

# Calculus III Notes

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## Fun stuff

## Chapter 12 Vectors and the geometry of space

### .1 12.1 Three-dimensional coordinate systems

#### 1. Rectangular (Cartesian) coordinate system

##### (a) 2D:

- Basics:  $xy$ -plane, orthogonal axis with standard orientation, 4 quadrants, coordinates of point  $(x, y)$ , projection onto axis, notation  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}$ .
- Distance between two points, Pythagoras, distance formula.

##### (b) 3D:

- $xyz$ -space, orthogonal axis with standard orientation, 8 octants, coordinates of point  $(x, y, z)$ , projection onto  $xy$ -plane  $(xz, yz)$ . Projection onto axis, notation  $\mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ .
- Distance between two points, Pythagoras twice, distance formula, proof in text.

#### 2. Graphs of equations

##### (a) 2D:

- Point
- Lines: Vertical, horizontal, sloped
- Quadratics, polynomials
- Circles from distance formula, standard form  $(x - 1)^2 + (y - 2)^2 = 4$ , complete the square if not in standard form.
- Lots more
- Try on own: Regions via inequalities  $y < x$ ,  $x^2 + y^2 > 9$ ,  $x/y < 1$ ,  $xy \geq 0$ .

##### (b) 3D:

- Point
- Planes: Vertical ( $x = 2$ ), horizontal ( $z = 1$ ), out at us ( $y = 0$ ).
- Try on own:  $x^2 + y^2 = 1$ ,  $x + y = 1$ ,  $z = x^2$ ,  $x < y$ .

- Spheres from the distance formula, standard form, complete the square if not in standard form.
- Showcase Geogebra.

3. Homework: 7, 9, 11, 13, 15, 17, 21, 23, 25-37 odd, 45

## .2 12.2 Vectors

1. Vector basics:  $\mathbb{R}^2$ , then  $\mathbb{R}^3$ .

- (a) Coordinate (location) vs vector (action such as displacement).
- (b) Vector has 2 attributes, magnitude (size) and direction (angle).
- (c) Location doesn't matter, standard position for comparison.
- (d) Vector components.

$$\vec{a} = \langle a_1, a_2 \rangle = \langle x, y \rangle$$

- (e) Vector from two points  $\vec{AB}$ . General formula.
- (f) Magnitude and direction. Need to adjust direction by  $180^\circ$  with arctangent formula for quadrants 2 and 3.

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}, \quad \theta = \arctan(y/x)$$

2. Vector operations: Geometry and algebra,  $\mathbb{R}^2$  then  $\mathbb{R}^3$

- (a) Addition: Parallelogram law, sum of components.
- (b) Scalar multiplication: Stretch / reverse, scale components.
- (c) Subtraction: Triangular law, subtract components, rewrite as

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

- (d) Bonus: Dot product

3. Theorem: Vector properties, all proven component-wise via properties of real number arithmetic, geometric intuition.

- (a) Commutative:  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- (b) Associative:  $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- (c) Addition identity:  $\vec{a} + \vec{0} = \vec{a}$
- (d) Addition inverse:  $\vec{a} + (-\vec{a}) = \vec{0}$
- (e) Scalar distribution:  $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
- (f) Vector distribution:  $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
- (g) Scalar associative:  $(cd)\vec{a} = c(d\vec{a})$
- (h) Scalar multiplication identity:  $1 \cdot \vec{a} = \vec{a}$

4. Unit vectors and standard basis

- (a)  $\mathbb{R}^2$ :  $\langle 1, 0 \rangle$ ,  $\langle 0, 1 \rangle$ , divide by length to make unit.

$$\vec{a} = \langle a_1, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle, \quad \pm \frac{1}{\|\vec{a}\|} \vec{a}$$

- (b)  $\mathbb{R}^3$ :  $\vec{i}, \vec{j}, \vec{k}$

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

(c)  $\mathbb{R}^2$ : Vector in terms of angle and magnitude.

$$\vec{a} = \|\vec{a}\| \langle \cos(\theta), \sin(\theta) \rangle$$

5. Application: Wire tension. Hang from a wire, wonder if will break. Know angles from ceiling. How much tension on each wire?

6. Homework: 3, 5, 7, 11, 13, 15, 17, 19, 21, 25, 29, 31, 35, 39, 45, 47

### .3 12.3 The dot product

1. Basics of the dot product:

(a) Definition:  $\mathbb{R}^2$ :  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$ ,  $\mathbb{R}^3$ :  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ ,

(b) Examples. Note result is a scalar, not a vector.

(c) Theorem: Properties of the dot product.

- $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot \vec{0} = 0$
- All are easily shown via the def of dot product. Show first two quick.

2. Meaning of the dot product.

(a) Theorem: For  $\theta$  the smallest angle between  $\vec{a}$  and  $\vec{b}$ .

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

(b) Proof: Law of cosines (generalized Pythagoras, after peek at proofs of LoC) and dot product properties.

(c) Why useful? Corollary:

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \rightarrow \theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right) \in [0, \pi]$$

(d) Corollary:  $\vec{a} \perp \vec{b}$  if and only if  $\vec{a} \cdot \vec{b} = 0$ .

(e) Example: Find angle between vectors. Show vectors perpendicular.  $\vec{0}$  is perpendicular to all vectors. Acute and obtuse cases.

3. Use of dot product, vector orientation.

(a) Direction angles and direction cosines.

(b)  $\mathbb{R}^3$ : Let  $\alpha$  be the angle between  $\vec{a}$  and  $\vec{i}$ . Likewise for angles  $\beta, \gamma$  and  $\vec{j}$  and  $\vec{k}$ .

(c)  $\cos(\alpha) = \frac{\vec{a} \cdot \vec{i}}{\|\vec{a}\| \|\vec{i}\|} = \frac{a_1}{\|\vec{a}\|}$ . Likewise for  $\cos(\beta)$ ,  $\cos(\gamma)$ .

(d) Theorem:

$$\frac{1}{\|\vec{a}\|} \vec{a} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

(e) Example: Find the direction angles of  $\vec{a} = \langle 1, 2, 3 \rangle$ .

4. Use of dot product 2, vector projection.

(a) Definitions:

- i. Scalar projection of  $\vec{b}$  onto  $\vec{a}$ :  $\text{comp}_{\vec{a}}(\vec{b})$
- ii. Vector projection of  $\vec{b}$  onto  $\vec{a}$ :  $\text{proj}_{\vec{a}}(\vec{b})$

(b) Find each using cosine of the angle between and dot product connection to  $\cos(\theta)$ .

(c) Theorem:

$$\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}, \quad \text{proj}_{\vec{a}}(\vec{b}) = \left( \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \right) \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a}$$

(d) Can see the projection is parallel to  $\vec{a}$ .

(e) Examples

5. Physics application, projection as a way to calculate work.

6. Dot product, cosine similarity, recommender systems. Coding demo.

7. Homework: 1, 3, 7, 9, 13, 15, 19, 23, 27, 29, 33, 39, 43, 45, 47, 61

## 4 12.4 The cross product

1. Basics of the cross product:

(a) Given two non-parallel vectors, find a third non-zero vector which is orthogonal to both. Will use this idea to define planes / tangent planes later on.

(b) Given  $\vec{a}, \vec{b}$  not parallel, want  $\vec{c}$  such that

$$\vec{a} \cdot \vec{c} = a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad \text{and} \quad \vec{b} \cdot \vec{c} = b_1c_1 + b_2c_2 + b_3c_3 = 0.$$

Eliminate  $c_3$  by multiplying two equations and subtracting to get

$$a_1b_3c_1 + a_2b_3c_2 - a_3b_1c_1 - a_3b_2c_2 = 0$$

which gives

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0.$$

Choose  $c_1 = (a_2b_3 - a_3b_2)$  and  $c_2 = (a_1b_3 - a_3b_1)$  which yields  $c_3 = (a_1b_2 - a_2b_1)$ .

(c) Definition: The cross product of  $\vec{a}$  and  $\vec{b}$  is

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1 \rangle.$$

Note, result is a vector where the dot product gives a scalar.

(d) Theorem:  $\vec{a} \times \vec{b}$  is orthogonal to both  $\vec{a}$  and  $\vec{b}$ . Proof just computes  $(\vec{a} \times \vec{b}) \cdot \vec{a}$ . Same for  $\vec{b}$ .

(e) Determinant notation:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

(f) Example: Find the cross product of two random vectors. Check that worked. What if vectors parallel? One zero?

(g) Orientation of  $\vec{a} \times \vec{b}$  and the right hand rule.

2. Information hidden in the cross product.

- (a) Theorem:  $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin(\theta)$ . See proof in text. Easy except for first part. Surprising at first, but can see just comes from the dot product result.
- (b) Corollary: Two nonzero vectors are parallel if and only if  $\vec{a} \times \vec{b} = \vec{0}$ .
- (c) Corollary:  $\|\vec{a} \times \vec{b}\|$  gives the area of the parallelogram formed by  $\vec{a}$  and  $\vec{b}$ . Draw parallelogram. Base times height.
- (d) Find the area of the triangle in  $\mathbb{R}^3$  formed by three random points.

### 3. Properties of the cross product.

- (a) Consider combinations of cross product of unit basis  $\vec{i}, \vec{j}, \vec{k}$ . Note in general  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$  because of right hand rule. Also since orthogonal basis,  $\sin(\pi/2) = 1$  and can see the result is unit. Parallelogram is a square.
- (b) Theorem: Properties of the cross product.
  - $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
  - $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
  - $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
  - $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
  - $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
  - $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
  - All are proven via the component-wise definition of the cross product.

### 4. Triple product, volume of parallelepiped.

- (a)  $3 \times 3$  determinant.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- (b) Theorem: The volume of the parallel-piped formed by  $\vec{a}, \vec{b}, \vec{c}$  is

$$\|\vec{a} \cdot (\vec{b} \times \vec{c})\| = \|\vec{a}\|\|\vec{b} \times \vec{c}\|\cos(\theta)$$

where  $\|\vec{b} \times \vec{c}\|$  is the area of the base and  $\|\vec{a}\|\cos(\theta)$  is the height. This comes from our dot product formula.

- (c) Corollary:  $\vec{a}, \vec{b}, \vec{c}$  are coplanar if and only if the triple product is zero.
- (d) Newton used this to derive Kepler's law of planetary motion.

### 5. Torque definition and magnitude.

### 6. Homework: 1, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 31, 33, 37, 49, 53

## .5 12.5 Equations of lines and planes

### 1. Equations of lines: Vector, parametric, symmetric.

- (a)  $\mathbb{R}^2$

- Familiar case:  $y = mx + b$ , Ex  $y = 2x + 1$ , graph it.
- Two step process: Get to the line via  $\vec{r}_0$ , traverse the line via  $\vec{v}$  which is parallel to the line.
- Ex:  $\vec{r}_0 = \langle 0, 1 \rangle$ ,  $\vec{v} = \langle 1, 2 \rangle$ , then

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle t, 1 + 2t \rangle.$$

Since  $t = x$ , we have  $y = 2x + 1$  again.

- Parameter  $t$  moves us along the line in a direction as  $t$  increases.
- Vector form is not unique.  $\vec{r} = \vec{r}_0 - t\vec{v}$  would give the same line, just traced backwards.

(b)  $\mathbb{R}^3$

- Vector equation: For  $\vec{v}$  parallel to the line and  $\vec{r}_0$  the vector from the origin to any point on the line,

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

- Draw picture.
- Parametric equations of a line: For parameter  $t$ ,

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$

PEs are not unique though they may draw the same line.

- Symmetric equations of a line: Solve for parameter  $t$ .

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

It is possible that  $a, b, c$  could be zero.

- (c) Example: Find the vector, parametric, and symmetric equations for the line thru two random points. Where does it intersect the  $xy$ -plane?  $xz$ ?  $yz$ ?
- (d) 3 possibilities for lines meeting now: parallel, intersecting, or skew (not parallel, not intersecting).
  - 3 lines, decide if pairs are parallel, intersecting, or skew. Graph in Geogebra.
- (e) Line segment from point  $(x_0, y_0, z_0)$  to  $(x_1, y_1, z_1)$ :

$$\vec{r} = (1 - t)\vec{r}_0 + t\vec{r}_1, \quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \vec{r}_1 = \langle x_1, y_1, z_1 \rangle, \quad 0 \leq t \leq 1.$$

## 2. Equations of planes: Vector, scalar, linear

- (a) Harder to define the direction of a plane. Normal (perpendicular) vector does the trick.
- (b) Vector equation of plane: For  $(x_0, y_0, z_0)$  a fixed point on the plane, any point  $(x, y, z)$  on the plane, and  $\vec{n} = \langle a, b, c \rangle$  a normal vector to the plane, we have that

$$\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$$

where  $\vec{p}_0 = (x_0, y_0, z_0)$  and  $\vec{p} = (x, y, z)$ . Draw picture to illustrate.

- (c) Scalar equation of plane: Compute  $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$ .

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- (d) Linear equation of plane: Combine constant terms of  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ .

$$ax + by + cz + d = 0$$

- (e) Example: Find the plane thru three random points. Uniquely possible if points are not colinear. Already have point, use cross product to get normal vector. Give all 3 forms. Plot the plane by computing the axis intercepts. Check with Geogebra.

## 3. Summary: In $\mathbb{R}^3$ ,

- (a) You need a point and a direction (parallel vector) to define a line.
- (b) You need a point and a normal vector to define a plane.
- (c) Examples: Group challenge.
  - Problems in text: 35, 37, 45, 51.

## 4. Homework: 1, 3, 5, 7, 11, 13, 15, 17, 19, 23, 29, 31, 35, 37, 39, 41, 45, 49, 51, 53, 55, 59, 63, 65

## .6 12.6 Cylinders and quadratic surfaces

1. Summary: Goal is to develop intuition for  $\mathbb{R}^3$ .

(a) We already considered two classes of surfaces in  $\mathbb{R}^3$ : Spheres and planes.

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2, \quad ax + by + cz + d = 0$$

(b) New surfaces for this section:

- Cylinders: Surfaces consisting of all lines (called *rulings*) parallel to a given line and passing thru a planar curve.
- Example:  $z = x^2$  is a parabolic cylinder. Parabolas are called vertical *traces*.
- Terminology: A *trace* is a curve of intersection of the surface with planes parallel to the coordinate planes  $(xy, xz, yz)$ .
- Quadratic surface: Any surface generated by the general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

- Example:  $x^2 + y^2 + z^2 = 1$  is a sphere.
- Quadratic surfaces:

2. Cylinders: Sketch the graph. What are the traces? What are the rulings?

(a) Example:  $x^2 + z^2 = 4$

(b) Example:  $y = z^2$

3. Quadratic surfaces: Sketch the traces, then the graph.

(a) Cone:  $z^2 = x^2 + y^2$ .

(b) Elliptic paraboloid:  $z = x^2 + y^2$

(c) Hyperbolic paraboloid:  $z = x^2 - y^2$

(d) Recall the formula for an ellipse of width  $2a$  and height  $2b$  centered at the origin. Circle is a special case.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(e) Show text table of 6 classes, won't test the hyper-stuff. Just basics.

4. Homework: 1,3,5,7,11,17,21,23,25,27

## Chapter 13 Vector functions

### .1 13.1 Vector functions and space curves

1. Finally we do calculus, basic case first:  $\vec{r}(t)$  is a vector-valued function. For input  $t$ , result is a vector.

(a) Need a function per component.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

(b) Example: Already know lines. Label direction.

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Knowing two points allows to draw the line. Show example.



(c) Example: Corkscrew. Label direction.

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

Helps to graph the projection onto the  $xy, xz, yz$  planes.

(d) Example: Try on own.

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

(e) Matching: Text problems 21-26.

(f) Find a vector function to describe the curve of intersection of cylinder  $x^2 + y^2 = 4$  and surface  $z = xy$ .

2. Limits and continuity: Everything component-wise.

(a) If  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ , then

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle.$$

(b)  $\vec{r}(t)$  is continuous at  $t = a$  if

$$\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$$

(c) Example from previous.

3. Homework: 1, 3, 5, 7, 9, 11, 15, 17, 27, 29, 31, 43, 49

## 13.2 Derivatives and integrals of vector functions

1. Derivatives of vector functions:

(a) Definition: For  $\vec{r}(t)$  any vector function, define the derivative as

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Draw picture in  $\mathbb{R}^3$  for some  $t$ . Result is a tangent vector at  $t$ . The unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

Also have a tangent line using this vector and point.

(b) Theorem: In  $\mathbb{R}^3$  for vector function  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ , we have

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Proof moves the difference quotient inside the vector function component-wise.

(c) Examples:

- $\mathbb{R}^2$  case, tangent vector to  $\vec{r}(t) = \langle t-2, t^2+1 \rangle$  when  $t=2$ . Draw picture. Tangent line also.
- Tangent vector for any line  $\vec{r}(t) = \vec{r}_0 + t\vec{v}$  is  $\vec{v}$  and the tangent line is the same line. This is the velocity vector for the line as we will see in the next section. Constant change with constant velocity.
- Find the tangent vector to  $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$  at  $t=2$ . Tangent line also. Geogebra.

2. Vector function differentiation rules.

(a) Theorem: For  $\vec{u}(t)$  and  $\vec{v}(t)$  differentiable vector functions,

- $\frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$

- $\frac{d}{dt} [c\vec{u}(t)] = c\vec{u}'(t)$
- $\frac{d}{dt} [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt} [\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

(b) Proof of dot product version, component-wise calculation.

3. Homework: 1, 3, 7, 13, 17, 21, 25, 43, 45, 47

### .3 13.3 Arc length and curvature

SKIP

### .4 13.4 Motion in space: Velocity and acceleration

1. Finally, velocity and speed.

(a) Definition: The velocity vector function  $\vec{v}(t)$  of position of particle curve  $\vec{r}(t)$  is given by

$$\vec{v}(t) = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Draw picture in  $\mathbb{R}^3$ . This give speed and direction.

(b) Definition: The speed of particle at position  $\vec{r}(t)$  is

$$\|\vec{v}(t)\| = \|\vec{r}'(t)\| = \frac{ds}{dt}/$$

(c) Definition: The acceleration is

$$\vec{a}(t) = \vec{v}'(t).$$

(d) Example: A parameterization of  $y = x^2$  is given by  $\vec{r}(t) = \langle 2t^2, 4t^4 \rangle$ . Plot the velocity and acceleration vectors for  $t = 1$ . Find the speed. Note the direction of the velocity vector is parallel to the old fashion tangent line.

2. Homework: 1, 3, 7, 9, 11, 15, 19,

3. Chapter review problems:

- (a) Concept check: 1-4, 8
- (b) T/F: 1-6, 11, 14
- (c) Exercises: 1-4, 9, 16-19

## Chapter 14 Partial derivatives

Here we return to calculus ideas to extend old idea (functions of one variable  $y = f(x)$ ) to 3 dimensional space (functions of two variables  $z = f(x, y)$ ).

- 2 dimensions: Get IROC for  $f(x)$  as  $\frac{df}{dx}$  via AROC as  $\frac{\Delta f}{\Delta x}$ . Graphs of  $y = f(x)$  have tangent lines. Key is idea of limit.

- 3 dimensions: Functions like  $f(x, y) = x^2 + y^2$  (and even  $f(x, y, z)$ ) should also have rates of change. Surface analogy. Key will still be limit.

Summary of chapter in 6 lines: Curve  $y = f(x)$  vs surface  $z = f(x, y)$ .

- $\frac{df}{dx}$  becomes two first order derivatives  $\frac{df}{dx}$  and  $\frac{df}{dy}$
- $\frac{d^2f}{dx^2}$  becomes four second order derivatives  $x^2, xy, yx, y^2$
- Linear approximation  $\Delta f \approx \frac{df}{dx}\Delta x$  becomes  $\Delta f \approx \frac{df}{dx}\Delta x + \frac{df}{dy}\Delta y$
- Tangent line  $y - y_0 = \frac{df}{dx}(x - x_0)$  becomes a tangent plane  $z - z_0 = \frac{df}{dx}(x - x_0) + \frac{df}{dy}(y - y_0)$ .
- Chain rule  $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$  becomes  $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$ .
- Max/min problem  $\frac{df}{dx}$  becomes the pair  $\frac{df}{dx}, \frac{df}{dy}$ .

## .1 14.1 Functions of several variables

### 1. Functions in $\mathbb{R}^2$

- (a)  $y = f(x)$  is a curve in the  $xy$ -plane.
- (b)  $x$  is the indep variable,  $y$  is the dependent variable.
- (c) Set of all  $x$  which  $f$  makes sense gives the domain, all obtainable  $y$  gives the range. Both are intervals.
- (d) Example:  $f(x) = \sqrt{x}$ .

### 2. Functions in $\mathbb{R}^3$

- (a)  $z = f(x, y)$  is a surface in  $\mathbb{R}^3$ .  $xy$  are independent and  $z$  is dependent. The domain is now a 2 dimensional region, and the range is still an interval. Simple extension, though all these ideas become harder.
- (b) Example:  $z = f(x, y) = \sqrt{x^2 + y^2}$ .
  - Need  $z \geq 0$  for range.
  - Level curves: For constant  $z = k$  we have circles  $k^2 = x^2 + y^2$ . These are circles, and they grow in diameter as  $z$  increases.
  - Resulting graph is a cone. Check in Geogebra.

### 3. Level curves:

- (a) Definition: The level curves of function  $f(x, y)$  are the curves with equations  $f(x, y) = k$  for constant  $k$  in the range of  $f$ .
- (b) Example: Find the level curves of  $f(x, y) = 2x + y$ . Level curves are lines  $k = 2x + y$  which are lines  $y = -2x + k$ . Graph in  $xy$ -plane. Result is a plane  $z = 2x + y$  giving  $2x + y - z = 0$ .
- (c) Note, different functions (surfaces) can have the same level curves. Compare  $f(x, y) = x^2 + y^2$  (paraboloid). Different locations though.
- (d) Examples: Try on own. Find domain and range. Sketch level curves. Describe surface.

$$z = \frac{y}{x}, \quad z = \sqrt{4 - x^2 - y^2}$$

- (e) Ideas extend to functions of 3+ variables as you think, harder to visualize.

$$f(x, y, z), \quad f(x_1, x_2, \dots, x_n)$$

4. Contour maps and calculus intuition: Show contour map of mountain with rivers.

- (a) Contours are drawn every 100 ft increase. What do you see?
- (b) Steep trails have close curves. Flat are far apart.
- (c) Creeks run perpendicular to level curves. Steepest direction is perpendicular.
- (d) Loops indicate peaks and troughs.
- (e) What if you walk along a level curve? No change in elevation.

5. Homework: 1, 7, 11, 13, 15, 19, 23, 25, 33, 35, 37, 41, 43, 49, 61, 63, 65

## .2 14.2 Limits and continuity

1. Limits in  $\mathbb{R}$

- (a) Intuition definition:  $\lim_{x \rightarrow a} f(x) = L$  if for  $x$  near  $a$ ,  $f(x)$  is near  $L$ . Draw picture. Idea is clear, but need precision to build a theory on.
- (b) Precise definition:  $\lim_{x \rightarrow a} f(x) = L$  if for any  $\epsilon > 0$  (no matter how near to  $L$ ), there exists a  $\delta > 0$  (near enough to  $a$ ) such that if  $|x - a| < \delta$ , then  $|f(x) - L| < \epsilon$ . Add  $\delta$  and  $\epsilon$  to graph.  $x$  window and  $y$  window. Technical definition which is hard to work with, instead prove theorems and build theory.
- (c) Techniques for computing limits:
  - Limit laws (solid foundation, grow complexity from basic functions).
  - Algebra tricks (multiply by conjugate, right / left limits, etc).
  - Squeeze theorem and indirect attacks.
  - Can direct substitute for continuous functions.
- (d) Why are limits important? Handling indeterminate form. Essence of calculus.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(x_i^*) \Delta x$$

$0/0$  and  $\infty \cdot 0$  indeterminate forms.

- (e) Examples:  $f(x) = x^2$ ,  $f'(3) = ?$ ,  $\lim_{x \rightarrow 0} \frac{|x|}{x}$ ,  $\lim_{x \rightarrow 2} \frac{\sqrt{x+2}-2}{x-2}$ .

2. Limits in  $\mathbb{R}^2$  and beyond

- (a) Intuition definition:  $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$  if for  $(x,y)$  near  $(a,b)$ ,  $f(x,y)$  is near  $L$ . Draw picture. Now we approach a point  $(a,b)$  from all directions, not just right/left. Precision again is needed.
- (b) Precise definition:  $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$  if for any  $\epsilon > 0$  (no matter how near to  $L$ ), there exists a  $\delta > 0$  (near enough to  $(a,b)$ ) such that if  $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ , then  $|f(x,y) - L| < \epsilon$ . Note the appearance of the distance formula, circle with center  $(a,b)$ . Again this definition is not practical.
- (c) Techniques for computing limits:
  - Limit laws from 1 dim generalize, but cannot separate  $x$  from  $y$ .
  - Squeeze theorem and indirect attacks.
  - Can direct substitute for continuous functions (polynomials, rationals in domain, etc).
  - Interesting case again will be indeterminate forms (next section for partial derivatives).
- (d) Same idea for 3+ dimensions.

3. Examples:

- (a) Table example in text. Hint how to explain a limit does not exist. Graph each in Geogebra.
- (b) Show  $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$  has no limit at  $(0, 0)$  by following paths  $x = 0$  and  $y = 0$  and getting different values. Similar to right left limits in  $\mathbb{R}$ . Graph in Geogebra.
- (c) Try on own: Show  $f(x, y) = \frac{xy}{x^2 + y^2}$  has no limit at  $(0, 0)$  by choosing two paths with different results. Graph in Geogebra.
- (d) Theorem: If  $f \rightarrow L_1$  as  $(x, y) \rightarrow (a, b)$  along path  $C_1$  and  $f \rightarrow L_2$  as  $(x, y) \rightarrow (a, b)$  along path  $C_2$  with  $L_1 \neq L_2$ , then  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  does not exist.
- (e) Show  $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0$  via the Squeeze theorem. Key step:

$$0 \leq \frac{3x^2|y|}{x^2 + y^2} = 3|y| \frac{x^2}{x^2 + y^2} \leq 3|y| \cdot 1$$

Can also do from definition. See text.

- (f) If point is in domain and function is continuous, can do direct substitution.  $\lim_{(x,y) \rightarrow (1,1)} \frac{3x^2y}{x^2+y^2} = 0$ .

4. Homework: 5, 9, 13, 17

### .3 14.3 Partial derivatives

1. One dimension review,  $\mathbb{R}$ :

- (a) For  $f(x)$ , change in  $x$  results in change in  $f$ . Then average rate of change  $\Delta f / \Delta x$  tends to instantaneous rate of change  $df/dx$  as  $\Delta x \rightarrow 0$ . That is,

$$\frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

- (b) Limits are foundation, but we built a theory of differentiation.

$$cf(x), f(x) + g(x), f(x)g(x), f(x)/g(x), f(g(x))$$

and also special functions such as logs, exponentials, trig, etc.

2. Two dimensions,  $\mathbb{R}^2$ :  $f(x, y)$

- (a) Analogy tangent plane to a surface. Strategy is to allow one variable to change at a time. If  $x$  can change for  $f(x, y) = x - yx$ , then  $\Delta f = \Delta x - y\Delta x$  and  $\Delta f / \Delta x = 1 - y$ . That is the  $x$  derivative of  $f(x, y)$  is  $1 - y$ . Hold  $y$  constant and differentiate  $f$  in  $x$ . Knowing both will lead to tangent planes (next section).
- (b) Definition: The partial derivative of  $f(x, y)$  with respect to  $x$  is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similar for  $f_y$ .

- (c) Notation: For  $f = f(x, y)$ ,

$$f_x = f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = D_x f$$

- (d) All our old differentiation rules hold since  $y$  is a constant.

- (e) Example:  $f(x, y) = 4 - x^2 - 3y^2$ .

- Compute  $f_x(1, 2), f_x, f_y(1, 2), f_y$ .
- Graph via Geogebra to get intuition around  $f_x, f_y$ . Note if we know  $f_x(1, 2), f_y(1, 2)$ , we can get a tangent plane (next section).

- Note local max at  $(0, 0)$ .
- Extend to four cases of second derivatives.

(f) Example:  $f(x, y) = x^3 + x^2y^3 - 2y^2$

- Try on own, all first and second order partials.
- Compare graph to  $f_x$  and  $f_y$ .

(g) Theorem:  $f_{xy} = f_{yx}$ , order of differentiation doesn't matter. Proof via the MVT.

(h) Example: Problem 9 in text.

3. Partial differential equations tour:

- [https://en.wikipedia.org/wiki/Partial\\_differential\\_equation](https://en.wikipedia.org/wiki/Partial_differential_equation)
- <https://web.stanford.edu/class/math220b/handouts/heateqn.pdf>

4. Homework: 5, 7, 9, 11, 13, 15, 21, 25, 33, 45, 51, 53, 61, 63, 81, 97

## .4 14.4 Tangent planes and linear approximations

1. Recall:  $y = f(x)$  version.

(a) The tangent line to  $y = f(x)$  at point  $(x_0, y_0)$  is

$$y - y_0 = f'(x_0)(x - x_0) \quad \rightarrow \quad y = L(x) = f'(x_0)(x - x_0) + y_0.$$

Give example for  $f(x) = x^2$  at  $x = 3$ .

(b) Linearization approximates  $f(x)$  by this line.

$$y = f(x) \approx L(x) = f'(x_0)(x - x_0) + y_0.$$

The closer to the tangent point, the better the approximation. Give example.

(c) Taylor series and Taylor's theorem continues this vein.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \dots$$

2. Extension to  $z = f(x, y)$ , tangent planes.

(a) Partial derivatives  $f_x, f_y$  give the slope of the tangent line to  $z = f(x, y)$  in the  $x, y$  directions. Draw picture. How to use this to find the tangent line thru a point  $(x_0, y_0, z_0)$ ? Need a point and a normal vector.

(b) Normal vector construction: Find vectors in direction of partial derivative lines.

- $f_x$  :,  $y$  held constant, if  $x$  increases 1 unit,  $z$  increases  $f_x$  units. Then,  $\vec{a} = \langle 1, 0, f_x \rangle$  is parallel to our line.
- $f_y$  :, likewise  $\vec{b} = \langle 0, 1, f_y \rangle$  works.
- The normal vector to the tangent plane is then

$$\vec{n} = \vec{a} \times \vec{b} = \langle -f_x, -f_y, 1 \rangle$$

(c) Vector form of tangent plane:

$$\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0 \quad \rightarrow \quad -f_x(x - x_0) - f_y(y - y_0) + (z - z_0) = 0$$

gives

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

Note the similarity to the tangent line for  $y = f(x)$ .

- (d) Example: Find the tangent line to the paraboloid  $z = 14 - x^2 - y^2$  at  $(x_0, y_0, z_0) = (1, 2, 9)$  Graph in geogebra. Both  $x, y$  tangent lines are on this plane. All tangent lines for all surface curves as well.
- (e) Try on own: Find the tangent plane to the sphere  $x^2 + y^2 + z^2 = 14$  at  $(1, 2, 3)$ . Can solve for  $z$  taking the positive root or use implicit differentiation with respect to  $x, y$ . Note the normal vector is in the same direction as the sphere radius when directed to our point.
- (f) Linearization of  $z = f(x, y)$  by the tangent plane.

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

Two dimensional Taylor series approximation. Can guess the extension to 3+ independent variables.

### 3. Differentiability of $f(x, y)$ :

- (a) Remind of differentiability in  $\mathbb{R}^2$ . Derivative exists. Differentiable implies continuous.
- (b) Def: We say  $f(x, y)$  is differentiable at point  $(a, b)$  if

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where  $\epsilon_1, \epsilon_2 \rightarrow 0$  as  $\Delta x, \Delta y \rightarrow 0$ . Basically says can approximate  $f$  well by the tangent line.

- (c) Theorem: If the partial derivatives  $f_x, f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$ .

### 4. Homework: 1, 3, 5, 11, 13, 19, 21

## .5 14.5 The chain rule

### 1. 1 dimension: $\frac{d}{dt}f(g(t))$ .

- (a) Goal is to differentiate function composition. Nested functions are common. Do  $g$  first, then  $f$  takes it from there.

$$\frac{d}{dt}f(x(t)) = f'(x(t))x'(t)$$

- (b) Compact notation:  $y = f(x)$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Right hand side collapses back if canceling were allowed.

- (c) The chain rule applied to integration is the substitution rule.

### 2. 2 dimensions, basic case: $\frac{d}{dt}f(x(t), y(t))$

- (a) Extend the dimension 1 case of the chain rule to get for  $z = f(x, y)$ :

$$\frac{dz}{dt} = \frac{d}{dt}f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Note the similarity to the 1 dimension case.

- (b) Example: For  $z = 3xy^2$ ,  $x = \cos(t)$ ,  $y = \sin(t)$ , compute  $\frac{dz}{dt}$ . Check by rewriting  $x, y$  in original. Graph in Geogebra, not traveling about the unit circle in  $xy$ . Consider  $t = 0, \frac{\pi}{2}$ . Rate of change along curve  $(x(t), y(t))$ .

### 3. 2 dimensions, standard case: $\frac{d}{dt}f(x(s, t), y(s, t))$

- (a) Repeat the above formula twice.

$$\frac{dz}{ds} = \frac{d}{ds}f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{d}{dt}f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

- (b) Example: For  $z = 3xy^2$ ,  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ , compute  $\frac{dz}{dr}$ . Try on own  $\frac{dz}{d\theta}$ ,  $\frac{d^2z}{dr^2}$
- (c) Second derivatives and converting to polar coordinates.  $z = f(x, y)$ ,  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$
- Compute  $f_{rr}$ ,  $f_{\theta\theta}$ .
  - Turns out  $f_{xx} + f_{yy} = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}$
  - This is the polar version of Laplace's equation.

4. Generalizes to any dimension. Show text formula. Work example 5.

5. Homework: 1, 3, 5, 7, 11, 13, 17, 21, 45, 49

## .6 14.6 Directional derivatives and gradient vectors

1. Directional derivatives: So far we calculate change for  $f(x, y)$  in the  $x$  direction ( $f_x$ ) or the  $y$  direction ( $f_y$ ), but of course  $f$  can change in any direction.

- (a) Recall our limit definitions:

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

We essentially hold  $y$  and  $x$  constant respectively. The directions we consider here are  $\vec{i}$  and  $\vec{j}$ . Note both are unit vectors.

- (b) Example: Find the change in  $f(x, y) = xy$  at point  $(3, 1)$  in the direction  $\vec{v} = \langle 1, 2 \rangle$ . Normalize our direction via the unit vector  $\vec{u} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$ . Then our change is from  $(3, 1)$  to  $(3 + h/\sqrt{5}, 1 + 2h/\sqrt{5})$  and

$$D_{\vec{u}}f(3, 1) = \lim_{h \rightarrow 0} \frac{f(3 + h/\sqrt{5}, 1 + 2h/\sqrt{5}) - f(3, 1)}{h} = \lim_{h \rightarrow 0} 7/\sqrt{5} + 2h/5 = 7\sqrt{5}.$$

Note  $h$  in the denominator because of the unit vector. Graph in Geogebra and compare to  $f_x$ ,  $f_y$ .

- (c) Definition: The directional derivative of  $f(x, y)$  at point  $(x_0, y_0)$  in the direction of unit vector  $\vec{u} = \langle a, b \rangle$  is

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Note,  $D_{\vec{i}}f = f_x$  and  $D_{\vec{j}}f = f_y$ . Also any unit vector can be expressed in terms of a direction angle  $\theta$  as

$$\vec{u} = \langle a, b \rangle = \langle \cos(\theta), \sin(\theta) \rangle$$

### 2. Computing directional derivatives

- (a) The above limit definition is messy to compute. Instead, we rewrite  $D_{\vec{u}}f$  in terms of  $f_x$  and  $f_y$ . This seems doable considering the tangent plane to a surface in  $\mathbb{R}^3$ .
- (b) Theorem: For  $f(x, y)$  differentiable in both  $x$  and  $y$  and  $\vec{u} = \langle a, b \rangle$  any unit vector in  $\mathbb{R}^2$ ,

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b.$$



(c) Proof: Define  $g(h) = f(x_0 + ah, y_0 + bh)$ . Then,

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}}f(x_0, y_0).$$

On the other hand, from the chain rule,

$$g'(h) = \frac{\partial f}{\partial h} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial h} = af_x + bf_y = af_x(x_0 + ah, y_0 + bh) + bf_y(x_0 + ah, y_0 + bh).$$

Evaluating  $g'(h)$  at zero and comparing to before gives the result.

(d) Example: Repeat above example  $f(x, y) = xy$  with new calculation.

(e) Example: Try on own for  $f(x, y) = xy^3 - x^2$  at  $(1, 2)$  in direction  $\theta = \frac{\pi}{3}$ . Check via Geogebra.

### 3. Gradient vectors:

(a) Example: Hint to bigger things.  $f(x, y) = 3x + y + 1$  at  $(1, 1)$ .

- $\vec{i}$  and  $\vec{j}$  directions.
- No change (level curve) direction. Find  $\vec{u} = \langle a, b \rangle$  such that

$$D_{\vec{u}}f = f_x a + f_y b = 0$$

gives  $\vec{u} = \frac{1}{\sqrt{10}} \langle -3, 1 \rangle$ ,

- Perpendicular to level curve gives steepest direction  $\vec{u} = \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$ . This matches  $\langle f_x, f_y \rangle$  at our point. Compute change and compare to  $f_x, f_y$ .
- Noting that the directional derivative is really a dot product, we see a new vector of import.

$$D_{\vec{u}}f = f_x a + f_y b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$

(b) Definition: For  $f(x, y)$ , the gradient of  $f$  is a vector-function of the form

$$\nabla f = \langle f_x, f_y \rangle$$

(c) Example: Compute gradient for previous example  $f(x, y) = xy^3 - x^2$  at  $(1, 2)$ . Reproduce previous result.

(d) Theorem: First importance of the gradient. For  $f$  differentiable, the maximum value of the directional derivative  $D_{\vec{u}}f$  is  $|\nabla f|$  and is in the direction of  $\nabla f$ .

(e) Proof: We use the law of cosines version of the dot product.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

where  $\theta$  is the angle between  $\vec{a}, \vec{b}$ . Then,

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos(\theta) = \|\nabla f\| \cos(\theta) \leq \|\nabla f\|$$

which occurs when  $\theta = 0$  meaning  $\vec{u}$  and  $\nabla f$  are in the same direction.

(f) Example: Apply previous theorem to  $f(x, y) = 3x + y + 1$  at  $(1, 1)$ ,  $f(x, y) = xy^3 - x^2$  at  $(1, 2)$ .

(g) Example: Try on own. Number 22 in text. Graph in Geogebra.

### 4. Extension to functions of three variables: $f(x, y, z)$ .

(a) Could be in  $\mathbb{R}^4$  in which case cannot visualize. Could be an implicit curve  $f(x, y, z)$  in  $\mathbb{R}^3$ .

(b) Definition of directional derivative in direction of unit vector  $\vec{u}$ .

$$D_{\vec{u}} = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

(c) Compute  $D_{\vec{u}}$  in terms of partial derivatives.

$$D_{\vec{u}} = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \nabla f \cdot \vec{u}$$

(d) Gradient of  $f$  is

$$\nabla f = \langle f_x, f_y, f_z \rangle.$$

(e) Examples are pretty well the same.

## 5. Tangent planes to level surfaces

(a) We already have tangent planes to surfaces of the form  $z = f(x, y)$  at point  $(x_0, y_0, z_0)$ .

$$z - z_0 = (f_x)_0(x - x_0) + (f_y)_0(y - y_0)$$

(b) This extends implicitly to a level surface  $F(x, y, z) = k$  at point  $(x_0, y_0, z_0)$ .

$$(F_x)_0(x - x_0) + (F_y)_0(y - y_0) + (F_z)_0(z - z_0) = 0$$

Note, the gradient vector  $\nabla F$  is our normal vector to the plane (and surface).

(c) The normal line then has symmetric equations

$$\frac{x - x_0}{(F_x)_0} = \frac{y - y_0}{(F_y)_0} = \frac{z - z_0}{(F_z)_0}.$$

(d) Example: Find the tangent plane to the ellipsoid  $x^2/4 + y^2 + z^2/9 = 3$  at point  $(-2, 1, -3)$ . Check result in Geogebra.

## 6. Summary of gradient vector: This section is rich. Summarize the key ideas.

(a) For  $f(x, y)$  (or  $f(x, y, z)$ ),  $\nabla f$  gives the direction of fastest increase of  $f$ .

(b)  $\|\nabla f\|$  is the fastest increase rate (slope).

(c)  $\nabla f$  is orthogonal to the level curve (or surface).

## 7. Homework: 5, 7, 9, 11, 15, 19, 23, 25, 27, 29, 37, 39, 41, 49

## .7 14.7 Maximum and minimum values

### 1. Recall functions of one variable...

(a) Draw  $f(x)$  with make an min values. Smooth and continuous on  $\mathbb{R}$ .

(b)  $f'(x) = 0$  (stationary points) gives locations of horizontal tangents.  $f''(x) = 0$  discerns the three cases.

- $f''(x) > 0$ , local min
- $f''(x) < 0$ , local max
- $f''(x) = 0$ , inflection point

(c) Two other cases for extrema: Singular points, end points.

(d) Absolute max and mins are ensured by the EVT: Continuous function  $f(x)$  on closed interval  $[a, b]$  must have a local max and local min.

### 2. Definitions for $f(x, y)$ .

(a) Local min at  $(a, b)$  with local min value  $f(a, b)$ . Likewise for max.

(b) Global min and max.

### 3. Extending calculus 1 results:

- (a) Theorem: If  $f(x, y)$  has a local max or min at  $(a, b)$  and  $f_x, f_y$  both exist at  $(a, b)$ , then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  ( $\nabla f = \vec{0}$ ).
  - (b) If  $\nabla f = \vec{0}$  at  $(a, b)$ , then this is called a stationary point. Not all critical points are local mins or maxes.
  - (c) Example: Find the stationary points of  $f(x, y) = 3x - x^3 - 2y^2 + y^4$ .
4. How to classify stationary points? Concavity is key, but we need to look in all directions.
- (a) 2 examples:  $x^2 + xy + y^2$  and  $x^2 + 10xy + y^2$ . Only  $(0, 0)$  stationary point for both. Both have two positive partials second ( $f_{xx} = f_{yy} = 2 > 0$ ). Graph in Geogebra to see different behavior.
  - (b) To classify, consider all the second directional derivatives at once. For  $f(x, y)$  and  $\vec{u} = \langle h, k \rangle$ ,

$$D_{\vec{u}}f = f_x h + f_y k.$$

$$D_{\vec{u}}^2 f = D_{\vec{u}}(f_x h + f_y k) = f_{xx} h^2 + 2f_{xy} h k + f_{yy} k^2 = f_{xx} \left( h + \frac{f_{xy}}{f_{xx}} k \right)^2 + \frac{k^2}{f_{xx}} (f_{xx} f_{yy} - f_{xy}^2)$$

where the last step follows by completing the square.

- (c) If we think concave up since  $f_{xx} > 0$  we would also need  $D = f_{xx} f_{yy} - f_{xy}^2 > 0$ . Likewise for concave down we need  $f_{xx} < 0$  but still  $D > 0$ .
- (d) Theorem: For  $(a, b)$  a stationary point of  $f(x, y)$  and

$$D = D(a, b) = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local min.
- If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local max.
- If  $D > 0$  then  $f(a, b)$  is a saddle point.

- (e) Check for 2 examples.
- (f) Example: Apply to first example.

5. Last, we extend the EVT

- (a) EVT: For  $f(x, y)$  continuous on closed, bounded region  $R$  in  $\mathbb{R}^2$ , then  $f$  has an absolute max and min in  $R$ .
- (b) How to find extrema? Abs max and mins must be at stationary points in  $R$  or on the boundary of  $R$ .
  - Find the stationary points in  $R$ .
  - Find the extreme values on the boundary via Calc 1.
  - Get the largest and smallest  $f$  values from parts 1 and 2.

- (c) Example: 34 in text.

6. Homework: 1, 3, 5, 7, 13, 15, 17, 23, 27, 31, 33

## .8 14.8 Lagrange multipliers

Skip.

## .9 Chapter 14 Review

1. Concept check: 1-18
2. True-False: 1-11
3. Exercises: 1-56

## Chapter 15 Multiple integrals

### .1 15.1 Double integrals over rectangles

1. Summary of past: Extend the definite integral of calculus 1 to 3 dimensions.

- (a)  $\int_a^b f(x) dx$  as area under the curve.
- (b) Compute via limit of Riemann sum. Classic calculus paradox.
- (c) Fundamental theorem of calculus.
- (d) Alternate view:  $\int_a^b f(x) dx$  as adding up 1D lengths to get 2D area.
- (e) Really about summation: Sum lines to area, areas to volumes (discs and washers), probability, force to work, line segment to arc length, arc length to surface area, etc

2. Basic case for  $z = f(x, y)$  in  $\mathbb{R}^3$ : Volumes over rectangular domains.

- (a) Find the volume of the solid between  $z = f(x, y)$  and the  $xy$ -plane over region  $R = [a, b] \times [c, d]$  (Cartesian product).
- (b) Partition  $R$  by  $\Delta x$  and  $\Delta y$  giving rectangular areas  $\Delta A$ .
- (c) Notation and limit of a Riemann sum.

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \lim_{m, n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

- (d) If the limit exists we say  $f$  is integrable over  $R$ .
- (e) Can approximate area as in Calc 1 by computing the finite sum, though only works for simple functions.

3. FTOC for calculation: Volume by accumulating area.

- (a) Slicing the solid in the  $x$  direction gives cross-sections with area

$$A(x) = \int_c^d f(x, y) dy.$$

This is a computable function of  $x$  for any  $y$  held constant.

- (b) Add up area to get volume. FTOC twice.

$$V = \int_a^b A(x) dx = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx$$

- (c) Of course slicing in  $y$  gives a similar formula resulting in Fubini's theorem which extends to more general regions as well.

$$\int_a^b \left[ \int_c^d f(x, y) dy \right] dx = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy$$

- (d) Examples: Divide and conquer to find the area under  $z = 1 + x^2 + y^2$  on  $[1, 2] \times [0, 1]$ .

4. Average value of functions:

- (a) Calc 1 version:

$$\frac{1}{b-a} \int_a^b f(x) dx$$

(b) Calc 3 version:

$$\frac{1}{A(R)} \iint_R f(x) \, dA$$

5. Homework: 1, 13, 15, 17, 19, 21, 25, 29, 31, 33, 35, 39, 47

## .2 15.2 Double integrals over general regions

1. Idea of general regions:

- (a) The domain of integration for  $z = f(x, y)$  doesn't have to be a rectangle. In general it can be any shape (denote as  $D$ ). Draw picture.
- (b) Can still do slicing in  $x$  or  $y$  direction. Result is 2 cases to choose from.
- (c) Volume from accumulating area in  $x$  (holding  $y$  constant). Draw domain picture.

$$\iint_D f(x, y) \, dA = \int_a^b A(x) \, dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) \, dy \, dx$$

- (d) Volume from accumulating area in  $y$  (holding  $x$  constant). Draw domain picture.

$$\iint_D f(x, y) \, dA = \int_c^d A(y) \, dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y) \, dx \, dy$$

- (e) Sometimes only one is an option, sometimes both can be used and need to think strategically.

2. Examples: Drawing the domain in the  $xy$ -plane is key.

- (a) Only one direction is easy. Find the volume under the surface  $z = x^2 + y$  on domain  $D$  bound by curves  $y = x + 1$  and  $y = x^2$ .
- (b) Divide and conquer by doing both at same time. Find the volume below the plane  $z = x - 2y$  and above the triangle with vertices  $(0, 0), (1, 1), (0, 1)$  in the  $xy$ -plane. Need to divide into two volumes in one direction leading to the below theorem.
- (c) Theorem: If  $D = D_1 \cup D_2$ , then

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA$$

3. If integration is hard, estimation will often do by capturing the solid inside and outside a cylinder.

- (a) Theorem: For  $m \leq f(x, y) \leq M$  on domain  $D$  with area  $A(D)$ , then

$$mA(D) \leq \iint_D f(x, y) \, dA \leq MA(D).$$

4.

5.

6. Homework:

- .3 15.3 Double integrals over polar coordinates
- .4 15.4 Applications of double integrals
- .5 15.5 Surface area
- .6 15.6 Triple integrals
- .7 15.7 Triple integrals in cylindrical coordinates
- .8 15.8 Triple integrals in spherical coordinates
- .9 15.9 Change of variable in multiple integrals

## Chapter 16 Vector calculus

- .1 16.1 Vector fields
- .2 16.2 Line integrals
- .3 16.3 The fundamental theorem of line integrals
- .4 16.4 Green's theorem
- .5 16.5 Curl and divergence
- .6 16.6 Parametric surfaces and their area
- .7 16.7 Surface integrals
- .8 16.8 Stoke's theorem
- .9 16.9 The divergence theorem
- .10 16.10 Summary

## Chapter 17 Second-order differential equations

- .1 17.1 Second-order linear equations
- .2 17.2 Nonhomogeneous linear equations
- .3 17.3 Applications of second-order differential equations
- .4 17.4 Series solutions