Example

Two friends must decide what movie to watch at the cinema. Alice would like to watch a sport movie and Bob would like to watch a comedy. Importantly, they would both rather spend their evening together than apart.

- The finite set of $N = 2$ players are the two neighbouring countries.
- The action sets are $\mathcal{A}_1 = \mathcal{A}_2 = \{\text{Sport}, \text{Comedy}\}$
- For two player games a common representation of the payoff functions is:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

Definition

A strategy for a player with action set $\mathcal{A}$ is a probability distribution over elements of $\mathcal{A}$.
Typically a strategy is denoted by $\sigma \in [0,1]_{\mathbb{R}}^{|\mathcal{A}|}$ so that:

$$\sum_{i=1}^{\mathcal{A}} \sigma_i = 1$$

As an example we have:

$$\sigma_r = (1/7, 6/7) \qquad \sigma_c = (1, 0)$$

This corresponds to Alice choosing sports $\frac{1}{7}$ of the time and Bob unilaterally choosing to sports.

Definition

For a given strategy σ, the support of σ: $\mathcal{S}(\sigma)$ is the set of actions $i \in \mathcal{A}$ for which $\sigma_i > 0$.

For our example we have:

$$\mathcal{S}(\sigma_r = (1/7, 6/7)) = \{\text{Sports}, \text{Comedy}\} \qquad \mathcal{S}(\sigma_c) = \{\text{Sports}\}$$

Definition

Average payoff:

- $u_r(\sigma_r, \sigma_c) = \sigma_r A \sigma_c^T$
- $u_c(\sigma_r, \sigma_c) = \sigma_r B \sigma_c^T$

For our example we have:

$$
\begin{aligned}
u_r((1/7, 6/7), (1, 0)) &= (1/7, 6/7) \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (1/7, 6/7) \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \frac{3}{7} + \frac{0}{7} = \frac{3}{7}
\end{aligned}
$$

and

$$
\begin{aligned}
u_r((1/7, 6/7), (1, 0)) &= (1/7, 6/7) \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= (1/7, 6/7) \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
&= \frac{2}{7} + \frac{0}{7} = \frac{2}{7}
\end{aligned}
$$

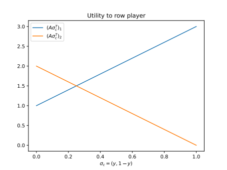
Definition

In a two player game $(A, B) \in \mathbb{R}^{m\times n^2}$ a strategy σ_r^* of the row player is a best response to a column players' strategy σ_c if and only if:

$$\sigma_r^* = \text{argmax}_{\sigma_r \in \mathcal{S}_1} \sigma_r A \sigma_c^T$$

Let us compute the best response σ_r^* as a function of $\sigma_c = (y, 1-y)$.

We have $A\sigma_c = \begin{pmatrix} 3y + 1 - y \\ 2(1-y) \end{pmatrix} = \begin{pmatrix} 2y+1 \\ 2-2y) \end{pmatrix}$



This gives:

$$\sigma_r^* = \begin{cases} (1, 0), & \text{if } y < 1/4 \\ (0, 1), & \text{if } y > 1/4 \\ \text{indifferent}, & \text{if } y = 1/4 \end{cases} \quad \text{similarly } \sigma_c^* = \begin{cases} (1, 0), & \text{if } x > 3/4 \\ (0, 1), & \text{if } x < 1/4 \\ \text{indifferent}, & \text{if } x = 3/4 \end{cases}$$

Theorem

In a two player game $(A, B) \in \mathbb{R}^{m\times n2}$ a strategy σ_r^* of the row player is a best response to a column players' strategy σ_c if and only if:

$$\sigma_{r^* i} > 0 \Rightarrow (A\sigma_c^T)_i = max_{k\in\mathcal{A}_2}(A\sigma_c^T)_k \text{ for all } i \in \mathcal{A}_1$$

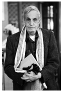
$(A\sigma_c^T)_i$ is the utility of the row player when they play their i^{th} action. Thus:

$$\sigma_r A\sigma_c^T = \sum_{i=1}^{m} \sigma_{r\,i}(A\sigma_c^T)_i$$

Let $u = \max_k(A\sigma_c^T)_k$ giving:

$$\begin{aligned}
\sigma_r A\sigma_c^T &= \sum_{i=1}^{m} \sigma_{r\,i}(u - u + (A\sigma_c^T)_i) && (1)\\
&= \sum_{i=1}^{m} \sigma_{r\,i} u - \sum_{i=1}^{m} \sigma_{r\,i}(u - (A\sigma_c^T)_i) && (2)\\
&= u - \sum_{i=1}^{m} \sigma_{r\,i}(u - (A\sigma_c^T)_i) && (3)
\end{aligned}$$

We know that $u - (A\sigma_c^T)_i \geq 0$, thus the largest $\sigma_r A\sigma_c^T$ can be is u which occurs if and only if $\sigma_{r\,i} > 0 \Rightarrow (A\sigma_c^T)_i = u$ as required.



John Nash[1] (1928 - 2015)
By Peter Badge / Typos1 - submission by way of Jimmy Wales, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=6977799

[1] John F. Nash. "Equilibrium points in *n*-person games". In: *Proceedings of the National Academy of Sciences* 36.1 (1950), pp. 48–49. DOI: 10.1073/pnas.36.1.48.

In his 25 page PhD thesis John Nash proved the following theorem.

Theorem

All Normal Form games have a set of strategies that are best responses to each other.[a]

[a] John F. Nash. "Equilibrium points in *n*-person games". In: *Proceedings of the National Academy of Sciences* 36.1 (1950), pp. 48–49. DOI: 10.1073/pnas.36.1.48.

The proof is incredibly elegant. He shows, using Kakutani fixed point theorem from the field of topology that a fixed point exists for a function that corresponds to the equilibrium condition.

kakutani1941generalization

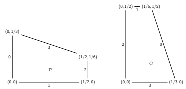
Definition

For a two player game $(A, B) \in \mathbb{R}_{>0}^{m \times n2}$ the row/column player best response polytope $\mathcal{P}/\mathcal{Q}$ is defined by:

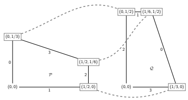
$$\mathcal{P} = \{x \in \mathbb{R}^m \mid x \geq 0; xB \leq 1\}$$

$$\mathcal{Q} = \{y \in \mathbb{R}^n \mid Ay \leq 1; y \geq 0\}$$

These best response polytopes is the bounded set of vectors that correspond to a scaled game where the best response of the row/column player is scaled to a value of 1.



Here are the polytopes for our game.
We label the polytopes according to the ordered defining half spaces. Because of the face that the ordering of $\mathcal{P}$ and $\mathcal{Q}$ we have a correspondence between the vertices of the polytope.
In $\mathcal{P}$ the label 0 corresponds to the first action of the row player's actions not being played. However, in $\mathcal{Q}$ the label 0 corresponds to the first action of the row player being a best response.



Once we have obtained the polytopes and labelled the vertices we can identify pairs of vertices (one in each polytope) that are **fully labeled**. A fully labeled vertex pair corresponds to pairs of strategies where either an action is not played or it is a best response to the strategies played by the other player.
In this case we have a game with three pairs of fully labelled vertices. Which we can convert to strategies (ie probability vectors):

- $\{(0, 1/3), (0, 1/2)\} \to \{(0, 1), (0, 1)\}$
- $\{(1/2, 0), (1/3, 0)\} \to \{(1, 0), (1, 0)\}$
- $\{(1/2, 1/6), (1/6, 1/2)\} \to \{(3/4, 1/4), (1/4, 3/4)\}$

L

emke-Howson Algorithm[a]

[a] C. E. Lemke and J. T. Howson Jr. “Equilibrium Points of Bimatrix Games”. In: *Journal of the Society for Industrial and Applied Mathematics* 12.2 (1964), pp. 413–423. DOI: 10.1137/0112033.

The Lemke Howson Algorithm presents an approach to systematically search vertices in pairs.

It allows us to unlock a technique called Integer Pivoting to efficiently find equilibria. Note however that it cannot guarantee to find all games.

A

game theoretic model of the behavioural gaming that takes place at the EMS - ED interface[a]

[a]Michalis Panayides, Vince Knight, and Paul Harper. "A game theoretic model of the behavioural gaming that takes place at the EMS - ED interface". In: *European Journal of Operational Research* 305.3 (2023), pp. 1236–1258. ISSN: 0377-2217. DOI: https://doi.org/10.1016/j.ejor.2022.07.001.

There are a number of examples of applications of Nash equilibria.

In this paper, we modelled the interface of emergency units and emergency vehicles. Ambulances are often blocked with patients because hospitals do not want to start a timer of taking them in the door. This blocks ambulances.

There are limitations to Nash equilibria: does the equilibria actually arise?

This will be the topic of my next seminar.