Calculus III Notes

Contents

Fun stuff

Chapter 12 Vectors and the geometry of space

.1 12.1 Three-dimensional coordinate systems

1. Rectangular (Cartesian) coordinate system
   (a) 2D:
   - Basics: xy-plane, orthogonal axis with standard orientation, 4 quadrants, coordinates of point (x, y), projection onto axis, notation $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y)|x, y \in \mathbb{R}\}$.
   - Distance between two points, Pythagoras, distance formula.

   (b) 3D:
   - xyz-space, orthogonal axis with standard orientation, 8 octants, coordinates of point (x, y, z), projection onto xy-plane (xz, yz). Projection onto axis, notation $\mathbb{R}^3 = \{(x, y, z)|x, y, z \in \mathbb{R}\}$.
   - Distance between two points, Pythagoras twice, distance formula, proof in text.
2. Graphs of equations
   (a) 2D:
   - Point
   - Lines: Vertical, horizontal, sloped
   - Quadratics, polynomials
   - Circles from distance formula, standard form $(x-1)^2 + (y-2)^2 = 4$, complete the square if not in standard form.
   - Lots more
   - Try on own: Regions via inequalities $y < x$, $x^2 + y^2 > 9$, $x/y < 1$, $xy \geq 0$.

   (b) 3D:
   - Point
   - Planes: Vertical $(x = 2)$, horizontal $(z = 1)$, out at us $(y = 0)$.
   - Try on own: $x^2 + y^2 = 1$, $x + y = 1$, $z = x^2$, $x < y$.

- Spheres from the distance formula, standard form, complete the square if not in standard form.
- Showcase Geogebra.

3. Homework: 7, 9, 11, 13, 15, 17, 21, 23, 25-37 odd, 45

.2 12.2 Vectors

1. Vector basics: $\mathbb{R}^2$, then $\mathbb{R}^3$.
   (a) Coordinate (location) vs vector (action such as displacement).
   (b) Vector has 2 attributes, magnitude (size) and direction (angle).
   (c) Location doesn't matter, standard position for comparison.
   (d) Vector components.
   $$\vec{a} = \langle a_1, a_2 \rangle = \langle x, y \rangle$$
   (e) Vector from two points $\vec{AB}$. General formula.
   (f) Magnitude and direction. Need to adjust direction by 180° with arctangent formula for quadrants 2 and 3.
   $$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}, \quad \theta = \arctan(y/x)$$
2. Vector operations: Geometry and algebra, $\mathbb{R}^2$ then $\mathbb{R}^3$
   (a) Addition: Parallelogram law, sum of components.
   (b) Scalar multiplication: Stretch / reverse, scale components.
   (c) Subtraction: Triangular law, subtract components, rewrite as
   $$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$
   (d) Bonus: Dot product
3. Theorem: Vector properties, all proven component-wise via properties of real number arithmetic, geometric intuition.
   (a) Commutative: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
   (b) Associative: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
   (c) Addition identity: $\vec{a} + \vec{0} = \vec{a}$
   (d) Addition inverse: $\vec{a} + (-\vec{a}) = 0$
   (e) Scalar distribution: $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
   (f) Vector distribution: $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
   (g) Scalar associative: $(cd)\vec{a} = c(d\vec{a})$
   (h) Scalar multiplication identity: $1 \cdot \vec{a} = \vec{a}$
4. Unit vectors and standard basis
   (a) $\mathbb{R}^2$: $\langle 1, 0 \rangle, \quad \langle 0, 1 \rangle$, divide by length to make unit.
   $$\vec{a} = \langle a_1, a_2 \rangle = a_1\langle 1, 0 \rangle + a_2\langle 0, 1 \rangle, \quad \pm\frac{1}{\|\vec{a}\|}\vec{a}$$
   (b) $\mathbb{R}^3$: $\vec{i}, \vec{j}, \vec{k}$
   $$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

(c) $\mathbb{R}^2$: Vector in terms of angle and magnitude.

$$\vec{a} = \|\vec{a}\|\langle\cos(\theta), \sin(\theta)\|$$

5. Application: Wire tension. Hang from a wire, wonder if will break. Know angles from ceiling. How much tension on each wire?

6. Homework: 3, 5, 7, 11, 13, 15, 17, 19, 21, 25, 29, 31, 35, 39, 45, 47

.3 12.3 The dot product

1. Basics of the dot product:

   (a) Definition: $\mathbb{R}^2$: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$, $\mathbb{R}^3$: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$,

   (b) Examples. Note result is a scalar, not a vector.

   (c) Theorem: Properties of the dot product.

   - $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
   - $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
   - $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c})$
   - $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}$
   - $\vec{a} \cdot \vec{0} = 1$
   - All are easily shown via the def of dot product. Show first two quick.

2. Meaning of the dot product.

   (a) Theorem: For θ the smallest angle between $\vec{a}$ and $\vec{b}$.

   $$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos(\theta)$$

   (b) Proof: Law of cosines (generalized Pythagoras, after peek at proofs of LoC) and dot product properties.

   (c) Why useful? Corollary:

   $$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} \quad \rightarrow \quad \theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right) \in [0, \pi]$$

   (d) Corollary: $\vec{a} \perp \vec{b}$ if and only if $\vec{a} \cdot \vec{b} = 0$.

   (e) Example: Find angle between vectors. Show vectors perpendicular. $\vec{0}$ is perpendicular to all vectors. Acute and obtuse cases.

3. Use of dot product, vector orientation.

   (a) Direction angles and direction cosines.

   (b) $\mathbb{R}^3$: Let α be the angle between $\vec{a}$ and $\vec{i}$. Likewise for angles β, γ and $\vec{j}$ and $\vec{k}$.

   (c) $\cos(\alpha) = \frac{\vec{a} \cdot \vec{i}}{\|\vec{a}\|\|\vec{i}\|} = \frac{a_1}{\|\vec{a}\|}$. Likewise for $\cos(\beta)$, $\cos(\gamma)$.

   (d) Theorem:

   $$\frac{1}{\|\vec{a}\|}\vec{a} = \langle\cos(\alpha), \cos(\beta), \cos(\gamma)\rangle$$

   (e) Example: Find the direction angles of $\vec{a} = \langle 1, 2, 3\rangle$.

4. Use of dot product 2, vector projection.

(a) Definitions:

i. Scalar projection of $\vec{b}$ onto $\vec{a}$: $\text{comp}_{\vec{a}}(\vec{b})$

ii. Vector projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}(\vec{b})$

(b) Find each using cosine of the angle between and dot product connection to $\cos(\theta)$.

(c) Theorem:

$$\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}, \qquad \text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}\right)\frac{1}{\|\vec{a}\|}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$$

(d) Can see the projection is parallel to $\vec{a}$.

(e) Examples

5. Physics application, projection as a way to calculate work.

6. Dot product, cosine similarity, recommender systems. Coding demo.

7. Homework: 1, 3, 7, 9, 13, 15, 19, 23, 27, 29, 33, 39, 43, 45, 47, 61

.4 12.4 The cross product

1. Basics of the cross product:

(a) Given two non-parallel vectors, find a third non-zero vector which is orthogonal to both. Will use this idea to define planes / tangent planes later on.

(b) Given $\vec{a}, \vec{b}$ not parallel, want $\vec{c}$ such that

$$\vec{a}\cdot\vec{c} = a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad \text{and} \quad \vec{b}\cdot\vec{c} = b_1c_1 + b_2c_2 + b_3c_3 = 0.$$

Eliminate c_3 by multiplying two equations and subtracting to get

$$a_1b_3c_1 + a_2b_3c_2 - a_3b_1c_1 - a_3b_2c_2 = 0$$

which gives

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0.$$

Choose $c_1 = (a_2b_3 - a_3b_2)$ and $c_2 = (a_1b_3 - a_3b_1)$ which yields $c_3 = (a_1b_2 - a_2b_1)$.

(c) Definition: The cross product of $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1\rangle.$$

Note, result is a vector where the dot product gives a scalar.

(d) Theorem: $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$. Proof just computes $(\vec{a}\times\vec{b})\cdot\vec{a}$. Same for $\vec{b}$.

(e) Determinant notation:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

(f) Example: Find the cross product of two random vectors. Check that worked. What if vectors parallel? One zero?

(g) Orientation of $\vec{a}\times\vec{b}$ and the right hand rule.

2. Information hidden in the cross product.

(a) Theorem: $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin(\theta)$. See proof in text. Easy except for first part. Surprising at first, but can see just comes from the dot product result.
(b) Corollary: Two nonzero vectors are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.
(c) Corollary: $\|\vec{a} \times \vec{b}\|$ gives the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$. Draw parallelogram. Base times height.
(d) Find the area of the triangle in $\mathbb{R}^3$ formed by three random points.

3. Properties of the cross product.

   (a) Consider combinations of cross product of unit basis $\vec{i}, \vec{j}, \vec{k}$. Note in general $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ because of right hand rule. Also since orthogonal basis, $\sin(\pi/2) = 1$ and can see the result is unit. Parallelogram is a square.
   (b) Theorem: Properties of the cross product.
   - $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
   - $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
   - $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
   - $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
   - $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
   - $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
   - All are proven via the component-wise definition of the cross product.

4. Triple product, volume of parallelpiped.

   (a) 3×3 determinant.
   $$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
   (b) Theorem: The volume of the parallel-piped formed by $\vec{a}, \vec{b}, \vec{c}$ is
   $$\|\vec{a} \cdot (\vec{b} \times \vec{c})\| = \|\vec{a}\|\|\vec{b} \times \vec{c})\| \cos(\theta)$$
   where $\|\vec{b} \times \vec{c})\|$ is the area of the base and $\|\vec{a}\| \cos(\theta)$ is the height. This comes from our dot product formula.
   (c) Corollary: $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if the triple product is zero.
   (d) Newton used this to derive Kepler's law of planetary motion.

5. Torque definition and magnitude.

6. Homework: 1, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 31, 33, 37, 49, 53

.5 12.5 Equations of lines and planes

1. Equations of lines: Vector, parametric, symmetric.

   (a) $\mathbb{R}^2$
   - Familiar case: $y = mx + b$, Ex $y = 2x + 1$, graph it.
   - Two step process: Get to the line via $\vec{r}_0$, traverse the line via $\vec{v}$ which is parallel to the line.
   - Ex: $\vec{r}_0 = \langle 0, 1 \rangle$, $\vec{v} = \langle 1, 2 \rangle$, then
     $$\vec{r} = \vec{r}_0 + t\vec{v} = \langle t, 1 + 2t \rangle.$$
     Since $t = x$, we have $y = 2x + 1$ again.

- Parameter t moves us along the line in a direction as t increases.
- Vector form is not unique. $\vec{r} = \vec{r_0} - t\vec{v}$ would give the same line, just traced backwards.

(b) $\mathbb{R}^3$

- Vector equation: For $\vec{v}$ parallel to the line and $\vec{r_0}$ the vector from the origin to any point on the line,
$$\vec{r} = \vec{r_0} + t\vec{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$
- Draw picture.
- Parametric equations of a line: For parameter t,
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$
PEs are not unique though they may draw the same line.
- Symmetric equations of a line: Solve for parameter t.
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$
It is possible that a, b, c could be zero.

(c) Example: Find the vector, parametric, and symmetric equations for the line thru two random points. Where does it intersect the xy-plane? xz? yz?

(d) 3 possibilities for lines meeting now: parallel, intersecting, or skew (not parallel, not intersecting).
- 3 lines, decide if pairs are parallel, intersecting, or skew. Graph in Geogebra.

(e) Line segment from point (x_0, y_0, z_0) to (x_1, y_1, z_1):
$$\vec{r} = (1-t)\vec{r_0} + t\vec{r_1}, \quad \vec{r_0} = \langle x_0, y_0, z_0 \rangle, \quad \vec{r_1} = \langle x_1, y_1, z_1 \rangle, \quad 0 \le t \le 1.$$

2. Equations of planes: Vector, scalar, linear

(a) Harder to define the direction of a plane. Normal (perpendicular) vector does the trick.

(b) Vector equation of plane: For (x_0, y_0, z_0) a fixed point on the plane, any point (x, y, z) on the plane, and $\vec{n} = \langle a, b, c \rangle$ a normal vector to the plane, we have that
$$\vec{n} \cdot (\vec{p} - \vec{p_0}) = 0$$
where $\vec{p_0} = (x_0, y_0, z_0)$ and $\vec{p} = (x, y, z)$. Draw picture to illustrate.

(c) Scalar equation of plane: Compute $\vec{n} \cdot (\vec{p} - \vec{p_0}) = 0$.
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(d) Linear equation of plane: Combine constant terms of $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
$$ax + by + cz + d = 0$$

(e) Example: Find the plane thru three random points. Uniquely possible if points are not colinear. Already have point, use cross product to get normal vector. Give all 3 forms. Plot the plane by computing the axis intercepts. Check with Geogebra.

3. Summary: In $\mathbb{R}^3$,

(a) You need a point and a direction (parallel vector) to define a line.

(b) You need a point and a normal vector to define a plane.

(c) Examples: Group challenge.
- Problems in text: 35, 37, 45, 51.

4. Homework: 1, 3, 5, 7, 11, 13, 15, 17, 19, 23, 29, 31, 35, 37, 39, 41, 45, 49, 51, 53, 55, 59, 63, 65

.6 12.6 Cylinders and quadratic surfaces

1. Summary: Goal is to develop intuition for $\mathbb{R}^3$.
   (a) We already considered two classes of surfaces in $\mathbb{R}^3$: Spheres and planes.
   $$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2, \quad ax + by + cz + d = 0$$
   (b) New surfaces for this section:
   - Cylinders: Surfaces consisting of all lines (called *rulings*) parallel to a given line and passing thru a planar curve.
   - Example: $z = x^2$ is a parabolic cylinder. Parabolas are called vertical *traces*.
   - Terminology: A *trace* is a curve of intersection of the surface with planes parallel to the coordinate planes (xy, xz, yz).
   - Quadratic surface: Any surface generated by the general equation
   $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$
   - Example: $x^2 + y^2 + z^2 = 1$ is a sphere.
   - Quadratic surfaces:
2. Cylinders: Sketch the graph. What are the traces? What are the rulings?
   (a) Example: $x^2 + z^2 = 4$
   (b) Example: $y = z^2$
3. Quadratic surfaces: Sketch the traces, then the graph.
   (a) Cone: $z^2 = x^2 + y^2$.
   (b) Elliptic paraboloid: $z = x^2 + y^2$
   (c) Hyperbolic paraboloid: $z = x^2 - y^2$
   (d) Recall the formula for an ellipse of width $2a$ and height $2b$ centered at the origin. Circle is a special case.
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
   (e) Show text table of 6 classes, won't test the hyper-stuff. Just basics.
4. Homework: 1,3,5,7,11,17,21,23,25,27

Chapter 13 Vector functions

.1 13.1 Vector functions and space curves

1. Finally we do calculus, basic case first: $\vec{r}(t)$ is a vector-valued function. For input t, result is a vector.
   (a) Need a function per component.
   $$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
   (b) Example: Already know lines. Label direction.
   $$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$
   Knowing two points allows to draw the line. Show example.

(c) Example: Corkscrew. Label direction.

$$\vec{r}(t) = \langle \cos(t), sin(t), t \rangle$$

Helps to graph the projection onto the xy, xz, yz planes.

(d) Example: Try on own.

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

(e) Matching: Text problems 21-26.

(f) Find a vector function to describe the curve of intersection of cylinder $x^2 + y^2 = 4$ and surface $z = xy$.

2. Limits and continuity: Everything component-wise.

(a) If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle.$$

(b) $\vec{r}(t)$ is continuous at $t = a$ if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

(c) Example from previous.

3. Homework: 1, 3, 5, 7, 9, 11, 15, 17, 27, 29, 31, 43, 49

.2 13.2 Derivatives and integrals of vector functions

1. Derivatives of vector functions:

(a) Definition: For $\vec{r}(t)$ any vector function, define the derivative as

$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Draw picture in $\mathbb{R}^3$ for some t. Result is a tangent vector at t. The unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

Also have a tangent line using this vector and point.

(b) Theorem: In $\mathbb{R}^3$ for vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, we have

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Proof moves the difference quotient inside the vector function component-wise.

(c) Examples:

- $\mathbb{R}^2$ case, tangent vector to $\vec{r}(t) = \langle t - 2, t^2 + 1 \rangle$ when $t = 2$. Draw picture. Tangent line also.
- Tangent vector for any line $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ is $\vec{v}$ and the tangent line is the same line. This is the velocity vector for the line as we will see in the next section. Constant change with constant velocity.
- Find the tangent vector to $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at $t = 2$. Tangent line also. Geogebra.

2. Vector function differentiation rules.

(a) Theorem: For $\vec{u}(t)$ and $\vec{v}(t)$ differentiable vector functions,

- $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}'(t) + \vec{u}'(t)$

- $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$
- $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}'(f(t))f'(t)$

(b) Proof of dot product version, component-wise calculation.

3. Homework: 1, 3, 7, 13, 17, 21, 25, 43, 45, 47

.3 13.3 Arc length and curvature

SKIP

.4 13.4 Motion in space: Velocity and acceleration

1. Finally, velocity and speed.

   (a) Definition: The velocity vector function $\vec{v}(t)$ of position of particle curve $\vec{r}(t)$ is given by

   $$\vec{v}(t) = \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

   Draw picture in $\mathbb{R}^3$. This give speed and direction.

   (b) Definition: The speed of particle at position $\vec{r}(t)$ is

   $$\|\vec{v}(t)\| = \|\vec{r}'(t)\| = \frac{ds}{dt}/$$

   (c) Definition: The acceleration is

   $$\vec{a}(t) = v'(t).$$

   (d) Example: A parameterization of $y = x^2$ is given by $\vec{r}(t) = \langle 2t^2, 4t^4 \rangle$. Plot the velocity and acceleration vectors for $t = 1$. Find the speed. Note the direction of the velocity vector is parallel to the old fashion tangent line.

2. Homework: 1, 3, 7, 9, 11, 15, 19,

3. Chapter review problems:

   (a) Concept check: 1-4, 8

   (b) T/F: 1-6, 11, 14

   (c) Exercises: 1-4, 9, 16-19

Chapter 14 Partial derivatives

Here we return to calculus ideas to extend old idea (functions of one variable $y = f(x)$) to 3 dimensional space (functions of two variables $z = f(x, y)$).

- 2 dimensions: Get IROC for $f(x)$ as $\frac{df}{dx}$ via AROC as $\frac{\Delta f}{\Delta x}$. Graphs of $y = f(x)$ have tangent lines. Key is idea of limit.

- 3 dimensions: Functions like $f(x,y) = x^2 + y^2$ (and even $f(x,y,z)$) should also have rates of change. Surface analogy. Key will still be limit.

Summary of chapter in 6 lines: Curve $y = f(x)$ vs surface $z = f(x,y)$.

- $\frac{df}{dx}$ becomes two first order derivatives $\frac{df}{dx}$ and $\frac{df}{dy}$
- $\frac{d^2f}{dx^2}$ becomes four second order derivatives x^2, xy, yx, y^2
- Linear approximation $\Delta f \approx \frac{df}{dx}\Delta x$ becomes $\Delta f \approx \frac{df}{dx}\Delta x + \frac{df}{dy}\Delta y$
- Tangent line $y - y_0 = \frac{df}{dx}(x - x_0)$ becomes a tangent plane $z - z_0 = \frac{df}{dx}(x - x_0) + \frac{df}{dy}(y - y_0)$.
- Chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ becomes $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}dydt$.
- Max/min problem $\frac{df}{dx}$ becomes the pair $\frac{df}{dx}, \frac{df}{dy}$.

.1 14.1 Functions of several variables

1. Functions in $\mathbb{R}^2$
   (a) $y = f(x)$ is a curve in the xy-plane.
   (b) x is the indep variable, y is the dependent variable.
   (c) Set of all x which f makes sense gives the domain, all obtainable y gives the range. Both are intervals.
   (d) Example: $f(x) = \sqrt{x}$.
2. Functions in $\mathbb{R}^3$
   (a) $z = f(x,y)$ is a surface in $\mathbb{R}^3$. xy are independent and z is dependent. The domain is now a 2 dimensional region, and the range is still an interval. Simple extension, though all these ideas become harder.
   (b) Example: $z = f(x,y) = \sqrt{x^2 + y^2}$.
      - Need $z \geq 0$ for range.
      - Level curves: For constant $z = k$ we have circles $k^2 = x^2 + y^2$. These are circles, and they grow in diameter as z increases.
      - Resulting graph is a cone. Check in Geogebra.
3. Level curves:
   (a) Definition: The level curves of function $f(x,y)$ are the curves with equations $f(x,y) = k$ for constant k in the range of f.
   (b) Example: Find the level curves of $f(x,y) = 2x + y$. Level curves are lines $k = 2x + y$ which are lines $y = -2x + k$. Graph in xy-plane. Result is a plane $z = 2x + y$ giving $2x + y - z = 0$.
   (c) Note, different functions (surfaces) can have the same level curves. Compare $f(x,y) = x^2 + y^2$ (paraboloid). Different locations though.
   (d) Examples: Try on own. Find domain and range. Sketch level curves. Describe surface.
   $$z = \frac{y}{x}, \quad z = \sqrt{4 - x^2 - y^2}$$
   (e) Ideas extend to functions of 3+ variables as you think, harder to visualize.
   $$f(x,y,z), \quad f(x_1, x_2, \ldots, x_n)$$

4. Contour maps and calculus intuition: Show contour map of mountain with rivers.
   (a) Contours are drawn every 100 ft increase. What do you see?
   (b) Steep trails have close curves. Flat are far apart.
   (c) Creeks run perpendicular to level curves. Steepest direction is perpendicular.
   (d) Loops indicate peaks and troughs.
   (e) What if you walk along a level curve? No change in elevation.
5. Homework: 1, 7, 11, 13, 15, 19, 23, 25, 33, 35, 37, 41, 43, 49, 61, 63, 65

.2 14.2 Limits and continuity

1. Limits in $\mathbb{R}$
   (a) Intuition definition: $\lim_{x\to a} f(x) = L$ if for x near a, $f(x)$ is near L. Draw picture. Idea is clear, but need precision to build a theory on.
   (b) Precise definition: $\lim_{x\to a} f(x) = L$ if for any $\epsilon > 0$ (no matter how near to L), there exists a $\delta > 0$ (near enough to a) such that if $|x-a| < \delta$, then $|f(x)-L| < \epsilon$. Add δ and ϵ to graph. x window and y window. Technical definition which is hard to work with, instead prove theorems and build theory.
   (c) Techniques for computing limits:
      - Limit laws (solid foundation, grow complexity from basic functions).
      - Algebra tricks (multiply by conjugate, right / left limits, etc).
      - Squeeze theorem and indirect attacks.
      - Can direct substitute for continuous functions.
   (d) Why are limits important? Handling indeterminate form. Essence of calculus.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \qquad \int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x$$

   $0/0$ and $\infty \cdot 0$ indeterminate forms.
   (e) Examples: $f(x) = x^2, f'(3) =?$, $\lim_{x\to 0} \frac{|x|}{x}$, $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2}$.
2. Limits in $\mathbb{R}^2$ and beyond
   (a) Intuition definition: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for (x,y) near (a,b), $f(x,y)$ is near L. Draw picture. Now we approach a point (a,b) from all directions, not just right/left. Precision again is needed.
   (b) Precise definition: $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for any $\epsilon > 0$ (no matter how near to L), there exists a $\delta > 0$ (near enough to (a,b)) such that if $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x,y) - L| < \epsilon$. Note the appearance of the distance formula, circle with center (a,b). Again this definition is not practical.
   (c) Techniques for computing limits:
      - Limit laws from 1 dim generalize, but cannot separate x from y.
      - Squeeze theorem and indirect attacks.
      - Can direct substitute for continuous functions (polynomials, rationals in domain, etc).
      - Interesting case again will be indeterminate forms (next section for partial derivatives).
   (d) Same idea for 3+ dimensions.
3. Examples:

(a) Table example in text. Hint how to explain a limit does not exist. Graph each in Geogebra.

(b) Show $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ has no limit at $(0,0)$ by following paths $x = 0$ and $y = 0$ and getting different values. Similar to right left limits in $\mathbb{R}$. Graph in Geogebra.

(c) Try on own: Show $f(x,y) = \frac{xy}{x^2+y^2}$ has no limit at $(0,0)$ by choosing two paths with different results. Graph in Geogebra.

(d) Theorem: If $f \to L_1$ as $(x,y) \to (a,b)$ along path C_1 and $f \to L_2$ as $(x,y) \to (a,b)$ along path C_2 with $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

(e) Show $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$ via the Squeeze theorem. Key step:

$$0 \leq \frac{3x^2|y|}{x^2+y^2} = 3|y|\frac{x^2}{x^2+y^2} \leq 3|y| \cdot 1$$

Can also do from definition. See text.

(f) If point is in domain and function is continuous, can do direct substitution. $\lim_{(x,y)\to(1,1)} \frac{3x^2y}{x^2+y^2} = 0$.

4. Homework: 5, 9, 13, 17

.3 14.3 Partial derivatives

1. One dimension review, $\mathbb{R}$:

(a) For $f(x)$, change in x results in change in f. Then average rate of change $\Delta f/\Delta x$ tends to instantaneous rate of change df/dx as $\Delta x \to 0$. That is,

$$\frac{df}{dx} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

(b) Limits are foundation, but we built a theory of differentiation.

$$cf(x), f(x)+g(x), f(x)g(x), f(x)/g(x), f(g(x))$$

and also special functions such as logs, exponentials, trig, etc.

2. Two dimensions, $\mathbb{R}^2$: $f(x,y)$

(a) Analogy tangent plane to a surface. Strategy is to allow one variable to change at a time. If x can change for $f(x,y) = x - yx$, then $\Delta f = \Delta x - y\Delta x$ and $\Delta f/\Delta x = 1 - y$. That is the x derivative of $f(x,y)$ is $1-y$. Hold y constant and differentiate f in x. Knowing both will lead to tangent planes (next section).

(b) Definition: The partial derivative of $f(x,y)$ with respect to x is

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Similar for f_y.

(c) Notation: For $f = f(x,y)$,

$$f_x = f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = D_x f$$

(d) All our old differentiation rules hold since y is a constant.

(e) Example: $f(x,y) = 4 - x^2 - 3y^2$.

- Compute $f_x(1,2), f_x, f_y(1,2), f_y$.
- Graph via Geogebra to get intuition around f_x, f_y. Note if we know $f_x(1,2), f_y(1,2)$, we can get a tangent plane (next section).

- Note local max at $(0,0)$.
- Extend to four cases of second derivatives.

(f) Example: $f(x,y) = x^3 + x^2y^3 - 2y^2$

- Try on own, all first and second order partials.
- Compare graph to f_x and f_y.

(g) Theorem: $f_{xy} = f_{yx}$, order of differentiation doesn't matter. Proof via the MVT.

(h) Example: Problem 9 in text.

3. Partial differential equations tour:
   - `https://en.wikipedia.org/wiki/Partial_differential_equation`
   - `https://web.stanford.edu/class/math220b/handouts/heateqn.pdf`
4. Homework: 5, 7, 9, 11, 13, 15, 21, 25, 33, 45, 51, 53, 61, 63, 81, 97

.4 14.4 Tangent planes and linear approximations

1. Recall: $y = f(x)$ version.

   (a) The tangent line to $y = f(x)$ at point (x_0, y_0) is

   $$y - y_0 = f'(x_0)(x - x_0) \quad \rightarrow \quad y = L(x) = f'(x_0)(x - x_0) + y_0.$$

   Give example for $f(x) = x^2$ at $x = 3$.

   (b) Linearization approximates $f(x)$ by this line.

   $$y = f(x) \approx L(x) = f'(x_0)(x - x_0) + y_0.$$

   The closer to the tangent point, the better the approximation. Give example.

   (c) Taylor series and Taylor's theorem continues this vein.

   $$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots$$

2. Extension to $z = f(x,y)$, tangent planes.

   (a) Partial derivatives f_x, f_y give the slope of the tangent line to $z = f(x,y)$ in the x, y directions. Draw picture. How to use this to find the tangent line thru a point (x_0, y_0, z_0)? Need a point and a normal vector.

   (b) Normal vector construction: Find vectors in direction of partial derivative lines.

   - f_x :, y held constant, if x increases 1 unit, z increases f_x units. Then, $\vec{a} = \langle 1, 0, f_x \rangle$ is parallel to our line.
   - f_y :, likewise $\vec{b} = \langle 0, 1, f_y \rangle$ works.
   - The normal vector to the tangent plane is then

   $$\vec{n} = \vec{a} \times \vec{b} = \langle -f_x, -f_y, 1 \rangle$$

   (c) Vector form of tangent plane:

   $$\vec{n} \cdot (\vec{p} - \vec{p_0}) = 0 \quad \rightarrow \quad -f_x(x - x_0) - f_y(y - y_0) + (z - z_0) = 0$$

   gives

   $$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

   Note the similarity to the tangent line for $y = f(x)$.

(d) Example: Find the tangent line to the paraboloid $z = 14 - x^2 - y^2$ at $(x_0, y_0, z_0) = (1, 2, 9)$ Graph in geogebra. Both x, y tangent lines are on this plane. All tangent lines for all surface curves as well.

(e) Try on own: Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$ at $(1, 2, 3)$. Can solve for z taking the positive root or use implicit differentiation with respect to x, y. Note the normal vector is in the same direction as the sphere radius when directed to our point.

(f) Linearization of $z = f(x, y)$ by the tangent plane.

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

Two dimensional Taylor series approximation. Can guess the extension to 3+ independent variables.

3. Differentiability of $f(x, y)$:

   (a) Remind of differentiability in $\mathbb{R}^2$. Derivative exists. Differentiable implies continuous.

   (b) Def: We say $f(x, y)$ is differentiable at point (a, b) if

   $$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

   where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Basically says can approximate f well by the tangent line.

   (c) Theorem: If the partial derivatives f_x, f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

4. Homework: 1, 3, 5, 11, 13, 19, 21

.5 14.5 The chain rule

1. 1 dimension: $\frac{d}{dt} f(g(t)$.

   (a) Goal is to differentiate function composition. Nested functions are common. Do g first, then f takes it from there.

   $$\frac{d}{dt} f(x(t)) = f'(x(t))x'(t)$$

   (b) Compact notation: $y = f(x)$

   $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

   Right hand side collapses back if canceling were allowed.

   (c) The chain rule applied to integration is the substitution rule.

2. 2 dimensions, basic case: $\frac{d}{dt} f(x(t), y(t))$

   (a) Extend the dimension 1 case of the chain rule to get for $z = f(x, y)$:

   $$\frac{dz}{dt} = \frac{d}{dt} f(x, y) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

   Note the similarity to the 1 dimension case.

   (b) Example: For $z = 3xy^2$, $x = \cos(t), y = \sin(t)$, compute $\frac{dz}{dt}$. Check by rewriting x, y in original. Graph in Geogebra, not traveling about the unit circle in xy. Consider $t = 0, \frac{\pi}{2}$. Rate of change along curve $(x(t), y(t))$.

3. 2 dimensions, standard case: $\frac{d}{dt} f(x(s, t), y(s, t))$

(a) Repeat the above formula twice.

$$\frac{dz}{ds} = \frac{d}{ds} f(x,y) = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{dz}{dt} = \frac{d}{dt} f(x,y) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

(b) Example: For $z = 3xy^2$, $x = r\cos(\theta), y = r\sin(\theta)$, compute $\frac{dz}{dr}$. Try on own $\frac{dz}{d\theta}, \frac{d^2z}{dr^2}$

(c) Second derivatives and converting to polar coordinates. $z = f(x,y)$, $x = r\cos(\theta), y = r\sin(\theta)$

- Compute $f_{rr}, f_{\theta\theta}$.
- Turns out $f_{xx} + f_{yy} = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_\theta$
- This is the polar version of Laplace's equation.

4. Generalizes to any dimension. Show text formula. Work example 5.

5. Homework: 1, 3, 5, 7, 11, 13, 17, 21, 45, 49

.6 14.6 Directional derivatives and gradient vectors

1. Directional derivatives: So far we calculate change for $f(x,y)$ in the x direction (f_x) or the y direction (f_y), but of course f can change in any direction.

   (a) Recall our limit definitions:

   $$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

   We essentially hold y and x constant respectively. The directions we consider here are $\vec{i}$ and $\vec{j}$. Note both are unit vectors.

   (b) Example: Find the change in $f(x,y) = xy$ at point $(3,1)$ in the direction $\vec{v} = \langle 1,2\rangle$. Normalize our direction via the unit vector $\vec{u} = \langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$. Then our change is from $(3,1)$ to $(3 + h/\sqrt{5}, 2h/\sqrt{5})$ and

   $$D_{\vec{u}} f(3,1) = \lim_{h\to 0} \frac{f(3 + h/\sqrt{5}, 1 + 2h/\sqrt{5}) - f(3,1)}{h} = \lim_{h\to 0} 7/\sqrt{5} + 2h/5 = 7\sqrt{5}.$$

   Note h in the denominator because of the unit vector. Graph in Geogebra and compare to f_x, f_y.

   (c) Definition: The directional derivative of $f(x,y)$ at point (x_0,y_0) in the direction of unit vector $\vec{u} = \langle a,b\rangle$ is

   $$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0,y_0)}{h}$$

   Note, $D_{\vec{i}} f = f_x$ and $D_{\vec{j}} f = f_y$. Also any unit vector can be expressed in terms of a direction angle θ as

   $$\vec{u} = \langle a,b\rangle = \langle \cos(\theta), \sin(\theta)\rangle$$

2. Computing directional derivatives

   (a) The above limit definition is messy to compute. Instead, we rewrite $D_{\vec{u}} f$ in terms of f_x and f_y. This seems doable considering the tangent plane to a surface in $\mathbb{R}^3$.

   (b) Theorem: For $f(x,y)$ differentiable in both x and y and $\vec{u} = \langle a,b\rangle$ any unit vector in $\mathbb{R}^2$,

   $$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

(c) Proof: Define $g(h) = f(x_0 + ah, y_0 + bh)$. Then,

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0).$$

On the other hand, from the chain rule,

$$g'(h) = \frac{\partial f}{\partial h} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} = af_x + bf_y = af_x(x_0 + ah, y_0 + bh) + bf_y(x_0 + ah, y_0 + bh).$$

Evaluating $g'(h)$ at zero and comparing to before gives the result.

(d) Example: Repeat above example $f(x, y) = xy$ with new calculation.

(e) Example: Try on own for $f(x, y) = xy^3 - x^2$ at $(1, 2)$ in direction $\theta = \frac{\pi}{3}$. Check via Geogebra.

3. Gradient vectors:

(a) Example: Hint to bigger things. $f(x, y) = 3x + y + 1$ at $(1, 1)$.

- $\vec{i}$ and $\vec{j}$ directions.
- No change (level curve) direction. Find $\vec{u} = \langle a, b\rangle$ such that

$$D_{\vec{u}} f = f_x a + f_y b = 0$$

gives $\vec{u} = \frac{1}{\sqrt{10}}\langle -3, 1\rangle$,

- Perpendicular to level curve gives steepest direction $\vec{u} = \frac{1}{\sqrt{10}}\langle 1, 3\rangle$. This matches $\langle f_x, f_y\rangle$ at our point. Compute change and compare to f_x, f_y.
- Noting that the directional derivative is really a dot product, we see a new vector of import.

$$D_{\vec{u}} f = f_x a + f_y b = \langle f_x, f_y\rangle \cdot \langle a, b\rangle$$

(b) Definition: For $f(x, y)$, the gradient of f is a vector-function of the form

$$\nabla f = \langle f_x, f_y\rangle$$

(c) Example: Compute gradient for previous example $f(x, y) = xy^3 - x^2$ at $(1, 2)$. Reproduce previous result.

(d) Theorem: First importance of the gradient. For f differentiable, the maximum value of the directional derivative $D_{\vec{u}} f$ is $|\nabla f|$ and is in the direction of ∇f.

(e) Proof: We use the law of cosines version of the dot product.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos(\theta)$$

where θ is the angle between $\vec{a}, \vec{b}$. Then,

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\|\|\vec{u}\| \cos(\theta) = \|\nabla f\| \cos(\theta) \leq \|\nabla f\|$$

which occurs when $\theta = 0$ meaning $\vec{u}$ and ∇f are in the same direction.

(f) Example: Apply previous theorem to $f(x, y) = 3x + y + 1$ at $(1, 1)$, $f(x, y) = xy^3 - x^2$ at $(1, 2)$.

(g) Example: Try on own. Number 22 in text. Graph in Geogebra.

4. Extension to functions of three variables: $f(x, y, z)$.

(a) Could be in $\mathbb{R}^4$ in which case cannot visualize. Could be an implicit curve $f(x, y, z)$ in $\mathbb{R}^3$.

(b) Definition of directional derivative in direction of unit vector $\vec{u}$.

$$D_{\vec{u}} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}$$

(c) Compute $D_{\vec{u}}$ in terms of partial derivatives.

$$D_{\vec{u}} = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \nabla f \cdot \vec{u}$$

(d) Gradient of f is

$$\nabla f = \langle f_x, f_y, f_z \rangle.$$

(e) Examples are pretty well the same.

5. Tangent planes to level surfaces

(a) We already have tangent planes to surfaces of the form $z = f(x,y)$ at point (x_0, y_0, z_0).

$$z - z_0 = (f_x)_0(x - x_0) + (f_y)_0(y - y_0)$$

(b) This extends implicitly to a level surface $F(x,y,z) = k$ at point (x_0, y_0, z_0).

$$(F_x)_0(x - x_0) + (F_y)_0(y - y_0) + (F_z)_0(z - z_0) = 0$$

Note, the gradient vector ∇F is our normal vector to the plane (and surface).

(c) The normal line then has symmetric equations

$$\frac{x - x_0}{(F_x)_0} = \frac{y - y_0}{(F_y)_0} = \frac{z - z_0}{(F_z)_0}.$$

(d) Example: Find the tangent plane to the ellipsoid $x^2/4 + y^2 + z^2/9 = 3$ at point $(-2, 1, -3)$. Check result in Geogebra.

6. Summary of gradient vector: This section is rich. Summarize the key ideas.

(a) For $f(x,y)$ (or $f(x,y,z)$), ∇f gives the direction of fastest increase of f.

(b) $\|\nabla f\|$ is the fastest increase rate (slope).

(c) ∇f is orthogonal to the level curve (or surface).

7. Homework: 5, 7, 9, 11, 15, 19, 23, 25, 27, 29, 37, 39, 41, 49

.7 14.7 Maximum and minimum values

1. Recall functions of one variable...

(a) Draw $f(x)$ with make an min values. Smooth and continuous on $\mathbb{R}$.

(b) $f'(x) = 0$ (stationary points) gives locations of horizontal tangents. $f''(x) = 0$ discerns the three cases.

- $f''(x) > 0$, local min
- $f''(x) < 0$, local max
- $f''(x) = 0$, inflection point

(c) Two other cases for extrema: Singular points, end points.

(d) Absolute max and mins are ensured by the EVT: Continuous function $f(x)$ on closed interval $[a, b]$ must have a local max and local min.

2. Definitions for $f(x,y)$.

(a) Local min at (a, b) with local min value $f(a, b)$. Likewise for max.

(b) Global min and max.

3. Extending calulus 1 results:

(a) Theorem: If $f(x,y)$ has a local max or min at (a,b) and f_x, f_y both exist at (a,b), then $f_x(a,b) = 0$ and $f_y(a,b) = 0$ ($\nabla f = \vec{0}$).

(b) If $\nabla f = \vec{0}$ at (a,b), then this is called a stationary point. Not all critical points are local mins or maxes.

(c) Example: Find the stationary points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$.

4. How to classify stationary points? Concavity is key, but we need to look in all directions.

(a) 2 examples: $x^2 + xy + y^2$ and $x^2 + 10xy + y^2$. Only $(0,0)$ stationary point for both. Both have two positive partials second ($f_{xx} = f_{yy} = 2 > 0$). Graph in Geogebra to see different behavior.

(b) To classify, consider all the second directional derivatives at once. For $f(x,y)$ and $\vec{u} = \langle h, k \rangle$,

$$D_{\vec{u}} f = f_x h + f_y k.$$

$$D_{\vec{u}}^2 f = D_{\vec{u}}\left(f_x h + f_y k\right) = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}^2\right)$$

where the last step follows by completing the square.

(c) If we think concave up since $f_{xx} > 0$ we would also need $D = f_{xx}f_{yy} - f_{xy}^2 > 0$. Likewise for concave down we need $f_{xx} < 0$ but still $D > 0$.

(d) Theorem: For (a,b) a stationary point of $f(x,y)$ and

$$D = D(a,b) = f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.
- If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.
- If $D > 0$ then $f(a,b)$ is a saddle point.

(e) Check for 2 examples.

(f) Example: Apply to first example.

5. Last, we extend the EVT

(a) EVT: For $f(x,y)$ continuous on closed, bounded region R in $\mathbb{R}^2$, then f has an absolute max and min in R.

(b) How to find extrema? Abs max and mins must be at stationary points in R or on the boundary of R.

- Find the stationary points in R.
- Find the extreme values on the boundary via Calc 1.
- Get the largest and smallest f values from parts 1 and 2.

(c) Example: 34 in text.

6. Homework: 1, 3, 5, 7, 13, 15, 17, 23, 27, 31, 33

.8 14.8 Lagrange multipliers

Skip.

.9 Chapter 14 Review

1. Concept check: 1-18

2. True-False: 1-11

3. Exercises: 1-56

Chapter 15 Multiple integrals

.1 15.1 Double integrals over rectangles

1. Summary of past: Extend the definite integral of calculus 1 to 3 dimensions.
   (a) $\displaystyle\int_a^b f(x)\ dx$ as area under the curve.
   (b) Compute via limit of Riemann sum. Classic calculus paradox.
   (c) Fundamental theorem of calculus.
   (d) Alternate view: $\displaystyle\int_a^b f(x)\ dx$ as adding up 1D lengths to get 2D area.
   (e) Really about summation: Sum lines to area, areas to volumes (discs and washers), probability, force to work, line segment to arc length, arc length to surface area, etc
2. Basic case for $z = f(x, y)$ in $\mathbb{R}^3$: Volumes over rectangular domains.
   (a) Find the volume of the solid between $z = f(x, y)$ and the $xy-$plane over region $R = [a, b] \times [c, d]$ (Cartesian product).
   (b) Partition R by Δx and Δy giving rectangular areas ΔA.
   (c) Notation and limit of a Riemann sum.
   $$\iint\limits_R f(x)\ dA = \int_a^b \int_c^d f(x, y)\ dy\ dx = \lim_{m,n\to\infty} \sum_{i=1}^n \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)\Delta A$$
   (d) If the limit exists we say f is integrable over R.
   (e) Can approximate area as in Calc 1 by computing the finite sum, though only works for simple functions.
3. FTOC for calculation: Volume by accumulating area.
   (a) Slicing the solid in the x direction gives cross-sections with area
   $$A(x) = \int_c^d f(x, y)\ dy.$$
   This is a computable function of x for any y held constant.
   (b) Add up area to get volume. FTOC twice.
   $$V = \int_a^b A(x)\ dx = \int_a^b \left[\int_c^d f(x, y)\ dy\right]\ dx$$
   (c) Of course slicing in y gives a similar formula resulting in Fubini's theorem which extends to more general regions as well.
   $$\int_a^b \left[\int_c^d f(x, y)\ dy\right]\ dx = \int_c^d \left[\int_a^b f(x, y)\ dx\right]\ dy$$
   (d) Examples: Divide and conquer to find the area under $z = 1 + x^2 + y^2$ on $[1, 2] \times [0, 1]$.
4. Average value of functions:
   (a) Calc 1 version:
   $$\frac{1}{b-a}\int_a^b f(x)\ dx$$

(b) Calc 3 version:

$$\frac{1}{A(R)} \iint\limits_R f(x)\ dA$$

5. Homework: 1, 13, 15, 17, 19, 21, 25, 29, 31, 33, 35, 39, 47

.2 15.2 Double integrals over general regions

1. Idea of general regions:

   (a) The domain of integration for $z = f(x, y)$ doesn't have to be a rectangle. In general it can be any shape (denote as D). Draw picture.

   (b) Can still do slicing in x or y direction. Result is 2 cases to choose from.

   (c) Volume from accumulating area in x (holding y constant). Draw domain picture.

$$\iint\limits_D f(x, y)\ dA = \int_a^b A(x)\ dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y)\ dy\ dx$$

   (d) Volume from accumulating area in y (holding x constant). Draw domain picture.

$$\iint\limits_D f(x, y)\ dA = \int_c^d A(y)\ dy = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)\ dx\ dy$$

   (e) Sometimes only one is an option, sometimes both can be used and need to think strategically.

2. Examples: Drawing the domain in the $xy-$plane is key.

   (a) Only one direction is easy. Find the volume under the surface $z = x^2 + y$ on domain D bound by curves $y = x + 1$ and $y = x^2$.

   (b) Divide and conquer by doing both at same time. Find the volume below the plane $z = x - 2y$ and above the triangle with vertices $(0, 0), (1, 1), (0, 1)$ in the $xy-$plane. Need to divide into two volumes in one direction leading to the below theorem.

   (c) Theorem: If $D = D_1 \cup D_2$, then

$$\iint\limits_D f(x, y)\ dA = \iint\limits_{D_1} f(x, y)\ dA + \iint\limits_{D_2} f(x, y)\ dA$$

3. If integration is hard, estimation will often do by capturing the solid inside and outside a cylinder.

   (a) Theorem: For $m \leq f(x, y) \leq M$ on domain D with area $A(D)$, then

$$mA(D) \leq \iint\limits_D f(x, y)\ dA \leq MA(D).$$

4. Homework: 1, 5, 7, 9, 13, 15, 17, 21, 23, 25, 31, 39, 45, 49, 51, 57, 59

.3 15.3 Double integrals over polar coordinates

1. When rotation is involved, rectangular coordinates are no longer nice. Switch to polar coordinates.

   (a) Example: $\iint\limits_R (3x+4y^2)\ dA$ for $R = \{(x, y) | 4 \leq x^2 + y^2 \leq 9\}$. Cannot divide into rectangles. Hard to separate curves.

(b) Polar coordinates basics: (x, y) replaced with (r, θ). Connection is trigonometry.

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad x^2 + y^2 = r^2$$

(c) Convert point $(3, 1)$ into polar coordinates. r is easy. $\theta = \arctan(y/x)$ if in quandrants 1 and 4. Shift θ to the right quadrant as for $(-3, -1)$.

(d) Above region is now easier to describe. Rectangle in terms of r, θ.

$$R = \{(r, \theta) | 2 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$$

(e) How to translate dA to be in terms of $d\theta$ and dr?

- For rectangles, the area of a small rectangle was $dA = dxdy$.
- For new curved rectangles, take the difference of the wedges which have area $\frac{1}{2}r^2\Delta\theta$. The centering of r in the wedgy-rectangle is needed to avoid Δr^2.

$$\Delta A = \frac{1}{2}(r + \Delta r/2)^2\Delta\theta - \frac{1}{2}(r - \Delta r/2)^2\Delta\theta = \frac{1}{2}(2r\Delta r)\Delta\theta = r\Delta r\Delta\theta$$

- Then,

$$\iint_R (3x + 4y^2)\ dA = \int_0^{2\pi}\int_2^3 (3r\cos(\theta) + 4r^2\sin^2(\theta))r\ dr\ d\theta$$

Recall the $\sin^2(\theta)$ term will require a calc 2 trig integral strategy via the half angle formula.

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

2. Example: Find the volume under the paraboloid $z = x^2 + y^2$ yet inside the cylinder $x^2 + y^2 = 2x$.

(a) Complete the square on the cylinder to graph it.

$$(x - 1)^2 + y^2 = 1$$

(b) Translate region into polar coordinates.

$$x^2 + y^2 = 2x \quad \rightarrow \quad r^2 = 2r\cos(\theta) \quad \rightarrow \quad r = 2\cos(\theta)$$

$$R = \{(r, \theta) | 0 \leq r \leq 2\cos(\theta), -\pi/2 \leq \theta \leq \pi/2\}$$

(c) Compute the integral.

$$\iint_R x^2 + y^2\ dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos(\theta)} r^2\ rdrd\theta = \cdots = 8\int_0^{\pi/2}\cos^4(\theta)\ d\theta == 8\int_0^{\pi/2}\left(\frac{1 + \cos(2\theta)}{2}\right)^2 d\theta = \frac{3\pi}{2}$$

3. Homework: 1, 3, 5, 7, 11, 17, 19, 25

.4 15.4 Applications of double integrals

Skip.

.5 15.5 Surface area

Skip.

.6 15.6 Triple integrals

1. Triple integrals: Continue the path.

   (a) Instead of small intervals (dx) or small boxes (dA), we not have small boxes (dV).

   (b) Integrating will be the easy part, setting up the integral is the challenge.

   (c) For $f(x, y, z)$ a continuous function on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$,

   $$\iiint\limits_B f(x, y, z)\ dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\Delta V = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dx\ dy\ dz$$

   (d) Fubini's theorem says the order of integration can be changed.

   (e) Below we will just integrate 1 to find volumes of solids. Replace 1 with any $f(x, y, z)$ with the same story.

2. Examples: Volumes of solids.

   (a) Box: Graph it. Two integrations gives area of cross section. Final integral adds area to get volume.

   $$\int_0^1 \int_0^3 \int_0^2 dx\ dy\ dz$$

   (b) Prism: Graph it. Should be half of box. Easiest to project onto the xy-plane first, then sort out the z bounds. Result is the following description of the solid.

   $$E = \{(x, y, z)| 0 \le x \le 2, \quad 0 \le y \le 3, \quad 0 \le z \le (3 - y)/3\}$$

   $$\int_0^2 \int_0^3 \int_0^{(3-y)/3} dz\ dy\ dx$$

   Note middle two integrals produce area of a cross section. Another view is that the inside integral is the area under integrand $f(x, y, z) = 1$ over the length of a line segment, then summed over the entire region in the xy-plane.

   $$\int_0^2 \int_0^3 \left(\int_0^{(3-y)/3} dz \right) dy\ dx$$

   (c) Try on own: Find the volume of the tetrahedron (4-sided pyramid) with corners 1s on the 3 axis. Follow previous example.

   $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx$$

3. Changing order of integration.

   (a) Try on own: Find the prism volume in order $dx\ dy\ dz$. Hint: Draw projection in $yz-$plane first.

   $$\int_0^1 \int_0^{3-3z} \int_0^2 dx\ dy\ dz$$

   (b) Try on own: Find the volume of the tetrahedron in order $dy\ dz\ dx$. Hint: Draw projection in the $xz-$plane first.

   $$\int_0^1 \int_0^{1-x} \int_0^{1-x-z} dy\ dz\ dx$$

   (c) Example 4 in text.

4. Homework: 1, 3, 9, 13, 19, 21, 27, 31, 35

.7 15.7 Triple integrals in cylindrical coordinates

1. Cylindrical coordinates:

   (a) These describe 3 dimensional solids which are rotationally symmetric about the z-axis. Similar to solids of revolution from Calc 1.

   (b) $(x, y, z) = (r, \theta, z)$, we trade the xy-plane rectangular coordinates for polar coordinates.

   (c) Conversion is same as before:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

$$r = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

   (d) Examples: Cylindrical point $(3, \pi/2, 2)$ to rectangular. Rectantular point $(2, -2, 1)$ to cylindrical.

2. Integrals in cylindrical coordinates: Hardest part is setting up the integral.

   (a) Nested integration where the inside is the polar integral of a cross section (integral).

$$\iiint\limits_R f(x,y,z)\; dV = \int_r^s \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos(\theta), r\sin(\theta), z)\; rdr\; d\theta\; dv$$

   Order of integration can change. Another order gives a new view.

$$\iiint\limits_R f(x,y,z)\; dV = \iint\limits_D \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\; dz \right)\; dA = \int_\theta \int_r \int_{g(x,y)}^{h(x,y)} f(r\cos(\theta), r\sin(\theta), z)\; dzr\; dr\; d\theta$$

   (b) Again, we focus on volume. Compute $\iiint\limits_R dV$ for $R = \{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \pi, 0 \le z \le 3\}$. Solid is half a cylinder.

   (c) Find the volume of the cone $r = 1 - z$, $0 \le r \le 1$. Three ways. Note the areas of each order (circle, shell, triangle).

$$\iint\limits_R \; dV = \int_0^1 \int_0^{2\pi} \int_0^{1-z} r\; dr\; d\theta\; dz = \int_0^1 \int_0^{2\pi} \int_0^{1-r} r\; dz\; d\theta\; dr = \int_0^{2\pi} \int_0^1 \int_0^{1-z} r\; dr\; dz\; d\theta$$

   (d) Find the volume of the solid which lies between the parabolid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$. Converting the bounds we have $z = r$ and $z = \sqrt{2 - r}$ and noting the intersection curve

$$r = \sqrt{4 - r} \quad \to \quad r^2 + r - 2 = 0 \quad \to r = 1, -2 \quad \to r = 1$$

   Then,

$$\iiint\limits_R \; dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{r}} dz\; rdr\; d\theta = \dots$$

3. Homework: 1, 3, 5, 7, 9, 11, 15, 17, 21, 23, 29

.8 15.8 Triple integrals in spherical coordinates

1. Spherical coordinate system

   (a) On Earth we use latitude and longitude. Allow dig in or fly out, and we need a third measurement, distance from center.

(b) (ρ, θ, ϕ) where $\rho \geq 0$ is distance from the origin, $0 \leq \theta \leq 2\pi$ is angle in $xy-$plane as before, and $0 \leq \phi \leq \pi$ is angle from z-axis.

(c) Describe the shapes.

- $\rho = 10$, $\theta = 1$, $\phi = 1$, ...

(d) Conversion to rectangular coordinates (x, y, z)

- In the xy-plane, $x = r\cos(\theta)$, $y = r\sin(\theta)$ though we discard r. Using the triangle formed by r and ρ, we have
$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta)$$
- Can see $z = \rho\cos(\phi)$ from another right triangle.
- Check that $x^2 + y^2 + z^2 = \rho^2$ as the distance formula will know.

2. Integration with spherical volumes

(a) Can show that
$$\iiint\limits_E f\ dV = \iiint\limits_E f\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi$$

(b) The resulting spherical box has edges $d\rho$, $\rho d\phi$ and horizontal edge becomes $\rho\sin(\phi)d\theta$. The product of the three is dV. This is really cubic distance from $\rho^2\ d\rho$.

(c) Example: Find the volume of a sphere of radiues R.
$$\iiint\limits_E\ dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R}\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi$$

(d) Example: Find the volume of the ice cream cone above cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.
$$\iiint\limits_E\ dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos(\phi)}\ \rho^2\sin(\phi)\ d\rho\ d\theta\ d\phi$$
Figure 11 in the text is helpful.

3. Homework:1, 3, 5, 7, 9, 11, 15, 17, 21, 23, 25, 41

.9 15.9 Change of variable in multiple integrals

1. Substituion from calculus 1:

(a) We reverse the chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.

(b) $\displaystyle\int F(g(x))g'(x)\ dx = \int F(u)\ du$

(c) Also can write as $\int F(u(x))\frac{du}{dx}\ dx = \int F(u)\ du$.

(d) Example: $\int x\sin(x^2)\ dx$

(e) Here we write things a bit backwards: $\displaystyle\int f(x)\ dx = \int f(x(u))\frac{dx}{du}\ du$

(f) As a change of variable, we have a stretching factor $J = \frac{dx}{du}$.

(g) Here we usually change variables to simplify the integrand, but now we consider the region of integration as well.

(h) Our aim is to generalize results such as $\displaystyle\int_R f(x, y)\ dydx = \int_S f(r\cos(\theta), r\sin(\theta))r\ drd\theta$. Saw the r from geometry before. Will see again.

2. Change of variables in $\mathbb{R}^2$: $x = g(u,v)$, $y = h(u,v)$.
   (a) For transformation $T(u,v) = (x,y)$ given by $x = g(u,v)$, $y = h(u,v)$. This maps from the uv-plane to the xy-plane. The reverse mapping T^{-1} also makes sense.
   (b) Example: $x = r\cos(\theta)$, $y = r\sin(\theta)$. Maps rectangular regions to circular ones.
   (c) We want to know how area changes with the new variables. That is relate $dxdy$ to $dudv$.
   (d) For rectangle in the uv-plane mapped to nonlinear region in the xy-plane, our rectangels $\delta u \delta v$ get mapped to nearly parallelograms $\delta u(g_u\vec{i} + h_u\vec{j})$ and $\delta v(g_v\vec{i} + h_v\vec{j})$. The cross product magnitude gives the area of such parallelograms.

$$\|\Delta u(g_u\vec{i}+h_u\vec{j})\times\Delta v(g_v\vec{i}+h_v\vec{j})\| = \|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ g_u & h_u & 0 \\ g_v & h_v & 0 \end{vmatrix}\|\Delta u\Delta v = \|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix}\|\Delta u\Delta v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}\Delta u\Delta v$$

   Limit and Riemann sum gives our integration result.
   (e) Theorem:

$$\iint\limits_R f(x,y)\;dx\;dy = \iint\limits_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\;du\;dv$$

   where the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

3. Examples:
   (a) For R the parallelogram with corners $(0,0),(2,1),(3,3),(1,2)$, computer $\iint\limits_R e^x\;dxdy$ by first transforming R to S the square with corners $(0,0),(0,3),(3,3),(3,0)$ in the uv-plane.
      - $u = 2x - y$ and $v = 2y - x$ does the trick. See where the edges of the parallelogram map.
      - Then $x = (2u+v)/3$ and $y = (u+2v)/3$ and we compute the jacobian $J = 1/3$.
      - Note $\iint\limits_R\;dxdy = \int_0^3\int_0^3 (1/3)\;dudv = 9/3 = 3$ and the area of R was 1/3 of new region S.
      - Last, $\iint\limits_R e^x\;dxdy = (3/2)(e^2-1)(e-1)$, though we could have used past techniques for region R.
      - Can see the opposite directoin would have tripled the area.
4. Triple integrals have 3×3 determinants. Rectangular prisms become parallelpipeds. See text for spherical coords derivation.
5. Homework: 1, 3, 5, 7, 9, 11, 15, 17

Chapter 16 Vector calculus

1. The big step of the class: FTOC for double and trible integrals.
2. For chapter 15, we mostly rely on FTOC from calculus 1, but there are high dimension versions.
3. 2 new ideas: Vector fields (vectors at all locations in the plane or 3-space) and line integrals (integration along a curve rather than an integral).
4. Our FTOC for double integrals will connect a double integration over a region to a single integration along a boundary curve (called Green's theorem).

.1 16.1 Vector fields

1. Foundation:

   (a) Def: A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\vec{F}(x, y)$. That is,
   $$\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}.$$
   Similar for $\mathbb{R}^3$.

   (b) Example: Position vector field $\vec{F}(x, y) = \langle x, y\rangle$.

   (c) Example: Spin vector field $\vec{F}(x, y) = \langle -y, x\rangle$. Each vector length is distance from the origin.

   (d) Graphing is a pain. Check out Geogebra...ish.

2. Gradient vector field:

   (a) $\nabla f(x, y)$ is really a vector field.

   (b) Important question: Which vector fields are gradient vector fields (called conservative vector feilds). That is, for vector field $\vec{F}$ is there a f such that $\vec{F} = \nabla f$? f is called the potential function for F.

   (c) The position vector field is conservative since $\vec{F} = \nabla f$ for $f(x, y) = x^2/2y^2/2$.

   (d) Can show the spin vector feild is not, but if you scale by $1/r^2 = 1/\sqrt{x^2 + y^2}$ it is for $f(x, y) = \arctan(y/x) = \theta$.

3. Homework: 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 35

.2 16.2 Line integrals

1. Line integral basics

   (a) Idea: A line integral is the integral along a curve rather than an interval. Interpretation is area, though physics has applications.

   (b) Draw picture for $z = f(x, y)$, Δs for change in arc length. Rectangles are still in the Riemann sum.

   (c) Definition: The line integral of f along curve C is
   $$\int_C f(x, y)\ ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s$$

   (d) Arc length is an integration problem from Chapter 10.
   $$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$
   for $x = x(t), y = y(t)$ a parameterization of curve C. This sum of line segment lengths (distance formula) in the limit.

   (e) Theorem: Argue along similar lines as arc length to get...
   $$\int_C f(x, y)\ ds = \int_a^b f(x, y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$
   which can also be written as
   $$\int_C f(\vec{r})\ ds = \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\ dt$$

(f) Note, for $f(x,0)$ this reduces to the interval version $\int_a^b f(x,0)\ dx$.

(g) Find the line integral of $f(x,y)=1$ along the upper half of the circle of radius 2. Easy to replace f with any function and just compute.

(h) Can also integrate in a single direction and translated to parameter t via the chain rule. Meaning to be mentioned shortly.

$$\int_C f(x,y)\ dx = \int_C f(x(t),y(t))x'(t)\ dt, \quad \int_C f(x,y)\ dy = \int_C f(x(t),y(t))y'(t)\ dt$$

Can combine into a single expression.

$$\int_C f(x,y)\ dx + \int_C g(x,y)\ dy = \int_C f(x,y)\ dx + g(x,y)\ dy$$

(i) Example 4 in text.

(j) Extensions to three dimensions not surprising.

$$\int_C f(x,y,z)\ ds = \int_a^b f(x,y,z)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\ dt = \int_a^b f(\vec{r}(t))\|\vec{r}'(t)\|\ dt$$

$$\int_C f(x,y,z)\ dx + g(x,y,z)\ dy + h(x,y,z)\ dz$$

2. Physics applications: Work and vector fields.

(a) In one dimension:

i. $W=F\cdot D$ for constant force, $W=\int_0^D F(x)\ dx$ for variable force.

(b) In two dimensions:

(c) $W=\vec{F}\cdot\vec{D}$ for constant force vector $\vec{F}$ (mag and dir) and direction vector $\vec{D}$.

(d) Line integrals give work done along a curve:

i. Constant force vector in direction of tangent vector $\vec{T}$.

$$W=\int_C \vec{F}\cdot\vec{T}\ ds = \int_C \vec{F}\cdot\frac{\vec{r}'}{\|\vec{r}'\|}\ ds = \int_C \vec{F}\cdot\frac{\vec{r(t)}'}{\|\vec{r(t)}'\|}\|\vec{r}'(t)\|\ dt = \int_C \vec{F}\cdot\vec{r(t)}'\ dt$$

ii. Variable force via a vector field $\vec{F}(x,y,z)=\vec{F}(\vec{r})$

$$W=\int_C \vec{F}(\vec{r})\cdot\vec{T}\ ds = \int_C \vec{F}(\vec{r(t)})\cdot\vec{r'(t)}\ dt$$

3. General vector fields

(a) Definition: The line integral of vector field $\vec{F}$ is

$$\int_C \vec{F}\ d\vec{r} = \int_C \vec{F}(\vec{r(t)})\cdot\vec{r'(t)}\ dt = \int_C \vec{F}\cdot\vec{T}\ ds$$

This is general for 2+ dimensions.

(b) Can connect line integrals for vector fields $\vec{F}=\langle P(x,y),Q(x,y)\rangle$ to line integral for functions giving reason to line integrals in x and y (and z).

$$\int_C \vec{F}\ d\vec{r} = \int_C \vec{F}(\vec{r(t)})\cdot\vec{r'(t)}\ dt = \int_C P(x,y)\ dx + Q(x,y)\ dy$$

Likewise for 3 variables.

(c) Example: Compute the work done by vector field $\vec{F} = \langle -y, x\rangle$ (spin field) along

- Straight line from $(1,0)$ to $(0,1)$. $y = 1-x$ gives $x = 1-t, y = t$ for $0 \le t \le 0$. Then,

$$\vec{F}(r(\vec{t})) = \langle -t, 1-t\rangle, \quad r'(\vec{t}) = \langle -1, 1\rangle$$

and

$$\int_C \vec{F}\ d\vec{r} = \int_C \vec{F}(r(\vec{t}) \cdot r'(\vec{t})\ dt = \int_0^1 (t + (1-t))\ dt = 1$$

- Quarter circle from $(0,1)$ to $(1,0)$. $x = \cos(t), y = \sin(t)$ for $0 \le t \le \frac{\pi}{2}$ works.

$$\vec{F}(r(\vec{t})) = \langle -\sin(t), \cos(t)\rangle, \quad r'(\vec{t}) = \langle -\sin(t), \cos(t)\rangle$$

and

$$\int_C \vec{F}\ d\vec{r} = \int_C \vec{F}(r(\vec{t}) \cdot r'(\vec{t})\ dt = \int_0^{\pi/2} (1)\ dt = \frac{\pi}{2}$$

- Note work done is positive for both, more for the second.
- Check out in Geogebra to see how more work is done along the circle.
- Direction reversed would give negative work done. Zero work done if directions are all perpendicular. Position vector field would give this.

4. Homework: 1, 3, 7, 11, 15, 17, 19, 21, 29

.3 16.3 The fundamental theorem of line integrals

1. Fundamental theorems

(a) FTOCP2: $\displaystyle\int_a^b F'(x)\ dx = F(b) - F(a)$. Area under the curve equates to endpoint evaluation.

(b) Theorem (FTOLI): For curve C defined by the vector function $\vec{r}(t)$, $a \le t \le b$ and differentiable function $f(x,y)$ such that ∇f is continuous on C, we have

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

(c) Proof: We use the chain rule and FTOCP2.

$$\int_C \nabla f \cdot d\vec{r} = \int_C \nabla f \cdot \vec{r}'(t)\ dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)\ dt = \int_a^b \frac{d}{dt} f(\vec{r}(t)\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

(d) Notes:

- For conservative vector fields $\vec{F} = \nabla f$, line integrals amount to endpoint evaluation at $A = (x(a), y(a))$ and $B = (x(b), y(b))$.
- Since C can be any curve, this implies path indendence so long as endpoints A and B are the same. That is

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

- For a closed curve (ends where you started), we have

$$\int_C \nabla f \cdot d\vec{r} = \oint_C \nabla f \cdot d\vec{r} = 0$$

- Integration can be completely avoided for $\displaystyle\int_C F \cdot d\vec{r}$ as long as we can verify F is conservative and find $F = \nabla f$ for potential f. How to test? Easy way is for $F = \langle M(x,y), N(x,y)\rangle = \nabla f$, we need $M_y = N_x$. This is the quick test.

(e) Example: Exercise 12 in text.

(f) Example: See again that the spin field is not conservative.

2. Conservation of energy:

   (a) Here we see why we say conservative and potential.

   (b) Force acting on a mass results in change in velocity. Newton's Law gives $F = ma = mv'$. Then over a path $\vec{r}$, $\vec{F}(\vec{r}(t)) = mr''(t)$ and

   $$W = \int_C \vec{F} \cdot \vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \frac{m}{2} \int_a^b \frac{d}{dt} [\vec{r} \cdot \vec{r}]\ dt = \frac{m}{2} \left(\|\vec{v}(b)\|^2 - \|\vec{v}(a)\|^2 \right)$$

   This last part is change in kenetic energy $\frac{m}{2}\|\vec{v}\|^2$, half mass times square of speed.

   (c) For $\vec{F}$ conservative it has potential f. $\vec{F} = -\nabla P$ gives P the potential energy of the object. Then,

   $$W = \int_C \vec{F} \cdot \vec{r} = \int_C -\nabla P \cdot \vec{r} = P(\vec{r}(a)) - P(\vec{r}(b))$$

   (d) Equating work done W we get conservation of total energy

   $$P(\vec{r}(a)) + \frac{m}{2}\|\vec{v}(a)\|^2 = P(\vec{r}(b)) + \frac{m}{2}\|\vec{v}(b)\|^2$$

3. Homework:1, 3, 5, 7, 11, 13, 15, 19, 23, 25, 29, 35

.4 16.4 Green's theorem

1. 2D version of FTOC, volume connected to integral, 150 years after Newton and Leibniz.

   (a) Equate a double integral over region R to a line integral along its boundary C. Some conditions need heeding.

   (b) Green's Theorem: Let C be a positively oriented (counter-clockwise traversal of R), smooth, simple closed curve in the plane and let D be the region bounded by C. Then,

   $$\int_C P\ dx + Q\ dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

   (c) Notation:

   $$\int_C P\ dx + Q\ dy = \oint_C P\ dx + Q\ dy$$

   (d) Can use to turn line integrals into area or area into line integrals.

2. Examples:

   (a) Compute $\oint_C x^4\ dx + xy\ dy$ where C is the triangle from $(0,0)$ to $(1,0)$ to $(0,1)$ and back.

   (b) Exer 9 in text.

   (c) Area application: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Use parameterization $x = a\cot(t), y = b\sin(t)$. Can choose any P, Q such that $Q_x - P_y = 1$. Choose $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$. Then area gives πab.

3. Homework:

Chapter 17 Second-order differential equations