1. (a) Give the definition of a normal form game. [2]

(b) Consider the normal form game with the following matrix representation:

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Consider the mixed strategies for the row player $\sigma_r = (x, 1-x)$ and the column player $\sigma_c = (y, 1-y)$. Sketch a plot of:

- Of the row player's utilities: $u_r((1,0),\sigma_c)$ and $u_r((0,1),\sigma_c)$. [1]
- Of the column player's utilities: $u_r(\sigma_r,(1,0))$ and $u_c(\sigma_r,(0,1))$. [1]

Using the plot, obtain the best responses of both players. [1]

(c) Give a proof of the following theorem:
In a two player game $(A, B) \in \mathbb{R}^{m\times n^2}$ a mixed strategy σ_r^* of the row player is a best response to a column player's strategy σ_c if and only if:

$$\sigma_{r\,i}^* > 0 \Rightarrow (A\sigma_c^T)_i = \max_k (A\sigma_c^T)_k \text{ for all } 1 \leq i \leq m$$

[5]

(d) Using the above theorem, identify the Nash equilibria for the game in question (b) and confirm your answer to question (b). [5]

(e) Consider the accompanying 2017 paper entitled "Measuring the price of anarchy in critical care unit interactions" by Knight et al.

(i) Give a general summary of the paper. [3]

(ii) What is the main theoretic result of the paper? [2]

(iii) Identify a specific modelling assumption made that limits the work. [2]

(iv) Propose an approach that could be used to overcome this limiting factor. [3]

2. (a) Give the definition of a repeated game. [2]

(b) Give the definition of strategy in a repeated game. [2]

(c) Show that in general the total size of the history space is given by:

$$\left|\bigcup_{t=0}^{T-1} H(t)\right| = \frac{1-(|S_1||S_2|)^T}{1-|S_1||S_2|}$$

Where S_1, S_2 are the strategy spaces for the stage game for both players and $H(t)$ is the history of play at stage t. [5]

(d) For the remainder of this question, consider the following stage game:

$$A = \begin{pmatrix} 3 & 6 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 7 & 7 \\ 20 & 1 & 0 \end{pmatrix}$$

Obtain all possible histories for the corresponding 2 stage repeated game. [3]

(e) Give a proof of the following theorem:

For any repeated game, any sequence of stage Nash profiles gives a Nash equilibrium. [4]

(f) Obtain all Nash equilibria for the 2 stage repeated game of question (d) that are sequences of stage Nash equilibria. [3]

(g) Obtain a Nash equilibrium that is not a sequence of stage Nash equilibria for the 2 stage repeated game of question (d). Justify this. [6]

3. (a) For a two player game $(A, B) \in {\mathbb{R}_{>0}^{m\times n}}^2$ give the definition of the row/column player best response polytope. [3]

(b) For the remainder of this question, consider the following game:

$$A = \begin{pmatrix} 3 & 6 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 7 & 7 \\ 20 & 1 & 0 \end{pmatrix}$$

Obtain the half space (inequalities) definition of the best response polytopes $\mathcal{P}$, $\mathcal{Q}$. [5]

(c) The vertices and corresponding labels that correspond to these polytopes are given by:

For $\mathcal{P}$:

$$\begin{aligned} (0, 0) &\text{ with labels: } \{0, 1\} \\ (1/7, 0) &\text{ with labels: } \{1, 3, 4\} \\ (0, 1/20) &\text{ with labels: } \{0, 2\} \\ (19/140, 1/20) &\text{ with labels: } \{2, 3\} \end{aligned}$$

For $\mathcal{Q}$:

$$\begin{aligned} (0, 0, 0) &\text{ with labels: } \{2, 3, 4\} \\ (0, 0, 1) &\text{ with labels: } \{0, 2, 3\} \\ (0, 1/6, 0) &\text{ with labels: } \{0, 2, 4\} \\ (1/3, 0, 0) &\text{ with labels: } \{0, 3, 4\} \end{aligned}$$

Confirm that the labels listed are correct. [4]

(d) Describe the vertex enumeration algorithm. [2]

(e) Use the vertex enumeration algorithm to find all equilibria of the game. [6]

(f) Describe the Lemke-Howson algorithm for two player games. [2]

(g) Use the Lemke-Howson algorithm to find **a** Nash equilibria for the game. [3]

4. (a) Give the definition of a Moran process on a game. [4]

(b) Consider a matrix $A \in \mathbb{R}^{m\times n}$ representing a game with two strategies.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let:

$$f_{1i} = \frac{a(i-1) + b(N-i)}{N-1}$$
$$f_{2i} = \frac{c(i) + d(N-i-1)}{N-1}$$

For the Moran process on this game, prove that the fixation probability x_i (of i individuals of the first type taking over the population) is given by:

$$x_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^{j} \frac{f_{2k}}{f_{1k}}}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^{j} \frac{f_{2k}}{f_{1k}}}$$

State any results you use.

[6]

(c) Consider the Markov process on the following game: $A = \begin{pmatrix} 0 & 2 \\ r & 0 \end{pmatrix}$

Use the above theorem to show that the fixation probability x_1 for $N = 3$ is given by:

$$\frac{1}{\frac{r^2}{4} + \frac{r}{4} + 1}$$

[6]

Obtain x_1 for $r = 2$.

[1]

(d) Show that a value of r that ensures $x_1 > 9/10$ satisfies:

$$r^2 + r - 4/9 < 0$$

[4]

Using this, obtain a condition for r that ensures $x_1 > 9/10$. [3]

Offer an interpretation for this. [1]