College Algebra Notes

Contents

Fun Stuff

1. Google AI experiments: `https://experiments.withgoogle.com/ai`
2. Babylonian tablet: `https://www.maa.org/press/periodicals/convergence/the-best-known-old-babyl`
3. Parabola in real world: `https://en.wikipedia.org/wiki/Parabola#Parabolas_in_the_physical_world`
4. Parabolic death ray: `https://www.youtube.com/watch?v=TtzRAjW6KO0`
5. Parabolic solar power: `https://www.youtube.com/watch?v=LMWIgwvbrcM`
6. Robots: `https://www.youtube.com/watch?v=mT3vfSQePcs`, riding bike, kicked dog, cheetah, back-flip, box hockey stick
7. Cat or dog: `https://www.datasciencecentral.com/profiles/blogs/dogs-vs-cats-image-classificat`
8. History of logarithm: `https://en.wikipedia.org/wiki/History_of_logarithms`
9. Log transformation: `https://en.wikipedia.org/wiki/Data_transformation_(statistics)`
10. Log plot and population: `https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=state:12000:06000:48000&ifdim=country&hl=en_US&dl=en&ind=false`
11. Yelp and NLP: `https://github.com/skipgram/modern-nlp-in-python/blob/master/executable/Modern_NLP_in_Python.ipynb` `https://www.yelp.com/dataset/challenge`
12. Polynomials and splines: `https://www.youtube.com/watch?v=O0kyDKu8K-k`, Yoda / matlab, `https://www.google.com/search?q=pixar+animation+math+spline&espv=2&source=lnms&tbm=isch&sa=X&ved=0ahUKEwj474fQja7TAhUB3YMKHY8nBGYQ_AUIBigB&biw=1527&bih=873#tbm=isch&q=pixar+animatio` `mesh+spline`, `http://graphics.pixar.com/library/`
13. Polynomials and pi/taylor series: Matlab/machin `https://en.wikipedia.org/wiki/Chronology_of_computation_of_%CF%80` `https://en.wikipedia.org/wiki/Approximations_of_%CF%80#Machin-lik` `formula` `https://en.wikipedia.org/wiki/William_Shanks`

Course Introduction

1. What is algebra? Complete the sentence: Algebra is
   - the math of equations.
   - the study of math symbols.
   - literally translated as the "reunion of broken parts"
2. Most important use of algebra is the idea of a function.

Chapter 1 Equations and graphs

1. Motivation: Housing data and curve fitting.
   (a) R code. Plot of Ames housing data.
   (b) On own: Describe what you see. What are the key features of the graph? What conclusions can you draw?
   (c) Axis labels super important, axis scales can differ, title super important, can also be misleading
   (d) Intuition: add extreme dots and interpret
   (e) What if reversed x,y axis? Same data new meaning.
   (f) Often no equation (general curve) to fit real data. Not even a function in this case.
2. Linear regression: best fit line
   (a) How to interpret line? Slope is value of each square foot
   (b) How to tell if line is any good?
   (c) What if above line? Below? Distance from line is important.
   (d) Can use more complex curves (polynomials, others). Time series and forecasting.
3. Chapter outline: Will teach from scratch, fast pace, deeper understanding.
   (a) Understand 2 dimensional space (cartesian plane, distance)
   (b) Curves in 2 space (circles, lines, basics)
   (c) Regions in 2 space (inequalities)
   (d) Solving equations and inequalities compared to graphs.

.1 1.1 The coordinate plane

1. Rectangular (Cartesian) coordinate system
   (a) Draw coordinate plane
   (b) What does it represent? 2D space (width, height), time and quantity (temp), anything else
   (c) x, y axis
   (d) $P = (x, y)$ point in space, over and up, not same as (y, x). Example point. What does it represent?
   (e) Title, units and labels super important (housing example)
   (f) Important features: Origin, quadrants, axis
   (g) Note, other coordinate systems (polar coords for rotation)
2. Curves and regions in the plane
   (a) Efficient to define many points at once. Connect visualization (intuition) with equation (calculation)
      - Equations represent curves in space. $x = 4$ (all points $(4, y)$, $y = -2, x = y$.
      - Inequalities represent regions in space. $y \geq 0$, $-1 < x <= 3$, , $y > 1$ and $x <= 0$
   (b) Try on own, check in Desmos: $x = 4$ and $y >= 0, y = -2$ and $-1 <= x < 0, xy = 0, xy > 0, y = |x|, x = |y|$
3. Pythagoras and the distance formula.
   (a) Example: 2 random points $P_1 = (1, 2)$, $P_2 = (-3, 4)$, draw triangle separate and remind of Pythagorean theorem

(b) General formula for any 2 points. $d(P_1, P_2)$, $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$.
(c) Pythagorean theorem (picture proof of why is true). Wiki page.
(d) Example: Do the three points $(-1, -3), (6, 1), (2, -5)$ form a right triangle?

4. Averages and the midpoint formula
   (a) Find the midpoint between $(1, 2)$ and $(-3, 4)$. Verify via the distance formula.
   (b) General case for any two points.
5. Summary of two formulas to memorize.
6. Homework: 5, 7, 13, 15, 17, 19, 23, 27, 33, 35, 39, 45

.2 1.2 Graph of Equations

1. Motivation: Equations and graphs
   (a) Gas at the pump.
   (b) Plot points, generalize to equation of line (considering all points at once). $x-$int, $y-$int.
   (c) What if: know tank capacity? Domain and range, add car wash?
2. Graph intuition: Try on own.
   (a) Complete a table for $x = -3, -2, -1, 0, 1, 2, 3$. Generalize to graph.
   (b) $y = x^2, 2x^2, 2x^2 - 4$ graph transformations, get to all parabolas $ax^2 + bx + c$.
3. Important graph features: $y = 2x^2 - 4$
   (a) Domain, range, intercepts, symmetry (x, y, rotational, how to check?)
   (b) Must there be intercepts / symmetry? No
   (c) Increasing and decreasing.
   (d) Trending: as x goes to $+-\infty$, what happens to y? Asymptotes
4. Graph symmetry:
   (a) Examples of each case.
   (b) Find intercepts, check for symmetry: $\frac{x^2}{9} + \frac{y^2}{4} = 1$, ellipse.
   (c) Show in desmos.
5. Important classes of equations and graphs
   (a) Lines (next section), parabolas (quadratic), polynomials (next chapters), absolute value (review now), root
   (b) Circles and ellipses
   (c) Rationals (chapter 3)
   (d) Exponential, logarithm (chapter 4)
   (e) Trigonometric (precalc)
   (f) More...functions are an important case.`https://en.wikipedia.org/wiki/List_of_mathematical_functions`
   (g) Serve 2 purposes:
      - simplification of reality which allow us to better understanding
      - examples which we can study completely to develop rich theory

6. Circles: Generalizing the distance formula.
   (a) Definition: A circle of radius $r > 0$ and center $C = (h, k)$ is the set of all points $P = (x, y)$ of distance r from C.
   (b) Draw picture. $d(P, C) = r$ gives
   $$(x-h)^2 + (y-k)^2 = r^2.$$
   (c) Example: Circle with center $C = (1, 2)$ and radius $r = 3$.
      - Write equation and verify 4 easy points work.
      - How to find intercepts, maybe only x in this case?
      - Expand out to show non-standard form. How to go back? Complete the square, later in course.
   (d) Try on own: Find equation of circle with two points on a diameter as $(-1, 3), (7, -5)$. Check equation in Desmos, tell to use Desmos in HW
7. Homework: 9, 13, 19, 23, 33, 49, 55, 59, 67, 69, 73, 77, 81, 95, 99

.3 1.3. Lines

1. Idea of a line
   (a) Draw line and curve.
   (b) How to carefully distinguish? Line has constant rate of change, known as the slope. (draw many cases of lines)
2. Example:
   (a) Line thru points $A = (1, 2)$ and $B = (3, -4)$.
   (b) What is the constant rate of change?
   $$m = \frac{y2 - y1}{x2 - x1} = -3 = \frac{\Delta y}{\Delta x} = \frac{rise}{run}$$
   (c) How to define in general for any point (x, y)? Slope still should hold.
   $$m = \frac{y - y1}{x - x1} = -3$$
   for any point on the line. Leads to point-slope form.
   (d) Find another point on line and double check slope is the same.
3. Equations and cases of lines:
   (a) Slopes cases and graphs: Positive, negative, zero, none
   (b) Equations of lines:
      - Point-slope form (reaffirms constant change)
      - Slope intercept (simple, know where you start)
      - Standard form $Ax + By + C = 0$ (most general, every line captured, many equations for same line, nice for theory)
      - Which form is best? Different intuition of each, different interpretation.
   (c) Find the equation of the line given two intercepts. Point and slope.
4. Example: Interpretation of lines.

(a) Getting gas and car wash
(b) General slope-intercept form: $y = mx + b$
(c) How much gas can you get if only have \$20?

5. Comparing lines
   (a) Parallel / perpendicular / intersecting lines.
   - Example: Find a line perpendicular to $y = 2x$. Show perpindicular.
   - General case for $y = mx + b$.

   (b) Example: Find the perpendicular bisector of segment AB where $A = (1, 4), B = (7 - 2)$. Find its x and y intercepts. Graph result. Check in Desmos.

6. Homework: 9, 17, 21, 23, 25, 29, 35, 37, 43, 47, 61, 63, 65, 67, 77, 81

.4 1.4 Solving quadratic equations

1. Equation solving:
   (a) Linear: $mx + b = 0$, $m \neq 0$ (easy to solve)
   (b) Quadratic: $ax^2 + bx + c = 0$, $a \neq 0$ (harder / richer)
   - Shows up in naturally many places: geometry, physics, optics, optimization, finance
   - Babylonian story and the solutions (YBC 7289)
   - Here we just focus on solving equations, graphs in Chapter 2-3

2. Solving quadratic equations $ax^2 + bx + c = 0$. Three methods:
   (a) Factor (always easiest, not always doable)
   (b) Complete the square (useful technique in surprising places)
   (c) Quadratic formula (memorize, can be tedious)

3. Example: Try on own. Solve $2x(x - 2) = x + 3$ for x.
   (a) 2 ways, factoring and quadratic formula
   (b) Check solutions work
   (c) Note why factoring separates into two equations ONLY if one side is zero.
   (d) Note factoring not always possible

4. Complete the square
   (a) Basic examples: $x^2 = 4$, $(x - 1)^2 = 4$, $3(x - 1)^2 = 4$, note the +- for two solutions.
   (b) What if not in this form? $3x^2 - 6x - 1 = a(x - h)^2 + k = 0$, needs some work to get there.
   (c) Idea: Rewrite $ax^+bx + c = 0$ (standard form) into $a(x - h)^2 + k = 0$ (vertex form)

$$3x^2 - 6x - 1 = 0 \leftrightarrow 3(x - 1)^2 = 4$$

   (d) First example: $x^2 + 6x - 7 = 0$. Note could have factored.
   (e) Less basic: $3x^2 - 6x - 1 = 0$.
   (f) Try on own: $4x^2 - 40x + 13 = 0$. Check via quadratic formula, doesn't factor easily. Note how similar to quadratic formula.
   (g) Quadratic formula derived: $ax^+bx + c = 0$ into $a(x - h)^2 + k = 0$. Wikipedia animation: `https://en.wikipedia.org/wiki/Completing_the_square`

5. Number of solutions of a quadratic equation

(a) 0-2 solution: Investigate each and discuss why. ($x^2 = 0, 1, -1$ and compare to completing the square process).

(b) Discriminant and the quadratic formula (give a table with number of solutions).

6. Homework: 7, 13, 17, 25, 25, 35, 37, 41, 57, 59, 61, 69, 75, 85

.5 1.5 Complex numbers

1. Quadratic equation with no real solution

   (a) $x^2 = -1$ not solvable with real numbers. Extending our real number systems allows for a solution.

   (b) Imaginary number: Define $i = \sqrt{-1}$.

   (c) Powers of $i, i^2, i^3, ...$ cycle. 4 cases.

2. Complex numbers and the complex plane.

   (a) Real numbers, imaginary numbers, add to get complex numbers in complex plane. Real and imaginary parts.

   (b) Graph in the complex plane.

   (c) Think of an extension of real numbers to allow more room.

   (d) Complex conjugate, pairs of numbers.

3. Complex arithmetic:

   (a) Add / subtract

   (b) Multiply

   (c) Divide, multiply by the conjugate

   (d) Note: Result is always of the form $a + bi$.

   (e) Example: Try on own

   $$\frac{(1+2i)(3-4i)}{5+6i}$$

   (f) Example: Try on own.

   - Solve for x: $x = 6 - 13/x$
   - Note a quadratic, check the discriminant for them to know expect complex.
   - Verify one of the solutions, can see where conjugate comes from

4. Applications of complex numbers: `https://en.wikipedia.org/wiki/Complex_number#Applications`

5. Homework: 7, 11, 19, 29, 35, 39, 43, 45, 47, 49, 55, 57, 63, 69

.6 1.6 Solving other types of equations

1. Ideas of this section: Handle equations with...

   (a) Factoring by grouping for polynomial equations

   (b) Fractional expressions

   (c) Mixed powers and radicals

   (d) Substitution

2. Factoring by grouping:

   (a) Example: $3x^3 - 5x^2 - 12x + 20 = 0$

   (b) Note, solving cubics not easy compared to quadratics.
   (c) Cubic formula for general solution.

3. Fractional expressions and extraneous solutions (solutions introduced thru alg operations, domain change, always check at end).

$$\frac{1}{x-6} + \frac{x}{x-2} = \frac{4}{x^2 - 8x + 12}$$

4. Fractional powers, remove the root, more extraneous solutions possible.
   (a) Example: $x + \sqrt{5x - 19} = -1$
   (b) Try on own: $\sqrt{2x-3} - \sqrt{x+7} + 2 = 0$
5. Substitution, simple but powerful technique.

$$x^6 - 3x^3 – 40 = 0$$

6. Homework: 7, 19, 21, 25, 27, 33, 35, 43, 47, 55, 57, 59, 67, 73

.7 1.7 Solving inequalities

1. Inequality basics:
   (a) $<, >, \geq, \leq$
   (b) Draw on number line, give interval notation: $x > -1$, $x \leq 2$, $-1 < x \leq 2$.
   (c) Emphasize the difference between [and (.
   (d) Intersection and union notation. And vs or.
2. Rules for inequalities: Add, subtract, multiply, divide
   (a) $x > 1, x + 2 > 1 + 2, 2x > 2, x/3 > 1/3$, draw number lines.
   (b) Negative multiplication / division exception: Reflection about zero. $2 > 1, -2 < -1$
3. Solving linear inequality:
   (a) Example: $9 + x/3 > 4 - x/2$
   (b) Use above operations. Avoid negative mult / division if possible.
   (c) Not easy to check, can use Desmos to visualize.
   (d) Double linear inequality. Think of as two inequalities (and), but can combine work into one.

$$2 < (6 - 5x)/3 \leq 5$$

   Check via Desmos.
4. Solving nonlinear inequalities.
   (a) Example: $x^2 - 3x <= 4$
   (b) Factor and consider cases, sign chart on number line, write as both interval and inequality, RHS must be zero. Don't divide by expressions involving x since we don't know the sign.
   (c) Try on own: $-3x^2 < -21x + 30$
5. Challenge problems.
   (a) $(x-5)(x-2)(x+1) > 0$. Zero on one side is key. Draw number line and sign chart.
   (b) Try on own: $(1 + x)/(1 - x) >= 1$

(c) Try on own: $(x+2)/(x+3) < (x-1)/(x-2)$

6. Modeling

   - Jobs: Number of employees x. I have 220 hours work to cover every week and each person works 40 hours per week. I pay 1000 per person per week and I have a budget for 7500 per week. What are the possible number of employees?
   - Car rental: plan A, 30 per day, 0.1 per mile, plan B, 50 per day, 0.05 per mile. For what range of miles will plan B save your money?
   - Projectile: A ball is thrown upward with an initial velocity of 20 ft/s from the top of a building 100 ft high. It's height h above the ground t seconds late will be $h = 100 + 20t - 20t^2$. During what time interval will the ball be at least 60 ft above the ground?

7. Homework: 19, 27, 33, 37, 39, 47, 51, 55, 59, 67, 81, 93

.8 1.8 Solving absolute value equations and inequalities

1. Idea of absolute value:
   (a) Distance from zero on the number line
   (b) Piecewise definition
   (c) Graph via PW definition
   (d) Important when thinking of size

2. Absolute value equations: Goal is to isolate and remove the absolute value.
   (a) Basic examples: $|x| = 2, |x-3| = 1$. Number line version of each. Plug in and check.
   (b) $3|2x-7| - 9 = 0$. Check via Desmos. What if +9? No solution possibility.
   (c) Try on own: $|x+3| = |2x+1|$. Check via Desmos.

3. Absolute value inequalities: Goal is to isolate and remove the absolute value.
   (a) Examples: $|x| < 2$, $|x| > 2$. Again explain by distance. Interval notation for the solution.
   (b) Example: $|3x+1| + 4 > 15$
   (c) Example: Try on own: $1 < |x-5| <= 3$
   (d) Nuance example: $(x-1)^2 > 4$. Variation: $(x-1)^2 < 4$
   (e) Challenge example: $|x+1| + |x-2| \geq 4$

4. Homework: 7, 9, 13, 17, 21, 25, 27, 31, 39, 45, 49, 51, 53, 55, 57

.9 Chapter 1 Review

Exam review problems:

1. Chapter 1 Review:

2. Concept check: 1-20

3. Exercises: 1-30, 35-94

4. Chapter 1 test: 1-14

Chapter 2 Functions

1. Function: the most important idea of this class.
   - Applications: Channel an action into equation
   - Math: Foundation of all math theory
2. Equations vs functions
   - Static: Solve $3 = 4x - 5$ for x
   - All cases: Study $y = 4x - 5$ for all (x, y), as x changes how is y influenced?
   - Key is generalization, thinking of this equation as an action performed on input x to generate output y.
   - Housing example and linear regression. Instead of understanding a single house instance, we can to generalize to the entire real estate situation.
3. Power of functions:
   - Practical view: Understanding and intuition. General rule (conceptual), formula (calculation), graph (intuition), super rich discussion
   - Math Theory foundation: Study many situations at once (classes of functions, lines, polynomials, exp; large groupings, smooth, continuous). Calculus is built on this, all smooth functions. More general makes for fuzzy understanding, but far reaching.
4. Chapter outline:
   - Idea of functions (concept, visualize)
   - First full function story (linear, quad/poly/rationals in chp 3, exp/logs in chp 4)
   - Combining / transforming simple functions to harness complexity
   - Reversing functions (inverses)
5. Application: Real world is complex, how to find a function that represents it? Machine learning is one way.
   - Classification: Cat, bird, bird song,
   - Forecasting: Stock market, covid
   - Many more

.1 2.1 Functions

1. Idea and definition of function:
   (a) Complete this sentence: A function $f(x)$ is....
   (b) What is the key idea? Why does it matter? Go as detailed as possible.
   (c) Definition: A function $f(x)$ is a rule which maps one input x to at most one output y, written $y = f(x)$.
   (d) Diagram to conceptualize: Domain, range, indep variable, dep variable
   (e) Most important part: One input to at most one output. Why? Want certainty in what will happen.
   (f) Think of as a machine: One thing in, at most one out.
   (g) Example: Email mapped to student. Want message to go to one person. Is reverse association possible?

2. Representations of functions:

   (a) Description (easy to understand, hard to use)
   (b) Table (practical and easy, not general)
   (c) Formula (good for calculation, not intuitive)
   (d) Graph (good for intuition, not practical)
   (e) Example: Line $y = 2x + 1$. 4 versions.
   (f) Best case is knowing all versions.

3. Example: Consider $y = f(x) = x^2 + 2x - 3$

   (a) Input to output: $f(2), f(a + 2)$.
   (b) Output to input: When is $f(x) = -3$? More at end of chapter.
   (c) Try on own: Simplify the difference quotient $(f(a + h) - f(a))/h$.

4. Piecewise functions:

   - Combining rules, three part example (const, $-2x + 1$, x^2).
   - Graph. Function evaluations.
   - Should remind of absolute value.

5. Finding domain and range:

   (a) $y = f(x) = x^2$, parabola, will study quadratics as transformations of this one in chapter 3.
   (b) Define domain and range. Draw diagram.
   (c) Find domain and range, check in desmos.
   (d) Domain usually easy, range take knowledge or intuition and often not doable.
   (e) Try on own: $f(x) = 3/\sqrt{4 - x}$. What can't x be? Check in desmos, graph inequality
   (f) Try on own: Find the domain of $f(x) = \frac{\sqrt{x^2-1}}{x+2}$ and write in interval notation. Range not easy. Check for zero division and negative roots, same for most of this class with functions.

6. Homework: 1, 3, 5, 7, 11, 17, 19, 21, 25, 29, 31, 33, 37, 39, 43, 49, 51, 55, 59, 69

.2 2.2 Graphs of functions

1. Graph features and stock prices.

   (a) Google tesla stock
   (b) Graph labels important
   (c) Is this a function? What is x, y? Domain / range?
   (d) Key features: Inc / dec, max / min (local and global)
   (e) Net change: $f(b) - f(a)$, average change: $(f(b) - f(a))/(b - a)$
   (f) Hard to find precise values with graph, better to have a formula $f(x)$. Not doable here.

2. Graph features: Definitions

   (a) Draw $y = f(x)$, general graph on side
   (b) Vertical line test
   (c) Domain / range
   (d) Inc / dec

   (e) Max / min, local and absolute

3. Example: Does the equation $x^2 - y = 7$ represent a function?
   (a) Is y a function of x? Is x a function of y? If yes write in function notation.
   (b) Graph function knowing basic $y = x^2$.
   (c) Domain and range
   (d) Try on own for $x^2 + y^2 = 9$. In some cases can solve for y as two separate functions. Easy if know the graph.

4. Graphs of functions: Visualizing a graph gives intuition of its behavior.
   (a) 2 ways to draw graphs:
      - Plot points, last resort, better left to computers
      - Understand classes of function (lines, parabolas, etc).
   (b) Important classes of functions:
      - Linear: $f(x) = mx + b$
      - Powers: x^n, even vs odd cases
      - Root: $\sqrt[n]{x}$
      - Reciprocal: $1/x^n$
      - Abs value: $|x|$
      - Greatest integer: $[\![x]\!]$
      - Will modify these later to grow complexity.

5. Homework: 5, 7, 9, 15, 19, 23, 35, 37, 45, 49, 51, 53, 57, 61, 77

.3 2.3 Information from graphs of functions

1. Recall: Key graph features.
   (a) $y = f(x)$, general graph on side
   (b) Vertical line test
   (c) Domain / range
   (d) Inc / dec
   (e) Max / min, abs and local
   (f) Example: Find all important features of $f(x) = sqrt9 - x^2$.
   (g) Example: Graph comparisons of $g(x) = x >, <, =, f(x) = sqrt9 - x^2$.
   (h) Try on own: Graph comparisons of $g(x) = x^2 >, <, =, f(x) = x + 2$.

2. Homework: 7, 9, 11, 13, 15, 23, 25, 31, 33, 43, 45,

.4 2.4 Average rate of change of a function

1. AROC idea:
   (a) Dis = rate (time), one decides the other, odometer vs spedometer.
   (b) What is the average velocity ($v(t)$, change in distance) if I go 20 miles in 15 mins? Note, we don't know a formula for v for all t.
   (c) Also called difference quotient.
   (d) Distinguish between net change and AROC.

2. AROC and function graphs:
   (a) $f(x) = x^2$ graph, AROC on [1,3]. Slope of secant line.
   (b) General case $f(x)$, 2 forms.
3. Lines redefined, the precise way.
   (a) Line $y = f(x) = mx + b$ is the function such that AROC is always constant.
   (b) $(f(b) - f(a))/(b - a) = \cdots = m$ for any a, b.
4. Calculus and the paradox of AROC:
   (a) First problem of calculus considers AROC approaching IROC. Secant lines approaching tangent lines.
   (b) Need a way around zero division. Invention of idea of limit is the key.
   (c) Desmos tangent and secant line
   (d) `https://en.wikipedia.org/wiki/Rate_(mathematics)#Rate_of_change`
   (e) 3blue1brown the essence of calculus
5. Homework: 7, 9, 11, 15, 19, 23, 25, 27,

.5 2.6 Transformation of functions

1. Motivation: Power of linear transformations.
   (a) List of 6 transformations for $f(x)$ and general combination.
   (b) Why bother? Can go from a basic function ($y = x^2$) to very complex transformation ($y = -2(x-3)^2 + 10$).
   (c) Grow tremendously the number of functions we can graph and understand.
   (d) Study each individually to gain intuition. Combine to understand complexity.
2. y-direction transformations for $f(x) = x^2$.
   (a) Summary: $y = f(x)$ vs $y = f(x)+k$ vs $y = af(x)$ vs $y == -f(x)$ vs all combined $y = -af(x)+k$.
   (b) Vertical shift:
      - $f(x) + k$, how is the output of f changed? Note, output changes, input location does not.
      - $y = f(x) = x^2, f(x) + 2, -3$.
      - Use Desmos tables. Draw +2 carefully on slide. Keep track of three points moving.
      - This balance of formula, table, graph is what grows your intuition in this class, rich diversity of perspective.
   (c) Try yourself:
      - Remaining $2, af(x), -f(x)$
      - Try in Desmos for various choices. Explain what happens.
      - Y multiplied (vertical stretch). Y flipped in sign (x-axis reflection)
3.
4.
5.
6. Types of transformation
   (a) Big table with the six.

(b) No rotation? Need trig.

(c) Why bother? Knowing a basic function allows us to graph a wide class. (Parabolas, etc)

7. Example: $y = f(x) = x^2$, $-2 \le x \le 1$, follow 3 main points

   (a) Build intuition via Desmos.

   (b) Vertical shift (ex: $y = f(x) - 2$ vs $y = f(x) + 2$)

   (c) Vertical scaling (stretch or compression) (ex: $y = 3f(x)$ vs $y = \frac{1}{3}f(x)$)

   (d) Vertical reflection (ex: $y = -f(x)$)

   (e) Horizontal shift (ex: $y = f(x-2)$ vs $y = f(x+2)$)

   (f) Horizontal scaling (stretch or compression) (ex: $y = f(3x)$ vs $y = f(\frac{1}{3}x)$)

   (g) Horizontal reflection (ex: $y = f(-x)$)

   (h) Combined function transformation: $y = af(bx+c) + d$

   (i) Does order of transformation matter? Yes.

   (j) Linear function is a transformation: Slope-intercept form

   (k) Parabolas can be reformed as combinations of transformations by completing the square: $ax^2 + bx + c = a(x-h)^2 + k$.

8. Examples:

   (a) $f(x) = \sqrt{x}$, $2\sqrt{x}$, $\sqrt{3x}$, $\sqrt{x-1}$, $\sqrt{x}+2$, $\sqrt{-x}$, $-\sqrt{x}+1$.

   (b) Hat function: $f(x) = x, 0 \le x \le 1, 2-x, 1 \le x \le 2$. Graph $y = 2f(x) + 1$. Which order is correct? Check by plugging in points.

   (c) Same function, $y = f(2x+4)$. Make more complex.

   (d) $f(x) = -2\sqrt{x-3} + 4, \quad f(x) = \frac{1}{3}\sqrt{3x+6} - 1$

   (e) $f(x) = -2(2x+4)^2 + 3$

   (f) $f(x) = -|x-3| - 3$

   - Identify the transformation
   - Divide into horizontal and vertical transformations
   - Do them one by one
   - The order matters

9. Graph Symmetry: Odd and even functions

   (a) Definition by graph (why do we care?)

   - y-axis and rotational symmetry give insight and convenience.
   - Is x-axis symmetry possible? Only for silly case $f(x) = 0$

   (b) Verify by formula

   - $f(-x) = f(x)$ and $f(-x) = -f(x)$

   (c) Typical odd function: odd power function

   (d) Typical even function: absolute value, even power

   (e) Example: Decide if odd, even, or neither.

$$f(x) = \sqrt{4-x^2}, \quad g(x) = 2x^3 - x, \quad h(x) = 2x^2 - x, \quad i(x) = x^3 + \frac{1}{x}$$

10. Intro Desmos project.

.6 2.7 Combining functions

1. Algebraic combinations

   - $f+g, \quad f-g, \quad f \cdot g, \quad f/g$
   - New notation is easy.
   - Domains are the main discussion: $D_f \cap D_g$ and avoid zero division.
   - Examples: $f(x) = \sqrt{x+2}$, $g(x) = \frac{x}{x+1}$

     (a) Compute $(f+g)(1)$, $(f/g)(0)$.

     (b) Find the domain of $(f+g), (f/g)$.

       - Two ways: Domain of f and g also $g \neq 0$ OR compute f/g keeping track of domian changes.

2. Composite functions

   (a) Motivation: Tax is a function of your income, your income is a function of your work hour, how does your tax change related to your work hour? $T = T(I), I = I(h)$ so $T = T(I(h))$, taxes are really a function of hours.

   (b) Definition: $(f \circ g)(x) = f(g(x))$

   (c) How to conceptualize? Draw picture. Think of this as relay race. What should the domain be?

   (d) Example: $f(x) = \frac{x}{x-2}, g(x) = \frac{1}{x}$.

   - Compute $f \circ g$ at $x = 3, \frac{1}{2}$. Latter is not in the domain even though $g(1/2)$ makes sense. Again, domain is the key discussion.
   - Find $f \circ g$ domain 2 different ways.
     - Need x in domain of g and $g(x)$ in the domain of f. (Preferred since it keeps the idea of function composition in mind.)
     - Compute $f \circ g$ keeping track of domain changes.

   (e) Example: $f(x) = \frac{x}{x^2-1}$ and $g(x) = 2x - 1$. Compute $f \circ g$ and $g \circ f$ and find the domain of each.

   - Order matters: $(f \circ g)(x) \neq (g \circ f)(x)$. Not like multiplication

   (f) Example: View as a composite function: $y = (2x+5)^3$. How many ways here? As many as you want really.

   (g) Example: Let's lean into inverse functions. $f(x) = 3x - 5$, $g(x) = \frac{1}{3}x + \frac{5}{3}$.

   - Show that $(f \circ g)(x) = x$ and mention $(g \circ f)(x) = x$. What does this mean?
   - Go back to set diagram. g undoes f and visa versa.
   - This is idea of inverse function.

.7 2.8 One to one functions and their inverse

1. Inverse functions: Same association, the opposite direction. Reverse of a function.

   (a) New mapping, given the output, find the input. Draw the set diagram. Output $\rightarrow$ input. Once you know one direction, and you know it is invertible, you should know both directions.

   (b) Big questions:

   - Why do we want to invert a function? Encryption/decryption, student id, currency, feet/meters, etc.
   - How to tell if a function is invertible?
   - If f is invertible, how to find it?
   - What is the relationship between a function and its inverse? (Domain/range, graph, etc)

2. **Motivating Example:** $f(x) = -2x + 1$
   - When is the output 1, -4, y? How do we know if there is only one answer here? Only one output for each input.
   - New function to give you x is our inverse function. $x = f^{-1}(y)$. Let's give some careful definitions.

3. Definitions:
   (a) One-to-one function: Function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (Same output never found twice).
   - Equivalently, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
   - Graphically, f passes the horizontal line test.

   (b) Inverse function: For f one-to-one, the inverse of f, written f^{-1} is the association which maps outputs of f to corresponding inputs. That is,

$$y = f(x) \quad \Longleftrightarrow f^{-1}(y) = x$$

   - Draw a picture to illustrate. Note the function composition story $(f \cdot f^{-1})(x) = (f^{-1} \cdot f)(x) = x$ for all x.
   - Note: There exists notational confusion.

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

4. How to find the inverse function f^{-1} of a given function f?
   - Check if it is one-to-one (HLT or carefully). Show careful version for $f(x) = -2x + 1$. If $f(x_1) = f(x_2)$, then $x_1 = x_2$.
   - Assume y is known, then find x. Solve the equation $y = f(x)$ for x. Result gives $x = f^{-1}(y)$.
   - Write it in the standard way: $y = f^{-1}(x)$.
   - Find the domain if necessary.
   - Verify your work using composition property: $(f \cdot f^{-1})(x) = (f^{-1} \cdot f)(x) = x$ for all x.

5. **Example:** Find the inverse function of $g(x) = x^2 - 1$, restricted domain $x > 0$. Why / where is the restriction needed?
   - Repeat above steps.
   - Graph each and relate the graphs.
   - Relate the domain and range.

$$\text{range of } f = \text{domain of } f^{-1}$$

$$\text{domain of } f = \text{range of } f^{-1}$$

6. **Student Examples:**
   - A bit more challenging, find inverse of $f(x) = \frac{x+2}{2x-1}$, list domain and range.
   - A bit more challenging, find inverse of $f(x) = 2\sqrt{x+4}$, list domain and range. Graph each together.

Chapter 3 Polynomial and rational functions

.1 3.1 Quadratic functions and models

1. Motivation:
   - Cannon ball, radar, head light, vortex of spinning water, `https://en.wikipedia.org/wiki/Parabola` see end)
   - Video parabolic death ray.
2. Quadratic functions
   (a) Standard form: $f(x) = ax^2 + bx + c$, $a \neq 0$
   (b) Vertex form: $f(x) = a(x-h)^2 + k$
   (c) Graph of parabolas
      - Axis of symmetry: $x = -\frac{b}{2a}$
      - Maximum/minimum location
      - Vertex
   (d) Why have 2 forms? Each useful for its own situtation.
      - Standard form good for solving equations: factor / quadradic forumula.
      - Vertex form good for graphing.
   (e) Lots of examples.
      i. Examples: $y = -2x^2 - 12x - 8$, $y = 2x^2 - 20x + 30$, $y = 2x\,(x-4) + 7$.
      ii. Complete the square then graph. Concave up / down.
      iii. Given the graph, find the equation. Vertex / intercept. 3 points.
         - Vertex intercept: $V = (1, 1)$, $y - int = 3$.
         - Vertex intercept: $V = (-1, 2)$, $x - int = 3$.
         - Intercepts: $x - int = -1, 2$, $y - int = -4$.
3. Applications:
   - Building a chicken fence around a corner of my dog fence. 200ft of fencing total. How to maximize area enclosed?
   - A rectangular gutter is formed by bending a 30 inch wide sheet into a 'u' shape. Find the height of such a gutter which maximizes the cross sectional area.
   - Selling Ipad: $5 discount, 10 more sale, currently $ 400, sell for 200. Maximize the profit

.2 3.2 Polynomial functions and their graphs

1. Motivation: Why polyomials? Computers / splines and Yoda / Taylor series (theory to replace any function with an infinite degree polynomial)
2. Polynomial function
   (a) Definition and notation: $P(x) = a_n x^n + ... + a_1 x + a_0$, $a_n \neq 0$
      - Coefficients
      - Degree/order of the polynomial
      - Leading term
      - Leading coefficient
      - Domain / range
   (b) Factorized form: $-2(x-4)^3(x-2)$. What's the degree? Leading coefficient?

3. Graph of polynomial
   (a) End behavior: $y \to \infty$ as $x \to \infty$
   (b) Zeros of polynomial
      - Real root
      - Multiplicity
      - Complex root (no root)
   (c) Sign table for graphing
      - Graph: $y = x^4 + x^3 - x^2$
      - Zeros correspond to factors.
      - Multiplicity of zero determines behavior around zero (cross or touch x-axis).
      - More examples: $y = x^4 - 2x^3 + 8x - 16$, $y = 2x^3 - x^2 - 18x + 9$, $y = -2x^4 - x^3 + 3x^2$.
      - Graph in desmos, ask to find a minimal degree polynomial: $p(x) = 2(x+3)(x+1)^2(x-2)$, note y-int is -12.
4. Intermediate value theorem and finding zeros of continuous functions (bisection method coding detour)
5. Local extrema of polynomials: a polynomial of degree n can have at most $n-1$ local extrema

.3 3.3 Dividing polynomials

1. What if factoring is not easy (no grouping or simple factors)? If you can find a zero, long division can be used.
2. Recall regular long division: $\frac{1234}{8} = 161 + \frac{6}{8}$? Review terminology: Dividend, divisor, quotient, remainder. Can rearrange as $1234 = 161(8) + 6$
3. Polynomial division is pretty well the same.
$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$
   or
$$P(x) = Q(x) \cdot D(x) + R(x)$$
   Talk about quotient, remainder, divisor
4. Long division examples: Divide $6x^3 - 3x^2 - 2x$ by $x - 3$. Check via multiplication. Divide $2x^3 - 7x^2 + 5$ by $x - 3$.
5. Synthetic division: It only works for linear factors, but it is just short hand for long division.
6. Remainder theorem: if $P(x)$ is divided by $x - c$, then $P(c) = R(c)$.
   - Explain why this works from the rearranged version of division.
   - Check with previous example. What if the remainder was 0? Then we found a zero and hence a factor!
7. Factor theorem: If c is a zero of $P(x)$ if and only if $x - c$ is a factor of $P(x)$
   - Example: Find all the zeros of $x^3 - 7x + 6 = 0$. Note $x = 1$ is a zero by inspection. Check via factor theorem.
   - So long division helps us factor as long as we can find a zero in the first place. Revisit bisection method.
8. Find a polynomial with specified zeros: Find a degree 3 polynomial with $x = 3, 2, 1$ and $P(0) = 6$.

.4 3.4 Real zeros fo polynomials

1. Rational zeros of polynomial

   - Rational zeros theorem: if the polynomial $P(x) = c_n x^n + \dots c_1 x + c_0$ has integer coefficients (where $c_n \neq 0$ and $c_0 \neq 0$), then every rational zero of P is of the form p/q (fraction in lowest terms) where p and q are integers and p is a factor of a_0, q is a factor of a_n.
   - Proof: Assume p/q is a rational zero. Then $P(p/q) = 0$ and rearranging yeilds

     $$p(a_n p^{n-1} + a_{n-1} p^{n-2} q + \dots a_1 q^{n-1}) = -a_0 q^n$$

     So p is a factor of the number on the left and since p/q is in lowest terms, a_0 must have a factor of p.
   - Process:
     (a) List all possible zeros and check if they work.
     (b) Once you find a zero. Divide and find zero remainder.
     (c) Repeat.
   - Example: finding rational zeros of $P(x) = 2x^3 + x^2 - 13x + 6$, $P(x) = 12x^3 - 20x^2 - 13x - 6$, $p(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$.

2. Decartes' rule of signs: OMIT

   - the number of positive real zeros of $P(x)$ is equal to the number of variations in sign in P(x) or is less than that by an even whole number
   - the number of negative real zeros of $P(x)$ is equal to the number of variations in sign in P(-x) or is less than that by an even whole number

3. Upper and lower bounds theorem: OMIT

   - If we divide $P(x)$ by $x - b$ with $b > 0$ using synthetic division and if the row that contains the quotient and remainder has no negative entry then b is and upper bound for the real zeros of $P(x)$
   - If we divide $P(x)$ by $x - a$ with $a < 0$ using synthetic division and if the row that contains the quotient and remainder has entries that are alternately nonpositive and nonnegative, then a is a lower bound for the real zeros of P
   - Show that all the zeros of the polynomial $P(x) = x^4 - 3x^2 + 2x - 5$ lie between -3 and 2
   - Does it make sense? Try take a big upper bound and small lower bound

4. Factoring any polynomial and graph the polynomial: OMIT

   $$x^4 - 6x^3 + 3x^2 + 26x - 24$$

   - Possible zeros
   - Decartes rule
   - Graph the polynomial

.5 3.5 Complex zeros and the fundamental theorem of algebra

1. The fundamental theorem of algebra

   (a) The fundamental theorem of algebra: Every polynomial with complex coefficients has at least one complex zero.

(b) Complete factorization theorem (another view of FTOA): If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers a, c_1..., c_n such that $P(x) = a(x - c_1)...(x - c_n)$. Here c_1... c_n are zeros of $P(x)$

2. Zeros of polynomial

   (a) Zero Theorem: a degree n polynomial has exactly n zeros

   (b) Zeros

   - Real zeros
   - Complex zeros (conjugate zeros): complex root always appear in pairs. If z is a zero, then $\bar{z}$ is also a zero of $P(x)$
   - Repeating zeros: multiplicity

3. Linear and quadratic factors: every polynomial with real coefficients can be factored in to a product of linear and irreducible quadratic factors with real coefficients.

   (a) Examples: $p(x) = x^4 + 3x^2 - 4$, $p(x) = x^5 + x^3 + 8x^2 + 8$. $p(x) = x^4 + x^3 + 7x^2 + 9x - 18$.

   (b) Recovering polynomial from roots: order 5 polynomial with roots ± 2, $1 + i$ and 0 while $P(1) = 1$

4. Graphing

.6 3.6 Rational functions

1. Definition;

   (a) A rational function is of the form $f(x) = p(x)/q(x)$ where p, q are polynomials.

   (b) Domain is all real numbers except the real zeros of q.

2. Graph

   (a) Motivating examples:

   - $f(x) = \frac{1}{x}$. Is this a rational function? Give a table to describe behavior near zero.
   - Does zero division imply a vertical asymptote exists there? No.

     i. $f(x) = \frac{x}{x}$.

     ii. $f(x) = \frac{x^2-9}{x-3}$.

     iii. Hole in place of asymptote. Need be in lowest terms to see vertical asymptotes.

   - $f(x) = \frac{3x+6}{x-1}$.

     i. Are we in lowest terms?

     ii. Divide top and bottom by highest order term in bottom for end behavior discussion. Also can use long division.

     iii. What if bottom HOT is x^2? (Divide by this HOT throughout to imagine behavior)

     iv. Top HOT x^2 (long division for oblique asymptote)?

   (b) Zero in the denominator: Two cases here.

   - Hole
   - Vertical asymptote

   (c) Asymptotes: Knowing these definitions is important.

     i. Vertical asymptote

     ii. Horizontal asymptote (leading terms or by polynomial division)

   (d) Drawing the graph of a rational function.

     i. Factor the top and the bottom

ii. Vertical asymptotes and holes

iii. Horizontal asymptotes or infinity

iv. x, y intercepts

v. Sketch the graph (possibility of intersection of horizontal asymptote)

$$y = \frac{x-2}{3x-1}, \quad y = \frac{x^2-4}{2x^2-4x}, \quad y = \frac{2x^2+7x-4}{x^2+x-2}$$

.7 3.7 Polynomial and rational inequalities

1. Mention section. Already done!
2. Solve by drawing graph.
   - $2x^3 + x^2 + 6 \geq 13x$
   - $\frac{(x-2)}{x-1} \leq 3$

Chapter 10 Systems of equations and inequalities

.1 10.1-10.2 Systems of linear equations in two variables

1. Motivation: Building a shed.
   - One company charges \$2000 plus \$15 per square foot.
   - One company charges \$5000 plus \$10 per square foot.
   - For what square footage will the companies match?
2. Motivation: Bottle feed a goat.
   - Formula 1 contains 5 mlg of calcium per ounce and 10 mlg of vitamin A per ounce.
   - Formula 2 contains 8 mlg of calcium per ounce and 2 mlg of vitamin A per ounce.
   - The goat needs 100 mlg of calcium and 60 mlg of vitamin A per day.
   - How much of each formula should we use without wasing?
3. System of linear equations
   (a) Definition
   (b) Solution by graph: intersection of lines
4. Solving system of linear equations
   (a) Substitution
   (b) Elimination
5. The number of solution
   (a) One solution
   (b) No solution
   (c) Infinitely many solutions

.2 10.2 Systems of linear equations in several variables

1. General linear system
   (a) Definition
   (b) Method of substitution
   (c) Method of elimination
      i. Triangular system
      ii. Method of elimination: transfer all system to an equivalent triangular system
         A. Equivalent system
         B. Steps
            - Add a nonzero multiple of one equation to another
            - Multiply an equation by a nonzero constant
            - Interchange the positions of two equations
2. Number of solutions of a linear system: count number of equations and number of variables
   (a) No solution: inconsistent
   (b) The system has exactly one solution
   (c) Infinitely many solution:

.3 10.4 Systems of nonlinear equations

1. System of nonlinear equations: definition and graph $y = x^2$ and $y = x_1$
2. Solving system of nonlinear equations
   (a) Substitution
   (b) Elimination: limited

$$y = x^2, \quad y = 2 - x^2$$

.4 10.5 System of inequalities

1. Graphing a (single) inequality
   (a) Move y on one side
   (b) (linear, quadratic, circle)
2. Graph the solution set of a system of inequalities
   (a) Nonlinear system
   (b) Linear system
   (c) Vertex
   (d) Bounded, bounded
3. Optimization: give one example, don't test

Chapter 4 Exponential and Logarithmic functions

.1 4.1 Exponential functions

1. Motivation: Compound interest example
   (a) Quick example
   (b) General formula and explanation of each variable
   $$A = P\left(1+\frac{r}{n}\right)^{nt}$$
   (c) Applied problem to find the amount given principal, compounding period, and rate.
2. Basic: Review laws of exponents! Refresher examples.
   (a) LoE: $a^0, a^1, a^m a^n, a^m/a^n, a^n b^n, (a/b)^n, a^{-n}$.
   (b) **Student Examples**: Simplify $\dfrac{\sqrt[3]{ab}\cdot b^2}{a^3\cdot b^{1/2}}$; $(-27)^{2/3}(4)^{-5/2}$; $\left(\dfrac{2x^{2/3}}{y^{1/2}}\right)\left(\dfrac{3x^{-5/6}}{y^{1/3}}\right)$
   (c) What exponent means: $2^3, 2^{-1}, 2^{1/2}, 2^{-4/3}$, good for any rational number, $2^\pi, 2^i$ needs calculus, but we have faith..
   (d) Solving exponential equations
      - **Student Examples**: Solve for x: $2^{-x}=8$; $8^{2x}=\dfrac{1}{2^{2-x}}$; $3(3^x)+9(3^{-x})=28$ (rewrite as same base and hidden quadratic)
3. Exponential function: $f(x)=a^x$
   (a) Definition: why $a>0$ and $a\neq 1$
   (b) Graphs
      - Concrete examples: $f(x)=2^x, 5^x, (1/3)^x = 3^{-x}$
      - Domain and range
      - $a^0=1$
      - Increasing/decreasing
      - Shape: depends on the a
      - Horizontal Asymptote
      - Note they are all one-to-one
4. Reading exponential function
   - Comparing base
   - General format: $b\cdot a^x$
   - Identify graphs with points and shift
5. Intuition / examples:
   - Exponential function grows fast (mark pen example)
   - Application: Student loan interest calculation, mortgage payment calculator.

.2 4.2 The Natural exponential functions

1. Motivation: Need for a single, uniform base.
   - Which one is bigger? (3^4 or 4^3)
   - The idea of a uniform base(base is not unique 2^{3x}, 4^x)
2. The natural base e
   (a) Rather than lots of bases a, we would like a uniform base with nice properties (the natural exponential). Called natural since it shows up in interesting way (instantaneous, large populations and reproduction, many times, many things, life isn't always discrete).
   (b) Continuous compound interest:
   - Invest \$1000 at 5% per year.
   $$1000 + (0.05)1000 = 1050$$
   - Same, twice a year, $\frac{5\%}{2}$ each time.
   $$1000 + (0.025)1000 + (0.025)(1000 + (0.025)1000) = 1000(1 + 0.05/2)^2 = 1050.625$$
   - Quarterly, $\frac{5\%}{4}$ each time.
   $$1000(1 + 0.05/4)^4 = 1050.945$$
   - Daily: 1051.267 (let students choose and guess here, per day second etc)
   - This seems to approach a limit / max.
   - Desmos: $(1 + \frac{0.05}{n})^{n/0.05}$.

   (c) Fact: modify above desmos, sort of growth rate 1.
   $$(1 + \frac{1}{n})^n \to e, \quad \text{when } n \to \infty$$
   where $e \approx 2.72$, Euler's number. Can show e is irrational as important as π, if not more. Shows up in applications all the time.
   (d) The natural exponential function f
   $$f(x) = e^x$$
3. Law of continuous growth formula
   $$q = q_0 e^{rt}$$
   - q_0: initial quantity
   - r: the growth rate
   - t: time
   - e: natural base

   (a) Note:
      i. $r > 0$: growth rate
      ii. $r < 0$: decay rate
      iii. r is better in terms of identifying the increasing and decreasing rate, no longer have cases with the base
      iv. "real" base: e^r

   (b) Continuous compound interest.
   (c) When to apply:
      i. grows/decays proportional to its current value

ii. continuously (instantaneously) changing

(d) Uniform base: transform $y = ae^{kt}$ to ab^t (still need logs to get here)

4. Applications
   - Continuous compound interest
   - Population growth
   - Radioactive decay (half life)
   - Anything that grow/decays at a percentage
   - How to understand continuous (not all the time, but can happen any time)
   - `https://www.google.com/publicdata/explore?ds=kf7tgg1uo9ude_&met_y=population&idim=state:06000:48000&hl=en&dl=en#!ctype=l&strail=false&bcs=d&nselm=h&met_y=population&scale_y=lin&ind_y=false&rdim=country&idim=state:06000:48000:12000&ifdim=country&hl=en_US&dl=en&ind=false`

.3 4.3-4.4 Logarithmic functions and log properties

1. Basics

   (a) Finding growth rate involves finding an input corresponding to a known output. The inverse of exponential function (all one-to-one here).

   (b) Graph $f(x) = a^x$ for $a > 1$ and $0 < a < 1$, automatically can draw f^{-1}. Name $f^{-1}(x) = \log_a(x)$.

   (c) Careful definition of logarithm (defined to be inverse).

   $$y = \log_a x \quad \text{if and only if} \quad x = a^y$$

   (d) The log as a function:

      i. Domain, range

      ii. Special point (1,0)

      iii. Special bases

      iv. Function composition of a^x and $\log_a(x)$.

         A. The logarithmic function with natural base: $\ln x$

         B. The common logarithmic function: $y = \log x$.

   (e) **Examples:**

      i. Compute $\log(1/100), \log_4(2), \log_5(1/5), \log^3(1), \log_8(4), \log_9(sqrt3)$ (easier to look at exponential form.

      ii. Solve for x: $\log_3(x+4) = \log_3(1-x)$ (one-to-one), $e^{2\ln(x)} = 9$ (inverses and domain restriction).

      iii. Find the domain and range: $\ln(\ln x)$.

2. Applications:

   (a) Originally for hand calculation because of log properties below. (Napier, slide rule, revolution of calculation)

   (b) Astronomical distance `https://en.wikipedia.org/wiki/Astronomical_system_of_units`

   (c) The Benford's law (first digit law) `https://en.wikipedia.org/wiki/Benford%27s_law`

   (d) Logarithmic transformation in data science: `https://en.wikipedia.org/wiki/Data_transformation_(statistics)`

   (e) Nature: `https://en.wikipedia.org/wiki/Logarithmic_spiral`

   (f) Solve exponential equation: $23x = 10$, $e^{2x} - 3e^x + 2 = 0$

3. Log properties:
   (a) $\log_a(xy)$, $\log_a(x/y)$
   (b) $\log_a(x^p)$
   (c) $\log_a x = \frac{\log_b(x)}{\log_b(x)}$ change of base
   (d) $a^{\log_a x} = x, \quad \log_a a^x = x$ inverse relation
   (e) These are just the laws of exponents written in logarithmic form. Write a^{s+t}, a^{st}, a^{-s} and draw parallels.
      - Prod to sum: Let $\log_a(x) = s, \log_a(y) = t$, then $a^s = x, a^t = y$.
      - $xy = a^s a^t = a^{x+t}$, rewrite in log form
      - $\log_a(xy) = s + t = \log_a(x) + \log_a(y)$
      - Rest are same idea.
   (f) As mentioned before, make calculation easier (product to sum, power to product, etc).

4. Typical problems
   (a) Express $\log_a \frac{x^3\sqrt{y}}{z^2}$ in terms of $\log x$, $\log y$, $\log z$
      i. Split $\cdot$ and /
      ii. Bring down the power
   (b) Express as one logarithm, opposite direction
   (c) Why are we doing this? Solving equations? Solve real life problem.
      i. The population of La Crosse 50000 in 2000, 55000 in 2010, what will it be in 2020 assuming continuous growth?
      ii. Which would you choose and why? Invest \$100 at 4% or \$500 at 3%? When do they equal? Depends on length of investment.
      iii. Google population of Florida, Cali, and Texas. Which is growing faster? Let them guess and explain why. Care about growth rate here, use log plot instead. Care about slope of this new line. Not a realistic fit globally though! Population of sad North Dakota

      $$y = Pe^{rt}, \ \ln(y) = \ln(P) + rt, \ z = c + rt$$

      `https://www.google.com/publicdata/directory` Possible project here, fit exponential, logistic growth, etc

5. Solving equations examples, these main ideas are all there is.
   (a) $8^{2x}(\frac{1}{4})^{x-2} = 4^{-x}$. Rewrite in same base.
   (b) $2^x = 3^{1-x}$, cannot rewrite in same base, use logarithm of any base. Many equivalent but different looking solutions. Nice bases to choose are 2,3.
   (c) $\log_3(-x) + \log_3(8 - x) = 2$. Beware of domain changes. Always need to check solution. Only $x = -1$ works here.

6. Transfer anything to base e: y = 2^x
   (a) Connection between continuous and discrete cases
   (b) Everything is continuous
   (c) One formula but restrict your x to be integer.
   (d) Groupwork handout, treat as take home quiz.
      i. Tips:
      ii. Remove the log
      iii. Check the domain

.4 4.5 Exponential and logarithmic equations

1. Exponential function
   - Basic: $3^{x+2} = 7$
   - Different base
   - Quadratic: $e^{2x} - e^x - 2 = 0$
   - Factor: $xe^x + x^2e^x = 0$
2. Logarithmic function
   (a) $\log_6(4x-5) = \log_6(2x+1)$
   (b) $\log_2(5+x) = 4$
   (c) $e^x = 4$
   (d) $\log(2x+3) = \log x + 1$, $\log_2 x + \log_2(x+2) = 3$
   (e) $2^x = 3^{2x-1}$
   (f) $\log_4(x) + \log_8(x) = 1$, change of basis formula.
   (g) $\ln(x^2) = (\ln x)^2$
3. Application problem (la crosse population, radiactive decay)