Electronic information processing and cybernetics - An algebrasation of the synthesis problem for circuits

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0 Introduction

The occasion for this work is a problem from automata theory: From a given set of building blocks an automaton, whose functionality is predetermined, shall be assembled. From the different, eventually existing solutions the cheapest shall be selected.

A building block $A \in \mathcal{U}$ is a physical, mostly electrical device with $Q(A)$ inputs and $Z(A)$ outputs. For each input a Set S of input signals is permitted, on which the building block reacts with output signals. We assume the following simplifications with regard to the issue, the following holds:

1. For each input of the elements of $\mathcal{U}$ a set of signals S is prescribed, and each element of S^n is allowed as input signal for A with $n = Q(A)$.
2. The set of output signals of $A \in \mathcal{U}$ lies in S^m with $m = Z(A)$.
3. If at time t the input signal $s \in S^n$ is applied to A, then the output signal at time t is uniquely determined by s. (We therefore neglect the finite propagation speed of signals).

Thus, the finite automaton is completely described by its function $\phi(A) : S^n \to S^m$. It is presumed that inputs and outputs of A are labeled with a fixed numbering from 1 to $Q(A)$, and respectively from 1 to $Z(A)$. The i-th input (output) is assigned to the i-th component of S^n (S^m).

An element of $\mathcal{U}$ is a circuit. If A and B are circuits with $Q(A)$, or $Q(B)$ inputs and $Z(A)$, or respectively $Z(B)$ outputs, then we build new circuits from A and B by integrating them to a new element $A \times B$ with $Q(A) + Q(B)$ inputs and $Z(A) + Z(B)$ outputs. We declare the i-th input of A as the i-th input of $A \times B$ and the i-th input of B as the $(Q(A) + i)$-th input of $A \times B$ (figure 1).

If $Z(A) = Q(B)$ we get from A and B a circuit $B \circ A$ by switching the i-th output of A to the i-th input of B.

A circuit of elements of $\mathcal{U}$ is a device, which is described inductively by the preceding explanations.

If $\phi(A)$ ($\phi(B)$) is the function of circuit A (B), then $\phi(A) \times \phi(B)$ is the function of $A \times B$, and $\phi(B) \circ \phi(A)$ for $Q(B) = Z(A)$ is the function of $B \circ A$.

The costs for the building blocks in $\mathcal{U}$ shall be defined by the function $L : \mathcal{U} \to N \cup 0$. We define:

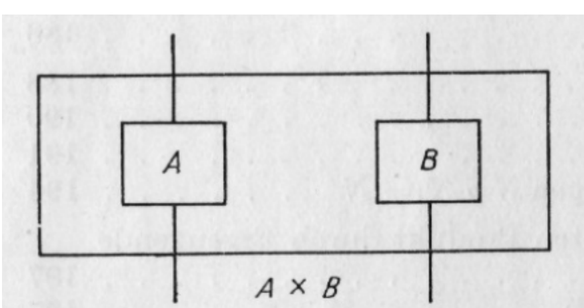

Figure 1:

$$
\begin{array}{rcl}
L(A \times B) & = & L(A) + L(B), \\
L(A \circ B) & = & L(A) + L(B).
\end{array}
$$

Hereby a price is assigned to each circuit.

Now, the task is the following: Given $f : S^n \to S^m$, find a circuit A with $\phi(A) = f$ and

$$L(A) = \min_{B \in \phi^{-1}(f)} \{L(B)\}.$$

If f does not fully map to S^n, but only to $R \subset S^n$, then the optimum shall be searched on $\cup_{g|R=f} \phi^{-1}(g)$.

The task is generalized in an obvious way, if $Q(f) = Z(F) = S^n$ and f, as it is often the case with finite automata, is determined just by a transformation of S^n,

In order to solve this problem it appears advantageous to know relations, which allow to generate from an element $A \in \phi^{-1}(f)$ all the elements from the class $\phi^{-1}(f)$.

In the first two sections of this work a theory of interconnection of automata will be developed, as already sketched out in the description of the task:

First, the topological notion of a *planar* network is introduced. The binary operators "$\circ$" and "$\times$" will be explained for these networks. One obtains an algebraic structure $\mathcal{R}$, which is a category with respect to "$\circ$" and a semi-group with respect to "$\times$". $\mathcal{R}$ turns out to be a generalisation of the D-category (category with direct products), which we want to call *X-Kategorie.*

$\mathcal{R}$ reflects the interconnection of automata, but does not reflect the possibility of different building blocks with the same number of inputs and outputs. This will be accommodated by assigning symbols of the alphabet $\mathcal{U}$ to inner points of the network, respecting the functions $Q : \mathcal{U} \to N \cup 0$ and $Z : \mathcal{U} \to N \cup 0$. One arrives at an X-Kategorie $\mathcal{F}(\mathcal{U})$, which possesses $\mathcal{U}$ as a free generator. The elements of $\mathcal{F}(\mathcal{U})$ correspond to the set of circuits which may be yielded from $\mathcal{U}$. The mapping ϕ, which assigns to each automaton its functions, becomes a functor $\phi : \mathcal{F}(\mathcal{U}) \to \mathcal{C}$, where $\mathcal{C}$ is the category of mappings of type $S^n \to S^m$.

The introduction of the quotient category $\mathcal{F}(\mathcal{U})/\mathcal{R}$ from $\mathcal{F}(\mathcal{U})$ to a relational system $\mathcal{R}$ forms the basis for the study of classes $\phi^{-1}(f)$. In section 3 $\phi^{-1}(f)$ will be studied for certain categories $\mathcal{F}(\mathcal{U})$ and distinguished functors ϕ: We consider only generators $\mathcal{U}$ with $\{U, V, D\} \subset\subset \mathcal{U}$ and $Z(A) = 1$ for $A \in \mathcal{U} \setminus \{U, V, D\}$ and functors ϕ, for which $\phi(U)$ is the mapping from S to S°, $\phi(V)$ the permutation of the components of S^2, and $\phi(D)$ the diagonalisation $S \to S^2$. Such functions are called *normal.* A relational system $\mathcal{R}$ will be given with the following property: For each normal ϕ it holds that $\phi(F) = \phi(G)$ for $F, G \in \mathcal{F}(\mathcal{U})$, iff $F \equiv G(\mathcal{R})$. $\mathcal{F}(\mathcal{U})/\mathcal{R}$ is a D-category and $\mathcal{U} \setminus \{U, V, D\}$ a free generator of the D-category.

Thus, the relational system allows to simplify the representations F of a function $\phi(F)$, which are possible without the knowledge of the elements in $\mathcal{U} \setminus \{U, V, D\}$. Figure 2 shows that there are proper simplifications in this system.

In this work we allow an arbitrary countable set for S, such that these results can also be of interest for computer programming. If one wants to formally simplify programs which are constructed from sub-programs, then the number of possible sub-programs renders the specific consideration of the computed function infeasible, but the rules for the transformation of programs have to be applicable uniformly for all sub-programs. Thus, that means that the allowed relations have to be the system $\mathcal{R}$ or an equivalent system.

For an orientation on the state of the art of the theory of circuit synthesis refer to [2], in relation to Streckenkpomplexen to [5] and category theory to [4] and [6]. For stimulating discussions and critical remarks I want to thank J. Dörr and D. Puppe. My thanks go to the Deutsche Forschungsgemeinschaft, who has supported this study with a grant of the FRITZ-THYSSEN-foundation.

1 The category of planar networks

1.1 The category $\mathcal{R}$ of planar networks

The following topological structure is understood as a network with n inputs and m outputs ($n, m = 0, 1, 2...$): A rectangle with sides g_1, g_2 and h_1, h_2 is given in the euclidean plane, where both g_i and h_i lie face to face to each other.

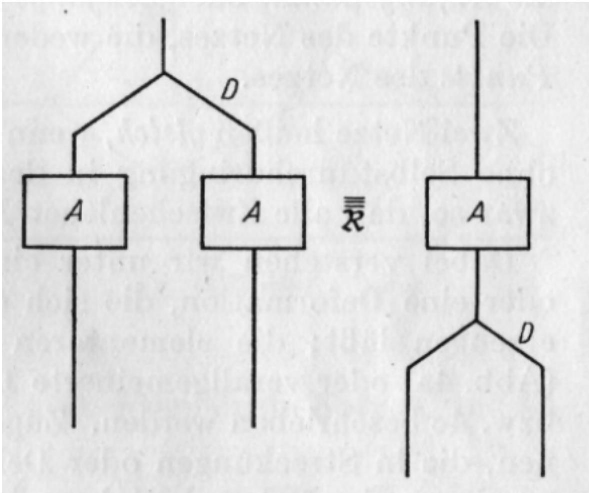

Figure 2:

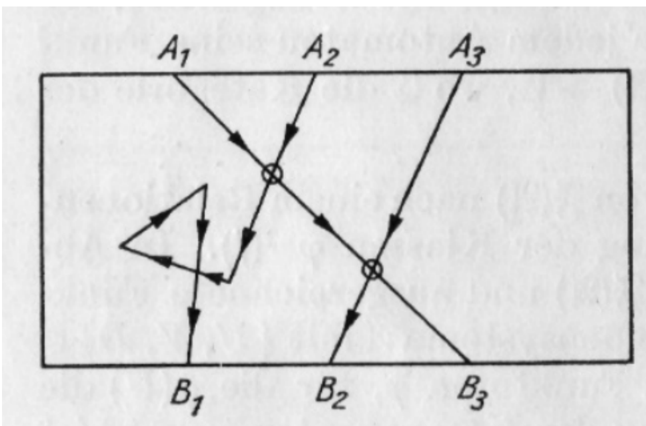

Figure 3:

$A_1 \dots A_n$ ($B_1 \dots B_n$) are mutually distinct points on g_1 (g_2), where the numbering of points runs from h_1 to h_2; the A_i (B_i) divide g_1 (g_2) into equidistant parts. Let A_i be the starting points and B_i be the endpoints of precisely one line segment of an oriented, finite, planar Streckenkomplex, which lies in the euclidean plane without crossings in the rectangle [1]. We further demand from the Streckenkomplex that each of its line segments simply lie above h_i and that the orientation of the line segments points from g_1 to g_2. Thus the case in figure 3 is excluded.

We call g_1, g_2, h_1, h_2 the network frame, $A_1, \dots, A_n$ ($B_1, \dots, B_m$) the *starting points* or inputs (*end points* or outputs) of the network. The points which are neither inputs nor outputs are called *inner points* of the network.

Two networks are called equal, if they can be - after overlaying their frames without self-penetration in the plane - deformed into one another, in such a way that all intermediate positions are networks.

We understand a *deformation* as an *elementary deformation* or a deformation, which may be generated by a chain of elementary deformations; the elementary deformations are triangular deformations (figure 4a) or generalised triangular deformations, as shown on figure 4b and 4c. Further the (trivial) deformations are allowed, which are stretches of the frame or euclidean motions of the network.

One recognises that the given equivalence definition is reflexive, symmetric and transitive, i.e. a division of the network into equivalence classes.

From now on we call this class a network and call networks in the conventional sense, when a distinction is important, *representative* of a network.

[1] In the euclidean sense it is a matter of piecewise linear curves.

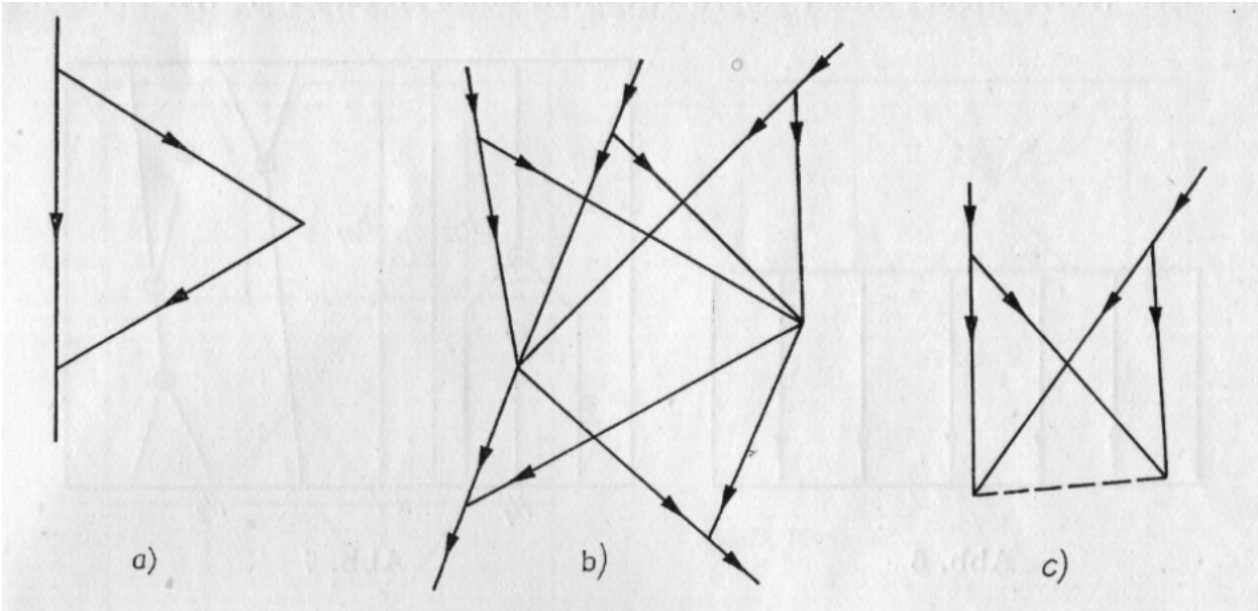

Figure 4:

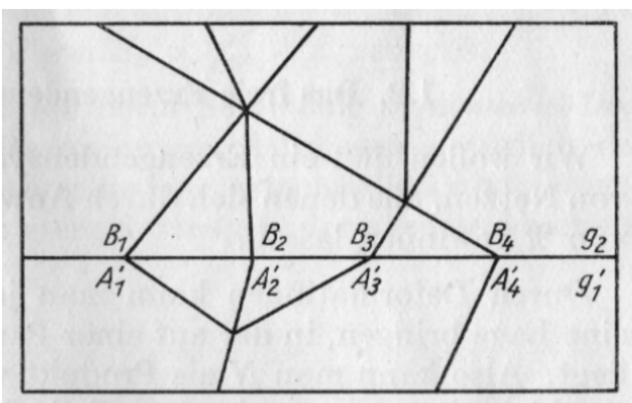

Figure 5:

Different representatives of a network have the same number of inputs and outputs, which we denote by $Q(N)$ and $Z(N)$.

Let N_1 and N_2 be networks with $Q(N_2) = Z(N_1)$, then we declare between N_1 and N_2 a composition over representatives $N_1^{'}$ and $N_2^{'}$ from N_1 and N_2:

We deform the frame of $N_2^{'}$ (in a trivial manner), such that we can join $N_2^{'}$ onto $N_1{}'$; i.e. we bring g_2 and $g_1^{'}$ to the same size and set both frames along g_2 and $g_1^{'}$ against each other (figure 5). We remove $g_2 = g_1^{'}$ and the points $A_i^{'} = B_i$ and get again a network representative, that we call $N_2 \circ N_1$. One sees that the definition is independent of the choice of representatives.

Further the following holds:

$$\begin{aligned} Q(N_2 \circ N_1) &= Q(N_1), \\ Z(N_2 \circ N_1) &= Z(N_2). \end{aligned}$$

As is shown easily, the product is associative; i.e. it holds for networks N_1, N_2, N_3 with $Q(N_2) = Z(N_1)$ and $Q(N_3) = Z(N_2)$ that

$$(N_3 \circ N_2) \circ N_1 = N_3 \circ (N_2 \circ N_1).$$

For each network with $Q(N) = n$ and $Z(N) = m$ there is exactly one network E_n and E_m with the property

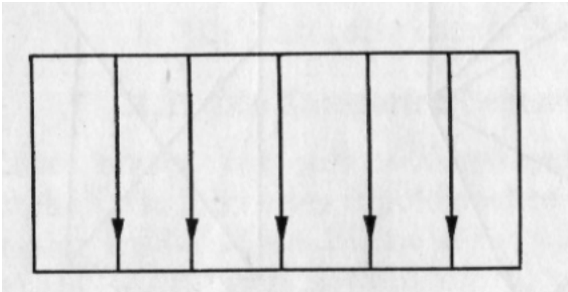

Figure 6:

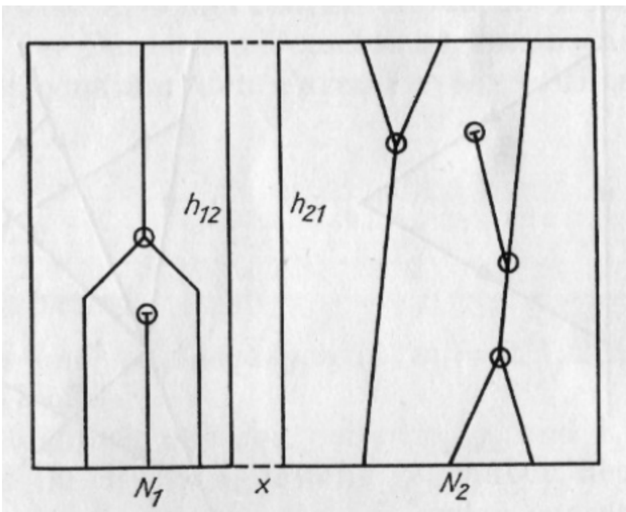

Figure 7:

$$N \circ E_n = E_m \circ N = N;$$

E_n is a network with n inputs, n outputs and n line segments (figure 6). The set of networks therefore forms a category with respect to "$\circ$".

We declare a further combination of networks, namely the *X-product*. Let $N_1^{'}$ and $N_2^{'}$ be representatives of the networks N_1 and respectively N_2, then we deform the frame to the same length and set them along h_{12} and h_{21} together (figure 7). Subsequently we delete $h_{21} = h_{12}$ and numbering the inputs from left to right. If we take care that the input- and output-points of the structure are again equidistant, then we have again a representative of a network N_3. We call N_3 the direct product of N_1 and N_2 and write $N_3 = N1 \times N_2$. One sees that the direct product is independent of the choice of the representative, associative, and that $E_k = E_1 \times \ldots \times E_1$ (k-times), and $E_0 \times N = N \times N_0 = N$ for every network N.

We call the set of networks with these two connections with $\mathcal{R}$. We summarize the results into
Theorem 1. *$\mathcal{R}$ forms a category with respect to "$\circ$" and a semi-group with respect to "$\times$".*