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Homework 4

1. $2x - 1 = \sin x$

(a) Find interval $[a, b]$ on which eq'n has root r, use IVT to prove r exists.

Let $f(x) = 2x - 1 - \sin(x)$, continuous.

$f(0) = -1$ and $f(\pi) = 2\pi - 1 > 0$

So, choose $[a, b] = [0, \pi]$. By IVT, since $f(x)$ is cts, $\exists$ at least one r on $[a, b]$.

Intuitively, the cts line must cross zero between the negative and positive values.

(b) Prove r is unique on $\mathbb{R}$.

Note $f'(x) = 2 - \cos(x) > 0$, so f is strictly increasing. Therefore it can only have one root.

It would need an extremum to cross zero again.

(c) Approx r to 8 decimal places using bisection code

See appended script.

Result approximation: 0.887862...
iterations: 24

The script found r between -1 and π as expected.

2. Use code from 1(c) approx the root at $x = 5$ of

$$f(x) = (x-5)^9$$

w/ $a = 4.82$, $b = 5.2$, $tol = 1e-4$

(a) Result approximation: 5.00007324...
iterations: 11

Found root with desired accurracy.

(b) Result approximation: 5.12875
iterations: 3

Found a root near $x=5$, but further from true root.

(c) The reduced accurracy in (b) results from catastrophic cancellation from the subtraction in the expanded form.

The same error does <u>not</u> result from the multiplication in (a).

3(a) Use Theorem 2.1 to find upper bound on the # iterations in bisection needed to find the root of x^3+x-4 on $[1,4]$ w/ accuracy 10^{-3}.

We need n # iterations such that the actual error (L.H.S. of Theorem 2.1) does not exceed 10^{-3}:

$$10^{-3} \geq \left(\frac{1}{2}\right)^n (4-1)$$

Bound given by

$$n = \lceil \log_{1/2}\left(\frac{10^{-3}}{3}\right) \rceil = \lceil 11.55.. \rceil = 12.$$

After no more than 12 iterations, we have error at or below 10^{-3}.

(b) Approximate root with bisection code to some accuracy. How does # iterations compare to n?

See appended code.

Approximation: 1.378662...
iterations: 11

The number of iterations is below the upper bounded, as expected.

4. Using Def^n 1, which of the following converges to x_*? If it converges, give order and, rate, for linear.

(a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $x_* = 2$

Let $f(x) = -16 + 6x + 12/x$

$f'(x) = 6 + \frac{-12}{x^2}$ and $|f'(2)| = 3$

Because $|f'(2)| > 1$, iteration does not converge.

Successive iterations will not be closer to 2, because the slope of $F(x)$ will result in large differences between iterates. Theoretically, λ would have to exceed 1, meaning increasing error, which would then make convergence impossible.

(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{(\frac{1}{3})}$

Let $f(x) = \frac{2}{3}x + \frac{1}{x^2}$,

$f'(x) = \frac{2}{3} + \frac{-2}{x^3}$ and $|f'(3^{\frac{1}{3}})| = \frac{2}{3} - \frac{2}{(3^{\frac{1}{3}})^3} = 0$

Since $|f'(x_*)| = 0$, the iteration converges.

$f''(x) = \frac{6}{x^4}$ and $f''(x_*) = \frac{6}{3^{\frac{4}{3}}} = \frac{2}{\sqrt{3}} \neq 0$

By 2.9, then, the order of convergence $\alpha = 2$.

The error reduces, meaning convergence, and the rate at which the error reduces, increases linearly.

(c) $x_{n+1} = \frac{12}{1+x_n}$, $x_* = 3$

Let $f(x) = \frac{12}{1+x}$

$f'(x) = \frac{-12}{(1+x)^2}$ and $f'(x_*) = \frac{-12}{4^2} = -\frac{3}{4} \neq 0$

$|f'(x_*)| < 1$, so it converges linearly ($\neq 0$) at rate 3/4.

The error of each iterate reduces at a constant rate, making the order $\alpha = 1$.

5. Find roots of $x - 4\sin(2x) - 3 = 0$ to 10 accurate digits.

(a) Plot $f = x - 4\sin(2x) - 3$. How many roots?

See appended plot. There are 5 crossings, meaning f has five roots.

(b) Programatically find roots of $f(x)$ using

$$x_{n+1} = -\sin(2x_n) + 5x_n/4 - 3/4$$

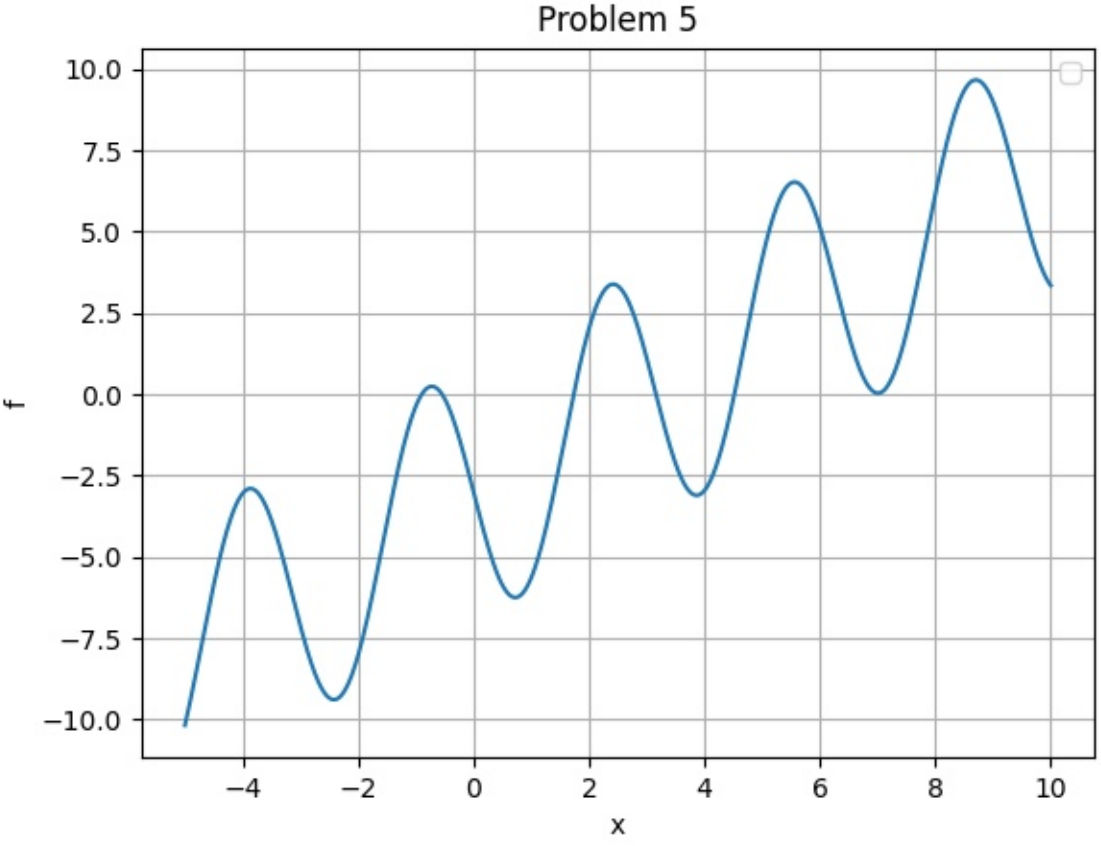
The code found $r_2 = -0.5444424006 8$
and $r_4 = 3.1618264865\ldots$ (the roots where f is decreasing)

It could not find r_1, r_3, r_5, the roots where f is increasing.

Let $g = -\sin(2x) + 5x/4 - 3/4$. At r_2 and r_4, $|g'| < 1$ but at r_1, r_3, r_5, $|g'| > 1$. So, at the latter roots $\lim_{n\to\infty} \frac{|x_{n+1} - x_*|}{|x_n - p|} > 1$, meaning the error increases, and the iteration cannot converge.

```
# PROBLEMS 1,2,3

import numpy as np

def driver():

# use routines

    f = lambda x: x**3 + x - 4

    #f = lambda x: x**9   - 45*x**8  +900*x**7  -10500*x**6
+78750*x**5 - 393750*x**4  +1312500*x**3   - 2812500*x**2  +3515625*x
- 1953125
    a = 1
    b = 4

    tol = 1e-3

    [astar,ier, count] = bisection(f,a,b,tol)
    print('the approximate root is',astar)
    print('the number of iterations was',count)
    print('the error message reads:',ier)
    print('f(astar) =', f(astar))

# define routines
def bisection(f,a,b,tol):

#    Inputs:
#     f,a,b       - function and endpoints of initial interval
#      tol  - bisection stops when interval length < tol

#    Returns:
#      astar - approximation of root
#      ier   - error message
#            - ier = 1 => Failed
#            - ier = 0 == success

#     first verify there is a root we can find in the interval

    fa = f(a)
    fb = f(b);
    if (fa*fb>0):
```

```
       ier = 1
       astar = a
       return [astar, ier, 0]

#   verify end points are not a root
    if (fa == 0):
      astar = a
      ier =0
      return [astar, ier, 0]

    if (fb ==0):
      astar = b
      ier = 0
      return [astar, ier, 0]

    count = 0
    d = 0.5*(a+b)
    while (abs(d-a)> tol):
      fd = f(d)
      if (fd ==0):
        astar = d
        ier = 0
        return [astar, ier, count]
      if (fa*fd<0):
         b = d
      else:
        a = d
        fa = fd
      d = 0.5*(a+b)
      count = count +1
#      print('abs(d-a) = ', abs(d-a))

    astar = d
    ier = 0
    return [astar, ier, count]

driver()
```


Problem 5
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
f
−4
−2
0
2
4
6
8
10
x

```python
# PROBLEM 5 (a)

import numpy as np
import matplotlib.pyplot as plt

f = lambda x: x - 4*np.sin(2*x) - 3

x = np.linspace(-5, 10, 1000)

y = f(x)

plt.plot(x, y)
plt.xlabel('x')
plt.ylabel('f')
plt.title('Problem 5')
plt.grid(True)
plt.legend()
plt.show()
```

```python
# PROBLEM 5 (b)

import numpy as np

def fixed_pt(x):
    return -np.sin(2 * x) + 5 * x / 4 - 3 / 4

x0 = 4.5
tol = 1e-11
Nmax = 1000

for i in range(Nmax):
    x1 = fixed_pt(x0)

    if abs(x1 - x0) < tol:
        break

    x0 = x1

print("Approx:", x1)
```