The Bhaskara-Aryabhata Approximation to the Sine Function

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Is it possible that well over a thousand years back, mathematicians knew of an approximation to the sine function that yields close to 99% accuracy, using a function that is simply a ratio of two quadratic functions? Such is the case, and the formula in question was found by the Indian mathematician Bhāskarā I: if $0 \leq x \leq 180$, then

$$\sin x^\circ \approx \frac{4x(180-x)}{40500 - x(180-x)}. \tag{1}$$

This article is about how one may find such a formula, and what makes it "tick."

Bhāskarā I (600–680) belonged to the school of mathematics established by the great Indian mathematician Āryabhaṭa (476–550). Āryabhaṭa lived in what has been called the golden age in India, when great advances were being made in fields as diverse as science, art, mathematics, astronomy, technology, and philosophy. The decimal numeration system and use of zero were developed during this period. Āryabhaṭa established a flourishing school of mathematics in northern India, but only one of his works has survived to modern times: the *Āryabhatīyā*, a terse compendium of results in arithmetic, algebra, areas of plane figures, volumes of solids, and astronomy—all set in Sanskrit verse. References to another work, the *Arya Siddhantha*, have been found in the works of later Indian mathematicians such as Varahamihira (500–587), Bhāskarā I himself, and Brahmagupta (598–670); but this work appears to be lost; see Plofker [**7**].

Bhāskarā I wrote valuable commentaries on Āryabhaṭa's work in mathematics and astronomy, and the lasting influence of Āryabhaṭa's work owes in no small measure to these expository works. (Historians have given him the designation "Bhāskarā I" in order to distinguish him from the later and much more famous Bhāskarā of Indian mathematics—Bhāskarā II, the twelfth century mathematician who wrote the lyrical work *Līlavati*.)

The *Āryabhatīyā* has a table of sine values, stated in a rather unfamiliar form. It is actually a table of first differences of chord lengths corresponding to different central angles, and stated in an alphabetic code invented by Āryabhaṭa himself. It also gives a recursive rule for computing these differences. The story of how the word used in that text for chord length, *jyā*, eventually morphed into the term used today, *sin*, over the course of a journey spanning six centuries and three continents, has been beautifully told by Eves; see [**2**, page 105].

Bhāskarā's formula (1) first appears in his book *Mahābhāskarīya*; he attributes it to Āryabhaṭa, but as there is no mention of the formula anywhere in the *Āryabhatīyā*, we shall refer to it as Bhāskarā's formula (though in the title of this article we do call it the Bhāskarā-Āryabhaṭa formula). As per the custom of the time, he stated the formula in stylized verse. Here is how it has been translated by Plofker in [**7**, p. 81]:

> The degree of the arc, subtracted from the total degrees of half a circle, multiplied by the remainder from that [subtraction], are put down twice. [In one place] they

Math. Mag. **84** (2011) 98–107. doi:10.4169/math.mag.84.2.98. © Mathematical Association of America

are subtracted from sky-cloud-arrow-sky-ocean [40500]; [in] the second place, [divided] by one-fourth of [that] remainder [and] multiplied by the final result [i.e., the trigonometric radius].

This prescription may be cast in a form more familiar to us. Let $f(x)$ be defined for real numbers x lying between 0 and 180, thus:

$$f(x) = \frac{4x(180-x)}{40500 - x(180-x)}.$$

(We ignore the bit about multiplication by the radius; this serves to give the chord length rather than the sine value. To be precise, if a chord has central angle θ in a circle of radius R, then its length is $2R\sin(\theta/2)$.) The approximation given by Bhāskarā I states that if $0 \le x \le 180$, then $\sin x^\circ \approx f(x)$.

From the form of f it is clear that $f(x) = f(180-x)$, so the formula captures the symmetry of the sine function about the 90° point. Here is a comparison of the values of $\sin x^\circ$ and $f(x)$, given to three significant figures, for some x-values:

x	0	15	30	45	60	75	90
$\sin x^\circ$	0	0.259	0.5	0.707	0.866	0.966	1
$f(x)$	0	0.260	0.5	0.706	0.865	0.965	1

We see a striking closeness between the two sets of values. It is clear that $f(x)$ yields a very good approximation to the sine function over the interval [0°, 180°]. See [**3**] for another such comparison of values.

As part of our study of this approximation, we give a heuristic derivation of the above function, and use various criteria to measure the degree of closeness of the intended approximation.

A simpler formulation

A change of origin and scale allows us to cast the problem in a more appealing way. We first note that

$$f(90-x) = \frac{4(90-x)(90+x)}{32400+x^2}, \qquad f(90-90x) = \frac{4(1-x^2)}{4+x^2}.$$

Since $\cos x^\circ = \sin(90-x)^\circ$, Bhāskarā's approximation (1) may be stated in an equivalent form as follows:

$$\text{For } -1 \le x \le 1,\ \cos 90x^\circ \approx \frac{4(1-x^2)}{4+x^2}.$$

For our purposes, a still more convenient form is obtained by switching to radian measure:

$$\text{For } -1 \le x \le 1,\ \cos\frac{\pi x}{2} \approx \frac{4(1-x^2)}{4+x^2}.$$

For convenience we shall refer to the function $B(x) := 4(1-x^2)/(4+x^2)$ as the *Bhāskarā function*. It is no good presenting the graphs of $C(x) := \cos \pi x/2$ and $B(x)$ on the same pair of axes, because the two graphs cannot be distinguished by the eye. We present them, instead, side by side (see FIGURE 1); their closeness is evident.

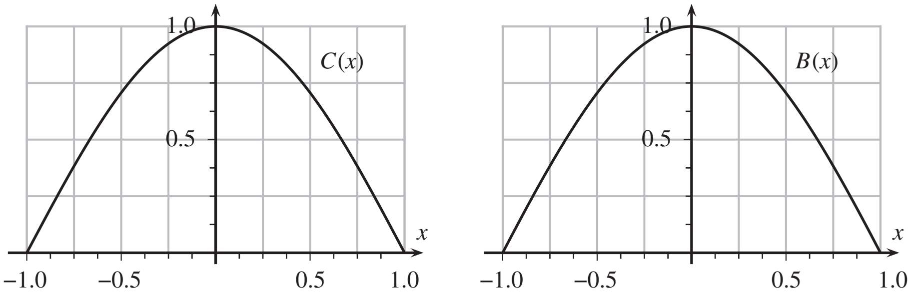

Figure 1 Graphs of $C(x) = \cos(\pi x/2)$ and $B(x) = 4(1 - x^2)/(4 + x^2)$

How good an approximation is it?

Various criteria may be used to assess how close two given functions are to each other. We make the following observations with regard to the two functions $C(x) = \cos \pi x/2$ and $B(x) = 4(1 - x^2)/(4 + x^2)$ defined on the interval $[-1, 1]$.

1. Both functions are even, and both are concave over $-1 \leq x \leq 1$. Their points of intersection with the two axes match exactly.
2. Other than $x = \pm 1$ and $x = 0$, the curves also intersect at $x = \pm 2/3$. (Indeed, these five values of x give all the points of intersection of the two curves.)
3. *Comparison of area.* The areas of the regions enclosed by the curves and the x-axis may be compared:

$$\int_{-1}^{+1} \cos \frac{\pi x}{2}\, dx = \frac{4}{\pi} \approx 1.27324,$$

$$\int_{-1}^{+1} \frac{4(1 - x^2)}{4 + x^2}\, dx = 20 \tan^{-1} \frac{1}{2} - 8 \approx 1.27295.$$

   The values compare favourably.
4. The slopes at the left endpoint are $C'(-1) = \pi/2 \approx 1.571$ and $B'(-1) = 1.6$.
5. FIGURE 2 shows the graph of $C(x) - B(x)$ over $-1 \leq x \leq 1$. We see that the maximum value of $|C(x) - B(x)|$ over this interval is roughly 0.0016. The plot has been made using *Mathematica*; use of its `FindRoot` function reveals that the maximum is achieved at $x \approx \pm 0.872$.

   FIGURE 3 shows the graph of the percentage error, $100(1 - B(x)/C(x))$, made in using $B(x)$ to estimate $C(x)$. Observe that the percentage error is largest for x close to ± 1. The use of L'Hôpital's rule shows that the percentage error tends to $|1 - 16/5\pi| \approx 1.9\%$ as $x \to \pm 1$. However, for $|x| < 0.9$, the error does not exceed 1%.
6. Using some of the known irrational values of the cosine function, we get moderately good rational approximations to these numbers, thus:

   - For $x = 1/2$ we get $C(x) = 1/\sqrt{2}$, $B(x) = 12/17$, hence $\sqrt{2} \approx 17/12$. The error is about 0.17%.
   - For $x = 1/3$ we get $C(x) = \sqrt{3}/2$, $B(x) = 32/37$, hence $\sqrt{3} \approx 64/37$. The error is about 0.13%.
   - For $x = 4/5$ we get $C(x) = (\sqrt{5} - 1)/4$, $B(x) = 9/29$, hence $\sqrt{5} \approx 65/29$. The error is about 0.24%.

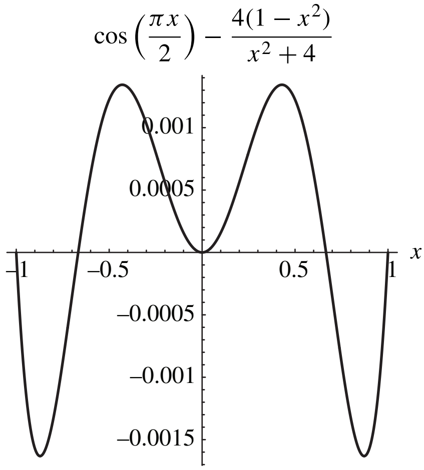

Figure 2 Graph of $C(x) - B(x)$ over $-1 \leq x \leq 1$

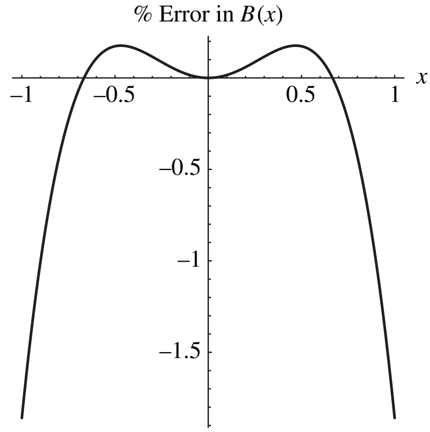

Figure 3 Graph of $100(1 - B(x)/C(x))$ over $-1 \leq x \leq 1$

The Padé approximant

To understand the relationship between the functions C and B a bit better, it helps to recall the basic facts about Padé approximants. Given a function f and integers $m, n \geq 0$, the *Padé approximant* of order (m, n) to f is that rational function

$$r(x) = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_m x^m}{1 + q_1 x + q_2 x^2 + \cdots + q_n x^n}$$

for which $f(x)$ and $r(x)$ agree at $x = 0$ up to the $(m+n)$-th derivative; the $m + n + 1$ coefficients are found using these $m + n + 1$ conditions. (See [**1**] for details.) Padé approximants are very useful in numerical work, as they provide an efficient tool for computing the values of otherwise intractable functions. For example, over the interval $-0.5 \leq x \leq 0.5$, the exponential function e^x is extremely well approximated by its Padé approximant of order (3, 3),

$$\frac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3},$$

the error never being more than 1 part in 10^7.

Let us now find the Padé approximant of order (2, 2) for the function $C(x) = \cos \pi x/2$. As the function is even, we need to include only the even powers of x in the approximant. Hence $r(x)$ has the form

$$r(x) = \frac{a + bx^2}{1 + cx^2},$$

where the constants a, b, c are to be found. The zeroth, second and fourth derivatives of $C(x)$ evaluated at $x = 0$ are

$$1, \qquad -\frac{\pi^2}{4}, \qquad \frac{\pi^4}{16}.$$

(The odd order derivatives of $C(x)$ and $r(x)$, evaluated at $x = 0$, are all 0, so we do not need to worry about them.) The zeroth, second, and fourth derivatives of $r(x)$

evaluated at $x = 0$ are

$$a, \qquad 2(b - ac), \qquad 24c(ac - b).$$

So we want

$$a = 1, \qquad 2(b - ac) = -\frac{\pi^2}{4}, \qquad 24c(ac - b) = \frac{\pi^4}{16}.$$

Solving these equations for a, b, c we get:

$$a = 1, \qquad b = -\frac{5\pi^2}{48}, \qquad c = \frac{\pi^2}{48}.$$

Thus the desired function is

$$r(x) = \frac{48 - 5\pi^2 x^2}{48 + \pi^2 x^2}.$$

We see that Bhāskarā's function $B(x)$ is close to being a Padé approximant to $\cos \pi x/2$; but its coefficients are slightly different. It is therefore reasonable to ask how these two rational approximants compare with each other.

By design, $r(x)$ agrees with $\cos \pi x/2$ at $x = 0$ for derivatives up to order 4. However it does not do so well on other fronts. The intersections of the graph of $r(x)$ with the x-axis occur at $x = \pm c$ where $c = \sqrt{48/5\pi^2}$, i.e., at $x \approx \pm 0.9862$, which falls a bit short of ± 1. The area enclosed by the curve and the x-axis is

$$\int_{-c}^{+c} \frac{48 - 5\pi^2 x^2}{48 + \pi^2 x} \, dx \approx 1.2664.$$

The discrepancy between this and the true value (1.27324) is larger than for Bhāskarā's function $B(x)$.

FIGURE 4 shows the graph of $C(x) - r(x)$ over $-1 \le x \le 1$. We see that the curve lies above the x-axis (i.e., $C(x) - r(x) \ge 0$ for $-1 \le x \le 1$) and has a very flat portion around $x = 0$; this is clearly a consequence of the equality of the first four derivatives of $C(x)$ and $r(x)$ at $x = 0$. But outside this region the curve rises more steeply, and the maximum value of $|C(x) - r(x)|$ over $-1 \le x \le 1$, achieved at $x = \pm 1$, is

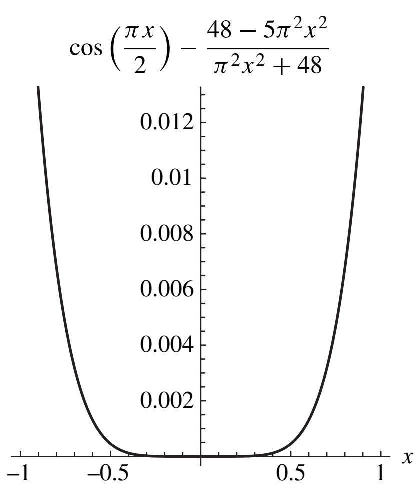

Figure 4 Graph of $\cos \pi x/2 - (48 - 5\pi^2 x^2)/(48 + \pi^2 x^2)$ over $-1 \le x \le 1$

much larger than the maximum value of $|C(x) - B(x)|$ over $-1 \le x \le 1$. (Indeed, $\max_{-1\le x\le 1} |C(x) - r(x)|$ is roughly 0.023, as compared to a maximum value of about 0.0016 for $|C(x) - B(x)|$.)

It is curious that Bhāskarā's function $B(x)$ out-performs the Padé approximant $r(x)$ on many counts.

Heuristic derivation of Bhaskara's function

We now show a heuristic way of arriving at Bhāskarā's function $B(x)$ as a rational approximation for $C(x) = \cos \pi x/2$ over the interval $-1 \le x \le 1$.

Since the graph of $C(x)$ over $-1 \le x \le 1$ is a concave arch passing through the points $(\pm 1, 0)$ and $(0, 1)$, a simple minded first approximation to $C(x)$ over the same interval is the function $1 - x^2$, whose graph shows the same features. But this function consistently yields an overestimate (except, of course, at $x = 0, \pm 1$); see FIGURE 5.

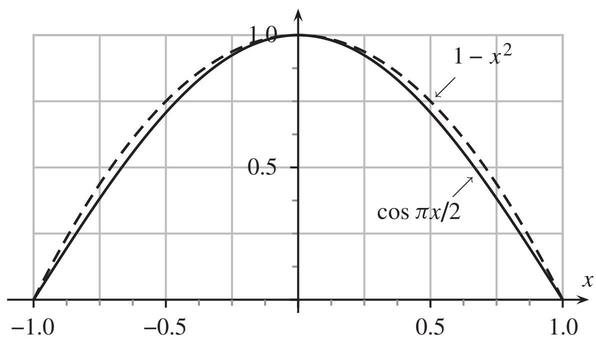

Figure 5 Graphs of $\cos(\pi x/2)$ and $1 - x^2$ over $-1 \le x \le 1$

In order to "fix" the overestimate, we examine the quotient

$$p(x) = \frac{1 - x^2}{\cos \pi x/2}$$

a little more closely. FIGURE 6 shows the graph of $p(x)$ for $-1 \le x \le 1$. Note that at $x = \pm 1$ the indeterminate form $0/0$ is encountered, but if we require p to be continuous at $x = \pm 1$ and use L'Hôpital's rule, then we get $p(\pm 1) = 4/\pi \approx 1.27$.

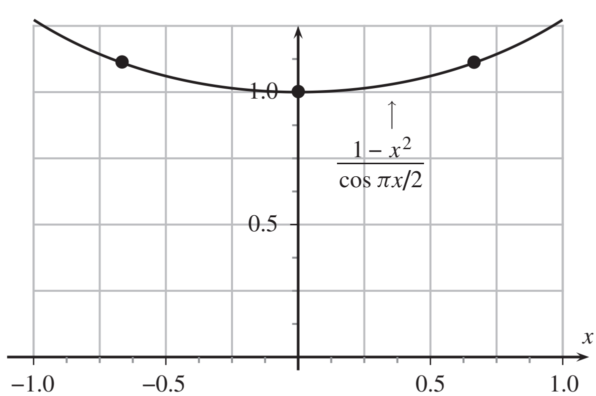

Figure 6 Graph of $(1 - x^2)/\cos(\pi x/2)$ over $-1 \le x \le 1$

(For fixing the overestimate, we could also study the reciprocal of $p(x)$, namely, the function $\cos(\pi x/2)/(1-x^2)$, rather than $p(x)$; but we opt for $p(x)$ as its graph has a more familiar shape and proves easier to approximate.)

The shape is strongly suggestive of a parabolic function, so let us look for such a function to fit the data. To this end we mark three points on the graph: (0, 1) and $(\pm 2/3, 10/9)$; conveniently for us, points with rational coordinates are available. For the parabola $y = d + ex^2$ to pass through them we must have $d = 1$ and $d + 4e/9 = 10/9$, giving $e = 1/4$. So the desired parabolic function is $y = 1 + x^2/4$, which means that we have the approximate relation

$$\frac{1-x^2}{\cos \pi x/2} \approx 1 + \frac{x^2}{4} \qquad (-1 \le x \le 1).$$

This leads right away to Bhāskarā's approximation, in which all the coefficients are rational numbers:

$$\cos \frac{\pi x}{2} \approx \frac{1-x^2}{1+x^2/4} = \frac{4(1-x^2)}{4+x^2}.$$

Adjustments using the Maclaurin series

Since the Maclaurin series for $C(x) = \cos \pi x/2$ about $x = 0$ is $1 - \pi^2 x^2/8 + \cdots$, while that of $B(x) = 4(1-x^2)/(4+x^2)$ about $x = 0$ is $1 - 5x^2/4 + \cdots$, the closeness of the two functions $C(x)$ and $B(x)$ for $-1 \le x \le 1$ may also be attributed to the approximate relation $\pi^2/8 \approx 5/4$ (which is equivalent to $\pi^2 \approx 10$).

Now the approximation $\pi^2 \approx 9.9$ is clearly better than $\pi^2 \approx 10$ (naturally, we prefer to use a rational approximation for π^2). Can we exploit this fact and improve on Bhāskarā's approximation, by replacing the '4' in that approximation by some suitable number a? The Maclaurin series for $a(1-x^2)/(a+x^2)$ about $x = 0$ is $1 - (1 + 1/a)x^2 + \cdots$, so we must solve the equation $1 + 1/a = 9.9/8$ for a; we get $a = 80/19$. This yields a new approximation:

$$B_1(x) := \frac{80(1-x^2)}{80+19x^2}.$$

Does this do better than Bhāskarā's approximation, $B(x)$? Contrary to expectation, it does not. It *does* do better on the interval $-0.5 \le x \le 0.5$; for example, for $x = 0.4$ we have:

$$\cos 0.2\pi \approx 0.8090, \qquad B(0.4) \approx 0.8077, \qquad B_1(0.4) \approx 0.8092.$$

But outside this interval, Bhāskarā's function continues to do better; for example, for $x = 0.8$ we have:

$$\cos 0.4\pi \approx 0.3090, \qquad B(0.8) \approx 0.3103, \qquad B_1(0.8) \approx 0.3125.$$

Approaches using the functional equation

Another approach, quite different in motivation, comes from the basic functional equation satisfied by the cosine function: $\cos 2x = 2\cos^2 x - 1$. This suggests that a possible criterion for closeness between a candidate function $g(x)$ and the cosine function $C(x)$, on the interval $[-1, 1]$, is the closeness between $g(x)$ and $2\,(g(x/2))^2 - 1$, on the same interval. Invoking the least-squares philosophy, we could look for a function g, within some well-defined class, which minimizes the quantity

$$\int_{-1}^{+1} \left(g(x) - 2(g(x/2))^2 + 1\right)^2 \, dx.$$

We shall stick to functions $g(x)$ of the type $(a + bx^2)/(c + x^2)$ which satisfy the boundary conditions $g(0) = 1$, $g(\pm 1) = 0$; these imply that $a = c$, $b = -a$. So our class consists of all the functions $g(x, a)$ of the type

$$g(x, a) := \frac{a(1 - x^2)}{a + x^2},$$

where a is a real number.

Analytically attempting to find the value of a that minimizes the function

$$k(a) := \int_{-1}^{+1} \left(g(x, a) - 2(g(x/2, a))^2 + 1\right)^2 \, dx$$

leads to decidedly unpleasant expressions, so we opt instead to do it numerically, using *Mathematica*. (We can use *Mathematica*'s `FullSimplify` command to get $k(a)$ in closed form, but the form is so forbidding that our aspirations for a closed form minimization quickly cool down.) FIGURES 7 and 8 display plots of $k(a)$ for $1 \le a \le 6$ and $3.8 \le a \le 4.8$, respectively, obtained using this CAS. We see that a minimum value of $k(a)$ occurs near $a = 4.3$. Use of the `FindRoot` function of *Mathematica* allows us to get this value more precisely; it is roughly 4.294.

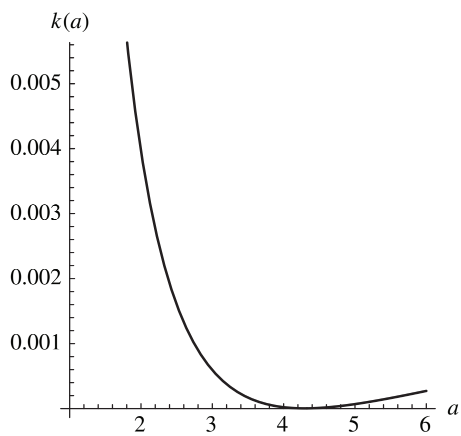

Figure 7 Plot of $k(a)$ for $1.0 \le a \le 6.0$

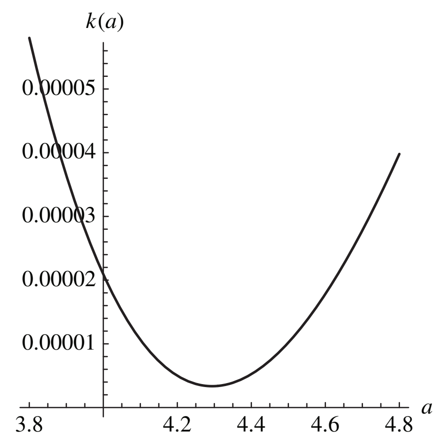

Figure 8 Plot of $k(a)$ for $3.8 \le a \le 4.8$

This idea can be extended by recalling another functional equation satisfied by the cosine function: $\cos 3x = 4\cos^3 x - 3\cos x$. Now we seek the value of a that minimizes the following function $j(a)$:

$$j(a) := \int_{-1}^{+1} \left(g(x, a) - 4(g(x/3, a))^3 + 3g(x/3, a)\right)^2 \, dx.$$

A plot of $j(a)$ for $4.0 \le a \le 4.8$ is shown in FIGURE 9. Once again, we use the `FindRoot` function to find more precisely the minimizing value of a; it is found to be roughly 4.366.

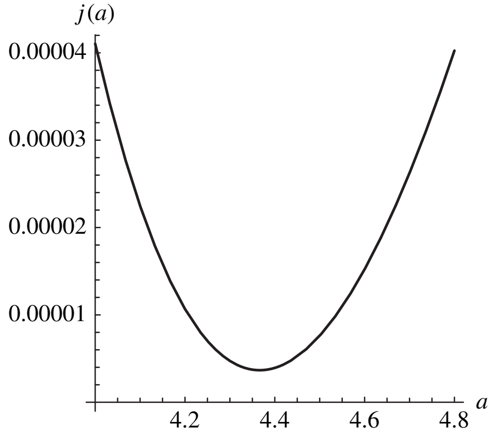

Figure 9 Plot of $j(a)$ for $4.0 \leq a \leq 4.8$

It is striking that the values of a that minimize $k(a)$ and $j(a)$ respectively are both close to 4; this means that the functions they yield are numerically not very far from Bhāskarā's function $B(x)$.

Remarks on the origin of the approximation

It is remarkable that Bhāskarā's approximation has fielded all the challenges we have thrown at it, and has walked away with credit!

There remains now the crucial question of the origin of the approximation. How did Bhāskarā I hit upon his formula? Unfortunately, we draw a blank here. Despite much thought having gone into this question, the origins remain obscure. Possible explanations have been offered, for example, in [**9**, p. 105], [**4**, pp. 121–136], [**5**, pp. 39–41], [**6**, pp. 39–41]; but these are essentially derivations from a modern viewpoint, much like the one in this article. In that respect none of them seems really satisfactory.

A feature common to early Indian mathematical writing is that justifications are rarely (if ever) given. (An exception is provided by the Kerala school of mathematics which flourished between the 14th and 16th centuries; see [**7**] and [**8**, pp. 291–306].) In the absence of definitive data, we may never know just how Bhāskarā I came upon this truly remarkable approximation; whether actually it is Āryabhaṭa's work and not Bhāskarā's; or how so non-intuitive a notion as one function approximating another might have arisen in that distant era. It is not the kind of relation that one would hit upon by chance, and one can only speculate about the depth of mathematical insight needed to find it.

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Summary In the seventh century AD the Indian mathematician Bhāskarā I gave a curious rational approximation to the sine function; he stated that if $0 \le x \le 180$ then $\sin x$ deg is approximately equal to $4x(180 - x)/(40500 - x(180 - x))$. He stated this in verse form, in the style of the day, and attributed it to his illustrious predecessor Āryabhaṭa (fifth century AD); however there is no trace of such a formula in Āryabhaṭa's known works. Considering the simplicity of the formula it turns out to be astonishingly accurate. Bhāskarā did not give any justification for the formula, nor did he qualify it in any way. In this paper we examine the formula from an empirical point of view, measuring its goodness of fit against various criteria. We find that the formula measures well, and indeed that these different criteria yield formulas that are very close to the one given by Bhāskarā.

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