{im}$ solve the system.

4. How do the eigenvalue real / imaginary parts contribute to the solution and overall behavior? Assume

$$\lambda = a \pm bi$$

(a) Sign of a assuming $b \neq 0$
   - $a > 0$ equilibrium is spiral source
   - $a < 0$ equilibrium is spiral sink
   - $a = 0$ equilibrium is center (only circular motion)

(b) b cases
   - $b > 0$ gives oscillation with frequency b
   - $b = 0$ says real eigenvalues, no oscillation

(c) Rotation direction: At $\vec{Y} = \langle 1, 0\rangle$, compute $y' = c$.
   - If $c > 0$, counterclockwise
   - If $c < 0$, clockwise

5. Examples:

(a) Center:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \vec{Y}$$

   - Eigenvalues: $\lambda = \pm 2i$
   - Eigenvector:
   - General solution:
   - When is one cycle completed? $t = \pi$ periodic
   - This is a circular orbit, but in general it can be an ellipse.

(b) Source:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 & 2 \\ -3 & 2 \end{bmatrix} \vec{Y}$$

(c) Eigenvalues: $\lambda = 1 \pm i\sqrt{5}$

(d) Eigenvector:

$$\vec{V} = e^{(1+\sqrt{5}i)t} \begin{bmatrix} 2 \\ 1 + i\sqrt{5} \end{bmatrix}$$

(e) General solution:

$$\vec{Y} = c_1 e^t \begin{bmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t) - \sqrt{5}\sin(\sqrt{5}t) \end{bmatrix} + c_2 e^t \begin{bmatrix} 2\sin(\sqrt{5}t) \\ \sqrt{5}\cos(\sqrt{5}t) + \sin(\sqrt{5}t) \end{bmatrix}$$

(f) Can still think about period and cycle as one revolution about the origin.

6. Stability:

(a) Which of the above are stable?

(b) Later we will go further to classify all stability cases of linear systems.

7. Homework: 1-24, 26

.5 3.5: Special cases: repeated and zero eigenvalues

1. Two special cases remain (degenerate systems). These play an importation role when studying system bifurcations.

   (a) Repeated real eigenvalues (may still have 2 linearly independent eigenvectors, or maybe only 1). Note repeat complex eigenvalues not possible since one complex eigenvalue implies there is a conjugate pair.

   (b) Zero eigenvalue (whole line of equilibrium solutions)

2. Repeated eigenvalue case

   (a) Complete case: still 2 linearly independent eigenvectors (actually infinitely many)

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{Y}$$

   - Note repeated eigenvalue are $\lambda = 2$ and can see that

$$\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

     are both eigenvectors (though there are many more).
   - General solution:

$$\vec{Y} = c_1 e^{\lambda t}\vec{V}_1 + c_2 e^{\lambda t}\vec{V}_2 = e^{\lambda t}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

     and there are infinite straight line solutions.
   - Pplane visual.
   - The sign of λ determines source or sink. No spiraling possible here.

   (b) Incomplete case: only 1 linearly independent eigenvector

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{Y}$$

   - Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 = 0$$

     gives $\lambda = -2$.
   - Eigenvector: $A\vec{V} = \vec{V}$ yields only

$$\vec{V} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   - Single solution: $Y = c_1 e^{-2t}\langle 1, 0\rangle$, but we know there should be two linearly independent solutions.
   - Switch gears and solve as a (partially) decoupled system.

$$\frac{dx}{dt} = -2x + y, \quad \frac{dy}{dt} = -2y$$

     Result is

$$y = y_0 e^{-2t}, \quad x = y_0 t e^{-2t} x_0 e^{-2t}$$

     giving our system general solution of

$$\vec{Y} = e^{-2t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-2t}\begin{bmatrix} y_0 \\ 0 \end{bmatrix} = e^{-2t}\vec{Y}_0 + y_0 t e^{-2t}\vec{V} = e^{-2t}\vec{Y}_0 + te^{-2t}(A - \lambda I)\vec{Y}_0$$

- This final solution form can be checked in general:

$$\vec{Y} = e^{\lambda t}\vec{Y}_0 + te^{\lambda t}(A - \lambda I)\vec{Y}_0$$

solves $\frac{d\vec{Y}}{dt} = A\vec{Y}$, and hence is the general solution. Plug into both sides and see.

(c) Theorem: For linear system

$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

with repeated eigenvalue λ and only one linearly independent eigenvectors, the general solution is

$$\vec{Y} = e^{\lambda t}\vec{Y}_0 + te^{\lambda t}\vec{V}$$

where

$$\vec{V} = (A - \lambda I)\vec{Y}_0.$$

- If $\vec{V} = \vec{0}$, then there are only straight line solutions as we saw in the first example.
- If $\vec{V} \neq \vec{0}$, then it is an eigenvector as in the second example.
- This theorem should be used from the start to solve any system with repeated eigenvalues. Our conversation stumbled across it.

(d) Example:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \vec{Y}$$

has repeated eigenvalue $\lambda = -1$. Then the general solution is

$$\vec{Y} = e^{-t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-t}(A + I)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = e^{-t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^{-t}\begin{bmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{bmatrix}$$

Check that it works to be sure.

3. Zero eigenvalue case: This case is simpler. One example will suffice.

(a) Example:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \vec{Y}$$

- Eigenvalues:

$$\det(A - \lambda I) = (4 - \lambda)(1 - \lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

$\lambda = 0, 5$. Note that implies $\det(A) = 0$ and this matrix is nonsingular and the nullspace has more than $\vec{0}$. Our eigenvector should live there too.

- Eigenvectors: $A\vec{V}_1 = 5\vec{V}_1$ gives

$$\vec{V}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

while $A\vec{V}_2 = \vec{0}$ gives

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- General solution:

$$\vec{Y} = c_1 e^{5t}\vec{V}_1 + c_2\vec{V}_2$$

- Every point along $\vec{V}_2$ will serve as an equilibrium point (unstable in this case, stable for $\lambda < 0$). All other solutions will follow lines parallel to $\vec{V}_1$.

4. Homework: 1-23

.6 3.6: Second-order linear equations

1. Here we give the complete unified story of the second-order linear equations:

   - Second order equation:

   $$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \Rightarrow \quad \frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

   where constants denote m (mass), b (damping constant), and k (spring strength). Engineer a leading coefficient of 1 for simplicity.

   - Equivalent first order linear system:

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$$

   where $\vec{Y} = \langle y, \frac{dy}{dt} \rangle = \langle y, v \rangle$.

   - It will turn out our characteristic equation from chapter 2 matches our eigenvalue equation from this chapter.

2. Example:

   $$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

   (a) Characteristic equation approach: Assume $y = e^{st}$, then the differential equation reduces to

   $$e^{st}(s^2 + 3s + 2) = 0 \quad \Rightarrow \quad s^2 + 3s + 2 = (s + 2)(s + 1) = 0 \quad \Rightarrow \quad s = -2, -1$$

   Then the general solution is

   $$y = c_1 e^{-2t} + c_2 e^{-t}.$$

   Note, the solution tends to zero without oscillation.

   (b) Plot the solution y in Desmos for $y(0) = 1$, $y'(0) = v(0) = 1$ giving $c_1 = 2, c_2 = -1$.

   (c) Linear system / eigenvalue approach: Let $\vec{Y} = \langle y, v \rangle$. Then,

   $$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{Y}$$

   . Then,

   $$\det(A - \lambda I) = -\lambda(-3 - \lambda) + 2 = \lambda^2 + 3\lambda + 2 = 0$$

   the same as the characteristic equation. $\lambda = -2, -3$ and the general solution is

   $$\vec{Y} = c_1 e^{-3t}\vec{V}_1 + c_2 e^{-2t}\vec{V}_2.$$

   where eigenvectors are found to be

   $$\vec{V}_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}, \quad \vec{V}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

   and we have the same y as found above.

   (d) Graph in pplane. Note equilibrium is a sink (stable).

3. Example:

   $$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0, \quad y(0) = 1, y'(0) = -4$$

(a) General solution: Characteristic equation is

$$s^2 - 4s + 13 = 0 \quad \Rightarrow \quad s = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$$

giving

$$e^{(-2+3i)t} = e^{-2t}(\cos(3t) + i\sin(3t))$$

for the general solution

$$y = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t).$$

(b) Initial condition gives $c_1 = 1, c_2 = -2$ with solution

$$y = e^{-2t}\cos(3t) - 2e^{-2t}\sin(3t).$$

Graph the result in Desmos.

(c) Give phase portrait for equivalent system

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -13 & 4 \end{bmatrix}\vec{Y}$$

(d) Can see the strength provided by a and b in the eigenvalue $a + bi$. This can be generalized in terms of m (mass), b (damping constant), and k (spring strength)

4. Summary of general harmonic oscillator cases:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \Rightarrow \quad ms^2 + bs + k = 0 \; quad \Rightarrow \quad s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

(a) $b = 0$ gives no damping, only oscillation due to purely imaginary eigenvalues.

$$\cos(\omega t), \sin(\omega t)$$

(b) $b^2 > 4mk$ gives overdamping, only exponential decay.

$$e^{s_1 t}, e^{s_2 t}$$

(c) $b^2 = 4mk$ gives critical damping, only exponential decay due to repeated root.

$$e^{s_1 t}, te^{s_1 t}$$

(d) $b^2 < 4mk$ gives underdamping, decay plus oscillation

$$e^{at}\cos(\omega t), e^{at}\sin(\omega t)$$

(e) Classify the above two examples via this simple test. Have seen many examples in past. More in the homework. Sample solutions for 3 cases:

- No damping

$$\frac{d^2y}{dt^2} + 0\frac{dy}{dt} + y = 0, \quad s = \pm i$$

- Overdamping

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0, \quad s = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

- Critical damping

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0, \quad s = -1$$

- Underdamping

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0, \quad s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

5. Homework: 1-33

.7 3.7: The trace-determinant plane

1. In the last section we unified the behavior of all harmonic oscillators. Now we tackle all first order linear systems.

$$\frac{d\vec{Y}}{dt} = A\vec{y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{Y}$$

with $\vec{Y} = \langle x, y \rangle$. Given A, we will answer two main questions.

   (a) Is the system stable? That is, if two solutions start close together, will they remain close together?

   (b) What is the global behavior of the system? Cases are
      - Sink, saddle, source, spirals, etc.

   (c) Both of these questions are determined by the eigenvalues of A, but a simpler approach is to consider the trace and determinant of A.

$$\text{tr}(A) = a + d, \quad \det(A) = ad - bc$$

2. The trace-determinant plane:

   (a) Eigenvalues from trace / determinant:

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

   So the trace and determinant determine λ via

$$\lambda = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}$$

   (b) Real / complex cases:
      - $\text{tr}(A)^2 > 4\det(A)$ gives two distinct real eigenvalues
      - $\text{tr}(A)^2 < 4\det(A)$ gives complex eigenvalues
      - $\text{tr}(A)^2 = 4\det(A)$ gives repeated real eigenvalues, barrier between real and complex cases.

   (c) Activity: Quiz bonus points. Give the trace / determinant plane with phase portraits covered up. Have them fill in the behavior of each then uncover. Options:
      - Sink, degenerate sink, spiral sink, saddle, line of stable fixed points, line of unstable fixed points, center, spiral source, degenerate source, source.

   Which ones are stable?

   (d) Stability cases:
      - Note also that $\det(A) = \lambda_1 \cdot \lambda_2$ because of the factoring of the characteristic equation. If real eigenvalues only, need $\text{tr}(A) < 0$(at least one negative eigenvalue) and $\det(A) > 0$ (product of eigenvalues same sign). If complex eigenvalues need $\text{tr}(A) < 0$.
      - Stable when $\text{tr}(A) < 0$ and $\det(A) > 0$.
      - Unstable otherwise.

   (e) This conversation is especially useful for bifurcation analysis which you will focus on in the homework.

3. Examples: Determine behavior and stability of each via the trace and determinant.

   (a) $A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$ is stable.

   (b) $A = \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}$ is unstable.

(c) $A = \begin{bmatrix} 0 & 4 \\ 5 & -6 \end{bmatrix}$ is unstable.

(d) $A = \begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix}$ is neutral.

4. Final visual: `https://i.stack.imgur.com/duPPi.png`

5. Homework: 1-12

.8 3.8: Linear systems in three dimensions

Jeff skip

1. What remains of the course?

   (a) What have we done?

   - First order equations, second order linear equations, 2×2 linear systems

   (b) What remains?

   - High order linear systems, $n \times n$ linear systems, of course these are tied together. I will only give basic idea, no homework.
   - Nonlinear systems, we will borrow techniques from linear systems to partially analyze.
   - New idea all together: Laplace transform, change our space from derivatives / functions to algebra and back.

2. High order linear equations: Consider third order, but ideas extend higher.

$$y''' + By'' + Cy' + Dy = 0$$

   (a) Transfer to a 3×3 linear system. Denote

$$\vec{Y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$$

   then we are left with the system

$$\frac{d\vec{Y}}{dt} = A\vec{Y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -D & -C & -B \end{bmatrix} \vec{Y}$$

   (b) Eigenvalues: Characteristic equation again.

$$\det(A - \lambda I) = -(\lambda^3 + B\lambda^2 + C\lambda + D) = 0$$

   Challenge here is cubic equations are harder to deal with.

   (c) Eigenvectors: For this case, eigenvectors have the special form

$$\vec{V} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

   Multiply $A\vec{V}$ and use the characteristic equation to see.

   (d) High dimensional systems are more difficult to analyze, though familiar features appear (sink, source, saddle, rotation, etc). See text for concrete examples.

3. Homework: None. Not on quiz / exam.

Chapter 4: Forcing and resonance

Jeff skip all chapter

.1 4.1: Forced harmonic oscillators

.2 4.2: Sinusoidal forcing

.3 4.3: Undamped forcing and resonance

.4 4.4: Amplitude and phase of the steady state

.5 4.5: The Tacoma Narrows Bridge

Chapter 5: Nonlinear systems

Here we touch on 2×2 nonlinear, autonomous systems of the form

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$$

where f and g are nonlinear functions of x and/or y. Of course analysis will be harder here, but we can borrow ideas of linear systems to do analysis at specific points (equilibrium points) / curves (isoclines).

.1 5.1: Equilibrium point analysis

1. Motivating example:

$$\frac{dx}{dt} = -2x + 2x^2 = f(x, y), \quad \frac{dy}{dt} = -3x + y + 3x^2 = g(x, y)$$

   (a) Phase portrait is complicated. Let's look closer.

   (b) Equilibrium points:

$$f(x, y) = 2x(-1 + x) = 0, \quad g(x, y) = -3x + y + 3x^2 = 0$$

   Two points work: $(0, 0), (1, 0)$. Zoom into these on phase portrait to see a saddle point and a source.

   (c) Idea of linearization: Replace nonlinear system with a linear one near equilibrium points.

2. Linearization of a nonlinear equation: The logistic equation.

$$\frac{dy}{dt} = y(1 - y) = f(y)$$

   (a) Equilibrium points: $f(y) = 0$ at $y = 0, 1$.

   (b) Stability: Draw the phase line to see stability (sink or source). Sketch direction field resulting from the phase portrait. Comparing to the graph of f, we just care if f is increasing or decreasing thru the equilbrium point.

   (c) Linearization approach to stability:

   - Consider $y = 1$. Approximate f by the tangent line thru this equilibrium point. Taylor series approximation gets us there.

$$f(y) = f(1) + f'(1)(y - 1) + \frac{f''(1)}{2!}(y - 1)^2 + \cdots$$
$$f(y) \approx f(1) + f'(1)(y - 1) = 0 + f'(1)(y - 1)$$

- Linearized equation centered at $y = 1$.

$$\frac{d(y-1)}{dt} = f'(1)(y-1) = -(y-1)$$

Note we shifted to be centered at zero to make easier to solve. Solution:

$$y - 1 = Ce^{-t}$$

This solution converges to zeros on the RHS and hence is stable.

- Linearized equation centered at $y = 0$.

$$\frac{d(y-0)}{dt} = f'(0)(y-0) = (y-0) \quad \Rightarrow \quad y = Ce^{t}$$

which is unstable (grows to ∞).

(d) General linearization of a single nonlinear equation centered at Y:

$$\frac{dy}{dt} = f(y) \quad \Rightarrow \quad \frac{d(y-Y)}{dt} = f'(Y)(y-Y)$$

Replace f with a linear approximation centered at the equilibrium. It is an approximation, though exact at the equilibrium which is our focus here. Here, the sign of f' at equilibrium determines stability ($f' > 0$ unstable, $f' < 0$ stable).

(e) This idea extends to 2 dimensions where nonlinear life is much harder otherwise.

3. Linearization of nonlinear system:

$$\frac{dx}{dt} = -2x + 2x^2 = f(x,y), \quad \frac{dy}{dt} = -3x + y + 3x^2 = g(x,y)$$

(a) Equilibrium points: $(x,y) = (0,0), (1,0)$.

(b) Linearization centered at $(1,0)$: Replace f (and g) with a linear function centered at $(1,0)$.

$$f(x,y) = f(1,0) + \frac{\partial f}{\partial x}(x-1) + \frac{\partial f}{\partial y}(y-0) + \cdots$$
$$f(x,y) \approx f(1,0) + \frac{\partial f}{\partial x}(x-1) + \frac{\partial f}{\partial y}(y-0) = \frac{\partial f}{\partial x}(x-1) + \frac{\partial f}{\partial y}(y-0)$$

Then, the linearized equation for x becomes

$$\frac{d(x-1)}{dt} = \frac{\partial f}{\partial x}(x-1) + \frac{\partial f}{\partial y}(y-0).$$

Likewise, for g we have

$$\frac{d(y-0)}{dt} = \frac{\partial f}{\partial x}(x-1) + \frac{\partial f}{\partial y}(y-0).$$

Combining the two, we have the system

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x}(1,0) & \frac{\partial f}{\partial y}(1,0) \\ \frac{\partial g}{\partial x}(1,0) & \frac{\partial g}{\partial y}(1,0) \end{bmatrix} \vec{Y} = A\vec{Y}$$

where

$$\vec{Y} = \begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

Note the shifting to center at $(1,0)$ was necessary to get our linear system matching on both sides.

(c) Definition: The Jacobian matrix is

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

(d) Stability analysis at $(1, 0)$: Differentiating, our system becomes

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \vec{Y} = A\vec{Y}$$

Take our trace-determinant plane analysis from last chapter.

- $\text{tr}(A) = 3 > 0$ and $\det(A) = 2 > 0$ so we are unstable.
- $\text{tr}(A)^2 > 4\det(A)$, so this is a source.

Alternatively eigenvalues / vectors can be found.

- Eigenvalues: 2,1
- Eigenvectors: $\lambda = 2$ gives $\langle 1, 3\rangle$ and $\lambda = 1$ gives $\langle 0, 1\rangle$.

(e) Repeat at second equilibrium point $(0, 0)$.

- Linearization:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix} \vec{Y} = A\vec{Y}$$

- Stability: $\text{tr}(A) = -1 < 0, \det(A) = -2 < 0$ so unstable and also a saddle:
- Eigenvalues are $\lambda = -2, 1$ with respective eigenvectors $\langle 1, 1\rangle$, $\langle 0, 1\rangle$.

(f) Bring back the phase portrait.

4. Example: Competing species model:

$$\frac{dx}{dt} = 2x(1 - x/2) - xy, \quad \frac{dy}{dt} = 3y(1 - y/3) - 2xy$$

(a) These are two separate logistic equations with added xy interaction terms. For the x equation, if y gets much larger than x, the population of x will decrease due to increased competition.

(b) Equilibriums:

- $\frac{dx}{dt} = 0$ when $x = 0$. This gives $y = 0, 3$ in equation 2.
- $\frac{dy}{dt} = 0$ when $y = 0$. This gives $x = 0, 2$ in equation 1.
- Note these are all points on the x and y axis. There must be solutions in the first quadrant (other quadrants make no sense) because of the existence / uniquiness theorem.

$$\begin{cases} x(2 - x - y) = 0 \\ y(3 - y - 2x) = 0 \end{cases}$$

Assuming $x, y \neq 0$,

$$\begin{cases} 2 - x - y = 0 \\ 3 - y - 2x = 0 \end{cases}$$

leading to $x = y = 1$.

(c) $(1, 1)$ is the only equilibrium allowing for coexistence. Let's see its behavior.

- Linearization at $(1, 1)$:

$$\frac{d\vec{Y}}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \vec{Y} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \vec{Y}$$

where $\vec{Y} = \langle x - 1, y - 1\rangle$.

- Eigenvalues: $\lambda = -1 \pm \sqrt{2}$ leads to a saddle point (unstable). Sad fate for one of the two species. Note on eigenvector divides the phase plane to decide which species will survive long term.

(d) Phase portrait.

5. Homework: 1-16

.2 5.2: Qualitative analysis

1. Nonlinear system:
$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y)$$
   (a) Linearization is only accurate near the equilibrium point (both $f = 0$ and $g = 0$) for which it is centered. Everything else in the phase portrait is a mystery.
   (b) Idea of nullclines: Consider $f = 0$ ($x-$nullcline) separate from $g = 0$ (y-nullcline). This will allow us to sharpen our knowledge of solution flow throughout the phase portrait.

2. Example: Competing species model (in first quadrant only)
$$\frac{dx}{dt} = f(x,y) = 2x(1-x/2) - xy, \quad \frac{dy}{dt} = g(x,y) = 3y(1-y/3) - 2xy$$
   (a) $x-$nullcline: Set $\frac{dx}{dt} = f(x,y) = 0$.
$$2x(1-x/2) - xy = x(2-x-y) = 0 \text{ gives } x = 0, y = -x+2$$
   These two curves are the $x-$nullclines.
   (b) $y-$nullcline: Set $\frac{dy}{dt} = g(x,y) = 0$.
$$3y(1-y/3) - 2xy = y(3-y-2x) = 0 \text{ gives } y = 0, y = -2x+3$$
   These two curves are the $y-$ nullclines.
   (c) Graph all 4 nullclines in the $xy-$plane.
      - Note, $x-$nullclines only allow change in y hence the direction vector field has to be vertical. Plug a point into $\frac{dy}{dt} = g(x,y)$ to see if point up (positive slope) or down (negative slope). Up/down direction cannot change unless cross a $y-$nullcline.
      - Likewise for $y-$nullclines.
      - Intersections of these 4 nullclines gives equilibrium points.
      - Note, these nullclines are not eigenvectors and in general will not be straight lines. They are any curve decided by f and g.
   (d) Nullclines divide the first quadrant into 4 regions (A, B, C, D).
      - Let A be the top left region. $x = 0$ is a solution curve. Also, boundaries of A point into A (up and left). Hence any solution starting in A has to stay in A trending up and right to the equilibrium point $(0, 3)$. Note, existence and uniqueness theorem says
      - Similar for region B (lower right).
      - Region C (lower left) is bordered below by solution curves and has three options: enter A, enter B, or approach equilibrium $(1, 1)$ via saddle point eigenvector.
      - Region D is similar to C though from above.

3. Example: Graph the nullclines and complete phase portrait in the first quadrant.
$$\frac{dx}{dt} = x(-x-y+40), \quad \frac{dy}{dt} = y(-x^2-y^2+2500)$$

(a) Nullclines:
   - $\frac{dx}{dt} = 0$ when $x = 0$ or $y = -x + 40$.
   - $\frac{dy}{dt} = 0$ when $y = 0$ or $x^2 + y^2 = 50^2$.
   - Find the horizontal / vertical vector field for each.

(b) Phase portrait:
   - Equilibrium: $(0, 0)$ is a source, $(0, 50)$ is a sink, $(40, 0)$ is a saddle.
   - Sketch and check in pplane.
   - It is clear here that nullclines really give the form of the phase portrait.

4. Homework: 1-14

.3 5.3: Hamiltonian systems

Jeff skip

.4 5.4: Dissipative systems

Jeff skip

.5 5.5: Nonlinear systems in three dimensions

Jeff skip

.6 5.6: Periodic forcing of nonlinear systems and chaos

Jeff skip

Chapter 6: Laplace transforms

.1 6.1: Laplace transforms

1. Here we see a new idea: Laplace transform

   (a) Idea:
   - Transform a differential equation into an algebraic equation ($y'(t)$ in time domain $\to sY(s)$ in frequency domain).
   - Solve algebra equation (for $Y(s)$ in frequency domain).
   - Transform back (to $y(t)$ in time domain).

   (b) Advantage of Laplace transform:
   - Process to make differential equations easier, though inverting the transformation is often the hardest part.
   - Can handle functions we couldn't before (step function (Heaviside function, turn on a signal switch), impulse function (delta function, sudden impulse of explosion or laser)).

2. Laplace transform intution: There are two sources of origin for Laplace trasform.

   (a) Mathematical: Generalized idea of a power series.
   - Given a sequence $\{a_n\}$, power series is of the form

$$\sum_{n=1}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots = f(x)$$

     which may converge to function f for certain x (interval of convergence).

- Examples:

$$\sum x^n = \frac{1}{1-x}, |x| < 1, \quad \sum \frac{1}{n!} x^n = e^x, \text{ for all } x, \quad \text{etc}$$

- Think of a power series as a transformation from discrete sequence a_n to function $f(x)$. Depending on a_n, this transformation may only make sense for some x values. Also, a_n and f have different meaning and usefulness.
- Laplace transform is the same transformation, only for continuous functions $a(t)$ (in place of a_n) to new functions $F(x)$ (we use s to denote signal space as seen below). Our summation becomes an improper integral then.

$$F(x) = \int_0^\infty a(t) x^t \, dx$$

Base x is not so nice here and it is good to change to base e instead.

$$F(x) = \int_0^\infty a(t) e^{\ln(x)t} \, dx$$

Since we need $0 < x < 1$ for convergence, we will have $\ln(x) < 0$. Do a change of variable naming $-s = \ln(x)$ and we get

$$F(s) = \int_0^\infty a(t) e^{-st} \, ds.$$

- The above line is the definition of the Laplace transform. Note that $a(t)$ cannot grow faster than an exponential otherwise we have no hope for convergence.
- Further, thru Euler's formula

$$e^{(ta+bi)t} = e^t(\cos(bt) + i\sin(bt))$$

makes base e ideal for complex numbers and handling signals over time.

(b) Signal processing / control theory:

- Signal processing has a wide range of application (audio, circuits, light, radar, telecommunications, etc) and took off during the tech boom of world war 2.
- Video intuition for Fourier transform (a special case of Laplace transform): `https://www.youtube.com/watch?v=spUNpyF58BY`

3. Definition of Laplace transform:

(a) The Laplace transform $Y(s)$ of function $y(t)$ is defined as

$$Y(s) = \mathcal{L}[y(t)] = \int_0^\infty y(t) e^{-st} \, dt$$

with domain all s for which the improper integral converges. In general s can be a complex number via Euler's formula.

(b) Terminology note. A transform differs from an operator. Operators take objects and map them to objects in a similar space (eg differential operator). Transformations change the space (in our case time t to signal s).

4. Examples: This transform is easy to compute. Just an improper integral from calculus.

(a) $y(t) = 1$ (akin to geometric series above)

$$\begin{aligned}
Y(s) = \mathcal{L}[1] &= \int_0^\infty 1e^{-st}\,dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\,dt \\
&= \lim_{b\to\infty} \frac{1}{-s} e^{-st}\Big|_0^b \\
&= \lim_{b\to\infty} \frac{1}{-s}(e^{-sb} - 1) \\
&= \frac{1}{s}
\end{aligned}$$

Note that we need $s > 0$ for this to work. Make sure to keep track of the domains of the Laplace transforms computed.

(b) $y(t) = e^{at}$ for some constant a since these are our prime targets for this course. Two ways to compute.

- Use formula as above:

$$Y(s) = \mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \cdots$$

  The book does this.

- Use $\mathcal{L}[1]$ result directly. For any function $f(t)$,

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a)$$

  Then, since $\mathcal{L}[1] = \frac{1}{s}$, $s > 0$, we have that

$$\mathcal{L}[e^{at}\cdot 1] = \frac{1}{s-a}, \; s > a.$$

  Multiplication by e^{at} inside the Laplace transform works as a horizontal shift.

(c) Will channel Euler's formula and the use the transform for $\mathcal{L}[e^{at}]$ to derive formulas for sine and cosine. Try on own if ambitious.

5. Laplace transform advantage with differential equations: Derivatives become powers of s and $\mathcal{L}$ is a linear transformation.

(a) Theorem: Laplace transform of derivatives.

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

Proof: This is just integration by parts in disguise.

$$\begin{aligned}
\mathcal{L}\left[\frac{dy}{dt}\right] &= \int_0^\infty \frac{dy}{dt}e^{-st}\,dt \\
&= \lim_{b\to\infty}\left(ye^{-st}\Big|_0^b - \int_0^b -sye^{-st}\,dt\right) \\
&= s\mathcal{L}[y] - y(0)
\end{aligned}$$

Here we need to assume again that y has no more than exponential growth in order for ye^{-st} to converge to zero.

(b) Example: Can iterate for high order derivatives.

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s\mathcal{L}\left[\frac{dy}{dt}\right] - y'(0) = s\left(s\mathcal{L}[y] - y(0)\right) - y'(0) = s^2\mathcal{L}[y] - sy(0) - y'(0)$$

(c) Example: Can now unravel powers of t: t^n.

$$\mathcal{L}[t] = \frac{1}{s}\left(\mathcal{L}[1] + y(0)\right) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \frac{1}{s}\left(\mathcal{L}[2t] + y(0)\right) = \frac{2}{s^3}$$

$$\mathcal{L}[t^n] = \frac{1}{s}\left(\mathcal{L}[2t] + y(0)\right) = \frac{n!}{s^{n+1}}$$

(d) Theorem: Linearity of Laplace transform.

$$\mathcal{L}\left[f(t) + g(t)\right] = \mathcal{L}\left[f(t)\right] + \mathcal{L}\left[g(t)\right], \quad \mathcal{L}\left[cf(t)\right] = c\mathcal{L}\left[f(t)\right]$$

Proof: These are just properties of the definite integral.

$$\begin{aligned}\mathcal{L}\left[f + g\right] &= \int_0^\infty (f+g)e^{-st}\ dt \\ &= \int_0^\infty fe^{-st}\ dt + \int_0^\infty ge^{-st}\ dt \\ &= \mathcal{L}\left[f\right] + \mathcal{L}\left[g\right]\end{aligned}$$

Likewise for the second result.

(e) Expect to be able to prove these in an exam.

6. Example: We now see why Laplace transform will be useful for linear differential equations.

$$\frac{dy}{dt} + 5y = e^{-t}, \quad y(0) = 2$$

(a) Previous techniques:

- Direction field. See which curve is ours from the initial condition.
- Nonhomogenous equation. Solution is of the form

$$y = y_h + y_p$$

where y_h is the general solution to the homogeneous equation and y_p is a particular solution found via integrating factor.

$$\frac{dy}{dt} + 5y = 0 \quad \to \quad y_h = Ce^{-5t}$$

$$\frac{dy}{dt} + 5y = e^{-t} \quad \to \quad \frac{d}{dt}(e^{5t}y) = e^{5t}e^{-t} \quad \to \quad y_p = e^{-5t}\int e^{4t}\ dt = \frac{1}{4}e^{-t}$$

Using the initial condition, our solution is

$$y = y_h + y_p = Ce^{-5t} + \frac{1}{4}e^{-t} = \frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}$$

(b) Apply Laplace transform to both sides of the equation and denote $Y(s) = \mathcal{L}[y]$.

$$\begin{aligned}
\mathcal{L}\left[\frac{dy}{dt} + 5y\right] &= \mathcal{L}\left[e^{-t}\right] \\
\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}\left[y\right] &= \mathcal{L}\left[e^{-t}\right] \\
s\mathcal{L}\left[y\right] - y(0) + 5\mathcal{L}\left[y\right] &= \frac{1}{s+1}, \quad s > -1 \\
(s+5)\mathcal{L}\left[y\right] - 2 &= \frac{1}{s+1}, \quad s > -1 \\
(s+5)\mathcal{L}\left[y\right] &= 2 + \frac{1}{s+1}, \quad s > -1 \\
\mathcal{L}\left[y\right] &= \frac{2s+3}{(s+5)(s+1)}, \quad s > -1 \\
Y(s) &= \frac{2s+3}{(s+5)(s+1)}, \quad s > -1
\end{aligned}$$

Notice we used no calculus here, just above two theorems.

(c) We found the transform $Y(s)$, now how to recover $y(t)$? Invert the Laplace transform.

7. Inverse Laplace transform:

(a) You can show (though not easily) that the Laplace transform is invertible. That is for each transform $F(s)$, there is a unique $f(t)$ which it maps back to. There is a formula for this reverse mapping:

$$f(t) = \mathcal{L}^{-1}[F] = \int_0^{\infty} e^{st}F(s)\, ds$$

which just reverses the exponential multiplication. Lucky for us we have no need for this formula.

(b) Note: Our favorite functions in this class are exponentials, sine/cosine, powers of t. Derivatives and integrals of these remain on the list. So if we know our basic Laplace transform formulas, we can just use them in reverse.

(c) Theorem: Because $\mathcal{L}$ is a linear operator, $\mathcal{L}^{-1}$ is also linear. That is,

$$\mathcal{L}^{-1}\left[F(s) + G(s)\right] = \mathcal{L}^{-1}\left[F(s)\right] + \mathcal{L}^{-1}\left[G(s)\right], \quad \mathcal{L}^{-1}\left[cF(s)\right] = c\mathcal{L}^{-1}\left[F(s)\right]$$

(d) Example: Find the inverse Laplace transform of

$$Y(s) = \frac{2s+3}{(s+5)(s+1)}.$$

- Note, this function reminds us of

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$$

- Use partial fraction decomposition to separate fraction into two parts.

$$Y(s) = \frac{2s+3}{(s+5)(s+1)} = \frac{A}{s+1} + \frac{B}{s+5} = \frac{1/4}{s+1} + \frac{7/4}{s+5}$$

Then,

$$y(t) = \mathcal{L}^{-1}(Y(S)) = \mathcal{L}^{-1}\left(\frac{1/4}{s+1} + \frac{7/4}{s+5}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{7}{4}\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = \frac{1}{4}e^{-t} + \frac{7}{4}e^{-5t}$$

This finishes the above example, same solution as before.

- Note the use of linearity here.

8. Homework: 1-25, 27

.2 6.2: Discontinuous functions

1. Advantage of Laplace transform:
   (a) We have seen how Laplace transform can turn a differential equation into an algebra problem. All calculus is transferred into transform formulas and derivative / linearity theorems.
   (b) Here we will see a second advantage: ability to handle difficult functions
      - step functions / turn on a switch or new disease (this section)
      - delta (impulse) functions / shock or sudden impact (later)

2. Heaviside function:
   (a) Basic step function.

$$H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

   (b) The switch can happen at any location $a > 0$ if we are considering $t \geq 0$.

$$H_a(t) = H(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$

   Graph and see. Looks like one side is heavier!

   (c) Can even switch on and off as the unit box function:

$$H_{ab}(t) = H_a(t) - H_b(t)$$

   (d) Multiplication by the Heaviside function allows you to switch on that function at location a. Graph:

$$y = f(t) = t^2 \quad \text{vs} \quad y = H_a(t)f(t)$$

   (e) Notation: Text uses u_a (unit step function) in place of H_a (Heaviside function).
   (f) Example: Forcing term only applied after $t = 2$.

$$\frac{dy}{dt} = -y + H_2(t)e^{-2(t-2)}, \quad y(0) = 1$$

   Think of this as two differential equations:

$$\frac{dy}{dt} + y = \begin{cases} 0, & t < 2 \\ e^{-2(t-2)}, & t \geq 2 \end{cases}$$

   where function e^{-2t} is switched on at $t = 2$.

3. Laplace transform of $H_a(t)$.
   (a) Note, $H(t) = 1$ for $t \geq 0$ and we already have $\mathcal{L}[1]$ from last time. So,

$$\mathcal{L}[H(t)] = \frac{1}{s}, \quad s > 0$$

(b) Use $\mathcal{L}[H(t)]$ to compute $\mathcal{L}[H_a(t)]$.

$$\begin{aligned}
\mathcal{L}[H_a(t)] &= \int_0^\infty H_a(t)e^{-st}\,dt \\
&= \int_a^\infty H(t-a)e^{-st}\,dt \\
&= \int_0^\infty H(\tilde{t})e^{-s(\tilde{t}+a)}\,d\tilde{t} \\
&= e^{-sa}\int_0^\infty H(\tilde{t})e^{-s\tilde{t}}\,d\tilde{t} \\
&= e^{-sa}\mathcal{L}[H(t)] \\
&= e^{-sa}\frac{1}{s} = \frac{e^{-sa}}{s}
\end{aligned}$$

(c) Theorem (shift theorem): The above calculation generalizes. If $\mathcal{L}[f] = F(s)$, then

$$\mathcal{L}[H_a(t)f(t-a)] = e^{-as}F(s)$$

where $H_a(t)f(t-a)$ turns on f (for $t > 0$) starting at $t = a$. Draw picture to illustrate.

(d) Formula for $\mathcal{L}[f(t-a)]$ in terms of $\mathcal{L}[f(t)]$:

- Note above $\mathcal{L}[1] = \mathcal{L}[H] = \frac{1}{s}$. This poses a problem for $\mathcal{L}^{-1}$. Which one is it? Kind of the same since $t \geq 0$.
- Gets muddy when you shift general functions f to give $f(t-a)$. Information before zero was gone, now it is there. Our formula

$$\mathcal{L}[H_a(t)f(t-a)] = e^{-as}F(s)$$

says erase the information before zero.

4. Example: Forcing term only applied after $t = 2$.

$$\frac{dy}{dt} = -y + H_2(t)e^{-2(t-2)}, \quad y(0) = 1$$

(a) Apply Laplace transform. Denote $\mathcal{L}[y] = Y(s)$.

$$\begin{aligned}
\mathcal{L}[\frac{dy}{dt}] &= \mathcal{L}[-y + H_2(t)e^{-2(t-2)}] \\
s\mathcal{L}[y] - y(0) &= -\mathcal{L}[y] + \mathcal{L}[H_2(t)e^{-2(t-2)}] \\
sY(s) - 1 &= -Y(s) + e^{-2s}\mathcal{L}[e^{-2t}] \\
sY(s) - 1 &= -Y(s) + e^{-2s}\frac{1}{s+2} \\
(s+1)Y(s) &= 1 + e^{-2s}\frac{1}{s+2} \\
Y(s) &= \frac{1}{s+1} + \frac{e^{-2s}}{(s+1)(s+2)}
\end{aligned}$$

(b) Invert the Laplace transform.

$$\begin{aligned}
\mathcal{L}[y] &= \frac{1}{s+1} + \frac{e^{-2s}}{(s+1)(s+2)} \\
y &= \mathcal{L}^{-1}[\frac{1}{s+1}] + \mathcal{L}^{-1}[\frac{e^{-2s}}{(s+1)(s+2)}]
\end{aligned}$$

(c) Partial fraction decomposition:

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{-1}{s+2}$$

(d) Finish inverting using shift theorem.

$$\begin{aligned} y &= \mathcal{L}^{-1}[\frac{1}{s+1}] + \mathcal{L}^{-1}[\frac{e^{-2s}}{s+1}] - \mathcal{L}^{-1}[\frac{e^{-2s}}{s+2}] \\ &= e^{-t} + H_2(t)e^{-2(t-1)} - H_2(t)e^{-2(t-2)} \\ &= e^{-t} + H_2(t)(e^{-2(t-1)} - e^{-2(t-2)}) \end{aligned}$$

(e) Note, our piecewise differential equation gave a piecewise solution.

$$y(t) = \begin{cases} e^{-t}, & t < 2 \\ e^{-t} + e^{-2(t-1)} - e^{-2(t-2)}, & t \geq 2 \end{cases}$$

5. Homework: 1-17

.3 6.3: Second-order equations

1. Summary of Laplace transform results so far:
   (a) Definition: $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\, dt$
   (b) $\mathcal{L}[1] = \frac{1}{s}, s > 0$
   (c) $\mathcal{L}[e^{at}]\frac{1}{s-a}, s > a$
   (d) $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, s > 0$
   (e) $\mathcal{L}[dy/dt] = s\mathcal{L}[y] - y(0)$, $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$ (differentiation)
   (f) $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g], \mathcal{L}[cf] = c\mathcal{L}[f]$ (linearity)
   (g) $\mathcal{L}[H(t)] = \frac{1}{s}, s > 0$
   (h) $\mathcal{L}[H_a(t)] = \frac{e^{-sa}}{s}, s > 0$
   (i) $\mathcal{L}[H_a(t)f(t-a)] = e^{-as}\mathcal{L}[f(t)]$ (t-axis shift)

2. We have seen advantages of the Laplace transform (simpler process, handle new kinds of functions), but here we will see again the drawback of having to invert the Laplace transform.

3. Second order linear equation with forcing:

$$m\frac{dy}{dt^2} + b\frac{dy}{dt} + ky = f(t) \quad \Rightarrow \quad \frac{dy}{dt^2} + p\frac{dy}{dt} + qy = f(t)$$

where m is mass, b is damping, k is spring stiffness, f is applied external force.

   (a) Solutions include terms like $\sin(\omega t), \cos(\omega t), e^{at}\sin(\omega t), e^{at}\cos(\omega t)$. We need these Laplace transforms.
   (b) Note, all of the four are of the form

$$e^{(a+ib)t} = e^{at}(\cos(bt) + i\sin(bt)$$

   and we know $\mathcal{L}[e^{at}]$ which holds even for complex numbers a.

(c) Laplace transform of $\cos(bt)$. Use the fact that

$$\cos(bt) = \frac{e^{ibt} + e^{-bit}}{2}$$

Then,

$$\mathcal{L}[\cos(bt)] = \frac{1}{2}\left(\mathcal{L}[e^{ibt}] + \mathcal{L}[e^{-bit}]\right) = \frac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right) = \frac{1}{2}\left(\frac{2s}{s^2+b^2}\right) = \frac{s}{s^2+b^2}$$

(d) Laplace transform of $\sin(bt)$. Similar to $\cos(bt)$,

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$$

(e) Multiplications of the form $e^{at}f(t)$ not surprisingly can be done straight from the definition of Laplace transform. Assume $\mathcal{L}[f] = F(s)$. Then,

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt = F(s-a).$$

Theorem: ($s-$axis shift) If $\mathcal{L}[f] = F(s)$. Then,

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

(f) Example: Compute the inverse Laplace transform of

$$Y(s) = \frac{s+1}{s^2+6s+10}.$$

Need to complete the square of the denominator as $(s-\alpha)^2 + b^2$ to see the shift and sine / cosine transform.

$$Y(s) = \frac{s+1}{s^2+6s+10} = \frac{s+1}{(s+3)^2+1} = \frac{s+3}{(s+3)^2+1} - 2\frac{1}{(s+3)^2+1}$$

Then,

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2+1}\right] = e^{-3t}\cos(t) - 2e^{-3t}\sin(t)$$

Note we can now handle any proper rational function $\frac{p(s)}{q(s)}$ since can always factor polynomial q into a product of linear and irreducible quadratic factors (fundamental theorem of algebra).

4. Example:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 2e^t, \quad y(0) = 3, y'(0) = 1$$

(a) Laplace transform:

$$\begin{aligned}
\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - 4\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] &= 2\mathcal{L}[e^t] \\
s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 5Y(s) &= 2\frac{1}{s-1} \\
s^2Y(s) - 3s - 1 - 4sY(s) + 12 + 5Y(s) &= \frac{2}{s-1}
\end{aligned}$$

(b) Solve for $Y(s)$:

$$(s^2 - 4s + 5)Y(s) = 3s - 11 + \frac{2}{s-1}$$
$$Y(s) = \frac{3s - 11}{(s-2)^2 + 1} + \frac{2}{(s-1)((s-2)^2+1)}$$

(c) Partial fraction decomposition:

$$Y(s) = \frac{1}{s-1} + \frac{2s-8}{(s-2)^2+1}$$

(d) Inverse Laplace transform:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{2s-8}{(s-2)^2+1}\right] \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 2\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+1}\right] - 4\mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+1}\right] \\ &= e^t + 2e^{2t}\cos(t) - 4e^{2t}\sin(t) \end{aligned}$$

Note no need to solve for unknown constants for initial conditions via system of equations.

5. Homework: 1-31

.4 6.4: Delta functions and impulse forcing

1. We have seen how Laplace transform allows us to solve differential equations with new functions. Here we add to that list.
   (a) Discontinuous functions, turning on a switch (two sections ago)
   (b) Delta function, sudden impulse (this section)
2. Impulse forcing:
   (a) Example:
   $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = f(t), \quad y(0) = 1, y'(0) = 1$$
   This is a spring mass equation with damping coefficient 2 and spring stiffness 5. f is an external force. Here we want to understand what happens if f is a sudden impact, a kick to the mass on the spring at a certain time t.
   (b) Unit impulse at zero: Approach this idea over smaller and smaller intervals.
   - $f(t) = F$ on the interval from $[a, b]$. A unit force is such that
   $$\int_{-\infty}^{\infty} f(t)\ dt = F(b-a) = 1$$
   - At zero: $f(t) = \frac{1}{h}$ over the interval $[0, h]$. Then
   $$f(t) = \frac{1}{h}\left[H(t) - H(t-h)\right] \quad \text{where } H \text{ is the Heaviside function}$$
   and
   $$\int_{-\infty}^{\infty} f(t)\ dt = \frac{1}{h}h = 1$$
   The smaller h is, the taller and skinnier this step function is.

- Taking the limit as $h \to 0$
$$\lim_{h\to 0} \frac{1}{h}\left[H(t) - H(t-h)\right]$$
gives the *Dirac delta function*.
$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$
such that
$$\int_{-\infty}^{\infty} \delta(t)\; dt = 1$$
- $\delta(t)$ is not really a function like we are used to (taking ∞ as an output makes no sense), but it is useful when the Laplace transform comes in.
- Also note the appearance of the difference quotient leading to the fact that
$$H'(t) = \delta(t).$$

(c) Laplace transform of the Delta function:
$$\mathcal{L}\left[\frac{1}{h}\left[H(t) - H(t-h)\right]\right] = \frac{1}{h}\left[\frac{1}{s} - \frac{e^{-hs}}{s}\right] = \frac{1}{s} - \frac{e^{-hs}}{hs} \to 1$$
via lHospital's rule. Then,
$$\mathcal{L}\left[\delta(t)\right] = 1$$
and we see a hint why the Delta function is important.

(d) Modified forcing:
- Unit forcing at time a: $\delta_a(t) = \delta(t-a)$ (just a horizontal shift so we know the Laplace transform)
$$\mathcal{L}[\delta_a(t)] = e^{-as}$$
- Forcing of strength A: $A\delta(t)$ (again a bit strange with the infinite value at $t = 0$, but the Laplace transform shows the strength change)
$$\mathcal{L}[A\delta(t)] = A$$

3. Example: This basic example illustrates the idea of impulse force.
$$\frac{d^2y}{dt^2} + y = A\delta_{\pi/2}(t), \quad y(0) = 1, y'(0) = 0$$
Spring-mass with no damping. Impulse force of strength A is applied at time $\pi/2$, the equilibrium point.

(a) Laplace transform:
$$s^2Y - s + Y = Ae^{-\frac{\pi}{2}s} \quad \Rightarrow \quad Y = \frac{s}{s^2+1} + \frac{Ae^{-\frac{\pi}{2}s}}{s^2+1}$$

(b) Inverse Laplace transform:
$$y(t) = \cos(t) + H(t-\pi/2)A\sin(t-\pi/2) = \begin{cases} \cos(t), & 0 \leq t \leq \frac{\pi}{2} \\ (1-A)\cos(t), & t \geq \frac{\pi}{2} \end{cases}$$

(c) Check that the final solution (displacement) is continuous as it should be.

(d) Graph in Desmos with slider for A. `https://www.desmos.com/calculator/7saayio0fw`

4.
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \delta_3(t), \quad y(0) = 1, y'(0) = 1$$

(a) Laplace transform:

$$s^2Y - sy(0) - y'(0) + 2sY - y(0) + 5Y = e^{-3s} \quad \to \quad Y = \frac{2 + s + e^{-3s}}{s^2 + 2s + 5}$$

(b) Inverse Laplace transform:

$$y = \mathcal{L}^{-1}\left[\frac{2 + s + e^{-3s}}{(s+1)^2 + 4}\right] = \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4} + \frac{1}{2}\frac{2e^{-3s}}{(s+1)^2+4}\right]$$

and then
$$y = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) + \frac{1}{2}H_3(t)e^{-(t-3)}\sin(2(t-3))$$

which is more cleanly written as

$$y = \begin{cases} e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t), & 0 \le t \le 3 \\ e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) + \frac{1}{2}e^{-(t-3)}\sin(2(t-3)) & t \ge 3 \end{cases}$$

5. Homework: 1-7

.5 6.5: Convolutions

1. Here we develop a new idea (convolution of two functions $f * g$) which is useful in two ways:
   (a) To help analyze inverse Laplace transforms of products.
   $$\mathcal{L}^{-1}[F(s)G(s)]$$
   (b) To derive a formula for any second order linear equation with forcing.
   $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t)$$
   (c) Bonus: Idea of convolution applies to a much more general setting than simply a tool for Laplace transform.

2. Idea: Laplace transform of products are difficult.
   (a) $\mathcal{L}[f(t)g(t)]$ has no nice formula in general.
   (b) We will derive a nice formula for $\mathcal{L}^{-1}[F(s)G(s)]$ as
   $$FG = \int_0^\infty (f * g)e^{-st}\, dt \quad \Rightarrow \quad \mathcal{L}^{-1}[FG] = f * g.$$
   (c) Motivation: Recall we motivated Laplace transform as a continuous analog of a power series.
   $$p(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad F(s) = \int_0^{infty} f(t)e^{-st}\, dt$$
   where we rewrote x^t as e^{-st}. For two power series
   $$p(x) = \sum a_n x^n, \quad q(x) = \sum b_n x^n$$

there is no nice way to find

$$\sum (a_n b_n) x^n$$

in terms of general p, q. Though, there is a way to find

$$p(x)q(x) = \left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x_n$$

called the Cauchy product of a power series. Its name is a *discrete convolution* and the formula is

$$c_n = \sum_{i=0}^{n} a_{n-i} b_i.$$

Offer bonus bounty to derive this discrete formula for convolution.

3. Convolution formula: This is a strange formula at first glance, but the same can be said for the Laplace transform.

$$f(t) * g(t) = \int_0^t f(t-u)g(u)\ du$$

is the new function of t such that

$$\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g] = F(s)G(s)$$

(will show this soon) and is useful when calculating the inverse transform of products

$$\mathcal{L}^{-1}[FG].$$

(a) Result: The convolution acts like a product in the sense that

$$f(t) * g(t) = g(t) * f(t).$$

This result is not clear at first glance thru the formula (need a substitution of $v = t - u$), but if you trust the Laplace transform result added to the uniqueness of the transform,

$$\mathcal{L}[f * g] = F(s)G(s) = G(s)F(s) = \mathcal{L}[g * f]$$

we see that this is just regular real number multiplication which we know is communative.

(b) Example: Let $f(t) = t^2$ and $g(t) = t$.

$$t * t^2 = \int_0^t (t-u)u^2\ du = \int_0^t tu^2 - u^3\ du = t\frac{u^3}{3} - \frac{u^4}{4}\Big|_0^t = \frac{t^4}{12}$$

Note

$$\mathcal{L}[t * t^2] = \mathcal{L}[t^4/12] = \frac{1}{12}\frac{4!}{s^5} = \frac{2}{s^5}.$$

But,

$$\mathcal{L}[t]\mathcal{L}[t^2] = \frac{1}{s^2}\frac{2}{s^3} = \frac{2}{s^5}$$

and our formula holds at least in this case.

(c) Example: This product is strange though.

$$1 * f(t) = \int_0^t f(u)\ du$$

Convolution with 1 is the antiderivative of f.

(d) Example: The convolution identity is $\delta(t)$.

$$\delta * f(t) = 1$$

4. Convolution formula derivation:

   (a) Denote $F = \mathcal{L}[f]$ and $G = \mathcal{L}[g]$. Then we want a formula for $\mathcal{L}^{-1}[FG]$.

$$
\begin{aligned}
F(s)G(s) &= \left[\int_0^\infty f(\tau)e^{-s\tau}\, d\tau\right]\left[\int_0^\infty g(u)e^{-su}\, du\right] \\
&= \int_0^\infty \int_0^\infty f(\tau)g(u)e^{-s(\tau+u)}\, du\, d\tau
\end{aligned}
$$

   This is a double integral over the first quadrant of the $u\tau$-plane. Again, our goal is a single Laplace transform of the mold

$$
F(s)G(S) = \int_0^\infty h(t)e^{-st}\, dt.
$$

   Do a change of variable calling $t = \tau + u$, $u = u$. Then,

$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty \int_0^\infty f(\tau)g(u)e^{-s(\tau+u)}\, du\, d\tau \\
&= \int_0^\infty \int_0^t f(t-u)g(u)e^{-st}\left|\frac{\partial(u,\tau)}{\partial(u,t)}\right|\, du\, dt \\
&= \int_0^\infty \int_0^t f(t-u)g(u)\, du\; e^{-st}\, dt \\
&= \int_0^\infty f(t) * g(t)\; e^{-st}\, dt
\end{aligned}
$$

   Note the Jacobian is computed as the following determinant.

$$
\left|\frac{\partial(u,\tau)}{\partial(u,t)}\right| = \left|\begin{array}{cc} \partial u/\partial u & \partial u/\partial t \\ \partial\tau/\partial u & \partial\tau/\partial t \end{array}\right| = \left|\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}\right| = 1
$$

   begin that $u = u$ and $\tau = t - u$. Also the first quadrant of the $u\tau$ plane $(0 < u < \infty, 0 < \tau < \infty)$ is achieved by u, t by dragging lines of slope negative 1 ($t = \tau + u$, $0 < u < t$) across the first quadrant in a diagonal fashion $(0 < t < \infty)$.

5. Example: It turns out this convolution formula is not all too practical. Compute the inverse Laplace transform of

$$
F(s)\frac{3s}{(s^2+1)(s^2+4)}.
$$

   (a) Convolution solution:

$$
F(s)\frac{3s}{(s^2+1)(s^2+4)} = 3\frac{s}{s^2+1}\frac{s^2+4}{=}3G(s)H(s)
$$

   Noting that $g(t) = L^{-1}[G] = \sin(t)$ $h(t) = L^{-1}[H] = \cos(2t)$, then

$$
\mathcal{L}^{-1}[F] = 3g(t) * h(t) = 3\int_0^t g(t-u)h(u)\, du = 3\int_0^t \sin(t-u)\cos(2u)\, du
$$

   which requires the a bit of trig integration.

   (b) Solution by partial fractions:

$$
f(t) = \cos(t) - \cos(2t)
$$

   so this must match the above definite integral (convolution $g * f$).

6. Convolution solution to general second order equation:

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t), \quad y(0) = y'(0) = 0$$

where f is a general forcing function.

(a) Take Laplace transform:

$$s^2Y + psY + qY = F \quad \Rightarrow \quad Y = F\frac{1}{s^2+ps+q} = FW$$

(b) $\mathcal{L}[y] = Y$ is a product $F(s)W(s)$ making y a convolution of the forcing function f with w.

$$y = f * w = \int_0^t f(t-u)w(u)\ du$$

How to interpret this?

(c) What is w?

- $W(s) = \frac{1}{s^2+ps+q}$ is called the *transfer function* for the system.
- $w(t)$ is the *weight function* for the system.
- How to interpret w? Note that $\mathcal{L}[\delta] = 1$. Then,

$$y'' + py' + qy = \delta, \quad y(0) = y'(0) = 0 \quad \Rightarrow \quad Y = \frac{1}{s^2+ps+q} = W(s)$$

giving $y = w$. We kick the mass starting at rest from equilibrium with unit impulse. The solution here is the response to this unit kick.

- Then,

$$y = f * w = \int_0^t f(t-u)w(u)\ du$$

is the combination of all these unit kicks as governed by f. Kick, kick, kick until accumulate solution y.

7. Bonus convolution idea:

(a) $f * g$ tells us to what degree f and g overlap across the domain of t.

(b) How the shape of one is modified by the other.

(c) `https://en.wikipedia.org/wiki/Convolution`

8. Homework: 1-9

.6 6.6: The qualitative theory of Laplace transform