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Stats/Prob Assignment

1. Given: $X \leftarrow P(x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$

$Y \leftarrow P(y) = \dfrac{e^{-\mu} \mu^y}{y!}$

Also given that X & Y are independent.

So, $P\{X \mid X+Y=n\} = \dfrac{P\{X, X+Y=n\}}{P\{X+Y=n\}} = \dfrac{P\{X=x\} \cdot P\{Y=n-x\}}{P\{X+Y=n\}}$ — ①

Assume, $Z = X+Y$

So, $M_Z(t) = E(e^{tZ}) = E(e^{tX}) \cdot E(e^{tY})$
$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

This is a Poisson$(\lambda+\mu)$ dis^n.

So, $P\{\underbrace{X+Y}_{Z}\} = \dfrac{e^{-(\lambda+\mu)} \cdot (\lambda+\mu)^Z}{Z!}$

$\therefore P(Z) = \dfrac{e^{-(\lambda+\mu)} (\lambda+\mu)^n}{n!}$

So, Eq^n ① becomes:

$$\frac{P\{X, X+Y=n\}}{P\{X+Y=n\}} = \left(\frac{e^{-\lambda} \cdot \lambda^x}{x!} \cdot \frac{e^{-\mu} \mu^{n-x}}{(n-x)!} \right) \times \frac{n!}{\left(e^{-(\lambda+\mu)} \cdot (\lambda+\mu)^n\right)}$$

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$$= \left(\frac{n!}{x!(n-x)!}\right) \cdot \left(\frac{\lambda^x \cdot \mu^{n-x}}{(\lambda+\mu)^n}\right)$$

$$= {}^nC_x \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{n-x} \left(\frac{\lambda}{\lambda+\mu}\right)^x$$

Given as π, Given as $(1-\pi)$

$$\boxed{Ans = {}^nC_x \cdot \pi^x \cdot (1-\pi)^{n-x}}$$

Hence Proved.

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(2) Given: X_1, X_2, X_3 ——

$$P\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2}$$

$$P(X_n = n) = \frac{1}{n^2}$$

(i) Checking if X_n converges in probability.

$$\lim_{n\to\infty} P\{|X_n - X| \geq \varepsilon\} = 0$$

or $X_n \xrightarrow{P} X$

or $\lim_{n\to\infty} P\{|X_n - X| < \varepsilon\} = 1$

'n' is a positive integer, then we can clearly see that as $n \to \infty$

$$P\left(X_n = \frac{1}{n}\right) \to 1$$

$$\& \quad P\left(X_n = n\right) \to 0$$

In other words,

$$\lim_{n\to\infty} P\left(\left|X_n - \frac{1}{n}\right| < \varepsilon\right) = 1 \text{ holds}$$

$$\therefore \quad X_n \xrightarrow{P} \frac{1}{n} = 0$$

(ii) Checking if X_n converges in quadratic mean

Now, X_n will converge in quadratic mean to X if

$$E(X_n - X)^2 \to 0$$

as

$$n \to \infty$$

OR

$$X_n \xrightarrow{q/m} X$$

here, $E(X_n - 0)^2 = E(X_n^2)$

where, $E(X_n^2) = 1 + \frac{1}{n^2} - \frac{1}{n^4}$

Now, $E(X_n^2) \to 1$ as $n \to \infty$.

Now clearly, $E(X_n - 0)^2 \to 0$ doesn't holds true.

$\therefore$ X_n doesn't converge in Q.M.

3. Given: $X \rightarrow$ continuous random Variable

$CDF = F$ & $PDF = f$.

$P(X>0) = 1$

$E(X)$ exists.

To show: $E(X) = \int_0^\infty P(X>x)\,dx$

Proof: $P(X \leq x) = F(x)$.

& $\int_0^x f(x)\,dx = F(x)$.

So,: $P(X \leq x) = \int_0^x f(x)\,dx$

$P(X>x) = \int_x^\infty f(x)\,dx$

$$\int_0^\infty P(X>x)\,dx = \int_0^\infty \left[\int_x^\infty f(x)\,dx\right] dx$$

OR

$$\int_0^\infty P(X>x)\,dx = \int_0^\infty \left[\int_x^\infty f(t)\,dt\right] dx \quad \rightarrow 0<x<t<\infty$$

or, $0<t<\infty$ & $0<x<t$

$$\int_0^\infty P(X>x)\,dx = \int_0^\infty \left[\int_0^t 1\cdot dx\right] f(t)\,dt$$

$$\int_0^\infty P(X>x)\,dx = \int_0^\infty t\cdot f(t)\,dt \quad -(1)$$

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Also, $E(X) = \int_{-\infty}^{\infty} t\cdot f(t)\,dt$

But, $P(X>x)=1 \Rightarrow P(X\leq x)=0$

$\therefore E(X) = \int_{0}^{\infty} t\cdot f(t)\,dt$

So, $\boxed{E(X) = \int_{0}^{\infty} P(X>x)\,dx} \rightarrow$ Hence Proved.

(4) KL divergence b/w 2 Gaussians is

$$D(q(x)\,||\,p(x)) = E_{y\sim q(y)}\left(\log\left[\frac{q(y)}{p(y)}\right]\right)$$

Now, $p(x) = \mathcal{N}(x|\mu,\Sigma)$ & $q(x) = \mathcal{N}(x|m,L)$

$$D(q(x)\,||\,p(x)) = E_{y\sim q(y)}\left(\log\left[\frac{q(y)}{p(y)}\right]\right)$$

$\because$ $p(x)$ & $q(x)$ are Gaussians

So, $p(x:\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right]$ — (1)

$q(x:m,L) = \frac{1}{(2\pi)^{n/2}|L|^{1/2}} \exp\left[-\frac{1}{2}(x-m)^T L^{-1}(x-m)\right]$ — (2)

$\therefore KL = \int q(x)\cdot \ln\left(\frac{q(x)}{p(x)}\right)dx = E_{q(x)}\left[\ln\frac{q(x)}{p(x)}\right]$ — (3)

Putting (1) & (2) in (3)

$$= E\left[\frac{1}{2}\ln\left|\frac{L}{E}\right| - \frac{1}{2}(x-\mu)^T E^{-1}(x-\mu) + \frac{1}{2}(x-m)^T L^{-1}(x-m)\right]$$

For scalar a, $a = \text{trace}(A)$
& $tr(ABC) = tr(BCA) = tr(CAB)$

Thus, $= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}E\left[tr\left(E^{-1}(x-\mu)(x-\mu)^T\right)\right]$

$$+ \frac{1}{2}E\left[tr\left[L^{-1}(x-m)(x-m)^T\right]\right]$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}tr\left(E^{-1}E\left[(x-\mu)(x-\mu)^T\right]\right) + \frac{1}{2}tr\left(L^{-1}E(xx^T - 2mx^T + mm^T)\right)$$

$\because$ $E(ax) = a\mu$
$E(x^T a x) = \mu^T a \mu + tr(aE)$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}tr(E^{-1}E) + \frac{1}{2}tr\left(L^{-1}(E + \mu\mu^T - 2m\mu^T + mm^T)\right)$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}x + \frac{1}{2}tr(L^{-1}E)$$

$$+ \frac{1}{2}tr\left(\mu^T L^{-1}\mu - 2\mu^T L^{-1}m + m^T L^{-1}m\right)$$

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$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{n}{2} + \frac{1}{2} tr(L^{-1}E) + \frac{1}{2}(m-\mu)^T L^{-1}(m-\mu)$$

$$\boxed{Ans = \frac{1}{2}\left[\ln\frac{|L|}{|E|} - n + tr(L^{-1}E) + (m-\mu)^T L^{-1}(m-\mu)\right]}$$

(5) (X, Y) has bivariate normal dis^n

$$f(X,Y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

To Show that $corr(X,Y) = \rho$ & $corr(X^2, Y^2) = \rho^2$

Now,

$$corr(X,Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$$

$$\& \; cov(X,Y) = E(XY) - E(X)E(Y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad - (1)$$

Let us take, $X = Z_1$ & $Y = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$

$$f(z_1, z_2) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left[\frac{-1}{2(1-\rho^2)} z_1^2 - 2\rho z_1[\rho z_1 + \sqrt{1-\rho^2} z_2] + \rho^2 z_1^2 + (1-\rho^2) z_2^2\right]$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left[\frac{-1}{2(1-\rho^2)} (1-\rho^2) z_1^2 + (1-\rho^2) z_2^2\right]$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left[\frac{-1}{2}(z_1^2 + z_2^2)\right]$$

Now, $Z_1 = Z_2 \sim N(0,1)$

$$\text{Var}(X) = \text{var}(Z_1) = 1$$
$$\text{var}(Y) = \rho^2 \text{var}(Z_1) + (1-\rho^2)\,\text{var}(Z_2)$$
$$= 1$$

$$\text{cov}(X, Y) = \text{cov}(Z_1, \rho Z_1 + \sqrt{1-\rho^2}\, Z_2)$$
$$= \rho\, \text{cov}(Z_1, Z_1) + \sqrt{1-\rho^2}\, \underbrace{\text{cov}(Z_1, Z_2)}_{0}$$
$$= \rho \cdot 1 + \sqrt{1-\rho^2} \cdot 0$$
$$= \rho$$

$\because Z_1 \,\&\, Z_2$ are independent

$$\therefore \boxed{\text{corr}(X, Y) = \rho}$$

Similarly, let, $X^2 = Z_1^2$

$$Y^2 = (\rho Z_1 + \sqrt{1-\rho^2}\, Z_2)^2$$

$$= \rho^2 Z_1^2 + (1-\rho^2) Z_2^2 + 2\rho\sqrt{1-\rho^2}\, Z_1 Z_2$$

$$\text{corr}(X^2, Y^2) = \text{cov}\big(Z_1^2, \rho^2 Z_1^2 + (1-\rho^2) Z_2^2 + 2\rho\sqrt{1-\rho^2}\, Z_1 Z_2\big)$$

$$= \rho^2 \underbrace{\mathrm{cov}(z_1^2, z_1^2)}_{1} + (1-\rho^2)\,\overbrace{\mathrm{cov}(z_1^2, z_2^2)}^{0}$$

$$+ 2\rho\sqrt{1-\rho^2}\,\underbrace{\mathrm{cov}(z_1^2, z_1 z_2)}_{0}$$

$$\boxed{\mathrm{corr}(X^2, Y^2) = \rho^2}$$

Hence proved.

Asked in group & then wrote the solⁿ

6. $X \sim \mathrm{Gamma}(n, 3)$

$$f(x) = \frac{1}{\sqrt{n} \cdot 3^n} \cdot x^{n-1} \cdot e^{-x/3}, \quad x > 0$$

'x' is a large integer

Now from CLT for $X_1, X_2, \ldots, X_n \sim \mathrm{Gamma}(n, 3)$ such that they are all i.i.d. r.v.s

$$\bar{X} = \frac{1}{n} \sum X_i$$

$$\& \ E(\bar{X}) = \frac{1}{n} E\left(\sum X_i\right) = \frac{1}{n} \cdot n\mu = \mu$$

$$\mathrm{var}(\bar{X}) = \frac{\sigma^2}{n}$$

$$\text{Then, } Y = \frac{\bar{X} - E(\bar{X})}{\sqrt{\mathrm{var}(\bar{X})}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$M_Y(t) = E(e^{tY})$$

$$E(e^{ty}) = E\left(e^{t\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}\right) = E\left(e^{t\left(\frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma}\right)}\right)$$

$$= E\left(e^{\frac{t(X_1-\mu)}{\sqrt{n}\sigma_1}}\right) \cdot e^{\frac{t(X_2-\mu)}{\sqrt{n}\sigma_2}}$$

$$= \prod_{i=1}^{n} E\left(e^{\frac{t(X_i-\mu)}{\sqrt{n}\sigma_i}}\right)$$

$$M_{\bar{x}}(t) = \left[E\left(e^{\frac{t(X_1-\mu)}{\sqrt{n}\sigma_1}}\right)\right]^n \quad \text{---} (1)$$

Say, $m(t) = E\left(e^{t(X-\mu)}\right) = e^{-\mu t} \cdot M(t)$

Then, $m(0) = E\left(e^{0(X-\mu)}\right) = E(1) = 1$

$$m'(0) = -\mu e^{-\mu t} M(t) + M'(t) e^{-\mu t}\Big|_{t=0}$$

$$m'(0) = -\mu + \mu = 0$$

$$m''(0) = \mu^2 e^{-\mu t} M(t) - \mu e^{-\mu t} M'(t) + M''(t) e^{-\mu t} - \mu e^{-\mu t} M'(t)\Big|_{t=0}$$

$$m''(0) = \sigma^2$$

Now, we know,

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a+\theta h)$$

(Taylor's formula)

At $a=0$,

$$f(h) = f(0) + \frac{h}{1!} f'(0) + \frac{h^2}{2!} f''(\theta h)$$

for, $f(h) = m(t)$

$$m(t) = m(0) + \frac{t}{1!} m'(0) + \frac{t^2}{2!} m''(\theta t)$$

$$m(t) = 1 + \frac{t^2}{2!} m''(\underbrace{\theta t}_{K})$$

then, $0 < K < t$

$$\therefore m(t) = 1 + \frac{\sigma^2 t^2}{2!} - \frac{\sigma^2 t^2}{2!} + \frac{t^2}{2!} M''(K)$$

$$= 1 + \frac{\sigma^2 t^2}{2!} + \left(\frac{M''(K) - \sigma^2}{2!}\right) t^2$$

From ①, $M_Y(t) = \left[E\left(m\left(\frac{t}{\sqrt{n}\sigma}\right)\right)\right]^n$

$$= \left[E\left[1 + \frac{\sigma^2 t^2}{2n\sigma^2} + \frac{(m''(K) - \sigma^2)}{2} \cdot \frac{t^2}{n\sigma^2}\right]\right]^n$$

Now, $\lim_{n\to\infty} M_X(t) = \lim_{n\to\infty} \left\{ 1 + \underbrace{\frac{\sigma^2 t^2}{2n\sigma^2} + \frac{(M''(K) - \sigma^2)}{2} \cdot \frac{t^2}{n\sigma^2}}_{\text{tends to } 0} \right\}^n$

So, 1^∞ format

$$\lim_{n\to\infty} M_X(t) = e^{\lim_{n\to\infty} \left\{ \frac{t^2}{2n} + \frac{(m''(K) - \sigma^2)}{2n\sigma^2} \cdot t^2 \right\} n}$$

$$= e^{\lim_{n\to\infty}\left\{\frac{t^2}{2} + \frac{m''(K) - \sigma^2}{2\sigma^2} \cdot t^2\right\}}$$

$$= e^{\frac{t^2}{2} + \frac{m''(K) - \sigma^2}{2\sigma^2} \cdot t}$$

Now, here, $-n < \frac{t}{\sqrt{n}\sigma} < n \Rightarrow \theta t = K$

As $n \to \infty$, $K \to 0$

$\left(\theta = \frac{1}{\sqrt{n}\sigma}\right)$

then, $m''(K) - \sigma^2 \longrightarrow m''(0) - \sigma^2 \longrightarrow 0$

As, $m''(0) \to \sigma^2$ as $n \to \infty$

Then, $M_Y(t) = e^{t^2/2}$.

$\therefore Y \xrightarrow{d} N(0,1)$

This holds true for all distributions including $X \sim Gamma(n, 3)$, given $n \to \infty$.

Since, n is very very large integer, CLT holds true.

$$f(x : n, 3) = \frac{1}{\sqrt{n}} \cdot \frac{1}{3^n} \cdot x^{n-1} \cdot e^{-x/3}$$

Mean $= \alpha\beta = 3n$, Variance $= \alpha\beta^2 = 9n$ (Approximately)

Then, $X \sim N(3n, 9n)$ for $n \to \infty$ by CLT

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