Assigned: Wednesday, March 11, 2015
Due: Wednesday, March 25, 2015

MTH 371: Homework 7
Matrix Conditioning and Tridiagonal Systems

GENERAL HOMEWORK GUIDELINES:

- On the very first page of your homework, provide your name, date, and homework number.
- Homework will be graded in part on neatness, organization, and completeness of solutions. Multiple pages MUST BE STAPLED.
- Attach all Scilab code, output, and plots to the page immediately following each problem. Also, clearly indicate the problem they correspond to.

1. Consider the linear system $H\vec{x} = \vec{b}$ for H the below n-dimensional Hilbert matrix.

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

   For each $n = 2,\ 5,\ 8,\ 11,\ 14,\ 17$, complete the following.

   (a) Set $\vec{x}$ as a n-dimensional vector of all 1's, and compute the resulting vector $H\vec{x} = \vec{b}$.

   (b) Use $\vec{b}$ from (a) to solve the system $H\vec{x}_c = \vec{b}$ using the Scilab command `xc = H\b`.

   (c) Compute the error between your computed solution $\vec{x}_c$ and the true solution $\vec{x}$ in the ℓ^2 vector norm. That is compute $\|\vec{x}_c - \vec{x}\|_2$.

   (d) Compute the condition number of the Hilbert matrix in the ℓ^2 norm, $\text{cond}_2(H)$, using the Scilab command `cond`.

   (e) List your results from (c) and (d) in table format. Discuss your findings.

   (f) Repeat work done in parts (a) through (e) by using a random matrix A via the Scilab `rand` command. Do this for at least 3 matrices and display your results in a table.

2. Gaussian elimination can be used for polynomial interpolation. This results in a matrix called the Vandermonde matrix which is notoriously ill-conditioned. Read the example provided on page 225 of the Cheney text. Write Scilab code to compute your own results. To test, repeat the process of parts (c), (d), and (e) of problem 1 for $n = 2,\ 5,\ 8,\ 11,\ 14,\ 17$.

3. (a) Compute the ℓ^2, ℓ^1, and ℓ^∞ norm of vector $\vec{v} = [4, 5, -6]^T$. Draw a picture in $\mathbb{R}^3$ to illustrate.

   (b) Compute the ℓ^1 and ℓ^∞ norm of

$$A = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

   Also compute the condition number for A in each norm.

4. Show for all vectors $\vec{v}$ in $\mathbb{R}^n$,
   (a) $\|\vec{v}\|_\infty \leq \|\vec{v}\|_2 \leq \sqrt{n}\|\vec{v}\|_\infty$
   (b) $\|\vec{v}\|_2 \leq \|\vec{v}\|_1$
   (c) $\|\vec{v}\|_1 \leq n\|\vec{v}\|_\infty$

   For each of the above inequalities, identify a nonzero vector $\vec{v}$ of dimension greater than 1 for which equality holds.

5. A tridiagonal system is a linear system of the form

$$\begin{bmatrix} d_1 & c_1 & & & & & & \\ a_1 & d_2 & c_2 & & & & & \\ & a_2 & d_3 & c_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & a_{i-1} & d_i & c_i & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & a_{n-2} & d_{n-1} & c_{n-1} \\ & & & & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

   where all blank entries are zero.

   (a) Read section 6.3 of the Cheney text, and write an EFFICIENT Scilab function

```
x=tri(n,a,d,c,b)
```

   which solves systems of this type.

   (b) How many floating point operations are performed when solving a $n \times n$ tridiagonal system using your code?

   (c) What happens to this algorithm if pivoting is required?

   (d) Solve the following system of 100 equations. Compare the numerical solution to the obvious exact solution.

$$x_1 + 0.5x_2 = 1.5$$

$$0.5x_{i-1} + x_i + 0.5x_{i+1} = 2.0 \quad (2 \leq i \leq 99)$$

$$0.5x_{99} + x_{100} = 1.5$$