Assignment 2A

~~1, Let~~

1, Let $P(A)$ be probability that Alice collects the tickets and $P(B)$ be probability that Bob collects tickets

Given: $\dfrac{P(A)}{P(B)} = \dfrac{1}{10}$

$$P(n_1, n_2, n_3, n_4, n_5 \mid A) = \prod_{i=1}^{5} P(n_i \mid A)$$

(Likelihood that tickets were collected by Alice over 5 samples) $(\mathcal{L}(A))$

$$= \frac{10^{(12+10+11+4+11)}\, e^{-5(10)}}{12!\ 10!\ 11!\ 4!\ 11!}$$

$\left[\cancel{P(X \mid A)} \sim \text{Poisson}(\lambda) \text{ where } X \text{ is number of tickets collected} \right.$
$\left. P(X \mid A = \lambda) \right]$ and $\lambda = 10$ in case of Alice

Similarly

$$\mathcal{L}(B) = \prod_{i=1}^{5} P(n_i \mid B)$$

$$= \frac{15^{(12+10+11+4+11)}\, e^{-5(15)}}{12!\ 10!\ 11!\ 4!\ 11!}$$

Since

$$\underbrace{P(A \mid n_1, n_2, n_3, n_4, n_5)}_{\text{(Posterior)}} = \frac{\overbrace{P(n_1, n_2, n_3, n_4, n_5 \mid A)}^{\text{(Likelihood)}}\ \overbrace{P(A)}^{\text{(Prior)}}}{P(n_1, n_2, n_3, n_4, n_5)}$$

$$\therefore \frac{P(A \mid n_1 \ldots n_5)}{P(B \mid n_1 \ldots n_5)} = \frac{\mathcal{L}(A)\, P(A)}{\mathcal{L}(B)\, P(B)}$$

Posterior odds

$$= \left(\frac{10^{48}\, e^{-50}}{15^{48}\, e^{-75}} \right) \left(\frac{1}{10} \right) \approx \underline{\underline{25.4}}$$

2) Given $X = \theta + \varepsilon \rightarrow$ noise

$\varepsilon \sim N(0,4)$

$\theta \sim N(5,9)$ (Prior)

$X|\theta \sim N(\theta, 4)$ (Since X depends upon θ)

As we know $f(\theta|x) = \dfrac{f(x|\theta)\, f_\theta(\theta)}{f_X(x)}$

(Posterior)

given $x = 6$

$$f(x|\theta) = \frac{1}{\sqrt{2\pi(4)}} e^{-\frac{(x-\theta)^2}{2(4)}}$$

$$= \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{8}}$$

$$f(6|\theta) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(6-\theta)^2}{8}}$$

$$f_\theta(\theta) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{(\theta-5)^2}{18}}$$

$$\therefore f(6|\theta) f_\theta(\theta) = \frac{1}{12\pi} e^{-\frac{(13x^2 - 148x + 424)}{72}}$$

$$\text{as } f_{X,\theta}(x,\theta) = f(x|\theta) f_\theta(\theta)$$

$$\therefore \text{and } f_X(x) = \int_{-\infty}^{\infty} f_{X,\theta}(x,\theta)\, d\theta$$

$$\therefore\ f_x(x) = \int_{-\infty}^{\infty} f(x|\theta)\, f(\theta)\, d\theta$$

$$f_x(6) = \int_{-\infty}^{\infty} f(6|\theta)\, f(\theta)\, d\theta$$

$$f(6|\theta)\, f(\theta) = \frac{1}{12\pi}\, e^{-\frac{(13\theta^2 - 148\theta + 424)}{72}}$$

$$= \frac{1}{12\pi}\, e^{-\frac{13\left(\theta^2 - 2\left(\frac{74}{13}\right)\theta + \left(\frac{74}{13}\right)^2 + \frac{36}{13^2}\right)}{72}}$$

$$= \frac{1}{12\pi}\, e^{-\frac{13}{72}\left(\left(\theta - \frac{74}{13}\right)^2\right) - \frac{36}{13\times 72}}$$

$$= \frac{1}{12\pi}\, e^{-\frac{13}{72}\left(\theta - \frac{74}{13}\right)^2 - \frac{1}{26}}$$

$$= \frac{e^{-\frac{13}{72}\left(\theta - \frac{74}{13}\right)^2} \times e^{-\frac{1}{26}}}{12\pi}$$

$$f_x(6) = \int_{-\infty}^{\infty} \frac{e^{-\frac{13}{72}\left(\theta - \frac{74}{13}\right)^2}\, e^{-\frac{1}{26}}}{12\pi}\, d\theta$$

put $\frac{13}{72}\left(\theta - \frac{74}{13}\right)^2 = t^2$

or $\sqrt{\frac{13}{72}}\left(\theta - \frac{74}{13}\right) = t$

$$\sqrt{\frac{13}{72}}\, d\theta = dt$$

$$= \frac{e^{-\frac{1}{26}}}{12\pi} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{\frac{72}{13}}\, dt$$

$$= \sqrt{\frac{72}{13}}\, \frac{e^{-1/26}}{12\pi} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \sqrt{\frac{72}{13}}\ \frac{e^{-1/26}}{12\pi}\ (\sqrt{\pi})$$

$$= \sqrt{\frac{72}{13}}\ \frac{e^{-1/26}}{12\sqrt{\pi}}$$

$$\therefore f(\theta|6) = \frac{e^{-\frac{13}{72}\left(\theta - \frac{74}{13}\right)^2}\ e^{-1/26}}{\left(\sqrt{\frac{72}{13}}\ e^{-1/26}\right)\left(\frac{12\pi}{12\sqrt{\pi}}\right)}$$

$$= \frac{e^{-\frac{13}{72}\left(\theta - \frac{74}{13}\right)^2}}{\sqrt{\frac{72}{13}}\ \sqrt{\pi}}$$

$$= \frac{e^{-\frac{(\theta - 74/13)^2}{2(36/13)}}}{\sqrt{2\pi\left(\frac{36}{13}\right)}}$$

The above equation is similar to pdf of $N(\mu, \sigma^2)$

which is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

when $\mu = \frac{74}{13}$ and $\sigma^2 = \frac{36}{13}$

$\therefore$ we can say $\theta|6 \sim N\left(\frac{74}{13}, \frac{36}{13}\right)$

b) Given $n = 4$
$\bar{x} = 6$

$$a = \frac{1}{\sigma^2_{prior}} = \frac{1}{9}$$

$$b = \frac{n}{\sigma^2} = \frac{4}{4} = 1$$

$$\mu_{post} = \frac{a\mu_{prior} + b\bar{x}}{a+b} = \frac{(1/9)(5) + (1)(6)}{1/9 + 1} = \frac{59}{10} = 5.9$$

$$\sigma^2_{post} = \frac{1}{1/9 + 1} = \frac{9}{10} = 0.9$$

thus posterior on $\theta \sim N(5.9, 0.9)$

c) As ~~pos~~ more data is received variance of posterior decreases since $\sigma^2_{post} \propto \frac{1}{n}$ thus we get more precise values.

We can say μ_{post} is weighted average of μ_{prior} and sample mean. As we increase the value of 'n' the sample mean will tend towards μ_{prior} as noise added has zero mean. Thus ~~overall~~ overall μ_{post} will tend towards μ_{prior}.

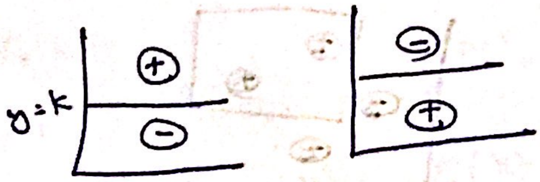
d) i) Using above formulae expected value of his true IQ $= \dfrac{\frac{100}{152} + \frac{80}{100}}{\frac{1}{152} + \frac{1}{100}} \approx$ ~~87~~ 87.93

ii) Expected value of her true IQ $= \dfrac{\frac{100}{152} + \frac{150}{100}}{\frac{1}{152} + \frac{1}{100}} \approx 130.15$

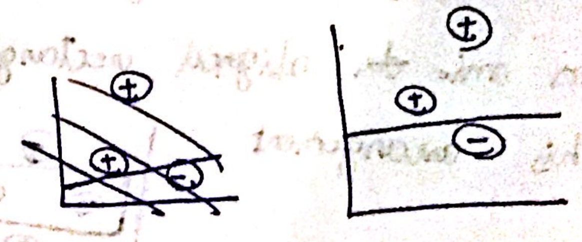
5)

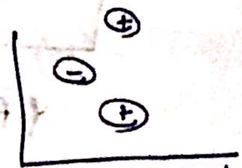
a) Constant function

Case for 2 points



Case for 3 points





(Cannot shatter)

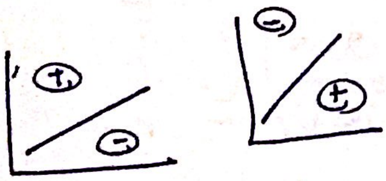
Thus its VC dim is atleast 2.

∴ VC dim = 2

b) Linear function in D dimension

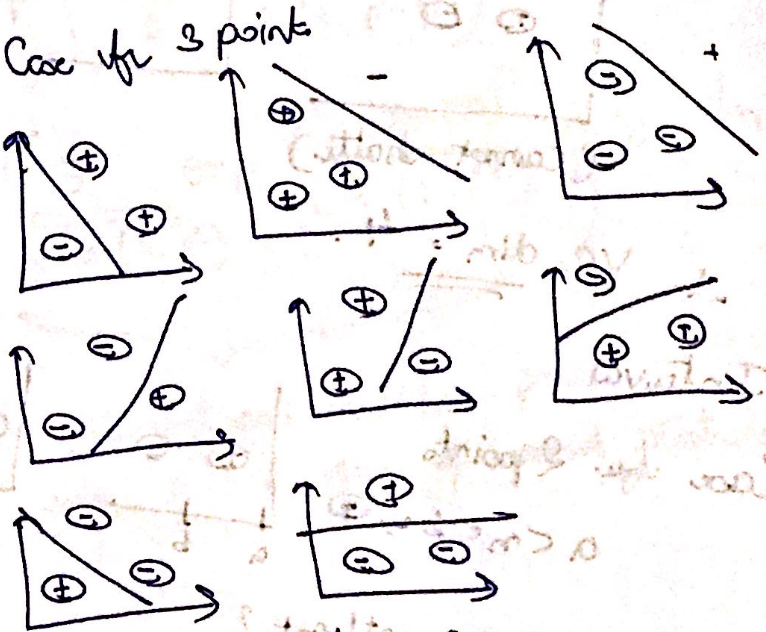
Consider A linear function in 2 dim (a line)

1) Case for 2 points



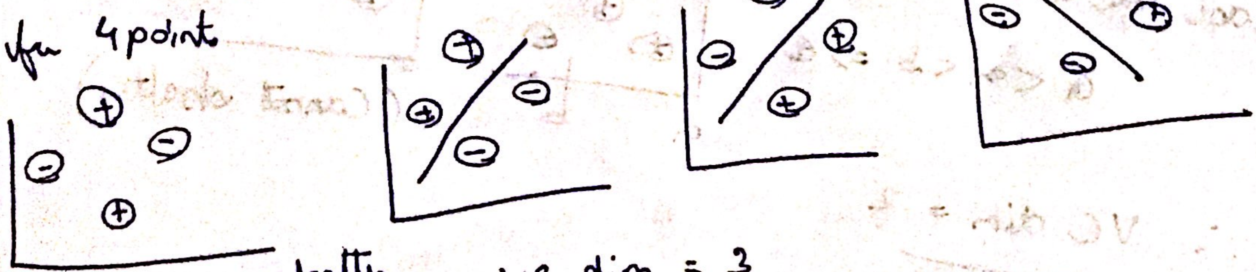
∴ VC dim is atleast 2

Case for 3 points



∴ Vc dim is atleast 3

Case for 4 points



As line cannot shatter these four point

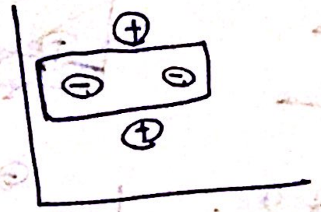
∴ VC dim = 3

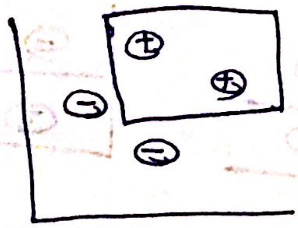
Similarly we can extend this to more dimensions

$\therefore$ VC dim of 'd' dimension linear function is $d+1$.

3) Axis aligned rectangles

An axis ~~the~~ aligned rectangle can shatter four points in this arrangement



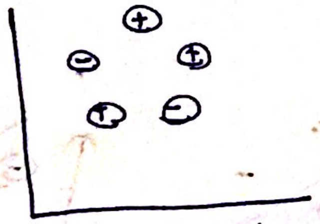


(All other combinations as well)

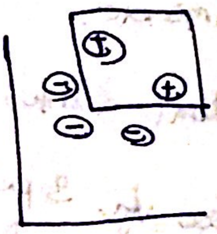
$\therefore$ VC dim is atleast 4.

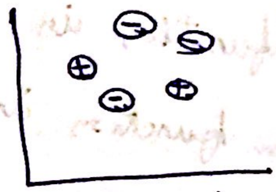
But it cannot shatter any arrangement of 5 points

Eg:



(Cannot shatter)





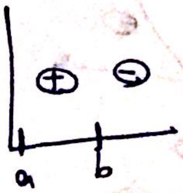
(Cannot shatter)

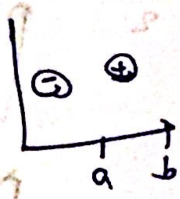
$\therefore$ VC dim = 4.

4) Interval

Case for 2 points

$a < x < b \Rightarrow +$

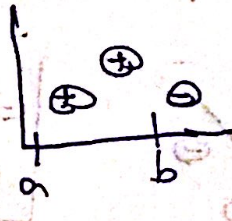


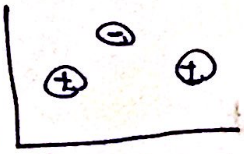


$\therefore$ VC dim is atleast 2

Case for 3 points

$a < x < b \Rightarrow +$





(Cannot shatter)

$\therefore$ VC dim = 2