Assigned: Wednesday, March 4, 2015
Due: Wednesday, March 11, 2015

MTH 371: Homework 6
Gaussian Elimination and Pivoting

GENERAL HOMEWORK GUIDELINES:

- On the very first page of your homework, provide your name, date, and homework number.
- Homework will be graded in part on neatness, organization, and completeness of solutions. Multiple pages MUST BE STAPLED.
- Attach all Scilab code, output, and plots to the page immediately following each problem. Also, clearly indicate the problem they correspond to.

1. Consider the linear system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Find the LU factorization of matrix A and use it to solve $A\vec{x} = \vec{b}$. That is, write $A\vec{x} = L(U\vec{x}) = \vec{b}$, then solve $L\vec{y} = \vec{b}$ and next $U\vec{x} = \vec{y}$. Show this by hand and clearly show every step of the construction of L and U as well as all steps for forward and backwards substitution.

2. Consider the linear system $A\vec{x} = \vec{b}$ for

$$A = \begin{bmatrix} 2 & 4 & -2 & -2 \\ 1 & 2 & 4 & -3 \\ -3 & -3 & 8 & -2 \\ -1 & 1 & 6 & -3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -4 \\ 5 \\ 7 \\ 7 \end{bmatrix}$$

   (a) Using partial pivoting, compute the LU decomposition of the matrix A with partial pivoting at each step resulting in $PA = LU$. Show all steps of this decomposition by hand.
   (b) Verify $PA = LU$ by matrix multiplication.
   (c) Use the decomposition from (a) to solve $A\vec{x} = \vec{b}$.

3. Write Scilab Functions for Gaussian elimination via LU decomposition for the general n-dimensional system $A\vec{x} = \vec{b}$. To do this, create functions which implement forward substitution, backward substitution, LU decomposition without pivoting, and LU decomposition with partial pivoting. Functions should be written so they can be called in Scilab by typing:
   (a) `y = ForwardSubs(L,b)` (forward substitution)
   (b) `x = BackwardsSubs(U,y)` (backwards substitution)
   (c) `[L,U] = LU(A)` (LU decomposition, no pivoting)
   (d) `[P,L,U] = PLU(A)` (LU decomposition with partial pivoting, $PA = LU$)

All functions should be stored in a dedicated .sci file, so 4 files total. Include this code with your homework submission. You should refer to the Gaussian elimination code you created in last week's homework.

4. Use the code developed in problem 3 to verify your work from problem 1. Write a .sce file which calls the needed functions. Print the results generated by your code and include this .sce file with your homework submission.

5. Use the code developed in problem 3 to verify your work from problem 2. Write a .sce file which calls the needed functions. Print the results generated by your code and include this .sce file with your homework submission.

6. The matrix factorization $LU = PA$ can be used to compute the determinant of A.
   (a) Explain why it must be true that $\det(L)\det(U) = \det(P)\det(A)$.
   (b) What is $\det(L), \det(U)$ and $\det(P)$ in general and why? Each of these should be easy to compute.
   (c) Modify your function of problem 3(d) `[P,L,U,sig] = PLU(A)` to also returns `sig` equal to $+1$ or -1 if `P` is an even or odd permutation.
   (d) Create a new function `mydet(A)` which calls function `PLU(A)` of part 6(c) and only uses entries of `U` and the value of `sig`. Test this function on the matrix of problem 2. Compare your result to the built in Scilab determinant function.

7. (a) The inverse of a matrix A can be defined s the matrix X whose columns $\vec{x}_j$ solve the equations
   $$A\vec{x}_j = \vec{e}_j$$
   where $\vec{e}_j$ is the jth unit basis vector. Explain why this is true.
   (b) Create your own function `myinv(A)` which calls `PLU(A)` from problem 3 *only once* and also utilizes `BackwardsSubs(U,y)`.
   (c) Test your function on problems 1 and 2 and compare to the built in Scilab function.