Math 371
Exam 2, May 1, 2015

Name: Solution

> You have until the end of the hour to complete this exam. Show all work, justify your solutions completely, simplify as much as possible. The only materials you should have on your desk are this exam and a pencil. If you have any questions, be sure to ask for clarification.

1. (10 points) Prove that a polynomial interpolant of degree at most n through the $(n+1)$ points $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$ must be unique.

Assume there are 2 such interpolants, $P(x)$ & $Q(x)$. Define $R(x) = P(x) - Q(x)$. Then R is a degree n polynomial as well. But

$$R(x_i) = P(x_i) - Q(x_i) = f(x_i) - f(x_i) = 0, \quad i = 0, 1, \ldots, n.$$

So R has $n+1$ total zeros. The only way this can be is if $R(x) \equiv 0$ (is the zero function). Then, $P(x) = Q(x)$ & we must have uniqueness.

2. (10 points) Do there exist numbers a, b, c and d such that the function

$$S(x) = \begin{cases} ax^3 + x^2 + cx, & -1 \le x \le 0 \\ bx^3 + x^2 + dx, & 0 \le x \le 1 \end{cases}$$

is a natural cubic spline which agrees with $f(x) = |x|$ at $x = -1, 0, 1$?

$$S'(x) = \begin{cases} 3ax^2 + 2x + c, & -1 \le x \le 0 \\ 3bx^2 + 2x + d, & 0 \le x \le 1 \end{cases}, \quad S''(x) = \begin{cases} 6ax + 2, & -1 \le x \le 0 \\ 6bx + 2, & 0 \le x \le 1. \end{cases}$$

$S''(-1) = 0 \Rightarrow -6a + 2 = 0 \Rightarrow \boxed{a = 1/3}$

$S''(1) = 0 \Rightarrow 6b + 2 = 0 \Rightarrow \boxed{b = -1/3}$

$S'(0^-) = S'(0^+) \Rightarrow \boxed{c = d}$

$S(-1) = 1 \Rightarrow -a + 1 - c = 1 \Rightarrow \boxed{c = -1/3}$

$S(0^-) = S(0^+) = 0$

$S(1) = 1 \Rightarrow b + 1 + d = 1 \Rightarrow \boxed{d = 1/3}$

So $c \neq d$. Therefore $S(x)$ cannot be a natural cubic spline.

3. (a) (10 points) Suppose $f(x)$ is a continuous function on the interval $[a, b]$. Use the first degree interpolating polynomial of f to derive the trapezoidal rule (for one subinterval).

$$P(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}.$$

$$\int_a^b f(x)\,dx \approx \int_a^b P(x)\,dx = \frac{f(a)}{a-b}\int_a^b (x-b)\,dx + \frac{f(b)}{b-a}\int_a^b (x-a)\,dx$$

$$= \frac{f(a)}{a-b}\left.\frac{(x-b)^2}{2}\right|_a^b + \frac{f(b)}{b-a}\left.\frac{(x-a)^2}{2}\right|_a^b$$

$$= \frac{f(a)}{a-b}\left(0 - \frac{(a-b)^2}{2}\right) + \frac{f(b)}{b-a}\left(\frac{(b-a)^2}{2} - 0\right) = \frac{b-a}{2}\left(f(a) + f(b)\right)$$

$$\therefore \int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right).$$

(b) (5 points) For the following function values, compute the n step *composite* trapezoidal rule approximation to $\int_0^{16} f(x)\,dx$ using $n = 1, 2,$ and 4 subintervals.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(x)$	5	20	14	9	0	10	14	17	17	12	7	4	3	3	11	16	5

$\boxed{n=1}$ $\int_0^{16} f(x)\,dx \approx \frac{16-0}{2}\left(f(0) + f(16)\right) = 8(5+5) = 80$

$\boxed{n=2}$ $h = \frac{16-0}{2} = 8$

$$\int_0^{16} f(x)\,dx \approx \frac{h}{2}\left(f(0) + 2f(8) + f(16)\right) = 4(5 + 2\cdot 17 + 5) = 4\cdot 44$$

$= 176$

$\boxed{n=4}$ $h = \frac{16-0}{4} = 4$

$$\int_0^{16} f(x)\,dx \approx \frac{h}{2}\left(f(0) + 2f(4) + 2f(8) + 2f(12) + f(16)\right)$$

$$= 2(5 + 2\cdot 0 + 2\cdot 17 + 2\cdot 3 + 5)$$

$$= 2\cdot 50 = 100.$$

4. Suppose function f generates the set of points $S = \{(1,5),(4,11),(5,1),(7,35)\}$.

(a) (10 points) Use Newton form to construct the minimum degree polynomial P which interpolates the points in S.

x	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
1	5	$\frac{11-5}{4-1}=2$	$\frac{-10-2}{5-1}=-3$	$\frac{9+3}{7-1}=2$
4	11	$\frac{1-11}{5-4}=-10$	$\frac{17+10}{7-4}=9$	
5	1	$\frac{35-1}{7-5}=17$		
7	35			

$$P(x) = f[1] + f[1,4](x-1) + f[1,4,5](x-1)(x-4) + f[1,4,5,7](x-1)(x-4)(x-5)$$

$$= 5 + 2(x-1) + (-3)(x-1)(x-4) + 2(x-1)(x-4)(x-5)$$

(b) (5 points) Given that $|f^{(4)}(x)| \leq 4$ on $[1,7]$, what can you say about error for $P(2)$?

$$|f(2) - P(2)| \leq \frac{|f^{(4)}(\xi)|}{4!}\left|(2-1)(2-4)(2-5)(2-7)\right| , \quad \text{some } \xi \in [1,7].$$

$$\leq \frac{4}{4!}\left|1(-2)(-3)(-5)\right| = 5$$

5. (a) (7 points) Use the method of undetermined coefficients to find constants A_0, A_1, A_2 such that the following quadrature rule is exact for all degree 2 polynomials.

$$\int_{-1}^{1} f(x)\,dx = A_0 f(-1) + A_1 f(0) + A_2 f(1)$$

Show exact for basis $1, x, x^2$.

$$\int_{-1}^{1} 1\,dx = \boxed{2 = A_0 + A_1 + A_2}$$

$$\int_{-1}^{1} x\,dx = \boxed{0 = -A_0 + A_2}$$

$$\int_{-1}^{1} x^2\,dx = \boxed{\frac{2}{3} = A_0 + A_2}$$

$\Rightarrow A_0 = A_2 = \frac{1}{3}, \quad A_1 = 2 - \frac{2}{3} = \frac{4}{3}$

So, $\int_{-1}^{1} f(x)\,dx = \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1)$

$= \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right)$

Simpson's Rule!

(b) (3 points) Show that the rule found in (a) is actually exact for all degree 3 polynomials, but not for degree 4 polynomials.

$\int_{-1}^{1} x^3\,dx = 0 = \frac{1}{3}(-1) + 0 + \frac{1}{3}$ ✓, $\quad \int_{-1}^{1} x^4\,dx = \frac{2}{5} \neq \frac{1}{3} + 0 + \frac{1}{3}$ ✗

6. (a) (10 points) Recall that the error for the trapezoidal rule on one subinterval is given by

$$\int_a^b f(x)\,dx - T_1(a,b) = -\frac{(b-a)^3}{12} f''(\xi)$$

where $T_1(a,b)$ denotes the trapezoidal rule you derived in part (a). Show that the error for the n step composite trapezoidal rule approximation is given by

$$\int_a^b f(x)\,dx - T_n(a,b) = -\frac{(b-a)}{12} f''(\eta) h^2$$

where $T_n(a,b)$ denotes the n step composite trapezoidal rule and h is the size of each subinterval.

Denote $h = \frac{b-a}{n}$ at subintervals $[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$, $x_i = a + ih$.

Then, $\displaystyle\int_a^b f(x)\,dx - T_n(a,b) = \sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right)\right)$

$$= \sum_{i=0}^{n-1} -\frac{h^3}{12} f''(\xi_i), \quad \xi_i \in [x_i, x_{i+1}]$$

$$= -\frac{h^2}{12}\left(\frac{b-a}{n}\right)\sum_{i=0}^{n-1} f''(\xi_i)$$

$$= -\frac{h^2}{12}\left(\frac{b-a}{n}\right)\cdot n f''(\xi), \quad \xi \in [a,b]$$ (By the intermediate value theorem)

$$= -\frac{h^2}{12}(b-a) f''(\xi).$$

(b) (10 points) How many subintervals are required to compute the composite Trapezoidal rule approximation for $\int_4^9 \frac{1}{x}\,dx$ to within 10^{-6}?

$f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$.

Then, $\displaystyle\max_{4 \le x \le 9} |f''(x)| = \frac{2}{4^3}$. We want n such that

$$\left|-\frac{(9-4)}{12} f''(\eta)(h^2)\right| \le \frac{1}{10^6} \Rightarrow \frac{5}{12}\cdot\frac{2}{4^3}\cdot\left(\frac{9-4}{n}\right)^2 \le \frac{1}{10^6}$$

$$\Rightarrow \frac{2\cdot 5^3}{4^4}\cdot\frac{1}{n^2} \le \frac{1}{10^6} \Rightarrow n \ge \sqrt{\frac{5^3 \cdot 10^6 \cdot 2}{4^4}} = \frac{5\cdot 10^3}{4^2}\sqrt{10}\sqrt{2} = \frac{5^3 \cdot 2\sqrt{2}}{4} ...$$

$$= \frac{5^3}{2}\sqrt{2}$$