APPM 4600
Homework 2

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Problem 1

Part a

Given the binomial expansion,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

As $x \to 0$, the higher-order terms grow in total at a significantly lower rate then x, so we have

$$(1+x)^n = 1 + nx + o(x).$$

Part b

To show that $x\sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$, we consider the function $f(x) = x\sin(\sqrt{x})$ and compare its growth rate to that of $g(x) = x^{3/2}$. We evaluate the derivatives of both functions:

$$f'(x) = \frac{d}{dx}\left(x\sin(\sqrt{x})\right) = \frac{\sqrt{x}\cos(\sqrt{x})}{2} + \sin(\sqrt{x})$$

$$g'(x) = \frac{d}{dx}\left(x^{3/2}\right) = 1.5x^{0.5}$$

We then examine the limit of the ratio of these derivatives as x approaches 0:

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{\frac{\sqrt{x}\cos(\sqrt{x})}{2} + \sin(\sqrt{x})}{1.5x^{0.5}}$$

Evaluating this limit yields:

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = 1$$

This result indicates that the growth rate of $f(x)$ is comparable to that of $g(x)$ as x approaches 0. Additionally, f's growth slows as x increases close to 0, so $x\sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$.

Part c

To show that $e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$, we evaluate the limit of the ratio of e^{-t} to $\frac{1}{t^2}$ as t approaches infinity:

$$\lim_{t\to\infty}\frac{e^{-t}}{\frac{1}{t^2}} = \lim_{t\to\infty}\frac{t^2}{e^t} = \lim_{t\to\infty}\frac{t^2}{e^t} = \lim_{t\to\infty}\frac{2}{e^t} = 0$$

After repeated applications of L'Hopital's Rule, the exponential term e^{-t} in the numerator ensures that the limit approaches 0, as the exponential decay outpaces any polynomial decay, satisfying the condition for $e^{-t} = o\left(\frac{1}{t^2}\right)$.

Part d

The integral is bounded by a constant multiple of ϵ: $k\epsilon$ which is greater than or equal to the largest slice of the integral (which is that closest to zero). Therefore, the integral is $O(\epsilon)$. Bounded by a constant multiple of ϵ.

Problem 2

Part a

The change in the solution Δx due to the perturbation in b can be calculated using the formula $\Delta x = A^{-1}\Delta b$:

$$\Delta x = \begin{bmatrix} 1-10^{10} & 10^{10} \\ -10^{10} & 1+10^{10} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} = \begin{bmatrix} -(10^{10}-1)\Delta b_1 + 10^{10}\Delta b_2 \\ -10^{10}\Delta b_1 + (10^{10}+1)\Delta b_2 \end{bmatrix}$$

Part b

The norm of A is 1, and the norm of A inverse is $2 \cdot 10^10$. See norm.py in the Homework 2 folder of the Git repository.

So the condition number is $1 \cdot 2 \cdot 10^{10} = 2 \cdot 10^{10}$.

Part c

The error in the solution is bounded above by $2 \cdot 10^{10} * P_r$, where P_r is the relative perturbation $\frac{||\Delta b||}{||b||}$.

If $\Delta b = \begin{bmatrix} p \times 10^{-5} \\ q \times 10^{-5} \end{bmatrix}$ then the upper bound is

$$2 \cdot 10^{10} \cdot \sqrt{(p^2+q^2)} \cdot 10^{-5} = 2\sqrt{(p^2+q^2)} \cdot 10^5$$

With the Euclidean norm, the behavior is not qualitatively different when the perturbations are the same. Given both components have same true value, equal perturbations are more likely.

Problem 3

Part a

Given the function $f(x) = e^x - 1$, the relative error can be bounded by the function's condition number $k_f(x)$:

$$\frac{|f(x) - \tilde{f}(x)|}{|f(x)|} \leq k_f(x) \frac{|x - \tilde{x}|}{|x|}$$

where the condition number $k_f(x)$ is defined as:

$$k_f(x) = \left| \frac{x f'(x)}{f(x)} \right|$$

For the given function $f(x) = e^x - 1$, the condition number becomes:

$$k_f(x) = \left| \frac{x e^x}{e^x - 1} \right|$$

Evaluating the limit as x approaches 0:

$$\lim_{x \to 0} \frac{x e^x}{e^x - 1} = \lim_{x \to 0} \frac{e^x + x e^x}{e^x} = \lim_{x \to 0} (1 + x) = 1$$

Part b

When x is very large:

$$\lim_{x \to \infty} \frac{x e^x}{e^x - 1} = \lim_{x \to \infty} (1 + x) = \infty$$

This indicates that the algorithm is stable due to the condition number when x is not large.

Part c

The algorithm gives 8 correct digits, this is expected because the error is on the same order of magnitude as the relative error for x and $f(x)$.

Part d

We seek a polynomial P_n such that:

$$\frac{|f - \tilde{f}|}{|f|} \leq 10^{-16}$$

The polynomial approximation $P(x)$ is given by:

$$P(x) = (e^{x_0} - 1) + (x - x_0)e^{x_0} + \frac{(x - x_0)^2 e^{x_0}}{2!} + \dots$$

The second-degree polynomial approximation P_2 is:

$$P_2 = f(0) + f'(0)x$$

Problem 4

Part a

Create vector t from 0 to pi incrementing by $pi/30$, vector y equal to cos(1). Write code that evaluates the specified sum and prints the result.

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The sum is: -23.915381134014112
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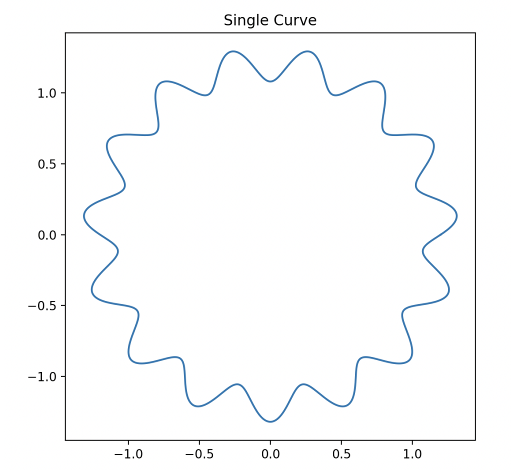
See t_y_sum.py in Git repository. The code simply creates vectors with np.arane() and evaluates with np.sum().

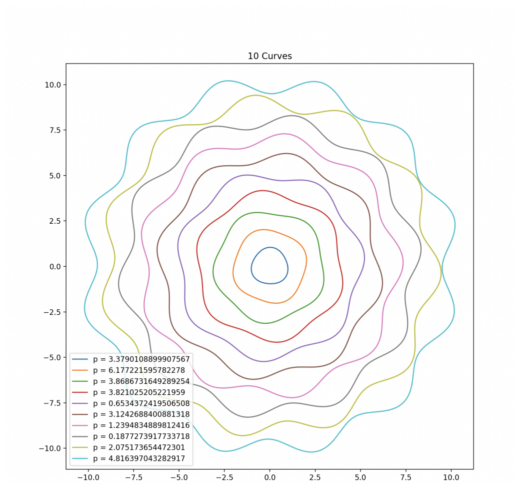
The true value is -2 (as can be trivially found with a single application of IBP).

The closest lowest error we can achieve using this algorithm requires us to to use a dt of pi/2. This error is due to cancellation from to the negative terms.

Part b

Plot a parametric curve for $x(\theta)$ with given parameters. Adjust the scale so that the axes have the same scale. In a second figure, plot 10 curves with varying parameters.





See wavy_circles.py in Git repository.