Calculus II Notes

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Fun Stuff

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4. Babylonian tablet: `https://www.maa.org/press/periodicals/convergence/the-best-known-old-babyl`
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Course Introduction

1. Syllabus highlights
   (a) Grades:
      i. Know the expectation / what you are getting into.
      ii. 15perc A (excellent), 35perc B (good), 35perc C (satisfactory),10perc D (passing), some F (failing)
      iii. Expect lower grades than you are used to. I was a student once upon a time. I know what it's like to give some effort in a class and still get an A/B. Night before study, good enough?
      iv. Turn in an exam / project. Did you do good work?
      v. Many will start off doing good / satisfactory work. Improve to something more. C is not the worst thing in existence. These letters say nothing of your capability.
   (b) What does good mean? Good means good. Good job! Excellent means you showed some flair.
   (c) Expect: More work, more expectation on good writing.
   (d) Math is a challenging subject. Not a natural thing to think or write in. It takes work and practice to be better. My goal is to train you to be better and give you ideas of where it can go.
   (e) Fact that you are here shows you are smart and capable. Your goal should be to improve.
   (f) Why do I do this? I do it out of respect for you. You are smart enough. I want you to gain something valuable here. I wouldn't do this job if I didn't think you were gaining something of value.
2. Grand scheme of things. Where does this class sit inside all of mathematics.
   (a) Basics: Algebra, arithmetic.
   (b) First steps: Geometry, functions. (us now)
   (c) Calculus: Math of change / infinity.
   (d) Linear algebra: Math of vectors. Anything with finite representation. Invention of computers fueled this one. Gateway to real math / applications.
   (e) Applied math. Any application you want. Physics, finance, marketing, material science, CFD, sports.
   (f) Abstract math. Create your own world of ideas. Number theory, analysis, algebra, topology, more.

.1 Day 1

1. What is Calculus? (Review of Calculus 1 ideas, let them complete in groups)
   (a) Write down a single sentence description.
      - Calculus is the mathematics of *change*.
   (b) Write down all important ideas. What are the top ideas?
      - Central ideas: Limit, derivative, integral
      - Limit
         i. continuity
         ii. limit laws
         iii. squeeze theorem
         iv. l'Hospital's rule
         v. IVT
      - Derivative:

   i. rules (power, product, quotient, chain, special functions (trig, exp, log, etc)
   ii. MVT
 - Integral:
   i. definite (area under curve)
   ii. indefinite (antiderivative)
   iii. Riemann sum
   iv. fundamental theorem of Calculus
   v. substitution

2. What is left for us? Continue the saga.
   (a) Integration techniques (more challenging than differentiation)
   (b) Generalize integration further (arc length and surface area)
   (c) New coordinate systems and representations of curves (polar coords and parametric eqns)
   (d) Encounter infinity
      i. Infinite areas (improper integral)
      ii. Infinite sum (series)
   (e) Power / Taylor series (replace functions with infinite degree polynomial)
   (f) Differential equations (a real purpose for integration)
      i. $y' = ky$, reverse differentiation with real meaning (y grows at a rate proportional to size)

.2 Motivating Calculus

1. Where does calculus sit within mathematics? Evolution of ideas:
   (a) Develop math tools:
      - Arithmetic (combining numbers)
      - Algebra (equations and solving for unknowns)
      - Functions (Machine that maps inputs to outputs, polynomials, logarithms, trigonometry, graphs)
      - Calculus (Solve paradoxes of processes, change, area, limit, infinity)
   (b) Branches following these tools:
      - Statistics and probability (chance and modeling uncertainty)
      - Linear algebra (data and high dimensional, discrete space)
      - Differential equations (translation of world into calculus, modeling)
      - Analaysis and abstract algebra (rigorous details and generalization of math)
      - Much more (number theory, computational, etc)
2. Two large application areas of calculus:
   (a) Optimization (will discuss soon)
   (b) Differential equations (mentioned above)
      - Links
   (c) More as well
3. The big picture of calculus (intuition here, details for the rest of the semester)
   (a) Area under a curve: area of a circle.
      - Consider a hard problem (which we already know). What is the area of a circle with radius R. Pick $R = 3$ for now.

- Lots of ways to chop it up to try (vertical rectangles, triangles, circular rings). Let's try circular rings with thickness dr (change in r).
- Take one ring at location r. Unroll the ring. Approximate by a rectangle.

$$\text{Ring area } = 2\pi r \; dr$$

- Stack all these rectangles vertically in the plane (plot $y = 2\pi r$).
- The smaller dr, the closer we are. Looks to approach the area of a triangle.

$$\text{Triangle area } = \frac{1}{2}bh = \frac{1}{2}32\pi 3 = \pi 3^2$$

- For general radius R, we get an area of πR^2.

(b) Process: Hard problem $\Rightarrow$ sum of many small values $\Rightarrow$ area under a graph.

- A bit of a paradox here. Rectangles disappear, infinitely many.

(c) Area under a curve: velocity / distance.

- Suppose a car speeds up then comes to a stop.
- Assume we know the velocity everywhere. Plot a velocity function that makes sense.
- $d = r \cdot t$, so we can compute the distance over small time intervals to approximate. The smaller the dt, the better the approximation.
- These are rectangles under the curve for v which we are summing.

(d) Area under a curve: general problem.

- Of course math is about pushing conversation beyond a single problem. We generalize to create a more powerful theory.
- Example: $y = x^2$. Find the area under the curve on $[0, 3]$ or in general $[0, x]$. Denote this area $A(x)$ also known as the *integral of* x^2.
- If we change the area slightly, call it dA, can approximate as

$$dA \approx x^2 dx \quad \Rightarrow \quad \frac{dA}{dx} \approx x^2$$

The smaller dx (and hence dA), the better the approximation.

- Derivative

$$\frac{dA}{dx} = f(x)$$

connects the function to the area under the curve (integral)

- This idea is the fundamental theorem of calculus. More later on.

(e)

Chapter 7: Techniques of integration

Project to watch visual calculus sequence for 3brown1blue: `https://www.youtube.com/playlist?list=PLZHQObOWTQDMsr9K-rj53DwVRMYO3t5Yr`

Give take home quiz on integration review (basic antiderivatives, chain rule, Riemann sum, Fundamental Theorem of Calculus).

We see that integration is much more challenging than differentiation. This chapter develops needed techinques.

.1 7.1 Integration by parts

1. Past methods for integration
   (a) Direct formula: $\frac{d}{dx}\sin(x) = -\cos(x)$, $\frac{d}{dx}x^n = nx^{n-1}$ and others
   (b) Sum formula: $\int (f(x) + g(x))\ dx$
   (c) Constant multiplication: $\int cf(x)\ dx$
   (d) Substitution: Reverse chain rule $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.
$$\int 2xe^{x^2}\ dx$$
   (e) What else can we reverse? Product rule? (Quotient rule is a disaster...) Try on own!
$$\int xe^x\ dx, \quad \int x\cos x\ dx, \quad \int x\sin x\ dx$$
   (f) New formula from the product rule:
$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$
$$\int \frac{d}{dx}(f(x)g(x))\ dx = \int (f'(x)g(x) + f(x)g'(x))\ dx$$
$$f(x)g(x) = \int f'(x)g(x)\ dx + \int f(x)g'(x)\ dx$$
$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$
2. **Theorem:** Integration by parts
$$\int f(x)g'(x)\ dx = f(x)g(x) - \int f'(x)g(x)\ dx$$
   (a) Example: Check it by differentiation to be sure.
$$\int xe^x\ dx$$
   (b) Steps:
      i. Assign f and g'.
      ii. Compute f' and g.
      iii. Substitute into formula
   (c) Comments:
      i. Difficulty: How to choose f and g'? Takes guess and check until intuition is developed.
         A. What if we chose $\int xe^x\ dx$ wrong?
         B. Always try to make it simpler.
      ii. Short hand notation as with substitution: Substitute $u = f(x), v = g(x)$, then via differentials $du = f'(x)dx, dv = g'(x)dx$. Then, the IBPs formula becomes
$$\int u\ dv = uv - \int v\ du$$
      Take care as this can be easier, though confusing.

3. Examples: A bit more challenging. Try on own.
   (a) $\int x^2 \sin(x)\ dx$ (twice IBPS)
   (b) $\int \ln(x)\ dx$ (hidden constant)
   (c) $\int \cos^2(x)\ dx$ (cos, cos, cyclic once Pythagorean id is used)
   (d) $\int \cos(x) e^x\ dx$ (cyclic reasoning)
4. Definite Integral Version: Fundamental Theorem of Calculus at work.
   (a) What if we have bounds / area under the curve? $\displaystyle\int_4^9 \frac{\ln(y)}{\sqrt{y}}\ dy$
   (b) Recall: FTOC says if $F'(x) = f(x)$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.
   (c) **Theorem:** Integration by parts for definite integrals.

$$\int_a^b f(x)g'(x)\ dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\ dx$$

   Recall: $f(x)\Big|_a^b = f(b) - f(a)$.
   (d) Create your own integration by parts problem.

.2 7.2 Trigonometric integrals

1. In the next section, we see a new substitution which transforms

$$\int x^2\sqrt{1-x^2}\ dx = \int \sin^2(\theta)\cos^2(\theta)\ d\theta.$$

via the unit circle and such trigonometric relationships. In light of this, we need techinques for integrating powers/products of trig functions.
   (a) *Everything* in this entire section is either *substitution* or *trig identites.*
   (b) Notational confusion:

$$\sin^n(x) = (\sin(x))^n \neq \sin(x^n), \qquad \sin^{-1}(x) = \arcsin(x) \neq 1/\sin(x) = \sec(x)$$

   I will always denote inverse sine as $\arcsin(x)$ to avoid confusion.
   (c) Basic integration formulas:

$$\int \cos(x)\ dx = \sin(x) + C, \quad \int \sin(x)\ dx = -\cos(x) + C$$

2. Combinations of sine and cosine: $\displaystyle\int sin^m(x)\cos^n(x)\ dx$ for m, n positive integers.
   (a) Example: $\displaystyle\int \sin^4(x)\cos(x)\ dx$, substitute $u = \sin(x)$.
   (b) Example: $\displaystyle\int \sin(x)\cos^4(x)\ dx$, substitute $u = \cos(x)$.
   (c) Example: $\displaystyle\int \sin^3(x)\cos^4(x)\ dx$, rewrite $\sin^2(x) = 1 - \cos^2(x)$, then substitute $u = \cos(x)$.

$$\int \sin^3(x)\cos^4(x)\ dx = \int \sin(x)\sin^2(x)\cos^4(x)\ dx = \int \sin(x)(1-\cos^2(x))\cos^4(x)\ dx$$

(d) Try on own:

   i. Example: $\int \cos^5(x)\ dx$, rewrite via Pythagoras and substitute $u = \sin(x)$.

   ii. Example: $\int \cos^2(x)\ dx$, integrate by parts (did this last time) OR use half angle formula.

(e) Two big formulas in trigonometry: Pythagoras ($\sin^2(x) + \cos^2(x) = 1$) which everyone knows, same as right triangle. Also, sum formula ($\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$). From the latter we get double/half angle formulas, take $u = v = x$.

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

Then, even powers of $\cos(x)$ (and $\sin(x)$) can be rewritten.

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)); \qquad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

(f) Examples:

   i. $\int \cos^2(x)\ dx$, use double angle formula, small substitution needed. Different form than the IBPs result, yet equivalent. Verify the trig identity.

$$\int \cos^2(x)\ dx = \int \frac{1}{2}(1 + \cos(2x))\ dx$$

   ii. $\int \sin^4(x)\ dx$, use double angle formula twice, small substitution needed.

(g) In summary, only two ideas: subsitution and trig identities.

   i. In conclusion, **EVERY** $\int \sin^m(x)\cos^n(x)\ dx$ can be integrated for m, n integers positive or negative.

   ii. Try substutution first (look ahead in mind, all possible if necessary).

   iii. Use trig identity (Pythagoras or half angle formulas, **MEMORIZE** both).

   iv. What if n, m are negative number or radical? Same ideas may work.

3. Sine and cosine are connected since their derivatives relate.

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Likewise for tangent and secant, we should have an analogous story. Why? Derivatives match and Pythagoras connects again.

$$\frac{d}{dx}\tan(x) = \sec^2(x), \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x),$$

$$\sin^2(x) + \cos^2(x) = 1 \quad \Rightarrow \quad \tan^2(x) + 1 = \sec^2(x)$$

4. Basic integratation formulas:

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \text{via subsituting } u = \cos(x)$$

$$\int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C \qquad \text{via subsituting } u = \sec(x) + \tan(x)$$

Mercator map application Strang for secant integral origins.

- Strang: `https://ocw.mit.edu/ans7870/resources/Strang/Edited/Calculus/Calculus.pdf`
- WIKI: `https://en.wikipedia.org/wiki/Mercator_projection`
- Paper: `https://www.maa.org/sites/default/files/pdf/cms_upload/0025570x15087.di021115.02p0115x.pdf`

5. Examples: For $\int \tan^m(x)\sec^n(x)\ dx$, we have same ideas as above.
   (a) $\int \tan^2(x)\ dx = \int (\sec^2(x) - 1)\ dx = \tan(x) - x + C$
   (b) $\int \tan^3(x)\ dx = \int \tan(x)(\sec^2(x) - 1)\ dx$ then separate and substitute $(u = \tan(x))$ / formula.
6. Examples: More challenging, try on own first.
   (a) $\int \sec^3(x)\ dx = \int \sec(x)\sec^2(x)\ dx = \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\ dx$, then Pythagoras and cyclic integral.
   (b) $\int \tan^3(x)\sec^4(x)\ dx$, assign $u = \tan(x)$ OR $u = \sec(x)$

   $$\int \tan^3(x)\sec^4(x)\ dx = \frac{1}{4}\tan^4(x) + \frac{1}{6}\tan^6(x) + C$$

   $$\int \tan^3(x)\sec^4(x)\ dx = \frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x) + C$$

   Same via Pythagorean theorem.
7. What if the trig functions argument is not x?
   (a) Example: $\int \sin(2x)\cos(2x)\ dx$. Substitute $u = 2x$ then carry on.
   (b) Example: $\int \sin(2x)\cos(5x)\ dx$. Of course there is another identity to use from the sum formula (product to sum formula).

   $$\sin(u)\cos(v) = \frac{1}{2}\left[\sin(u - v) + \sin(u + v)\right]$$

   Then, $\int \sin(2x)\cos(5x)\ dx = \int \frac{1}{2}\left[\sin(-3x) + \sin(8x)\right]\ dx$ separate, then substitute. Now you've seen it, later **I won't test it**.

.3 7.3 Trigonometric substitution

1. Our motivation for last time was to transform integrals of the form

$$\int x^2\sqrt{1 - x^2}\ dx = \int \sin^2(\theta)\cos^2(\theta)\ d\theta$$

into trig integrals. Why? To leverage trig identities and symmetry. How? Substitute $x = \sin(\theta)(dx = \cos(\theta)\ d\theta$ then Pythagoras is there again.

$$\int x^2\sqrt{1 - x^2}\ dx = \int \sin^2(\theta)\sqrt{1 - \sin^2(\theta)}\cos(\theta)\ d\theta = \int \sin^2(\theta)\cos^2(\theta)\ d\theta$$

2. A basic example:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\cos(\theta)}\cos(\theta)\,d\theta = \int 1\,d\theta = \theta + C = \arcsin(x) + C$$

Here, $x = \sin(\theta), dx = \cos(\theta)\,d\theta, \theta = \arcsin(x)$. This same formula was found via function inverses and log differentiation in Calculus 1. An obscure formula is easily derived now.

3. Trigonometric substitution overview (backwardsish from regular substitution)

   (a) Remove the radical via trig substitution then Pythagoras

      i. $\sqrt{1-x^2}$, $\quad x = \sin(\theta)$, $\quad \sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)}$

      ii. $\sqrt{1+x^2}$, $\quad x = \tan(\theta)$, $\quad \sqrt{1+x^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)}$

      iii. $\sqrt{x^2-1}$, $\quad x = \sec(\theta)$, $\quad \sqrt{x^2-1} = \sqrt{\sec^2(\theta)-1} = \sqrt{\tan^2(\theta)}$

      iv. All boils down to

      $$\sin^2(\theta) + \cos^2(\theta) = 1, \qquad \tan^2(\theta) + 1 = \sec^2(\theta).$$

      v. Does $\sqrt{x^2} = x$ always? Not for x negative. So, $\sqrt{1-x^2} \to \sqrt{\cos^2\theta} = |\cos\theta|$

   (b) Domain issues are involved (required for you to write each time!)

      i. $\theta = \arcsin(x)$ requires domain restriction $-\pi/2 \le \theta \le \pi/2$ (draw graph, unit circle).

      ii. $\theta = \arctan(x)$ requires domain restriction $-\pi/2 \le \theta \le \pi/2$

      iii. $\theta = \text{arcsec}(x)$ requires domain restriction $0 \le \theta \le \pi/2$ and $\pi \le \theta \le 3\pi/2$

   (c) Insert absolute value and domain restriction into the substitution for above problem. Why does $|\cos(\theta)| = \cos(\theta)$ here so we need not worry about the absolute value?

4. Examples: A bit more sophisticated. Only transfer to trig integral. They will handle in groupwork.

   (a) Constant not 1 inside radical. Factor then subsitute $\frac{x}{2} = \sin(\theta), \frac{1}{2}dx = \cos(\theta)d\theta$.

   $$\int \sqrt{4-x^2}\,dx = 2\int \sqrt{1-\left(\frac{x}{2}\right)^2}\,dx = 2\int \sqrt{1-\sin^2(\theta)}2\cos(\theta)\,d\theta = 4\int \cos^2(\theta)\,d\theta$$

   Alternatively, substitute $x = 2\sin(\theta)$ from the get go to get the same.

   $$\int \sqrt{2^2-x^2}\,dx = \int \sqrt{2^2-2^2\sin^2(\theta)}2\cos(\theta)\,d\theta = 4\int \cos^2(\theta)\,d\theta$$

   (b) Reference triangle required for back-substitution ($x = \tan(\theta), dx = \sec^2(\theta)\,d\theta$).

   $$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|\sqrt{1+x^2} + x| + C$$

   (c) Complete the square, a useful technique.

   $$\int \frac{1}{\sqrt{2x-x^2}}\,dx = ?$$

   This isn't of the form we need. Any quadratic in standard form can be rewritten in vertex form by competing the square.

   $$ax^2 + bx + c = a(x-h)^2 + k$$

   $$2x - x^2 = -x^2 + 2x = -(x^2-2x) = -(x^2-2x+1) + 1 = -(x-1)^2 + 1 = 1 - (x-1)^2$$

   Show Wikipedia visual: `https://en.wikipedia.org/wiki/Completing_the_square`
   Give random example and see who can do it fastest. Show can check easily.

Then, use this form instead with our integral and substitute $u = x - 1, du = dx$.

$$\int \frac{1}{\sqrt{2x - x^2}}\, dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\, dx = \int \frac{1}{\sqrt{1-u^2}}\, du$$

Which trig substitution is this now? $u = \sin(\theta)$.

5. Overview of trigonometric substitution

   (a) Always try u substitution first. Sometimes easier ways.

   $$\int 2x\sqrt{x^2+1}\, dx$$

   (b) Complete the square if needed.

   (c) Assign a trig substitution (along with the domain of θ).

   (d) Match the constant for substitution if not 1.

   (e) Integrate via trig integral ideas of sectin 7.2.

   (f) Replace θ by x, may need reference triangle here.

6. Groupwork challenge (expect something of this caliber on the exam).

   (a) Transfer to trigonometric integral, use 7.2. Simplish solution hits to substitute from the get-go ($u = x^2 + 1$). Much easier.

   $$\int \frac{x^3}{(x^2+1)^{3/2}}\, dx = \frac{1}{\sqrt{1+x^2}} + \sqrt{1+x^2} + C$$

   (b) Half angle formula needed. Better yet, area would have made life simple. Just the upper hemisphere of circle of radius 3.

   $$\int_0^3 \sqrt{9-x^2}\, dx$$

   (c) Complete the square.

   $$\int \frac{x}{\sqrt{3-2x-x^2}}\, dx = -\sqrt{3-2x-x^2} - \sin^{-1}(\frac{x+1}{2}) + C$$

   (d) Complete the square.

   $$\int \frac{1}{x^2+10x+16}\, dx$$

   This last one hints at a better way. How to reverse fraction addition? This is partial fraction decomposition of the next section.

   $$\frac{1}{x^2+10x+16} = \frac{1}{(x+8)(x+2)} = \frac{?}{x+8} + \frac{?}{x+2}$$

.4 7.4 Integration of Rational Functions by Partial Fractions

1. The rational of rational function comes from ratio...

   $$f(x) = \frac{p(x)}{q(x)}$$

   is a rational function where $p(x)$ and $q(x)$ are polynomials. Our goal is to have a method integrate any rational function.

2. Integral of proper rational function (degree of top is less than degree of bottom).

   (a) Linear denominator

      i. Already know:
      $$\int \frac{1}{(ax+b)}\,dx = \frac{1}{a}\ln|ax+b| + C$$
      via substitution. So linear function division is doable.

      ii. Production of distinct linear factors. How to reverse fraction to reduce to previous case? It is reasonable to think that $\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{A}{2} + \frac{B}{3}$ giving $1 = 3A + 2B$. Use this idea.
      $$\frac{3x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$
      How to find A, B? Two ways.

         A. Add fractions and compare coefficients.
         $$\frac{3x+1}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

         B. Add fractions, clear fractions on both sides, choose nice x values.
         $$3x+1 = A(x+3) + B(x-1), \quad x = -3, 1$$

         Either way $A = 1, B = 2$ and we are set to integrate.
      $$\int \frac{3x+1}{(x-1)(x+3)}\,dx = \int \left(\frac{1}{x-1} + \frac{2}{x+3}\right)dx = \ln|x-1| + 2\ln|x+3| + C$$

      iii. Example: Try on own, no need to integrate. Can always add fractions to check.
      $$\frac{4x-2}{x^2-1} = \frac{1}{x-1} + \frac{3}{x+1}$$
      More challenging examples. Divide and conquer.
      $$\frac{15x-20}{(2x-1)(x+2)(x-3)} = \frac{2}{2x-1} - \frac{2}{x+2} + \frac{1}{x-3}$$
      and
      $$\frac{4x^2-6x-22}{(x^2-1)(x-5)} = \frac{3}{x-1} - \frac{1}{x+1} + \frac{2}{x-5}$$

      iv. Repeating linear factors with multiplicity
      $$\frac{1}{(x-1)^2}, \quad \frac{x+1}{(x-1)^2} = \frac{1}{x-1} + \frac{2}{(x-1)^2}, \quad \frac{6x+2}{(x-1)^2(x+3)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+3}$$

      v. Factoring is hard: Factor $x^3 + 3x^2 + x - 5$ knowing that $x = 1$ is a zero.

   (b) Quadratic denominator

      i. It is not always possible to have linear factors (and avoid complex numbers).

         A. Reducible quadratic factor can be factored: $x^2 + 4x + 3 = (x+1)(x+3)$. Linear terms.

         B. Irreducible quadratic factor cannot be factored: $x^2 + 1 = ?$ (How to tell? Check the discriminant $b^2 - 4ac < 0$ which indicates complex zeros). This is where arctangent lives from last time.

      ii. Is the denominator irreducible? Check. Need to complete the square here and substitute $u = x + 1$.
      $$\int \frac{2x+4}{x^2+2x+2}\,dx = \int \frac{2x+4}{(x+1)^2+1}\,dx = \int \frac{2(u-1)+4}{u^2+1}\,dx = \int \frac{2u}{u^2+1}\,du + \int \frac{2}{u^2+1}\,du$$

iii. Example: $\int \frac{10}{(x-1)(x^2+9)}$

(c) Tips

i. Often split into 2 parts, ln and arctangent.

ii. Match the u substitution for the ln part by taking derivative of the denominator

iii. Basic formula for the tangent part

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{1}{a}x)$$

iv. Complete the square if necessary

(d) Do we need to consider irreducible cubics? No.

i. **The fundamental theorem of ALGEBRA**
Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots.
`https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra`

ii. This implies every polynomial can be factored into a product of linear and quadratic factors (complex zeros come in conjugate pairs).

3. Integration of improper rational functions: If the degree of the numerator is the same or exceeds that of the denomiantor, PFD fails. How to handle improper rational functions?

(a) Think regular improper fractions: $\frac{12}{5}, \frac{63}{4}$.

(b) Long division to polynomial + proper rational function

i. Linear factors

ii. Irreducible quadratic factors

iii. Factors may repreat.

Try

$$\frac{x^2}{x^2-1}$$

4. Create your own examples (don't compute the PFD coefficients)

5. Special cases: Radicals converged to rationals via substitution.

$$\int \frac{\sqrt{x+1}}{x}\,dx = 2\sqrt{x+1} + \ln(1-\sqrt{x+1}) - \ln(\sqrt{x+1}+1) + C$$

What about

$$\int \frac{\sqrt[4]{x+1}}{x}\,dx, \quad \int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$$

.5 7.5 Strategy for integration

Tip

1. Simplify if possible

2. Direct formula

3. Look for direct substitution

4. Try techniques from 7.1-7.4. What are the giveaways of each technique? Make a list.

5. If no techniques are obvious, think about a substitution to transform.

Group work

.6 7.7 Approximate integration

Show idea, but do not test.

1. Motivation: Necessity (no technique to compute exactly) vs practicality (coding software) vs theory / history (`https://en.wikipedia.org/wiki/Numerical_integration#History`)
2. Techniques:
   (a) Riemann sum (midpoint rule / pw constant approximation)
   (b) Trapezoidal rule (tangent line approximation) `https://www.desmos.com/calculator/d9rmt4wfoa`
   (c) Simpsons rule (quadratic approximation) `https://www.desmos.com/calculator/cdgj6pgeni`
   (d) Script to show comparison (main considerations are accuracy vs calculation)
   (e) Idea leads into Taylor series

.7 7.8 Improper integral

1. Here we extend integration to unbounded regions. Reasons:
   (a) Applications
      - Probability: Bell curve and normal distributions (`https://en.wikipedia.org/wiki/Normal_distribution`)
      - Physics: Escape velocity (`https://en.wikipedia.org/wiki/Escape_velocity`)

   (b) Mathematics and theory
      - Study the nuances of infinity
      - Find the limitations of the Riemann integral (`https://en.wikipedia.org/wiki/Lebesgue_integration#Intuitive_interpretation`, `en.wikipedia.org/wiki/Henri_Lebesgue#Lebesgue's_theory_of_integration`) as discovered in 1900s thru studying probability / Fourier series / transform.
      - Fourier: `https://en.wikipedia.org/wiki/Fourier_series`
      - Orbitness: `https://www.youtube.com/watch?v=QVuU2YCwHjw`
2. Back to infinity. What if our area of integration is unbounded? For example, two possibilities here. Draw graph.
$$\int_1^\infty \frac{1}{x}, \quad \int_0^1 \frac{1}{x}$$
   (a) Infinite interval
   (b) Discontinuous integrand
   (c) Intuition says such a region must be infinite, though that isn't always the case.
3. Case 1: Improper integral with infinite interval
$$\int_0^\infty f(x)\ dx, \quad \int_{-\infty}^0 f(x)\ dx, \quad \int_{-\infty}^\infty f(x)\ dx$$
   (a) Is it always infinity?
   (b) If $f(x) > 0$ is it always infinity?
   (c) Try several examples by plug in ∞. Disclaimer: sloppy mathematics here. Not indeterminant forms here are cringeworthy.
$$\int_0^\infty e^x\ dx, \quad \int_0^\infty e^{-x}\ dx, \quad \int_0^\infty x\ dx, \quad \int_{-\infty}^\infty x\ dx$$

4. Better to think of this as a limit of a definite integral (which we already understand well).
   (a) Example:
   $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{x}\Big|_1^t\right) = 1$$
   (b) Definition
      i.
      $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$
      $$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$
      ii. If the limit exists, the improper integral is called convergent. If the limit doesn't exist, it's called divergent
      iii. If both
      $$\int_{-\infty}^b f(x)\,dx, \quad \int_a^\infty f(x)\,dx$$
      are convergent, then
      $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$
      where a can be any number you choose. Note the subtly of a limit law here. Need both limits to exist to pass the limit.
      $$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$
      only if BOTH limits exist (are finite).

5. Examples: Draw the graph. Let them vote convergent/divergent beforehand.
$$\int_1^\infty \frac{1}{x^3}\,dx, \quad \int_1^\infty \frac{1}{x}\,dx, \quad \int_1^\infty \sqrt{x}\,dx, \quad \int_{-\infty}^0 xe^x\,dx, \quad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$$
For each, hint at comparison theorem ideas using intuition of $\int_1^\infty \frac{1}{x^2}\,dx$.

6. Lecture on infinitesimal
   - The ghost step
   - Cavalieri's principle
   - The wrong case
   - The difference between 0, > 0, limit equal 0
   - The theory of infinitesimal calculus

7. Tips
   (a) Must use limit notation (don't plug in ∞ as we already know from experience with l'Hospitals rule..) Silly example: $\lim_{x\to\infty}(x^2 - x)$.
   (b) $-\infty$ to ∞ must be split into two parts by definition.
   (c) Convergent or divergent is most important discussion from a math view. If convergent, resulting constant mostly used for application.

8. Important case: Positive and decreasing is not enough. Speed of decay is key.

$$\int_1^\infty \frac{1}{x}\,dx = \infty$$

compared to $\int_1^\infty \frac{1}{x^2}\,dx = 1$. In general we can look at all powers of x to think about rates of decay.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^p}\,dx = \lim_{a\to\infty} \frac{-p}{x^{p-1}}$$

   (a) Convergent if $p > 1$
   (b) Divergent if $p < 1$
   (c) Divergent if $p = 1$ (interesting barrier for rate of decay)

9. How to understand this using area under a curve?
   - $0 \cdot \infty$ is indeterminant
   - The ghost step
   - Cavalieri's principle
   - The difference between infinitely small and zero

10. The comparison theorem: If cannot easily integrate, can still sometimes tell if convergent / divergent.

    Suppose f, g are continuous functions and $f(x) \geq g(x) \geq 0$ for $x > a$ (draw a picture), then
    - If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent
    - If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent

    There was a Calculus 1 analogy of this for bounded intervals. Mostly we are comparing rates of decay.

11. Examples: Use the comparison theorem to say divergent or convergent. First use intuition, then bound in the correct direction. Try

$$\int_2^\infty \frac{1+e^{-x}}{x}\,dx, \quad \int_\pi^\infty \frac{1}{\sqrt[3]{x^2+1}}\,dx, \quad \int_1^\infty \frac{e^x}{e^{2x}+3}\,dx$$

12. Case 2: Discontinuous (integrand) integrand

$$\int_0^1 \frac{1}{x}\,dx$$

    Draw the graph, how does this compare to infinite interval case? Same, just sideways. How to handle carefully? Cannot compute and substitute. Try. Fail (sorta). Limit needed again. Expect this to be infinite as before by comparing the inverse function.

    (a) Definition (let them write down the careful version on own first). Note the necessity of limit direction here.
        i. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

        ii. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

iii. If f is discontinuous at c and both

$$\int_a^c f(x)\,dx \quad \text{and} \quad \int_c^b f(x)\,dx \quad \text{are convergent}$$

then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

is also convergent.

13. Example: Where is the discontinuity? Does your intuition say converge or diverge? Vote. The inverse of $\frac{1}{\sqrt{x-2}}$ is $\frac{1}{x^2}+2$ so expect convergent.

$$\int_2^5 \frac{1}{\sqrt{x-2}} = 2\sqrt{3}$$

Relate to $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ and reverse the above theorem.

14.
$$\int_{-1}^1 \frac{1}{x}\,dx, \quad \int_{-\infty}^\infty \frac{1}{x}\,dx = \qquad 0 \quad \text{or} \quad \text{divergent?}$$

Key is the definition we choose to accept. Difference between

$$\lim_{t\to\infty}\int_{-t}^t \frac{1}{x}\,dx = \lim_{t\to\infty}\int_{-t}^0 \frac{1}{x}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{x}\,dx?$$

The left hand side is the Cauchy principle value as seen in certain generalizations.

15. Tips
   (a) Use limit notation
   (b) Split the terms so that each term is well defined
   (c) Be extremely sensitive to $\dfrac{1}{x^p}$
   (d) Comparison theorem is the first choice if only care about converge / diverge.

Chapter 8 Further application of integration

Here we see some uses of integration which aren't directly area under the curve.

- Already did volumes of revolution. (Section 6.2)
  1. Archimedes and the volume of the sphere.
     `http://math.gmu.edu/~rsachs/math400/History%20Method%20of%20Archimedes%20Gould.pdf`
  2. Cavaleri and volume of sphere
     `http://www.matematicasvisuales.com/english/html/history/cavalieri/cavalierisphere.html`
- Arc length: Sum of line segment (distance)
- Surface area (of revolution): Sum of cone section (circle)
- Physics and engineering (pressure, force, center of mass): Sum of pressure, low dim moments) `https://en.wikipedia.org/wiki/Work_(physics)`

- Economics (surplus): Sum of demand https://en.wikipedia.org/wiki/Economic_surplus
- Biology (Blood flow): Sum cylinder (circle) https://en.wikipedia.org/wiki/Cardiac_output
- Probability: Sum of individual probabilities

Key for each is to compute via summation of continuous values

.1 8.1 Arc length

1. Arc length: Find the arc length of $y = x^2$ between $(0,0)$ and $(2,4)$.
   (a) Draw picture, label $n+1$ points $P_0, \ldots, P_n$ spaced equally in x by Δx.
   (b) Approximate with line segment lengths, take limit as number of segments goes to infinity. This familiar process results in a definite integral.
   (c) Pythagoras gives us distance formula.
   $$L = \lim_{n\to\infty} \sum_{i=1}^{n} \left|\overline{P_{i-1}P_i}\right| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta x^2 + \Delta y^2}$$
   $$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x = \int_0^2 \sqrt{1 + \left(f'(x)\right)^2}\ dx$$
   Trig substitution naturally appears due to Pythagoras.
   (d) Theorem: The arc length of $y = f(x)$ on interval $[a, b]$ is given by
   $$L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\ dx$$
   This is the only new thing in this section, but the idea that we can bend Riemann sums to do new things shows the reach of calculus.
2. Example: Check formula with something simpler. Find the arc length of $y = 3x + 1$ from $x = 0$ to $x = 4$.
   $$L = \int_0^4 \sqrt{1 + 3^2}\ dx = 4\sqrt{10}$$
   Graph it. Pythagoras agrees.
3. Examples: Divide and conquer.
   (a) Finish what we started... Find the arc length of $y = x^2$ from $(0,0)$ to $(2,4)$.
   (b) Find the length of a quarter of the perimeter of the unit circle.
4. Integrals in y: For $x = g(y)$, we have an almost identical formula:
   $$L = \lim_{n\to\infty} \sum \sqrt{\Delta x^2 + \Delta y^2} \lim_{n\to\infty} \sum \sqrt{\frac{\Delta x}{\Delta y}^2 + 1}\ \Delta y = \int_a^b \sqrt{(g'(y))^2 + 1}\ dy$$
   (a) Repeat above example for $y = f(x) = x^2$ but instead $x = g(y) = \sqrt{y}$ from $(0,0)$ to $(2,4)$.
5. Parametric curves: This idea and most calculus ideas extend nicely to *parametric curves* which show up naturally in places like physics among others. This is usually a better way to describe geometry and doesn't restrict us to curves which are functions. Though, they are more difficult to conceptualize and often compute.
   (a) Examples:

i. Familiar: Unit circle $x^2+y^2=1$ is given by $x=\cos(t), y=\sin(t)$ for parameter $t\geq 0$. Just as in precalculus.

ii. Silly yet new: Parabola $y=x^2$ is given by $x=t, y=t^2$ for parameter $-\infty<t<\infty$.

iii. Desmos demonstration.
`https://www.desmos.com/calculator/qepemyowpv`

(b) Arc length formula transforms. For $x=g(t), y=h(t)$ from $t=a$ to $t=b$,

$$L=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\Delta x^2+\Delta y^2}=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\ \Delta t=\int_a^b\sqrt{g'(t)+h'(t)}\ dt$$

(c) Can now do the full unit circle even though not a function. One revolution is given by $x=\cos(t), y=\sin(t)$ for parameter $0\leq t\leq 2\pi$.

$$L=\int_0^{2\pi}\sqrt{\sin^2(t)+\cos^2(t)}\ dt=2\pi$$

More on this in chapter 10.

6. This section is just one formula (though three versions). Challenges are

   (a) Setup the integral by drawing a graph (if possible).

   (b) Compute integral using chapter 7 techniques.

.2 8.2 Area of a surface of revolution

1. Recall, solids of revolution have a nice formula for volume (draw picture for simple example).

$$V=\int_a^b A(x)\ dx=\int_a^b \pi r^2\ dx=\int_a^b \pi f(x)^2\ dx$$

where $A(x)$ is the area of a cross section (circle). This is summation of circular areas over an interval. Surface area of revolusions is the same idea in lower dimension (sum circle circumference to get surface area).

2. Surface of revolution:

   (a) Formula for $y=f(x)$ on $[a,b]$ rotated about the $x-$ axis.

$$S=\int_a^b 2\pi r L\ dx=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\ dx$$

   where

   - r: rotation radius
   - L: arc length

   (b) Similar formula if rotated about the $y-$axis. Consider $x=g(y)$ for y in interval $[c,d]$.

$$S=\int_c^d 2\pi r L\ dy=\int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\ dy$$

3. This is the same idea as volume of revolution (and Reimann sum for that matter). Read the careful proof on own in text. Want proof on quiz / exam?

4. Just another formula, intuition makes it easy to remember. Examples:

   (a) Find the surface area generated by by $y=x^2$ from $(1,1)$ to $(2,4)$ rotated about the y axis (Solution: $\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$)

(b) You try: What's the surface area of a sphere with radius 1? Divide and conquer x direction and y direction.

5. Tips:

   (a) Sometimes have choice of dx or dy. Think which is easier to integrate. Write both down if needed.

   (b) Make sure formula is consistent with x or y

   (c) Implicit formula, u - substitution, perfect square

6. Gabriel's horn: Evangelista Torricelli, (1608-1647), Italian physicist and mathematician (Carrol paper). Pre-Newton, Torricelli brought about a troubling paradox on the boundary between the finite and infinite.

   (a) Rotate curve $y = \frac{1}{x}, x \geq 1$ about the x-axis. Find the volume / surface area of the solid.

   (b) Volume:
   $$V = \int_1^\infty A(x)\, dx = \int_1^\infty \frac{\pi}{x^2}\, dx = \pi$$

   (c) Surface Area:
   $$A = \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}}\, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4}+1}{x^3}\, dx$$
   which diverges by the comparison test $\frac{\sqrt{x^4}+1}{x^3} \geq \frac{1}{x}$ on $[1, \infty)$.

   (d) Paper: `https://www.jstor.org/stable/pdf/10.4169/math.mag.86.4.239.pdf?casa_token=JVd8NiDg6ugAAAAA:wbjOXan7IUUVJVNXJxg-PVfgG4q0xQgwtMfEkD3QLTe5D9T4mdJqtKQLe8OEsOkvjEmWR`

   So, GH has finite volume π, but infinite surface area. Rephrase, can fill the horn with less than 4 gallons of paint, but painting the outside is impossible. Can show the converse is false, no possible shape with infinite volume and finite surface area.

.3 8.3 Applications to physics and engineering

.4 8.4 Applications to economics and biology

.5 8.4 Probability

ADD THIS

Chapter 10 Parametric equations and polar coordinates

So far we have considered calculus in 2 dimensions for

- functions of x: $y = f(x)$
- implicit functions of x: $f(x, y) = 0$ where $y = y(x)$ (i.e. $x^2 + y^2 = 1$, implicit differentiation).

Functions of x often fall short. Physics has no shortage of examples:

- Newton planetary motion: `https://en.wikipedia.org/wiki/Newton%27s_theorem_of_revolving_orbits`

Here we consider two new ways to describe planar curves:

- Parametric equations: Tracing a curve via new parameter t (often denotes time)
- Polar coordinates: Curves described by an angle θ and radius r. Rotation.

.1 10.1 Curves defined by parametric equations

1. Motivation: Earth rotating about the sun following an ellipse (sun at one focus). This is given by Kepler's law of planetary motion, Newton later proved it. Here x is not the driving information. Instead, time t (or distance traveled) makes sense. $t = 365$ gives one full rotation. Variables $x = x(t)$ and $y = y(t)$ tracing this curve should each be variables of t. `https://en.wikipedia.org/wiki/Kepler%27s_laws_of_planetary_motion`

2. Definition: For parameter t,
$$x = f(t), \quad y = g(t)$$
are parametric equations. As t varies, the point $(x, y) = (f(t), g(t))$ traces a parametric curve. Note:
   - Parameter t can be restricted (ie $a \leq t \leq b$).
   - Parametric curves may not be functions of x or y.
   - Often simply denote $x = x(t), y = y(t)$.

3. Examples:
   - $x(t) = t - 1$, $y(t) = 2t + 4$, $-3 \leq t \leq 2$. Make table for t values. Eliminate parameter t. Not always possible to eliminate t.
   - Try on own: $x(t) = t^2 - 3$, $y(t) = 2t + 1$, $-2 \leq t \leq 3$
   - Try on own: $x(t) = 4\cos(t)$, $y = 4\sin(t)$, $0 \leq t \leq 2\pi$. Need Pythagoras to eliminate t in this case.

4. Notes:
   - There are often many parametrizations of the same curve. Show off in desmos.
     - $x(t) = 4\cos(t)$, $y(t) = 4\sin(t)$, $0 \leq t \leq 2\pi$
     - $x(t) = 4\cos(2t)$, $y(t) = 4\sin(2t)$, $0 \leq t \leq \pi$ (different speed)
     - $x(t) = 4\sin(t)$, $y(t) = 4\cos(t)$, $0 \leq t \leq 2\pi$ (different start / end point)
   - Can always parametrize if given an equation though the way the curve is traced may differ.
     - $y = x^2 + 1$: $x(t) = t, y(t) = t^2 + 1$
     - $y = x^2 + 1$: $x(t) = 2 + t, y(t) = t^2 + 4t + 5$

5. Cycloid: Famous example from geometry. Curve traced by a point on a circle rolling along the $x-$ axis.
   - `https://en.wikipedia.org/wiki/Cycloid`
   - Let the point P start at the origin (rotation angle $\theta = 0$ and the circle have radius r. As θ increases the circle rolls along the positive $x-$ axis. Can show
$$x = r(\theta - \sin(\theta), \quad y = r(1 - cos(\theta))$$
   - It is possible to eliminate parameter θ and write as a single Cartesian equation, though not pretty.

6. History and the brachistochrone problem:
   - `https://en.wikipedia.org/wiki/Brachistochrone_curve`
   - `https://www.youtube.com/watch?v=Cld0p3a43fU`

7. Homework: 1-28, 37-38

.2 10.2 Calculus with parametric curves

Here we extend calculus ideas (tangent line, area, arc length, etc) to parametric curves.

Working example: Cycloid for the unit circle.

$$x(\theta) = \theta - \sin(\theta), \quad y(\theta) = 1 - \cos(\theta)$$

1. Tangent line: Find the tangent line to the cycloid when $\theta = \frac{\pi}{2}$.

   (a) Need $\frac{dy}{dx}$ for a general parametric curve $x = x(t), y = y(t)$. Noting that y is also a function of x, the chain rule is the key.

   $$\frac{dy}{dt} = \frac{d}{dt}y(x) = \frac{dy}{dx}\frac{dx}{dt}$$

   rearranges as

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

   where we need $\frac{dx}{dt} \neq 0$ (no vertical tangents).

   (b) Graph and find tangent line. Should have positive slope.

   (c) Note, to find horizontal tangents we need $\frac{dy}{dx} = 0$ leading to only $\frac{dy}{dt} = 0$. Horizontal tangents for the cycloid are at $\pi, 3\pi$, etc as expected.

   (d) Need a limiting idea to examine vertical tangents.

   $$\lim_{\theta \to 2\pi^+} \frac{dy}{dx} = \infty$$

   (e) High order derivatives are also easy.

   $$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

2. Area: Find the area under one arch of the cycloid.

   (a) Again, the chain rule is needed (substitution for integration). For $y = y(x) \geq 0$ on $[a, b]$, substitute $x = x(t)$ with $\alpha \leq t \leq \beta$ giving $dx = x'(t)dt$

   $$A = \int_a^b y(x)\ dx = \int_\alpha^\beta y(t)x'(t)\ dt$$

   (b) Back to the cyloid, the area of one arch is then

   $$A = \int_0^{2\pi} y(t)x'(t)\ dt = \int_0^{2\pi} (1 - \cos(t)(1 - \cos(t))\ dt = 3\pi$$

   via the half angle formula for cosine.

   (c) Can show for the cycloid of a circle of radius r, the area under one arch is 3 times the area of the circle.

   $$A = 3\pi r^2$$

3. Arc length

   (a) Already have the formula from chapter 8.

   $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\ dt$$

(b) Show the length of the unit cycloid arch is 8. In general for a circle of radius r, the cycloid arch is $8r$.

4. Skip surface area.
5. Another example: $x = t^2, y = t^3 - 3t$.
   (a) Find the two tangent lines at point $(3, 0)$.
   (b) Find vertical and horizontal tangents.
   (c) Find where concave up / down.
   (d) Sketch the curve.
6. Homework: 1-8, 11-19, 25-36, 41-44

.3 10.3 Polar coordinates

In the last sections we extend the idea of function to parametric curves. Here we look at an alternate coordinate system designed to handle rotation / circular motion.

1. Idea of polar coordinates:
   (a) For many situations, distance r and direction (θ) are more natural than horizontal (x) and vertical (y). Ie. Navigation (ship, airplane), astronomy (planets spinning and rotating).
   (b) Cartesian coordinates (x, y) vs polar coordinates (r, θ).
   - Draw the plane with point $(1, \sqrt{3})$. Two ways to describe. How are they connected? Trigonometry and right triangles.
   - In general, for any point (x, y),
   $$x = r\cos(\theta), \quad y = r\sin(\theta).$$
   - Given r and θ, easy to find x and y. The other way around is messier. Pythagoras gives r.
   $$r^2 = x^2 + y^2$$
   Arctangent gives θ (almost). Issue is the restricted domain for tangent to make invertible.
   $$\tan(\theta) = \frac{y}{x}, \quad \theta = \arctan(y/x), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

   (c) Examples: Note r and θ both are allowed to be negative.
   - Convert to Cartesian coordinates: $(r, \theta) = (1, -\frac{5\pi}{6})$, $(-2, \frac{3\pi}{4})$. One unique solution for each.
   - Convert to polar coordinates, many solutions for each due to periodicity and negative r: $(1, -1)$, $(0, -2)$. Note the above formula fails.
2. Graphs of polar curves: We know Cartesian graphs $(y = f(x))$. Now we change our thinking to polar graphs $(r = F(\theta))$.
   (a) Simple examples: $r = 2$, $r = -2$, $\theta = 1$, $r = \frac{1}{\cos(\theta)}$ (straight line $x = 1$), $r = \theta$ (spiral(s) of Archimedes).
   - Note how they are traced for θ increasing.
   - It is good to see that r and θ are orthogonal just as x and y are.
   (b) Example: $r = \cos(\theta)$. Have them guess what it should be. Approaches.
   - Imagine θ increasing on $[0, 2\pi]$. Better yet to graph $r = \cos(\theta)$ in $\theta - r$ plane. Circle is traced once on $[0, \pi]$.

- Rewrite in Cartesian coordinates: $r^2 = r\cos(\theta)$ gives a circle $x^2 + y^2 = x$ and complete the square to standard form.
- Parametric equations: $x = r\cos(\theta) = \cos^2(\theta)$ and $y = r\sin(\theta) = \cos(\theta)\sin(\theta)$ though this doesn't help much with the graph.

(c) Example: Try on own via the graph in $\theta - r$ plane.

- Cardiod: $r = 1 + \sin(\theta)$.
- Rose: $r = \cos(2\theta)$.

(d) Useful to take advantage of symmetry with graphs.For $r = F(\theta)$,

- θ and $-\theta$ give same equation: x-axis symmetry.
- r and $-r$ give same equation: rotational symmetry.
- θ and $\pi - \theta$ give same equation: y-axis symmetry.

(e) Use Desmos: `https://www.desmos.com/calculator/uu1erqkbey`

3. Tangent lines: Assume $r = F(\theta)$ and revert to parametric equation results with parameter θ.

(a) $x = r\cos(\theta) = F(\theta)\cos(\theta)$ and $y = r\sin(\theta) = F(\theta)\sin(\theta)$ gives

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

(b) For cardiod $r = 1 + \sin(\theta)$ find the tangent line $\theta = \frac{\pi}{3}$. Location of vertical tangent lines (note need to show $\frac{dy}{dx} \to \pm\infty$, zero division is not enough if top is also zero. Handle the pole this way.)?

4. Interesting:

- Mars as seen from Earth. `https://www.google.com/search?q=mars++path+as+seen+from+earth&rlz=1C1JZAP_enUS695US695&tbm=isch&sxsrf=ACYBGNSJ_XktM-A984LtyJH4bUp5LMHNxw:1578943964452&source=lnms&sa=X&ved=0ahUKEwiB7vvQqIHnAhVKbc0KHREWBGQQ_AUICigB&biw=1163&bih=510&dpr=1.65#imgrc=WHFeMQWy5IPmcM:`
- Mercury wandering sun: `https://planetpailly.com/2018/07/25/things-i-dont-understand-merc`
- Cardiod refraction: `https://www.tandfonline.com/doi/pdf/10.4169/amer.math.monthly.122.5.452?needAccess=true`
- Cardiod: `https://en.wikipedia.org/wiki/Cardioid`

5. Homework: 1-49, 55-63

.4 10.4 Areas and lengths in polar coordinates

1. Area:

(a) Example: Find the area enclosed by $r = \cos(\theta)$.

(b) Unfortunately the area under polar curve $r = F(\theta)$ is not simply $\int_a^b F(\theta)\, d\theta$. Why not?

(c) Dividing our region into small slices from θ to $\theta + \Delta\theta$, we see shapes are wedges (not rectangles). The area of this wedge with interior angle $\Delta\theta$ is a fraction of the area of an entire circle.

$$A_w = \frac{\Delta\theta}{2\pi}\pi r^2 = \frac{\Delta\theta}{2} r^2$$

Summing all these areas via a Riemann sum gives

$$A = \int_a^b \frac{1}{2} r^2\, d\theta = \int_a^b \frac{1}{2}(F(\theta))^2\, d\theta.$$

(d) Back to our example, it is tempting to just use $[0, 2\pi]$ as the integration interval.

$$A = \int_0^{2\pi} \pi r^2 \, d\theta = \int_0^{2\pi} \pi \cos^2(\theta) \, d\theta = \frac{\pi}{2}$$

This is incorrect. We should have $\frac{\pi}{4}$ for the area of a circle of radius 1/2. Issue, the circle was traced twice. The correct interval should have been $[0, \pi]$ as we saw last section.

(e) Find the area inside circle $r = \cos(\theta)$ but outside circle $r = \frac{1}{2}$. Take the upper curve area minus the lower curve area.

$$A = \int_a^b \frac{1}{2}(\cos^2(\theta) - (1/2)^2) \, d\theta$$

Finding intersections of polar curves is just as easy as Cartesian curves $y = f(x)$. Solve $\cos(\theta) = \frac{1}{2}$ to get $\frac{\pi}{3}$ and $-\frac{\pi}{3}$. Note θ needed to trace the curve correctly. Then,

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(\cos^2(\theta) - (1/2)^2) \, d\theta$$

(f) Example: Try on own. Find the area between the cardiod $r = 1+\cos(\theta)$ and circle $r = 1$. Bounds end up as $[-\pi/2, \pi/2]$.

2. Arc length:

(a) Here we just use the parametric result

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Simplifying,

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

giving us

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

(b) Example: Find the arc length of $r = \cos(\theta)$. Make sure to only go around once $(0 \leq \theta \leq \pi)$. Results is π as expected.

(c) Example: Find the length of the cardiod $r = 1 + \cos(\theta)$. Via symmetry,

$$L = 2\int_0^{\pi} \sqrt{2 + 2\cos(\theta)} \, d\theta = 4\int_0^{\pi} \cos(\theta/2) \, d\theta = 8$$

3. Homework: 1-12, 17-21, 23-33, 35-42, 45-48

.5 10.5 Conic sections

.6 10.6 Conic sections in polar coordinates

Chapter 11 Infinite sequences and series

Purpose of this chapter

- A general way to study most all functions. Make new connections $\sin(x), \cos(x), e^x$ are close relatives and any nice function can be rewritten as a polynomial (of infinite degree). (Desmos and $\frac{1}{1+x}$, $\sin(x)$.

- A general way to study derivative and integral. How many integrals can you do?
- A rigorous way to define, compute and understand infinity or infinitesimal (1/infinity). (Desmos π computation?)
- A way to do approximation well. What is the first 100 digits of π? Compute $\int_0^1 e^{-x^2}\,dx$ accurate to 5 digits. Use telescope data for exoplanet detection. Predict weather / stock market trends. Transfer data to function. So much more.
- Basically we push the boundary of calculus, though the sacrifice is abstraction.

.1 11.1 Sequences

1. What is a sequence?
   (a) Intuition: A list of numbers where order matters (stock market prices, weather tempurature, my value).
   (b) Definition: An infinite sequence of numbers is a function whose domain is the set of positive integers.
   (c) Notation: Set notation $\{a_1, a_2, \dots, a_n, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$. a_n is called the nth term.
2. Writing out a list of numbers is not the way to go. We want a formula.
   (a) Explicit formula
   $$a_n = \frac{(-1)^{n+1}}{2n-1}, \quad a_1 = 1, a_2 = -\frac{1}{3}, \dots$$
   (b) Recursive formula
   $$a_n = \frac{a_{n-1}}{n+1}, a_1 = 1, \quad a_2 = \frac{1}{3}, \dots$$
3. Example: Write a formula for each. Can you get them all?
   (a) $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ Explicit
   (b) $\{-\frac{1}{4}, \frac{3}{9}, -\frac{5}{16}, \frac{7}{25}, \dots\}$ Explicit
   (c) $\{1, 1, 2, 3, 5, 8, \dots\}$ Recursive, explicit is unpretty but possible!
   Ted talk Art Benjamin: `https://www.youtube.com/watch?v=SjSHVDfXHQ4`
   Closed form and other goodness: `https://en.wikipedia.org/wiki/Fibonacci_number`
   How fast does Fibonacci grow? Linear, polynomial, exponential? Akin to exponential $F_n \approx \frac{\phi^n}{\sqrt{5}}$ for golden ration $\phi = \frac{1+\sqrt{5}}{2}$.
   (d) $\{2, 5, 8, 11, 14, \dots\}$ Explicit and recursive, called arithmetic sequence
   (e) $\{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots\}$ Explicit and recursive, called geometric sequence
   Archimedes and quadrature of parabola, geometric sum formula: `https://en.wikipedia.org/wiki/The_Quadrature_of_the_Parabola`
   (f) $\{3, 1, 4, 1, 5, \dots\}$ Neither possible (as far as we know), Pi movie (every system has order which can be expressed mathematically)

   Explicit vs recursive formula, which is better?
   (g) Create your own sequence groupwork! 1 bonus point if can stump the chumps, 1 if solve. Mine: Look-say sequence. Has a formula believe it or not. Conway.
   $$1, 11, 21, 1211, 111221, 312211, 13112221, \dots$$
   Conway himself: `https://www.youtube.com/watch?v=ea7lJkEhytA`

(h) Note: Formulas arent always possible even if pattern is there. Sometimes seemingly no pattern exists.

4. Function vs sequence.

   (a) Generally, functions have continuous variables (eg distance), sequences are discrete (eg time).
   (b) A sequence is a special type of function, so all we know of functions transfers.
   (c) Graphing a sequence $\{a_n\} = \{f(n)\}$, dots only on positive integers.
   (d) CAN take limit $\lim_{n\to\infty} a_n$ (end behavior of sequence, La Crosse population stabalization)
   (e) CANNOT take finite limits, differentiate, integrate, etc. Why? But we sometimes can interplay $\{a_n\} = \{f(n)\}$ connection and analyze f instead.

5. Limit of a infinite sequence: $\lim_{n\to\infty} a_n$, a few possibilities.

   (a) Convergent (finite limit): $\{a_n = \frac{n-1}{n}\}$ (write out sequence) gives $\lim\limits_{n\to\infty} a_n = 1$. Precisely,

   For every $\epsilon > 0$ (small), there is an integer N such that if $n > N$, then $|a_n - L| < \epsilon$.

   Draw picture to convince. Similar to $\delta - \epsilon$ definition from Calc 1.

   (b) Divergent: 2 cases

      i. Infinite limit: $a_n = n$, $\lim\limits_{n\to\infty} a_n = \infty$. Can you manage precision?

      For every $M > 0$ (large), there is an integer N such that if $n > N$, then $a_n > M$.

      Similar for limit to $-\infty$.

      ii. Rambler: $a_n = (-1)^n$ does not fit into either of above and also diverges.

   (c) Note: The first n terms doesn't matter. We are after end behavior only.

6. Convergent/divergent:

   (a) Examples: Try on own. Find the end behavior.

$$\frac{1}{n}, \quad \frac{n^2}{3n^2-1}, \quad \frac{\ln(n)}{n}$$

   First two can use old tricks.

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad p > 0, \quad \lim_{n\to\infty} \frac{n^2}{3n^2-1} = \lim_{n\to\infty} \frac{1}{3 - 1/n^2} = \frac{1}{3}$$

   We are tempted to use l'Hospital for third, but remember cannot differentiate a sequence. Need more.

   (b) Theorem: If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.

   Feel free to use any techniques for function $f(x)$ as in Calc 1. Who remembers any?

      i. Limit laws
      ii. Squeeze theorem, sometimes called comparison theorem.
      iii. l'Hospital's Rule
      iv. Absolute convergence (If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$. Really the squeeze theorem.
      v. If $\lim\limits_{n\to\infty} a_n = L$ and $f(x)$ is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$. Pass the limit inside the function.
      vi. Just cite this theorem when using this fact.

vii. Theorem converse (reverse) is not true. Can you think of an example where $\lim_{n\to\infty} a_n = L$ but $\lim_{x\to\infty} f(x) \neq L$? $a_n = \sin(\pi n)$ works.

(c) Examples: Careful solutions for the last two.

i. $\lim_{n\to\infty} \frac{\ln(n)}{n}$ =? What does your intuition say? Need l'Hospital. Let x be a continuous variable. Then,

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \quad \left(\frac{\infty}{\infty} IF\right) = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

By the above theorem, $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$. Note the careful footwork to avoid differentiating a sequence.

(d) Examples: Groupwork!

i. $\lim_{n\to\infty} \frac{n}{n+1} = 1$

ii. $\lim_{n\to\infty} \frac{n}{\sqrt{n+10}} = \infty$

iii. $\lim_{n\to\infty} \left(\frac{2n}{3n+1} + \frac{1}{\sqrt[3]{n}} - 3\right) = -\frac{7}{3}$

iv. $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ (Squeeze theorem)

v. $\lim_{n\to\infty} \sin\left(\frac{n\pi + 1}{\sqrt[2]{n^3}}\right) = 0$ (Continuous function theorem)

vi. $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$

vii. $\lim_{n\to\infty} \sqrt[n]{2} = 1$ (Logarithm / e continuous function)

viii. $\frac{e^n}{n!}, \frac{n^n}{n!}$ (Squeeze theorem?)

ix. $\lim_{n\to\infty} \frac{n!}{n^n}$ (Squeeze theorem $0 < \frac{n!}{n^n} < \frac{1}{n}$)

7. Monotone convergence

(a) Definition: $\{a_n\}$ is monotone increasing if $a_n < a_{n+1}$ for all $n \geq 1$. Similar for monotone decreasing. $\{a_n\}$ is bounded above if there exists a M such that $a_n < M$ for all $n \geq 1$. Similar for bounded below. If bounded above and below, we say $\{a_n\}$ is bounded.

(b) Theorem: Every bounded, monotone sequence is convergent. Draw a picture of inc / dec cases to convince. Much like the squeeze theorem, this is a nice way to show convergence of difficult sequences.

(c) Example:

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0(\text{monotone decreasing, bounded above and below}), \quad \lim_{n\to\infty} \sqrt[n]{n} = 1, \quad \lim_{n\to\infty} \frac{n^n}{3^n \cdot n!},$$

(d) How to tell increasing or decreasing

- $a_n - a_{n-1}$ positive or negative
- a_n/a_{n-1} bigger or smaller than 1
- Derivative positive or negative

(e) We care about monotone convergence theorem because it is used to show most of remaining results in this chapter. Tis a powerful tool in abstract mathematics.

8. Reasoning: Find the end behavior.

(a) $1, 1, 1, 1, 1, 1, 1, 1, 1, \cdots$

(b) $1, 2, 3, 4, 5, 6, \ldots$

(c) $-1, 1, -1, 1, -1, 1 \ldots$

(d) $1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1$

9. Fact: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ Boring, but can lead somewhere interesting.

$$a_n = 1 + \frac{1}{2 + \frac{1}{2+\cdots}}$$

Mention the limit of this continued fraction which must be rational is an irrational number, $\sqrt{2}$. How? Let $\lim_{n\to\infty} a_n = x$. Then,

$$x = 1 + \frac{1}{1+x}$$

and $x = \sqrt{2}$ via algebra.

.2 11.2 Series

1. There are two interesting (and related) views of sequences:
   - End behavior (done)
   - Summing terms (remainder of chapter)

2. Definitions:

   (a) A partial sum S_n of sequence $\{a_n\}$ is given by

   $$S_n = a_1 + a_2 + a_3 \ldots = \sum_{i=1}^{n} a_i$$

   This is also called a finite series. Note, this is a special type of sequence itself.

   (b) Examples: Compute the nth partial sums. Explicit form is the preference here, transfer series to sequence.

   i. Arithmetic sequence: $a_n = 3n - 1$ (sum properties / formulas from Calc 1)

   ii. Geometric sequence: $a_n = \frac{1}{2}^n$ (pictureness, note not a way to compute in general.)

   iii. Telescoping sum (special case, important, transfer series to sequence): $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

   (c) An infinite series s of sequence $\{a_n\}$ is

   $$s = \sum_{n=1}^{\infty} a_n$$

   If the limit converges (is finite), then s is called the sum of the series. Note, this is the limit of a partial sum.

   $$s = \lim_{n\to\infty} S_n.$$

   Getting an explicit form of S_n makes finding s easy. Revisit above 3 examples.

3. Is it possible that s (a sum of infinite numbers) is finite? We already have examples where it is!

   (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$ (draw picture justification).

   (b) Riemann sum is very similar, yet not quite same.

   (c) $1 + 1 + 1 + \cdots = \infty$ (of course cannot always be finite, what condition might we need? For sure decreasing sequence to sum.)

(d) $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$ (Harmonic series, will show decreasing is not enough, not surprising in light of improper integrals, will need to decrease fast enough.

(e) $1 - 1 + 1 - 1 + 1 - 1 + \cdots = ?$ (not all cases are clear)

4. Geometric series (most important series)

(a) Definition: A geometric series is of the form

$$\sum_{n=1}^{\infty} ar^{n-1}$$

for a, r constants. r is the common ratio ($\dfrac{a_n}{a_{n-1}} = r$ for all n).

(b) Above example fits this mold: $a_n = \frac{1}{2}^n = \frac{1}{2}\left(\frac{1}{2}\right)^n$.

(c) Why geometric? Can always draw nice pictures to interpret as area as above and in Archimedes quadrature of parabola. See text.

(d) When does the geometric series converge? Can compute the partial sum via algebraic fanciness to create a telescoping sum.

$$S_n = a + ar + \cdots + ar^{n-1}$$

$$rS_n = ar + ar^2 + \cdots + ar^n$$

$$S_n - rS_n = a - ar^n \quad \Rightarrow \quad S_n = \frac{a - ar^n}{1 - r}$$

$$s = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1 - r}$$

Only need $\lim\limits_{n\to\infty} r^n$ finite which requires $|r| < 1$.

(e) Theorem: The geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1}$ is

- convergent if $|r| < 1$
- divergent if $|r| \geq 1$
- The sum when convergent is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{1}{1 - r}$$

(f) Examples: Does each yield a geometric series? If yes, does each converge?

i. $3, 1, 1/3, 1/9, \ldots$

ii. $a_n = 2^{2n}\pi^{1-n}$ (If cannot see, use ratio to check.)

iii. $0.\overline{7} = 0.77777\cdots = \frac{7}{10} + \frac{7}{100} + \ldots$ (can express any repeating decimal as a fraction this way, $0.999\cdots = 1$, chickawha?)

iv. $1 + 1 + 1 \ldots$

v. $1 - 1 + 1 - 1 \cdots = \frac{1}{2}$? (Note quite, on the end of convergence.)

vi. $1 + 2 + 4 + 8 \cdots = \frac{1}{1-2}$? (Don't forget to check convergence condition.)

(g) Power series: Represent a function as an infinite degree polynomial. Powerful (heh) idea here.

$$\frac{1}{1 - x} = \sum_{n=1}^{\infty} x^n = 1 + x + x^2 + \ldots, \qquad |x| > 1$$

Can perform operations to generate more:

i. Differentiate:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots$$

ii. Substitute: Let $x = -x^2$.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \dots$$

iii. Integrate:

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

In a sense, this is the most practical definition of logarithm we have so far. Can compute $\ln(2) = -\ln(1/2)$ by hand (approximately). Key questions: how fast does it converge? when does it even converge? $|x| < 1$. This gives concrete motivation for series.

Details to be checked here: can we pass limit to infinite number of terms? More in section 11.8.

5. Given a series, a key question is does it converge or not. Here we have the simplest of all tests, always our first resort.

   Divergence test:

   (a) If the series $\sum a_n$ is convergent, then $\lim_{n\to a_n} = 0$.
   (b) If $\lim_{n\to} a_n \neq 0$, then $\sum a_n$ is divergent
   (c) Example: The reverse is not true. $a_n = \ln(1+1/n)$ has $\lim_{n\to a_n} = 0$, but this telescoping sequence diverges to infinity.

   $$\sum_{n=1}^{k} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{k} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{k} (\ln(n+1) - \ln(n)) = \ln(k+1)$$

   Note, the divergence test can only show divergence. Never convergence.

6. Harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \dots$

   (a) Proof that harmonic series diverges: Idea is to compare to a simpler series.

   $$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\cdots > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{8}\right)+\left(\frac{1}{16}+\dots\right)\cdots > 1+\frac{1}{2}+\frac{1}{2}+\dots$$

   The last smaller series diverges by the divergence test. By the comparison test, the harmonic series must also. Another view is to use $-\ln(1-x)$ series for $x = 1$.
   (b) Surprising, yet not. Think back to $\int_1^\infty \frac{1}{x}\,dx$ which also diverged. This inspires the next section.
   (c) Many a paradox to be had: If an ant travels on a rubber band at 1 inch per minute and the band stretches at 1 foot per minute after each step, will the ant get to the other side? Indeed it will eventually.
   (d) Show how slow this thing diverges via desmos. Carefulness next section for speed of divergence.
   (e) Called the harmonic series since the sum of harmonics.

7. Theorem: Series are limits of sequences, so our limit laws of last section apply. Assuming $\sum a_n$ and $\sum b_n$ both exist (are finite), then

   $$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n, \quad \sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n$$

   Be sure to check that limits exist!

(a) What about $\sum_{n=1}^{\infty} a_n b_n$? You think on it.

(b) Show the existence of god. Something from nothing. Grandi's series.

$$0 = (1-1)+(1-1)+(1-1)+\cdots = 1+(-1+1)+(-1+1)-1+\cdots = 1$$

0 = 1 via zero division for fun: $x=1$, $x^2-1=0$, $(x+1)(x-1)=0$, $x+1=0$, $2=0$, $1=0$.

Similar voodoo occurs if laws of series are ignored. Where did Numberphile go wrong? `https://plus.maths.org/content/infinity-or-just-112`

.3 11.3 The integral test

1. Here we abandon computing infinite series. Instead, for sections 11.3-11.6 we simply answer, does the series converge or not? These are tests for convergence, and we have many ideas.

2. Examples:

   (a) Last time we showed the harmonic series $\sum \frac{1}{n}$ diverges. This was shown via trick, but we also mentioned comparing to an improper integral. That is the essence of this section. Draw the picture here to remind and do carefully. Show

   $$\sum \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\, dx = \infty$$

   by drawing area under curve inside rectangles adding to infinite sum.

   (b) Repeat for $\sum \frac{1}{n^2}$. Why draw inside here instead of outside? Because gut says convergence. Can show $\sum \frac{1}{n^2} = \frac{\pi}{6}$. Euler and the Basil Problem.

   (c) For positive series, comparison to improper integrals can show convergence or divergence of the resulting series.

3. Theoerm: The integral test

   Suppose a_n has only positive terms. Let $a_n = f(n)$ with $f(x)$

   - continuous on $[1,\infty)$
   - positive on $[1,\infty)$
   - decreasing on $[1,\infty)$.

   Then,

   - if $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent
   - if $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

4. Notes:

   (a) Need to check three hypothesis each time we use this.

   (b) $\sum a_n$ and $\int f(x)\, dx$ converge/diverge together. We do not have equality: $\sum a_n \neq \int f(x)\, dx$.

5. Example: Does $\sum \frac{1}{1+n^2}$ converge or diverge?

   (a) Use divergence test first. Inconclusive. Why?

   (b) Can we use the integral test? Check the three hypothesis on $f(x) = \frac{1}{1+x^2}$. First two easy, use derivative to show third carefully.

   (c) Can we easily integrate? Eyeball.

   (d) Integrate to convergence.

6. p-series: For what p does $\sum \frac{1}{n^p}$ converge?
   (a) Silly cases first. $p < 0$ diverges by divergence test.
   (b) $p = 0$, same.
   (c) $p > 0$, $f(x) = \frac{1}{x^p}$ passes the three and we know $\int \frac{1}{x^p}$ converges from our past.
   (d) Theorem: The p-series converges for $p > 1$ and diverges otherwise. This is our second most important series result so far behind geometric series.

7. Examples: Try on own. Take 2, what does this integral test yield?

$$\frac{\ln n}{n}, \frac{n^2}{e^{n^3}}, \frac{1}{n^2 + 2n + 2}, \frac{\sin n}{n^2}$$

8. Error estimate: Our series and integral don't match in value, yet we can estimate a series in this way.
   (a) Assume a_n is such that the integral test can be used.
   (b) Denote $s = \sum a_n$ and partial sum $s_n = \sum_{k=1}^{n} a_k$. When is s_n close to s?
   (c) Check if the remainder is small: $R_n = s - s_n$
   (d) Suppose $a_n = f(n)$, and $f(x)$ is
      - Continuous on $[n, \infty)$
      - Positive on $[n, \infty)$
      - Decreasing on $[n, \infty)$

      then

$$\int_{n+1}^{\infty} f(x)\ dx \le s - s_n \le \int_{n}^{\infty} f(x)\ dx$$

      Draw a picture here, rectangles inside for one, outside for other.

9. Examples:
   (a) How close is the sum of the first 100 terms of $\sum \frac{1}{n^2}$ to the true sum?

$$\int_{101}^{\infty} \frac{1}{x^2}\ dx \le s - s_{100} \le \int_{100}^{\infty} \frac{1}{x^2}\ dx$$

   (b) Can modify to check rate of divergence as well. How big is the sum of the first 100 terms of divergent harmonic series $\sum \frac{1}{n}$? 1000?

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \le 1 + \int_{1}^{n} \frac{1}{x}\ dx = 1 + \ln(n)$$

      Now, when will the ant get there?

.4 11.4 The comparison test

1. Theorem: The comparison test

   Let a_n, b_n are series with positive terms and $a_n \le b_n$. Then
   (a) if $\sum a_n$ diverges then $\sum b_n$ diverges.
   (b) if $\sum b_n$ converges then $\sum a_n$ converges.

   Why does this make sense? Remember, a positive series converges if it goes to zero fast enough. If a_n is too slow, b_n must be also. If b_n fast enough, a_n also is. Similar to integral comparison test. See text for proof. Key is monotone bounded.

2. Examples:

   (a) $\sum \frac{5}{n^2+n+3}$ converges by comparison to $p=2$ series

   (b) $\sum \frac{\ln k}{k}$ diverges by comparison to harmonic

   (c) $\sum \frac{k \cos^2 k}{1+k^3}$ converges by comparison to $p=2$ series

   (d) $\sum \frac{1}{\sqrt[3]{n^4+1}}$ converges by $p = \frac{4}{3}$ series

   (e) $\sum \frac{e^{1/n}}{n}$ diverges by comparison to harmonic

   (f) $\sum \frac{1}{2^n - 1}$ not easy to bound, think should converge compared to $\sum \frac{1}{2^n}$.

   (g) Be sure to check the hypothesis. Difficulty is having intuition, then finding a suitable nice series to compare to. Harmonic, geometric, and p-series are our candidates.

3. How else can we compare two series other than bounding? $a_n = \frac{1}{2^n-1}$ has the same rate of end behavior as $b_n = \frac{1}{2^n}$. That is, $\lim a_n$ and $\lim b_n$ grow at the same rate. Then, $\lim \frac{a_n}{b_n}$ must be constant.

4. Theorem: The limit comparison test:

   Suppose $\sum a_n$, $\sum b_n$ are series with positive terms. If

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{(same end behavior speed)}$$

   where $c > 0$ is a finite number. Then either both series converge or both diverge.

5. Examples:

   (a) $\sum \frac{1}{2^n - 1}$ converges

   (b) $\sum \frac{(k+1)(2k-2)}{\sqrt[3]{k^8-1}}$ diverges

   (c) $\sum \sin\left(\frac{1}{n}\right)$ diverges compared to harmonic. Trig limit from the past.

   $$\lim_{n\to\infty} n \sin(1/n) = \lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

6. Notes:

   (a) Leading terms are key.

   (b) What if $c = \infty$, $c = 0$? All we know is rates differ. Either both converge, only one, or neither. Give me examples of each. Test is inconclusive.

   (c) If $\sum a_n$, $\sum b_n$ is convergent and a_n, $b_n > 0$, then $\sum a_n b_n$ converges. Why? Is converse true?

7. Thar be monsters at the edge of the world. Doe each converge or diverge? Give intuition first.

   (a) $\sum \frac{1}{n \ln(n)}$ diverges by integral test

   (b) $\sum \frac{1}{n(\ln(n))^2}$ converges by integral test

(c) $\sum \frac{1}{1+n^{1/n}}$ diverges?

(d) $\sum \frac{1}{n^{1+1/\ln n}}$ diverges by limit comparison with harmonic

(e) $\sum \frac{1}{e^n - n^e}$ converges by

.5 11.5 Alternating series

1. Inspiration: Parallel parking ham video.
2. Thus far, we have tests for positive series only. The geometric is all that we allowed to be negative. What about others? A first case is the alternating series.
3. Definition: $\sum a_n$ is an alternating series if $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n$ for $b_n > 0$.
4. Theorem: Alternating series test

   If the alternating series $\sum a_n = \sum (-1)^n b_n$ satisfies

   (a) b_n is decreasing for all n ($b_n < b_{n-1}$)

   (b) $\lim_{n\to\infty} b_n = 0$

   then $\sum a_n$ is convergent. Again, proof consists of monotone and bounded. Draw picture for idea of proof, parallel parking-esque.
5. Examples: This theorem is easy to use. How to show decreasing? May need derivative if inequality is not easy.

   (a) $\sum \frac{(-1)^n}{n}$ alternating harmonic converges

   (b) $\sum (-1)^n \sin(\frac{1}{n})$

   (c) $\sum (-1)^n \cos(\frac{1}{n})$

   (d) $\sum (-1)^n \frac{n^2}{n^3+1}$
6. Alternating series remainder theorem:

   If $s = \sum a_n = \sum (-1)^n b_n$ is convergent, then

   $$|R_n| = |s - s_n| \le b_{n+1}$$

   . Reason, s lies between consecutive terms by alternatingness. Then, $|s - s_n| \le |s_{n+1} - s_n| = b_{n+1}$. Cute...
7. Example: How many digits are required to estimate the series to 2 digits?

   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

   Can show (in HW, also in 11.9) that $\sum \frac{(-1)^n}{n} = \ln(2)$. Of course it does...

.6 11.6 Absolute convergence and the ratio and root tests

1. Absolute convergence

   (a) Definition: A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

   (b) Theorem: The absolute convergence theorem:

   If a series is absolutely convergent , then it is convergent.

      i. Why? $|a_n|$ goes to zero slower than a_n. So this is the comparison test in disguise. More carefully, we have
      $$0 \leq a_n + |a_n| \leq 2|a_n|$$
      and
      $$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|.$$

      ii. Why is it important? Make any series positive to access all previous tests.

   (c) Can you find anything that is convergent but not absolutely convergent? Alternating harmonic.

   (d) Definition: A series is conditionally convergent if it is convergent but not absolutely convergent.

   (e) Examples: Is each absolutely, conditionally, or not convergent?

      i. $\sum \frac{(-1)^n}{n^2}$ Abs convergent

      ii. $\sum \frac{(-1)^n n}{n+1}$ Not convergent

      iii. $\sum \frac{\cos n}{n^2}$ Abs convergent

      iv. $\sum \frac{\sin n}{n}$

      v. $\sum \frac{(-1)^n}{n}$

   (f) What is the advantage of absolute convergence over conditional?

      i. Theorem: If $\sum a_n$ is absolutely convergent, can add the series in any order to result in same sum.

      ii. Theorem: If $\sum a_n = s, \sum b_n = t$, can multiply series $\left(\sum a_n\right)\left(\sum b_n\right) = st$.

      iii. This sheds light on $\sum (-1)^n$. Can get most any sum you want by rearranging.

2. The ratio and root tests

   (a) Theorem: The ratio test:

   For series $\sum a_n$, if
   $$\lim_{n\to} \left|\frac{a_{n+1}}{a_n}\right| = L$$
   and

      i. if $L < 1$, $\sum |a_n|$ is convergent thus $\sum a_n$

      ii. if $L > 1$ or $L = \infty$, $\sum a_n$ is divergent

      iii. if $L = 1$, inconclusive

   (b) Theorem: The root test

   If
   $$\lim_{n\to} \sqrt[n]{|a_n|} = L$$
   and

i. if $L < 1$, $\sum |a_n|$ is convergent thus $\sum a_n$

ii. if $L > 1$ or $L = \infty$, $\sum a_n$ is divergent

iii. if $L = 1$, inconclusive

(c) Comments:

i. Why does it work? This should remind of geometric series. We are checking if $a_n \approx L^n$ for large n.

ii. Result is stronger then usual, we get absolute convergence instead of just convergence.

iii. Test is often inconclusive, so not a silver bullet. Yet it is simpler than other tests.

(d) Examples: This thing is easy to use. What does your intuition say first?

i. Friendly faces

$$\sum r^n, \quad \sum \frac{1}{n^p}$$

Inconclusive for $r = 1$, any p.

ii. Ratio test

$$\sum \frac{(-1)^n n^3}{3^n}, \quad \sum \frac{n!}{n^n}, \quad \sum \frac{1}{n!} = e$$

iii. Root test

$$\sum \left(\frac{2n+3}{3n-2}\right)^n, \quad \sum \left(\frac{n+1}{n}\right)^{n^2}, \quad \sum \left(1 + \frac{1}{n}\right)^n$$

(e) Can now handle some recursive sequences within series via ratio test, so that is pretty sweet.

$$\sum a_n, \quad a_n = (\frac{n^2+1}{2n^2-2n})a_{n-1}$$

(f) Example: Bonus offerings

$$\sum \frac{2n!}{n!5^n}, \quad \sum \frac{n^n}{n!3^n}, \quad \sum \frac{n}{\sqrt{n^3+2}}$$

.7 11.7 Strategy for testing series

1. Groupwork handout.

.8 11.8 Power series

1. The idea of the remainder of the chapter is that we can represent many functions as series (infinite degree polynomial) instead.

   (a) Recall our geometric series:

   $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

   (b) So, if in algebra we continued the journey of $c, mx + b, ax^2 + bx + c, \dots$, we would (?) have found all other functions.

   (c) Why bother? Can make life easier (or harder). It expands the mathematical world to allow for new maneuvers.

2. Definition: A power series centered at a is given by

   $$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 \dots$$

   for c_n some sequence. Note,

(a) c_n: coefficients (numbers)

(b) a: the center. ($a = 0$ sometimes as with above geometric series)

(c) Is this a power series $\sum_{n=1}^{\infty} x^{2n}$?

(d) A power series is a function (infinite order polynomial) which sometimes has a tidy/familiar formula. What formula? Stay tuned.

(e) It's a general collection of series (ie geometric series)

(f) It's convergent/divergent for different x (ie geometric series needs $|x| < 1$), though the function may make sense elsewhere.

3. Write each in power series form. Ie what is a, c_n?

$$\sum \frac{(x+1)^n}{n!}, \quad \sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \sum \frac{(x-2)^n}{e^n+1}$$

4. We are masters of series convergence. When (for what x) does each converge? ($x = 0$ always)

$$\sum_{n=1}^{\infty} x^n, \quad \sum_{n=1}^{\infty} \frac{x^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$

(a) Imagine x as some number. Ratio test jumps out. Ration test is limited so end points are needed.

(b) Not surprisingly, geometric series appears naturally.(give random examples)

$$\sum_{n=1}^{\infty} \frac{nx^n}{3^{n+1}}$$

(c) The center $a \neq 0$

$$\sum_{n=1}^{\infty} \frac{n(x-2)^n}{3^{n+1}}, \sum_{n=1}^{\infty} \frac{n(x+1)^n}{3^{n+1}}$$

Now why is a called the center? Interval of convergence is always symmetric about it.

5. Theorem: For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities for the interval of convergence for x:

- The series converges only when $x = a$. Why?
- The series converges for all x.
- There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. $|x-a| = R$ needs checking any anything goes.

(a) R is called the radius of convergence (for how many x the series is convergent).

(b) R doesn't include the ending points.

(c) The interval of divergence

(d) Check the previous examples to rephrase in our new language.

(e) Try this one if still nervous: $\sum \frac{(-1)^n(x-2)^n}{n!}$.

6. Advanced topic

1. If

$$\lim_{n\to\infty} \sqrt[n]{c_n} = c$$

where $c \neq 0$. Then the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is $R = 1/c$

2. If

- $\sum_{n=0}^{\infty} c_n x^n$, $R = 2$
- $\sum_{n=0}^{\infty} d_n x^n$, $R = 3$

What is the radius of convergence of

$$\sum_{n=0}^{\infty}(c_n + d_n)x^n$$

3. Suppose the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is R. Then what is the radius of convergence of

$$\sum_{n=0}^{\infty} c_n x^{2n}$$

.9 11.9 Representations of functions as power series

1. Goal: Given a function, how can we rewrite as a power series (infinite degree polynomial)?
2. Modify existing formula to derive new ones. Already saw this a bit with geometric series.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n, \quad |x| < 1$$

(a) Examples: Substitution, constant multiple, distribution (allowed since absolutely convergent).

$$\frac{1}{1+x^2}, \quad \frac{1}{2+x} = \frac{1/2}{1+x/2}, \quad \frac{x^3}{2+x}, \frac{1}{(1-x)^2} = \frac{1}{1-x}\cdot\frac{1}{1-x}$$

i. How does the radius of convergence change? It shifts with the substitution.
ii. Why is it called radius of convergence and why does it limit our series? $\frac{1}{1-x}$ can only be represented on $|x| < 1$. Makes sense since we collide with a discontinuity at $x = 1$. But, $\frac{1}{1+x^2}$ also is limited to $|x| < 1$ despite being defined everywhere. It turns out power series naturally extend to the complex plane and complex numbers are pulling the strings out of site. That is, $\frac{1}{1+i^2}$ gives a discontinuity. Draw circle in complex plane. We will see a deeper connection to complex numbers in the next section with Euler's formula.

(b) Theorem: Differentiation and integration of power series

If the power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function

$$f(x) = \sum_{n=1}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a - R, a + R)$ and

- $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
- $\int f(x)\ dx = \sum_{n=1}^{\infty} \frac{1}{n+1} c_n(x-a)^{n+1} + C$
- The radius of convergence stay the same

i. Key is that the interval of convergence comes for free. Always write this. Important.

ii. Examples: Differentiate, integrate.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}, \quad \ln(1-x) = -\int \frac{1}{1-x}\ dx, \quad \arctan(x) = \int \frac{1}{1+x^2}\ dx$$

When we integrate, what is C? We should only have one constant here. Check $x = a$ at center to see. What is the radius of convergence for each?

iii. Interesting to see all these functions from geometric series alone. What else can we get?

3. Applications of power series.

(a) Historic detour: π calculation goodness.

Noting that $\arctan(1) = \frac{\pi}{4}$, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

a definition of π called Leibniz formula! Unfortunately it is very slow to converge (5000th term is still size of 0.0001) so it isn't practical for π digits. Yet, it inspired the following journey.

i. Start with Archimedes and polygons bounding a circle found 34 digits.

ii. Halley while waiting for his comet modified the formula and chose $x = \frac{1}{\sqrt{3}}$ giving 71 correct digits of $\frac{\pi}{6}$.

iii. Machin and followers edited the formula to increase speed of convergence. Why does this work? Try and see!

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) =?$$

iv. The saga is long... `https://en.wikipedia.org/wiki/Approximations_of_%CF%80`. I think the key wonder is whether the digits of π do occur randomly ($\frac{1}{10}$th of the time). No one knows the answer here.

(b) Approximation:

$$\int_0^{0.5} \frac{1}{1+x}$$

(c) What about $\sin(x)$, $\cos(x)$, e^x?

i. These do not follow from the geometric series exactly, but they are three close relatives as next section will show.

ii. e^x we can handle now. Define $f(x) = e^x$ as the function such that $f' = f$. Assuming f has a power series,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n x^n = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

Comparing coefficients,

$$c_0 = c_1, \quad c_1 = 2c_2, \quad c_2 = 3c_3, \ldots$$

But, $f(0) = c_0 = 1$. Then,

$$c_1 = 1, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{3 \cdot 2}, \quad \ldots \quad , c_n = \frac{1}{n!} \quad \ldots$$

At last, a familiar series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Radius of convergence? $R = \infty$.

(d) It is worth mentioning here that there exist other types of series for representing functions. Power series are simplest. Fourier series are instead

$$f(x) = \sum a_n \sin(nx)$$

and spread out across the entire domain of the function and are built for periodic behavior. Power series on the other hand do very well at the center of the series.

.10 11.10 Taylor and Maclaurin series

1. Last section hints that we need a better way to find power series for a function. Taylor series gives a method to find any smooth function 's power series.

2. Idea of Taylor: Assume function f has a power series with center $x = a$.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Our goal is to make the left and right hand derivatives match at $x = a$. This yields

$$c_n = \frac{f^{(n)}(a)}{n!}$$

3. Theorem: If $f(x)$ has a power series representation at a, then it is given by the Taylor series formula is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R$$

   (a) R: radius of convergence still.
   (b) $0! = 1$ and $n!$ factorial notation
   (c) When $a = 0$, this is called a Maclaurin series.
   (d) This is just a formula in the end, though complicated to look at.

4. Example: Find the Maclaurin series of each.

   (a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

   (b) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

   (c) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(d) $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

(e) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \dfrac{x^n}{n}$

(f) Memorize age 762, table 1. Be able to derive when tortured.

5. Example: Find the Taylor series of each.
   (a) Above at $a = 1$, $a = \frac{\pi}{2}$
   (b) $f(x) = \sqrt{x}, a = 16$
   (c) $f(x) = \ln(x), a = 2$

6. A curious example: What happens to the power series with Fibonacci coefficients (0,1,1,2,3,5,...)? Use recusion $F_n = F_{n-2} + F_{n-1}$ to reveal.

$$s(x) = \sum_{n=0}^{\infty} F_n x^n$$

$$s(x) = 0+x+\sum_{n=2}^{\infty} F_n x^n = x+\sum_{n=2}^{\infty} F_{n-2}x^n+\sum_{n=2}^{\infty} F_{n-1}x^n = x+x\sum_{n=0}^{\infty} F_n x^n+x^2\sum_{n=0}^{\infty} F_n x^n = x+xs(x)+x^2s(x)$$

$$s(x) = \frac{x}{1-x-x^2}$$

7. A key idea is to truncate a power series to give a (finite degree) polynomial approximation of a function.

   Taylor's Remainder Theorem: Let $f(x) = T_n(x) + R_n(x)$
   (a) $T_n(x)$, nth degree Taylor polynomial of $f(x)$
   (b) $R_n(x)$ remainder (of the infinite series)

   If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq R$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad for \quad |x-a| \leq R$$

8. Note: Taylor's remainder theorem is used for two main purposes.
   (a) Theoretical: To show (carefully) a Taylor series converges to a function.
   (b) Practical: To assess how well a Taylor polynomial approximates a function. This has a wide range of uses.

9. Applications: Approximation and computer calculation.
   (a) Difficult number calculations. Can you find $\sqrt{2}$? $\sin(1)$?Your calculator can and Taylor polynomials is a way. Let's use

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

   How accurate is it when $-0.3 \leq x \leq 0.3$? Via Taylor's theorem, we can bound the remainder by

$$R_5 \leq \frac{1}{7!}|x|^7$$

   On our interval, $R_5 \leq \frac{0.3^7}{7!} = 4.3 * 10^{-8}$. Show desmos to see. Using symmetry / periodicity of the sine curve, can calculate $\sin(x)$ for any x value desires. This is similar to how your calculator functions.

(b) Analysis: How well does the difference quotient approximate f'? Consider $x = a + h$. Then,

$$f(a+h) \approx f(a) + hf'(a)$$

fits our mold

$$f(x) \approx f(a) + f'(a)x$$

and Taylor's theorem gives

$$|R_2| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

So, we depend on second derivative behavior and a squared term. Can approximate any derivative this way as long as we are mindful of error.

10. Euler's formula: Complex numbers tie up the loose ends left by real numbers. Recall, imaginary unit $i = \sqrt{-1}$. Then, via our power series results,

$$e^{i\theta} = 1 + (i\theta) + \frac{i^2\theta^2}{2!} + \cdots = 1 + (i\theta) - \frac{\theta^2}{2!} - \frac{i\theta}{3!} + \ldots$$

Separate real and imaginary parts.

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \ldots\right) + \left(i\theta - i\frac{\theta^3}{3!} + \ldots\right) = cos(\theta) + i\sin(\theta)$$

So, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. This is Euler's formula.

(a) First, this formula is bedaffling. Complex exponential is a complex number of this form $(a + bi)$. $e^{2\pi i} = 1$.

(b) Can draw this easily in complex plane via trigonometry. Unit circle connection.

(c) $e^{\pi i} = -1$, then $\ln(-1) = \pi i$ is imaginary. That is why we missed it in precalculus. This gives a easy way to make more room in the reals.

11. Summary: The idea of Taylor polynomial is that any smooth function $f(x)$ can be approximated by a polynomial under restrictions.

(a) Smooth: all the derivatives of $f(x)$ are nice (finite)

(b) Restrictions: only works on a interval, highest accuracy near center.

- You can shift (so it will work for any x)
- Approximate all the $f(x)$ in $(a - R, a + R)$ using all the informations at a

(c) Approximate (nth degree)

- The more you choose, the more accurate it will be
- The remainder theorem is nice, insurance yet not always practical.
- The closer x to the center a, the smaller the remainder will be.
- One more term means one more derivative
- Euler's method (Newton's method in Calculus I). Peek at the analysis.
  `https://en.wikipedia.org/wiki/Newton%27s_method`
  `https://en.wikipedia.org/wiki/Euler_method`

(d) All the information of the derivatives of $f(x)$ is contained in $f(x)$

(e) This is a top idea of this class, and in a sense what our entire discussion of series is about.

.11 11.11 Applications of Taylor polynomials

Chapter 9 Differential equations

.1 9.1 Modeling with differential equations

1. Differential equations
   (a) Definition: equation with derivatives
   (b) Population modeling
   Modification (HIV, fish harvest)
   (c) Spring-mass system
   (d) Order of differential equation
2. Initial value problem: differential equation + initial value

   Initial values
3. Solvability:
   (a) Existence: non-solvable
   $$y' = \frac{1}{y}, \quad y(0) = 0$$
   (b) Uniqueness: not unique
   $$y' = x\sqrt{y}, \quad y(0) = 0, \quad y = \frac{x^4}{16}$$
4. Homework: page 584, 1, 2, 5, 6(a), 11, 12, 13

.2 9.2 Direction fields and Euler's method

How to solve equation?

1. Guess and verify (limitation: seriously?)
2. Solve by maths (limitation: anti-derivative)
3. Direction field (use d-field software), $y' = x + y$, $y' = x^2 + y$, $y' = -\frac{x}{y}$ (limitation first order)
4. Euler's method (limitation first order)
5. Generalize one of above. Taylor series