Math 371
Final Exam, May 13, 2014 Name: Solution

You have two hours to complete this exam. Show all work, justify your solutions completely, simplify as much as possible. The only materials you should have on your desk are this exam and a pencil. If you have any questions, be sure to ask for clarification.

1. (12 points) (a) State the polynomial interpolation error theorem.

For $f \in C^{n+1}[a,b]$ & $P(x)$ the polynomial interpolant of
2 $f(x)$ of degree at most n through $(n+1)$ points
$x_0 < x_1 < \dots < x_n$, then for each $x \in [a,b]$ there exists a $\xi(x) \in [a,b]$ st.
(in $[a,b]$)
$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

(b) Prove the polynomial interpolation error theorem.

For $x = x_i$, $i = 0, \dots, n$, both sides are zero & we are done.
Assume $x \neq x_i$ & denote $w(x) = (x-x_0)(x-x_1)\dots(x-x_n)$. Consider
x fixed & define $c = \dfrac{f(x)-p(x)}{w(x)}$. Define function g in
10 variable t as

$$g(t) = f(t) - P(t) - c\,w(t).$$

Then, $g \in C^{n+1}[a,b]$ since $f \in C^{n+1}[a,b]$ & p, w are polynomials.
Also, $g(x_i) = f(x_i) - p(x_i) - c\,w(x_i) = 0$, $i = 0, 1, \dots, n$
& $g(x) = f(x) - p(x) - c\,w(x) = (f(x)-p(x)) - \left(\frac{f(x)-p(x)}{w(x)}\right)w(x) = 0.$

So, g has $(n+2)$ zeros on $[a,b]$.

By Rolle's theorem, $g^{(n+1)}$ has 1 zero on $[a,b]$,
call it $\xi = \xi(x) \in [a,b]$.

$$\Rightarrow g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \underbrace{p^{(n+1)}(\xi)}_{0} - c\,\underbrace{w^{(n+1)}(\xi)}_{(n+1)!} = 0$$

$$\Rightarrow c = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\Rightarrow f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

$\square$

2. (12 points) Let function $g(x) = x^2(2-x)$.

(a) Complete the following statement. A sequence x_n converges to number L quadratically if

there exists a constant K such that

$|x_{n+1} - L| \le K|x_n - L|^2$, all n sufficiently large.

(b) Write down a fixed point iteration for finding the fixed points of g.

$x_{n+1} = g(x_n)$, $n = 0, 1, \ldots$, x_0 given

(c) Find the fixed points of g algebraically.

$x = g(x) \to x = \cancel{x}x^2(2-x) \to x = 2x^2 - x^3$

$\to x^3 - 2x^2 + x = 0 \to x(x^2 - 2x + 1) = 0 \to x(x-1)^2 = 0 \to x = 0$ or $x = 1$.

(d) To which of the fixed points found in (c) will the iteration in (b) converge quadratically? Why?

Require $g'(x) = 0$ at fixed point x.

$g'(x) = 4x - 3x^2 = x(4-3x) = 0$. So only $x = 0$ is quadratic.

(e) Find a function f with roots the same as the fixed points of g.

$x = x^2(2-x) \to f(x) = x(x-1)^2$ has roots

$x = 0$ & $x = 1$.

(f) Write down Newton's method for finding the roots of function f from part (e).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \to x_{n+1} = x_n - \frac{x_n(x_n-1)^2}{(x_n-1)(3x_n-1)}, \quad n = 0, 1, \ldots \quad x_0 \text{ given}$$

(g) To which of the roots found of function f will the iteration in (f) converge quadratically? Why?

Require $f'(x) \neq 0$ at root x.

So, only $x = 0$ is quadratic (again).

3. (12 points) Find the LU decomposition without partial pivoting for the following matrix. Show all steps!

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 5 & 1 & 1 \end{bmatrix}$$

Rewrite in row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 5 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 = \cancel{R_2} R_3 - 5R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -9 & -14 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - \frac{9}{5}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -5 \end{bmatrix}.$$

So $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 9/5 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -5 \end{bmatrix}$

are such that $A = LU$.

4. (12 points) Prove that a polynomial interpolant of degree at most n through the $(n+1)$ points $\{(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\}$ must be unique.

Let $P(x)$ & $Q(x)$ both interpolate f & be of degree at most n. Then, define

$$R(x) = P(x) - Q(x).$$

So, R is also of degree at most n & a polynomial. Then, for each x_i, $i = 0, 1, \ldots, n$,

$$R(x_i) = P(x_i) - Q(x_i)$$
$$= f(x_i) - f(x_i) = 0$$

So, R has $(n+1)$ total zeros. By the Fundamental Theorem of Algebra,

$$R(x) \equiv 0.$$

Therefore $P(x) = Q(x)$ & we have uniqueness.

5. (12 points) (a) A freshman calculus student was amazed to learn that the prime numbers have no predictable pattern. Mess with their head by using divided differences to produce a polynomial P for which $P(n)$ gives the nth prime number for $n = 1, 2, 3, 4, 5$. (Recall, the first prime number is 2).

$f(1)=2,\ f(2)=3,\ f(3)=5,\ f(4)=7,\ f(5)=11.$

$$P(x)=f[1]+f[1,2](x-1)+f[1,2,3](x-1)(x-2)+f[1,2,3,4](x-1)(x-2)(x-3)$$
$$+f[1,2,3,4,5](x-1)(x-2)(x-3)(x-4)$$

x	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot,\cdot]$
1	$f[1]=2$				
		$f[1,2]=\frac{3-2}{2-1}=1$			
2	$f[2]=3$		$f[1,2,3]=\frac{2-1}{3-1}=\frac{1}{2}$		
		$f[2,3]=\frac{5-3}{3-2}=2$		$f[1,2,3,4]=\frac{0-\frac{1}{2}}{4-1}=-\frac{1}{6}$	
3	$f[3]=5$		$f[2,3,4]=0$		$f[1,2,3,4,5]=\frac{\frac{1}{3}+\frac{1}{6}}{5-1}=\frac{1}{8}.$
		$f[3,4]=\frac{7-5}{4-3}=2$		$f[2,3,4,5]=\frac{1-0}{5-2}=\frac{1}{3}$	
4	$f[4]=7$		$f[3,4,5]=\frac{4-2}{5-3}=1$		
		$f[4,5]=\frac{11-7}{5-4}=4$			
5	$f[5]=11$				

So, $p(x)=2+(x-1)+\frac{1}{2}(x-1)(x-2)+\frac{-1}{6}(x-1)(x-2)(x-3)+\frac{1}{8}(x-1)(x-2)(x-3)(x-4)$

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(b) Assuming the existence of such a miraculous function f for which $f(n)$ gives the nth prime number, provided the error formula for $f(6) - P(6)$.

$$f(6)-P(6)=\frac{f^{(5)}(\xi)}{5!}(6-1)(6-2)(6-3)(6-4)(6-5),\quad \text{Some } \xi\in[1,6].$$

6. (12 points) (a) List all 8 conditions for $S(x)$ to be a natural cubic spline for a function f through points $(-1, f(-1))$, $(0, f(0))$, $(1, f(1))$.

$S(-1) = f(-1)$, $S(0) = f(0)$, $S(1) = f(1)$ (4 total conditions)

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$S'(0^-) = S'(0^+)$, $S''(0^-) = S''(0^+)$ (2 conds.)

$S''(-1) = 0$, $S''(1) = 0$ (2 conds)

So, $4+2+2=8$ grand total conditions.

(b) Do there exist constants a, b, c, d such that the function

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$$S(x) = \begin{cases} ax^3 + x^2 + cx, & -1 \le x \le 0 \\ bx^3 + x^2 + dx, & 0 \le x \le 1 \end{cases}$$

is a natural cubic spline for $f(x) = |x|$ through points $(-1, 1)$, $(0, 0)$, $(1, 1)$.

$S(-1) = \boxed{-a+1-c=1}$, $S(0^-) = S(0^+) = 0$ ✓

$S(1) = \boxed{b+1+d=1}$.

$S'(x) = \begin{cases} 3ax^2 + 2x + c, & -1 \le x \le 0 \\ 3bx^2 + 2x + d, & 0 \le x \le 1 \end{cases}$, $S''(x) = \begin{cases} 6ax + 2, & -1 \le x \le 0 \\ 6bx + 2, & 0 \le x \le 1 \end{cases}$.

$S'(0^-) = S'(0^+) \to \boxed{c=d}$, $S''(0^-) = S''(0^+)$ ✓

$S''(-1) = \boxed{-6a+2=0}$, $S''(1) = \boxed{6b+2=0}$

$a = 1/3$, $b = -1/3$, $c = -a = -1/3$, $d = -b = 1/3$

Then, $c \neq d$. So, not possible.

7. (12 points) Recall that the error for the trapezoidal rule on one subinterval is given by

$$\int_a^b f(x)\,dx - T_1(a,b) = -\frac{(b-a)^3}{12} f''(\xi)$$

where $T_1(a,b)$ denotes the trapezoidal rule you derived in part (a). Show that the error for the n step composite trapezoidal rule approximation is given by

$$\int_a^b f(x)\,dx - T_n(a,b) = -\frac{(b-a)}{12} f''(\eta)h^2$$

where $T_n(a,b)$ denotes the n step composite trapezoidal rule and h is the size of each subinterval.

Denote $h = \frac{b-a}{n}$ & subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ for $x_i = a + ih$. Then,

$$\int_a^b f(x)\,dx - T_n(a,b) = \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] \right)$$

$$= \sum_{i=0}^{n-1} \left(-\frac{h^3}{12} f''(\xi_i) \right), \quad \xi_i \in [x_i, x_{i+1}]$$

$$= \frac{-h^2}{12} \left(\frac{b-a}{n} \right) \sum_{i=0}^{n-1} f''(\xi_i)$$

$$= -\frac{h^2}{12}(b-a) f''(\xi), \quad \xi \in [a,b] \quad \left(\text{By the IVT.} \right)$$

$\square$

8. (12 points) (a) Find the needed constants A_0, A_1, x_0, x_1 so that the quadrature rule of the form

$$\int_{-1}^{1} f(x)\, dx \approx A_0 f(x_0) + A_1 f(x_1)$$

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is exact for degree 3 polynomials. What is the name of this quadrature rule? (Hint: Assume $A_0 = 1$.)

Want exactness for $f(x) = 1, x, x^2, x^3$.

$f(x)=1$: $2 = 1 + A_1 \to \boxed{A_1 = 1}$

$f(x)=x$: $0 = x_0 + x_1 \to x_0 = -x_1$

$f(x)=x^2$: $2/3 = x_0^2 + x_1^2 \to x_0^2 = 1/3$

$\to \boxed{x_0 = -1/\sqrt{3},\ x_1 = 1/\sqrt{3}}$

$f(x)=x^3$: $0 = x_0^3 + x_1^3$ ✓

So,

$\int_{-1}^{1} f(x)\,dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$.

This is a Gaussian quadrature rule.

(b) Use part (a) to find constants B_0, B_1, t_0, t_1 so that the quadrature rule of the form

$$\int_{-1}^{1} t g(t)\, dt \approx B_0 g(t_0) + B_1 g(t_1)$$

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which is exact for degree 3 polynomials.

Let $f(t) = t\underline{g}(t)$. Then, via the above,

$\int_{-1}^{1} t\underline{g}(t)\,dt = \int_{-1}^{1} f(t)\,dt = f(-1/\sqrt{3}) + f(1/\sqrt{3})$

$= -\frac{1}{\sqrt{3}}\underline{g}\left(-\frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\underline{g}\left(\frac{1}{\sqrt{3}}\right)$.

9. (12 points) (a) Give the Taylor series expansion for $f(x-h)$ and $f(x+h)$ with center x up til error $O(h^5)$.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + O(h^5)$$

4 $$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + O(h^5)$$

(b) Use part (a) to derive the central difference approximation formula for $f'(x)$. What is the order of the error?

From (a),

$$f(x+h) - f(x-h) = 2hf'(x) + 2\frac{h^3}{3!}f'''(x) + O(h^5)$$

4 $$\rightarrow f'(x) = \underbrace{\frac{f(x+h)-f(x-h)}{2h}}_{\text{Central diff.}} + \underbrace{O(h^2)}_{\text{Error.}}$$

(c) Use your results in part (b) along with Richardson Extrapolation to create an approximation of $f'(x)$ which is $O(h^4)$.

Denote $\phi_0(h) = \dfrac{f(x+h)-f(x-h)}{2h}$. Then,

$$f'(x) = \phi_0(h) + \frac{h^2}{6}f''(x) + O(h^4)$$

4 Also, $$f'(x) = \phi_0(h/2) + \frac{1}{4}\frac{h^2}{6}f''(x) + O(h^4)$$

$$\rightarrow (1-4)f'(x) = \phi_0(h) - 4\phi_0(h/2) + O(h^4).$$

$$\rightarrow f'(x) = \underbrace{\frac{4}{3}\phi_0(h/2) - \frac{1}{3}\phi_0(h)}_{} + O(h^4)$$

10. (12 points) Consider the following initial value problem (IVP).

$$\begin{cases} y' = f(t,y), & a \le t \le b \\ y(a) = y_a. \end{cases}$$

(a) Give the Taylor series expansion for $y(t+h)$ with center t up til error $O(h^3)$.

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + O(h^3)$$

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$$= y(t) + hf(t,y) + \frac{h^2}{2}\left(\frac{d}{dt}f(t,y)\right) + O(h^3)$$

$$= y(t) + hf(t,y) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f\right) + O(h^3).$$

(b) Derive Euler's method for the above IVP. What is the local error and global error?

Truncate (a) at $O(h^2)$

LE: $O(h^2)$

3 E.M. $\begin{cases} w_0 = y_a \\ w_{i+1} = w_i + hf(t_i, w_i) \end{cases}$

GE: $O(h)$

(c) Derive an order 2 Taylor method for the above IVP. What is the local error and global error?

Truncate (a) at $O(h^3)$

3

LE: $O(h^3)$

Order 2 TM $\begin{cases} w_0 = y_a \\ w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t_i} + \frac{\partial f}{\partial y_i}f_i\right) \end{cases}$

GE: $O(h^2)$.

(d) Write down the methods from (b) and (c) for the following IVP.

$$\begin{cases} y' = 2ty, & 0 \le t \le 1 \\ y(0) = 1. \end{cases}$$

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Euler	2^{nd} Taylor
$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + h2t_iw_i \end{cases}$	$\begin{cases} w_0 = 1 \\ w_{i+1} = w_i + h2t_iw_i + \frac{h^2}{2}(2w_i + 2t_i^2w_i) \end{cases}$