APPM 4600
Homework 1

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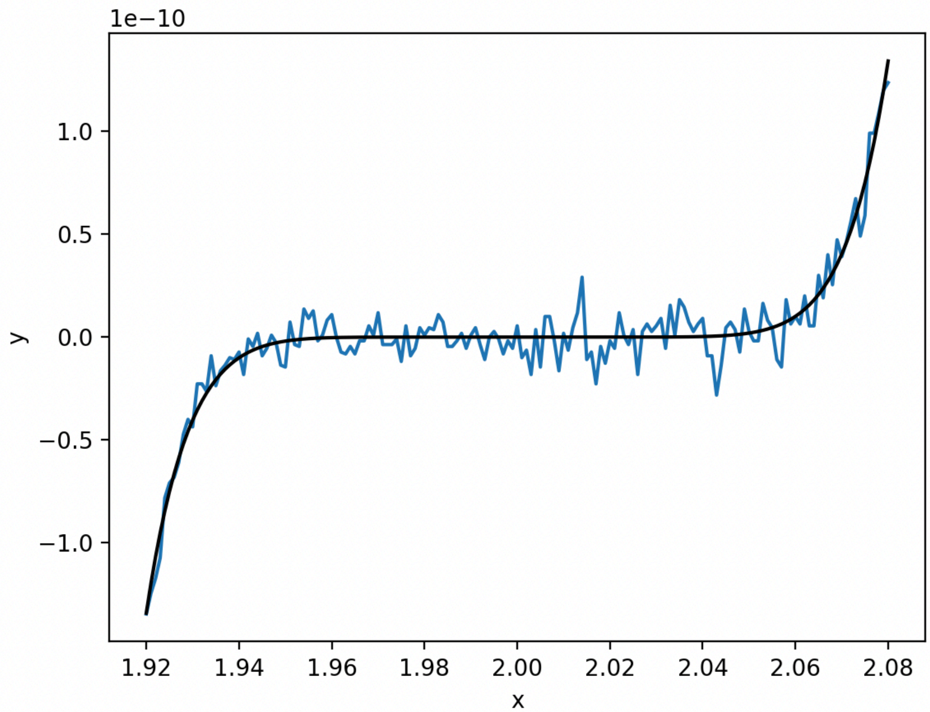
Problem 1

Prompt: Plot the polynomial $p(x)$ and its expanded and factored forms. Explore the differences and their sources.

Parts i and ii

The expanded form is shown in blue, and the factored form is shown in black.



Part iii

The black plot of the factored form is more correct. We know this based on our familiarity with functions of the form $(x-a)^b$. The blue plot of the expanded form has more numerical error from the addition of floating point numbers of various magnitudes, while the factored form avoids this.

Problem 2

Rewrite the expressions so they can be evaluated such as to avoid cancellation.

Part i

Evaluate for x close to zero:

$$\sqrt{x+1}-1=\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}=\frac{x}{\sqrt{x+1}+1}$$

Part ii

To evaluate for x close to y, use double-angle identities:

$$\begin{aligned}\sin(x)-\sin(y)&=2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})\\&=2\cos(\frac{x+y}{2})[\sin(x/2)\cos(-y/2)+\cos(x/2)\sin(-y/2)]\end{aligned}$$

Part iii

Multiply and divide by the conjugate of the numerator:

$$\begin{aligned}\frac{1-\cos x}{\sin x}&=\frac{(1-\cos x)(1+\cos x)}{\sin x(1+\cos x)}=\frac{1-\cos^2 x}{\sin x+\sin x\cos x}=\frac{\sin^2 x}{\sin x+\sin x\cos x}\\&=\frac{\sin x}{1+\cos x}\end{aligned}$$

Problem 3

Find the second degree Taylor polynomial about 0 for $f(x) = (1 + x + x^3)^3 \cos(x)$.

This is in the form

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

We find the derivatives:

$$f' = (1 + 3x^2)\cos x - (1 + x + x^3)\sin x$$
$$f'' = \cos x(5x - 1 - x^3) + \sin x(-2 - 6x^2)$$

So, $P_2(x) = 1 + x + x^2/2$.

Part a

Approximate $f(0.5)$ using $P_2(0.5)$.

$$P_x(0.5) = 1.3750...$$

Find an upper bound for the error $|f(0.5) - P_2(0.5)|$ and compare it to the actual error.

An upper bound for the error comes from the error term of the Taylor polynomial, and is expressed:

$$\frac{|f'''(c)|}{3!}|0.5|^3$$

where c is the value that maximizes $|f'''(c)|$ on the interval $[0, 0.5]$. Since $f'''(x) = (3-9x^2)\cos x + (1-17x+x^3)\sin x$, in this case $c = 0$, so the bound is

$$\frac{3}{6}|0.5|^3 = 0.0625...$$

The actual error is

$$|f(0.5) - P_2(0.5)| = |\frac{18}{8}\cos\frac{1}{2} - 1.3750| = 0.0510...$$

This is below the bound, as expected.

Part b

Find a bound for $|f(x) - P_2(x)|$ for the approximation in general.

In general, the error when approximating f at x is bounded by

$$\frac{|f'''(c)|}{3!}|x|^3$$

where c is the value that maximizes $|f'''(c)|$ on the interval $[0, x]$.

Part c

Approximate $\int_0^1 f(x)dx$ using $\int_0^1 P_2(x)dx$.

$$\int_0^1 1 + x + \frac{x^2}{2}dx = x + x^2 + \frac{x^3}{3}|_0^1 = \frac{4}{3}$$

Part d

Estimate the error in the integral.

We can estimate this without directly computing the true integral by using the error term again. Since integration is a linear operator, the error is bounded above by

$$\int_0^1 \frac{|f'''(c)|}{3!}|x|^3 dx$$

where c maximizes $|f'''(c)|$ on $[0, x]$.

Since $c = 0$ is the maximizer for all sub-intervals of $[0, 1]$, this simplifies to

$$\frac{1}{2}\int_0^1 |x|^3 dx = \frac{1}{2}\frac{x^4}{4}|_0^1 = \frac{1}{8}$$

Problem 4

Consider $x^2 - 56x + 1$.

Part a

Compute the relative errors for the two roots using the quadratic formula, assuming you can calculate the square root to 3 significant digits.

The absolute error for both is $\pm\frac{1}{4}10^{-3}$. Using the formula we find the roots are $28 \pm 3\sqrt{87}$. We can compute these are approximately 0.0178628 and 55.9821.

Using

$$\text{Relative Error} = \frac{\text{Approximate}}{\text{Actual}}$$

we find the relative errors are about 0.014 and 4.47e-6, respectively.

Part b

The "bad" root is $28 - 3\sqrt{87}$, corresponding to

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We multiply and divide by the conjugate of the numerator:

$$\begin{aligned} x &= \frac{(-b - \sqrt{b^2 - 4ac})(-b + \sqrt{b^2 - 4ac})}{2a(-b + \sqrt{b^2 - 4ac})} \\ &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{2a(-b + \sqrt{b^2 - 4ac})} \\ &= \frac{2c}{-b + \sqrt{b^2 - 4ac}} \end{aligned}$$

This algorithm allows us to calculate the smaller root with more lower relative error.

Problem 5

Part a

Find upper bounds on the error $|\Delta y|$ and the relative error $|y|/|y|$ and find when the relative error is large.

When x_1 is large and x_2 is small, the absolute error is bounded by $|\Delta x_1 - \Delta x_2|$ and the relative error is bounded by

$$\frac{|\Delta x_1 - \Delta x_2|}{|x_1 - x_2|}$$

Part b

Using the sum-to-sum product identities, we can manipulate $\cos(x + \delta) - \cos(x)$ into

$$-2\sin\left(x + \frac{\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$$