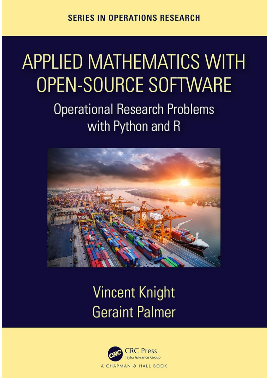
Evolutionary Game Theory

Vince Knight

2024-03-04 until 2024-03-29


SERIES IN OPERATIONS RESEARCH
APPLIED MATHEMATICS WITH OPEN-SOURCE SOFTWARE
Operational Research Problems with Python and R
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NASH STREET

Two Thirds of the Average

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- Closest to two thirds of the average of all picked numbers wins.

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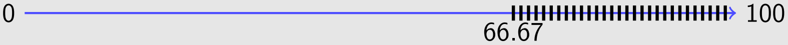
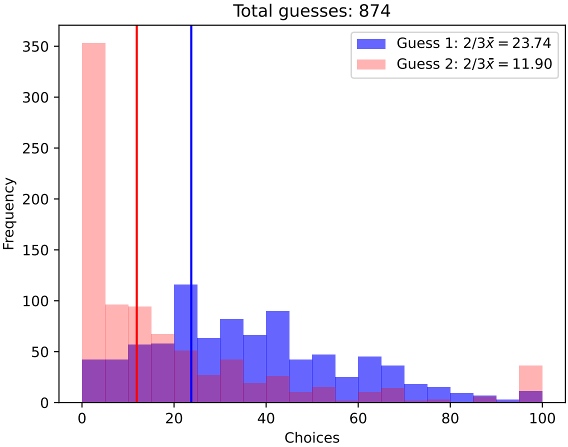
If we consider all possible strategies the game is equivalent to picking a number on this line:

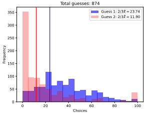


The biggest possible winning guess is $2\frac{100}{3} \approx 66.67$ which only occurs if all other guesses are larger than that.
Thus all choices bigger than $2\frac{100}{3}$ are **dominated**. If we continue to repeat this process we see that there are only two valid equilibrium:

- All players guess 0.
- All players guess 1.

Now that we have rationalised this behaviour, let us play again.


Total guesses: 874
Guess 1: $2/3\bar{x} = 23.74$
Guess 2: $2/3\bar{x} = 11.90$
Frequency
Choices
0
50
100
150
200
250
300
350
0
20
40
60
80
100

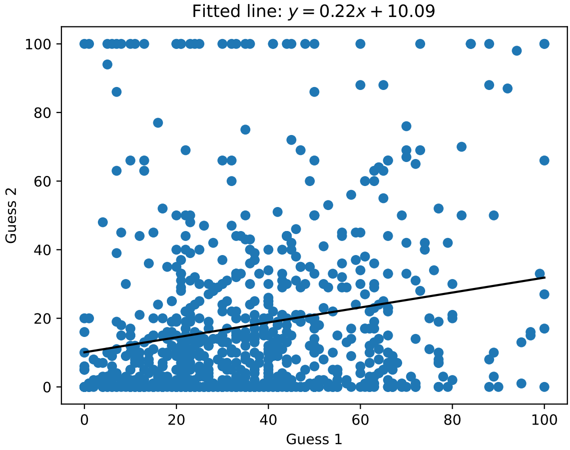


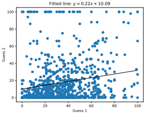
We see that over the two guesses the behaviour shifts.

Note that with the first guesses: most people were above the winning guess.

For the second guess: most people were below the winning guess.

Will we eventually arrive at the equilibrium?


Fitted line: $y = 0.22x + 10.09$
Guess 2
Guess 1
0
20
40
60
80
100
0
20
40
60
80
100



One approach would be to model the macro behaviour with that fitted line: if we assume that the $n+1$th guess is .22 of the nth guess we could imagine that this would eventually converge. We'd need more data here of course.

This does not necessarily work, perhaps, as most players in the second guess were below the winning guess, a third guess would in fact be higher! So perhaps the linear regression would not longer be accurate.

Let us base our analysis on biological evolution.

Definition

Considering an infinite population of individuals each of which represents an action from $\mathcal{A}$, we define the population profile as a vector $x \in [0,1]_{\mathbb{R}}^{|\mathcal{A}|}$. Note that:

$$\sum_{i\in\mathcal{A}} x_i = 1$$

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For the two thirds game, our vector $x \in \mathbb{R}^{101}_{[0,1]}$.

Now: for evolution we need fitness.

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For two thirds game a sensible fitness function would be something inversely proportional to the winning guess:

$$f_i(x) = \frac{1}{1 + \left(i - \frac{2}{3}\sum_{i=0}^{N} i x_i\right)^2}$$

Note we add 1 to the denominator to deal with division by zero that can occur in some populations.

Now we need an equation to model the evolutionary process.

Definition

Replicator Dynamics Equation

$$\frac{dx_i}{dt} = x_i(f_i(x) - \phi) \text{ for all } i$$

where:

$$\phi = \sum_{i=0}^{N} x_i f_i(x)$$

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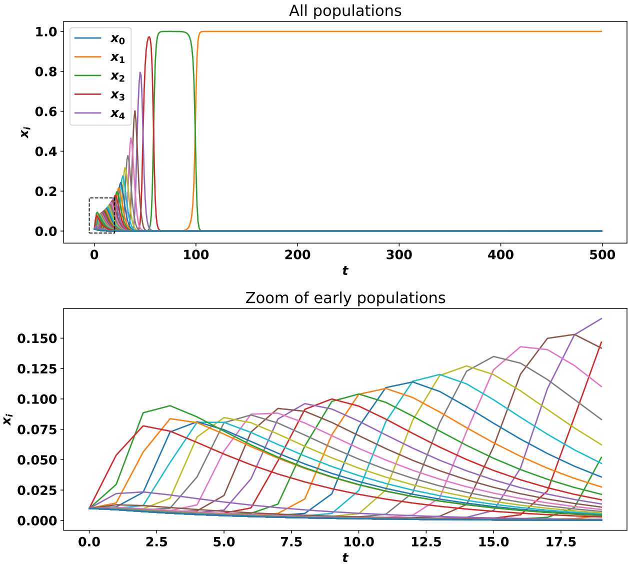
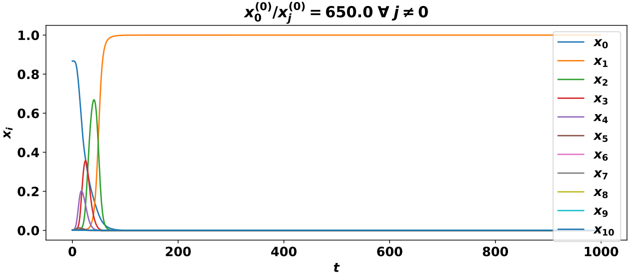
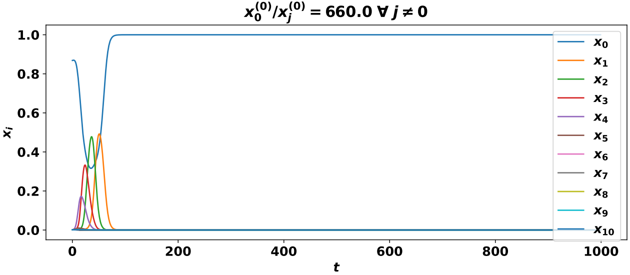
This differential equation ensures that individuals with a fitness above the average ϕ have positive derivative and individuals with a fitness below the average ϕ have negative derivate.

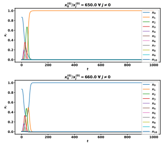
For the case of the two thirds game we can solve the differential equations numerically efficiently.


All populations
x_0
x_1
x_2
x_3
x_4
x_i
t
Zoom of early populations
x_i
t

We see that over time, the population emerges to all guessing 1.
So everyone wins.
Note that everyone guessing 0 also is stable.

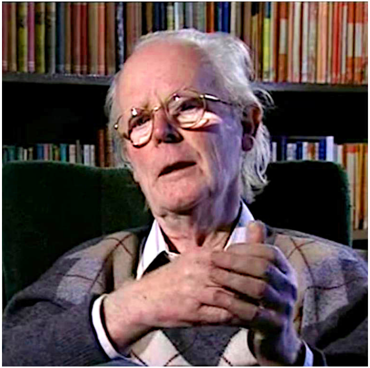

$x_0^{(0)}/x_j^{(0)} = 650.0\ \forall\ j \neq 0$
x_i
t
0.0
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
x_9
x_{10}


$x_0^{(0)}/x_j^{(0)} = 660.0\ \forall\ j \neq 0$
x_i
t
0.0
0.2
0.4
0.6
0.8
1.0
0
200
400
600
800
1000
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
x_9
x_{10}



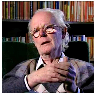
With a different starting population we can also emerge to that stable population.

Both these populations are stable.



John Maynard-Smith[1] (1920 - 2004)

[1] J.M. Smith. *The Theory of Evolution*. A Pelican original. Penguin, 1977. ISBN: 9780140204339.



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This leads to John Maynard-Smith. He is often considered the father of evolutionary game theory.

He defined the concept of an Evolutionary Stable Strategy.

We will use the concept of a particular type of evolutionary game to describe this.

Definition

In a population game when considering a pairwise contest game we assume that individuals are randomly matched and play some game with utility matrices A, A^T. For a population profile x this gives a compact expression for the fitness:

$$f = Ax$$

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As an example, let us consider the Hawk Dove game which is attributed to Maynard-Smith. Two types of behaviour: $\{\text{Aggressive}, \text{Not Aggressive}\}$ referred to as Hawk or Dove.

- If a Dove and Hawk meet the Hawk takes the resources getting a utility of v, the dove gets 0.
- If two Doves meet they share the resources both getting $\frac{v}{2}$
- If two Hawks meet there is a fight over the resources (with an equal chance of winning) and the winner takes the resources while the loser pays a cost $c > v$

This corresponds to:

$$A = \begin{pmatrix} \frac{v-c}{2} & v \\ 0 & \frac{v}{2} \end{pmatrix}$$

Definition

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This extension of the notion of fitness allows us to consider individuals as strategies. Perhaps more precisely strategies as individuals.

In this case we can relate evolutionary stability to a subset of Nash equilibria which is where we are going.

Definition

A strategy σ^* is called an **Evolutionary Stable Strategy** if there exists an $0 < \bar{\epsilon} < 1$ such that for every $0 < \epsilon < \bar{\epsilon}$ and every $\sigma \neq \sigma^*$ σ^* is:

$$u(\sigma^*, x_\epsilon) > u(\sigma, x_\epsilon)$$

Where x_ϵ is the post entry population where a proportion ϵ of the population are σ.

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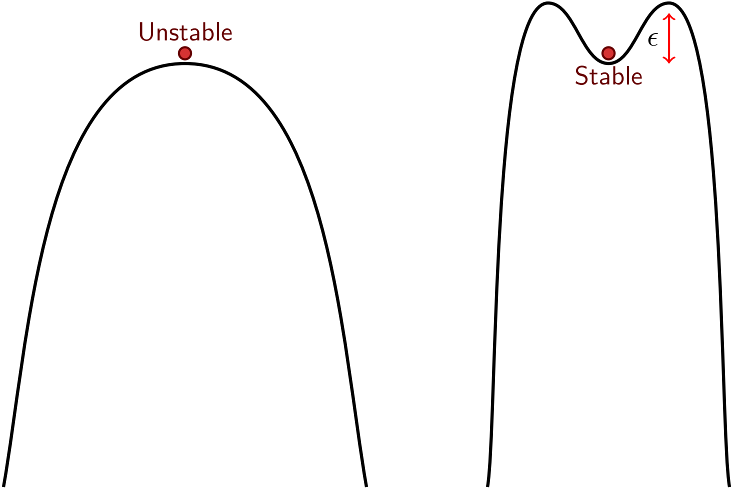
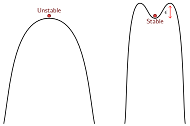
Where x_ϵ is the post entry population where a proportion ϵ of the population are σ.

This corresponds to the idea that for a small enough change of the population that an evolutionary stable strategy will reject the change.
This is similar to what we discussed with the two thirds of the average game.
The strategy 0 would be considered to be evolutionarily stable because there exists a small enough change of the population from everyone playing 0 that returns to everyone playing 0.


Unstable
Stable
ϵ



This corresponds to the idea of both of those two pictures are stable. However, only one of them can be pushed with **any** force without the marble falling.

Theorem

If σ^* is an ESS in a pairwise contest population game then for all $\sigma \neq \sigma^*$:
1. $u(\sigma^*, \sigma^*) > u(\sigma, \sigma^*)$ OR 2. $u(\sigma^*, \sigma^*) = u(\sigma, \sigma^*)$ and $u(\sigma^*, \sigma) > u(\sigma, \sigma)$
Conversely, if either (1) or (2) holds for all $\sigma \neq \sigma^*$ in a two player normal form game then σ^* is an ESS.

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This result gives us an efficient way of computing ESS. The first condition is in fact almost a condition for Nash Equilibrium (with a strict inequality), the second is thus a stronger condition that removes certain Nash equilibria from consideration. This becomes particularly relevant when considering Nash equilibrium in mixed strategies.

To find ESS in a pairwise context population game we:

1. Write down the associated two-player game;
2. Identify all symmetric Nash equilibria of the game;
3. Test the Nash equilibrium against the two conditions of the Theorem.

For the Hawk-Dove game this can be used to show that $\sigma^* = (\frac{v}{c}, 1 - \frac{v}{c})$ is an ESS.

An evolutionary game theoretic model of rhino horn devaluation[a]

[a] Nikoleta E. Glynatsi, Vincent Knight, and Tamsin E. Lee. "An evolutionary game theoretic model of rhino horn devaluation". In: *Ecological Modelling* 389 (2018), pp. 33–40. ISSN: 0304-3800.

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There are numerous examples of applications of evolutionary game theory. In this particular paper, my co-authors and I modelled the practice of devaluing a Rhino's horn (either by dying it or cutting it off) to dissuade poachers.

The logic behind this idea is that there is a best response by poachers which is to not kill a Rhino for its horn. However, using evolutionary game theory lets us understand if this is actually likely to happen.

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