① $Z = X + Y$ $\nearrow$ $x > 0$

$f(x, z) = f(x, y)$

$= e^{-\lambda+\mu} \frac{\lambda^x}{x!} \frac{\mu^{z-x}}{(z-x)!}$

$g(x,z) = e^{-(\lambda+\mu)} \binom{z}{x} \frac{\lambda^x}{z!} \mu^{z-x}$

$f_z(z) = \sum_{x=0}^{z} g(x, z)$ $x \geq 0$

$= e^{-(\lambda+\mu)} \frac{\mu^z}{z!} \sum_{0}^{z} \binom{z}{x} (\lambda/\mu)^x$

$Z = X + Y$ $y > 0$, $X > 0$

$Z - X > 0$

$\Rightarrow X < Z$ $0 < X < Z$

$= e^{-(\lambda+\mu)} \frac{\mu^z}{z!} \left(1 + \frac{\lambda}{\mu}\right)^z$

$$= e^{-(\lambda+\mu)} \frac{\mu^z}{z!} \left(1+\frac{\lambda}{\mu}\right)^z$$

$$f_z(z) = \frac{e^{-(\lambda+\mu)} (\mu+\lambda)^z}{z!}$$

$$f(x|z) = \frac{f(x,z)}{f(z)} = \frac{\cancel{e^{-(\lambda+\mu)}} \binom{z}{x} \frac{\lambda^x}{z!} \mu^{z-x}}{\frac{\cancel{e^{-(\lambda+\mu)}} (\mu+\lambda)^z}{\cancel{z!}}}$$

$$= \binom{z}{x} \left(\frac{\lambda}{\mu}\right)^x \left(\frac{\mu}{\mu+\lambda}\right)^z$$

$$\frac{\pi}{1-\pi} = \frac{\lambda}{\mu}$$

$$f(x|z=n) = \binom{n}{x} \left(\frac{\pi}{1-\pi}\right)^x (1-\pi)^n$$

$$\boxed{= \binom{n}{x} \pi^x (1-\pi)^{n-x}}$$

2/

X_n

$$p(X_n) = \begin{cases} \frac{1}{n^2} & X_n = n \\ 1-\frac{1}{n^2} & X_n = \frac{1}{n} \\ 0 & \text{else} \end{cases}$$

$$E(X_n) = \frac{1}{n} + \frac{1}{n}\left(1-\frac{1}{n^2}\right) = \frac{2}{n} - \frac{1}{n^3}$$

$$E X_n^2 = 1 + \frac{1}{n^2}\left(1-\frac{1}{n^2}\right) = 1 + \frac{1}{n^2} - \frac{1}{n^4}$$

$$Var(X_n) = 1 + O\left(\frac{1}{n^2}\right)$$

$n \to \infty \qquad Var(X_n) \to 1 \qquad E(X_n) \to 0$

let ~~us~~ X be the r.v. to which X_n converges to, ~~then~~ $EX = \mu$, $Var X = \sigma^2$

let $\mu = 0$

$E[X_n] \to 0$

$\therefore \; E X_n \to \mu$

but $Var(X_n) \not\to 0$

$\therefore$ it doesn't converge in quadratic mean

let X be $f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{else} \end{cases}$

$\mu = 0 \qquad var(X) = 0$

$$|X_n - X| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right|$$

so $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$

$\frac{1}{n} \geq \epsilon \quad = 0 \qquad B$

$\forall \epsilon > 0 \qquad \therefore$ proved convergence in prob.

3/

$$\left\{\begin{array}{ll} \cancel{f(0) = 1} & \\ \cancel{x \geq 0} & \cancel{f(x)} \\ \cancel{\text{else}} & \cancel{0} \end{array}\right.$$

$$f(x) = \left\{\begin{array}{ll} \div & x > 0 \\ 0 & \text{else} \end{array}\right.$$

$$E(X) = \int_0^\infty x f(x)\, dx$$

$$P(X > x) = 1 - F(x)$$

$$\int_0^\infty 1.(1 - F(x))\,dx = \left[x.(1 - \cancel{F(x)})\right]_0^\infty + \int_0^\infty x f(x)\,dx$$

(by parts)

$$= E(X)$$

4/

p q

μ, ϵ m, L

$$D_{KL}(p \| q) = \int_{-\infty}^{\infty} p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-\frac{(x-\mu)^2}{2\epsilon^2}} \cdot \left[\frac{-(x-\mu)^2}{2\epsilon^2} + \frac{(x-m)^2}{2L^2} + \log\frac{L}{\epsilon}\right]$$

$$= \frac{-1}{2} + \int_{-\infty}^{\infty} \frac{(x-m)^2}{2L^2} (\quad\cdot\quad)\, dx$$

$\operatorname{var} X = EX^2 - (EX)^2$

$$= \frac{-1}{2} + \int_{-\infty}^{\infty} \frac{x^2 - 2xm + m^2}{2L^2} (\quad)\, dx$$

$$= \frac{-1}{2} + \frac{1}{2L^2}(\epsilon^2 + \mu^2) - \frac{2m\mu}{L^2} + \frac{m^2}{2L^2}.$$

$$= \frac{(\epsilon^2 - L^2 + \mu^2 + m^2 - 2m\mu)}{2L^2} + \log\frac{L}{\epsilon} = \frac{\epsilon^2 - L^2 + (\mu - m)^2}{2L^2} + \log\frac{L}{\epsilon}$$

PAGE No.
DATE: / / 202

5. $\text{Corr}(X,Y) = E(XY) - E X E Y$

$$\frac{-1}{2(\;)}\; x^2 - 2xy + y^2 + 2xy(1 - \rho)$$

$$\frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left(\frac{-1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

$$|\Sigma| = 1 - \rho^2$$

assuming $\mu_x = \mu_y = 0$ & $\sigma_x = \sigma_y = 1$

$$x = \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix}$$

$$\frac{-1}{2}\begin{bmatrix} x - \mu_x & y - \mu_y \end{bmatrix}\begin{bmatrix} \sigma_x & r \\ r & \sigma_y \end{bmatrix}\begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & +r \\ +r & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} yr & xr \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2rxy$$

$$\begin{bmatrix} x + yr & y + xr \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 2xyr + y^2$$

$\therefore \rho = -r$

$r = -\rho$

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

$\therefore \text{Corr}(X,Y) = \rho$

$\mu_x = \mu_y = 0 \quad \sigma_x = \sigma_y = 1$

5/

$$M(t) = \exp\left(\frac{1}{2} t^T \Sigma t\right)$$

$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

$\cancel{\therefore M(t) = \exp\left(\frac{1}{2}\, t\right.}$

$$t^T \Sigma t = \begin{bmatrix} t_x & t_y \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$= \begin{bmatrix} t_x + \rho t_y & t_y + \rho t_x \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$\cancel{t_x^2 (1+\rho}$

$\cancel{t_x^2 + \rho t_y^2 + t_y t_x}$

$$t_x^2 + \rho t_y t_x + t_y^2 + \rho t_y t_x$$

$$t^T \Sigma t = t_x^2 + t_y^2 + 2\rho t_y t_x$$

$$\therefore M(t) = \exp\left(\frac{t_x^2 + t_y^2}{2} + \rho t_y t_x\right)$$

$$\frac{\partial M(t)}{\partial t_x} = (t_x + \rho t_y) \exp\alpha$$

$$\frac{\partial^2 M(t)}{\partial t_x^2} = \left[1 + (t_x + \rho t_y)^2\right] \exp\alpha$$

$$\frac{\partial^3 M(t)}{\partial t_x^2 \partial t_y} = \left[2\rho^2 t_y + 2\rho t_x + \left[1 + (t_x + \rho t_y)^2\right] \times (t_y + \rho t_x)\right.$$

$$\frac{\partial^3 M(t)}{\partial t_x^2 \partial t_y} = \Big[\underbrace{2\rho(\rho t_y + t_x) + (t_y + \rho t_x) + (t_x + \rho t_y)^2 (t_y + \rho t_x)}_{\beta}$$

$$\left.\frac{\partial^4 M(t)}{\partial t_x^2 \partial t_y^2}\right|_0 = (\beta' + \beta\alpha') \exp\alpha$$

PAGE No
DATE: / 202

$\alpha = 0$

$\beta = 0$ $\quad \cancel{\partial \beta^2}$

$$\beta' = \frac{\partial \beta}{\partial t_y} = 2\rho^2 + 1$$

$$\therefore \left.\frac{\partial^4 M(t)}{\partial t_x^2 \partial t_y^2}\right|_{t_x = t_y = 0} = 2\rho^2 + 1$$

$\therefore E(X^2 Y^2) = 2\rho^2 + 1$

$EX^2 = EY^2 = 1$

$Var X^2 = Var Y^2 = 2\sigma^4 \qquad \sigma = 1$

$= 2$

$$\therefore Corr(X^2, Y^2) = \frac{Cov(X^2, Y^2)}{\sqrt{Var X^2 \, Var Y^2}}$$

$$= \frac{E(X^2 Y^2) - EX^2 EY^2}{2}$$

$$= \frac{2\rho^2 + 1 - 1}{2} = \rho^2$$

$\therefore$ proved.

6/ ~~$\Gamma(n, 3)$~~ $X = \sum_{i=0}^{n} X_i$

$X_i \sim \exp(\lambda)$
$\sum X_i \sim \text{Gamma}(n, \lambda^{-1})$ } poisson process

here $\lambda^{-1} = 3$ $\lambda = \frac{1}{3}$

$X \sim \exp(\frac{1}{3})$ by CLT as $n \to \infty$

CLT

$$\frac{\sum X_i - n\mu}{\sqrt{n\sigma^2}} \sim \text{Normal}(0,1)$$

~~$\mu = \lambda$ $\sigma^2 = \lambda^2$~~

~~$z = \frac{X_i - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{X_i - \lambda}{\frac{\lambda}{\sqrt{n}}}$~~

$\therefore X \sim N(n\lambda, \sqrt{n\lambda^2})$
$X \sim N\left(\frac{n}{3}, \sqrt{\frac{n}{9}}\right)$

it's like A lot of samples being taken from the exp dist and their sum

& it becomes normal by CLT.