Math 371
Quiz 6, April 3, 2015 Name: Solution

1. Compute the minimal degree Lagrange form interpolating polynomial $p(x)$ for $f(x) = \sqrt{x+1}$ through nodes $x = 0, 3, 8$.

$$p(x) = f(0)\frac{(x-3)(x-8)}{(0-3)(0-8)} + f(3)\frac{(x-0)(x-8)}{(3-0)(3-8)} + f(8)\frac{(x-0)(x-3)}{(8-0)(8-3)}$$

$$= 1\,\frac{(x-3)(x-8)}{24} + 2\,\frac{x(x-8)}{-15} + 3\,\frac{x(x-3)}{40}$$

$$= \frac{1}{24}(x-3)(x-8) - \frac{2}{15}x(x-8) + \frac{3}{40}x(x-3).$$

2. Use the polynomial interpolation error formula to bound the error of p's approximation of $f(1)$. Compute the exact error of p's approximation of $f(1)$ by using your work from problem 1.

$$f(x) = \sqrt{x+1}, \quad f'(x) = \frac{1}{2}(x+1)^{-1/2}, \quad f''(x) = -\frac{1}{4}(x+1)^{-3/2}, \quad f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2}$$

$$|f(1) - p(1)| = \left|\frac{f^{(3)}(\xi)}{3!}(1-0)(1-3)(1-8)\right|, \quad \text{some } \xi \in [0,8]$$

$$\le \left|\frac{f^{(3)}(0)}{3!}\,1(-2)(-7)\right| \quad (\text{since } f^{(3)} \text{ is decreasing})$$

$$= \frac{3/8}{6}(2)(7) = \frac{21}{24} = \frac{7}{8}.$$

Exact error is

$$|f(1) - p(1)| = \left|\sqrt{2} - \left(\frac{1}{24}(-2)(-7) - \frac{2}{15}1(-7) + \frac{3}{40}1(-2)\right)\right|$$

$$= \left|\sqrt{2} - \left(\frac{7}{24} + \frac{14}{15} - \frac{3}{20}\right)\right| = \left|\sqrt{2} - \frac{7}{24} - \frac{14}{15} + \frac{3}{20}\right|.$$