classmate

Date

Page

We want:

① $$\frac{P(X=x \cap X+Y=n)}{P(X+Y=n)}$$ ~~P(X+Y~~

For denominator

~~P(X+Y=n)~~ PDF of $X+Y \sim e^{-\lambda+\mu} \frac{(\lambda+\mu)^{u}}{u!}$

where $u = x+y$

② $\therefore P(X+Y=n) = e^{-\lambda+\mu} \frac{(\lambda+\mu)^n}{n!}$

For numerator

$$P(X=x \cap Y=n-x) = \left(e^{-\lambda} \frac{\lambda^x}{x!}\right) \times \left(\frac{e^{-\mu} \mu^{n-x}}{(n-x)!}\right)$$

③ $$= e^{-\lambda+\mu} \frac{\lambda^x \mu^{n-x}}{x!(n-x)!}$$

- Sustitute ② & ③ into ①

$$\rightarrow \frac{\cancel{e^{-(\lambda+\mu)}} \lambda^x \mu^{n-x}}{x!(n-x)!} \times \frac{n!}{\cancel{e^{-\lambda+\mu}} (\lambda+\mu)^n}$$

$$= \binom{n}{x} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{n-x}$$

$$= \binom{n}{x} \pi^x (1-\pi)^{n-x}$$

classmate

Date

Page

2) i) By definition : sequence $X_n \xrightarrow{P} c$ if

$$\lim_{n\to\infty} P(|X_n - c| < \epsilon) = 1 \quad \text{for all } \epsilon \geq 0$$

~~Suppose~~ We claim that sequence converges to $c=0$

$$\therefore P(|X_n - 0| < \epsilon) = \begin{cases} 0 & \text{if } 0 \leq \epsilon \leq \frac{1}{n} \\ \cancel{1-\frac{1}{n^2}} & \cancel{\text{if } \frac{1}{n} < \epsilon < n} \\ 1-\frac{1}{n^2}+\frac{1}{n^2} & \text{if } \epsilon > n \end{cases}$$

~~for all $\epsilon \geq 0$~~

$$\lim_{n\to\infty} P(\ \ldots\ ) = \begin{cases} 1-\frac{1}{\infty^2} & \text{if } 0 < \epsilon \leq \frac{1}{\infty}\ \infty \\ 1 & \text{if } \epsilon > \infty \end{cases}$$

$$= \begin{cases} 1 & \cancel{\text{if } \epsilon \leq 0} \\ 1 & \text{if } \epsilon > \infty \end{cases} = 1 \quad \text{for all } \epsilon$$

$\therefore$ Sequence converges to 0 in probability

ii) $E(|X_n - 0|^2) = E(X_n^2)$

$$= \left(\frac{1}{n}\right)^2 \times \left(1-\frac{1}{n^2}\right) + n^2 \times \left(\frac{1}{n^2}\right)$$

$$= \frac{1}{n^2} - \frac{1}{n^4} + 1 \quad \therefore \lim_{n\to\infty}\left(E(|X_n - 0|^2)\right) = 1 \neq 0$$

$\therefore$ No convergence in quadratic mean

By definition

3) ① $E(X) = \int_{-\infty}^{\infty} f(x) \times x \, dx$

$P(X>x) = 1 - F(x) \Rightarrow F(0) = 0 \Rightarrow \int_{-\infty}^{0} f(x) = 0 \Rightarrow$

$\Downarrow$

$f(x)=0$ for $x<0$

$\cancel{\int_0^{\infty} P(X>x) = \int_0^{\infty} 1 - F(x)}$

~~Applying by parts on ①~~

$\cancel{\Rightarrow F(x) \times x \Big|_0^{\infty} - \int_0^{\infty} F(x)\, dx}$

$\cancel{P(X>x) = \int_x^{\infty} f(x)\, dt}$

$\cancel{\Rightarrow \int_0^{\infty} P(X>x)\,dx = \int_0^{\infty}\int_0^{\infty} f(x)\,dx\,dx}$

$P(X>x) = \int_x^{\infty} f(t)\, dt$

$\int_0^{\infty} P(X>x)\,dx = \int_0^{\infty}\int_x^{\infty} f(t)\,dt\,dx = \int_0^{\infty}\int_0^{t} f(t)\,dx\,dt$

$= \int_0^{\infty} f(t) \times t\, dt = E(X)$

4) $D_{KL}(P\|Q) = E_{x\sim P}\left[\log P(x) - \log Q(x)\right]$

$D_{KL}(P\|Q) = E_P\left[\log\frac{1}{\sqrt{2\pi E}} - \frac{(x-\mu)^2}{2E} - \log\left(\frac{1}{\sqrt{2\pi L}}\right) + \left(\frac{x-m}{2L}\right)^2\right]$

$= E\left[x^2\left(\frac{1}{2L}-\frac{1}{2E}\right) + x\left(\frac{\mu}{E}-\frac{m}{L}\right) + \frac{m^2}{2L} - \frac{\mu^2}{2E} + \frac{1}{2}\log\left(\frac{L}{E}\right)\right]$

(since $E(x^2) = \mu^2+\sigma^2$)

$= \left[(L+\mu^2)\left[\frac{1}{2L}-\frac{1}{2E}\right] + \mu\left(\frac{\mu}{E}-\frac{m}{L}\right) + \frac{m^2}{2L} - \frac{\mu^2}{2E} + \frac{1}{2}\log\left(\frac{L}{E}\right)\right]$

$= \frac{1}{2} - \frac{L}{2E} + \frac{\mu^2}{2L} - \frac{\mu^2}{2E} + \frac{\mu^2}{E} - \frac{\mu m}{L} + \frac{m^2}{2L} - \frac{\mu^2}{2E} + \frac{1}{2}\log\left(\frac{L}{E}\right)$

$= \frac{1}{2} - \frac{L}{2E} + \frac{\mu^2}{2L} - \frac{\mu m}{L} + \frac{m^2}{2L} + \frac{1}{2}\log\left(\frac{L}{E}\right)$

$= \frac{1}{2} - \frac{L}{2E} + \frac{1}{2}(\mu-m)^2 + \frac{1}{2}\log\left(\frac{L}{E}\right)$

classmate

Date

Page

5) $E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) x\,dx\,dy$

$= 0$ (odd function) $= E(Y)$

$\mathrm{Corr}(X,Y) = E(XY) - E(X)E(Y)$

$\cancel{\mathrm{Corr}(X,Y) = E(XY)}$

Let $t = x+y$, $u = x-y$

$\hookrightarrow x = \frac{u+t}{2}$; $y = \frac{t-u}{2}$

$\hookrightarrow J = \left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\| = \frac{1}{2}$

$\rightarrow f(t,u) = \frac{1}{4\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[(x+y)^2\left(\frac{\rho+1}{2}\right) + (x-y)^2\left(\frac{1-\rho}{2}\right)\right]\right)$

$\rightarrow f(t,u) = \frac{1}{4\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{4}\left[\frac{t^2}{1-\rho} + \frac{u^2}{1+\rho}\right]\right)$

a) $E(XY) = E\left(\frac{t^2-u^2}{4}\right) = \frac{1}{4}\left(E(t^2) - E(u^2)\right)$

$$\Rightarrow E(t^2) = \iint \frac{1}{4\pi(1-\rho^2)^{\frac12}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{4}\left[\frac{t^2}{1-\rho}\right]\right)\times\exp\left(\frac{-1}{4}\left[\frac{u^2}{1+\rho}\right]\right) t^2\,du\,dt$$

$$= \frac{1}{4\pi(1-\rho^2)^{\frac12}}\int_{-\infty}^{\infty}\exp\left(\frac{-1}{4}\left[\frac{t^2}{1-\rho}\right]\right)\times t^2\times \sqrt{\pi}\times 2\sqrt{1+\rho}\,dt$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{1-\rho}}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{t}{c}\right)^2\right)\times\left(\frac{t}{c}\right)^2\times c^2\,dt$$

where $c = 2\sqrt{1-\rho}$

(this u substitution has nothing to do with the previous u)

$$= \frac{1}{2\sqrt{\pi}\sqrt{1-\rho}}\times c^3\int_{-\infty}^{\infty}u^2\exp(-u^2)\,du = \frac{1}{2\sqrt{\pi}}\times 4(1-\rho)\times\Gamma\left(\frac{3}{2}\right)$$

$$= 2(1+\rho)$$

$E(u^2) = 2(1+\rho)$ (by symmetry)

$$\therefore E(XY) = \frac{1}{4}\left(2+2\rho-(2-2\rho)\right) = \rho = \operatorname{cor}(X,Y)$$

b) $E(X^2) = \frac{1}{2}E(X^2+Y^2) = \frac{1}{2}E\left(\frac{t^2+u^2}{2}\right) = \frac{1}{4}E(t^2+u^2)$

$$= \frac{1}{4}\left(2(1+\rho)+2(1-\rho)\right) = 1 = E(Y^2)$$

$$E(X^2Y^2) = E\left(\left(\frac{t^2-u^2}{4}\right)^2\right) = \frac{1}{16}E(t^4+u^4-2u^2t^2)$$

$$E(t^4) = \frac{1}{4\pi(1-\rho^2)^{\frac12}}\iint \exp\left(\frac{-1}{4}\left[\frac{t^2}{1-\rho}\right]\right)\times\exp\left(\frac{-1}{4}\left[\frac{u^2}{1+\rho}\right]\right) t^4\,du\,dt$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{1-\rho}} \int_{-\infty}^{\infty} \frac{t^4}{c^4} \exp\left(-\frac{t^2}{c^2}\right) \times c^4 \, dt \qquad \left(\text{where } c = 2\sqrt{1-\rho}\right)$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{1-\rho}} \times c^4 \times 2 \int_{0}^{\infty} u^2 \exp(-u) \times \frac{c}{2\sqrt{u}} \, du \qquad \left(\text{let } u = \frac{t^2}{c^2}\right)$$

$$\hookrightarrow du = \frac{2t\,dt}{c^2} \Rightarrow dt = \frac{c}{2\sqrt{u}}$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{1-\rho}} \times c^5 \int_{0}^{\infty} u^{3/2} \exp(-u) \, du$$

$$= \frac{1}{2\sqrt{\pi}\sqrt{1-\rho}} \times c^5 \times \frac{3\sqrt{\pi}}{4} = \frac{3}{8\sqrt{\pi}\sqrt{1-\rho}} \times 2^5 \times \left(\sqrt{1-\rho}\right)^5 = \frac{12}{\sqrt{\pi}} \times (1+\rho)^2$$

$$\Rightarrow E(u^4) = 12 \times (1-\rho)^2 \qquad \text{(by symmetry)}$$

$$\cancel{Cov(u,t) = E(ut) - E(u)E(t) = E(x^2 - y^2) - [E(x) - E(y)][E(x) + E(y)] = 0 - 0 \times 0 = 0}$$

$$E(u^2 t^2) = \frac{1}{4\pi(1-\rho^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} t^2 \exp\left(-\frac{1}{4}\left[\frac{t^2}{1-\rho}\right]\right) dt \int_{-\infty}^{\infty} u^2 \exp\left(-\frac{1}{4}\left[\frac{u^2}{1+\rho}\right]\right) du$$

$$= \frac{1}{4\pi(1-\rho^2)^{\frac{1}{2}}} \times \left[2(1+\rho) \times 2\sqrt{\pi}\sqrt{1-\rho}\right] \times \left[2(1-\rho) \times 2\sqrt{\pi}\sqrt{1+\rho}\right]$$

$$= 4(1-\rho^2)$$

$$\therefore E(X^2Y^2) = \frac{1}{16}\left[12 \times (1+\rho)^2 + 12(1-\rho)^2 - 2\times 4(1-\rho^2) \right]$$

$$= \frac{1}{16}\left[24 + 24\rho^2 - 8 + 8\rho^2 \right] = 1 + 2\rho^2$$

$$\therefore \operatorname{cov}(X^2, Y^2) = E(X^2Y^2) - E(X^2)E(Y^2) = 1 + 2\rho^2 - 1 = 2\rho^2 \ :($$

6) Let $x_i \sim \text{Gamma}(n, 3)$ then $MGF_{x_i}(t) = (1-nt)^{-3}$

define $\bar{X} = \dfrac{x_1 + x_2 \dots x_k}{k}$

$$MGF_{\bar{x}}(t) = E\left[\exp\left(\frac{t(x_1+x_2 \dots x_k)}{k}\right)\right]$$

$$= E\left[\exp\left(\frac{tx_1}{k}\right) \times \exp\left(\frac{tx_2}{k}\right) \dots \exp\left(\frac{tx_k}{k}\right)\right]$$

$$= E\left[\exp\left(\frac{tx_1}{k}\right)\right]^k = \left[MGF\left(\frac{t}{k}\right)\right]^k$$

$$= \left(1 - \frac{nt}{k}\right)^{-3k}$$

$$= \left(1 - \frac{nt}{k}\right)^{-3k}$$

(if k is large) form $\approx 1 + \left(-\frac{nt}{k}\right) \times -3k + -\left(\frac{n^2t^2}{k^2}\right) \times \frac{(-3k)(-3k-1)}{2}$

$$= 1 + 3nt - \frac{n^2t^2}{2} \times (-3)\left(-3 - \frac{1}{k}\right)$$

(if k is large) $\approx 1 + 3nt - \frac{9}{2}n^2t^2$

$$\approx \exp\left(3nt - \frac{9}{2}n^2t^2\right) \longrightarrow \boxed{\mu = 3n, \; \sigma^2 = 9n^2}$$