

1. (a) Give the definition of a normal form game. [2]  
 (b) Consider the normal form game with the following matrix representation:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Consider the mixed strategies for the row player  $\sigma_r = (x, 1 - x)$  and the column player  $\sigma_c = (y, 1 - y)$ . Sketch a plot of:

- Of the row player's utilities:  $u_r((1, 0), \sigma_c)$  and  $u_r((0, 1), \sigma_c)$ . [1]
- Of the column player's utilities:  $u_c(\sigma_r, (1, 0))$  and  $u_c(\sigma_r, (0, 1))$ . [1]

Using the plot, obtain the best responses of both players. [1]

- (c) Give a proof of the following theorem:

In a two player game  $(A, B) \in \mathbb{R}^{m \times n^2}$  a mixed strategy  $\sigma_r^*$  of the row player is a best response to a column player's strategy  $\sigma_c$  if and only if:

$$\sigma_{ri}^* > 0 \Rightarrow (A\sigma_c^T)_i = \max_k (A\sigma_c^T)_k \text{ for all } 1 \leq i \leq m$$

[5]

- (d) Using the above theorem, identify the Nash equilibria for the game in question (b) and confirm your answer to question (b). [5]  
 (e) Consider the accompanying 2017 paper entitled "Measuring the price of anarchy in critical care unit interactions" by Knight et al.  
 (i) Give a general summary of the paper. [3]  
 (ii) What is the main theoretic result of the paper? [2]  
 (iii) Identify a specific modelling assumption made that limits the work. [2]  
 (iv) Propose an approach that could be used to overcome this limiting factor. [3]

2. (a) Give the definition of a repeated game. [2]  
 (b) Give the definition of strategy in a repeated game. [2]  
 (c) Show that in general the total size of the history space is given by:

$$\left| \bigcup_{t=0}^{T-1} H(t) \right| = \frac{1 - (|S_1||S_2|)^T}{1 - |S_1||S_2|}$$

Where  $S_1, S_2$  are the strategy spaces for the stage game for both players and  $H(t)$  is the history of play at stage  $t$ . [5]

- (d) For the remainder of this question, consider the following stage game:

$$A = \begin{pmatrix} 3 & 6 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 7 & 7 \\ 20 & 1 & 0 \end{pmatrix}$$

Obtain all possible histories for the corresponding 2 stage repeated game. [3]

- (e) Give a proof of the following theorem:  
 For any repeated game, any sequence of stage Nash profiles gives a Nash equilibrium. [4]  
 (f) Obtain all Nash equilibria for the 2 stage repeated game of question (d) that are sequences of stage Nash equilibria. [3]  
 (g) Obtain a Nash equilibrium that is not a sequence of stage Nash equilibria for the 2 stage repeated game of question (d). Justify this. [6]

3. (a) For a two player game  $(A, B) \in \mathbb{R}_{>0}^{m \times n^2}$  give the definition of the row/column player best response polytope. [3]
- (b) For the remainder of this question, consider the following game:

$$A = \begin{pmatrix} 3 & 6 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 7 & 7 \\ 20 & 1 & 0 \end{pmatrix}$$

Show that the game is degenerate. [1]

- (c) Obtain the half space (inequalities) definition of the best response polytopes  $\mathcal{P}$ ,  $\mathcal{Q}$ . [5]

- (d) The vertices and corresponding labels that correspond to these polytopes are given by:

For  $\mathcal{P}$ :

$(0, 0)$  with labels:  $\{0, 1\}$   
 $(1/7, 0)$  with labels:  $\{1, 3, 4\}$   
 $(0, 1/20)$  with labels:  $\{0, 2\}$   
 $(19/140, 1/20)$  with labels:  $\{2, 3\}$

For  $\mathcal{Q}$ :

$(0, 0, 0)$  with labels:  $\{2, 3, 4\}$   
 $(0, 0, 1)$  with labels:  $\{0, 2, 3\}$   
 $(0, 1/6, 0)$  with labels:  $\{0, 2, 4\}$   
 $(1/3, 0, 0)$  with labels:  $\{0, 3, 4\}$

Confirm that the labels listed are correct.

- [4]
- (e) Describe the vertex enumeration algorithm. [2]
- (f) Make use of the vertex enumeration algorithm to find all pure equilibria of the game. [5]
- (g) Describe the Lemke-Howson algorithm for two player games. [2]
- (h) Make use of the Lemke-Howson algorithm to find a Nash equilibria for the game. [3]

4. (a) Give the definition of a Moran process on a game. [4]  
 (b) Consider a matrix  $A \in \mathbb{R}^{m \times n}$  representing a game with two strategies.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let:

$$f_{1i} = \frac{a(i-1) + b(N-i)}{N-1}$$

$$f_{2i} = \frac{c(i) + d(N-i-1)}{N-1}$$

For the Moran process on this game, prove that the fixation probability  $x_i$  (of  $i$  individuals of the first type taking over the population) is given by:

$$x_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^j \frac{f_{2k}}{f_{1k}}}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^j \frac{f_{2k}}{f_{1k}}}$$

State any results you use.

[6]

- (c) Consider the Markov process on the following game:  $A = \begin{pmatrix} 0 & 2 \\ r & 0 \end{pmatrix}$

Use the above theorem to show that the fixation probability  $x_1$  for  $N = 3$  is given by:

$$\frac{1}{\frac{r^2}{4} + \frac{r}{4} + 1}$$

[6]

Obtain  $x_1$  for  $r = 2$ .

[1]

- (d) Show that a value of  $r$  that ensures  $x_1 > 9/10$  satisfies:

$$r^2 + r - 4/9 < 0$$

[4]

Using this, obtain a condition for  $r$  that ensures  $x_1 > 9/10$ .

[3]

Offer an interpretation for this.

[1]