

Calculus III Notes

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Fun stuff

Chapter 12 Vectors and the geometry of space

.1 12.1 Three-dimensional coordinate systems

1. Rectangular (Cartesian) coordinate system

(a) 2D:

- Basics: xy -plane, orthogonal axis with standard orientation, 4 quadrants, coordinates of point (x, y) , projection onto axis, notation $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}$.
- Distance between two points, Pythagoras, distance formula.

(b) 3D:

- xyz -space, orthogonal axis with standard orientation, 8 octants, coordinates of point (x, y, z) , projection onto xy -plane (xz, yz) . Projection onto axis, notation $\mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$.
- Distance between two points, Pythagoras twice, distance formula, proof in text.

2. Graphs of equations

(a) 2D:

- Point
- Lines: Vertical, horizontal, sloped
- Quadratics, polynomials
- Circles from distance formula, standard form $(x - 1)^2 + (y - 2)^2 = 4$, complete the square if not in standard form.
- Lots more
- Try on own: Regions via inequalities $y < x$, $x^2 + y^2 > 9$, $x/y < 1$, $xy \geq 0$.

(b) 3D:

- Point
- Planes: Vertical ($x = 2$), horizontal ($z = 1$), out at us ($y = 0$).
- Try on own: $x^2 + y^2 = 1$, $x + y = 1$, $z = x^2$, $x < y$.

- Spheres from the distance formula, standard form, complete the square if not in standard form.
- Showcase Geogebra.

3. Homework: 7, 9, 11, 13, 15, 17, 21, 23, 25-37 odd, 45

.2 12.2 Vectors

1. Vector basics: \mathbb{R}^2 , then \mathbb{R}^3 .

- (a) Coordinate (location) vs vector (action such as displacement).
- (b) Vector has 2 attributes, magnitude (size) and direction (angle).
- (c) Location doesn't matter, standard position for comparison.
- (d) Vector components.

$$\vec{a} = \langle a_1, a_2 \rangle = \langle x, y \rangle$$

- (e) Vector from two points \vec{AB} . General formula.
- (f) Magnitude and direction. Need to adjust direction by 180° with arctangent formula for quadrants 2 and 3.

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}, \quad \theta = \arctan(y/x)$$

2. Vector operations: Geometry and algebra, \mathbb{R}^2 then \mathbb{R}^3

- (a) Addition: Parallelogram law, sum of components.
- (b) Scalar multiplication: Stretch / reverse, scale components.
- (c) Subtraction: Triangular law, subtract components, rewrite as

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

- (d) Bonus: Dot product

3. Theorem: Vector properties, all proven component-wise via properties of real number arithmetic, geometric intuition.

- (a) Commutative: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- (b) Associative: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- (c) Addition identity: $\vec{a} + \vec{0} = \vec{a}$
- (d) Addition inverse: $\vec{a} + (-\vec{a}) = \vec{0}$
- (e) Scalar distribution: $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
- (f) Vector distribution: $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
- (g) Scalar associative: $(cd)\vec{a} = c(d\vec{a})$
- (h) Scalar multiplication identity: $1 \cdot \vec{a} = \vec{a}$

4. Unit vectors and standard basis

- (a) \mathbb{R}^2 : $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$, divide by length to make unit.

$$\vec{a} = \langle a_1, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle, \quad \pm \frac{1}{\|\vec{a}\|} \vec{a}$$

- (b) \mathbb{R}^3 : $\vec{i}, \vec{j}, \vec{k}$

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

(c) \mathbb{R}^2 : Vector in terms of angle and magnitude.

$$\vec{a} = \|\vec{a}\| \langle \cos(\theta), \sin(\theta) \rangle$$

5. Application: Wire tension. Hang from a wire, wonder if will break. Know angles from ceiling. How much tension on each wire?

6. Homework: 3, 5, 7, 11, 13, 15, 17, 19, 21, 25, 29, 31, 35, 39, 45, 47

.3 12.3 The dot product

1. Basics of the dot product:

(a) Definition: \mathbb{R}^2 : $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$, \mathbb{R}^3 : $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$,

(b) Examples. Note result is a scalar, not a vector.

(c) Theorem: Properties of the dot product.

- $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot \vec{0} = 0$
- All are easily shown via the def of dot product. Show first two quick.

2. Meaning of the dot product.

(a) Theorem: For θ the smallest angle between \vec{a} and \vec{b} .

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

(b) Proof: Law of cosines (generalized Pythagoras, after peek at proofs of LoC) and dot product properties.

(c) Why useful? Corollary:

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \rightarrow \theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right) \in [0, \pi]$$

(d) Corollary: $\vec{a} \perp \vec{b}$ if and only if $\vec{a} \cdot \vec{b} = 0$.

(e) Example: Find angle between vectors. Show vectors perpendicular. $\vec{0}$ is perpendicular to all vectors. Acute and obtuse cases.

3. Use of dot product, vector orientation.

(a) Direction angles and direction cosines.

(b) \mathbb{R}^3 : Let α be the angle between \vec{a} and \vec{i} . Likewise for angles β, γ and \vec{j} and \vec{k} .

(c) $\cos(\alpha) = \frac{\vec{a} \cdot \vec{i}}{\|\vec{a}\| \|\vec{i}\|} = \frac{a_1}{\|\vec{a}\|}$. Likewise for $\cos(\beta)$, $\cos(\gamma)$.

(d) Theorem:

$$\frac{1}{\|\vec{a}\|} \vec{a} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

(e) Example: Find the direction angles of $\vec{a} = \langle 1, 2, 3 \rangle$.

4. Use of dot product 2, vector projection.

(a) Definitions:

- i. Scalar projection of \vec{b} onto \vec{a} : $\text{comp}_{\vec{a}}(\vec{b})$
- ii. Vector projection of \vec{b} onto \vec{a} : $\text{proj}_{\vec{a}}(\vec{b})$

(b) Find each using cosine of the angle between and dot product connection to $\cos(\theta)$.

(c) Theorem:

$$\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}, \quad \text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \right) \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a}$$

(d) Can see the projection is parallel to \vec{a} .

(e) Examples

5. Physics application, projection as a way to calculate work.

6. Dot product, cosine similarity, recommender systems. Coding demo.

7. Homework: 1, 3, 7, 9, 13, 15, 19, 23, 27, 29, 33, 39, 43, 45, 47, 61

4 12.4 The cross product

1. Basics of the cross product:

(a) Given two non-parallel vectors, find a third non-zero vector which is orthogonal to both. Will use this idea to define planes / tangent planes later on.

(b) Given \vec{a}, \vec{b} not parallel, want \vec{c} such that

$$\vec{a} \cdot \vec{c} = a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad \text{and} \quad \vec{b} \cdot \vec{c} = b_1c_1 + b_2c_2 + b_3c_3 = 0.$$

Eliminate c_3 by multiplying two equations and subtracting to get

$$a_1b_3c_1 + a_2b_3c_2 - a_3b_1c_1 - a_3b_2c_2 = 0$$

which gives

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0.$$

Choose $c_1 = (a_2b_3 - a_3b_2)$ and $c_2 = (a_1b_3 - a_3b_1)$ which yields $c_3 = (a_1b_2 - a_2b_1)$.

(c) Definition: The cross product of \vec{a} and \vec{b} is

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1 \rangle.$$

Note, result is a vector where the dot product gives a scalar.

(d) Theorem: $\vec{a} \times \vec{b}$ is orthogonal to both \vec{a} and \vec{b} . Proof just computes $(\vec{a} \times \vec{b}) \cdot \vec{a}$. Same for \vec{b} .

(e) Determinant notation:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

(f) Example: Find the cross product of two random vectors. Check that worked. What if vectors parallel? One zero?

(g) Orientation of $\vec{a} \times \vec{b}$ and the right hand rule.

2. Information hidden in the cross product.

- (a) Theorem: $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin(\theta)$. See proof in text. Easy except for first part. Surprising at first, but can see just comes from the dot product result.
- (b) Corollary: Two nonzero vectors are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.
- (c) Corollary: $\|\vec{a} \times \vec{b}\|$ gives the area of the parallelogram formed by \vec{a} and \vec{b} . Draw parallelogram. Base times height.
- (d) Find the area of the triangle in \mathbb{R}^3 formed by three random points.

3. Properties of the cross product.

- (a) Consider combinations of cross product of unit basis $\vec{i}, \vec{j}, \vec{k}$. Note in general $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ because of right hand rule. Also since orthogonal basis, $\sin(\pi/2) = 1$ and can see the result is unit. Parallelogram is a square.
- (b) Theorem: Properties of the cross product.
 - $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
 - $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
 - $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
 - $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
 - $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
 - $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
 - All are proven via the component-wise definition of the cross product.

4. Triple product, volume of parallelepiped.

- (a) 3×3 determinant.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- (b) Theorem: The volume of the parallel-piped formed by $\vec{a}, \vec{b}, \vec{c}$ is

$$\|\vec{a} \cdot (\vec{b} \times \vec{c})\| = \|\vec{a}\|\|\vec{b} \times \vec{c}\|\cos(\theta)$$

where $\|\vec{b} \times \vec{c}\|$ is the area of the base and $\|\vec{a}\|\cos(\theta)$ is the height. This comes from our dot product formula.

- (c) Corollary: $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if the triple product is zero.
- (d) Newton used this to derive Kepler's law of planetary motion.

5. Torque definition and magnitude.

6. Homework: 1, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 31, 33, 37, 49, 53

5. 12.5 Equations of lines and planes

1. Equations of lines: Vector, parametric, symmetric.

- (a) \mathbb{R}^2

- Familiar case: $y = mx + b$, Ex $y = 2x + 1$, graph it.
- Two step process: Get to the line via \vec{r}_0 , traverse the line via \vec{v} which is parallel to the line.
- Ex: $\vec{r}_0 = \langle 0, 1 \rangle$, $\vec{v} = \langle 1, 2 \rangle$, then

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle t, 1 + 2t \rangle.$$

Since $t = x$, we have $y = 2x + 1$ again.

- Parameter t moves us along the line in a direction as t increases.
- Vector form is not unique. $\vec{r} = \vec{r}_0 - t\vec{v}$ would give the same line, just traced backwards.

(b) \mathbb{R}^3

- Vector equation: For \vec{v} parallel to the line and \vec{r}_0 the vector from the origin to any point on the line,

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

- Draw picture.
- Parametric equations of a line: For parameter t ,

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$

PEs are not unique though they may draw the same line.

- Symmetric equations of a line: Solve for parameter t .

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

It is possible that a, b, c could be zero.

- (c) Example: Find the vector, parametric, and symmetric equations for the line thru two random points. Where does it intersect the xy -plane? xz ? yz ?
- (d) 3 possibilities for lines meeting now: parallel, intersecting, or skew (not parallel, not intersecting).
 - 3 lines, decide if pairs are parallel, intersecting, or skew. Graph in Geogebra.
- (e) Line segment from point (x_0, y_0, z_0) to (x_1, y_1, z_1) :

$$\vec{r} = (1 - t)\vec{r}_0 + t\vec{r}_1, \quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle, \quad \vec{r}_1 = \langle x_1, y_1, z_1 \rangle, \quad 0 \leq t \leq 1.$$

2. Equations of planes: Vector, scalar, linear

- (a) Harder to define the direction of a plane. Normal (perpendicular) vector does the trick.
- (b) Vector equation of plane: For (x_0, y_0, z_0) a fixed point on the plane, any point (x, y, z) on the plane, and $\vec{n} = \langle a, b, c \rangle$ a normal vector to the plane, we have that

$$\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$$

where $\vec{p}_0 = (x_0, y_0, z_0)$ and $\vec{p} = (x, y, z)$. Draw picture to illustrate.

- (c) Scalar equation of plane: Compute $\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- (d) Linear equation of plane: Combine constant terms of $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

$$ax + by + cz + d = 0$$

- (e) Example: Find the plane thru three random points. Uniquely possible if points are not colinear. Already have point, use cross product to get normal vector. Give all 3 forms. Plot the plane by computing the axis intercepts. Check with Geogebra.

3. Summary: In \mathbb{R}^3 ,

- (a) You need a point and a direction (parallel vector) to define a line.
- (b) You need a point and a normal vector to define a plane.
- (c) Examples: Group challenge.
 - Problems in text: 35, 37, 45, 51.

4. Homework: 1, 3, 5, 7, 11, 13, 15, 17, 19, 23, 29, 31, 35, 37, 39, 41, 45, 49, 51, 53, 55, 59, 63, 65

.6 12.6 Cylinders and quadratic surfaces

1. Summary: Goal is to develop intuition for \mathbb{R}^3 .

(a) We already considered two classes of surfaces in \mathbb{R}^3 : Spheres and planes.

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2, \quad ax + by + cz + d = 0$$

(b) New surfaces for this section:

- Cylinders: Surfaces consisting of all lines (called *rulings*) parallel to a given line and passing thru a planar curve.
- Example: $z = x^2$ is a parabolic cylinder. Parabolas are called vertical *traces*.
- Terminology: A *trace* is a curve of intersection of the surface with planes parallel to the coordinate planes (xy, xz, yz) .
- Quadratic surface: Any surface generated by the general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

- Example: $x^2 + y^2 + z^2 = 1$ is a sphere.
- Quadratic surfaces:

2. Cylinders: Sketch the graph. What are the traces? What are the rulings?

(a) Example: $x^2 + z^2 = 4$

(b) Example: $y = z^2$

3. Quadratic surfaces: Sketch the traces, then the graph.

(a) Cone: $z^2 = x^2 + y^2$.

(b) Elliptic paraboloid: $z = x^2 + y^2$

(c) Hyperbolic paraboloid: $z = x^2 - y^2$

(d) Recall the formula for an ellipse of width $2a$ and height $2b$ centered at the origin. Circle is a special case.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(e) Show text table of 6 classes, won't test the hyper-stuff. Just basics.

4. Homework: 1,3,5,7,11,17,21,23,25,27

Chapter 13 Vector functions

.1 13.1 Vector functions and space curves

1. Finally we do calculus, basic case first: $\vec{r}(t)$ is a vector-valued function. For input t , result is a vector.

(a) Need a function per component.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

(b) Example: Already know lines. Label direction.

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Knowing two points allows to draw the line. Show example.

(c) Example: Corkscrew. Label direction.

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

Helps to graph the projection onto the xy, xz, yz planes.

(d) Example: Try on own.

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

(e) Matching: Text problems 21-26.

(f) Find a vector function to describe the curve of intersection of cylinder $x^2 + y^2 = 4$ and surface $z = xy$.

2. Limits and continuity: Everything component-wise.

(a) If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \vec{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle.$$

(b) $\vec{r}(t)$ is continuous at $t = a$ if

$$\lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$$

(c) Example from previous.

3. Homework: 1, 3, 5, 7, 9, 11, 15, 17, 27, 29, 31, 43, 49

.2 13.2 Derivatives and integrals of vector functions

1. Derivatives of vector functions:

(a) Definition: For $\vec{r}(t)$ any vector function, define the derivative as

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Draw picture in \mathbb{R}^3 for some t . Result is a tangent vector at t . The unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

Also have a tangent line using this vector and point.

(b) Theorem: In \mathbb{R}^3 for vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, we have

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Proof moves the difference quotient inside the vector function component-wise.

(c) Examples:

- \mathbb{R}^2 case, tangent vector to $\vec{r}(t) = \langle t-2, t^2+1 \rangle$ when $t=2$. Draw picture. Tangent line also.
- Tangent vector for any line $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ is \vec{v} and the tangent line is the same line. This is the velocity vector for the line as we will see in the next section. Constant change with constant velocity.
- Find the tangent vector to $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at $t=2$. Tangent line also. Geogebra.

2. Vector function differentiation rules.

(a) Theorem: For $\vec{u}(t)$ and $\vec{v}(t)$ differentiable vector functions,

- $\frac{d}{dt} [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$

- $\frac{d}{dt} [c\vec{u}(t)] = c\vec{u}'(t)$
- $\frac{d}{dt} [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt} [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt} [\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt} [\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

(b) Proof of dot product version, component-wise calculation.

3. Homework: 1, 3, 7, 13, 17, 21, 25, 43, 45, 47

.3 13.3 Arc length and curvature

SKIP

.4 13.4 Motion in space: Velocity and acceleration

1. Finally, velocity and speed.

(a) Definition: The velocity vector function $\vec{v}(t)$ of position of particle curve $\vec{r}(t)$ is given by

$$\vec{v}(t) = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Draw picture in \mathbb{R}^3 . This give speed and direction.

(b) Definition: The speed of particle at position $\vec{r}(t)$ is

$$\|\vec{v}(t)\| = \|\vec{r}'(t)\| = \frac{ds}{dt}/$$

(c) Definition: The acceleration is

$$\vec{a}(t) = \vec{v}'(t).$$

(d) Example: A parameterization of $y = x^2$ is given by $\vec{r}(t) = \langle 2t^2, 4t^4 \rangle$. Plot the velocity and acceleration vectors for $t = 1$. Find the speed. Note the direction of the velocity vector is parallel to the old fashion tangent line.

2. Homework: 1, 3, 7, 9, 11, 15, 19,

3. Chapter review problems:

- (a) Concept check: 1-4, 8
- (b) T/F: 1-6, 11, 14
- (c) Exercises: 1-4, 9, 16-19

Chapter 14 Partial derivatives

Here we return to calculus ideas to extend old idea (functions of one variable $y = f(x)$) to 3 dimensional space (functions of two variables $z = f(x, y)$).

- 2 dimensions: Get IROC for $f(x)$ as $\frac{df}{dx}$ via AROC as $\frac{\Delta f}{\Delta x}$. Graphs of $y = f(x)$ have tangent lines. Key is idea of limit.

- 3 dimensions: Functions like $f(x, y) = x^2 + y^2$ (and even $f(x, y, z)$) should also have rates of change. Surface analogy. Key will still be limit.

Summary of chapter in 6 lines: Curve $y = f(x)$ vs surface $z = f(x, y)$.

- $\frac{df}{dx}$ becomes two first order derivatives $\frac{df}{dx}$ and $\frac{df}{dy}$
- $\frac{d^2f}{dx^2}$ becomes four second order derivatives x^2, xy, yx, y^2
- Linear approximation $\Delta f \approx \frac{df}{dx}\Delta x$ becomes $\Delta f \approx \frac{df}{dx}\Delta x + \frac{df}{dy}\Delta y$
- Tangent line $y - y_0 = \frac{df}{dx}(x - x_0)$ becomes a tangent plane $z - z_0 = \frac{df}{dx}(x - x_0) + \frac{df}{dy}(y - y_0)$.
- Chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ becomes $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$.
- Max/min problem $\frac{df}{dx}$ becomes the pair $\frac{df}{dx}, \frac{df}{dy}$.

.1 14.1 Functions of several variables

1. Functions in \mathbb{R}^2

- (a) $y = f(x)$ is a curve in the xy -plane.
- (b) x is the indep variable, y is the dependent variable.
- (c) Set of all x which f makes sense gives the domain, all obtainable y gives the range. Both are intervals.
- (d) Example: $f(x) = \sqrt{x}$.

2. Functions in \mathbb{R}^3

- (a) $z = f(x, y)$ is a surface in \mathbb{R}^3 . xy are independent and z is dependent. The domain is now a 2 dimensional region, and the range is still an interval. Simple extension, though all these ideas become harder.
- (b) Example: $z = f(x, y) = \sqrt{x^2 + y^2}$.
 - Need $z \geq 0$ for range.
 - Level curves: For constant $z = k$ we have circles $k^2 = x^2 + y^2$. These are circles, and they grow in diameter as z increases.
 - Resulting graph is a cone. Check in Geogebra.

3. Level curves:

- (a) Definition: The level curves of function $f(x, y)$ are the curves with equations $f(x, y) = k$ for constant k in the range of f .
- (b) Example: Find the level curves of $f(x, y) = 2x + y$. Level curves are lines $k = 2x + y$ which are lines $y = -2x + k$. Graph in xy -plane. Result is a plane $z = 2x + y$ giving $2x + y - z = 0$.
- (c) Note, different functions (surfaces) can have the same level curves. Compare $f(x, y) = x^2 + y^2$ (paraboloid). Different locations though.
- (d) Examples: Try on own. Find domain and range. Sketch level curves. Describe surface.

$$z = \frac{y}{x}, \quad z = \sqrt{4 - x^2 - y^2}$$

- (e) Ideas extend to functions of 3+ variables as you think, harder to visualize.

$$f(x, y, z), \quad f(x_1, x_2, \dots, x_n)$$

4. Contour maps and calculus intuition: Show contour map of mountain with rivers.

- (a) Contours are drawn every 100 ft increase. What do you see?
- (b) Steep trails have close curves. Flat are far apart.
- (c) Creeks run perpendicular to level curves. Steepest direction is perpendicular.
- (d) Loops indicate peaks and troughs.
- (e) What if you walk along a level curve? No change in elevation.

5. Homework: 1, 7, 11, 13, 15, 19, 23, 25, 33, 35, 37, 41, 43, 49, 61, 63, 65

.2 14.2 Limits and continuity

1. Limits in \mathbb{R}

- (a) Intuition definition: $\lim_{x \rightarrow a} f(x) = L$ if for x near a , $f(x)$ is near L . Draw picture. Idea is clear, but need precision to build a theory on.
- (b) Precise definition: $\lim_{x \rightarrow a} f(x) = L$ if for any $\epsilon > 0$ (no matter how near to L), there exists a $\delta > 0$ (near enough to a) such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$. Add δ and ϵ to graph. x window and y window. Technical definition which is hard to work with, instead prove theorems and build theory.
- (c) Techniques for computing limits:
 - Limit laws (solid foundation, grow complexity from basic functions).
 - Algebra tricks (multiply by conjugate, right / left limits, etc).
 - Squeeze theorem and indirect attacks.
 - Can direct substitute for continuous functions.
- (d) Why are limits important? Handling indeterminate form. Essence of calculus.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \quad \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(x_i^*) \Delta x$$

$0/0$ and $\infty \cdot 0$ indeterminate forms.

- (e) Examples: $f(x) = x^2$, $f'(3) = ?$, $\lim_{x \rightarrow 0} \frac{|x|}{x}$, $\lim_{x \rightarrow 2} \frac{\sqrt{x+2}-2}{x-2}$.

2. Limits in \mathbb{R}^2 and beyond

- (a) Intuition definition: $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ if for (x,y) near (a,b) , $f(x,y)$ is near L . Draw picture. Now we approach a point (a,b) from all directions, not just right/left. Precision again is needed.
- (b) Precise definition: $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ if for any $\epsilon > 0$ (no matter how near to L), there exists a $\delta > 0$ (near enough to (a,b)) such that if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$. Note the appearance of the distance formula, circle with center (a,b) . Again this definition is not practical.
- (c) Techniques for computing limits:
 - Limit laws from 1 dim generalize, but cannot separate x from y .
 - Squeeze theorem and indirect attacks.
 - Can direct substitute for continuous functions (polynomials, rationals in domain, etc).
 - Interesting case again will be indeterminate forms (next section for partial derivatives).
- (d) Same idea for 3+ dimensions.

3. Examples:

- (a) Table example in text. Hint how to explain a limit does not exist. Graph each in Geogebra.
- (b) Show $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ has no limit at $(0, 0)$ by following paths $x = 0$ and $y = 0$ and getting different values. Similar to right left limits in \mathbb{R} . Graph in Geogebra.
- (c) Try on own: Show $f(x, y) = \frac{xy}{x^2 + y^2}$ has no limit at $(0, 0)$ by choosing two paths with different results. Graph in Geogebra.
- (d) Theorem: If $f \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along path C_1 and $f \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along path C_2 with $L_1 \neq L_2$, then $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ does not exist.
- (e) Show $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0$ via the Squeeze theorem. Key step:

$$0 \leq \frac{3x^2|y|}{x^2 + y^2} = 3|y| \frac{x^2}{x^2 + y^2} \leq 3|y| \cdot 1$$

Can also do from definition. See text.

- (f) If point is in domain and function is continuous, can do direct substitution. $\lim_{(x,y) \rightarrow (1,1)} \frac{3x^2y}{x^2 + y^2} = 0$.

4. Homework: 5, 9, 13, 17

.3 14.3 Partial derivatives

1. One dimension review, \mathbb{R} :

- (a) For $f(x)$, change in x results in change in f . Then average rate of change $\Delta f / \Delta x$ tends to instantaneous rate of change df/dx as $\Delta x \rightarrow 0$. That is,

$$\frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

- (b) Limits are foundation, but we built a theory of differentiation.

$$cf(x), f(x) + g(x), f(x)g(x), f(x)/g(x), f(g(x))$$

and also special functions such as logs, exponentials, trig, etc.

2. Two dimensions, \mathbb{R}^2 : $f(x, y)$

- (a) Analogy tangent plane to a surface. Strategy is to allow one variable to change at a time. If x can change for $f(x, y) = x - yx$, then $\Delta f = \Delta x - y\Delta x$ and $\Delta f / \Delta x = 1 - y$. That is the x derivative of $f(x, y)$ is $1 - y$. Hold y constant and differentiate f in x . Knowing both will lead to tangent planes (next section).
- (b) Definition: The partial derivative of $f(x, y)$ with respect to x is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similar for f_y .

- (c) Notation: For $f = f(x, y)$,

$$f_x = f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = D_x f$$

- (d) All our old differentiation rules hold since y is a constant.

- (e) Example: $f(x, y) = 4 - x^2 - 3y^2$.

- Compute $f_x(1, 2), f_x, f_y(1, 2), f_y$.
- Graph via Geogebra to get intuition around f_x, f_y . Note if we know $f_x(1, 2), f_y(1, 2)$, we can get a tangent plane (next section).

- Note local max at $(0, 0)$.
- Extend to four cases of second derivatives.

(f) Example: $f(x, y) = x^3 + x^2y^3 - 2y^2$

- Try on own, all first and second order partials.
- Compare graph to f_x and f_y .

(g) Theorem: $f_{xy} = f_{yx}$, order of differentiation doesn't matter. Proof via the MVT.

(h) Example: Problem 9 in text.

3. Partial differential equations tour:

- https://en.wikipedia.org/wiki/Partial_differential_equation
- <https://web.stanford.edu/class/math220b/handouts/heateqn.pdf>

4. Homework: 5, 7, 9, 11, 13, 15, 21, 25, 33, 45, 51, 53, 61, 63, 81, 97

.4 14.4 Tangent planes and linear approximations

1. Recall: $y = f(x)$ version.

(a) The tangent line to $y = f(x)$ at point (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0) \quad \rightarrow \quad y = L(x) = f'(x_0)(x - x_0) + y_0.$$

Give example for $f(x) = x^2$ at $x = 3$.

(b) Linearization approximates $f(x)$ by this line.

$$y = f(x) \approx L(x) = f'(x_0)(x - x_0) + y_0.$$

The closer to the tangent point, the better the approximation. Give example.

(c) Taylor series and Taylor's theorem continues this vein.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \dots$$

2. Extension to $z = f(x, y)$, tangent planes.

(a) Partial derivatives f_x, f_y give the slope of the tangent line to $z = f(x, y)$ in the x, y directions. Draw picture. How to use this to find the tangent line thru a point (x_0, y_0, z_0) ? Need a point and a normal vector.

(b) Normal vector construction: Find vectors in direction of partial derivative lines.

- f_x :, y held constant, if x increases 1 unit, z increases f_x units. Then, $\vec{a} = \langle 1, 0, f_x \rangle$ is parallel to our line.
- f_y :, likewise $\vec{b} = \langle 0, 1, f_y \rangle$ works.
- The normal vector to the tangent plane is then

$$\vec{n} = \vec{a} \times \vec{b} = \langle -f_x, -f_y, 1 \rangle$$

(c) Vector form of tangent plane:

$$\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0 \quad \rightarrow \quad -f_x(x - x_0) - f_y(y - y_0) + (z - z_0) = 0$$

gives

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

Note the similarity to the tangent line for $y = f(x)$.

- (d) Example: Find the tangent line to the paraboloid $z = 14 - x^2 - y^2$ at $(x_0, y_0, z_0) = (1, 2, 9)$ Graph in geogebra. Both x, y tangent lines are on this plane. All tangent lines for all surface curves as well.
- (e) Try on own: Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$ at $(1, 2, 3)$. Can solve for z taking the positive root or use implicit differentiation with respect to x, y . Note the normal vector is in the same direction as the sphere radius when directed to our point.
- (f) Linearization of $z = f(x, y)$ by the tangent plane.

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

Two dimensional Taylor series approximation. Can guess the extension to 3+ independent variables.

3. Differentiability of $f(x, y)$:

- (a) Remind of differentiability in \mathbb{R}^2 . Derivative exists. Differentiable implies continuous.
- (b) Def: We say $f(x, y)$ is differentiable at point (a, b) if

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$. Basically says can approximate f well by the tangent line.

- (c) Theorem: If the partial derivatives f_x, f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

4. Homework: 1, 3, 5, 11, 13, 19, 21

.5 14.5 The chain rule

1. 1 dimension: $\frac{d}{dt}f(g(t))$.

- (a) Goal is to differentiate function composition. Nested functions are common. Do g first, then f takes it from there.

$$\frac{d}{dt}f(x(t)) = f'(x(t))x'(t)$$

- (b) Compact notation: $y = f(x)$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

Right hand side collapses back if canceling were allowed.

- (c) The chain rule applied to integration is the substitution rule.

2. 2 dimensions, basic case: $\frac{d}{dt}f(x(t), y(t))$

- (a) Extend the dimension 1 case of the chain rule to get for $z = f(x, y)$:

$$\frac{dz}{dt} = \frac{d}{dt}f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Note the similarity to the 1 dimension case.

- (b) Example: For $z = 3xy^2$, $x = \cos(t)$, $y = \sin(t)$, compute $\frac{dz}{dt}$. Check by rewriting x, y in original. Graph in Geogebra, not traveling about the unit circle in xy . Consider $t = 0, \frac{\pi}{2}$. Rate of change along curve $(x(t), y(t))$.

3. 2 dimensions, standard case: $\frac{d}{dt}f(x(s, t), y(s, t))$

- (a) Repeat the above formula twice.

$$\frac{dz}{ds} = \frac{d}{ds}f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{d}{dt}f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

- (b) Example: For $z = 3xy^2$, $x = r \cos(\theta)$, $y = r \sin(\theta)$, compute $\frac{dz}{dr}$. Try on own $\frac{dz}{d\theta}$, $\frac{d^2z}{dr^2}$
- (c) Second derivatives and converting to polar coordinates. $z = f(x, y)$, $x = r \cos(\theta)$, $y = r \sin(\theta)$
- Compute f_{rr} , $f_{\theta\theta}$.
 - Turns out $f_{xx} + f_{yy} = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}$
 - This is the polar version of Laplace's equation.

4. Generalizes to any dimension. Show text formula. Work example 5.

5. Homework: 1, 3, 5, 7, 11, 13, 17, 21, 45, 49

.6 14.6 Directional derivatives and gradient vectors

1. Directional derivatives: So far we calculate change for $f(x, y)$ in the x direction (f_x) or the y direction (f_y), but of course f can change in any direction.

- (a) Recall our limit definitions:

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

We essentially hold y and x constant respectively. The directions we consider here are \vec{i} and \vec{j} . Note both are unit vectors.

- (b) Example: Find the change in $f(x, y) = xy$ at point $(3, 1)$ in the direction $\vec{v} = \langle 1, 2 \rangle$. Normalize our direction via the unit vector $\vec{u} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$. Then our change is from $(3, 1)$ to $(3 + h/\sqrt{5}, 1 + 2h/\sqrt{5})$ and

$$D_{\vec{u}}f(3, 1) = \lim_{h \rightarrow 0} \frac{f(3 + h/\sqrt{5}, 1 + 2h/\sqrt{5}) - f(3, 1)}{h} = \lim_{h \rightarrow 0} 7/\sqrt{5} + 2h/5 = 7\sqrt{5}.$$

Note h in the denominator because of the unit vector. Graph in Geogebra and compare to f_x , f_y .

- (c) Definition: The directional derivative of $f(x, y)$ at point (x_0, y_0) in the direction of unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Note, $D_{\vec{i}}f = f_x$ and $D_{\vec{j}}f = f_y$. Also any unit vector can be expressed in terms of a direction angle θ as

$$\vec{u} = \langle a, b \rangle = \langle \cos(\theta), \sin(\theta) \rangle$$

2. Computing directional derivatives

- (a) The above limit definition is messy to compute. Instead, we rewrite $D_{\vec{u}}f$ in terms of f_x and f_y . This seems doable considering the tangent plane to a surface in \mathbb{R}^3 .
- (b) Theorem: For $f(x, y)$ differentiable in both x and y and $\vec{u} = \langle a, b \rangle$ any unit vector in \mathbb{R}^2 ,

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

(c) Proof: Define $g(h) = f(x_0 + ah, y_0 + bh)$. Then,

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}}f(x_0, y_0).$$

On the other hand, from the chain rule,

$$g'(h) = \frac{\partial f}{\partial h} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial h} = af_x + bf_y = af_x(x_0 + ah, y_0 + bh) + bf_y(x_0 + ah, y_0 + bh).$$

Evaluating $g'(h)$ at zero and comparing to before gives the result.

(d) Example: Repeat above example $f(x, y) = xy$ with new calculation.

(e) Example: Try on own for $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in direction $\theta = \frac{\pi}{3}$. Check via Geogebra.

3. Gradient vectors:

(a) Example: Hint to bigger things. $f(x, y) = 3x + y + 1$ at $(1, 1)$.

- \vec{i} and \vec{j} directions.
- No change (level curve) direction. Find $\vec{u} = \langle a, b \rangle$ such that

$$D_{\vec{u}}f = f_x a + f_y b = 0$$

gives $\vec{u} = \frac{1}{\sqrt{10}} \langle -3, 1 \rangle$,

- Perpendicular to level curve gives steepest direction $\vec{u} = \frac{1}{\sqrt{10}} \langle 1, 3 \rangle$. This matches $\langle f_x, f_y \rangle$ at our point. Compute change and compare to f_x, f_y .
- Noting that the directional derivative is really a dot product, we see a new vector of import.

$$D_{\vec{u}}f = f_x a + f_y b = \langle f_x, f_y \rangle \cdot \langle a, b \rangle$$

(b) Definition: For $f(x, y)$, the gradient of f is a vector-function of the form

$$\nabla f = \langle f_x, f_y \rangle$$

(c) Example: Compute gradient for previous example $f(x, y) = xy^3 - x^2$ at $(1, 2)$. Reproduce previous result.

(d) Theorem: First importance of the gradient. For f differentiable, the maximum value of the directional derivative $D_{\vec{u}}f$ is $|\nabla f|$ and is in the direction of ∇f .

(e) Proof: We use the law of cosines version of the dot product.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

where θ is the angle between \vec{a}, \vec{b} . Then,

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos(\theta) = \|\nabla f\| \cos(\theta) \leq \|\nabla f\|$$

which occurs when $\theta = 0$ meaning \vec{u} and ∇f are in the same direction.

(f) Example: Apply previous theorem to $f(x, y) = 3x + y + 1$ at $(1, 1)$, $f(x, y) = xy^3 - x^2$ at $(1, 2)$.

(g) Example: Try on own. Number 22 in text. Graph in Geogebra.

4. Extension to functions of three variables: $f(x, y, z)$.

(a) Could be in \mathbb{R}^4 in which case cannot visualize. Could be an implicit curve $f(x, y, z)$ in \mathbb{R}^3 .

(b) Definition of directional derivative in direction of unit vector \vec{u} .

$$D_{\vec{u}} = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

(c) Compute $D_{\vec{u}}$ in terms of partial derivatives.

$$D_{\vec{u}} = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \nabla f \cdot \vec{u}$$

(d) Gradient of f is

$$\nabla f = \langle f_x, f_y, f_z \rangle.$$

(e) Examples are pretty well the same.

5. Tangent planes to level surfaces

(a) We already have tangent planes to surfaces of the form $z = f(x, y)$ at point (x_0, y_0, z_0) .

$$z - z_0 = (f_x)_0(x - x_0) + (f_y)_0(y - y_0)$$

(b) This extends implicitly to a level surface $F(x, y, z) = k$ at point (x_0, y_0, z_0) .

$$(F_x)_0(x - x_0) + (F_y)_0(y - y_0) + (F_z)_0(z - z_0) = 0$$

Note, the gradient vector ∇F is our normal vector to the plane (and surface).

(c) The normal line then has symmetric equations

$$\frac{x - x_0}{(F_x)_0} = \frac{y - y_0}{(F_y)_0} = \frac{z - z_0}{(F_z)_0}.$$

(d) Example: Find the tangent plane to the ellipsoid $x^2/4 + y^2 + z^2/9 = 3$ at point $(-2, 1, -3)$. Check result in Geogebra.

6. Summary of gradient vector: This section is rich. Summarize the key ideas.

(a) For $f(x, y)$ (or $f(x, y, z)$), ∇f gives the direction of fastest increase of f .

(b) $\|\nabla f\|$ is the fastest increase rate (slope).

(c) ∇f is orthogonal to the level curve (or surface).

7. Homework: 5, 7, 9, 11, 15, 19, 23, 25, 27, 29, 37, 39, 41, 49

.7 14.7 Maximum and minimum values

1. Recall functions of one variable...

(a) Draw $f(x)$ with make an min values. Smooth and continuous on \mathbb{R} .

(b) $f'(x) = 0$ (stationary points) gives locations of horizontal tangents. $f''(x) = 0$ discerns the three cases.

- $f''(x) > 0$, local min
- $f''(x) < 0$, local max
- $f''(x) = 0$, inflection point

(c) Two other cases for extrema: Singular points, end points.

(d) Absolute max and mins are ensured by the EVT: Continuous function $f(x)$ on closed interval $[a, b]$ must have a local max and local min.

2. Definitions for $f(x, y)$.

(a) Local min at (a, b) with local min value $f(a, b)$. Likewise for max.

(b) Global min and max.

3. Extending calculus 1 results:

- (a) Theorem: If $f(x, y)$ has a local max or min at (a, b) and f_x, f_y both exist at (a, b) , then $f_x(a, b) = 0$ and $f_y(a, b) = 0$ ($\nabla f = \vec{0}$).
- (b) If $\nabla f = \vec{0}$ at (a, b) , then this is called a stationary point. Not all critical points are local mins or maxes.
- (c) Example: Find the stationary points of $f(x, y) = 3x - x^3 - 2y^2 + y^4$.
4. How to classify stationary points? Concavity is key, but we need to look in all directions.
- (a) 2 examples: $x^2 + xy + y^2$ and $x^2 + 10xy + y^2$. Only $(0, 0)$ stationary point for both. Both have two positive partials second ($f_{xx} = f_{yy} = 2 > 0$). Graph in Geogebra to see different behavior.
- (b) To classify, consider all the second directional derivatives at once. For $f(x, y)$ and $\vec{u} = \langle h, k \rangle$,

$$D_{\vec{u}}f = f_x h + f_y k.$$

$$D_{\vec{u}}^2 f = D_{\vec{u}}(f_x h + f_y k) = f_{xx} h^2 + 2f_{xy} h k + f_{yy} k^2 = f_{xx} \left(h + \frac{f_{xy}}{f_{xx}} k \right)^2 + \frac{k^2}{f_{xx}} (f_{xx} f_{yy} - f_{xy}^2)$$

where the last step follows by completing the square.

- (c) If we think concave up since $f_{xx} > 0$ we would also need $D = f_{xx} f_{yy} - f_{xy}^2 > 0$. Likewise for concave down we need $f_{xx} < 0$ but still $D > 0$.
- (d) Theorem: For (a, b) a stationary point of $f(x, y)$ and

$$D = D(a, b) = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local min.
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local max.
- If $D > 0$ then $f(a, b)$ is a saddle point.

(e) Check for 2 examples.

(f) Example: Apply to first example.

5. Last, we extend the EVT

- (a) EVT: For $f(x, y)$ continuous on closed, bounded region R in \mathbb{R}^2 , then f has an absolute max and min in R .
- (b) How to find extrema? Abs max and mins must be at stationary points in R or on the boundary of R .
- Find the stationary points in R .
 - Find the extreme values on the boundary via Calc 1.
 - Get the largest and smallest f values from parts 1 and 2.

(c) Example: 34 in text.

6. Homework: 1, 3, 5, 7, 13, 15, 17, 23, 27, 31, 33

.8 14.8 Lagrange multipliers

Skip.

.9 Chapter 14 Review

1. Concept check: 1-18
2. True-False: 1-11
3. Exercises: 1-56

Chapter 15 Multiple integrals

.1 15.1 Double integrals over rectangles

1. Summary of past: Extend the definite integral of calculus 1 to 3 dimensions.

(a) $\int_a^b f(x) dx$ as area under the curve.

(b) Compute via limit of Riemann sum. Classic calculus paradox.

(c) Fundamental theorem of calculus.

(d) Alternate view: $\int_a^b f(x) dx$ as adding up 1D lengths to get 2D area.

(e) Really about summation: Sum lines to area, areas to volumes (discs and washers), probability, force to work, line segment to arc length, arc length to surface area, etc

2. Basic case for $z = f(x, y)$ in \mathbb{R}^3 : Volumes over rectangular domains.

(a) Find the volume of the solid between $z = f(x, y)$ and the xy -plane over region $R = [a, b] \times [c, d]$ (Cartesian product).

(b) Partition R by Δx and Δy giving rectangular areas ΔA .

(c) Notation and limit of a Riemann sum.

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \lim_{m, n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

(d) If the limit exists we say f is integrable over R .

(e) Can approximate area as in Calc 1 by computing the finite sum, though only works for simple functions.

3. FTOC for calculation: Volume by accumulating area.

(a) Slicing the solid in the x direction gives cross-sections with area

$$A(x) = \int_c^d f(x, y) dy.$$

This is a computable function of x for any y held constant.

(b) Add up area to get volume. FTOC twice.

$$V = \int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx$$

(c) Of course slicing in y gives a similar formula resulting in Fubini's theorem which extends to more general regions as well.

$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

(d) Examples: Divide and conquer to find the area under $z = 1 + x^2 + y^2$ on $[1, 2] \times [0, 1]$.

4. Average value of functions:

(a) Calc 1 version:

$$\frac{1}{b-a} \int_a^b f(x) dx$$

(b) Calc 3 version:

$$\frac{1}{A(R)} \iint_R f(x) \, dA$$

5. Homework: 1, 13, 15, 17, 19, 21, 25, 29, 31, 33, 35, 39, 47

.2 15.2 Double integrals over general regions

1. Idea of general regions:

- (a) The domain of integration for $z = f(x, y)$ doesn't have to be a rectangle. In general it can be any shape (denote as D). Draw picture.
- (b) Can still do slicing in x or y direction. Result is 2 cases to choose from.
- (c) Volume from accumulating area in x (holding y constant). Draw domain picture.

$$\iint_D f(x, y) \, dA = \int_a^b A(x) \, dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) \, dy \, dx$$

- (d) Volume from accumulating area in y (holding x constant). Draw domain picture.

$$\iint_D f(x, y) \, dA = \int_c^d A(y) \, dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y) \, dx \, dy$$

- (e) Sometimes only one is an option, sometimes both can be used and need to think strategically.

2. Examples: Drawing the domain in the xy -plane is key.

- (a) Only one direction is easy. Find the volume under the surface $z = x^2 + y$ on domain D bound by curves $y = x + 1$ and $y = x^2$.
- (b) Divide and conquer by doing both at same time. Find the volume below the plane $z = x - 2y$ and above the triangle with vertices $(0, 0), (1, 1), (0, 1)$ in the xy -plane. Need to divide into two volumes in one direction leading to the below theorem.
- (c) Theorem: If $D = D_1 \cup D_2$, then

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA$$

3. If integration is hard, estimation will often do by capturing the solid inside and outside a cylinder.

- (a) Theorem: For $m \leq f(x, y) \leq M$ on domain D with area $A(D)$, then

$$mA(D) \leq \iint_D f(x, y) \, dA \leq MA(D).$$

4. Homework: 1, 5, 7, 9, 13, 15, 17, 21, 23, 25, 31, 39, 45, 49, 51, 57, 59

.3 15.3 Double integrals over polar coordinates

1. When rotation is involved, rectangular coordinates are no longer nice. Switch to polar coordinates.

- (a) Example: $\iint_R (3x + 4y^2) \, dA$ for $R = \{(x, y) | 4 \leq x^2 + y^2 \leq 9\}$. Cannot divide into rectangles. Hard to separate curves.

(b) Polar coordinates basics: (x, y) replaced with (r, θ) . Connection is trigonometry.

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad x^2 + y^2 = r^2$$

(c) Convert point $(3, 1)$ into polar coordinates. r is easy. $\theta = \arctan(y/x)$ if in quadrants 1 and 4. Shift θ to the right quadrant as for $(-3, -1)$.

(d) Above region is now easier to describe. Rectangle in terms of r, θ .

$$R = \{(r, \theta) | 2 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$$

(e) How to translate dA to be in terms of $d\theta$ and dr ?

- For rectangles, the area of a small rectangle was $dA = dx dy$.
- For new curved rectangles, take the difference of the wedges which have area $\frac{1}{2}r^2\Delta\theta$. The centering of r in the wedgy-rectangle is needed to avoid Δr^2 .

$$\Delta A = \frac{1}{2}(r + \Delta r/2)^2\Delta\theta - \frac{1}{2}(r - \Delta r/2)^2\Delta\theta = \frac{1}{2}(2r\Delta r)\Delta\theta = r\Delta r\Delta\theta$$

- Then,

$$\iint_R (3x + 4y^2) dA = \int_0^{2\pi} \int_2^3 (3r \cos(\theta) + 4r^2 \sin^2(\theta)) r dr d\theta$$

Recall the $\sin^2(\theta)$ term will require a calc 2 trig integral strategy via the half angle formula.

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

2. Example: Find the volume under the paraboloid $z = x^2 + y^2$ yet inside the cylinder $x^2 + y^2 = 2x$.

(a) Complete the square on the cylinder to graph it.

$$(x - 1)^2 + y^2 = 1$$

(b) Translate region into polar coordinates.

$$x^2 + y^2 = 2x \quad \rightarrow \quad r^2 = 2r \cos(\theta) \quad \rightarrow \quad r = 2 \cos(\theta)$$

$$R = \{(r, \theta) | 0 \leq r \leq 2 \cos(\theta), -\pi/2 \leq \theta \leq \pi/2\}$$

(c) Compute the integral.

$$\iint_R x^2 + y^2 dA = \int_{-\pi/2}^{\pi/2} \int_0^{\cos(\theta)} r^2 r dr d\theta = \dots = 8 \int_0^{\pi/2} \cos^4(\theta) d\theta = 8 \int_0^{\pi/2} \left(\frac{1 + \cos(2\theta)}{2} \right)^2 d\theta = \frac{3\pi}{2}$$

3. Homework: 1, 3, 5, 7, 11, 17, 19, 25

.4 15.4 Applications of double integrals

Skip.

.5 15.5 Surface area

Skip.

15.6 Triple integrals

1. Triple integrals: Continue the path.

- (a) Instead of small intervals (dx) or small boxes (dA), we now have small boxes (dV).
- (b) Integrating will be the easy part, setting up the integral is the challenge.
- (c) For $f(x, y, z)$ a continuous function on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$,

$$\iiint_B f(x, y, z) dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_i^*, y_j^*, z_k^*) \Delta V = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

- (d) Fubini's theorem says the order of integration can be changed.
- (e) Below we will just integrate 1 to find volumes of solids. Replace 1 with any $f(x, y, z)$ with the same story.

2. Examples: Volumes of solids.

- (a) Box: Graph it. Two integrations gives area of cross section. Final integral adds area to get volume.

$$\int_0^1 \int_0^3 \int_0^2 dx dy dz$$

- (b) Prism: Graph it. Should be half of box. Easiest to project onto the xy -plane first, then sort out the z bounds. Result is the following description of the solid.

$$E = \{(x, y, z) | 0 \leq x \leq 2, \quad 0 \leq y \leq 3, \quad 0 \leq z \leq (3 - y)/3\}$$

$$\int_0^2 \int_0^3 \int_0^{(3-y)/3} dz dy dx$$

Note middle two integrals produce area of a cross section.

- (c) Try on own: Find the volume of the tetrahedron (4-sided pyramid) with corners 1s on the 3 axis. Follow previous example.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx$$

3. Changing order of integration.

- (a) Try on own: Find the prism volume in order $dx dy dz$. Hint: Draw projection in yz -plane first.

$$\int_0^1 \int_0^{3-3z} \int_0^2 dx dy dz$$

- (b) Try on own: Find the volume of the tetrahedron in order $dy dz dx$. Hint: Draw projection in the xz -plane first.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-z} dy dz dx$$

- (c) Example 4 in text.

4. Homework: 1, 3, 9, 13, 19, 21, 27, 31, 35

- .7 15.7 Triple integrals in cylindrical coordinates
- .8 15.8 Triple integrals in spherical coordinates
- .9 15.9 Change of variable in multiple integrals

Chapter 16 Vector calculus

- .1 16.1 Vector fields
- .2 16.2 Line integrals
- .3 16.3 The fundamental theorem of line integrals
- .4 16.4 Green's theorem
- .5 16.5 Curl and divergence
- .6 16.6 Parametric surfaces and their area
- .7 16.7 Surface integrals
- .8 16.8 Stoke's theorem
- .9 16.9 The divergence theorem
- .10 16.10 Summary

Chapter 17 Second-order differential equations

- .1 17.1 Second-order linear equations
- .2 17.2 Nonhomogeneous linear equations
- .3 17.3 Applications of second-order differential equations
- .4 17.4 Series solutions