Assigned: Wednesday, February 25, 2014
Due: Wednesday, March 4, 2014

MTH 371: Homework 5
Fixed Point Methods and Linear Algebra

GENERAL HOMEWORK GUIDELINES:

- On the very first page of your homework, provide your name, date, and homework number.
- Homework will be graded in part on neatness, organization, and completeness of solutions. Multiple pages MUST BE STAPLED.
- Attach all Scilab code, output, and plots to the page *immediately following* each problem.
- Clearly label all plots (title, x-axis, y-axis, legend). Use the "subplot" when needed

1. The goal of this problem is to solve the cubic equation $f(x) = x^3 + 6x^2 - 8 = 0$ via fixed point iterations.
   (a) Use the Intermediate Value Theorem to show f has a root on $[1, 2]$.
   (b) Show function $g_1(x) = x^3 + 6x^2 + x - 8$ has a fixed point which is a zero of f.
   (c) Show function $g_2(x) = \sqrt{\dfrac{8}{x+6}}$ has a fixed point which is a zero of f.
   (d) Show function $g_3(x) = \sqrt{\dfrac{8-x^3}{6}}$ has a fixed point which is a zero of f.
   (e) Picking a starting point of $x_0 = 1.5$ graph a cobweb plot of the fixed point iteration.
   (f) Using the fixed point theorem from class, analyze the convergence of fixed point iterations in (b), (c), and (d) for any starting point. Hint: One should diverge and the others will converge.
   (g) Use Scilab to test your conclusions in part (f).

2. Fixed point iteration: Consider the iteration $x_n = \frac{1}{2}\left(x_{n-1} + \frac{c}{x_{n-1}}\right)$ for constant $c > 0$.
   (a) Prove this iteration converges at least quadratically.
   (b) What will the iteration converge to? Why?
   (c) Write a script in Scilab to confirm your results of parts (a) and (b).
   (d) Relate this iteration to Newton's Method.

3. Find a solution p to $x^4 + 2x^2 - x - 3 = 0$ using the following methods.
   - A fixed point iteration for $g_1(x) = (3 + x - 2x^2)^{1/4}$. First verify that a solution to the above equation is a fixed point for g_1. Use initial guess $x_0 = 0.5$.
   - A fixed point iteration for $g_2(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1}$. First verify that a solution to the above equation is a fixed point for g_2. Use initial guess $x_0 = 0.5$.
   - The Secant Method. Use the code developed above with initial guesses $x_0 = 0.5, x_1 = 2$.
   - Newton's Method. Use the code developed above with initial guess $x_0 = 0.5$.

   For each, compute until you reach absolute error less than 10^{-14}. Use Wolfram alpha to find the exact root to at least 14 decimal digits for calculation of the absolute error. Also, compute an approximation to the rate of convergence for each as was discussed in class (rate $r_n = \frac{\log(e_{n+1}/e_n)}{\log(e_n/e_{n-1})}$ where $e_n = |p - x_n|$ is the absolute error at iteration n).

(a) For each method, print the approximation, absolute error, and computed rate of convergence. Use a table format and label the columns.

(b) How many steps did each method take? What does the rate of convergence stabilize to for each method?

(c) For each method, how does the convergence rate compare to the number of correct digits gained at each step? Generalize what happens for linear, quadratic, and other rates of convergence.

4. Consider the linear system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

(a) Solve this system by using regular Gaussian elimination without pivoting. Show this by hand and clearly show every step as well as all steps for backwards substitution. Verify that your $\vec{x}$ solution solves this system.

(b) Write a Scilab script which computes the solution using Gaussian elimination without pivoting. Compare your results to what you found in part (a).