Numerical analysis in the twentieth century

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Numerical analysis can be defined as the branch of mathematics interested in *constructive methods*. By constructive, we mean methods which allow to construct effectively, that is to obtain numerically, the solution of mathematical problems. A discussion of possible definitions is given in [249]. It is difficult to find who coined this appellation but a symposium with the title *Problems for the Numerical Analysis of the Future* was held at UCLA, July 29–31, 1948. It was published by the *National Bureau of Standards* as volume 15 of the *Applied Mathematics Series*. However, this was not the first conference on this subject since a previous one with the title *Conference on Advanced Computation Techniques* (84 participants) was held at MIT, October 29–31, 1945 [9]. It seems that the first book with this expression in its title was by Douglas Rayner Hartree (1897–1958), a mathematical physicist from Cambridge, England, in 1952 [128]. He took part in the symposium at UCLA where he gave a talk on *Some unsolved problems in numerical analysis*. Of course, there were books on numerical methods a long time before Hartree's. Let us mention, for example, those of J. Vieille [262] dating from 1852 and where some methods for solving nonlinear equations are described and the more complete one by Robert Fernand Bernard, Viscount de Montessus de Ballore (1870–1937) and Robert d'Adhémar [190] in 1911. There is also a chapter on numerical and graphical methods for quadrature and differential equations by C. Runge and F.A. Willers in the *Encyklopädie der Mathematischen Wissenschaften* [219]. Also, in 1798, Joseph Louis Lagrange (1736–1813) published a paper with the words *analyse numérique* in its title (but with the meaning of "number theory") [168].

Anyway, numerical analysis is concerned with the design and the study of *algorithms*, which can be defined as a set of rules allowing to perform numerical computations. It is well known that this word comes out from the name of the arabic mathematician Mohammed Ibn Musa Abu Djefar Al-Khwarizmi (about 780–850), whose book on algebra began, in its latin translation, by *Algoritmi dixit...*

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this project was to present the historical development of numerical analysis during the last century and to review the current research in selected domains of numerical analysis. The volumes are

Vol. I:	Approximation Theory Luc Wuytack, Jet Wimp, eds.
Vol. II:	Interpolation and Extrapolation Claude Brezinski, ed.
Vol. III:	Linear Algebra, Apostolos Hadjidimos, Henk van der Vorst, Paul Van Dooren, eds.
Vol. IV:	Optimization and Nonlinear Equations Layne T. Watson, Michael Bartholomew–Biggs, John Ford, eds.
Vol. V:	Quadrature and Orthogonal Polynomials Walter Gautschi, Francisco Marcellan, Lothar Reichel, eds.
Vol. VI:	Ordinary Differential Equations and Integral Equations Christopher Baker, Giovanni Monegato, John Pryce, Guido Vanden Berghe, eds.
Vol. VII:	Partial Differential Equations David Sloan, Stefan Van de Walle, Endre Suli, eds.

Of course, it is impossible, in this volume, to cover all aspects of numerical analysis and some quite important ones have been omitted. However, we do hope that the historical and survey papers herein will give a good idea of the development of numerical analysis during the twentieth century and will also serve as a basis for new researches. Predictions for the next fifty years are given in [251].

The scenery

Mathematical analysis was developed throughout many centuries by those mathematicians who needed to solve real world problems. The first problems were connected with the measurement of fields (as the computation of the square root) and weights (as ascending continued fractions). Then, more difficult problems were related to astronomy and, for example, the differential and integral calculus were developed for that purpose. Soon, mathematicians realized that the real world problems they had to solve were far too complicated to be handled by their analytical methods. So, they invented special types of methods for solving these problems *numerically*. Thus, numerical analysis was first the apanage of scientists (mostly astronomers at the beginning) and applied mathematicians. This is why some numerical methods bear the names of such astronomers as Newton, Euler, Gauss, Jacobi, Lagrange, Adams, etc., and other applied scientists as Cholesky who was a cartographer in the French army and was killed during World War I. More recently, the development of some methods for the numerical integration of differential equations was pushed by problems in ballistics. One should, at this stage, make a distinction between numerical analysis whose purpose is the study of algorithms and their properties, and *applied mathematics* which is concerned in transforming a physical problem into a mathematical one, and is also often called *modeling*. Both domains could be gathered under the name of *scientific computing*. It was John von Neumann (1903–1957) who first recognized the importance

of computing in science, and computers had and still have a great impact on the development of numerical analysis [186].

The work of a numerical analyst has to go through different stages. First, starting from the physical problem, he has to put it into equations. The various interesting variables (the unknowns of the problem) have to be identified and the equations they satisfied have to be written down. When these equations are too complicated, they must be simplified by neglecting the terms whose influence is small compared to that of the other terms. In many situations, these equations are ordinary or partial differential equations, or integral equations. This phase, most of the time accomplished with the help of the specialists of the problem, is called *modelisation*. Then comes the *mathematical analysis* of these equations. It is necessary to study their theoretical properties such as the conditions under which there is existence and uniqueness of the solution. If several solutions are possible, one has to characterize the solution corresponding to the physical reality. The solution of these equations are continuous functions of some parameters such as time, pressure, temperature, and so on. In practice, it will only be possible to compute the solution at discrete values of the parameters. So, the continuous problem is replaced by an approximate discrete one. This is the phase of *discretization*. If this problem is still too complicated, it has to be simplified, for example by linearizing the equations. This approximate problem has then to be solved by a method of *numerical analysis*. This method has to be turned into an *algorithm* to be used on a computer. The data have to be stored in the computer (which could be an important and difficult problem) and the algorithm has to be programmed in the most efficient way, taking into consideration the conditioning of the mathematical problem and the propagation of rounding errors due to the finite precision of computers (that is the numerical stability of the algorithm). If an iterative method is used, a test for stopping the iterations has to be implemented. In any case, the accuracy of the solution given by the computer has to be controlled. The computer program must be *portable*, which means that it could be ran on various computers without modifications, and *robust*, that is giving the same results. Finally, the numerical results have to be analyzed and compared with the reality.

Numerical analysis covers a wide range of domains, which goes from arithmetic, probability and statistics to pure mathematics. For example, the understanding of the propagation of rounding errors in numerical computations deals with computer's arithmetic and some approaches to their study require a knowledge of probability and use statistical evaluation. The design of algorithms suitable for parallel computation is obviously closely related to computers' architecture. On the other hand, some parts of numerical analysis heavily rely on the mathematics introduced by Bourbaki (although some members of the group were not very much in favor of applied mathematics) and, in particular, on functional analysis (see [59, 175, 151]). This is the case of approximation theory and of the numerical methods for the solution of partial differential equations (see the early work of Jacques–Louis Lions [176]). Some other domains of numerical analysis are connected to complex function theory, or linear algebra, or geometry. Complexity theory has deep roots into pure mathematics also.

Of course, the impetus to numerical analysis was given by the development of modern computers just before and during World War II and the introduction of high level programming lan-

guages. Alan Mathison Turing (1912–1954) (see [140, 253]) and John von Neumann (1903–1957) were pioneers in this domain. Numerical analysis methods are to be programmed on computers and, vice versa, the analysis of the numerical results can lead to new ideas, new algorithms or even new theoretical results. This is exactly what happened with the discovery of fractals and chaotic behavior. So, there is a constant back and forth movement between the computer and the mathematician. Thus, numerical analysis can also be defined as the mixture of mathematical analysis and numerical computation. In fact, as stated by Peter Wynn [277], *numerical analysis is very much an experimental science.*

For an early history of numerical analysis see [116] and [47], where some more recent contributions are also analyzed. Apart from those contained in this volume, other historical papers dealing with recent topics in numerical analysis can be found in [194]. The reference [135] is devoted to the work conducted at the *Institute for Numerical Analysis* of the *National Bureau of Standards* during the period 1947 to 1956 where, in particular, Lanczos methods and the conjugate gradient algorithm were developed. The history of continued fractions and Padé approximants is presented in [31]. For the history of interpolation, see [32].

The actors

Let us try to trace the history of the first courses in numerical analysis in each country and to give the names of the mathematicians who were pioneers in the teaching of numerical analysis and, so, were much influential in its development and spreading. Obviously, it is a difficult (and perilous) task to establish such a list and we would like to apologize in advance for errors and omissions. We also would like to thanks all colleagues who sent us informations (their names are mentioned into parenthesis). Many other informations were found on the web.

Argentina

In Argentina at the *School of Sciences, Universidad de Buenos Aires*, there was a course on the numerical solution of ordinary differential equations in 1962. It was taught by Pedro Zadunaisky and Victor Peyrera was his assistant. The textbook was P. Henrici's book on the numerical solution of ordinary differential equations [131]. Earlier, there must have been courses by Manuel Sadosky, who had written a basic book on numerical methods in Spanish. A course on numerical linear algebra by Alexander M. Ostrowski (1893–1986) in 1959 and one on numerical methods by Lothar Collatz (1910–1990) were also given around the same time.

(*Contribution by Victor Peyrera*)

Austria

Edmund Hlawka gave a two–years-lecture on "Algebra" at the Universität Wien and a numerical pendant on solving algebraic and transcendental equations at the Technische Hochschule Wien (half a year) in 1946–1948.

Since approximately 1955, there existed a 2–year curriculum "Computational Techniques" (in German, "Rechentechnik") at the Technical University of Vienna which taught a mixture of early computer science and numerical analysis. It could either be taken as a supplement to one of the regular courses of study, or as a stand–alone study. After satisfying all requirements, the students got a document qualifying them as "Certified Computational Technicians" (in German, "Gepruefter Rechentechniker"). This must have been one of the earliest official curricula in electronic computing worldwide. There had been an IBM 650 at the University since 1958, the first "real" computer in Austria. Before that, the courses were based on punched card machinery. In 1964, an IBM 7040 arrived which remained the most powerful computer in Austria for a good number of years. This pioneering introduction of mechanical computing into academic teaching was due to Rudolf Inzinger (1907–1980), a professor in mathematics. He was also very modern in other ways: he solicited computational projects from the industry against payment which was used for the enhancement of the computing facilities. This was quite unusual in Academia around 1960. The title for that chair at the Technical University of Vienna was "Computational Techniques" and it was changed to "Numerical Mathematics" upon the request of Hans J. Stetter when he took the position in 1965.

From 1959 to 1961 (?), Walter Knoedel gave a lecture entitled "Numerische Behandlung von algebraischen und transzendenten Gleichungen" each winter term and "Numerische Behandlung von linearen Gleichungssystemen und algebraischen Eigenwertaufgaben" each summer term at the "Technische Hochschule Wien". In the winter term of 1958, he gave a lecture on "Numerische Verfahren der Algebra".

At the University of Innsbruck, it was Gerhard Wanner (now Professor in Geneva, Switzerland) who gave the first lectures in numerical analysis during the winter term 1969–1970 (it is worth to mention that one of his students at that time was Ernst Hairer).

The first course in numerical mathematicsz at the Technical University of Graz were given by Helmut Florian during the winter semester of 1964–1965. The title of the course was "Einführung in die Numerische Mathematik".

(*Contributions by Walter Knoedel, Alexander Ostermann, Karl Perktold and Hans J. Stetter*)

Belgium

A course in numerical analysis was created at the *Université Catholique de Louvain* in October 1960. It was taught by Jean Meinguet and contained the classical topics: approximation, interpolation, integration and numerical solution of equations. A second course, on the numerical solution of partial differential equations by finite differences and finite elements methods, began in 1971 and it was also taught by Meinguet until his retirement in 1995. In 1956, he was also one of the first Belgian to use computers for the solution of scientific and technical problems.

In Leuven University, the first courses in numerical analysis which were taught under that name must have been given by Prof. Ludo Buyst at the end of the 1960's and early 1970's.

He was teaching mainly in the engineering faculty, but attracted also students from mathematics. Of course, computer programming and numerical methods went hand in hand and a (not obligatory) "Practicum" in numerical analysis was also given where the students first learned how to do calculations with the sliding rule. The contents of the more advanced lecture notes were iterative methods for nonlinear equations. A separate course was on (polynomial) interpolation which treated mainly finite differences and some ideas about numerical differentiation and integration. In a course on linear algebra, numerical methods for linear systems of equations (Gauss, etc.) were studied. Some of these topics were hidden in the computer programming course. An extensive course existed soon after that on approximation (orthogonal polynomials, least squares approximation, etc.) This one contained much more theorems than the others, which were mainly computational algorithms. The numerical methods for differential equations came somewhat later (first handwritten notes around 1970). Early 1970's there were two courses for the engineering students: 1st year: error analysis, interpolation, differentiation, integration, 2nd year: iterative methods nonlinear equations, eigenvalues, optimization.

Hugo Van de Vel was the professor giving numerical courses (solution of partial differential equations) especially for mathematics students. This must have been shortly after L. Buyst started. Hugo's courses were more of a theoretical kind: more convergence theorems than practical algorithms. Also astronomy (more or less part of mathematics) was a group that did numerical computations (Prof. Bertiau and later P. Smeyers). FORTRAN was taught in the exercises of the astronomy course.

(*Contributions by Jean Meinguet and Ronald Cools*)

Brazil

The fisrt courses in numerical analysis were initially taught at the Engineering Schools under the denomination "Numerical Calculus", due to its practical importance for the engineers. Of course at the time there was not such strong emphasis on convergence of the methods as is required in Numerical Analysis proper, but this does not mean that these questions where ignored. "Numerical Calculus" was started as a regular course approximately at the same time in two places:
1. At the Polytechinc School of the University of Sao Paulo, in Sao Paulo, under Professor Monteiro de Camargo,
2. At the School of Engineering (founded in 1897) of the Federal University of Rio Grande do Sul, in its capital city Porto Alegre, in March 1956, under Professor Manuel da Silva Neto. Its full name was "Calculo Numerico, Grafico e Mecanico" and the program followed closely the classical book of Willers [272] translated from German. The following year the book by Mario George Salvadori (born in 1907) and Melvin L. Baron (1927–1997) [226] was adopted for use with differences in the numerical solution of ordinary and partial differential equations. There was also a French book, with a title like "Cours de Nomographie", edited by Dunod and about 80 pages long, that was also used as complement reference for graphical methods. The program covered the basic topics of numerical analysis. Slowly graphical methods were set aside, with, perhaps, only one lecture devoted to it. Pedro Nowosad,

originally an engineer, started his career at the School of Engineering in Porto Alegre as an Assistant to Professor Silva Neto in 1957. In 1965, he got his PhD in Mathematics from the Courant Insititute at NYU, encouraged by Prof. Silva Neto who always had a very keen notion of the importance of mathematics for engineering.

Courses in numerical analysis were also given at the State University at Campinas – UNICAMP in 1969. The professors were Odelar Leite Linhares (who got his doctorate in Zürich) and Ivam de Queiroz Barros (who, at the time, taught both at the State University in São Paulo – USP, as well as at UNICAMP).

(*Contributions by João Frederico C.A. Meyer, A.C. Moura and Pedro Nowosad*)

Bulgaria

The first course in numerical analysis was given in the academic year 1959–1960 at Sofia University, Faculty of Physics and Mathematics. The lecturer was Blagovest Hristov Sendov, assistant in the Chair of Algebra. At that time, in Bulgaria, there was only mechanical calculators and one analog computer. Nevertheless, students had exercices on all basic numerical methods. Starting from 1961, Sendov was engaged in the construction of the first original Bulgarian computer which was operational in 1963.

(*Contribution by Blagovest H. Sendov*)

Cuba

In Cuba, numerical analysis was included for the first time in the curriculum in mathematics at the University of Havana in 1963. The teacher was an invited professor of the Carolina University of Prague named Stanislav Malon, who stayed three years and initiated these studies in the branch of applied mathematics. The contents of Numerical Analysis I were : introduction to the method of numerical analysis, solution of nonlinear equations, interpolation and numerical linear algebra. The textbook used that of Berezin and Zhidkov [17].

(*Contribution by Maria Victoria Mederos*)

Czechoslovakia

The situation between the wars in Czechoslovakia was such that there were two universities (Prague, Brno) in the Czech part, one (Bratislava) in Slovakia.

There were also Technical "Universities" in Prague and (maybe later) in Brno.

Numerical mathematics was taught at the Czech Technical University in Prague; there were professors V. Láska, V. Hruška who even published a book in Czech, Theory and Practice of Numerical Computing, around 1935. There also was a group of mathematical methods at the Škoda–Works in Pilsen, lead by M. Hampl.

During the war 1939–1945, the Czech universities (all) were closed by the German authorities. Some professors held (illegally and privately) seminars in their homes, others were persecuted.

After the war, the universities were reopened. Mathematics was taught at faculties of natural sciences of Charles University in Prague and Masaryk University in Brno. Emphasis was in pure mathematics: algebra, topology, analysis, geometry.

The idea to supplement research in applications led to the creation of the Academy of Sciences which would not only be a body of academicians (member of the Academy) but also a number of institutes.

Thus, in 1950, the "Central Institutes" of Mathematics, of Physics etc. were founded; these became institutes of the Czechoslovak Academy of Sciences in 1952.

In mathematics, starting in July 1950, around 8–10 postdocs became "scientific aspirants" (that was a Russian system): they included (later well known) V. Pták, M. Fiedler, J. Mařík, O. Vejvoda, M. Zlámal, I. Babuška, J. Hájek, F. Šik, J. Kurzweil, Z. Nádeník, etc.

Some lectures were in linear algebra, numerical mathematics (Vl. Knichal), statistics, computer science (A. Svoboda, who came from the U.S. to Czechoslovakia to introduce and support research in computer science; he left Czechoslovakia after 1958 or so).

In numerical mathematics, a part of the aspirants became interested (Babuška, Fiedler, Pták, Mařík) and started research in this area.

As teaching of numerical mathematics at the Charles University concerns, it was started in September 1956. Miroslav Fiedler had a lecture (for students in their third year) in the winter semester 1956–1957 in numerical algebra, O. Vejvoda continued in the spring semester 1957 by numerical methods in differential equations.

After a few years, J. Mařík took over lecturing in numerical mathematics; he also published some materials for students.

Around 1960, the faculty of natural sciences was split, and the Faculty of Mathematics and Physics was created. A part of the faculty was then Department of Numerical Mathematics, first led (externally) by M. Hampl; later, Ivo Marek became the head.

See also [10].

(*Contribution by Miroslav Fiedler and the help of Zdenek Strakos*)

Denmark

It seems that the first course in numerical analysis was taught in 1961 at the Technical University of Denmark by Peter Naur, at that time an associate professor. Naur, who was originally an astronomer, later became professor of computer science at Copenhagen University.

(*Contribution by Per Christian Hansen*)

Finland

Ernst Leonard Lindelöf (1870–1946) was professor at Helsinki University and his lectures appeared as a series of five volumes [174]. A characteristic feature is his interest in numerical computations. In the first volume, he started from Lagrange interpolation and then

moved to Taylor's formula [87]. Evert Johannes Nyström (1895–1960), a former student of Lindelöf and a professor of applied mathematics at the Helsinki University of Technology, also gave a course with the title "Applied mathematics" in the spring of 1948. It contained interpolation, least squares approximation, numerical integration, nomography, mechanical harmonic analyzator, and graphical integration. Later this type of courses has been given under different type of titles but mostly under "Numerical Methods" if translated word by word. Pentti Laasonen (1914–2000) also gave courses with this title at Helsinki Technical University in the 1960's.

It seems that the first thesis on a numerical analysis subject was defended at the University of Helsinki in 1914 by Johan Helo (until 1906 Helenius; 1889–1966) [130] under the supervision of Ernst Lindelöf. He studied the convergence of Newton's method for the solution of an equation and continued to work on the subject during the seven following years. Then, he studied law and started a political career after 1928 [87].

(*With contributions by Timo Eirola and Olavi Nevanlinna*)

France

A course in numerical analysis, with the title "Applied Analysis", was created at the Institut Polytechnique de Grenoble by Jean Kuntzmann (1912–1992) in 1947. The topics covered were finite differences, differential equations, zeros of functions, systems of linear equations, etc. It was the first course of this type taught in a French university and Grenoble was the only center for numerical analysis in France for some time. Kuntzmann published mimeographed notes for students as soon as 1950 [162]. The practical work on computers was conducted by Jean Laborde (1912–1997) [164]. Kuntzmann created the Laboratoire de Calcul in 1951. He also became one of the founder and the first editor–in–chief of the French journal of numerical analysis *Chiffres* (now M2AN) [163]. Noël Gastinel (1925–1984) came to Grenoble in 1957 and Pierre–Jean Laurent in 1958. Gastinel was much influential in the development of numerical analysis in France [76].

At the University of Toulouse, courses in numerical analysis were taught to graduate students in mathematics and engineering by Émile Durand, a physicist who used an IBM 650 computer around 1957 for the solution of his own problems. Another course was given by Laudet around the end of the 1950's.

The first chair of numerical analysis was created at the Faculté des Sciences of the Université de Paris in 1959 for René de Possel (1905–1974), one of the first member of the Bourbaki group. He was succeeded by Jacques–Louis Lions in 1966. Lions was one the first in France to be interested by problems coming out from industry, by their modelisation, their mathematical treatment by functional analysis tools and by their algorithmic solution. The school he founded does not need to be presented.

Another influential people for the development of numerical analysis in France was Henri Mineur (1899–1954), the first head of the Astrophysics Laboratory in 1939, who wrote a book on numerical analysis in 1952 [188]. Louis Pierre Couffignal (1902–1966) must also be mentioned. He was the director of the Laboratoire de Calcul Numérique at Institut

Blaise Pascal in Paris from 1946 to 1957. This laboratory was the follower of the analogic computing laboratory created by Joseph J. Pérès (1890–1962) and Lucien C. Malavard as soon as 1932 [192]. Let us mention that Malavard was involved in the development of the Turbosail wind–propulsion system with Jacques–Yves Cousteau (1910–1997) in 1982. Maurice Parodi (1907–1992) was another pioneer in numerical analysis with his works on the computation of eigenvalues.

(*With contributions by Françoise Chatelin and Pierre–Jean Laurent*)

Germany

The first chair of applied mathematics was taken by to Carl David Tolmé Runge (1856–1927) in Göttingen in 1904. His Habilitation (1883) was on the numerical solution of algebraic equations. Friedrich Adolf Willers (1883–1959), later in Dresden, had his first position 1928 in Freiberg, Sachsen. He was a student of Runge (Ph.D. in 1908, Göttingen). He gave most likely courses in numerical analysis.

The first woman involved in the teaching of numerical analysis was probably Hilda Geiringer–Pollaczek (1893–1973); see [212]. At that time she was "Assistentin" of Richard Martin Edler von Mises (1883–1953) (and his wife from 1943) at Berlin University (now called Humboldt Universität) and conducted the "Numerisches Praktikum" there in the years before the emigration in 1933. The students used mechanical desk calculators. One of the students who took this course was Lothar Collatz in 1934. Collatz had his first position in Karlsruhe from 1935 to 1943. The first lecture he gave was in summer 1937 in Karlsruhe with the title "Statik und Festigkeitslehre für Architekten". In the same year, he also gave a course in "Nomographie". In the summer 1938, he gave a course on "Numerische Methoden (Angewandte Mathematik B)".

Special courses on numerical analysis were given in 1955 by Friedrich Ludwig Bauer when he was lecturer in mathematics at the Technische Hochschule München. Then, he went to Mayence in 1958 were he gave regular introductory courses in numerical analysis as part of the curriculum, before moving to Munich in 1962.

One of the central figures in numerical mathematics and computers at that time in Germany was Alwin Walters (1898–1967) at Technische Hochschule Darmstadt. The names of Bertram (Saarbruecken), Unger (Bonn), Lehmann (Dresden) and Romberg (Heidelberg) have also to be mentioned. Robert Sauer, who, together with Klaus Samelson (1918–1980), may be regarded as an early founder of computer science in Munich, gave a course in numerical analysis at Munich University of Technology in 1961. There have been also courses at the University of Frankfurt, given by people from the "Wetterdienst" at Offenbach (the German agency for weather reports and investigations).

(*Contributions by Friedrich L. Bauer, Martin Brokate, Rita Meyer–Spasche, Gerhard Opfer and Hans Joseph Pesch*)

Great Britain

Great Britain was certainly the country where the teaching of numerical analysis was the most developed and where courses were given at an earlier time.

Around the beginning of this century, Edmund Taylor Whittaker (1873–1956) created a Computation Laboratory at the University of Edinburgh, where various branches of numerical analysis were taught that had previously been taught systematically in no British university. His book [268] with George Robinson, a Canadian mathematician then on his staff, remained for 40 years a major text on numerical analysis. In the Preface, it is written

> *The present volume represents courses of lectures given at different times during the years 1913–1923 by Professor Whittaker to undergraduate and graduate students in the Mathematical Laboratory of the University of Edinburgh, and may be regarded as a manual of teaching and practice of the Laboratory... The manuscript of the lectures has been prepared for publication by Mr. Robinson, who has performed the whole of the work of numerical verification and has contributed additional examples.*

The book covers interpolation difference formulae, determinants and linear equations, the numerical solution of algebraic and transcendental equations, numerical integration and summation, normal frequency distributions, the method of least squares, practical Fourier analysis, the smoothing of data, correlation, the search for periodicities, the numerical solution of differential equations and some further problems.

Whittaker was the first in Scotland to teach a course in numerical analysis, and probably he was the first in the whole U.K.

Another leading figure in numerical analysis in Edinburgh was Alexander Craig Aitken (1895–1967). Born and educated in New Zealand, he came to Edinburgh in 1923 for his Ph.D. under Whittaker and he stayed there for the remainder of his life. In 1925, he was appointed to a lectureship in Actuarial Mathematics. In his lessons, one of the topics was interpolation and numerical integration [3, p. 83]. In 1936, he became Reader in Statistics. When Whittaker retired in 1946, Aitken took his Chair, a position he held until his own retirement in 1965 [269]. Aitken taught a course entitled "Mathematical Laboratory" at least from 1952, but it certainly goes back further.

In its lectures at the London School of Economics in 1926 and 1927, just after he returned to England (see the Subsection on the USA below), Leslie John Comrie (1893–1950) stressed the importance of machines [274]. In [53], he wrote

> *In many cases they* [relay and electronic computers] *will enable numerical mathematical analysis, with its wide range of choice in parameters, to replace experiments with costly models.*

Of course, as in the USA, the war effort was essential in the development of numerical analysis [225].

A numerical analysis course was given by John Todd at King's College in London in 1946. In particular, he was teaching Cholesky method (see below).

The first stored–program electronic digital computer, called *the baby*, started at the University of Manchester on June 21, 1948.

When he arrived at Oxford University in 1958, Leslie Fox (1918–1992) began to lecture on numerical linear algebra to the first year students [101].

There was a Senior Honours course in numerical analysis at the University in St Andrews in 1953–1954. It was given by A.R. Mitchell who moved to Dundee in 1967.

A numerical analysis course was given in Aberdeen in 1958–1959 by Fred W. Ponting. There was a lot of use of mechanical desk machines, and the book by D.R. Hartree was followed [128]. Ponting said that it was the first time a course in numerical analysis had been given in Aberdeen. In those days there were only the four ancient Scottish Universities (St Andrews, Glasgow, Aberdeen and Edinburgh) plus Queen's College Dundee (a college of St Andrews, which became the University of Dundee in 1966) and the Glasgow institution which became Strathclyde University.

(*With contributions by Mary Croarken, Roger Fletcher, Philip Heywood, George Phillips, Garry J. Tee, John Todd and Alistair Watson*)

Greece

It seems that the Mathematics Department of the Patras University offered the very first courses in numerical analysis. It was in 1967–1968, the name of the course was "Numerical Analysis and Special Functions", and it was given by the Chair of (Theoretical) Mechanics, Konstantinos Goudas. The first professor in numerical analysis was appointed only in 1974–1975 and he was Kosmas Iordanidis.

The Mathematics Department of the University of Athens offered its first numerical analysis courses in 1970–1971. The Chair responsible was that of Applied Mathematics (with contents numerical analysis and computer programming) and the professor selected in 1971 was Nikolaos Apostolatos.

The Mathematics Department of the Ioannina University was the third to follow in 1971–1972. Apostolos Hadjidimos was appointed in 1972 in the Chair of numerical analysis. It was the very first chair in a Greek University with that name.

The Mathematics Department of the University of Thessaloniki gave its first numerical analysis courses also in 1971–1972. The professor who gave the course was a professor of mathematical analysis. The Chair of numerical analysis was created in the late 70's and the professor was Elias Houstis.

Then, it was the Technical University of Athens. It must have given the first courses in numerical analysis by the mid 70's when Alexis Bacopoulos was elected around that time. The Chair of numerical analysis belonged to one of the Engineering Schools (most probably to the Civil Engineering one).

The University of Crete, which was created in the late 70's, appointed its first numerical analysis professor in 1982. He was Vassilios Dougalis.

In 1981, the Ministry of Education passed a law for the Greek Universities which replaced the old one used since 1932 (for the old one, Konstantinos Caratheodory (1873–1950) was the man responsible and the system of Chairs was very close to the French–German one). Chairs were abolished by the new law and the American system in the ranking of the academic staff was introduced. Consequently it was much easier to appoint academic staff at lower ranks who could teach, say, numerical analysis.

Nowadays numerical analysis or similar courses are offered in all Greek Universities and not only by and in the Mathematics Departments.

(*Contributions by Apostolos Hadjidimos and George Miminis*)

Hungary

Nobody knows exactly by whom, and when, the first course in numerical analysis was given in Hungary. By different persons' opinion, it was most probably given by a great Hungarian professor of Geometry by the name of György Hajós (1912–1972). He taught a course called numerical analysis after the second world war around 1947–1948 at the University of Budapest which is now called Budapest University of Technology and Economy (Budapesti Mûszaki és Gazdaságtudományi Egyetem). At that time, its name was something like Hungarian Royal Joseph Palatine University of Technology and Economy (Magyar Királyi József nádor Mûszaki és Gazdaságtudományi Egyetem). "Nádor", which can be translated by "palatine", was the highest rank in the old feudal government.

(*Contribution by Maria Vicsek*)

Italy

The first course of numerical analysis seems to have been taught at Regia Scuola di Ingegneria di Pisa by Gino Cassinis (1885–1964) as early as 1925. He published a book on the subject [45].

A course with the title "Calcoli Numerici e Grafici" was also given by Mauro Picone (1885–1977) in 1932 at the "Scuola di Scienze Statistiche e Attuariali" in Rome, a school due to the initiative of Guido Castelnuovo (1865–1952) and founded by Francesco Paolo Cantelli (1875–1966). This course was not among those offered to the students of the university since mathematicians like Gaetano Fichera (1922–1996) said that it had not the characteristics for appearing in the list of optional courses for a Master degree in mathematics [196]. In fact, Picone's concern in applied mathematics arose much earlier from problems encountered during World War I and it can be seen even in his pure mathematics researches just after his "Laurea". Picone was interested in theoretically justified and efficient mathematical methods which can counterbalance the modest computational instruments at his disposal. Even if the expression "numerical analysis" was not used, he defined a domain of research between classical analysis and numerical computing [7, 123]. Picone was also the founder of the "Istituto Nazionale per le Applicazioni del Calcolo" in Naples in 1927 (transferred to Rome in 1937). It was the first institute in the world devoted to numerical mathematics and Picone was a world pioneer in establishing such an institute. Picone assistant from 1940 to 1948 was Aldo Ghizzetti (1908–1992) [94].

(*Contributions by Giampietro Allasia, Michiel Bertsch, Andrea Celli, Luigi Gatteschi, Enrico Magenes, Donatella Marini and Alfio Quarteroni and the help of Michela Redivo Zaglia*)

Japan

In Japan, numerical analysis was treated in practical mathematics courses and, so, there were no course of numerical analysis in the Universities before 1970.

Keiichi Hayashi (1879–1957) was Professor of the Faculty of Engineering of Kyushu Imperial University (Now, Kyushu University). He took up his position around 1910 and he studied numerical analysis. He published five mathematical tables in German.

The first course of numerical analysis was given in 1970 by Ichizo Ninomiya (born in 1921) at the Faculty of Engineering of Nagoya University.

(*Contribution by Naoki Osada*)

Mexico

The first course of numerical analysis was given by Pablo Barrera Sanchez at the National University of Mexico (UNAM) in the early 70's.

(*Contribution by Pablo Barrera Sanchez*)

New Zealand

The first course, with the title "Statistical and Numerical Mathematics", was taught in 1963 by John C. Butcher at the University of Canterbury. The part devoted to numerical analysis mainly dealt with the numerical solution of ordinary differential equations [215, p. 144].

(*Contribution by Garry J. Tee*)

Norway

It seems that it was Werner Romberg (born in 1909) who introduced numerical analysis in Norway. In the autumn of 1938, he came to the University of Oslo where, in different positions, he stayed until the end of 1948 (except for a period in Uppsala, Sweden, during World War II). During these years, he worked with Prof. Hylleraas and, for a shot period, with Prof. Holtsmark. In 1949, Romberg went to the Norwegian Institute of Technology in Trondheim as an associate professor of physics. In 1960, he became professor in applied mathematics. In 1968, he returned to the University of Heidelberg in Germany as a professor in "Numerische Methoden der Naturwissenschaften" [126].

Poland

The first course in numerical analysis was given by Stefan Paszkowski in 1962 at the University of Wrocław. He was, maybe, preceded by one or two year by Andrzej Kielbasinski at Warsaw University.

(*Contribution by Stefan Paszkowski*)

Portugal

The first course on numerical analysis at the University of Porto was given by Jaime Rios de Sousa for students in sciences and engineering.

(*Contribution by Filomena Dias d'Almeida*)

Spain

Some topics in numerical analysis were taught by Sixto Rios, a statistician in Madrid and by Ernesto Gardeñes in Barcelona around 1970. The first chair of numerical analysis was taken by Mariano Gasca in Bilbao in 1972. In 1973, a course of numerical analysis became obligatory for all students in mathematics.

(*Contribution by Mariano Gasca*)

Sweden

Numerical analysis as an academic subject in Sweden has three roots:
1) Courses at the Royal Institute of Technology and Chalmers Institute of Technology within the general subject of applied mathematics. At the Royal Institut, they started in the thirties, on the initiative of Prof. Liljeström (originally a physicist), extended by Prof. C.G. Esseen, better known for his contributions to probability theory. He gave well planned courses in numerical methods, which contained linear algebraic systems, single nonlinear equations, interpolation and numerical integration, Runge–Kutta's and Simpson's rule for ordinary differential equations and nomography. During 1957, the preparations started for an associate professorship in Applied Mathematics, in particular Numerical Analysis. Germund Dahlquist got this position in 1959. It was vacant in 1958, and a physicist named N. Åslund, who has a good hand with computing, acted protem and developed some courses, also containing an introduction to digital and analog computers, their logic construction and the elements of programming (Åslund later became professor of physics in Stockholm). Practical programming, including the running of programs written by students, started in 1959. Some of the years from 1954–1958, Dahlquist organized courses in programming for the Swedish electronic computer BESK, and organized the work on a program library for basic numerical methods etc., but there were no regular courses on numerical methods.

During the sixties, activities grew also at other universities and institutes of technology. In 1963, full professorships were created in Stockholm (Dahlquist, jointly for Royal Institute of Technology and the University), Gothenburg (Heinz Otto Kreiss at Chalmers), and in 1965 in Lund (Carl–Erik Fröberg). A few years later, Kreiss moved to Uppsala, and at the end of the decade Owe Axelsson and Åke Björck became professors in Gothenburg and Linköping, respectively. From 1959, Björck and Dahlquist worked together in Stockholm with teaching (elementary courses for 500 students and a second course for 30 students) and research in numerical analysis. The first Swedish version of their textbook was printed in 1969 (the English version of 1974 is much larger [64]).
2) Independently C.–E. Fröberg started, around 1956 (?) courses in numerical analysis at the Department of Theoretical Physics, at Lund university, and published in 1959 (?) a

first mimeographed version of a textbook that he later considerably extended. A Swedish version (415 pages) was published in 1962, and at least two considerably modernized versions were published in English in the seventies and eighties [105]. Fröberg, who got his Ph.D. in theoretical physics, became associate professor of numerical analysis in 1956 and full professor in 1965. Around 1960, Fröberg's courses covered more stuff than the courses of Björck and Dahlquist in Stockholm, but they had a much larger number of students.
3) Carl Harald Cramér (1893–1985) became professor in actuarial mathematics and mathematical statistics in Stockholm in 1929. When Dahlquist studied the subject in 1942–1943, he had to read parts of Whittaker and Robinson [268] and a Danish version of Steffensen's book on Interpolation. There was no teaching in numerical methods, but he remembers that, in the exam, one of the problems was to derive a 5th order accurate Newton–Cotes formula.

(*Contribution by Germund Dahlquist*)

Switzerland

A course with the title *Calcul Approximatif* was given as soon as 1877–1879 at the University of Geneva by Charles Galopin–Schaub (deceased in 1901), then a Privat Docent. He also wrote a small book on the subject [107]. His course contains a first part on the theory of approximate computations, a second part on various approximation methods (such as series and continued fractions), a third part dealing with least squares, a fourth part on various numerical methods for the solution of transcendental equations and a fifth part with applications to analysis and mechanics.

The department of applied mathematics (Institut für Angewandte Mathematik) of the Eidgenössische Technische Hochschule Zürich (ETHZ) was founded in January 1948 by Eduard Stiefel (1909–1978) with the help of Heinz Rutishauser (1918–1970) and Ambros Paul Speiser. Stiefel immediately began to lecture on numerical analysis and a Z4 computer was rented to Konrad Zuse (1910–1995) in 1949 [285]. At that time, with also Peter Henrici (1923–1987) who got his Ph.D. in mathematics in 1952 under the direction of Stiefel, ETHZ was a renowned international center for numerical analysis [124].

A course in numerical analysis was taught at the University of Basel in 1950 by Eduard Batschelet (1914–1979). At the University of Lausanne and then at École Polytechnique Fédérale de Lausanne, Charles Blanc (born in 1910), a specialist of operations research, also oriented the teaching of mathematics toward applied mathematics and numerical analysis. In 1958, he participated to a meeting at UNESCO where the French word "informatique" was coined.

(*Contributions by Olivier Besson, Jean Meinguet and Walter Gautschi*)

USA

Numerical analysis was not much developed in the USA before World War II and there were very few courses on this topics.

Leslie John Comrie was born in 1893 at Pukekohe in New Zealand and he died at London in 1950. He studied in this Department, and graduated M.A. (Honours in Chemistry) in 1916. Throughout the second quarter of this century, Comrie was acknowledged as the leader in scientific computation. In 1923–1924, he was Assistant Professor of Mathematics and Astronomy at Swarthmore College, Philadelphia, and in 1924–1925 he was Assistant Professor of Astronomy at Northwestern University, Chicago. At both posts, Comrie introduced courses on computation into the standard degree curriculum [180]. He returned to England in 1925 [58].

James B. Scarborough gave a course of Engineering Mathematics in 1925 at the U.S. Naval Academy in Annapolis, Maryland, and published a book on the subject [227].

Those two courses must have been amongst the earliest courses in numerical analysis to be given in the USA.

Researches in numerical analysis began with the war effort [185, 205, 206, 217], in particular at the National Bureau of Standards [244, 245]. Part of the impetus was due to the emigrated mathematicians [83, 19]. The emphasis was put on the computation of tables and the journal *Mathematical Tables and other Aids to Computation* (MTAC) was created in January 1943 by Raymond Clare Archibald (1875–1957) for this purpose [143]. It is very instructive to look at the first volumes of this journal to see the shift from the construction of tables to numerical analysis. Another important topic was the computation of special functions which culminated with the publication of the Handbook [1] under the supervision of Milton Abramowitz (1915–1958) and Irene A. Stegun (born in 1919) (a new version of it is under construction, see http://dlmf.nist.gov/).

After the war, the subject of numerical analysis was taken up by universities (see, for example, the impact of war on the development of operations research [191]). As explained by Mina Spiegel Rees (1902–1997) [207]

> *Before World War II, there had been relatively little emphasis in American universities on "applied mathematics" which was strongly represented at a number of German centers, particularly Goettingen, and at other continental universities as well as on British campuses. After the war, with some of the world's most distinguished emigre mathematicians on our campuses, it was possible to contemplate a strong development in these fields in the U.S. The ONR* [Office of Naval Research] *seized the opportunity to support emerging groups like that at New York University under Richard Courant* [(1888-1972)]*whose work at Goettingen had made his name magic through the world. There were also individuals, such as* [Gabor] *Szegö* [(1895–1985)] *and Pólya at Stanford, about whom strong groups could be expected to grow....*
>
> *A number of very able young people were trained at the Institute* [for Numerical Analysis at UCLA]*, and large number of older mathematicians learned the new approaches that computers required. When the Institute was terminated as a government agency, many of these people went to new positions across the country*

and spread the word about the computer revolution to many universities that had not yet been involved.

A course in numerical analysis was given by Kaiser S. Kunz at Harvard around 1945–1946. He also wrote a book [165]. A similar course was given by William Edmund Milne (1890–1971) around 1949 at the Oregon State College. His book [187] contains an annotated bibliography on early texts on numerical mathematics. In a paper of 1946, it is mentioned that courses on practical mathematics were taught at Brown and New York Universities [88]. Gene Howard Golub, in 1953, followed a course on computing given by John Purcell (Jack) Nash at the University of Illinois. Then, he became assistant in the Digital Computing Laboratory there; see [267]. A seminar on numerical analysis was organized by Wolfgang Richard Wasow (1909–1993) in the 1957–1958 academic year at the University of Wisconsin in Madison. The theme was a study of iterative methods for matrix problems.

(*With contributions by Mary Croarken, Gerald Hedstrom, David R. Kincaid, Garry J. Tee and David Young*)

USSR

In 1935, a Department of Approximate Methods of Analysis was created at the Steklov Mathematical Institute of the Russian Academy of Sciences in Moscow. Its first head was A.M. Zhuravskii.

It seems that the first course in numerical analysis was given in 1948 at the University of Leningrad (now again St Petersburg) by Leonid Vitalyevich Kantorovich (1912–1986). At that time he was interested in showing the broad possibilities for using the ideas of functional analysis in numerical mathematics.

A chair of computational mathematics was created at the University of Leningrad in 1951 and Vladimir Ivanovich Krylov (born in 1902) was appointed. Then, the position was taken up by Mark Konstantinovich Gavurin (1911–1992). The book by Vera Nikolaevna Faddeeva (1906–1983) [89] was very influential and Dmitrii Konstantinovich Faddeev (1907–1989) was also an important contributor to numerical linear algebra [159].

In 1948, a Department of Numerical Mathematics at a Research Institute in Mechanics of the Moscow State University was founded by Lazar Arionovich Lyusternik. The names of Andrei Nikolaevich Tikhonov (1906–1993) and Aleksandr Andreevich Samarskii must also be mentioned.

In the early 1950's, Sergei Lvovich Sobolev (1908–1989) began to work on computational mathematics. At the same time, the M1 computer was developed in the Laboratory of Electrosystems at the Institute of Energy of the USSR Academy of Science under the direction of I.S. Brouk. In 1952, Sobolev became head of the first department of computational mathematics in the Soviet Union (at Moscow State University).

More recently, Gurii̇ Ivanovich Marchuk was much influential in the development of numerical analysis in the USSR as the head if the Institute of Numerical Mathematics of the Russian Academy of Sciences.

See also [10].

(*With contributions by Vera Kublanovskaya and Maria Vicsek*)

The Netherlands

At Delft University of Technology (TUD), on page 108 of the Program for the Academic Year 1947–1948, the following course is listed:

Prof. Dr. S.C. van Veen: Applied Analysis (Toegepaste Analyse)
A. Interpolation methods. Mechanical quadrature. Graphical differentiation and integration. Numerical solution of algebraic and transcendental equations. Theory and application of planimeters and harmonic analyzers.
B. Approximate and graphical solution of ordinary differential equations of first and second order, with boundary conditions. Determination of eigenvalues and eigenfunctions. Method of Ritz–Galerkin. Approximation formulas for elliptic integrals.

Although not called numerical analysis, it was really the subject of the course. Prof. van Veen changed the name of the course to Numerical Analysis in the academic program for for 1950–1951 (page 138), without changing the contents. It was an elective for students in the third and fourth year of various engineering disciplines. Professor van Veen was a highly regarded analyst in The Netherlands. He taught for many years at TUD. He retired around 1965.

In 1956, the course was taken over by prof. Dr. R. Timman, who changed the contents (academic program 1956–1957, page 198):

Prof. Dr. R. Timman: Numerical Analysis
A. Interpolation methods, numerical integration and differentiation. Solution methods for algebraic and transcendental equations with one unknown. Integration methods for ordinary differential equations (2 hours in the first semester).
B. Solution methods for linear systems of equations, iteration methods, numerical determination of eigenvalues and eigenvectors of matrices. Solution methods for partial differential equations, relaxation methods and methods of characteristics (2 hours in the second semester).

Timman taught various other courses in applied mathematics. The first professor who devoted himself solely to numerical analysis was Prof. Dr. E. van Spiegel, who arrived in 1958. Pieter Wesseling succeeded him in 1977.

The first course of numerical analysis in Amsterdam was given by Prof. Dr. Adriaan van Wijngaarden.

He was born in 1916 and, for most of his career, had his main position at the Mathematical Centre (now CWI), Amsterdam, from 1947 until he retired in 1981.

He was appointed as part–time professor at the University of Amsterdam in 1952 and presented his inaugural address on 27 October 1952 entitled “Rekenen en vertalen” (i.e. Computing and translating).

He started his course on numerical analysis at the University of Amsterdam (presumably) in September or October 1952. The name of his course (presumably) was "Numerieke Wiskunde" (i.e. numerical analysis). He gave his course every year (weekly on Friday morning). He put much emphasis on algorithms, programming, and later on programming languages, in particular Algol 60 and Algol 68, and design of languages.

Students assignments had the form of solving a certain problem by writing a program and then to show the program and the results in an oral examination.

Adriaan van Wijngaarden retired from the University of Amsterdam in 1983 and died in 1987 at the age of 70.

(*Contributions by Dirk Dekker and Pieter Wesseling*)

Venezuela

In Venezuela, the first course in numerical analysis was taught by Victor Pereyra in the winter semester of 1967. It was at the School of Physics and Mathematics, Faculty of Sciences, Universidad Central de Venezuela, Caracas.

(*Contribution by Godela Scherer*)

Yugoslavia

The first course of numerical analysis at the University of Ljubljana (Slovenia) was taught by France Krizanic in the academic year 1961–1962. Slovenia was until 1990 part of Yugoslavia. From then on, the course was taken over by Zvonimir Bohte (lecturer at that time) who taught the subject for the next 35 years.

(*Contribution by Zvonimir Bohte*)

Landmarks

Recently, a list of the ten top algorithms was published [82] and a list of the classic papers in numerical analysis was given by L.N. Trefethen [250]. In numerical analysis, not only algorithms are important but some concepts as well. This is the reason why we prefer to speak about landmarks and give a (non limitative) list of those we consider as the most important ones.

1901: Runge–Kutta methods

The history of methods for the numerical integration of ordinary differential equations does not began with the twentieth century. It seems that Newton was the first to solve differential equations by a numerical process and every numerical analyst knows, for example, Euler method which goes back to 1768. There were other methods, mostly obtained by astronomers for solving celestial mechanics problems (the return of Halley's comet, for example). Their history can be found in [116] and [111].

Here, we will only discuss the case of Runge–Kutta methods. Runge published a paper in 1895 [218] where he gave two second–order two–stage methods, one based on the midpoint rule

(a generalization of Simpson method for computing definite integrals) and the other one on the trapezoidal rule. In 1900, Karl Heun (1859–1929) [136] derived a restricted class of Runge–Kutta methods (among them, those of Runge) and determined their coefficients. The general class was introduced by Martin Wilhelm Kutta (1867–1944) in 1901 [166]. He wrote down the system of nonlinear equations that must be satisfied by the coefficients to attain a specific order. In particular, he obtained the well–known fourth–order four–stage formula and a fifth–order six–stage one.

An extensive historical perspective on the numerical integration of ordinary differential equations can be found in [247]. For the developments of numerical methods for ordinary differential equations in the twentieth century, consult [44]. Applications to Volterra integral equations is reviewed in [11] and retarded differential equations are discussed in [12].

1903: Cholesky decomposition

André Louis Cholesky (1875–1918) was a French military engineer. He was occupied in cartography where angles and lengths have to fulfill constrains expressing that the sum of the angles in a triangle have a value greater that π depending of the sphericity of the Earth and that the lengths remain the same independently of the order in which the measures are taken. This procedure leads to a linear system with more unknowns than equations which has to be solved by a least squares procedure. Cholesky never published his work. So, it is difficult to date exactly his method but seems to have been found when he was in Tunisia between 1902 and 1905. It was first presented in a paper by Commandant Benoit [16] in 1924. Cholesky method was taught by John Todd in his numerical analysis course at King's College in London as soon as 1946 [245]. In [242], it is said

> *In 1946 one of us (J.T.) offered a course at King's college, London (*KCL*) on Numerical Mathematics. While we had had some war time experience in numerical mathematics, including characteristic values of matrices, we had had little to do with the solution of systems of linear equations. In order to see how this topic should be presented, we made a survey of Math. Rev. (at that time easy!) and found a review (*MR *7 (1944), 488), of a paper by Henry Jensen [146], written by E. Bodewig. Jensen stated "Cholesky's method seems to possess all advantages". So it was decided to follow Cholesky and, since the method was clearly explained, we did not try to find the original paper.*
>
> *Leslie Fox, then in the newly formed Mathematics Division of the (British) National Physical Laboratory (*NPL*), audited the course and apparently found the Cholesky Method attractive for he took it back to* NPL*, where he and his colleagues studied it deeply. From these papers the Cholesky (or sometimes Choleski) Method made its way into the tool boxes of numerical linear algebraists via the textbooks of the 1950s.*

In the B.A. Honours and B.Sc Special Examinations in "Mathematics, Advanced Subjects – Numerical Methods" for the internal students of King's College in 1947, Todd gave an exercise on the numerical application of Cholesky's method to a 4×4 Hilbert matrix.

Cholesky method can also be found in papers by Paul Summer Dwyer (born 1901), the first one being [85] where it was called the "square root method" and related to a method due to Tadeusz Banachiewicz (1882–1954) [13].

The first analysis of Cholesky's method was written by Leslie Fox, Harry Douglas Huskey and James H. Wilkinson. Its numerical stability was simultaneously studied by Alan Turing [252].

As noticed in [178], Otto Toeplitz (1881–1940) proved in 1907 that an Hermitian matrix can be factorized into a product LL^* where L is lower triangular [246].

A biography of Cholesky can be found in [34] and an analysis of his method is given in [47].

1926: Aitken Δ^2 process

Many methods used in numerical analysis are iterative. Quite often, the convergence is slow and, so, the effectiveness of the method is reduced. Of course, it is sometimes possible to modify the construction of the sequence and to obtain a faster convergence. However, such a procedure highly depends on the process by which the sequence is constructed. Another way for accelerating the convergence is to transform the sequence (S_n) into another one, (T_n), which can, under some assumptions, converge faster to the same limit.

One of the most popular acceleration method is certainly Aitken Δ^2 process defined by

$$T_n = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n}, \quad n = 0, 1, \ldots$$

The method was given by Alexander Craig Aitken (1895–1967) in 1926 [2] for accelerating the convergence of Bernoulli method for computing the dominant zero of a polynomial. Aitken pointed out that the same method was obtained by Hans Eduard von Naegelsbach (1838– ?) in 1876 in his study of Furstenau method for solving nonlinear equations [193]. The process was also given by James Clerk Maxwell (1831–1879) in his book of 1873 [181]. However, neither Naegelsbach nor Maxwell used it for the purpose of convergence acceleration. In fact, Aitken process was known to Takakazu Seki (1642?–1708), often considered the greatest Japanese mathematician. In his book *Katsuyō Sanpō*, vol. IV, he used this process to compute the value of π, the length of a chord and the volume of a sphere. This book was written around 1680 but only published in 1712 by his disciple Murahide Araki. Parts of it can be found in [138]. For more details, see [35]. A biography and autobiography of Aitken can be found in [3]. He was also a great mental calculator as explained in [236, pp. 266–275].

What makes Aitken process so popular is that it accelerates the convergence of all linearly converging sequences, that is sequences such that $\exists a \neq 1$

$$\lim_{n\to\infty} \frac{S_{n+1} - S}{S_n - S} = a.$$

Aitken process was the starting point for the study of other powerful convergence acceleration methods. The first generalization was the sequence transformation due to Daniel Shanks (1917–1996) [233] and the ε–algorithm of Peter Wynn [276] (which allows to implement it in an easy way). Shanks transformation is related to Padé approximation, another important topic in applied mathematics [31, 33]. On the history of convergence acceleration methods, see [35].

Convergence acceleration is based on the idea of extrapolation and it is related to the knowledge of an asymptotic expansion of the error [264]. Some acceleration devices also lead to new methods outside the domain of convergence acceleration since they are connected to fixed point iterations (Steffensen method comes out from Aitken process, for example) and other topics.

The theoretical and practical study of these methods now form an independent chapter in numerical analysis; see, for example, [273, 75, 36].

1934: Rémès algorithms

It can be said that numerical methods for the best approximation of functions (that is involving the minimization of a norm) started in 1934.

Let $f, \phi_1, \ldots, \phi_n$ be functions in $C[a, b]$, the set of continuous functions on $[a, b]$. Chebyshev approximation consists in finding

$$\phi(x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

which minimizes

$$\|f - \phi\|_\infty = \max_{x \in [a,b]} |f(x) - \phi(x)|.$$

For the case $\phi_i(x) = x^{i-1}$, it was proved in 1905 by Émile Félix Édouard Borel (1871–1956) that $a_1, \ldots, a_n$ is a solution if and only if there exist $n+1$ points in $[a, b]$ where the norm is attained with alternating signs [28]. Moreover, this solution is unique. This result was extended to functions forming a Chebyshev system (which means that every linear combination of $\phi_i, \ldots, \phi_n$ has at most $n-1$ zeros in $[a, b]$) by John Wesley Young (1879–1932) in 1907 [282]. In 1918, Alfréd Haar (1885–1933) proved that a necessary and sufficient condition for the uniqueness of the solution is that the ϕ_i's form a Chebyshev system [125]. An algorithm for the solution of this problem was given by George Pólya (1887–1985) in 1913 [203] but a counterexample to its convergence was found by Jean Descloux in 1963 [78].

In 1934, Evgeny Yakovlevich Remez (1896–1975) (or Rémès as he himself spell his name in French) proposed two algorithms for the numerical solution of this problem [209, 210]. Its first algorithm applies to the general Chebyshev best approximation problem. However, its convergence can be extremely slow. The second algorithm exploits the alternation property and solves a sequence of discrete problems. It was proved by László Veidinger in 1960 [261] that it converges quadratically if f and the ϕ_i's are twice continuously differentiable.

Nowadays, approximation theory form an important branch of numerical analysis. A detailed account of its history is given in [266]; see also [41, 32].

1946: The Monte Carlo method

The idea of the Monte Carlo method came to the mind of Stanislaw Marcin Ulam (1909–1984) when he was playing solitaires after an illness. Instead of making complicated combinatorial computations about his chances, he had the idea of playing one hundred times and counting the number of successful plays [86, 256]. The method was developed by him, John von Neumann

and Nicholas Metropolis (1915–1999) [184]. It consists in simulating a physical or mathematical system by a sampling operation whose expectation or variance gives the behavior of the system. For example, the solution of a partial differential equation which is too difficult to compute by a more classical method (for example, the neutron transport equation or Schrödinger equation) can be obtained by a stochastic process whose probability function asymptotically satisfies the equation. Of course, Monte Carlo methods need the construction of random number generators.

1946: Splines

Spline interpolation consists of piecewise polynomial interpolation. Thus, a spline function is a piecewise polynomial whose values and the values of some of its first derivatives coincide at the interpolation points. Although the subject had been studied as early as 1906 by George David Birkhoff (1884-1944) [23], its development really starts in 1946 with two papers by Isaac Jacob Schoenberg (1903-1990) [228, 229]. In [230], he reported the history of his discovery (see also [32]) and said

> *A spline is a simple mechanical device for drawing smooth curves. It is a slender flexible bar made of wood or some other elastic material. The spline is placed on the sheet of graph paper and held in place at various points by means of certain heavy objects (called "dogs" or "rats") such as to take the shape of the curve we wish to draw.*

Spline functions have many applications in computer aided design. The story began when the French automobile company *Citroën* asked Paul de Casteljau in 1958 to develop a system for helping engineers to design cars. Casteljau had the idea of using Bernstein polynomials for this purpose but, due to secrecy, he was not allowed to publish his results. However the secret was not total since, in the early sixties, Pierre Bézier (1910-1999), an engineer at *Renault*, heard that Citroën was developing such a project and he initiated a new approach to curves and surfaces approximation. It consists of piecewise polynomials controlled by polygonal lines. The system UNISURF used by Renault was described in several publications and this is why such curves and surfaces are now known as *Bézier curves* and *surfaces*. It was R. Forrest who proved in 1972, while translating Bézier's book [18], that both approaches were identical since Bézier curves and surfaces can be formulated in terms of Bernstein polynomials [99].

Parametric splines, that is splines with certain differentiability constrains, where introduced in Computer Aided Geometric Design (CAGD) by J. Ferguson in 1963 who was working at Boeing Co [93]. These curves were studied simultaneously by Carl de Boor and W. Gordon at General Motors for the purpose of interpolating data. B–splines were introduced by de Boor in 1972 [71]. They are connected to Bézier curves.

The case of two variables began to be treated in the sixties by a tensor product approach. In 1959, Garrett Birkhoff (1911–1996) began his consulting work with General Motors Research. The problem was the representation of automobile surfaces in order to exploit the new numerically controlled milling machines for the cutting of dies needed for the stamping of outer and inner panels. It was necessary to determine the free parameters in a flexible mathematical model so as to fit closely to measurements taken from the finished physical model of the car. Birkhoff quickly

recommend to use cubic splines for the representation of smooth curves. He was familiar with their use through his contact with the David Taylor Model Basin and also through the work done at Boeing. Together with Henry Garabedian, he developed a bivariate generalization of cubic spline interpolation for interpolating a C^1 surface to a given rectangular mesh of cubic splines [22]. This was a pioneer work in splines which led to many further developments [259, 281]. De Casteljau constructed a surface from a net of control points [73, p.18] and Ferguson gave a method for interpolating vector valued surface data by composite bicubic surfaces [93]. Bézier surfaces were introduced in the late sixties and they still remain, together with B–splines surfaces, one of the most popular schemes [26]. The triangular polynomial patches of de Casteljau [72, p.10] are widely used in the finite elements method.

Radial basis functions, which are now very much in flavor, can be considered as a generalization of splines to several dimensions [204].

Many historical notes on splines can be found in [231]; see also [32]. On the history of multivariate polynomial interpolation, see [108].

1947: The simplex algorithm

A *linear programming problem* consists in finding $x \in \mathbb{R}^n$ which minimizes (c, x) and, simultaneously, satisfies the constrains $x \geq 0$ and $Ax = b$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are given. Such problems arise in planning activities in economics.

The constrains define a convex polyhedral set and the linear programming problem can be solved by a descent procedure along its outside edges. This was the approach suggested independently by Jean Baptiste Joseph Fourier (1768–1830) in 1824 [100], Charles de la Vallée Poussin (1866–1962) in 1911 [74] and Frank Lauren Hitchcock (1875–1957) in 1941 [139]. The *simplex algorithm* for solving linear programming problems was proposed by George Bernard Dantzig in 1947 [66]. It follows the same idea but looking at the problem in the geometry of the columns of the matrix A rather than its rows.

Linear programming was also independently studied by L.V. Kantorovich. In his Nobel prize presentation speech in economics in 1975 he said
(see www.nobel.se/economics/laureates/1975/kantorovich-autobio-html)

> *The thirties was also important for me as I began my first economics. The very starting point was rather accidental. In 1938, as professor of the university, I acted as a consultant for the Laboratory of the Plywood Trust in a very special extreme problem. Economically, it was a problem of distributing some initial raw materials in order to maximize equipment productivity under certain restrictions. Mathematically, it was a problem of maximizing a linear function on a convex polytope. The well–known general recommendation of calculus to compare the function values in the polytope vertices lost its force since the vertices number was enormous even in very simple problems.*
>
> *But this accidental problem turned out to be very typical. I found many different economic problems with the same mathematical form: work distribution for equipment, the best use of sowing area, rational material cutting, use of complex resources, distribution of transport flows. This was reason enough to find an efficient method of*

solving the problem. The method was found under influence of ideas of functional analysis as I named the "method of resolving multipliers".

In 1939, the Leningrad University Press printed my booklet called The Mathematical Method of Production Planning and Organization which was devoted to the formulation of the basic economic problems, their mathematical form, a sketch of the solution method, and the first discussion of its economic sense. In essence, it contained the main ideas of the theories and algorithms of linear programming. The work remained unknown for many years to Western scholars. Later, Tjalling Koopmans, George Dantzing, et al, found these results and, moreover, in their own way. But their contributions remained unknown to me until the middle of the 50s.

I recognized the broad horizons offered by this work at an early stage. It could be carried forward in three directions:

1) The further development of methods of solving these extremal problems and their generalization; their application to separate classes of problems;

2) A mathematical generalization of these problems such as, non–linear problems, problems in functional spaces, the application of these methods to extremal problems of mathematics, mechanics and technical sciences;

3) The spreading of the method of description and analysis from separate economic problems to general economic systems with their application to planning problems on the level of an industry, a region, the whole national economy as well as the analysis of the structure of economic indices....

Some activity took place in the first two directions (the results were published partly immediately, partly after the war), but the third one lured me most. I hope that the reasons were clarified enough in my Nobel lecture.

The studies were interrupted by the war. During the war, I worked as Professor of the Higher School for Naval Engineers. But even then I found time to continue my deliberations in the realm of economics. It was then that I wrote the first version of my book. Having returned to Leningrad in 1944, I worked at the University and at the Mathematical Institute of the USSR Academy of Sciences, heading the Department of Approximate Methods. At that time, I became interested in computation problems, with some results in the automation of programming and in computer construction.

My economics studies progressed as well. I particularly wish to mention the work done in 1948–1950 at the Leningrad Carriage-Building Works by geometrist V. A. Zalgaller under my guidance. Here the optimal use of steel sheets was calculated by linear programming methods and saved material. Our book of 1951 summarized our experience and gave a systematic explanation of our algorithms including the combination of linear programming with the idea of dynamic programming (independently of R. Bellman).

On the history of the simplex method, see [67]. See also [173] and [5, pp. 61–79].

More recently, a new algorithm for linear programming was proposed by Narendra Karmarkar [153]. It was claimed to be faster than the simplex algorithm but, however, it was received with scepticism by some researchers because of lack of evidence for this property [96, p. 185].

1950: Nonlinear programming

The birth of nonlinear programming is dated from 1950 with the publication of a paper by Harold W. Kuhn and Albert W. Tucker (1905–1995) [160]. They gave a necessary sufficient condition for the existence of an optimal solution to the problem of minimizing $f(x)$ subject to the constraints $g_i(x) \leq 0$ for $i = 0, \ldots, m$ and $x \in X \subseteq \mathbb{R}^n$. This paper attracted much attention while a similar result obtained by William Karush in his Master's Thesis in 1939 [154] under the supervision of Lawrence M. Graves at the University of Chicago and by Fritz John (1910–1995) in 1948 [147] were almost totally ignored (John's paper was even rejected).

For a complete discussion, see [155] and [156].

1950: Successive overrelaxation method

After having discretized, by finite differences, a linear partial differential equation (usually of elliptic type), one has to solve a system of linear equations. For a discretization of fifty points in both direction, a system of 2500 equations is obtained. Around 1945, this was much too large for applying a direct method and *relaxation methods* (such as those of Liebmann, Richardson, Jacobi, Gauss–Seidel and Southwell), were used. However, their convergence was slow, as Leslie Fox showed in his D. Phil. Thesis in 1942 under Sir Richard Vynne Southwell (1888-1970) [101]. When David Young asked Garrett Birkhoff for a thesis subject, he was offered to work on this topic. He made two key observations: for systems with a *consistently ordered* matrix, the eigenvalues of the Gauss–Seidel method are the squares of those of the Jacobi method and, instead of relaxing (that is modifying the ith component of the approximate solution for annihilating the ith component of the residual), it is better to introduce a parameter and *overrelax*. For some type of matrices, Young found the expression for the optimal value of the overrelaxation parameter (that is the value minimizing the spectral radius of the iteration matrix). It is connected to the square of the spectral radius of the Jacobi iteration matrix. The method, also studied simultaneously by Stanley P. Frankel [104], was called *successive overrelaxation* (SOR). Young defended his thesis in 1950 [279]. On the history of SOR, see [280].

Following this early work, a whole theory was built and other methods were found and studied. The method was successfully applied to the solution of many partial differential equations, which certainly had quite a great impact on the future development of numerical analysis. On these methods, see the book of Richard Stephen Varga, whose first edition goes back to 1962 and is certainly one of the most frequently quoted book of the numerical analysis literature [260].

1951: Continuation methods

For solving a system of nonlinear equations one needs tools for localizing the area of the expected solution and for constructing acceptable data for starting an effective iteration process. Numerical

continuation methods include a variety of topics such as imbedding methods, homotopy methods, parameter variation methods, incremental methods, piecewise linear methods. The basic concept of the so–called incremental methods goes back to E. Lahaye in 1934 [169]. The use of continuation by differentiation started about 1952 by Dimitrii Fedorovich Davidenko (born in 1922) [69]. Several variants of continuation methods were among others investigated in the 1970's by Herbert Bishop Keller (born in 1923), Eugene L. Allgower and Werner Carl (born in 1927) Rheinboldt (see [211] for references). Software packages (e.g. HOMPACK, MANPACK, ALCON, BIFPACK) exist that are based on continuation methods. For a survey we refer to the paper by Rheinboldt [211]. See [6] for an extensive study.

1952: Lanczos method

A *projection method* for solving the $n \times n$ system $Ax = b$ consists in constructing a sequence of iterates $x_k \in \mathbb{R}^n$ such that

$$\begin{aligned} x_k - x_0 &\in \mathcal{K}_k \\ r_k = b - Ax_k &\perp \mathcal{L}_k \end{aligned}$$

where $\mathcal{K}_k$ and $\mathcal{L}_k$ are two subspaces of dimension k of $\mathbb{R}^n$. When $\mathcal{K}_k = K_k(A, r_0) = \text{span}\,(r_0, Ar_0, \ldots, A^{k-1}r_0)$, the method is called a *Krylov subspace* method. If, in addition, $\mathcal{L}_k = K_k(A^T, y)$ where y is an (almost) arbitrary vector, we find the method obtained by Cornelius Lanczos (1893–1974) in 1952 [171]. It was Aleksei Nikolaevich Krylov (1863–1945) who first used this type of subspace in 1931 to construct the characteristic polynomial of a matrix [157]. Lanczos encountered the mathematical problem when he was doing research for the Boeing Aircraft Company. Around this time, he wrote to Albert Einstein (1879–1955) [112]

> *The reason I am interested very much in dealing with methods of approximation is not the practical applicability of the solution but rather the fact that a very "economical" solution is possible only if it is very "adequate" too. To get a solution in a few steps means nearly always that one has found a method which is consistent with the intrinsic nature of the problem.*

Einstein replied

> *Your remark concerning the significance of the adapted approximation methods is very enlightening, and I am convinced that it represents not merely a practical method but also is a promising mathematical viewpoint.*

This was really a prophetic view!

In an interview given in 1974 [152], only some time before he died, Lanczos was asked

> *What would you say has been the most important, the most fundamental and essential aspect of your sixty years of work?*

He answered

I believe my most important contribution was in the fields of mathematics, to be precise, in numerical analysis – my discovery of a method now known as the Lánczos method. It is very little used today, because there are now a number of other methods, but it was particularly interesting in that the analysis of the matrix could be carried out, that is, all the eigenvectors could be obtained by a simple procedure.

The main advantage of projection methods is that they only require products of vectors either by A and A^T, and the computation of scalar products. So, they are cheap and easy to implement.

When the matrix A is symmetric positive definite, the vector $x = A^{-1}b$ is the unique vector minimizing the strictly convex quadratic functional

$$J(u) = \frac{1}{2}(u, Au) - (u, b).$$

This minimum can be reached by a descent method such as the *conjugate gradient* algorithm proposed by Magnus Rudolph Hestenes (1906–1991) and Eduard Ludwig Stiefel (1909–1978) in 1952 [134]. Hestenes was working at the *Institute for Numerical Analysis* (INA) located at UCLA. In June and early July 1951, he derived the conjugate gradient algorithm. When, in August, Stiefel arrived at INA from Switzerland for a congress, the librarian gave him the routine written by Hestenes. Shortly after, Stiefel came to Hestenes office and said about the paper "this is my talk!". Indeed, Stiefel had invented the same algorithm from a different point of view. So, they decided to write a joint paper on the algorithm and its properties [134]. In the meantime, Lanczos, an Hungarian born physicist (he worked with Einstein) and mathematician, was a member of INA. He soon observed that the conjugate gradient algorithm could be derived from his biorthogonalization algorithm for computing eigenvalues of matrices [170]. Then, he derived from it his *method of minimized iterations* for solving a system of linear equations [171]. Let us also mention that Lanczos method is also connected to a generalization of Aitken Δ^2 process called the topological ε–algorithm [30, pp. 184–189] and, so, to Padé approximants.

The historical development of gradient methods has been described in several publications; see [133], the preface of [132], and [135, pp. 48–65]. An annotated bibliography with historical comments can be found in [119]. For more recent developments, see [224].

Krylov subspace methods have a finite termination property, that is $\exists k \leq n$ (the dimension of the system) so that $r_k = 0$ and $x_k = x$. This is, in particular, true for Lanczos method and the conjugate gradient algorithm. In fact, this property played against these methods for quite a long time since, indeed, if n is very large, it is not suitable to be obliged to perform so many iterations to obtain the solution of the problem. However, around 1960, John K. Reid [208] observed that, in many cases, the conjugate gradient algorithm gives a very good approximation of x in a few iterations. This remark was the rebirth of the conjugate gradient algorithm. In his 1952 paper, Lanczos treated the case of any matrix while the conjugate gradient algorithm was restricted to symmetric positive definite ones. This algorithm was extended to the general case and put into an algorithmic form by Roger Fletcher in 1974 [95]. It is the *biconjugate gradient* algorithm (BiCG) which allows to implement Lanczos method. Other algorithms for this implementation exist. They can be derived by purely linear algebra techniques or obtained from the theory of *formal orthogonal polynomials* as reviewed in [38]. Then, the BiCG gave rise to a vast literature

and was widely used. It can suffer from drawbacks as the lost of orthogonality (due to the finite precision of the computations), division by zero (breakdowns) or by a number close to zero (near–breakdown). Such problems could be tackled by various techniques, some of them based again on formal orthogonal polynomials; see, for example, [37].

More recently, transpose–free variants of Lanczos methods were defined: the CGS in 1989 by Peter Sonneveld [237] and the BiCGSTAB in 1992 by Henk Van der Vorst [257]. The GMRES obtained by Yousef Saad and Martin H. Schultz in 1986 [223] is also a Krylov subspace method. Together with BiCGSTAB, it is the most used method for solving linear systems.

Let us mention that a topic which has been much developed these last decades is that of orthogonal polynomials. It has important applications in interpolation, best approximation, Padé approximation, quadrature and cubature, wavelets, linear algebra, etc. A review on their applications and the numerical problems posed by their computation in given in [110]. For cubature and quadrature, see [57, 172].

1953: Interior point methods

In mathematical programming interior–point methods had a huge impact on theory, practice and computation since the development of the simplex method. Various interior-point methods have been considered in the 1950s and were investigated extensively in the 1960s. It was however a paper by N. Karmarkar in 1984 that made them of interest for many researchers [153]. Working with the primal problem in linear programming was easier to implement or had better complexity bounds. A new area was inaugurated by N. Megiddo in 1987 [182] giving a framework for primal–dual algorithms. It allowed extensions to convex programming and linear complementarity. Also new algorithms with interesting properties could be obtained, e.g. a predictor–corrector algorithm by S. Mehrotra in 1989 [183]. More information on interior–point methods can be found in the survey paper by Florian A. Potra and S.J. Wright in [275].

1955: Romberg method

It is well known that, for a sufficiently smooth function f, the asymptotic expansion of the error of the trapezoidal rule for computing an approximate value of

$$I = \int_a^b f(x)\,dx$$

is given by the Euler–Maclaurin formula. In 1955, Werner Romberg gives a procedure for eliminating the successive terms in this asymptotic expansion and, thus, improving the accuracy of the trapezoidal rule [216]. It is based on the extrapolation process due to Lewis Fry Richardson (1881–1953) [213, 214], itself coming out from the Neville–Aitken scheme (1932) for computing the interpolation polynomial. For more details about the history of this process, see [84] and [35, pp. 451–453].

This method can now be found in every textbook on numerical analysis without any reference to Romberg's original paper, a proof of true fame! Besides its usefulness, Romberg method also showed to non specialists that convergence acceleration methods can be quite powerful. So, it had

an important impact on the development of this domain of numerical analysis. The E–algorithm, which is the most general convergence acceleration procedure presently known, is an extension of Romberg method; see [35]. A French translation of Romberg's original paper can be found in [46, pp. 501–503] and an English one in [47, pp. 451–453].

1956: The finite element method

It can be quite difficult to assign a year to the birth of an idea. This is particularly true for the *finite element method* for solving partial differential equations (PDEs). Until the mid fifties and beginning of sixties, PDEs were discretized by finite differences on rectangular grids. When the domain is not a rectangle some problems for following its shape arise and it will be easier to use triangles instead of rectangles. Finite elements are based on triangles. However, the finite element method, which uses triangles, is more than that and contains several ingredients. It is based on a triangularization of the domain, on a basis of functions with finite support, on the assemblage of the triangles together, and on a variational formulation of the problem in the spirit of the Rayleigh–Ritz method. The paper by M.J. Turner et al. [255] of 1956 is usually considered as the beginning of finite elements. It deals with a local approximation and uses an assembly strategy. However, the results were derived without the use of a variational formulation. Finite elements really start with the paper by Ray William Clough, Jr. (born in 1920) in 1960 [51] on linear plane elasticity problems. From 1960 to 1965, a variety of finite elements were proposed and studied, mostly in the engineering community. Spline functions, which form an essential part of the finite element method, were developed around the same time. The mathematical theory was slower to emerge from this intense activity. The first papers in this direction are those of Richard S. Varga [258] and Garrett Birkhoff (1911–1996) et al. [21] in 1966; see [4, pp.1–15]. Two dimensional problems began to be studied in 1968 and the paper by M. Zlamal [284] attracted the attention of many mathematicians who started working on the topic. One could mention, in particular, a convergence result obtained by Philippe G. Ciarlet in 1968 [49]. By 1972, the finite element method had become a major area in numerical analysis and applied mathematics, see [50].

The history of finite elements is given in [198, 199, 283]. See also [243].

1957: The fortran optimizing compiler

The first high level programming language was FORTRAN (FORmula TRANslation) which was developed between 1953 and 1956 by John Backus and his team. The first compiler was written in 1957 for the IBM 704 computer and it contained 25000 instructions. The immediate success of FORTRAN was due to the fact that it was very close to mathematics and it certainly helped the popularization of numerical analysis methods in the natural sciences community. Nowadays, computer scientists find that, although several versions with many improvements followed, it remains a language with many weaknesses. However, there are so many FORTRAN codes everywhere in the world that nobody will even try to rewrite them in a better language, and it will certainly still remain in use for a long time.

On Backus, see [234, pp. 5–20].

1957: Error analysis

When making numerical computations, there is an important problem which was soon realized by pioneers. It concerns the finite precision of floating–point computations. In a computer, each arithmetical operation is only performed with a finite number of digits (as by hand where, for example, only a certain number of 3 are kept when dividing 1 by 3). These *rounding errors* can accumulate in a catastrophic way and the result obtained could have nothing to do with the true one. In that case, the algorithm suffers from *numerical instability*. Upper bounds on these errors can be obtained. Such inequalities were first derived by James Hardy Wilkinson (1919–1986) [270] (see [200]). It is also possible to study these errors by a *backward error analysis*, an approach which consists in finding which perturbation on the data would have given the same results in exact arithmetic. This approach was implicitly contained in the papers of von Neumann and Herman H. Goldstine in 1947 [263] and Turing [252] in 1948. It was explicit in a report by J. Wallace Givens, Jr. (1910–1993) in 1954 [113], but it was mainly developed by Wilkinson. His book on the subject [271] was a classic for many years (see [137] for a more recent reference).

However, this approach is a pessimistic one since, hopefully, rounding errors can compensate. A more recent approach, due to Michel La Porte and Jean Vignes [167], is a statistical one. It is based on stochastic arithmetic and it provides a more realistic knowledge of rounding errors. The software CADNA allows to implement this approach [48]. Of course, in practice, it is not enough to evaluate rounding errors and it is better to control or even to try to avoid them. For this purpose, several ways have been explored: correction of the arithmetic [202], p–adic arithmetic [121, 122], multiprecision arithmetic, and interval arithmetic with guaranteed and sharp inclusion regions for the exact result [161].

There is another notion related to the finite precision of the computer: it is the concept of *condition number* which measures the sensitivity of the exact result of a mathematical problem to variations in the data. This concept should not be confused with the numerical stability of an algorithm for solving the problem. Although, this concept was clear for a long time, the term was first used by Turing [252].

1959: The QR factorization

Triangularization of a matrix by unitary similarity transformations seems to have been used for the first time in the book by Herbert Westren Turnbull (1885–1961) and Aitken in 1932 [254] in the proof of a result by Issai Schur (1874–1941) [232]. These transformations also appear in the book of William Feller (1906–1970) and George Elmer Forsythe (1917–1972) [92] in 1951. However, the systematic use of orthogonal transformations to reduce matrices to a simpler form was initiated by Givens in 1958 [114]. Reflectors were first used as an efficient tool for generating orthogonal matrices by Alston Scott Householder (1904–1993) in 1958 [141, 142].

In 1961–1962, J.F.G. Francis [103] and Vera N. Kublanovskaya [158] independently turned Householder's factorization into a method for computing the eigenvalues of a matrix. It is the QR–algorithm which is in the style of the LR–algorithm obtained in 1955 by Heinz Rutishauser (1918–1970) [222]. See [120] for historical comments.

Whatever the form of an eigenvalue problem, the QR algorithm is likely to be useful. For

example, for generalized eigenvalue problems $Ax = \lambda Bx$, the QZ algorithm [189] is a variant of QR. Another variant is used to calculate the singular value decomposition (SVD) of a matrix (see below) and QR can also be helpful for large eigenvalue problems . The application to linear least squares problem is due to Gene Howard Golub in 1965 [117]. An extensive treatment is given in [24]. The use of QR–methods to solve least squares problems is mentioned in [197]. QR–based codes are used in the software packages EISPACK and LINPACK. A parallel version of QR can be found in ScaLAPACK.

For technical and historical details, see [239] and the paper of D.S. Watkins [265].

1960: Multigrid methods

The idea of multigrid was introduced and analyzed by H. Brakhage [29] and R.P. Fedorenko [91], but only got more attention in the 1970's. The central principle is based on the interplay of smoothing and coarse grid correction. These techniques and be applied to a variety of partial differential equations from Poisson equations to Navier–Stokes equations, from two–dimensional square domains to three–dimensional unstructured airfoil grids. Several forms of multigrid methods appear in the literature. We refer to the paper by K. Stueben [241] in this volume for a review of algebraic multigrid. Multigrid methods has proved itself as a powerful tool for fast and efficient computations.

1961: Fractals and chaos

Until the middle of the 19th century, mathematicians were thinking that a continuous function had a derivative at almost all points. In 1872, Karl Theodor Wilhelm Weierstrass (1815–1897) defined a continuous function having no derivative at any point. In 1883, Georg Ferdinand Ludwig Philipp Cantor (1845–1918) gave the example of a segment of length one to which an infinite number of segments of total length one was subtracted and so that a non denumerable set of points remained. The mathematicians of this time were forced to admit these examples but thought that they showed the limits of classical analysis and they considered them, as Jules Henri Poincaré (1854–1912) said, only as monsters for the museum of horrors. Let us remind that, when correcting an error in the work for which he obtained the Prize offered by Oskar II, the King of Sweden and Norway, in 1889, Poincaré discovered the chaotic behavior of the solution of some differential equations when the initial conditions are changed.

A curve has dimension 1 since one parameter is sufficient to locate each of its points (the distance from a point of the curve chosen as the origin). A surface has dimension 2 since two parameters are needed for defining each of its points, and so on. In 1890, Giuseppe Peano (1858–1932) constructed a curve (of dimension 1) passing through every point in a square (of dimension 2). So, the definition of the dimension from the number of parameters required was caduc. In 1906, Niels Fabian Helge von Koch (1870–1924) gave a closed curve (known as the *snowflake*) with infinite length enclosing a finite area. So, the question arises to assign a dimension to such objects. If you have a rule, you need two of them to obtain a rule with a double length. If you have a square, you need four of them to have a square with a side of double length, and so on. More generally, for multiplying by a the side of an object of dimension d, you need $c = a^d$ similar

objects where d is the dimension. Thus $d = \log c / \log a$. This number d is called the *fractal dimension* and, if it is not an integer, the object is called a *fractal*. Applying this definition to the snowflake leads to $d = 1.2618\ldots$

Lewis Fry Richardson (the same responsible for Richardson extrapolation method) remarked that if you measure the length L of a coast, or that of a boarder between two countries, the result depends on the length a of the instrument chosen for the measurement. Indeed, with a smaller rule you can measure smaller details. He found by experiments that $L(a) = a^{1-d}$ with $d = 1.25$ for the coast of Britany and $d = 1.15$ for the boarder between Portugal and Spain. Applying the same technique to the snowflake gives again $d = 1.2618\ldots$

The notion of fractal dimension was the subject of quite intensive researches when physicists found that many objects, such as clouds, have a fractal dimension. In fact, fractals are everywhere in nature [14].

Another problem is the dynamics of the fixed point iterations $x_{k+1} = F(x_k)$, $k = 0, 1, \ldots$ with x_0 given. Such iterations model the evolution of a system from the time t to the time $t + dt$. So, to predict its evolution over a long time, many iterations are needed. In practice, the law F is only known approximately and only stable phenomena (that is those not very sensitive to small changes in F) are of interest. These iterations can exhibit a quite chaotic behavior. Let us consider the simple iterations $x_{k+1} = x_k^2 + c = F_c(x_k)$ in the complex plane. F_c has two fixed points $(1 \pm \sqrt{1 - 4c})/2$. If $c \neq 1/4$, one of them is attractive and the other one repulsive. According to the choice of x_0, the sequence (x_n) can converge or not. Let $\mathcal{A}_c(\infty)$ be the set of $x_0 \in \mathbb{C}$ such that $|x_k|$ tends to infinity with k and let J_c be the frontier of $\mathcal{A}_c(\infty)$. J_c is called the *Julia set* of F_c. Obviously, J_c depends on the value of the complex number c. The first researches about such topics go back to the beginning of the twentieth century with the works of Gaston Maurice Julia (1893–1978) in 1918 [149], Pierre Joseph Louis Fatou (1878–1929) in 1919–1920 [90], and others. However, these pioneers were not able to see the full complexity (and beauty) of the problem which was only realized with the first numerical computations; see [201]. In fact, to the surprise of the first experimenters, J_c can change very much with c and it can be very complicated, in fact it can be a fractal. Let $\mathcal{K}_c$ be the complement of $\mathcal{A}_c(\infty)$ in the complex plane. It was proved that either $\mathcal{K}_c$ is a connected set, or it consists of an infinite number of distinct points. It is a connected set if and only if it contains the origin. The *Mandelbrot set* $\mathcal{M}$ of F_c is the set of values of c so that $\mathcal{K}_c$ is connected. The set $\mathcal{M}$ can be a fractal. Many of these notions were introduced and studied, from 1961, by Benoît Mandelbrot who wrote many books on the subject (see, for example, [179]).

On chaos and fractals, see [115]. An interview of Mandelbrot can be found in [4, pp. 206–225].

1963: A–stability

The first paper on stiff differential equations is due to C.F. Curtiss and J.O. Hirschfelder in 1952 [60]. The concept of A–stability, a very fundamental one for the numerical integration of stiff ordinary differential equations, was introduced by Germund Dahlquist around 1963 [62]. He showed that the order of an A–stable linear explicit multistep method cannot exceed two. It was also proved that explicit Runge–Kutta methods cannot be A–stable. In fact, as explained in [63], the whole story began in 1951 when Dahlquist and Rutishauser met at a GAMM meeting at

Freiburg im Breisgau and discovered they found quite similar results on the numerical (in)stability of the leapfrog method [61, 221].

Two ways of research emerge from these results: the search for requirements less limitative than A–stability, and the study of implicit methods; see [111]. The concept of A–stability is a major one. It has generated many researches and an enormous literature on the subject is available; see, for example, [127, 144].

1964: Symbolic computation

Symbolic computing, also known as *computer algebra*, consists in the manipulation of mathematical formulae as done by hand with a pencil and a paper. Although the roots of computer algebra go back to the 1950's, the first system, named, ALPAK, was developed in 1964 at Bell Laboratories. It was followed by ALTRAN in 1968, REDUCE in 1968, and MACSYMA in 1970, which was the most influential of these packages. MAPLE was conceived in November 1980 and soon became very popular. Nowadays, symbolic computing is often coupled with numerical algorithms for a better efficiency. Computer algebra systems are based on deep mathematics such as Gröbner bases, introduced in 1965 by Bruno Buchberger [40] and integration in finite terms, due to Joseph Liouville (1809–1882) [177, pp. 351–422].

1965: The fast Fourier transform

Let $f_n = f(nT/N)$ be the values of a given function f at equidistant points. The discrete Fourier transform of f is the function $\widehat{f}$ whose values at the points $\omega_k = k/T$ are given by $\widehat{f}(\omega_k) = TF_k$ with

$$F_k = \frac{1}{N} \sum_{i=0}^{N-1} f_i \omega_N^{-ik}$$

with $\omega_N = e^{i2\pi/N}$.

The computation of $F_0, \ldots, F_{N-1}$ by these formulae requires $(N-1)^2$ complex multiplications and $N(N-1)$ complex additions if the quantities ω_N^j are known. This is too much for a real time application.

It was remarked by John Wilder Tukey (1915–2000) in 1963 that if N has the form $N = n_1 n_2$, then the series can be expressed as an n_1–term series, each term being itself a subseries with n_2 terms. This trick, which is the basic idea for the so–called *fast Fourier transform* (FFT), reduces the number of operations from N^2 to $N(n_1 + n_2)$. Moreover, in some cases, the process can be iterated and the number of operations still reduced to $N \log N$. Indeed, if $n_1 = 2$ and if we set $n_2 = m$, we have

$$F_k = \frac{1}{2}(F_k^e + \omega_N^{-k} F_k^o)$$

with the even and odd parts respectively given by

$$F_k^e = \frac{1}{m} \sum_{i=0}^{m-1} f_{2i} \omega_{N/2}^{-ik}$$

$$F_k^o = \frac{1}{m}\sum_{i=0}^{m-1} f_{2i+1}\omega_{N/2}^{-ik}.$$

So

$$F_{k+m} = \frac{1}{2}(F_k^e - \omega_N^{-k}F_k^o)$$

and the procedure can be repeated if m is even.

Soon after, James William Cooley wrote a three–dimensional FFT program. The main difficulty was to save storage by overwriting data using a clever indexing scheme. Because of the many applications of the technique in industry, the possibility of a patent was discussed but it was decided to put the FFT into the public domain. Cooley asked Tukey to be a coauthor of the paper which appeared in 1965 [56].

The idea of the FFT did not emerge from nothing. Early contributors are mentioned in [54] (see also the references quoted there) and, among them, Carl Friedrich Gauss (1777–1855) [109], H. König and C. Runge in 1924 [220], and G.C. Danielson and C. Lanczos in 1942 [65] (see [55]. In particular, Gauss discovered the FFT in 1805, at the age of 28, two years before Fourier's paper [129].

The FFT has very many applications in the physical sciences, in particular in signal processing, and many computations would not have been possible without it, even with the fastest present computers.

1965: The singular value decomposition

Let $A \in \mathbb{R}^{n\times m}$ be a rectangular matrix of rank r. There exist real numbers $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and two orthogonal matrices $V = [v_1, \ldots, v_m] \in \mathbb{R}^{m\times m}$ and $U = [u_1, \ldots, u_n] \in \mathbb{R}^{n\times n}$ such that

$$\begin{array}{llll} Av_i = \sigma_i u_i, & i = 1, \ldots, r & A^T u_i = \sigma_i v_i, & i = 1, \ldots, r \\ Av_i = 0, & i = r+1, \ldots, m & A^T u_i = 0, & i = r+1, \ldots, n. \end{array}$$

Thus, the vectors $v_1, \ldots, v_m$ are the eigenvectors of A^TA and $u_1, \ldots, u_n$ are those of AA^T corresponding to the nonzero eigenvalues $\sigma_1^2, \ldots, \sigma_r^2$ of A^TA and AA^T.

Let Σ be the $n \times m$ matrix

$$\Sigma = \begin{pmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{pmatrix}$$

with $\hat{\Sigma}$ =diag$(\sigma_1, \ldots, \sigma_r)$. Thus

$$A = U\Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T.$$

This formula is called the *singular value decomposition* (SVD) of A and the numbers σ_i are called its *singular values*. It is a much valuable tool in numerical linear algebra and its discovery goes back to Eugenio Beltrami (1835–1900) [15] in 1873 and Marie Ennemond Camille Jordan (1838–1922) [148] in 1874. See [238], for the complete early history.

From the numerical point of view, an algorithm for computing the singular value decomposition is due to Gene Howard Golub and William Kahan in 1965 [118]; it is based on the QR factorization.

1965: Quasi–Newton methods

Let us consider the system of nonlinear equations $f(x) = 0$ where $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$. Newton method consists of the iterations

$$x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k), \quad k = 0, 1, \ldots$$

where $f'(x_k)$ is the Jacobian matrix of f at the point x_k. So, Newton method requires the analytical expressions of n^2 partial derivatives, sometimes an impossible task. In quasi–Newton methods, $f'(x_k)$ (or its inverse) is replaced by an approximation and the iterations take one the forms

$$\begin{aligned} x_{k+1} &= x_k - H_k^{-1} f(x_k), \\ x_{k+1} &= x_k - C_k f(x_k). \end{aligned}$$

These methods were introduced by William Davidon [70] at the end of the sixties for the minimization of a convex nonlinear functional (thus for the solution of an unconstrained optimization problem). If the functional is quadratic, this algorithm, when applied with exact minimization, is equivalent to the conjugate gradient algorithm as showed by Roger Fletcher and Mike J.D. Powell [97]. It is known as the Davidon–Fletcher–Powell algorithm (DFP). Soon after, another algorithm was proposed by Fletcher and Colin M. Reeves [98]. These methods gained full recognition after the methods proposed by Charles G. Broyden in 1965 [39] and the condition obtained by John E. Dennis and Jorge J. Moré in 1974 for their superlinear convergence [77]. In these methods, the approximation of the Jacobian (or of its inverse) is updated at each iteration by a rank–one or a rank–two modification. See [42] for some historical account.

There exist many methods of this type. They are very much in favor in optimization; see, for example, [96]. Their historical development is analyzed in [278]. Truncated Newton methods are reviewed in [195].

1969: Fast matrix manipulations

Let us consider the product of two $n \times n$ matrices $A = BC$. The elements of A are given by the classical formula

$$a_{ij} = \sum_{k=1}^{n} b_{ik} c_{kj}, \quad i, j = 1, \ldots, n.$$

So, the computation of A requires n^3 multiplications and $n^2(n-1)$ additions. This is not optimal and another formula was proposed by Volker Strassen in 1969 [240]. It requires $(7/8)n^3$ multiplications and $(7/8)n^3 + (11/4)n^2$ additions. If $n = 2^k$, the algorithm can be used repeatedly and the total number of arithmetical operations comes down to $4.7n^{2.807}$. This procedure can be used in direct methods for the solution of systems of linear equations and the number of operations goes from $\mathcal{O}(n^3)$ to $\mathcal{O}(n^{2.807})$ or even less.

This branch of numerical analysis is called *complexity*. It is an important challenge to find algorithms for reducing the number of arithmetical operations (such as in the FFT), to demonstrate that they have a lowest bound that cannot be passed and to find the corresponding algorithm if it exists. On this topic, see [248].

1976: High–performance computing and software packages

As stated in the introduction of [80], the first commercially available high–performance computer was the CRAY 1, delivered at Los Alamos in 1976. It was a vector machine. The first commercially viable parallel computer was the ALLIANT in 1985 but more highly parallel machines only appeared on the market in 1988. These architectures had, and still have, a quite important impact on the search and development of numerical algorithms, in particular in numerical linear algebra.

The increasing availability of advanced–architecture computers has had a significant effect on scientific computing, including algorithm research and software development especially in numerical linear algebra. The EISPACK [235], and LINPACK [79] software libraries were designed for supercomputers used in the 1970's and early 1980's. MATLAB was elaborated as a teaching tool during the period 1977–1984 by Cleve B. Moler at the University of New Mexico. The development of LAPACK [8] in the late 1980's was intended to make the EISPACK and LINPACK libraries run on shared memory, vector supercomputers. In 1991 began the development of SCALAPACK [25] for distributed memory concurrent computers.

For more details on the interaction between algorithms and software, see [81] in this volume. See also [27].

1982: Wavelets

Fourier analysis has proved to be very useful in many branches of applied sciences. However, an expansion into a Fourier series is non local, which means that, for a given value of the variable, it needs the summation of an infinite number of terms. This drawback disappears if the basis functions have a compact support. However, it is difficult to construct such a basis. The drawback partly disappears if the basis functions are rapidly (that is exponentially) decreasing outside a compact interval. In that case, a small number of terms will be sufficient for obtaining $f(x)$ with a high precision. The aim of wavelets is to construct such basis. There is another drawback to which wavelets remedy: the Fourier transform hides information on time. Indeed, it gives the frequencies contained in the signal but does not say anything about the time when these frequencies are emitted. However, information on time is not lost but only hidden in the phases. Thus, a sum of sine and cosine functions can be amplified or cancelled out if they are simultaneous in time or not. Moreover, it is not possible to compute the phases from the Fourier coefficients with enough accuracy. To remedy such drawbacks, windows were introduced into Fourier analysis by Dennis Gabor (1900–1979) in 1946 [106] (Nobel prize winner in physics in 1971 for the discovery of holography). They consist in decomposing the signal into frequencies for various intervals. The size of the window does not change but it is successively filled up by oscillations with various frequencies. But, if the window is too narrow, only picks and discontinuities are localized and the analysis becomes blind to low frequencies whose period is too big for the window. On the other side, if the window is large, it is not possible to be precise in time when a discontinuity or a pick arises because information is too much diluted. Wavelets do not have these drawbacks and are satisfactory both in frequency and time. Instead of keeping the size of the window fixed and to vary the frequencies, the key idea consists of the converse: to keep the frequencies fixed and change the size of the window. Of course, this idea was in the air for some time and it was,

in fact, used independently in various branches of applied sciences. However, the starting can be attributed to Jean Morlet, a French geophysicist, in 1982. From then, the theory and applications of wavelets developed very rapidly and an enormous literature on the subject is available. Among the numerous references, one can mention [68].

For an account of the subject and its historical developments, see [43] and [145]. A review is given in [52]. Fourier series and wavelets are presented, together with their history, in [150].

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