Assigned: Wednesday, March 27, 2015
Due: Wednesday, April 3, 2015

MTH 371: Homework 8
Lagrange Interpolation

GENERAL HOMEWORK GUIDELINES:

- On the very first page of your homework, provide your name, date, and homework number.
- Homework will be graded in part on neatness, organization, and completeness of solutions. Multiple pages MUST BE STAPLED.
- Attach all Scilab code, output, and plots to the page immediately following each problem. Also, clearly indicate the problem they correspond to.

1. (a) Construct the minimum degree Lagrange form interpolating polynomial $P(x)$ for $f(x) = \sin(x)$ passing through points

$$(0, \sin(0)), \quad \left(\frac{\pi}{4}, \sin\left(\frac{\pi}{4}\right)\right), \quad \left(\frac{\pi}{2}, \sin\left(\frac{\pi}{2}\right)\right).$$

   (b) Plot the polynomial $P(x)$ from (a) and the function $f(x) = \sin(x)$ on the same graph for $0 \le x \le \frac{\pi}{2}$. Also, using the `subplot` command, plot the absolute error $|f(x) - P(x)|$. Use 200 equally spaced data points for x (in Scilab: `x = linspace(0,%pi/2,200)`).

   (c) Use the polynomial from part (a) to estimate both $\sin(\pi/3)$ and $\sin(\pi/6)$. What is the error in each approximation?

   (d) Using the polynomial interpolation error theorem from class, compute

$$\max_{0 \le x \le \pi/2} |f(x) - P(x)|.$$

   Use Scilab to compute the maximum error on the 200 equally spaced points in part (b), and compare this to the theoretical error bound you just found.

2. Write a Scilab function (`.sci` file) for polynomial interpolation using Lagrange form. The input for this function should be an array of x-values, `x`, with $(n+1)$ distinct points $x_0, x_1, x_2, \ldots, x_n$ and also an array of function values, `f`, with $(n+1)$ function values $f_0, f_1, f_2, \ldots, f_n$. Your function should return a Scilab polynomial `P` such that $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

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P = LagrangeInt(x,f)
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   To test your fantastic new function, write a script (`.sce` file) which plots the Lagrange degree $n = 4$ interpolation polynomial using the `subplot` command for each of the following. Include the original function in each plot and use 200 x-values. You should use the `horner` command in Scilab to evaluate a polynomial at x-values.

   (a) $f(x) = \sin(x), \ 0 \le x \le 2\pi$
   (b) $g(x) = \cos(x), \ 0 \le x \le 2\pi$
   (c) $h(x) = \ln(x), \ 0.5 \le x \le 2$
   (d) $i(x) = e^x, \ 0 \le x \le 2$

3. Prove that a polynomial interpolant of degree at most n through the $(n+1)$ points $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$ must be unique.
   Hint: Assume that there are two such polynomials, P and Q, and argue that they must be identical. Consider the function $R(x) = P(x) - Q(x)$.

4. Consider the Runge function $f(x) = \dfrac{1}{1+25x^2}$ on $[-1, 1]$. Graph the following all on the same plot. Label your graph and include a legend. Use the function in problem 2!
   (a) Graph $y = f(x)$ on $[-1, 1]$ using 100 data points (in Scilab: `x = linspace(-1, 1, 100)`).
   (b) Graph the degree 10 polynomial interpolant of function f through 11 equally spaced nodes in $[-1, 1]$ (in Scilab: `x = linspace(-1, 1, 11)`) using the same data points as in (a).
   (c) Graph the degree 10 polynomial interpolant of function f through the 11 non-equally spaced Chebyshev points $x_j = \cos\left(\dfrac{\pi j}{10}\right)$, $j = 0, 1, \ldots, 10$ using the same data points as in (a).

5. (a) Verify that the polynomials

$$p(x) = 5x^3 - 27x^2 + 45x - 21, \qquad q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$$

   both interpolate the points in the below table. Feel free to use Scilab to do this.

x	1	2	3	4
y	2	1	6	47

   Above you showed that interpolation polynomials are unique. Explain why this problem does not contradict that.

   (b) Verify that the polynomials

$$p(x) = 3+2(x-1)+4(x-1)(x+2), \qquad q(x) = 3\frac{(x+2)x}{3} - 3\frac{(x-1)x}{6} - 7\frac{(x-1)(x+2)}{-2}$$

   both interpolate the points in the below table. Feel free to use Scilab to do this.

x	1	-2	0
y	3	-3	-7

   Above you showed that interpolation polynomials are unique. Explain why this problem does not contradict that.

6. To evaluate a polynomial in Scilab, the `horner` function is used. Research Horner's method (either in the textbook, library, or a simple internet search) and write a paragraph discussion of this method.