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# Improved Computation of Cubic Natural Splines with Equi-Spaced Knots

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**Abstract.** An improved algorithm is given for the computation of the coefficients of the interpolating polynomials for cubic natural splines with equi-spaced knots. By solving the continuity equation recursively, a gain in computation efficiency is obtained and the requirement of previous techniques for exact computation is eliminated.

**1. Introduction.** Considerable interest has developed over the last several years in the use of spline functions for interpolation, largely as a result of the discovery of their extremal properties [1]. Cubic splines have become popular because they combine a fair degree of approximation (continuity of the function and its first two derivatives at the knots) with relative ease of determination of the spline parameters (only a second-order difference equation need be solved). Two recent papers have shown that the parameters may be even more simply determined for the case of natural splines defined at equi-spaced knots ([2], [3]). Unfortunately, the techniques given in both these papers suffer from the defect that they require exact computation. To retain any accuracy requires multiple precision computation once the number of data points exceeds the limiting value $n_L = (n_s/\log 4) - 2$, where $n_s$ is the number of significant decimal digits carried by the computer employed. In the present paper, an improved algorithm is presented which does not require exact computation and which also displays an improved efficiency in determining the spline parameters.

**2. Definitions.** The function $S(x) \in C^2$ defined on the $n$ knots $x_i$ by $S(x_i) = Y_i$ is a cubic natural spline if it is represented as a cubic polynomial in each interval $x_i \leqq x \leqq x_{i+1}$ $(i = 1(1)n - 1)$ and if the second derivatives vanish at the endpoints:

$$S''(x_1) = S''(x_n) = 0$$

(we follow the notation of Hoskins [3] with slight modification). The function $S(x)$ is then defined within each interval $[x_i, x_{i+1}]$ by the expression

$$\begin{aligned} S(x) = {} & (Y_i - m_i)\left(\frac{x_{i+1} - x}{h}\right) + (Y_{i+1} - m_{i+1})\left(\frac{x - x_i}{h}\right) \\ & + m_i\left(\frac{x_{i+1} - x}{h}\right)^3 + m_{i+1}\left(\frac{x - x_i}{h}\right)^3 \end{aligned} \tag{2.1}$$

where $h$ is the uniform knot spacing and $m_i \equiv (h^2/6)S''(x_i)$. Once the $m_i$ are determined, the spline is defined uniquely in terms of $h$ and the local values $(x_i, x_{i+1}, Y_i,$

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$Y_{i+1}$, $m_i$, $m_{i+1}$), a point of some practical importance. The $m_i$ reveal themselves as those solutions of the so-called continuity equation

$$m_{i+1} + 4m_i + m_{i-1} = \delta^2 Y_i \qquad (i = 2(1)n - 1) \tag{2.2}$$

which satisfy the boundary conditions

$$m_1 = m_n = 0. \tag{2.3}$$

**3. Solution of the Continuity Equation.** The continuity equation (2.2) is a simple second-order, linear, inhomogeneous difference equation with constant coefficients and may be solved by the standard technique of variation of parameters [4]. However, doing so results in a solution involving a summation of terms whose magnitudes grow exponentially with $n$. Though the solutions presented in [2], [3] were not derived in this manner, they retain the exponential character and suffer from the same accuracy loss due to truncation as $n \to n_L$. This problem is known to arise in the use of implicit techniques for solving partial difference equations [5], and its solution by a recursive technique is both stable and accurate. To this end, define

$$m_i = e_i m_{i+1} + f_i \tag{3.1}$$

and substitute in the continuity equation (2.2). The identification of the $(e_i, f_i)$ parameters is immediate on comparing with (3.1), and results in the expressions

$$e_i = -(4 + e_{i-1})^{-1}, \tag{3.2}$$

$$f_i = -e_i(\delta^2 Y_i - f_{i-1}). \tag{3.3}$$

Substituting the first of the boundary conditions (2.3) into (3.1) and requiring that the $(e_i, f_i)$ be independent of the $m_i$ establishes that

$$e_1 = f_1 = 0, \tag{3.4}$$

whence the $(e_i, f_i)$ may be evaluated by an ascending recursion from $i = 2$ to $n$. The second of the boundary conditions (2.3) may then be used with (3.1) to evaluate $m_i$ in a descending recursion from $i = n$ to 1, which begins $m_{n-1} = f_n$. This technique is known to be stable [5]. Although the $f_i$ are characteristic of the set $Y_i$ and must be recalculated when a new set is introduced, the $e_i$ may be calculated once for all, and the first seven are listed in the accompanying table. A comparison of this table with that of Hoskins [3] is revealing. The entries in his table vary as $4^{i-1}$ and begin to lose

TABLE

| $i$ | $e_i$ |
|---|---|
| 1 | −0.2500 0000 |
| 2 | −0.2666 6667 |
| 3 | −0.2678 5714 |
| 4 | −0.2679 4258 |
| 5 | −0.2679 4872 |
| 6 | −0.2679 4916 |
| 7 | −0.2679 4919 |

significance when $i$ surpasses the value $n_L$ defined in Section 1; while the entries in the $e_i$ table here remain of order unity throughout. (Note that as $i \to \infty$, $e_i \to -(2 - \sqrt{3})$, one of the two characteristic roots of the difference equation (2.2); $e_i$ is the $i$th approximant to the continued fraction of form (3.2) for the quantity $-(2 - \sqrt{3})$.)

Thus, the $m_i$ are determined and the expression (2.1) may be used to interpolate $Y(x)$, its derivatives and integrals.

**4. Computational Efficiency.** We consider briefly the relative efficiencies of the current algorithm and that of Hoskins [3], as realized on a digital computer. Although an accurate estimate of program efficiency requires a count of all the operations, both algorithms have roughly the same balance of adds and multiplies, so we have enumerated only the multiplications to compare the relative efficiencies. Apart from the calculation of the $e_i$ (which may be input as constants), the realization of the scheme of Section 3 above requires $(n - 1)$ multiplications to calculate both $f_i$ and $m_i$, for a total of $2(n - 1)$ operations. The scheme of Hoskins requires $(n - 2)$ operations to evaluate each of his formulas (H3.13), (H2.2), and $(n - 1)$ operations to recover his $M_i$ from the calculated quantities $(a_{n-3}(h^2/6)M_i)$ as shown in his Section 4, for a total of $3n - 5$ operations. Thus, the current algorithm is about 50% faster than Hoskins' algorithm with no limitation in calculation time or accuracy as $n$ becomes large. In evaluating Hoskins' scheme, we assumed that his table $a_i$ is built into the program (as the $e_i$ above) and that the factor of 4 required in evaluating the continuity equation (H2.2) is obtainable by a simple binary shift. If the latter is not the case, an additional $(n - 1)$ operations may have to be introduced, making the present scheme roughly twice as fast as his.

**5. Conclusions.** The algorithm presented here is a simple, computationally stable and accurate technique of improved efficiency for generating the parameters for cubic natural splines defined on equi-spaced knots.

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