College Algebra Notes

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Fun Stuff

1. Google AI experiments: `https://experiments.withgoogle.com/ai`
2. Babylonian tablet: `https://www.maa.org/press/periodicals/convergence/the-best-known-old-babyl`
3. Parabola in real world: `https://en.wikipedia.org/wiki/Parabola#Parabolas_in_the_physical_world`
4. Parabolic death ray: `https://www.youtube.com/watch?v=TtzRAjW6KO0`
5. Parabolic solar power: `https://www.youtube.com/watch?v=LMWIgwvbrcM`
6. Robots: `https://www.youtube.com/watch?v=mT3vfSQePcs`, riding bike, kicked dog, cheetah, backflip, box hockey stick
7. Cat or dog: `https://www.datasciencecentral.com/profiles/blogs/dogs-vs-cats-image-classificat`
8. History of logarithm: `https://en.wikipedia.org/wiki/History_of_logarithms`
9. Log transformation: `https://en.wikipedia.org/wiki/Data_transformation_(statistics)`
10. Log plot and population: `https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=state:12000:06000:48000&ifdim=country&hl=en_US&dl=en&ind=false`
11. Yelp and NLP: `https://github.com/skipgram/modern-nlp-in-python/blob/master/executable/Modern_NLP_in_Python.ipynb` `https://www.yelp.com/dataset/challenge`
12. Polynomials and splines: `https://www.youtube.com/watch?v=O0kyDKu8K-k`, Yoda / matlab, `https://www.google.com/search?q=pixar+animation+math+spline&espv=2&source=lnms&tbm=isch&sa=X&ved=0ahUKEwj474fQja7TAhUB3YMKHY8nBGYQ_AUIBigB&biw=1527&bih=873#tbm=isch&q=pixar+animatio mesh+spline`, `http://graphics.pixar.com/library/`
13. Polynomials and pi/taylor series: Matlab/machin `https://en.wikipedia.org/wiki/Chronology_of_computation_of_%CF%80` `https://en.wikipedia.org/wiki/Approximations_of_%CF%80#Machin-lik formula` `https://en.wikipedia.org/wiki/William_Shanks`

Course Introduction

.1 0.1 Day 1

1. Syllabus highlights
   (a) Grades:
      i. Know the expectation / what you are getting into.
      ii. 15perc A (excellent), 35perc B (good), 35perc C (satisfactory),10perc D (passing), some F (failing)
      iii. Expect lower grades than you are used to. I was a student once upon a time. I know what it's like to give some effort in a class and still get an A/B. Night before study, good enough?

iv. Turn in an exam / project. Did you do good work?

v. Many will start off doing good / satisfactory work. Improve to something more. C is not the worst thing in existence. These letters say nothing of your capability.

(b) What does good mean? Good means good. Good job! Excellent means you showed some flair.

(c) Expect: More work, more expectation on good writing.

2. What is algebra? Complete the sentence: Algebra is
   - the math of equations.
   - the study of math symbols.
   - literally translated as the "reunion of broken parts"

   Most important use of algebra is the idea of a function.

3. Explain why learn math
   - It's in daily life (coupon stacking, mortgage options, retirement plan)
     (a) Coupons: Cost 100 dollars. Better to do 20% off first, then save $10, or other way. What about $1000?
     (b) Student loan: 20year at 5% or 30 year at 3%?
     (c) Mortgage: Down payment vs rate.
   - Key ideas
     - Ability to abstract (working on cars)
     - Attention to details (report with 100+ pages)
     - Confidence with numbers (show an excel file)
     - Don't back down from challenge

4. Explain why no calculator (running or driving)

5. Explain why they need to show their work (steps is for bullet points)

.2 0.2 Exam 1 Fallout

1. Again, what to expect from this class?
   - Lower grades.
   - What does an A,B,C,D,F mean?
   - What was your studying regime? Are you taking anything away?

2. Again, what do we hope to gain? Become better thinkers. Key ideas:
   - Ability to abstract
   - Attention to details
   - Confidence with numbers
   - Don't back down from challenge

3. What should you do to improve? My steps for success.
   - Write better solutions. If your writing is disorganized and hard to follow, your thoughts are too. Homework as if you are completing an exam.
   - Take away with all work done. Finish a problem, what are the main ideas to remember?
   - 2 hours for every lecture is enough. More is too much. You are wasting time.
   - Identify what you are missing if you didn't do well. Make an action plan.

Chapter 1 Equations and graphs

.1 1.1 The coodinate plane

1. Motivation: Demo housing data and ask why we need coordinate plane.
   (a) Axis labels are important. What if they weren't there?
   (b) Axis scales can differ.
   (c) What if I reverse x, y? Same data, new meaning.
   (d) No equation in the real world. Can always approximate though.
2. Rectangular (Cartesian) coordinate system
   (a) Definitions
      - Ordered pair: (x, y) is not the same as (y, x).
      - x, y axis
      - Units and labels
      - Origin
      - Quadrants
      - Other coordinate systems (polar)

   (b) Examples
      - Draw points
      - Define many points efficiently as a graph.
        - $x = 4$ (add $y \geq 0$)
        - $y = -2$ (add $-2 \leq x < 1$)
        - $-1 < x \leq 3$
        - $y > 1$ and $x \leq 0$
        - $xy = 0$
        - $xy > 0$
        - $\frac{x}{y} \leq 0$
        - $y = x$ (add $y > 0$)
        - $y = -x$ (add $x < 0$)
        - $y = |x|$
        - $x = |y|$
3. Distance formula: $d(A, B) = ?$
   (a) Find the distance from $(1, 2)$ to $(-3, 4)$.
   (b) Is the formula always positive?
   (c) Pythagorean theorem (picture proof of why is true)
   (d) Example: Do the three points $(-1, -3), (6, 1), (2, -5)$ form a right triangle?
4. The midpoint formula
   (a) Think of as averaging. Visualize on the 1D real axis.
   (b) Find the midpoint between $(1, 2)$ and $(-3, 4)$. Verify via the distance formula.
   (c) Are these formulas you need to memorize?

.2 1.2 Graph of Equations

1. Equation with two variables
   - Why we care: Relate gas and money paid. How does adding a car wash effect?
   - Ordered pairs: All the points that satisfies the equation.
2. Sketching the graph of an equation
   (a) Plot points first via a table. Make them vow that this is not the best way.
   (b) Make a guess of the shape
   (c) How to show "goes forever"?
   (d) Examples:
      - Horizontal/vertical line examples.
      - $y = x$, $y = 2x$, $y = 2x + 1$, $y = |x|$.
3. Important features
   (a) Domain and range: Illustrate for above vertical, horizontal, and slant lines, also $y = 1/x$
   (b) x and y intercepts
      - Illustrate for $y = 2x + 1$.
      - How about $y = x^2$, $y = 2x^2$, $y = 2x^2 - 4$?
      - $x = y^2$?
   (c) Increasing/decreasing (on above parabola $y = 2x^2 - 4$)
   (d) Symmetry (show for above parabolas)
      - x axis: (x, y) and $(x, -y)$ both on graph.
      - y axis: (x, y) and $(-x, y)$ both on graph.
      - Origin (rotational): (x, y) and $(-x, -y)$ both on graph. Illustrate for $y = x^3$.
   (e) Trending (end behavior)
      - As $x \to \infty$, $y \to ?$ and visa versa.
      - Horizontal and vertical asymptote (briefly mention)
4. Basic functions
   (a) Lines: Talk about reasoning (slope and y-intercept), draw the y increment per unit x increment, increasing and decreasing
   (b) Parabola: talk about nonnegative, increasing, decreasing, and symmetry
   (c) Circle: talk about distance formula
      - Give the formula of a circle: $(x - h)^2 + (y - k)^2 = r^2$
      - Special case: Unit circle
      - Equation of a semi circle? Upper half? Right half?
      - Radical function (semi parabola)
      - Tricky questions
        - Know two points on a diameter
        - Know tangent relation (to a line $y = 2x + 1$ with center at the origin)
        - Need to complete the square: $x^2 + y^2 + 8x - 10y + 37 = 0$ has center $C = (-4, 5)$, $r = 2$.
        - A point is inside/outside a circle

.3 1.3. Lines

1. Linear Equation
   (a) Why we care? (Hourly pay, gas price, distance traveled, name your example) Draw a picture.
   (b) Draw a general line with two points labeled carefully. How do we distinguish straight lines from other curves? Constant rate of change. Need one point for uniquiness.
   (c) How to define steepness? Slope. Doesn't matter where you start or how far you go. Use ratio to normalize steepness.
   (d) Where to start? y-intercept.
2. Graph: Give a equation and illustrate $y = 2x + 1$, $y = -\frac{1}{2}x - 3, 2x + 4y = 6$
   (a) Give a table to show idea. Key is constant rate of increase.
   (b) m: slope (go by triangle per unit step) steepness, positive/negative, increasing/decreasing

   $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

   (c) b: y-intercept
   (d) General slope-intercept form: $y = mx + b$
3. Finding the slope-intercept equation: need a slope and a y-intercept
   (a) Given slope and y-intercept
   (b) Given two points
   (c) Given two intercepts
   (d) Given a slope and a point
4. Forms for equation of a line:
   (a) Slope-intercept form: $y = mx + b$.
   (b) Point-slope form: $y - y_1 = m(x - x_1)$ comes from generalizing above slope definition.

   $$m = \frac{y - y_1}{x - x_1}$$

   (c) Standard form: $ax + by = c$ (Why do we need standard form? Higher dimension)
5. Remarks:
   (a) $y = mx$: line through origin
   (b) Horizontal/vertical line.
   (c) Parallel / perpendicular lines.
      - Draw $y = x$, ask what is perpindicular to it?
      - Draw $y = 2x$, ask what is perpindicular to it? Draw a 1,2 triangle and 2,1 triangle rotated. Middle has to be 90 degress.
      - What about $y = mx$? Prove $m_p = -\frac{1}{m}$. Draw two perpindicular lines through the origin. Take $x = 1$ and draw two points. Show the resulting right triangle with these two points and the origin has to maintain the Pythangorean theorem.
   (d) Perpendicular bisector of a line segment.
6. Examples:
   (a) Find the line through point $(5, 4)$ and parallel/perpendicular to $3x + 2y = 7$.
   (b) Where do the perpindicular lines intersect?
   (c) How to check if indeed perpendicular? Pythagorean theorem can be used.

.4 1.4 Solving quadratic equations

1. Quadratic equation
   (a) Definition
   - Quadratic polynomial: $ax^2 + bx + c$, $a \neq 0$ (why a cannot be zero?)
   - Quadratic equation
   - Babylonian story and the solutions (YBC 7289)
2. Solving quadratic equations
   (a) Three methods:
   - Factor (always easiest, not always doable)
   - Complete the square (useful technique)
   - Quadratic formula (can be tedious)

   (b) Factor: Solve $2x(x-2) = x+3$ for x. Check the solution(s).
   - RHS must be 0 before factoring. Why? $pq = 0$ says $p = 0$ or $q = 0$. If $pq = 5$, what can we say? Nothing.
   - Note, if we had done this at $2x^2 - 5x = 3 \Rightarrow x = 1, 4$, wrong solution.
   - Not always easy to do. $x^2 + x - 1 = 0$. How to factor?

   (c) Complete the square
   - $x^2 = 3$, then $x = \pm\sqrt{3}$. Note the two cases here.
   - $(x+2)^2 = 3$
   - General case:
     - $x^2 + 6x - 7 = 0$. After, note that we could have factored.
     - $4x^2 - 40x + 13 = 0$.
     - $3x^2 + 6x + 1 = 0$.
     - Need to complete the square: $x^2 + y^2 + 8x - 10y + 37 = 0$ has center $C = (-4, 5)$, $r = 2$.
   - Quadratic formula: Do above example with completing the square and the formula together.
   - Derive quadratic formula. Visual: `https://en.wikipedia.org/wiki/Completing_the_square`

   (d) Steps
   i. Rewrite it into standard form
   ii. Make sure the RHS is 0
   iii. Try factoring first
   iv. Use quadratic formula
   v. Practice completing the square. Useful technique for later (graphing quadratics).
3. Number of solutions of a quadratic equation
   - 0-2 solution: Investigate each and discuss why. ($x^2 = 0, 1, -1$ and compare to completing the square process).
   - Discriminant and the quadratic formula (give a table with number of solutions).
4. Try on own: Solve 2 ways, complete the square and quadratic formula. QF team has to check their answer.
$$3x(x+4) + 10 = 3 \qquad (x = -2 \pm \sqrt{5/3})$$
5. Application: Design a poster: a 6 (wide) by 8 (long) inch sheet of paper is to be used for posters. The margin at the sides and the top are to have the same width and the bottom margin is to be twice as wide as the other margins. Find the width of the margins if the printed area is to be 20 square inch. (check the meaningful answers)

6. Golden ratio:
   (a) Draw rectangle with sides $1, x$, inner rectangle with sides $1, 1-x$.
   $$\frac{1}{x} = \frac{1-x}{1}$$
   (b) Compute
   $$1 + \cfrac{1}{1 + \cfrac{1}{1+\frac{1}{\vdots}}}$$
7. Graphs of quadratics:
   (a) Parabola, projectile motion, angry birds, satelite dish, mirror death ray, etc.

.5 1.5 Complex numbers

1. Quadratic equation with no real solution
   (a) Why $x^2 = -1$ not solvable? Square root of negative number. If we extend our real number system, we can still have solutions.
   (b) Imaginary number
   $$i = \sqrt{-1}, i^2 = -1$$
   Power of i: i^7,i^{-7}, i^{92}
2. Complex numbers
   (a) Real number
   (b) Imaginary number
   (c) Real part + imaginary part ($a + bi$ is the general form)
      - $2x + 3i = 4 + 3yi$. Find x, y
   (d) Sum and difference of complex numbers: $(3 + i) - (4 + 2i)$
   (e) Product of complex number: $(3 + i)(4 + 2i)$, $\quad i(2 - 7i)^2$
   (f) Quotient of complex number: $\dfrac{3+i}{4+2i}$, $\quad \dfrac{\sqrt{-36}\sqrt{-49}}{1-\sqrt{-16}}$
      - Multiply by the conjugate.
      - How about i^{-13}? Fraction or $i^{-13}i^{16}$.
   (g) Solving quadratic equations: $x = 6 - \dfrac{13}{x}$, $\quad x^3 = -64$ (note, 3 solutions with this last on, not just one).
      - Check the discriminant ahead to verify complex solutions expected.
      - Check the solution.

.6 1.6 Solving other types of equations

1. Ideas of this section: Handle equations with...
   (a) Factoring by grouping for polynomial equations
   (b) Fractional expressions
   (c) Mixed powers and radicals
   (d) Substitution

2. Extraneous solutions are the major trap. Idea:

$$x = 1 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = -1, 1$$

3. Factoring by grouping: Finding common terms is key. Solutions are always easy to check!

$$3x^3 - 5x^2 - 12x + 20 = 0$$

4. Fractional type: Beware of domain changes and extraneous solutions. Always be checkin.

$$\frac{3}{x} - \frac{2}{x-3} = \frac{-12}{x^2-9}\text{(only one solution, one extraneous)}, \quad \frac{1}{x-6} + \frac{x}{x-2} = \frac{4}{x^2-8x+12}$$

5. Radical type: Key is to remove the radicals.
   (a) Beware: $x = 1 \to x^2 = 1$, squaring gives extraneous solutions. Need to check if in domain. Only happens with even powers, not odd. Why?
   (b) $x^{3/2} = x^{1/2}$. Squaring is easiest, can also set equal to zero and factor. Dividing by both sides gives a third option, but beware of zero division.
   (c) $x + \sqrt{5x+19} = -1 \quad (x = 9$, extraneous solution here)
   (d) $\sqrt{2x-3} - \sqrt{x+7} + 2 = 0$
   (e) $\sqrt[3]{x} = 2\sqrt{x}$

6. Hidden quadratic: Do subsitution

$$x^6 - 3x^3 - 40 = 0, \quad x + 2x^{1/2} - 3 = 0$$

.7 1.7 Solving inequalities

1. Inequality basics:
   (a) $<, >, \geq, \leq$
   (b) Draw on number line, give interval notation: $x > -1$, $x \leq 2$, $-1 < x \leq 2$.
   (c) Emphasize the difference between [and (.
   (d) Intersection and union notation. And vs or.

2. Rules for inequalities: example, $x + 1$, $2x > 4$
   (a) $A \leq B \Leftrightarrow A \pm C \leq B \pm C$
   (b) If $C > 0$ then $A \leq B \Leftrightarrow CA \leq CB$
   (c) If $C < 0$ then $A \leq B \Leftrightarrow CA \geq CB$
   (d) If $A > 0$, $B > 0$, then $A \leq B \Leftrightarrow \frac{1}{A} \geq \frac{1}{B}$
   (e) If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$
   (f) If $A \leq B$ and $B \leq C$ then $A \leq C$

3. Solving linear inequality:
   (a) Solve for x: $9 + \frac{x}{3} \geq 4 - \frac{x}{2}$.
      - Can no longer easily check our solution.
      - Performing operations on both sides, +, -, *, / all work.
      - Multiply/divide by -1 changes the direction of the inequality.
      - Why? Show with $-2 < 5$.

(b) Solve for x: $5 \geq \frac{6-5x}{3} > 2$

- Double inequality can handle just the same. Just think of as two separate inequalities.

4. Solving nonlinear inequality: $x^2 - 3x \leq 3$

   (a) Procedure:
   - Move all terms to one side
   - Key here is to keep track of the sign of each factor
   - Method of sign chart
   - The right hand must be zero
   - Write solution in interval / inequality notation
   - Try on own: $-3x^2 < -21x + 30$, solution is $(-\infty, 2) \cup (5, \infty)$.

   (b) Examples:
   $$x(x-1)^2(x+3)^3 > 0 \quad , x(x-1) \geq 2,$$

   (c) Solving quotient: Could clear the fraction if keep track of sign cases of denominator
   $$\frac{1+x}{1-x} \geq 1$$

   (d) More: $\frac{x^2 - x}{x^2 + 2x} \leq 0$, solution is $(-2, 0) \cup (0, 1]$.

5. Modeling
   - Jobs: Number of employees x. I have 220 hours work to cover every week and each person works 40 hours per week. I pay 1000 per person per week and I have a budget for 7500 per week. What are the possible number of employees?
   - Car rental: plan A, 30 per day, 0.1 per mile, plan B, 50 per day, 0.05 per mile. For what range of miles will plan B save your money?
   - Projectile: A ball is thrown upward with an initial velocity of 20 ft/s from the top of a building 100 ft high. It's height h above the ground t seconds late will be $h = 100 + 20t - 20t^2$. Durign what time interval will the ball be at least 60 ft above the ground?

.8 1.8 Solving absolute value equations and inequalities

1. Absolute value.

   (a) Simple examples. What is it for?

   (b) Distance from zero.

   (c) As a piecewise formula.

2. Absolute value equations: Goal is to remove the absolute value.

   (a) Motivation: $|x| = 2, |x-4| = 2$. Think distance from zero.

   (b) Solve $3|x-7| - 9 = 0$ for x. Isolate the absolute value first.

3. Absolute value inequalities

   (a) Examples: $|x| < 2$, $|x| > 2$. Again explain by distance

   (b) Interval notation for the solution.

   (c) Example: $|2x+1| + 2 \geq 3$

   (d) Example: $0 < |x-5| \leq \frac{1}{2}$

   (e) Example: $(x-1)^2 > 4$. Variation: $(x-1)^2 < 4$
   (f) Example: $|x+1| + |x-2| \geq 4$

4. Inequality groupwork handout. Take home quiz.

5. No webwork.

Chapter 2 Functions

.1 2.1 Functions

1. Intuition and basics
   (a) Definition: to be a function, one input is assigned to a unique output. (IMPORTANT!)
   (b) Real life example of correspondence of one object with another: gas/price, email/name, mortgage/downpayment, number of students paying attention/time, house/price, image/classification, machine learning!
   (c) It's just a rule which relates two things.
   (d) This is the most powerful math idea you have yet to come across. It is a vessel which can contain a lot of information.

2. Representation of a function (student paying attention example)
   (a) Description (easy to understand, hard to use)
   (b) Table (practical but not general)
   (c) Graph (big picture, but no details)
   (d) Formula ($S = f(t)$, most general way, good luck writing it down in this case, finding these in the real world is hard (regression, ML, etc))

3. Mathematical representation of function
   (a) Definition: A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$ in a set B.
   (b) Notation: $y = f(x)$
   (c) Terminology
      - x: independent variable, y dependent variable
      - x: input, $y = f(x)$: output
      - A: domain, B: range
      - Key requirement: exactly one

4. Function in math is usually expressed as an equation:
   (a) $f(x) = x^2 + x - 1$
      - Equation defines the whole picture.
      - Fix one variable, find the other one.
      - The difference between y and $f(x)$. y is the second variable, $f(x)$ is the x relation.
   (b) Compute each for the above f.
      - $f(3) = ?$, $f(-\sqrt{2})$, $f(a+b)$, $f(a) + f(b)$, difference quotient
   (c) Again, key restriction: one x cannot go to multiple y (name and email address, both direction, person/name, person/ID, one to one function)
   (d) Example of piecewise functions.

(e) What if we are not written in function notation? Does the equation define y as a function of x? x as a function of y? Solve for x, y and see if only one possible.

$$3x + 2y = 5, \quad 3x^2 - y = 5$$

5. Domain and range
   - $f(x) = x^2$ domain and range
   - Find the domain of $f(x) = \dfrac{3}{\sqrt{x-4}}$, $g(t) = \dfrac{\sqrt{t^2-1}}{t+2}$. Domain is easy. What can go wrong? Range is usually hard.
   - Domain needs to be specified for any function (example of a line vs a segment) otherwise it is implied
   - Range is determined by the domain (give range for above line vs segment)
   - Finding the domain of a given function (if the domain is not specified)
     - Denominator cannot be zero
     - Even radical function cannot be less than zero inside

.2 2.2-2.3 Graphs of functions

1. The graph of a function motivation
   (a) Given a random graph (Google stock price, population of Wisconsin, etc), what are the key features?
   - How to tell it is a function? Vertical line test. Same as the definition of a function.
   - Domain and range?
   - Important features: increasing / decreasing / local and abs peaks / local and abs bottoms / rates
   - Get function evaluations via the graph.
   - Axis labels are important

   (b) Draw on own: Give real life examples and let them think about what the graph should look like (commuting distance from home function one way vs round trip, repeat for velocity, think of own)

2. The graph of a function precise version
   (a) Vertical line test: Same as definition of function
   (b) Increasing / decreasing on an interval
   (c) Local max / min and absolute max / min
   (d) Net change from a to b: $f(b) - f(a)$.
   (e) Given an equation, how to graph? Plotting points is bad, made for computers. Categorize into well understood classes (lines, parabolas, circles, ect)
   - Linear functions: $y = mx + b$
   - Constant function: $y = c$ (horizontal line, what about vertical lines?)
   - Power functions: $y = x^2$, $y = x^3$, and so on.
   - Root functions: $y = \sqrt{x}$, $y = \sqrt[3]{x}$, and so on.
   - Recriprocal functions: $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and so on.
   - Absolute value: $y = |x|$
   - Greatest integer: $y = [x]$.
   - Unit circle is not a function. Why?

- Basic operations like vertical shift / stretch / reflections: $y = x^2 + 2, y = 3x^2, y = -x^2$.

(f) Graph $f(x) = \sqrt{16 - x^2}$, give domain, range, inc, dec, max, mins

(g) Desmos function circus

3. Piecewise defined functions

   (a) Random example with lines. Find domain, range, inc, dec, max, mins
   (b) Reverse it: Given a graph of straight lines, have them write down the PW formula.
   (c) $y = |x|, y = |x|/x, y = |x| + x$
   (d) Floor function: $y = [[x]]$

4. Solving equations and inequalities via graphs.

   (a) $x^2 = x + 2$ via graph of two functions vs algebra.
   (b) $x^2 > x + 2$ via graph of two functions vs algebra.

.3 2.4 Average rate of change of a function

1. Average rate of change of f over interval (a, b): $\dfrac{f(b) - f(a)}{b - a}$

   (a) Examples: Average velocity driving to work (15 miles in 20 mins), change of temperature, profit growth
   (b) Compare to net change. What is the difference? Rate is key.
   (c) Draw the graph and interpret as slope of secant line
   (d) Two forms of difference quotient: $\dfrac{f(b) - f(a)}{b - a}, \dfrac{f(x + h) - f(h)}{h}$

2. Linear function rephrased: function with constant rate of change.

3. Instantaneous rate of change: Three cases

   (a) Straight: rate of change is always the same
   (b) Concave up: rate of change is increasing
   (c) Concave down: rate of change is decreasing

.4 2.5 Linear function and models

1. Linear model: unit price + initial cost

   (a) Gas price and car wash. What will 20 dollars get me?
   (b) Drain my fish 120 gallon tank half way and fill at rate of 15 gals per minute. When is it full?
   (c) Dog tries to escape. Runs away at 10 feet per second. Gets a 6 second head start. I run at 15 feet per second. Will she make it to the woods?

2. Linear regression training slides.

   (a) Assume should be a linear function (may be wrong).
   (b) Test model quality.
   (c) High dimension is no problem.
   (d) Coefficient meaning.

.5 2.6 Transformation of functions

1. Types of transformation
   (a) Big table with the six.
   (b) No rotation? Need trig.
   (c) Why bother? Knowing a basic function allows us to graph a wide class. (Parabolas, etc)
2. Example: $y = f(x) = x^2$, $-2 \leq x \leq 1$, follow 3 main points
   (a) Build intuition via Desmos.
   (b) Vertical shift (ex: $y = f(x) - 2$ vs $y = f(x) + 2$)
   (c) Vertical scaling (stretch or compression) (ex: $y = 3f(x)$ vs $y = \frac{1}{3}f(x)$)
   (d) Vertical reflection (ex: $y = -f(x)$)
   (e) Horizontal shift (ex: $y = f(x - 2)$ vs $y = f(x + 2)$)
   (f) Horizontal scaling (stretch or compression) (ex: $y = f(3x)$ vs $y = f(\frac{1}{3}x)$)
   (g) Horizontal reflection (ex: $y = f(-x)$)
   (h) Combined function transformation: $y = af(bx + c) + d$
   (i) Does order of transformation matter? Yes.
   (j) Linear function is a transformation: Slope-intercept form
   (k) Parabolas can be reformed as combinations of transformations by completing the square: $ax^2 + bx + c = a(x - h)^2 + k$.
3. Examples:
   (a) $f(x) = \sqrt{x}$, $2\sqrt{x}$, $\sqrt{3x}$, $\sqrt{x-1}$, $\sqrt{x} + 2$, $\sqrt{-x}$, $-\sqrt{x} + 1$.
   (b) Hat function: $f(x) = x, 0 \leq x \leq 1, 2 - x, 1 \leq x \leq 2$. Graph $y = 2f(x) + 1$. Which order is correct? Check by plugging in points.
   (c) Same function, $y = f(2x + 4)$. Make more complex.
   (d) $f(x) = -2\sqrt{x-3} + 4, \quad f(x) = \frac{1}{3}\sqrt{3x+6} - 1$
   (e) $f(x) = -2(2x + 4)^2 + 3$
   (f) $f(x) = -|x - 3| - 3$
      - Identify the transformation
      - Divide into horizontal and vertical transformations
      - Do them one by one
      - The order matters
4. Graph Symmetry: Odd and even functions
   (a) Definition by graph (why do we care?)
      - y-axis and rotational symmetry give insight and convenience.
      - Is x-axis symmetry possible? Only for silly case $f(x) = 0$
   (b) Verify by formula
      - $f(-x) = f(x)$ and $f(-x) = -f(x)$
   (c) Typical odd function: odd power function
   (d) Typical even function: absolute value, even power
   (e) Example: Decide if odd, even, or neither.

$$f(x) = \sqrt{4 - x^2}, \quad g(x) = 2x^3 - x, \quad h(x) = 2x^2 - x, \quad i(x) = x^3 + \frac{1}{x}$$

5. Intro Desmos project.

.6 2.7 Combining functions

1. Algebraic combinations
   - $f + g, \quad f - g, \quad f \cdot g, \quad f/g$
   - New notation is easy.
   - Domains are the main discussion: $D_f \cap D_g$ and avoid zero division.
   - Examples: $f(x) = \sqrt{x+2}$, $g(x) = \frac{x}{x+1}$
     (a) Compute $(f+g)(1)$, $(f/g)(0)$.
     (b) Find the domain of $(f+g), (f/g)$.
       – Two ways: Domain of f and g also $g \neq 0$ OR compute f/g keeping track of domian changes.

2. Composite functions
   (a) Motivation: Tax is a function of your income, your income is a function of your work hour, how does your tax change related to your work hour? $T = T(I), I = I(h)$ so $T = T(I(h))$, taxes are really a function of hours.
   (b) Definition: $(f \circ g)(x) = f(g(x))$
   (c) How to conceptualize? Draw picture. Think of this as relay race. What should the domain be?
   (d) Example: $f(x) = \frac{x}{x-2}, g(x) = \frac{1}{x}$.
   - Compute $f \circ g$ at $x = 3, \frac{1}{2}$. Latter is not in the domain even though $g(1/2)$ makes sense. Again, domain is the key discussion.
   - Find $f \circ g$ domain 2 different ways.
     – Need x in domain of g and $g(x)$ in the domain of f. (Preferred since it keeps the idea of function composition in mind.)
     – Compute $f \circ g$ keeping track of domain changes.

   (e) Example: $f(x) = \frac{x}{x^2-1}$ and $g(x) = 2x - 1$. Compute $f \circ g$ and $g \circ f$ and find the domain of each.
   - Order matters: $(f \circ g)(x) \neq (g \circ f)(x)$. Not like multiplication

   (f) Example: View as a composite function: $y = (2x+5)^3$. How many ways here? As many as you want really.
   (g) Example: Let's lean into inverse functions. $f(x) = 3x - 5$, $g(x) = \frac{1}{3}x + \frac{5}{3}$.
   - Show that $(f \circ g)(x) = x$ and mention $(g \circ f)(x) = x$. What does this mean?
   - Go back to set diagram. g undoes f and visa versa.
   - This is idea of inverse function.

.7 2.8 One to one functions and their inverse

1. Inverse functions: Same association, the opposite direction. Reverse of a function.
   (a) New mapping, given the output, find the input. Draw the set diagram. Output $\rightarrow$ input. Once you know one direction, and you know it is invertible, you should know both directions.
   (b) Big questions:
   - Why do we want to invert a function? Encryption/decryption, student id, currency, feet/meters, etc.
   - How to tell if a function is invertible?
   - If f is invertible, how to find it?
   - What is the relationship between a function and its inverse? (Domain/range, graph, etc)

2. **Motivating Example:** $f(x) = -2x + 1$
   - When is the output 1, -4, y? How do we know if there is only one answer here? Only one output for each input.
   - New function to give you x is our inverse function. $x = f^{-1}(y)$. Let's give some careful definitions.

3. Definitions:
   (a) One-to-one function: Function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (Same output never found twice).
   - Equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
   - Graphically, f passes the horizontal line test.

   (b) Inverse function: For f one-to-one, the inverse of f, written f^{-1} is the association which maps outputs of f to corresponding inputs. That is,

   $$y = f(x) \quad \Longleftrightarrow f^{-1}(y) = x$$

   - Draw a picture to illustrate. Note the function composition story $(f \cdot f^{-1})(x) = (f^{-1} \cdot f)(x) = x$ for all x.
   - Note: There exists notational confusion.

   $$f^{-1}(x) \neq \frac{1}{f(x)}$$

4. How to find the inverse function f^{-1} of a given function f?
   - Check if it is one-to-one (HLT or carefully). Show careful version for $f(x) = -2x + 1$. If $f(x_1) = f(x_2)$, then $x_1 = x_2$.
   - Assume y is known, then find x. Solve the equation $y = f(x)$ for x. Result gives $x = f^{-1}(y)$.
   - Write it in the standard way: $y = f^{-1}(x)$.
   - Find the domain if necessary.
   - Verify your work using composition property: $(f \cdot f^{-1})(x) = (f^{-1} \cdot f)(x) = x$ for all x.

5. **Example:** Find the inverse function of $g(x) = x^2 - 1$, restricted domain $x > 0$. Why / where is the restriction needed?
   - Repeat above steps.
   - Graph each and relate the graphs.
   - Relate the domain and range.

   $$\text{range of } f = \text{domain of } f^{-1}$$

   $$\text{domain of } f = \text{range of } f^{-1}$$

6. **Student Examples:**
   - A bit more challenging, find inverse of $f(x) = \frac{x+2}{2x-1}$, list domain and range.
   - A bit more challenging, find inverse of $f(x) = 2\sqrt{x+4}$, list domain and range. Graph each together.

Chapter 3 Polynomial and rational functions

.1 3.1 Quadratic functions and models

1. Motivation:
   - Cannon ball, radar, head light, vortex of spinning water, `https://en.wikipedia.org/wiki/Parabola` see end)
   - Video parabolic death ray.
2. Quadratic functions
   (a) Standard form: $f(x) = ax^2 + bx + c$, $a \neq 0$
   (b) Vertex form: $f(x) = a(x-h)^2 + k$
   (c) Graph of parabolas
      - Axis of symmetry: $x = -\frac{b}{2a}$
      - Maximum/minimum location
      - Vertex
   (d) Why have 2 forms? Each useful for its own situtation.
      - Standard form good for solving equations: factor / quadradic forumula.
      - Vertex form good for graphing.
   (e) Lots of examples.
      i. Examples: $y = -2x^2 - 12x - 8$, $y = 2x^2 - 20x + 30$, $y = 2x(x-4) + 7$.
      ii. Complete the square then graph. Concave up / down.
      iii. Given the graph, find the equation. Vertex / intercept. 3 points.
         - Vertex intercept: $V = (1,1)$, $y - int = 3$.
         - Vertex intercept: $V = (-1,2)$, $x - int = 3$.
         - Intercepts: $x - int = -1, 2$, $y - int = -4$.
3. Applications:
   - Building a chicken fence around a corner of my dog fence. 200ft of fencing total. How to maximize area enclosed?
   - A rectangular gutter is formed by bending a 30 inch wide sheet into a 'u' shape. Find the height of such a gutter which maximizes the cross sectional area.
   - Selling Ipad: \$5 discount, 10 more sale, currently \$ 400, sell for 200. Maximize the profit

.2 3.2 Polynomial functions and their graphs

1. Motivation: Why polyomials? Computers / splines and Yoda / Taylor series (theory to replace any function with an infinite degree polynomial)
2. Polynomial function
   (a) Definition and notation: $P(x) = a_n x^n + ... + a_1 x + a_0$, $a_n \neq 0$
      - Coefficients
      - Degree/order of the polynomial
      - Leading term
      - Leading coefficient
      - Domain / range
   (b) Factorized form: $-2(x-4)^3(x-2)$. What's the degree? Leading coefficient?

3. Graph of polynomial
   (a) End behavior: $y \to \infty$ as $x \to \infty$
   (b) Zeros of polynomial
      - Real root
      - Multiplicity
      - Complex root (no root)
   (c) Sign table for graphing
      - Graph: $y = x^4 + x^3 - x^2$
      - Zeros correspond to factors.
      - Multiplicity of zero determines behavior around zero (cross or touch x-axis).
      - More examples: $y = x^4 - 2x^3 + 8x - 16$, $y = 2x^3 - x^2 - 18x + 9$, $y = -2x^4 - x^3 + 3x^2$.
      - Graph in desmos, ask to find a minimal degree polynomial: $p(x) = 2(x+3)(x+1)^2(x-2)$, note y-int is -12.

4. Intermediate value theorem and finding zeros of continuous functions (bisection method coding detour)

5. Local extrema of polynomials: a polynomial of degree n can have at most $n-1$ local extrema

.3 3.3 Dividing polynomials

1. What if factoring is not easy (no grouping or simple factors)? If you can find a zero, long division can be used.

2. Recall regular long division: $\frac{1234}{8} = 161 + \frac{6}{8}$? Review terminology: Dividend, divisor, quotient, remainder. Can rearrange as $1234 = 161(8) + 6$

3. Polynomial division is pretty well the same.
$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$
or
$$P(x) = Q(x) \cdot D(x) + R(x)$$
Talk about quotient, remainder, divisor

4. Long division examples: Divide $6x^3 - 3x^2 - 2x$ by $x - 3$. Check via multiplication. Divide $2x^3 - 7x^2 + 5$ by $x - 3$.

5. Synthetic division: It only works for linear factors, but it is just short hand for long division.

6. Remainder theorem: if $P(x)$ is divided by $x - c$, then $P(c) = R(c)$.
   - Explain why this works from the rearranged version of division.
   - Check with previous example. What if the remainder was 0? Then we found a zero and hence a factor!

7. Factor theorem: If c is a zero of $P(x)$ if and only if $x - c$ is a factor of $P(x)$
   - Example: Find all the zeros of $x^3 - 7x + 6 = 0$. Note $x = 1$ is a zero by inspection. Check via factor theorem.
   - So long division helps us factor as long as we can find a zero in the first place. Revisit bisection method.

8. Find a polynomial with specified zeros: Find a degree 3 polynomial with $x = 3, 2, 1$ and $P(0) = 6$.

.4 3.4 Real zeros fo polynomials

1. Rational zeros of polynomial

   - Rational zeros theorem: if the polynomial $P(x) = c_n x^n + \ldots c_1 x + c_0$ has integer coefficients (where $c_n \neq 0$ and $c_0 \neq 0$), then every rational zero of P is of the form p/q (fraction in lowest terms) where p and q are integers and p is a factor of a_0, q is a factor of a_n.
   - Proof: Assume p/q is a rational zero. Then $P(p/q) = 0$ and rearranging yeilds

     $$p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \ldots a_1 q^{n-1}) = -a_0 q^n$$

     So p is a factor of the number on the left and since p/q is in lowest terms, a_0 must have a factor of p.
   - Process:
     (a) List all possible zeros and check if they work.
     (b) Once you find a zero. Divide and find zero remainder.
     (c) Repeat.
   - Example: finding rational zeros of $P(x) = 2x^3 + x^2 - 13x + 6$, $P(x) = 12x^3 - 20x^2 - 13x - 6$, $p(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$.

2. Decartes' rule of signs: OMIT

   - the number of positive real zeros of $P(x)$ is equal to the number of variations in sign in P(x) or is less than that by an even whole number
   - the number of negative real zeros of $P(x)$ is equal to the number of variations in sign in P(-x) or is less than that by an even whole number

3. Upper and lower bounds theorem: OMIT

   - If we divide $P(x)$ by $x - b$ with $b > 0$ using synthetic division and if the row that contains the quotient and remainder has no negative entry then b is and upper bound for the real zeros of $P(x)$
   - If we divide $P(x)$ by $x - a$ with $a < 0$ using synthetic division and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P
   - Show that all the zeros of the polynomial $P(x) = x^4 - 3x^2 + 2x - 5$ lie between -3 and 2
   - Does it make sense? Try take a big upper bound and small lower bound

4. Factoring any polynomial and graph the polynomial: OMIT

   $$x^4 - 6x^3 + 3x^2 + 26x - 24$$

   - Possible zeros
   - Decartes rule
   - Graph the polynomial

.5 3.5 Complex zeros and the fundamental theorem of algebra

1. The fundamental theorem of algebra

   (a) The fundamental theorem of algebra: Every polynomial with complex coefficients has at least one complex zero.

(b) Complete factorization theorem (another view of FTOA): If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers a, c_1..., c_n such that $P(x) = a(x - c_1)...(x - c_n)$. Here c_1... c_n are zeros of $P(x)$

2. Zeros of polynomial

   (a) Zero Theorem: a degree n polynomial has exactly n zeros

   (b) Zeros

   - Real zeros
   - Complex zeros (conjugate zeros): complex root always appear in pairs. If z is a zero, then $\bar{z}$ is also a zero of $P(x)$
   - Repeating zeros: multiplicity

3. Linear and quadratic factors: every polynomial with real coefficients can be factored in to a product of linear and irreducible quadratic factors with real coefficients.

   (a) Examples: $p(x) = x^4 + 3x^2 - 4$, $p(x) = x^5 + x^3 + 8x^2 + 8$. $p(x) = x^4 + x^3 + 7x^2 + 9x - 18$.

   (b) Recovering polynomial from roots: order 5 polynomial with roots ± 2, $1 + i$ and 0 while $P(1) = 1$

4. Graphing

.6 3.6 Rational functions

1. Definition;

   (a) A rational function is of the form $f(x) = p(x)/q(x)$ where p, q are polynomials.

   (b) Domain is all real numbers except the real zeros of q.

2. Graph

   (a) Motivating examples:

   - $f(x) = \frac{1}{x}$. Is this a rational function? Give a table to describe behavior near zero.
   - Does zero division imply a vertical asymptote exists there? No.
     i. $f(x) = \frac{x}{x}$.
     ii. $f(x) = \frac{x^2-9}{x-3}$.
     iii. Hole in place of asymptote. Need be in lowest terms to see vertical asymptotes.
   - $f(x) = \frac{3x+6}{x-1}$.
     i. Are we in lowest terms?
     ii. Divide top and bottom by highest order term in bottom for end behavior discussion. Also can use long division.
     iii. What if bottom HOT is x^2? (Divide by this HOT throughout to imagine behavior)
     iv. Top HOT x^2 (long division for oblique asymptote)?

   (b) Zero in the denominator: Two cases here.

   - Hole
   - Vertical asymptote

   (c) Asymptotes: Knowing these definitions is important.

     i. Vertical asymptote
     ii. Horizontal asymptote (leading terms or by polynomial division)

   (d) Drawing the graph of a rational function.

     i. Factor the top and the bottom

ii. Vertical asymptotes and holes
iii. Horizontal asymptotes or infinity
iv. x, y intercepts
v. Sketch the graph (possibility of intersection of horizontal asymptote)

$$y = \frac{x-2}{3x-1}, \quad y = \frac{x^2-4}{2x^2-4x}, \quad y = \frac{2x^2+7x-4}{x^2+x-2}$$

.7 3.7 Polynomial and rational inequalities

1. Mention section. Already done!
2. Solve by drawing graph.
   - $2x^3 + x^2 + 6 \geq 13x$
   - $\frac{(x-2)}{x-1} \leq 3$

Chapter 10 Systems of equations and inequalities

.1 10.1-10.2 Systems of linear equations in two variables

1. Motivation: Building a shed.
   - One company charges \$2000 plus \$15 per square foot.
   - One company charges \$5000 plus \$10 per square foot.
   - For what square footage will the companies match?
2. Motivation: Bottle feed a goat.
   - Formula 1 contains 5 mlg of calcium per ounce and 10 mlg of vitamin A per ounce.
   - Formula 2 contains 8 mlg of calcium per ounce and 2 mlg of vitamin A per ounce.
   - The goat needs 100 mlg of calcium and 60 mlg of vitamin A per day.
   - How much of each formula should we use without wasing?
3. System of linear equations
   (a) Definition
   (b) Solution by graph: intersection of lines
4. Solving system of linear equations
   (a) Substitution
   (b) Elimination
5. The number of solution
   (a) One solution
   (b) No solution
   (c) Infinitely many solutions

.2 10.2 Systems of linear equations in several variables

1. General linear system
   (a) Definition
   (b) Method of substitution
   (c) Method of elimination
      i. Triangular system
      ii. Method of elimination: transfer all system to an equivalent triangular system
         A. Equivalent system
         B. Steps
            - Add a nonzero multiple of one equation to another
            - Multiply an equation by a nonzero constant
            - Interchange the positions of two equations
2. Number of solutions of a linear system: count number of equations and number of variables
   (a) No solution: inconsistent
   (b) The system has exactly one solution
   (c) Infinitely many solution:

.3 10.4 Systems of nonlinear equations

1. System of nonlinear equations: definition and graph $y = x^2$ and $y = x_1$
2. Solving system of nonlinear equations
   (a) Substitution
   (b) Elimination: limited

$$y = x^2, \quad y = 2 - x^2$$

.4 10.5 System of inequalities

1. Graphing a (single) inequality
   (a) Move y on one side
   (b) (linear, quadratic, circle)
2. Graph the solution set of a system of inequalities
   (a) Nonlinear system
   (b) Linear system
   (c) Vertex
   (d) Bounded, bounded
3. Optimization: give one example, don't test

Chapter 4 Exponential and Logarithmic functions

.1 4.1 Exponential functions

1. Motivation: Compound interest example
   (a) Quick example
   (b) General formula and explanation of each variable

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

   (c) Applied problem to find the amount given principal, compounding period, and rate.

2. Basic: Review laws of exponents! Refresher examples.

   (a) LoE: $a^0, a^1, a^m a^n, a^m/a^n, a^n b^n, (a/b)^n, a^{-n}$.
   (b) **Student Examples**: Simplify $\dfrac{\sqrt[3]{ab}\cdot b^2}{a^3\cdot b^{1/2}}$; $(-27)^{2/3}(4)^{-5/2}$; $\left(\dfrac{2x^{2/3}}{y^{1/2}}\right)\left(\dfrac{3x^{-5/6}}{y^{1/3}}\right)$
   (c) What exponent means: $2^3, 2^{-1}, 2^{1/2}, 2^{-4/3}$, good for any rational number, $2^{\pi}, 2^{i}$ needs calculus, but we have faith..
   (d) Solving exponential equations
      - **Student Examples**: Solve for x: $2^{-x} = 8$; $8^{2x} = \dfrac{1}{2^{2-x}}$; $3(3^x) + 9(3^{-x}) = 28$ (rewrite as same base and hidden quadratic)

3. Exponential function: $f(x) = a^x$
   (a) Definition: why $a > 0$ and $a \neq 1$
   (b) Graphs
      - Concrete examples: $f(x) = 2^x, 5^x, (1/3)^x = 3^{-x}$
      - Domain and range
      - $a^0 = 1$
      - Increasing/decreasing
      - Shape: depends on the a
      - Horizontal Asymptote
      - Note they are all one-to-one

4. Reading exponential function
   - Comparing base
   - General format: $b \cdot a^x$
   - Identify graphs with points and shift

5. Intuition / examples:
   - Exponential function grows fast (mark pen example)
   - Application: Student loan interest calculation, mortgage payment calculator.

.2 4.2 The Natural exponential functions

1. Motivation: Need for a single, uniform base.
   - Which one is bigger? (3^4 or 4^3)
   - The idea of a uniform base(base is not unique 2^{3x}, 4^x)
2. The natural base e
   (a) Rather than lots of bases a, we would like a uniform base with nice properties (the natural exponential). Called natural since it shows up in interesting way (instantaneous, large populations and reproduction, many times, many things, life isn't always discrete).
   (b) Continuous compound interest:
   - Invest \$1000 at 5% per year.
   $$1000 + (0.05)1000 = 1050$$
   - Same, twice a year, $\frac{5\%}{2}$ each time.
   $$1000 + (0.025)1000 + (0.025)(1000 + (0.025)1000) = 1000(1 + 0.05/2)^2 = 1050.625$$
   - Quarterly, $\frac{5\%}{4}$ each time.
   $$1000(1 + 0.05/4)^4 = 1050.945$$
   - Daily: 1051.267 (let students choose and guess here, per day second etc)
   - This seems to approach a limit / max.
   - Desmos: $(1 + \frac{0.05}{n})^{n/0.05}$.

   (c) Fact: modify above desmos, sort of growth rate 1.
   $$(1 + \frac{1}{n})^n \to e, \quad \text{when } n \to \infty$$
   where $e \approx 2.72$, Euler's number. Can show e is irrational as important as π, if not more. Shows up in applications all the time.
   (d) The natural exponential function f
   $$f(x) = e^x$$
3. Law of continuous growth formula
   $$q = q_0 e^{rt}$$
   - q_0: initial quantity
   - r: the growth rate
   - t: time
   - e: natural base

   (a) Note:
      i. $r > 0$: growth rate
      ii. $r < 0$: decay rate
      iii. r is better in terms of identifying the increasing and decreasing rate, no longer have cases with the base
      iv. "real" base: e^r

   (b) Continuous compound interest.
   (c) When to apply:
      i. grows/decays proportional to its current value

ii. continuously (instantaneously) changing

(d) Uniform base: transform $y = ae^{kt}$ to ab^t (still need logs to get here)

4. Applications
   - Continuous compound interest
   - Population growth
   - Radioactive decay (half life)
   - Anything that grow/decays at a percentage
   - How to understand continuous (not all the time, but can happen any time)
   - `https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&idim=state:06000:48000&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&sca y=lin&ind_y=false&rdim=country&idim=state:06000:48000:12000&ifdim=country&hl=en_US&dl=en&ind=false`

.3 4.3-4.4 Logarithmic functions and log properties

1. Basics

   (a) Finding growth rate involves finding an input corresponding to a known output. The inverse of exponential function (all one-to-one here).

   (b) Graph $f(x) = a^x$ for $a > 1$ and $0 < a < 1$, automatically can draw f^{-1}. Name $f^{-1}(x) = \log_a(x)$.

   (c) Careful definition of logarithm (defined to be inverse).

   $$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

   (d) The log as a function:

      i. Domain, range

      ii. Special point (1,0)

      iii. Special bases

      iv. Function composition of a^x and $\log_a(x)$.

         A. The logarithmic function with natural base: $\ln x$

         B. The common logarithmic function: $y = \log x$.

   (e) **Examples:**

      i. Compute $\log(1/100), \log_4(2), \log_5(1/5), \log^3(1), \log_8(4), \log_9(sqrt3)$ (easier to look at exponential form.

      ii. Solve for x: $\log_3(x+4) = \log_3(1-x)$ (one-to-one), $e^{2\ln(x)} = 9$ (inverses and domain restriction).

      iii. Find the domain and range: $\ln(\ln x)$.

2. Applications:

   (a) Originally for hand calculation because of log properties below. (Napier, slide rule, revolution of calculation)

   (b) Astronomical distance `https://en.wikipedia.org/wiki/Astronomical_system_of_units`

   (c) The Benford's law (first digit law) `https://en.wikipedia.org/wiki/Benford%27s_law`

   (d) Logarithmic transformation in data science: `https://en.wikipedia.org/wiki/Data_transformation_(statistics)`

   (e) Nature: `https://en.wikipedia.org/wiki/Logarithmic_spiral`

   (f) Solve exponential equation: $23x = 10$, $e^{2x} - 3e^x + 2 = 0$

3. Log properties:
   (a) $\log_a(xy)$, $\log_a(x/y)$
   (b) $\log_a(x^p)$
   (c) $\log_a x = \frac{\log_b(x)}{\log_b(x)}$ change of base
   (d) $a^{\log_a x} = x, \quad \log_a a^x = x$ inverse relation
   (e) These are just the laws of exponents written in logarithmic form. Write a^{s+t}, a^{st}, a^{-s} and draw parallels.
      - Prod to sum: Let $\log_a(x) = s, \log_a(y) = t$, then $a^s = x, a^t = y$.
      - $xy = a^s a^t = a^{x+t}$, rewrite in log form
      - $\log_a(xy) = s + t = \log_a(x) + \log_a(y)$
      - Rest are same idea.
   (f) As mentioned before, make calculation easier (product to sum, power to product, etc).
4. Typical problems
   (a) Express $\log_a \frac{x^3\sqrt{y}}{z^2}$ in terms of $\log x$, $\log y$, $\log z$
      i. Split $\cdot$ and $/$
      ii. Bring down the power
   (b) Express as one logarithm, opposite direction
   (c) Why are we doing this? Solving equations? Solve real life problem.
      i. The population of La Crosse 50000 in 2000, 55000 in 2010, what will it be in 2020 assuming continuous growth?
      ii. Which would you choose and why? Invest \$100 at 4% or \$500 at 3%? When do they equal? Depends on length of investment.
      iii. Google population of Florida, Cali, and Texas. Which is growing faster? Let them guess and explain why. Care about growth rate here, use log plot instead. Care about slope of this new line. Not a realistic fit globally though! Population of sad North Dakota
      $$y = Pe^{rt}, \;\; \ln(y) = \ln(P) + rt, \;\; z = c + rt$$
      `https://www.google.com/publicdata/directory` Possible project here, fit exponential, logistic growth, etc
5. Solving equations examples, these main ideas are all there is.
   (a) $8^{2x}(\frac{1}{4})^{x-2} = 4^{-x}$. Rewrite in same base.
   (b) $2^x = 3^{1-x}$, cannot rewrite in same base, use logarithm of any base. Many equivalent but different looking solutions. Nice bases to choose are 2,3.
   (c) $\log_3(-x) + \log_3(8 - x) = 2$. Beware of domain changes. Always need to check solution. Only $x = -1$ works here.
6. Transfer anything to base e: y = 2^x
   (a) Connection between continuous and discrete cases
   (b) Everything is continuous
   (c) One formula but restrict your x to be integer.
   (d) Groupwork handout, treat as take home quiz.
      i. Tips:
      ii. Remove the log
      iii. Check the domain

.4 4.5 Exponential and logarithmic equations

1. Exponential function
   - Basic: $3^{x+2} = 7$
   - Different base
   - Quadratic: $e^{2x} - e^x - 2 = 0$
   - Factor: $xe^x + x^2e^x = 0$
2. Logarithmic function
   (a) $\log_6(4x - 5) = \log_6(2x + 1)$
   (b) $\log_2(5 + x) = 4$
   (c) $e^x = 4$
   (d) $\log(2x + 3) = \log x + 1$, $\log_2 x + \log_2(x + 2) = 3$
   (e) $2^x = 3^{2x-1}$
   (f) $\log_4(x) + \log_8(x) = 1$, change of basis formula.
   (g) $\ln(x^2) = (\ln x)^2$
3. Application problem (la crosse population, radiactive decay)