Calculus I Notes

Contents

Review

Prerequisite take home quiz assigned, refresh and keep track of what is important. Give short discussion / highlights below.

1. Day 1: What is a function? Answer in a way which explains to someone who doesn't know in the best way you can. Inspire via Feynman method: `https://www.youtube.com/watch?v=FrNqSLPaZLc`
2. Functions
   - Idea, def, domain/range, graph, vertical line test
   - What is a function good for? Why is one output so important?
3. Function graphs
   - Intercepts, odd/even function, function transformations, increasing/decreasing, asymptotes
4. Composite of functions, think of as combining multiple functions (first step, second, etc)
5. Inverse function (how to reverse a function? always possible?)
   - Horizontal line test, function composition with original, graph relations.
6. Simple functions (the logic of new concept, what is real world, appproximate real world, computers, etc)
   (a) Constant / linear / quadratic function
   (b) Polynomials (simple, computers inspire)
   (c) Rational functions
   (d) Root functions
   (e) Trigonometric functions (circular motion, everywhere)
   (f) Inverse trig functions
   (g) Exponential functions (growth / decay)
   (h) Logarithmic functions
7. Motivating examples: Graph, domain, range, compose.
   (a) $f(x) = -2x - 1$
   (b) $g(x) = x^2 + 3$, restrict to make invertible.
   (c) Piecewise combination of the two about $x = 0$. Domain, range invertible?
   (d) $h(x) = \frac{1}{x}$

Chapter 0

.1 0.0 Motivating Calculus

1. Where does calculus sit within mathematics? Evolution of ideas:
   (a) Develop math tools:
      - Arithmetic (combining numbers, quantify)
      - Algebra (equations and solving for unknowns, abstract)
      - Geometry (visualize, structure, intuition)
      - Functions (Machine to capture a process, polynomials, logarithms, trigonometry, graphs)
      - Calculus (Solve paradoxes of processes, change, area, limit, infinity)

   (b) Math fields (lots):
      - Linear algebra (data, matrices, high dimensional, discrete space)
      - Probability and statistics (chance, randomness, quantify uncertainty)
      - Differential equations (translation of world into calculus, modeling)
      - Analaysis (rigor, generalization, theory)
      - Much more (number theory, computational, hybrid, etc)

   (c) All the calculuses:
      - Calc 1: Main story of calculus, derivative connect to integral, limit is foundation, fundamental question of indeterminnant form
      - Calc 2: Full story of integration, generalize beyond functions, infinite series / power series big new idea
      - Calc 3: Extension to 3+ dimensional space, closer to the real world (eng, physics)

   (d) Calculus 1 contents:
      - Paradox of calculus (zero division and the tangent line, infinite accumulation and area under a curve)
      - Limit (solution to paradox, foundation of calculus)
      - Derivative (change, deep full story, applications)
      - Integral (area, accumulation)
      - Newton and Liebnitz connected last two via FTOC.

2. Two large application areas of calculus:
   (a) Optimization (will discuss soon)
      - `https://en.wikipedia.org/wiki/Mathematical_optimization`
      - `https://www.uwlax.edu/globalassets/offices-services/urc/jur-online/pdf/2016/meyers-jack-daniel.mth.pdf`

   (b) Differential equations (mentioned above)
      - `https://en.wikipedia.org/wiki/Differential_equation`
      - `https://en.wikipedia.org/wiki/List_of_named_differential_equations`

   (c) More as well

3. The big picture of calculus (intuition here, details for the rest of the semester)
   (a) Area under a curve: area of a circle.
      - Consider a hard problem (which we already know). What is the area of a circle with radius R. Pick $R = 3$ for now.

- Lots of ways to chop it up to try (vertical rectangles, triangles, circular rings). Let's try circular rings with thickness dr (change in r).
- Take one ring at location r. Unroll the ring. Approximate by a rectangle.

$$\text{Ring area } = 2\pi r\ dr$$

- Stack all these rectangles vertically in the plane (plot $y = 2\pi r$).
- The smaller dr, the closer we are. Looks to approach the area of a triangle.

$$\text{Triangle area } = \frac{1}{2}bh = \frac{1}{2}32\pi 3 = \pi 3^2$$

- For general radius R, we get an area of πR^2.

(b) Process: Hard problem $\Rightarrow$ sum of many small values $\Rightarrow$ area under a graph.

- A bit of a paradox here. Rectangles disappear, infinitely many.

(c) Area under a curve: velocity / distance.

- Suppose a car speeds up then comes to a stop.
- Assume we know the velocity everywhere. Plot a velocity function that makes sense.
- $d = r \cdot t$, so we can compute the distance over small time intervals to approximate. The smaller the dt, the better the approximation.
- These are rectangles under the curve for v which we are summing.

(d) Area under a curve: general problem.

- Of course math is about pushing conversation beyond a single problem. We generalize to create a more powerful theory.
- Example: $y = x^2$. Find the area under the curve on $[0, 3]$ or in general $[0, x]$. Denote this area $A(x)$ also known as the *integral of* x^2.
- If we change the area slightly, call it dA, can approximate as

$$dA \approx x^2 dx \quad \Rightarrow \quad \frac{dA}{dx} \approx x^2$$

  The smaller dx (and hence dA), the better the approximation.
- Derivative

$$\frac{dA}{dx} = f(x)$$

  connects the function to the area under the curve (integral)
- This idea is the fundamental theorem of calculus. More later on.

(e)

Chapter 2

.1 Introduction

1. Calculus and paradox

   - Zeno paradox (Achilles and tortoise, tortoise always wins, infinite times when tortoise ahead) `https://en.wikipedia.org/wiki/Zeno%27s_paradoxes`
   - 1=0.999999 (∞ as a process) `https://en.wikipedia.org/wiki/0.999...`

$$1 = 1 \cdot \frac{1}{3} = 1 \cdot (0.\overline{3}) = 1 \cdot (0.333...) = 0.999... = 0.\overline{9}$$

- $D = rt$ (inst veloc), newton quote `https://en.wikipedia.org/wiki/History_of_calculus`

$$D = rt \to r = \frac{D}{t}$$

  What is this as $t \to 0$?

- Achimedes and reductio ad absurdum: Practical solutions to be had: `https://en.wikipedia.org/wiki/The_Quadrature_of_the_Parabola`
- Used and criticised thru history, idea of limit formalized in 19th century, let to revolution in mathematical analysis.

2. Outline of chapter
   - Motivation: Tangent / velocity problem, paradox
   - Approach: Limit of a function, idea of solution
   - Techniques: Limit laws (structure), delta eps (rigor), infinity (more paradox)
   - Continuity: Big math idea applies to all functions
   - Derivative definition, develop deep in chapter 2

.2 2.1 The Tangent and Velocity Problems

1. Motivation: Playing the stock market
   - Calculus stock over time
   - When to buy and sell? How to tell what will happen next?
   - Average rate of change is easy (AROC) but gets weird as interval gets smaller.

$$\frac{\Delta S}{\Delta t}$$

   - Instantaneous rate of change makes sense with intuition, but not with calculation. 6/2 vs 6/0 vs 0/0.
   - Paradox of 0/0.

2. Motivation: Distance and velocity
   - My commute to work, plot velocity as I see on spedometer.
   - Can you draw distance? Δv vs Δd. Fast and slow Δd.
   - Using distance graph, how to get velocity? IROC at midpoint?
   - Connection: Average velocity.

$$d = rt \quad \to \quad r = \frac{d}{t}$$

   - Paradox of instantaneous velocity. 0/0.

3. AROC, IROC, and the difference quotient:
   - Graph general function $y = f(x)$ and label $x = a, b$.
   - Def of diff quotient.

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

   - Graph, secant line slope.
   - Connection to IROC. Can never get to IROC, our first paradox of calculus.

- Secant line trends to a tangent line.

4. Example: Try on your own.
   - $f(x) = x^2$, AROC over $[1, 2]$.
   - Try to approx IROC at $x = 2$. By hand, use calculator / computer.
   - Graph.
   - Compute AROC and draw secant line.
   - Use desmos.

5. Example: Alternate form of difference quotient.
   - a and b
   - a and $a + h$.
   - Graph to compare.
   - Second better for calculation.

.3 2.2 The Limit of a Function

1. Limit idea and notation Seems silly and weird and confusing.
   (a) Definition in words. For x near a, $f(x)$ is near L.
   $$\lim_{x \to a} f(x) = L$$
   (b) Important that L is finite here.
   (c) Reading notation: the limit of $f(x)$, as x approaches a, equals L.
   (d) Draw picture, careful language, how to read notation, idea only here, fuzzy and not careful.
   (e) Distinction between limit and f(a), may differ or same. Show can move f(a) in picture. Near does not mean equal.
   (f) Possible limit doesn't exist. Show picture.

2. Return to IROC:
   (a) Example from last section: IROC at $x = 2$ for $f(x) = x^2$
   (b) Limit of diff quotient, undefined at zero.
   (c) Plot diff quotient in desmos, show can remove zero division by factoring and simplifying, called removable discontinuity.
   (d) Limit def of IROC

3. Limit existence
   (a) Draw cases where exists, continuous, removable discontinuity
   (b) Draw cases where doesn't, jump discontinuity, asymptote (L must be finite), oscillatory case

4. Example: Piecewise function. Try on own.
   (a) Graph on own, and figure out limits everywhere in its domain. Where do limits not exist?
   $$f(x) = \begin{cases} 2 - x^2, & -1 \le x < 0 \\ 2 - x, & 0 < x \le 1 \\ 2x, & 1 < x < 2 \end{cases}$$

5. One sided limit.
   (a) Draw picture with jump disc.
   (b) Right and left side limit notation. Again, $f(a)$ doesn't matter.

$$\lim_{x \to a^{+/-}} f(x) = L$$

   (c) If they differ, regular limit doesn't exist. If same, regular limit is the same and agrees. Sometimes decomposing a limit into two sides is a good strategy.

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

   implies

$$\lim_{x \to a} f(x) = L$$

   and reverse as well.

6. Example: Previous problem. Explore one sided limits.
   (a) Graph on own, and figure out limits everywhere in its domain. Where do limits not exist?

$$f(x) = \begin{cases} 2 - x^2, & -1 \leq x < 0 \\ 2 - x, & 0 < x \leq 1 \\ 2x, & 1 < x < 2 \end{cases}$$

7. Infinite limits
   (a) Motivating examples: $f(x) = 1/x, 1/x^2$
   (b) Def of $\lim_{x \to a} = + - \infty$
   (c) Right / left limits can be one-sided, if agree get regular limit.
   (d) Have seen this before: VAs, bottom zero, top not
   (e) If limit is infty, still say limit DNE
   (f) Example: How to reason sign of infinity? Check in desmos.

$$f(x) = \frac{2 - x}{x + 1}, \quad g(x) = \frac{x^2 - 2x - 8}{x^2 - 5x + 6} \quad x \to 2$$

.4 2.3 Calculating Limits Using the Limit Laws

1. Current ways to calculate limit
   (a) graph (imprecise, unreliable)
   (b) calculator (impractical, not intuitive)
   (c) reasoning (fuzzy)
   (d) Need a precise approach for any function $f(x)$
2. Path of math
   (a) Precise foundation: Basic building block.
      - Soon will be $\delta - \epsilon$ def of limit, short version in next section
   (b) Build theory (skip to here for now): Prove more complicated, useful results.
      - Theorems, limit laws as base, combine these to handle very complex functions.

3. Limit laws (analytic / computational technique, practical)

   (a) Basics, for a, c constants.

   $$\lim_{x\to a} x = a, \quad \lim_{x\to a} c = c$$

   (b) Limit laws if *both limits exist (right, left agree and finite)* (SUBTLE) and c is a constant, then

      i. $f+g$
      ii. $f-g$
      iii. cf
      iv. $f\cdot g$
      v. $\frac{f}{g}$ if $\lim_{x\to a} g(x) \neq 0$
      vi. $f(x)^n$
      vii. $\sqrt[n]{f(x)}$

   (c) These laws match your reasoning, but need to be shown carefully using $\delta-\epsilon$ def of limit.

   (d) Why do we care about these laws? Practical.

      i. $\lim\limits_{x\to 2}\left(2x^2-x+2\right)$, reference corresponding limit law at each step.
      ii. Note need to simplify algebra first otherwise zero division: $\lim\limits_{x\to 2}\dfrac{x^2+4x-12}{x^2-2x}$. Note $x\neq 2$ for the simplification steps and we don't care since limitness.
      iii. Check each in Desmos.

   (e) Return to IROC in previous section, $f(x) = x^2+1$ at $x=1$.

   (f) Powerful.

      i. Theorem: For $p(x)$ any polynomial and $r(x)$ any rational function, we can use direct substitution to evaluate limits.

      $$\lim_{x\to a} p(x) = p(a), \quad \lim_{x\to a} r(x) = r(a)$$

      provided a is in the domain of the rational function.

4. Challenge examples: Try on own first. Check in Desmos.

   (a) $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}$ (mult by conjugate)
   (b) $\lim_{x\to -6}\frac{2x+12}{|x+6|}$ (use def to remove abs val)
   (c) $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$
   (d) $\lim_{x\to 0} x\sin(1/x)$ (challenge, need squeeze theorem)

5. Squeeze theorem: The indirect attack.

   (a) Statement: if $f(x)\leq g(x)\leq h(x)$when x is near a (except at a) and

   $$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

   then

   $$\lim_{x\to a} g(x) = L$$

   (b) Draw picture of idea. Ask to draw on own first.
   (c) Hard to know when to use. Bounding sine is the key giveaway here and future.
   (d) Useful for proving other theorems in the future.

.5 2.4 The precise definition of a limit

1. Recall idea of limit:
   (a) Try to write down on own. Draw picture.
   (b) $\lim_{x\to a} f(x) = L$, near wording.
   (c) Note again $x \neq a$ and $f(x) \neq L$.
   (d) Issue: Fuzzy idea, lacks precision. How near is near?
2. Example: Motivation, try on own.
   (a) Design a circular plate. Boss cares about area. How off can the radius be?
   (b) Area 100 inches square +- 1 square inch. How off can the radius be?
   (c) Introduce function, use absolute value.
   $$A(r) = 100 \pm 1 \to |A(r) - 100| \leq 1$$
   (d) Draw graph of f and translate to L and a. Graph is parabola.
   (e) Boss comes back with +- 0.5 inch. Do in general once and for all. Update previous calculation and graphs.
3. $\delta - \epsilon$ definition of limit.
   (a) $\lim_{x\to a} f(x) = L$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that if
   $$|x - a| < \delta$$
   then
   $$|f(x) - L| < \epsilon$$
   (b) Draw picture. x window, f window.
   (c) Connect to previous example.
   (d) Key is no matter hos small ϵ is, can always find a δ.
4. Example: Prove limit of a random line.

.6 2.5 Continuity

1. Idea of a continuous function
   (a) Seen before. Only have a fuzzy definition.
   (b) Graph can be drawn without lifting pencil, no jumps, holes, asymptotes, etc.
   (c) Not precise enough (sin(1/x)), Dirichlet function
   (d) Need to be precise if want to build a theory on this idea (most ubiquitous math idea from this class)
2. Precise definition of continuous function:
   (a) Function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.
   (b) Three things are involved here:
      i. limit exists (two sides)
      ii. function value defined
      iii. they are equal

(c) Value: If can show a class of functions is continuous (ie polynomials), then limit calculation is easy (same as function evaluation)

3. Def, discontinuous at a point $x = a$.
   (a) Happens if one or more of three conditions fails
   (b) Try and find what fails for each: Make table
      - Removable discontinuity
      - Jump discontinuity
      - Infinite discontinuity
      - High oscillation ($y = \sin(1/x)$)
4. Definition: Continuous on an interval
   (a) A function is continuous on an interval if continuous at every x value in the said interval. Many types of intervals:
   $$(a,b), \quad (a,b], \quad (a,\infty), \quad (-\infty,\infty), \quad \ldots$$
   (b) Right / left continuity can be used here if endpoints are included. Just check right / left limit.
   (c) Continuous functions are continuous everywhere in their domain.
5. Example: Graph crazy piecewise function (removable, jump, infinite, not in domain).
   - Where is f discontinuous?
   - Where is f left / right continuous?
   - On what interval is f continuous.
6. Combining basic functions
   (a) Theorem: The following are continuous functions *in their domian.* (not surprising that they are familiar functions, but each needs showing carefully, text does this)
      - Polynomials
      - Rational functions (not, only in it's domain)
      - Root functions
      - Trigonometric functions
      - Inverse trig functions
      - Exponential functions
      - Logarithmic functions
   (b) Theorem: if f and g are continuous at a and c is a constant, then the following functions are also continuous at a:
   $$f \pm g, cf, f \cdot g, f/g, \quad if \quad g(a) \neq 0$$
   These are just the five limit laws!
   (c) Example: Where is the following function $f(x)$ continuous?
   $$\frac{\ln(x-1) + \sin(x)}{x^2 - -x - 2}$$
7. Function composition:
   (a) Recall, function composition.
   (b) Theorem: If $g(x)$ is continuous at $x = a$ and f is continuous at $g(a)$, then
   $$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

(c) Note, this theorem implies continuity of composition of two continuous functions.

(d) Random example: $\sin(e^x)$

8. Intermediate Value Theorem
   (a) Draw picture of continuous function.
   (b) Theorem: For f continuous on $[a, b]$ with $f(a) \neq f(b)$. For any number L between $f(a)$ and $f(b)$, there exists an N such that $f(c) = L$ for $a < c < b$.
   (c) Draw picture.
   (d) Seem obvious. Useful when you don't have a good handle on f.
   (e) Named theorem means important. Know this result since shows up in surprising places.
9. Bisection method:
   (a) Show $F(x) = x^3 + x^2 - 1$ has a root on $[0, 1]$.
   (b) Picture to explain why. How to approximate?
   (c) Bisection demo in Excel

.7 2.6 Limits at infinite: horizontal asymptotes

1. Example: $f(x) = 1/x$, draw graph.
   (a) Know VAs. Guess what the HA version should be.
   (b) Limit notation easy, how to think about it carefully.
   (c) Caution around infinity
2. Def:
   (a) Let f be a function defined on some interval (a, ∞), then
   $$\lim_{x\to\infty} f(x) = L$$
   means that the value of f(x) can be made arbitrarily close to L by taking x sufficiently large.
   (b) Similar for $-\infty$. Note two directions do not have to agree as we always see with rational functions.
   (c) Careful δ, ϵ version, draw picture.
3. Definition:
   (a) The line $y = L$ is called a horizontal asymptote of the curve $f(x)$ if
   $$\lim_{x\to\infty} f(x) = L \quad or \quad \lim_{x\to-\infty} f(x) = L$$
   (b) Who cares? End behaviour and such. UWL ash tree, ecology population asymptotics.
4. Theorem: Basic limits at ∞:
   (a) $1/x, 1/x^2, ..., 1/x^r, r > 0$, look at $x \to + - \infty$
   (b) e^x, a^x
   (c) $\arctan(x)$
   (d) Check in desmos
5. Examples: Imagine limit process as first step.
   (a) $\lim_{x\to\infty} \dfrac{x^4 - x^2 + 1}{(x+1)^3(2x-1)}$

- Divide by HOT in denom
- Check in Desmos

(b) $\lim_{x\to\infty} \sqrt{4x^2+3x} - 2x$

- Hint: Conjugate

(c) $\lim_{x\to\infty} \frac{\sin(x)}{x^2}$

- Squeeze theorem still works.

(d) $\lim_{x\to 0-} e^{1/x}$

- Substitution, silly simple idea, powerful technique.
- Function composition and continuity not usable.

6. Indeterminate forms and the essence of calculus.
   (a) 7 cases, our goal will be to convert everything to ratio cases.
   (b) Comparing infinity: x–x^2; x/x^2, transfer to basic case.
   (c) $\infty - \infty$ is strange: Grandi's series and god

.8 2.7-2.8 Derivatives and rates of change

1. Recall: AROC as an approximation of IROC
   (a) Previous definition of IROC as limit of IROC. Note, this is a definition.
   (b) Picture. Interpret as secant lines approaching tangent lines.
   (c) x/a vs x/h versions.
   (d) We now know limit tackles this process
2. Definition: Derivative at a point, $f'(a)$ and prime notation.
   (a) IROC, tangent line slope, derivative all the same
   (b) 2 main formulas. Which is better? h formula has an advantage for limit calculation.
   (c) Notation: $f'(x) = df/dx = (d/dx)f(x)$, Newton, Leibniz, operator.
   (d) Examples:
      - From the past: $f(x) = x^2$ at $a = 2$. Both ways. Graph result.
      - Try on own: $f(x) = 1/x$ at $a = 3$. Find tangent line.
      - Try on own: $f(x) = \sqrt{x}$ at $a = 4$. Find tangent line. Note, inverse of previous problem. Should know what to expect. What if we allow that point to change? Consider the graph. Desmos.
3. Definition: Derivative as a function. Two versions, h version standard.
   (a) Generalize previous deriv at a point to a new function.
   (b) Key questions:
      - How is f related to f'?
      - Is differentiation reversible?
      - Are all functions differentiable?
   (c) Notation: $f'(x) = df/dx$
   (d) Can go higher, $f''(x) = ...$
4. Example:

(a) $f(x) = x^2$, compute $f'(x)$ and draw both together. Connect two. Check individual points.
(b) f to f' is unique, f' to f not.

5. Examples:
   (a) Draw two graphs. Ask to graph f'
   (b) Wavy function, cubic. Shift up, same $f'(x)$.
   (c) $f(x) = x^3 - x$, compare f' and f''.
   (d) Corner function, absolute value. $f'(0)$ does not exist.
   (e) Show carefully that $f(x) = |x|$ is not differentiable at $a = 0$. Compute right and left limits. In short f' is not continuous at $a = 0$.

6. When differentiation fails:
   (a) Corners, vertical tangent, discontinuity
   (b) Theorem: If f is diff at a, then f is continuous at a (so diff is stronger than continuity). Venn diagram of functions.

Chapter 3 Differentiation rules

1. Motivation:
   - Difference quotient is a pain, need to keep building a theory to make diff easier (more efficient, abstraction powerful).
   - Why? Understand functions better, translate real world change to equation (DEs), optimization, etc
   - Demo differential equation simulation: CFD, Frozen

2. Chapter outline:
   - Easy way to diff simple functions (think limit laws)
   - Polys, exps, rationals, trig, also combos of these.
   - Extend to curves which are not functions (implicit curves)
   - Apply to two problems: Beginning DEs, related rates and GPS.

.1 Derivatives of polynomials and exponential functions

1. Tackle basic functions:
   (a) $f(x) = c, mx + b, x^2, x^n, e^x, a^x$
   (b) Apply difference quotient to each. H version easiest.
   (c) Combine simple function difference quotients using limit laws: $f \pm g, f \cdot g, f/g, f^n$, $\sqrt[n]{f}$.

2. Examples: Try on own.
   (a) $c, mx + b, ax^2 + bx + c$
   (b) Can see limit law usefulness with last two.

3. Theorem: Power rule
   (a) $\frac{d}{dx} x^n = nx^{n-1}$
   (b) Try to prove, can see the challenge with h version.

(c) Use $x-a$ version instead using factor formula

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})$$

then limit laws after indeterminate form is removed.

(d) Proof holds for n any positive integer. Will extend later to any number n including irrationals.

(e) Revisit $ax^2 + bx + c$ using this result hinting at limit laws again.

(f) Examples: $x^5, \sqrt{x^3}, \frac{1}{x^3}, x^0$.

4. Theorem: Limit laws applied to derivatives.

   (a) $\frac{d}{dx}(cf(x)) = cf'(x)$

   (b) $\frac{d}{dx}(f(x) + -g(x)) = f'(x) + g'(x)$ Prove this one to illustrate limit law use.

   (c) Can treat $\frac{d}{dx}$ as a operator (like limits or multiplication in a way)

   (d) Revisit above quadratic example again. Finally we can differentiate without directly using limits. This allows us to tackle any polynomial easily.

   (e) Warning: Note, no simple diff rule for prod, quot.

   $$(fg)' \neq f'g', \quad (f/g)' \neq f'/g'$$

   Try on own: Create random examples to show not same. Check limit def of product to see the complication.

5. Exponential functions: Recap from algebra.

   (a) Basic defs. Natural number, integer, rational, irrational, zero. Laws of exponents.

   (b) a^x, different a

   (c) e^x importance, compound interest desmos

   (d) Search eulers number, more important than pi?

6. Theorem: Derivative of exponentials.

   (a) Difference quotient for general a^x

   (b) Definition: $F'(0)$ limit is 1 for e.

   (c) Desmos graph.

   (d) Will have to wait for other exponentials

   (e) Note can diff e^x many times, unchanged.

.2 3.2 The product and quotient rules

1. Already noted that $(fg)' \neq f'g'$ and likewise $(f/g)' \neq f'/g'$, so what are they?

2. Geometry and intuition:

   (a) Can think of product $f(x)g(x)$ as the area of a rectangle.

   (b) Let x change to $x + \Delta x$, then f, g change by $\Delta f, \Delta g$.

   (c) So the change in the rectangle's area is

   $$\Delta(f \cdot g) = (f + \Delta f)(g + \Delta g) - fg = f\Delta g + g\Delta f + \Delta f \Delta g$$

   $$\frac{\Delta(f \cdot g)}{\Delta x} = f\frac{\Delta g}{\Delta x} + g\frac{\Delta f}{\Delta x} + \Delta f\frac{\Delta g}{\Delta x}$$

   (d) Take $\Delta x \to 0$. Wild.

3. Product rule:

   (a) Theorem (product rule) If both f and g are differentiable, then

   $$\frac{d}{dx}(f \cdot g) = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)$$

   or more compactly

   $$(f \cdot g)' = f'g + g'f$$

   (b) Show a rigorous proof. Add and subtract same term to get diff quotients. The power of adding zero.

   (c) $(x-1)(x+1)$ easier to distribute, second derivative of x^2e^x.

4. Quotient rule:

   (a) Theorem (quotient rule):

   $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

   or

   $$(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$$

   (b) Can prove via same trick as with power rule. See text.

   (c) Show the proof by finding $(1/g)'$ first via difference quotient, then apply product rule.

   (d) $\frac{x^2-1}{x^3+6}$, find the second derivative of e^x/x.

   (e) Can now show carefully $(x^{-n})' = -nx^{-n-1}$ via the quotient rule. This further generalizes the power rule.

5. Start creating a list of differentiation formulas. Will need to memorize all these.

.3 3.3 Derivatives of Trigonometric functions

1. Trig review

   (a) Sine and cosine, right triangles, unit circle, graphs.

   (b) Other 4, tangent is other essential.

2. Basic trig derivatives

   (a) Sine and cosine

   - $\frac{d}{dx}\sin(x)$, difference quotient troubles.
   - Leverage sum formula for sine

   $$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

   - Back to difference quotient:

   $$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h) - sin(x)}{h} = \cos(x)\lim_{h\to 0}\frac{\sin h}{h} + \sin()\lim_{h\to 0}\frac{\cos h - 1}{h}$$

   - Indirect attach for below. Back to unit circle and apply squeeze theorem.

   $$\lim_{h\to 0}\frac{\sin h}{h}$$

   Image to focus on

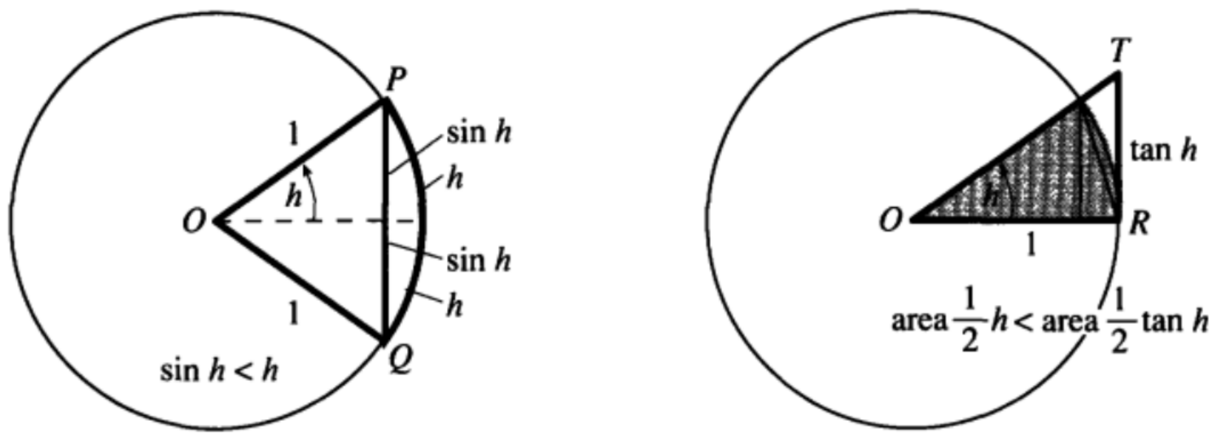

Fig. 2.11 Line shorter than arc: $2\sin h < 2h$. Areas give $h < \tan h$.

which produces

$$\sin(h) < h, \quad h < \tan(h).$$

- Tackle cosine limit via the conjugate.
- Ask them to prove cosine derivative on own using the same idea and

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h).$$

(b) Other 4, get from quotient rule connecting to sine and cosine.

(c) Note x must be in term of radians here, degrees differ in result by constant.

(d) 2 important limits to know.

3. Theorem: Derivative of all the trig functions ($\cos x$ is in homework 20, find others by yourself). Show these except cosine via quotient rule.

$$\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x), \quad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x), \quad \frac{d}{dx}\sec(x) = \sec(x)\tan(x), \quad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

4. Examples:

(a)

$$\frac{d}{dx}\frac{\sec x \sin x}{e^x + \tan x}$$

(b) Find the second derivative of $\sec x$, note Pythagorean identities to be applied. Many equivalent answers possible.

(c) Find the 99th derivative of $\sin x$

5. Above limit results can be used in weird ways.

(a)

$$\lim_{\theta\to 0}\frac{\sin(7\theta)}{3\theta} = \lim_{\theta\to 0}\frac{\sin(7\theta)}{7\theta}\frac{7\theta}{3\theta} = \frac{7}{3}$$

(b) Find

$$\lim_{\theta\to 0}\frac{\sin(4x)}{\sin(6x)} = \frac{2}{3}$$

(c) Mention limit law use and substitution ideas here.

.4 3.4 The chain rule

1. Take stock: Goal is to diff any function f(x) by...
   (a) growing a list of basic functions (trig is next, really sine and cosine are only new ones)
   (b) combining functions in various ways, new combination here is function composition
   (c) Short review of function composition
2. Composition of rates of change
   (a) A cheetah is 10x as fast as me. I am 2x as fast as my chicken. How much faster is the cheetah than my chicken? 20x as fast.
   (b) Example of temperature of La Crosse, temperature in the room, temperature in my storage case.
   (c) Explanation of chain idea: change in daytime light - changes - temperature - changes - growth of apple tree - changes - size of apple - changes - size of worm population
   (d) Back to classic function composition diagram. Rate of change in $f \circ g$ at x is the same as ROC of g at x times ROC of f at $g(x)$.
3. Theorem: Chain rule, for f and g differentiable, $f \circ g$ is also differentiable and

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

   (a) Proof idea:

$$\frac{d}{dx} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h}$$

   then change of variable and done. See text for technical details.
   (b) Leibniz notation is convenient.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

4. Examples:
   (a) $(x^2 + 2x + 3)^1 00$
   (b) $\tan^3(\sin(x) + 1)$
   (c) $2^x = e^{2\ln(x)}$ leading towards below.
   (d) Challenge is identifying f and g for composition $f(g(x))$.
5. This is a versatile new technique. Quotient rule revisited:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx} f(x)(g(x))^{-1}$$

6. General exponential functions
   (a) $2^x = e^{\ln(2)x}$, now differentiate via the chain rule.
   (b) Theorem:

$$\frac{d}{dx} a^x = a^x \ln(a)$$

   (c) Example: $2^{3^{x^2}}$

.5 3.5 Implicit differentiation

1. What remains? Extend the chain rule to further our reach.
   (a) Inverse functions (log for exp, inv trig, others), next section
   (b) Curves which are not functions (circle, trajectories, others), this section, tangent lines should still make sense
2. Example: Find the equation of the tangent line to $x^2 + y^2 = 1$ at point $(1/\sqrt{2}, 1/\sqrt{2})$.
   (a) Check that point is actually on the curve. Need dy/dx at this point then done.
   (b) Assume $y = y(x)$ locally to this point, apply chain rule
   (c) Note, could have solved for y first in this case, try on own.
3. Example: Folium of Descartes
   (a) $x^3 + y^3 = 6xy$, not a function, cannot solve for y
   (b) Wiki page story, history of calculus
   (c) Find tangent line at (3,3). Horizontal tangents?
4. Power rule revisited
   (a) Can extend the power rule to rational exponents.
   (b) $y = x^p/q$ gives $y^q = x^p$, diff both sides and solve.
   (c) What about irrational powers?
5. Inverse functions:
   (a) Recall: Inverse function idea
      - General case
      - Simple example: $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$. Graph together. Domain and range swap.
      - Desmos graphs
   (b) Differentiating inverse functions
      - $f(x) = sqrt(x)$ derivative connected to $f^{-1}(x) = x^2$. Refer to the graph.
      - General case via implicit differentiation: $d/dx f^{-1}(x) = 1/f'(f^{-1}(x))$
      - Note how graph tangent line slope changes when reflected.
6. Derivatives of inverse trigonometric functions
   (a) Example: Inverse sine
      - Review of inverse sine (hint at other trig functions, main 4 most important)
      - Restricted sine is invertible.
      - Use implicit differentiation to compute. Check that agrees with previous general inverse function formula. Key is to use a right triangle to eliminate y.
      - Key is domain restriction, make sure to write down.
   (b) Domain restrictions for main 4 trig functions.
      - $\arcsin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
      - $\arccos(x)$, $0 \leq x \leq \pi$
      - $\arctan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$
      - $\text{arcsec}(x)$, $0 \leq x \leq \pi$

(c) 4 derivative formulas. Remember these.

$$\begin{aligned}
\frac{d}{dx}\arcsin(x) &= \frac{1}{\sqrt{1-x^2}}, \quad -1 \le x \le 1 \\
\frac{d}{dx}\arccos(x) &= \frac{-1}{\sqrt{1-x^2}}, \quad -1 \le x \le 1 \\
\frac{d}{dx}\arctan(x) &= \frac{1}{1+x^2} \\
\frac{d}{dx}\text{arcsec}(x) &= \frac{1}{x\sqrt{x^2-1}}
\end{aligned}$$

(d) Example: Use two methods to find the derivative ((students) chain rule, (me) draw triangle and simplify as algebraic expression).

$$y = \sin(\cos^{-1} x)$$

.6 3.6 Derivative of logarithmic functions

1. Review of logs:
   (a) Definition, keep track of domain and range
   (b) Log properties
   (c) Historic motivational interestingness: `https://en.wikipedia.org/wiki/History_of_logarithms`
2. Derivatives of logarithms:
   (a) Already can differentiate exponentials. Use implicit differentitation to get at it.
   (b) Result:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

   and

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

   (c) Examples:
      i. $\frac{d}{dx}(\ln(x^2 e^x)/(x+1))$, leverage log props, can introduce logs when needed, just remove later, see below.
      ii. Theorem: $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, force the domains to match, graph in desmos
3. Logarithmic differentiation: Introduce logs to leverage sweet properties.
   (a) Example: $y = x^x$, then $y' =$? (no such rule)
   (b) Example: $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$, then $y' =$? (quotient, product, chain rule madness)
   (c) Summary of steps:
      i. Identify the situation (lots of multiplication, quotient, and powers)
      ii. Take log on both sides (if possible) and simplify using the log properties.
      iii. Differentiate implicitly with respect x
      iv. Solve for y'
      v. What if $y = f(x) < 0$ for some x? Use absolute value.

$$|y| = |f(x)|, \quad \ln(|y|) = \ln(|f(x)|), \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{f(x)}f'(x), \quad \frac{dy}{dx} = \dots$$

   (d) Example: Finally, the full power rule: $y = x^n$, n any real number, log differentiation.

4. Important results to know:

   (a) Theorem:

   $$\lim_{x\to 0}(1+x)^{1/x} = e$$

   Reason: $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$. So,

   $$1 = f'(1) = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\ln((1+h)^{1/h})$$

   Then, because exponential functions are continuous,

   $$\lim_{x\to 0}\ln(1+x)^{1/x} = 1 \quad \Rightarrow \quad e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0}(e^{\ln(1+x)^{1/x}}) = \lim_{x\to 0}(1+x)^{1/x}$$

   (b) Corollary: Take $n = \frac{1}{x}$ above,

   $$e = \lim_{n\to\infty}(1+\frac{1}{n})^n \quad \text{(holy compound interest Batman!)}$$

   (c) Continuous compounded interest: PERT all ova the place.

   $$\lim_{n\to\infty} P(1+\frac{r}{n})^{nt} = Pe^{rt}$$

   (d) Wikipedia continuous growth

   (e) Note, this leans on the previous limit

   $$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

.7 3.8 Exponential growth and decay

1. Differential equations: Translation of change into calculus
   See the many examples:

   (a) https://en.wikipedia.org/wiki/Differential_equation

   (b) https://people.maths.ox.ac.uk/trefethen/pdectb.html

   (c) Pixar research, Frozen video demo

   (d) Def: A differential equation is an equation involving derivatives where the unknown is a *function*. (analogous to algebraic equations)

2. DEs and Exponential growth: Population grows at rate proportional to size.

   (a) $\frac{dy}{dx} = ry$, r a postitive constant, $y(0)$ initial condition.

   (b) Example: $r = 1, y(0) = 10$. Graph and interpret. $r = 2, -3$?

   (c) General solution $y = y(0)e^{rt}$

   (d) Trouble is, don't usually know r. Need to find this from data.

   (e) La Cross population growth. Find population now and 10 years ago. Project population in 10 year. Plot result in desmos. Google real trends.

   (f) Issue: Exponential growth is unrealistic long term. Can modify rule.

3. Improved population growth. Assume a carry capacity L.

   (a) $y << L$ increase fast then slow down, $y >> L$ decrease fast then slow down, $y \approx L$ little change, $y \approx 0$ little change.

   (b) Harder to solve by hand, but this one is doable, often not possible.

   (c) Approximate with slope field. Google dfield.

   (d) Google logistic growth.

.8 3.9 Related rates

1. Key idea: Which rates are related?
   - Snowball melting youtube. List all things changing. Which are connected?
   - `https://www.youtube.com/watch?v=LNEBZ8ekU18`
   - Volume of sphere formula. Time as variable.
   - Example: Suppose snowball is melting at $5cm^3$ per minute. How fast is the diameter shrinking when $r = 4cm$?
   - Steps: Picture, assign variables, rates as derivatives comb with data, equation relating all vars, implicit differentiation.
2. Example: A ladder 10 ft long is sliding against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6ft from the way?
   (a) Drawing picture is key, Pythagorean theorem is connection.
   (b) What is the changing rate of θ?
3. Example: Boat pulled to dock by rope 1 ft above the bow of boat. If the rope is pulled at 1 ft / sec, how fast is the boat approaching the dock when 8 ft from doc?
4. Global positioning system story of related rates. Student at MIT. `http://www.pcworld.com/article/2000276/a-brief-history-of-gps.html`

.9 3.10 Linear approximations and differentials

1. Motivation and idea:
   (a) Practical questions: What is $\sqrt{4.1}$, $\sin(46°)$?
   (b) Idea: Use the value of a function around a known $f(a)$ in a smart way.
   (c) Think of
   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
   When x is close to a they basically satisfies the relationship.
2. Linear approximation: the linear (tangent line) approximation of f at a is
   $$L(x) = f(a) + f'(a)(x - a)$$
   Also known as the linearization of f at a. Compare to limit of difference quotient above.
   (a) Idea: $f(x)$ is "locally" a line (around a), draw picture of f and L
   (b) This is an approximation and may not be accurate at all
      - depending on the original shape
      - depending on how close your x is to a
3. **Examples:** Find the linearization of $\sqrt{x}$ at 4
   (a) Use it to approximate $\sqrt{4.1}$, $\sqrt{4.5}$, $\sqrt{6}$ and compare to the real value.
   (b) Find $\sin(44°)$, do the same thing.
   (c) In physics, $\sin x \approx x$ when x is small. This is linearization.
4. Differentials:
   $$dy = f'(x)dx$$

(a) What is this? Reminds of $\frac{dy}{dx} = f'(x)$. What if treat as a ration?

(b) Find the differential of x^2 at $x = 2$. Pick different dx and graph.

(c) Difference between dy and $\triangle y$. Actually, $dy = \delta L$.

(d) This is close to the original conceptualization of calculus.

5. **Example:** A sphere was measured and its radius was found to be 45 inches with a possible error of no more that 0.01 inches. What is the maximum possible error in the volume if we use this value of the radius?

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \Delta V \approx dV = 4\pi r^2 dr$$

6. Can we replace $f(x)$ locally by a quadratic equation?

   (a) Doable? (Yes, need first and second derivatives to match)

   (b) More work? (Yes)

   (c) Better accuracy? (Yes)

   (d) Any polynomial? (Taylor polynomial, calculus 2)

   (e) Why bother replacing functions with polynomials? (Biggest take-away of the section)

   - Approximation of hard calculations
   - Polynomials are nicer functions than anything else, so live in a better place.

   (f) Can we use things other than polynomials? Sure thing (Fourier series) for periodic functions (light, sound, universe of waves).

.10 3.11 Hyperbolic functions

1. Motivation:

   (a) Think about a heavy flexible cable suspended between two points at the same height (the golden gate bridge, telephone cable). This is called a catenary. What is that curve? Not quite a parabola. `https://www.google.com/search?q=catenaries&espv=2&biw=1680&bih=921&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwj5552K2ejKAhVCFR4KHRoADp8QsAQIQw#tbm=isch&q=catenary&imgrc=ES8GEHgRx3OpXM%3A`

$$\frac{e^x + e^{-x}}{2}$$

   (b) What's the derivative?

2. The family of hyperbolic functions, such parallels with regular trigonomety here.

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

   (a) Regular division and such gives the rest. $\tanh(x) =, \ldots$.

   (b) Just as the points $(\cos(t), \sin(t))$ form a circle with a unit radius, the points $(\cosh(t), \sinh(t))$ form the right half of the equilateral hyperbola $x^2 - y^2 = 1$.

   (c) For some applications, this is the correct geometry (special relativity).

3. Hyperbolic identities:

   (a) Odd, even:

$$\sinh(-x) = -\sinh(x), \quad \cosh(-x) = \cosh(x)$$

(b) The "Pythagorean" identities:

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 = sech^2 x$$

(c) The sum formula:

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sin(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sin(y)$$

(d) Double angle formula:

$$\sinh(2x) = 2\sinh x \cosh x$$

4. Derivatives of hyperbolic functions (show this)

$$(\sinh x)' = \cosh x$$

5. Inversere hyperbolic function (show this, substitution, hidden quadratic)

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

6. What you need to know:
   (a) Know they come from application. Be aware.
   (b) You don't have to memorize anything but the definition of the hyperbolic sine and cosine.
   (c) Feel free to check the book when you do the homework
   (d) I may test it as an exercise of derivatives.

Chapter 4 Applications of differentiation

1. We've seen some applications in Ch3, but the list is long. Sometimes seem more mathy than useful.
   (a) https://en.wikipedia.org/wiki/Differential_calculus#Applications_of_derivatives
   (b) https://en.wikipedia.org/wiki/Mathematical_optimization
2. Chapter outline:
   (a) Deeper function understanding: Graphing with detail
   (b) Optimization: Max and min values
   (c) Theory: MVT
   (d) Beginnings: Reversing differentiation, called integration

.1 4.1 Maximum and minimum values

1. Extreme values of functions: Local min and max, absolute min and max.
   (a) Main application: Optimization, largest, cheapest, fastest.
   (b) Def of abs min / max, local min / max
   (c) Eg. Local min if $f(c) <= f(x)$ for all x near c
   (d) Draw picture to illustrate.
   (e) Possible locations: Zero derivatives, corners, discontinuities, endpoints, inflection pts
2. Extreme value theorem (EVT)
   (a) How to ensure EVs happen? Avoid the bad scenarios: holes, asymptotes.

(b) EVT: If $f(x)$ is continuous on closed interval $[a,b]$, then $f(x)$ must attain EV on $[a,b]$.

(c) Note, does not say where it is or how to find, just that it exists.

3. How to find extreme values?

   (a) Must occur at a critical number.

   (b) Cases for critical numbers: Endpoints, stationary points, singular points.

4. Example: Find the absolute max and min by checking the critical numbers. Use Desmos to check.

   (a) $f(x) = x^3 + x^2 - x$ on $[-2, 2]$.

   (b) $f(x) = x^{\frac{2}{3}}$, no interval then add open / closed. Change to $x^{\frac{1}{3}}$

   (c) $f(x) = x + 2\cos(x)$ on $[0, 2\pi]$

.2 4.2 The mean value theorem

1. Big picture of the MVT:

   (a) Math theory detour, useful for proofiness rather than application

   (b) Big picture: Connect IROC and AROC, no limits

   (c) Most used calculus result in math world

2. Rolle's Theorem: Let function $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$. Then there's a number c in (a,b) such that $f'(c) = 0$.

   (a) Ask to draw picture and see why true.

   (b) More than one c possible.

   (c) Why is closed interval important? Diff? $f(a) = f(b)$?

3. Example: Show that $x^3 + x - 1 = 0$ has only one real solution.

   (a) Using IVT on $[0,1]$ to show existence.

   (b) What if had 2 zeros on $(0,1)$, $f(a) = f(b) = 0$? Then Rolle's theorem says there is c in (a,b) such that $f'(0) = 0$. But, $f'(x) = 3x^2 + 1$. So, can only have 1 zero.

4. Mean Value Theorem: Let function $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there's a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   (a) Ask to draw picture and see why true.

   (b) More than one c possible.

   (c) Why is closed interval important? Diff? $f(a) = f(b)$?

   (d) Average rate of change equals inst rat of change.

   (e) When you are driving, there'll always be a moment that your instantaneous velocity is the same as the average velocity.

   (f) Suppose you are driving from La Crosse to Madison: 150 mile, 1.5 hours. What should the speeding ticket be written for? 100mile/h

5. Example: Find an upper bound on difference $\cos(1) - \cos(1.1)$.

   (a) Apply MVT via rearrangement. Check hypothesis first.

   (b) Connected to linearization.

6. Theorems: The power of MVT
   (a) If $f'(x) = 0$ on (a, b), then f is constant on (a, b).
   $$0 = f'(x) = (f(x_1) - f(x_2))/(x_1 - x_2)$$
   for all x_1, x_2 on (a, b), then $f(x1) = f(x2)$.
   (b) If $f'(x) = g'(x)$ on (a, b), then $f(x) = g(x) + C$ for some constant C
   $$h(x) = f(x) - g(x) \quad \Rightarrow \quad h'(x) = 0$$
   and $h(x) = C$ by above theorem.

.3 4.3 How derivatives affect the shape of a graph

1. Example
   (a) Graph $f(x)$, $f'(x)$ and $f''(x)$ for $f(x) = x^3 + x^2 - x$
   (b) How is f' related to f, f'' to f', f'' to f?
   (c) Desmos
2. First derivative $f'(x)$
   (a) **Increasing/decreasing test**
   - If $f'(x) > 0$, then f is increasing ($a < b$ gives $f(a) < f(b)$)
   - If $f'(x) < 0$, then f is decreasing ($a < b$ gives $f(a) > f(b)$)

   (b) **The first derivative test** Suppose c is a critical number for f (possible local max/min). Let them fill in blank.
   - If $f'(x)$ changes from positive to negative at c, then .... f has a local max at c.
   - If $f'(x)$ changes from negative to positive at c, then .... f has a local min at c.
   - If $f'(x)$ does not change sign at c, then .... f has no local max or min at c. (called a saddle point)
3. Example: Draw number line to find inc/dec. Find min/maxs. Draw on own.
   (a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
   (b) How to graph f? Have a pretty good picture. What else can we add for detail? Where are turning points? Zeros?
4. Second derivative $f''(x)$, above example. Let them fill in blank.
   (a) **Concavity test**
   - If ... $f''(x) > 0$, then f is concave up
   - If ... $f''(x) < 0$, then f is concave down

   (b) **The second derivative test:**
   - $f'(c) = 0$, $f''(c) > 0$, local min
   - $f'(c) = 0$, $f''(c) < 0$, local max

   (c) Used to find local max/min, easier than first derivative test.
   (d) If $f''(x) = 0$, it's inconclusive. Why? Think of a graph. This is an inflection point.
   (e) When can't the second derivative test be used? If f'' does not exist (corner)
5. **Examples:** Graph sketching
   (a) Finish above example. Add inflection point.
   (b) Try on own: $f(x) = x^3 - 3x^2 - 9x + 4$, .

.4 4.4 Indeterminate forms and L'Hospital Rule

1. Recall: Indeterminate form, the reason for limits.
   (a) Limit of difference quotient. 0/0 IF. Limit idea invented to handle this problem.
   (b) Already have algebraic techniques: $f(x) = x^2, f'(1) =?$ $f(x) = \sqrt{x}, f'(1) =?$
   (c) Our techniques are not enough: $f(x) = \ln(x), f'(1) =?$ Used the inverse relation to handle in past.
   (d) Key: Indeterminate forms can be ANYTHING. Modify above example to show 2, 200, π, ∞, 0, etc.
   (e) Types of indeterminate form:
      - Quotient $0/0$
      - Quotient ∞/∞
      - Product $0 \cdot \infty$
      - Difference $\infty - \infty$
      - Exponent $0^0, 1^\infty, \infty^0$
      - Strategy: Rewrite all as first two quotients.

2. Theorem: (l'Hospital's Rule) If f and g are differentiable around $x = a$, and $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

   if $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists or is $\pm\infty$
   (a) Proof idea (0/0 IF case): Close to $x = a$, replace $f(x)$ with linearization $f'(a)(x - a)$ (tangent line approximation). Draw graph. Likewise for g. Cancel $(x - a)$ factor to get result.
   (b) Tangent line tells rate to 0 or inf. This is the tug of war.
   (c) Note: LR works for one sided limits also

3. Examples: Check on own. Check if LR hypothesis holds first.
   (a) $\lim_{x\to 1}(x^2 - 1)/(x - 1)$
   (b) $\lim_{x\to 1} ln(x)/(x - 1)$
   (c) Can get crazy: Which grows faster, x^{1000} or e^x? How to tell? Look at the ratio: $\lim_{x\to\infty} \dfrac{x^{1000}}{e^x}$
   (d) Old limits are easier:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{h\to 0} \frac{\sin h}{h} = 2/3, \quad \lim_{x\to 0} \frac{\sin 2x}{\sin 3x} = 2/3$$

   (e) Beware of temptation:

$$\lim_{x\to\infty} \frac{\sqrt{x^2 - 1}}{x} =?$$

4. Other indeterminate forms: Idea is to always transfer to $0/0$ or ∞/∞ form.
   (a) $\lim_{x\to 0^+} (1/x - 1/(e^x - 1))$ (add fractions)
   (b) $\lim_{x\to 0} x^x$ (log and exp, then log prop)
   (c) $\lim_{x\to inf} (1 + 1/x)^x$
   (d) Mention it's super important: indeterminant form is the most common case we want to work on. (derivative, def integral, etc)

.5 4.5 Summary of curve sketching

1. Guidelines for curve sketching (we've already covered this!):
   (a) Find the omain
   (b) Locate x and y intercepts
   (c) Does f have symmetry (even or odd)?
   (d) Asymptotes (horizontal, vertical, oblique)
   (e) Where is f increasing / decreasing?
   (f) Find local mins and maxes (critical pts and 1st or 2nd derivative test)
   (g) Concavity and points of inflection
   (h) Put all together to get a fantastic picture
2. **Examples:**
   (a) $f(x) = \frac{1+2x^2}{1-x^2}$ (horizontal and vertical asymptotes)
   (b) $g(x) = \frac{-3x^2+2}{x-1}$ (oblique asymptote, need long division)

   Read the section and finish the homework!

.6 4.7 Optimization problmes

1. Idea:
   (a) Find min/max of a target function $f(x)$ subject to some sort of constraint ($a \leq x \leq b$).
   (b) Same as abs min / max problem.
   (c) Check critical points (stationary, singular, endpoints)
   (d) Difficulty is translating the problem into math (function, relating variables, etc)
2. Chicken fence next to dog area:Try on own
   (a) 200 ft of fence, on corner of dog fence (10ft and 20ft sides). What is the maximum area enclosed?
   (b) Generalize to steps as with related rates.
3. Optimization problem strategy:
   (a) Make sure it's an optimization problem (-est, most, least)
   (b) Draw a picture to help
   (c) Find the variable y that you want to minimize/maximize, introduce other notation
   (d) Find the changing variable x
   (e) Write $y = f(x)$ as a function of x, eliminate other variables if needed
   (f) Identify an closed interval for x (why necessary? Extreme Value Theorem)
   (g) Find the extreme value of y
   (h) Answer the original question in words
4. **Examples:**
   (a) A cylindrical can is required to hold 1 liter of oil. Design the can to minimize the use of material.

$$S = 2\pi r^2 + 2\pi rh, \text{ (eliminate } h\text{, can also use implicit diff)}$$

(b) Find the point on the curve $y = 2x - 1$ closest to the point (3, 2).

$$d = \sqrt{(x-2)^2 + (y-2)^2}, \text{ (can eliminate or implicit diff)}$$

(c) What's the area of the biggest rectangle that can be inscribed inside a unit circle?

$$A = 2xy = 2x\sqrt{1-x^2}$$

5. So many applications here, especially in business.
   - Wiki page
   - UWL journal ug research paper

.7 4.8 Newton's method

1. **Motivating Example** Solve $x^3 - 3x + 1 = 0$
   - Pick place to start: $x_0 = 0$
   - Find the linearization at (x_0, y_0)
   - Follow linearization to get zero which approximates f's zero.
   - Show Desmos right away
   - Write down the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   - Do the iteration by hand
   - Do $x_0 = 7$, 4.

2. Idea of Newton's method

   (a) Again, replace f by linearization (calculus), sweet move.

   (b) Does it always work? What could go wrong? We need more than just a formula.

      i. Could be no root at all (IVT to check existence)

      ii. Could hit zero derivative, shoot to infinity (zero division)

      iii. Find wrong root.

      iv. Even though the formula stays the same, the result depends on the initial value x_1

      v. Slows down at roots with multiplicity.

      vi. Sometimes it just doesn't work (MVT, diverges to infinity)

$$x^{1/3} = 0, \quad x_1 = 1.$$

      vii. Under the right assumptions (continuous and differentiable around the root, choose x_1 close enough), can prove Newton's method is fast and effective.

3. Find the solution of $\cos x = x$ using the Newton's method

   (a) Draw a picture to see how many solutions are there

   (b) Find the iteration method

   (c) For which initial value does it fail

   (d) Assign the initial value

   (e) Compute the result

   (f) Mention fixed point methods if interested $x_n = \cos(x_{n-1})$

4. Mind-blowing awesomeness:

   (a) http://octave-online.net/

   (b) $\sqrt{2}$ via $x^2 = 2$.

   (c) π via $\sin(x) = 0$ fast, $\cos(x) = 1$ slow. Why? Multiplicity of root.

   (d) R pseudocode:

```
# newton's method
options(digits=16)

f <- function(x){x^2-2}
fp <- function(x){2*x}

x <- 1
for (i in c(1:10)){
  x <- x - f(x)/fp(x)
  print(x)
}
```

   (a) How to improve on Newton? Taylor series to higher order

   (b) Fractals and complex numbers

   - Zoomin: https://www.youtube.com/watch?v=0jGaio87u3A
   - Applications: https://en.wikipedia.org/wiki/Fractal#Applications_in_technology
   - Nature: https://www.google.com/search?q=fractal+nature&espv=2&biw=1309&bih=781&tbm=isch&tbo=u&source=univ&sa=X&ved=0ahUKEwjS677XsIbMAhUMMSYKHSwkBC0QsAQIGw

.8 4.9 Anti-derivatives

1. Motivation: Goal is to reverse differentiation. Key here is lack of uniqueness.

   (a) Many physical laws quantify change, but computing the underlying quantities is most interesting.

   (b) Saw this already with DEs for exponential growth.

   (c) Conservation law.

   (d) Free fall $a(t) = 9.8m/s^2$, can get velocity and distance. Free fall with drag $a(t) = 9.8 - kv$ (gravity const - drag).

   (e) Kepler's laws of planetary motion spurred Newton to work on Calculus and support theory of physics.

   (f) Any physical (or other) law.

2. Def: $f(x)$ is an antiderivative of $f(x)$ on interval I if $f'(x) = f(x)$ for all x in I

   (a) Example: $2x$ has antiderivative x^2

   (b) Note, not unique here. Any $f(x) + C$ works for C an arbitrary constant. Graph multiple antiderivatives.

   (c) Think of derivative as an operator we aim to reverse.

3. Def: The collection of all antiderivatives of $f(x)$ is denoted $\int f(x)\ dx$. That is, $\int f(x)\ dx = F(x) + C$ where $F(x)$ is any antiderivative and C an arbitrary constant.

   (a) New idea to capture all reverse derivatives.

   (b) Example: $\int 2x\ dx$

(c) Will explain notation in Chapter 5.

4. Theorem: Properties of the indefinite integral come from differentiation. They are easily checked through that lens.

   (a) $\int cf(x)\ dx = c\int f(x)\ dx$

   (b) $\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$

   (c) This allows us to treat $\int\ dx$ as an operator much like $\frac{d}{dx}$.

5. Examples:

   (a) Integration rules (no need to memorize, just reasoning): $k, x^n, \frac{1}{x}, \sin(x), \cos(x), \sec^2(x), \csc^2(x)$

   (b) $\sec(x)\tan(x), \csc(x)\cot(x), e^x, e^{kx}, a^x, \frac{1}{1+x^2}$

   (c) $\int(1 + \sin(x) + 4x^2 + 2^x)\ dx$

   (d) Not always easy: $2x\cos(x^2)$, $\tan(x)$, xe^x

   (e) Find $f(x)$ such that $f''(x) = x^2$ and $f(3) = 1$, $f'(4) = 1$.

Chapter 5

1. Second paradox of calculus: Area under the curve

   (a) Area under the curve.

   (b) Why bother?
      - Lots of applications, anything involving accumulation.
      - Probs to chance, velocity to disp, force to work, Calc 2 etc

   (c) Approach? Mirror tangent line
      - Approximation, limiting process, indet form. Again limit is key.
      - Deep connection to derivative.
      - FTOC: Newton connected IROC to AUC, Leibnitz did AUC to IROC

2. Chapter outline:

   (a) Area under curve.

   (b) FTOC

   (c) Basic integration techniques.

.1 5.1 Areas and distances

1. Motivation: Classic problem of physics: $d = rt$

   (a) $s = s(t)$, we know $\frac{ds}{dt} = v$. What about the reverse connection? Our car knows...maybe.

   (b) Constant velocity case, 60mph for 4 hrs. Graph. Distance is AUC.

   (c) Changing velocity, can approximate velocity and approximate AUC. Smaller subintervals the better. See the tug of war.

   (d) Think of summing up velocity to get distance.

2. Example: Approximating AUC

   (a) $f(x) = x^2 + 1$ on $[0, 2]$

(b) Approximate by simple shapes. 4 rectangles of equal base.

(c) Lots of ways to choose sample point: Left, right, midpoint.

(d) Desmos and Riemann sum. More rectangels. Take the limit to get the whole way. Trouble is how to formalize this process.

3. Example: Approximating AUC

   (a) $f(x) = x^2 + 1$ on $[0, 2]$

   (b) Chop into equal width subintervals. n total.

   (c) Choose right endpoint as sample point x_i. Other options are left and midpoint.

   (d) Simplify Riemann sum via f.

   (e) Need special summation formulas involving i, i^2, i^3, more?

   - Def of summation notation, simple example.
   - Theorem: Special summation formulas for i, i^2, i^3.
   - Theorem: Properties of summation (ca_i, $(a_i + b_i)$)

   (f) Finish previous example.

4. Def: AUC as limit of Riemann sum

   (a) Draw general picture.

   (b) Clarify notation, $[a, b], f(x), x_i, x_i^*, \Delta x$.

   (c) Challenge, need to simplify the summation to compute limit. Usually hard.

5. Example: Try on own. Find AUC of $f(x) = -2x + 6$ on $[0, 4]$. Check via geometry. Note the signed AUC.

6. Area thru history: Archimedes and quadrature of parabola, volume of sphere, Cavaliri principle

.2 5.2 The definite integral

1. Goal:

   (a) Definition and properties for calculation

2. Definite integral of f from a to b

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

   Provided the limit exists.

   (a) AUC as limit of Riemann sum, graph, rectangles, sample point, same calculations as previous section

   (b) Package as an operation on function f(x)

   (c) 3 main cases for sample point x_i^*, but can be anything reall.

   (d) Notes: Result is a number, Δx to dx as in differentials, think of as operator

   (e) Terminology: Integrand, upper and lower limit, bounds

3. Properties of the definite integral: Signed AUC

   (a) Pos, neg

   (b) Even, odd functions

   (c) Signed area under the curve, can see this from the definition

4. Properties of the definite integral: Limit laws and area ideas
   (a) $f(x) + -g(x), cf(x)$
   (b) $\int_a^a$
   (c) $\int_a^b = -\int_b^a$
   (d) $\int_a^b = \int_a^c + \int_c^b$
5. Example: Try on own. Geometry first. Riemann sum next.
   (a) $\int_{-1}^{1}(2x-1)\ dx$
   (b) If cannot use geometry, need summation formulas. Bottleneck here.
   (c) Limits can fail to exist, but DI usually exist.
   (d) Limitations of the Riemann sum if you go too deep.

.3 5.3 The fundamental theorem of Calculus

1. Biggest idea of this course...
   (a) Tangent line and area problems are completely connected (reverses of eachother).
   (b) Results from each field flow into eachother. Area becomes much easier.
   (c) This is Newton and Liebnitzs primary conribution
2. The Fundamental Theorem of Calculus (at last!)
   (a) Part 1: For $g(x) = \int_a^x f(t)\ dt$, $g'(x) = f(x)$ (key connection)
   (b) Part 2: $\int_a^b f(x)\ dx = F(b) - F(a)$ (most useful)
   (c) Part 1 connects definite integration and differentiation. Part 2 makes definite integrals easier (boils down to antiderivative problem).
3. Example: Idea of FTOC part 1.
   (a) Accumulation function: $g(x) = \int_0^x 2t\ dt$ (turn area into function)
   (b) Compute $g'(x)$. (differentiate area to get curve)
   (c) General picture.
   (d) Main point is area and derivative are connected.
4. Example: FTOC part 2 is the computing game changer.
   (a)
   $$\int_0^2 (x^2+1)\ dx = \frac{8}{3} + 2 \quad \text{(same as before)}$$
   (b) Give many other examples. Anything antiderivative we can handle is fair game.
5. **Proof of FTOC:** If f is continuous on [a,b], define
   $$g(x) = \int_a^x f(t)\ dt$$
   (a) **Part 1:**
   - Assume $g(x)$ is differentiable (can show this), then we have access to difference quotient.
   $$g'(x) = \frac{d}{dx}\int_a^x f(t)\ dt = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\ dt$$

- Show that $g'(x) = f(x)$ via use of the Squeeze Theorem. If $m \leq f(x) \leq M$ on $[x, x+h]$, then

$$m \leq \frac{1}{h} \int_x^{x+h} f(t)\ dt \leq M$$

(b) **Part 2:**

- From part 1, we have one antiderivative of f.

$$g(x) = \int_a^x f(t)\ dt, \quad \text{where } g'(x) = f(x)$$

- Then any antiderivative is $F(x) = g(x) + C$ (where C changes when a change).
- Then,

$$F(b) - F(a) = g(b) - g(a) = g(b) - 0 = \int_a^b f(t)\ dt$$

6. Another example: $\displaystyle\int_{-1}^{3} (x-1)(2x+1)dx$.

7. A historic controversy: Issac Newton vs Gottried Leibniz

   - 1666: Newton start to work on calculus (manuscript)
   - 1674: Leibniz started to work on calculus
   - 1684: Leibniz published calculus
   - 1687: Newton's 1st publication about calculus
   - 1693: Newton's publication of fluxion
   - 1696: L'Hospital published his work and quote Leibniz's work
   - 1699: the controversy began (the royal society)
   - 1704: Newton's full work
   - 1711: the controversy broke out

.4 5.4 Indefinite integrals

1. Indefinite integrals, again...

   - Indef vs dev integral, all antiders vs AUC, properties agree
   - Connection via FTOC, thus same notation.

2. Examples: Thanks to FTOC, finding definite integrals is just as easy. More ideas involved when it comes to area though. Even though cannot integrate directly, can still manage.

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{(some notation here)}$$

   (a) $\displaystyle\int_{-1}^{2} (x - 2|x|)\ dx$

   (b) $\int_0^{\frac{3\pi}{2}} |\sin x|\ dx$

   (c) $\int_{-\pi/2}^{\frac{3\pi}{2}} |\sin x|\ dx$

3. Net change theorem: Context for the FTOC

   (a) FTOC: $f'(x)$ is rate of anything, $f(b) - f(a)$ is net change

(b) Example: $\int_0^5$ velocity dt = distance travelled

(c) $\int_0^5$ oil drip rate dt = oil lost

(d) $\int_0^5$ (birth rate – death rate) dt = populatio nchange

(e) Adding up IROC gives net change

.5 5.5 The substitution rule

1. Challenge of integration:
   - Turns out integration is much harder than differentiation. Often there is no technique (not possible)
   - Ex: $e^{x^2}dx$, $\int \ln(sin(x^2))dx$, etc
   - Wiki nonelemntary integral. Gaussian. Normal distribution
2. Hints at the idea of substitution:

   (a) Level 1: give direct examples

   $$\int 2x\cos(x^2)\ dx, \quad \int 2\sin x\cos x\ dx$$

   (b) Level 2: modified by a constant

   $$\int e^{2x}\ dx, \quad \int \frac{\tan^{-1} x}{1+x^2}\ dx, \quad \int \frac{x}{1+x^2}\ dx$$

   (c) Level 3: not so obvious, but doable.

   $$\int x^5\sqrt{x^2+1}\ dx$$

3. Chain rule

   (a) $\int f'(g(x))g'(x)dx = \int f(u)du$ by renaming $u = g(x)$.

   (b) Proof idea: From chain rule,

   $$\int f'(g(x))g'(x)\ dx = f(g(x)) + C = f(u) + C = \int f'(u)\ du$$

   (c) Once used, hopefully we can integrate f. After integrate, substitute back.

4. **Examples:**

   (a) Do all the easy ones again with this structure.

   (b) Harder:

   $$\int x^2\cos(x^3+2)\ dx, \quad \int \tan x\ dx$$

   (c) Harder yet:

   $$\int x^5\sqrt{x^2+1}\ dx, \quad \int \sin^4 x\cos^3 x\ dx$$

   Here we see the power of undoing such a simple rule. This opens the door to integrating many more functions. How about definite integrals?

5. **Theorem:** Substitution Rule for Definite Integrals
   If we assume

- g' is continuous on $[a, b]$
- f is continuous on the range of $u = g(x)$

then we have

$$\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

6. **Examples:**

   (a) $\displaystyle\int_0^4 \sqrt{3x+4}\ dx = \frac{112}{9}$

   (b) $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}dx = \frac{1}{14}$

   (c) $\displaystyle\int_1^e \frac{\ln x}{x}dx = \frac{1}{2}$

   (d) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}dx = \frac{1}{2}$

   (e) $\displaystyle\int_{-1}^1 \sqrt{1-x^2}dx = \frac{\pi}{2}$ again via geometry. Can we actually compute? Yes, trig sub. Calc 2 will revisit...mwhuahahahahaha....

7. Integral of Symmetric Functions Suppose $f(x)$ is continuous on $(-a, a)$

   (a) If f is even, then
   $$\int_{-a}^a f(x)\ dx = 2\int_0^a f(x)\ dx$$

   (b) If f is odd, then
   $$\int_{-a}^a f(x)\ dx = 0$$

8. More area intuitiveness. Show for any integrable f,
   $$\int_a^b f(x+c)\ dx = \int_{a+c}^{b+c} f(x)\ dx$$

Chapter 6: Applications of Integration

Here we bend the idea of Riemann sum to see how versitile accumulation is.

.1 6.1 Area between curves

1. Area between curves.

   (a) Find the area between f and g on interval $[a, b]$.

   (b) Can use separate integrals for f and g, but too complicated.

   (c) Instead construct a new one from a Riemann sum.
   $$A = \int_a^b (f(x) - g(x))\ dx$$

2. Examples: Challenge can be finding the bounds of integration.

   (a) Find the are enclosed by $y = x, y = 4 - x, x = 0$.

(b) Find the area between $y = x^2$ and $y = 2 - x^2$. Repeat for interval $[-2, 2]$.

(c) Generalize formula to

$$A = \int_a^b |f(x) - g(x)| \, dx$$

(d) Find the area between $y = x^3$ and $y = x$. Could have used symmetry.

3. Integrate with respect to y.

$$A = \int_c^d (f(y) - g(y)) \, dy, \quad A = \int_c^d |f(y) - g(y)| \, dy$$

(a) Repeat for last example.

.2 6.2 Volumes

1. Idea:

   - Last section and chapter, we intuitively find area by summing length. Here we advance one dimension.
   - Compute volume by summing area.
   - First example: Volume of a sphere of radius r.

$$V = \int_{-r}^{r} A(x) \, dx = \int_{-r}^{r} \pi y(x)^2 \, dx$$

   - Look carefully to see why this works. Approximate by simple solids.

2. Definition:

   (a) Limit of sum of simple volumes (cylinders, etc), draw with previous picture.

   (b) Formula

$$V = \lim_{n \to \infty} \sum A(x_i^*) \Delta x = \int_a^b A(x) \, dx$$

3. Examples: Solid of revolution

   (a) Find the volume generated by rotating the region $y = \sqrt{x}$, $0 \le x \le 1$ with respect to the x-axis

   (b) General formula for disc rotation.

$$V = \int_a^b A(x) \, dx = \int_a^b \pi (f(x)^2) \, dx$$

   (c) Extension to washers: Find the volume generated by rotating the region contained by $y = x^3$, $y = x$, $x \ge 0$ with respect to the x axis.

   (d) General formula for washer rotation.

$$V = \int_a^b A(x) \, dx = \int_a^b \pi \left[f(x)^2 - g(x)^2 \right] \, dx$$

   (e) Rotate about $x = -1$, $y - axis$.

4. Can handle other shapes as well.

   (a) Pyramid of square base length 2, height 3. Cross sections are squares.

.3 6.3 Volumes by cylindrical shells

1. When you want to change the direction of integration, but you cannot.
   (a) Example: Rotate $f(x) = x - x^2$ about the y-axis. Washer method is challenging. Use the x or y direction.

.4 6.4 Work

1. Work: force times distance
2. Unit: ft-lb, Joule (m times N)
3. Formula: work done in moving the object from a to b

$$\int_a^b f(x)\, dx$$

   $f(x)$: force

4. Hooke's law: a force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to 15 cm. How much work is done in stretching the spring from 15cm to 18 cm?
5. A tank has the shape of an inverted circular cone with height 10 m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3)
6. A 10 ft chain weights 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it's level with the upper end.

.5 6.5 Average value of a function

1. Discrete average value
2. Average value via Riemann sum
3. Average value of a function formula)
4. The mean value theorem of integrals: if f is continuous on [a,b] then there exists a number c in [a,b] such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

   that is,

$$\int_a^b f(x)\, dx = f(c)(b-a)$$

5. Understanding from physics (average speed = dispacement / time)

Chapter 9 Differential Equations

.1 9.1 Modeling with differential equations

1. Motivation
   (a) Exponential growth
   (b) Logistic function

$$y' = ky(1 - y/R), \quad y = \frac{R}{1+e^{-kx}}$$

(c) Physics

2. Differential equation

    (a) Definition: equations with derivatives

    (b) Order of differential equations

    (c) Definition of solution

3. Solving differential equations

    (a) Analytical

    (b) Direction field

    (c) Numerical: Euler's method

4. Initial value problem