1. (a) Give the definition of Normal Form Game. [2]

(b) Consider the Normal Form Game with the following matrix representation:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Consider the mixed strategies for the row/column players $\sigma_r = (x, 1-x)/\sigma_c(y, 1-y)$. Sketch a plot of:

- Of the row player's utilities: $u_r((1,0),\sigma_c)$ and $u_r((0,1),\sigma_c)$. [1]
- Of the column player's utilities: $u_r(\sigma_r,(1,0))$ and $u_c(\sigma_r,(0,1))$. [1]

Using the plot obtain the best responses of both players. [1]

(c) Give a proof of the following theorem:
In a two player game $(A, B) \in \mathbb{R}^{m\times n^2}$ a mixed strategy σ_r^* of the row player is a best response to a column players' strategy σ_c if and only if:

$$\sigma_{r\,i}^* > 0 \Rightarrow (A\sigma_c^T)_i = \max_k (A\sigma_c^T)_k \text{ for all } 1 \leq i \leq m$$

[5]

(d) Using the above theorem, confirm the findings of question 2. [5]

(e) Consider the accompanying 2017 paper entitled "Measuring the price of anarchy in critical care unit interactions" by Knight et al.

(i) Give a general summary of the paper. [3]

(ii) What is the main theoretic result of the paper? [2]

(iii) Identify a specific modelling assumption made that limits the work. [2]

(iv) Propose an approach that could be used to overcome this limit factor. [3]

2. (a) Give the definition of repeated game. [2]

(b) Give the definition of strategy in a repeated game. [2]

(c) Obtain all possible histories for the 2 stage repeated game with the following stage game:

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 1 & 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 2 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$

[3]

(d) Show that in general the total size of the history space is given by:

$$\left| \bigcup_{t=0}^{T-1} H(t) \right| = \frac{1 - (|S_1||S_2|)^T}{1 - |S_1||S_2|}$$

Where S_1, S_2 are the strategy spaces for the stage game for both players and $H(t)$ is the history of play at stage t. [5]

(e) Give a proof of the following theorem:

For any repeated game, any sequence of stage Nash profiles gives a Nash equilibrium. [4]

(f) Obtain all Nash equilibria for the repeated game of question 3 that are sequence of stage Nash equilibria. [3]

(g) Obtain a Nash equilibrium that is not a sequence of stage Nash equilibria for the repeated game of question 3. Justify this. [6]

3. (a) For a two player game $(A, B) \in {\mathbb{R}_{>0}^{m\times n}}^2$ give the definition of the row/column player best response polytope. [3]

(b) Obtain the half space (inequalities) definition of the best response polytopes for the following game:

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 1 & 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 2 & 1 \\ 1 & -1 & 5 \end{pmatrix}$$

[5]

(c) The vertices that correspond to these polytopes are given by:

$$\ldots$$

Show that they have labels:

$$\ldots$$

[4]

(d) Describe the vertex enumeration algorithm. [2]

(e) Use the vertex enumeration algorithm to find all equilibria of the game. [5]

(f) Describe the Lemke-Howson algorithm for two player games. [2]

(g) Use the Lemke-Howson algorithm to find a Nash equilibria for the game. [4]

4. (a) Give the definition of a Moran process on a game. [4]

(b) Consider a matrix $A \in \mathbb{R}^{m\times n}$ representing a game with two strategies.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let:

$$f_{1i} = \frac{a(i-1) + b(N-i)}{N-1}$$
$$f_{2i} = \frac{c(i) + d(N-i-1)}{N-1}$$

For the Moran process on this game, prove that the fixation probability x_i (of i individuals of the first type taking over the population) is given by:

$$x_i = \frac{1 + \sum_{j=1}^{i-1}\prod_{k=1}^{j}\frac{f_{2k}}{f_{1k}}}{1 + \sum_{j=1}^{N-1}\prod_{k=1}^{j}\frac{f_{2k}}{f_{1k}}}$$

[6]

(c) Consider the Markov process on the following game: $A = \begin{pmatrix} 3 & 2 \\ r & 3 \end{pmatrix}$ Use the above theorem to obtain the fixation probabilities for each strategy for $r = 2$ and $N = 6$. [5]

(d) What value of r ensures that the fixation probability for the second strategy is $> .9$ (for $N = 6$)? [5]

(e) Consider a population with 3 strategies corresponding to the following game with neutral drift (all strategies have equal fitness).

Assuming $N = 3$ list all possible states of the underlying Markov chain. [2]

For a given state $(i_1, i_2, i_3$ obtain the probability of transitioning to a state with $i_1 + 1$ individuals of the first type. [3]