Math 371
Exam 1, March 13, 2015

Name: Solution

You have until the end of the hour to complete this exam. Show all work, justify your solutions completely, simplify as much as possible. The only materials you should have on your desk are this exam and a pencil. If you have any questions, be sure to ask for clarification.

1. (10 points) (a) Convert the binary number 1011.11 to decimal.

$$1011.11_2 = 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 + 1\cdot 2^{-1} + 1\cdot 2^{-2}$$
$$= 8 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} = 11.75.$$

(b) Convert the decimal number 27 to binary.

$$27 = 16 + 8 + 2 + 1 = 1\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^1 + 1\cdot 2^0$$
$$= 11011_2.$$

(c) Add the above two numbers using binary arithmetic.

$$\begin{array}{r} 1011.11 \\ +\,11011.00 \\ \hline 100110.11 \end{array}$$

Result: 100110.11_2

2. (5 points) Consider a number system which can only store numbers of the form $\pm 1.b_1 b_2 \times 2^E$ for $E = -1, 0, 1$. Exactly, what is machine epsilon in this system and why?

ε is the gap between 1 & next largest representable number.
$1 = 1.00_2 \times 2^0$ & next largest is $1.01_2 \times 2^0 = 1 + \tfrac{1}{4} = \tfrac{5}{4}$.
So, $\varepsilon = \tfrac{5}{4} - 1 = \tfrac{1}{4}$ in this system.

3. (10 points) (a) Compute by hand the 4th degree Taylor polynomial $P(x)$ for function $f(x) = \sin(x)$ around $a = 0$.

$f(x) = \sin(x)$
$f'(x) = \cos(x)$
$f''(x) = -\sin(x)$
$f'''(x) = -\cos(x)$
$f^{(4)}(x) = \sin(x)$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 0 + x + 0 - \frac{x^3}{3!} + 0$$
$$= x - \tfrac{1}{6}x^3.$$

(b) Use Taylor's theorem to compute the maximum error of $|f(x) - P(x)|$ on $-0.3 \le x \le 0.3$.

Note, $f^{(5)}(x) = \cos(x)$. Then,

$$|f(x) - P(x)| = \left|\frac{f^{(5)}(\xi)}{5!}x^5\right| = \frac{|\cos(\xi)\, x^5|}{5!} \le \frac{1\cdot 0.3^5}{5!} = \frac{(0.3)^5}{5!}.$$

Some $\xi \in [-0.3, 0.3]$

4. (10 points) If you use the nth degree Taylor polynomial of $f(x) = e^x$ centered at $x_0 = 0$ to approximate e, what should n be to guarantee accuracy within absolute error 10^{-9}.

Note $f(1) = e$, so let $f(x) \approx P_n(x)$ on $[0,1]$.

We want $|f(1) - P_n(1)| \leq 10^{-9}$.

$$\rightarrow |f(1) - P_n(1)| = |R_n(1)| = \left|\frac{e^{\xi}}{(n+1)!} 1^{n+1}\right| \leq \max_{0 \leq \xi \leq 1} \left|\frac{e^{\xi}}{(n+1)!}\right| = \frac{e}{(n+1)!}$$

Want $\frac{e}{(n+1)!} < 10^{-9}$, so need n such that

$$(n+1)! > e \cdot 10^9.$$

5. (10 points) (a) Let the rootfinding problem $f(x) = 0$ have solution $x = p$. Also, let p_n be the nth term found by the bisection method. Show that if we want the absolute error less than some error tolerance TOL, i.e. $|p - p_n| < TOL$, we need $n > \log_2\left(\frac{b-a}{TOL}\right)$.

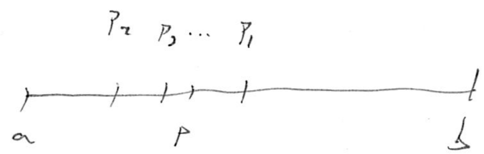
At each step, bisection halves interval $[a,b]$. Then,

$$|p - p_1| < \frac{b-a}{2}$$

$$|p - p_2| < \frac{b-a}{2^2}$$

$$\vdots$$

$$|p - p_n| < \frac{b-a}{2^n}.$$



So, for $|p - p_n| < TOL$, require $\frac{b-a}{2^n} < TOL$

$$\rightarrow 2^n > \frac{b-a}{TOL} \rightarrow n > \log_2\left(\frac{b-a}{TOL}\right).$$

(b) If the bisection method converges, what rate does it converge at? What does this mean precisely?

Bisection converges linearly. That is,

$$|p - p_n| \leq K |p - p_{n-1}|^{1}$$ ← 1st power.

6. (12 points) Consider the fixed point problem $x = g(x) = \frac{1}{2}\left(x + \frac{3}{x}\right)$.

(a) State a fixed-point iteration for this problem.

$x_n = g(x_{n-1}) \rightarrow x_n = \frac{1}{2}\left(x_{n-1} + \frac{3}{x_{n-1}}\right)$, x_0 given.

(b) Rewrite this fixed point problem as a root-finding problem.

$x = g(x) \rightarrow x = \frac{1}{2}\left(x + \frac{3}{x}\right) \rightarrow 2x^2 = x^2 + 3 \rightarrow x^2 - 3 = 0$.

So, $f(x) = x^2 - 3 = 0$ is a root-finding problem.

(c) State Newton's method for the root-finding problem in (b).

$f(x) = x^2 - 3$, $f'(x) = 2x$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow x_{n+1} = x_n - \frac{(x_n^2 - 3)}{2x_n}$, x_0 given.

7. (13 points) Let $g(x) = \frac{1}{10}(x^2 + x + 8)$.

(a) Find the smallest positive fixed point of g.

$x = g(x) \rightarrow x = \frac{1}{10}(x^2 + x + 8) \rightarrow 10x = x^2 + x + 8 \rightarrow x^2 - 9x + 8 = 0$

$\rightarrow (x-8)(x-1) = 0 \rightarrow x = 1, 8$.

So, $p = 1$ is the smallest fixed point of g.

(b) Using the Fixed Point Theorem from class, show that starting with any $x_0 \in [0, 4]$, the sequence $x_n = g(x_{n-1})$ will converge to the smallest fixed point of g.

$g'(x) = \frac{1}{10}(2x + 1) = \frac{x}{5} + \frac{1}{10} > 0$ on $[0,4]$. So, g is always increasing on $[0,4]$. $g(0) = 8/10 = 4/5$, $g(4) = \frac{16+4+8}{10} = 14/5$.

So, $g: [0,4] \rightarrow [4/5, 14/5] \subset [0,4]$. Also, $|g'(x)| = \left|\frac{x}{5} + \frac{1}{10}\right| \le \left|\frac{4}{5} + \frac{1}{10}\right| < 1$ on $[0,4]$. By the Fixed Point Theorem, $x_n = g(x_{n-1})$ converges at least linearly to $p = 1$ for any $x_0 \in [0,4]$.

(c) What is the rate of convergence of the fixed point iteration in (b)? Can it be quadratic?

$g''(x) = \frac{1}{5} \neq 0$ for $x = p = 1$; $g'(p) = 3/10 \neq 0$.

So, we cannot ensure quadratic convergence. Only linear RoC is guaranteed.

8. (20 points) Consider the following system, $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

(a) Perform Gaussian elimination WITHOUT pivoting to solve this system. Use an augmented matrix and show all steps.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ -2 & 1 & -1 & 2 \\ 4 & -1 & 2 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & -1 & 3 & -2 \\ 0 & 3 & -6 & 7 \end{array}\right] \xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

Backwards substitution:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= -2 \\ -x_2 + 3x_3 &= -2 \\ 3x_3 &= 1 \end{aligned} \implies \begin{aligned} x_3 &= 1/3 \\ x_2 &= -(-2 - 3(1/3)) = 3 \\ x_1 &= -2 + x_2 - 2x_3 = -2 + 3 - 2/3 = 1/3. \end{aligned}$$

Then, $\vec{x} = \begin{bmatrix} 1/3 \\ 3 \\ 1/3 \end{bmatrix}$.

(b) Find the LU decomposition of matrix A without pivoting, and use this decomposition to solve this system. Feel free to use work from part (a).

$A = LU$. From (a) $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

So, $A\vec{x} = \vec{b} \Rightarrow LU\vec{x} = \vec{b}$.

Solve $L\vec{y} = \vec{b}$, then $U\vec{x} = \vec{y}$.

$\boxed{L\vec{y} = \vec{b}}$ $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \Rightarrow$ $y_1 = -2$, $y_2 = 2 + 2 \cdot 2 = -2$, $y_3 = -1 - 4(-2) + 3(-2) = 1$ $\Rightarrow \vec{y} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$.

$\boxed{U\vec{x} = \vec{y}}$ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ Solved in (a)! $\vec{x} = \begin{bmatrix} 1/3 \\ 3 \\ 1/3 \end{bmatrix}$.

9. (1 extra credit point) π-day bonus! State π correct to 6 decimal digits.

$\pi = 3.141592\ldots$