Assignment 1

1) $X \leftarrow P_x(m) = \dfrac{e^{-\lambda}\lambda^m}{m!}$

$Y \leftarrow P_y(y) = \dfrac{e^{-\mu}\mu^y}{y!}$

Distribution of X given that $X+Y=n$ can be shown by

$P(X=m \mid X+Y=n)$

or $P(X=m \mid Y=n-X) = \dfrac{P(X,Y)}{P(X+Y=n)}$ — ③

$$P(X+Y=n) = \sum_{m=0}^{n} P_x(m)\,P_y(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left(\frac{e^{-\mu}\mu^{n-m}}{(n-m)!}\right)$$

$$= e^{-(\lambda+\mu)} \sum_{m=0}^{n} \left(\frac{1}{n!}\right)\left(\frac{n!}{m!\,(n-m)!}\right)\lambda^m \mu^{n-m}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \sum_{m=0}^{n} {}^nC_m\,\lambda^m \mu^{n-m}$$

$$= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!} \quad \text{(Binomial expansion)} \quad — ①$$

$P(X,Y) = P_x(X=m)\,P_y(Y=y)$ [Both are independant random variables]

$$= P_x(X=m)\,P_y(Y=n-m)$$

$$= \left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left(\frac{e^{-\mu}\mu^{n-m}}{(n-m)!}\right)$$

$$= \frac{e^{-(\lambda+\mu)}}{n!}\left(\frac{n!}{m!\,(n-m)!}\right)\lambda^m\mu^{n-m} = \frac{e^{-(\lambda+\mu)}}{n!}\binom{n}{m}\lambda^m\mu^{n-m} \quad — ②$$

Using 1, 2 & 3

$$P(X=m \mid Y=n-X) = \frac{\dfrac{\cancel{e^{-(\lambda+\mu)}}}{\cancel{n!}} \binom{n}{m} \lambda^m \mu^{n-m}}{\dfrac{\cancel{e^{-(\lambda+\mu)}} (\lambda+\mu)^n}{\cancel{n!}}}$$

$$= \binom{n}{m} \frac{\lambda^m \mu^{n-m}}{(\lambda+\mu)^n} = \binom{n}{m} \frac{\lambda^m \mu^{n-m}}{(\lambda+\mu)^m (\lambda+\mu)^{n-m}}$$

$$= \binom{n}{m} \left(\frac{\lambda}{\lambda+\mu}\right)^m \left(\frac{\mu}{\mu+\lambda}\right)^{n-m}$$

$$= \binom{n}{m} \pi^m (1-\pi)^{n-m}, \quad \pi = \frac{\lambda}{\lambda+\mu}$$

Q3) given: $P(X>0)=1$

↳ This implies $\int_{-\infty}^{0} f_X(x)\,dx = 0$

$(\because P(X\leq 0)=0)$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{0}^{\infty} x f_X(x)\,dx$$

$$= x\int_{0}^{x} f_X(x)\,dx - \int\left(\int f_X(x)\,dx\right)dx \Big|_0^\infty$$

$$= x F_X(x) - \int F_X(x)\,dx$$

$$= x(1-P(X>x)) - \int (1-P(X>x))\,dx \Big|_0^\infty$$

$$= x - xP(X>x) - x + \int P(X>x)\,dx \Big|_0^\infty$$

$$= -xP(X>x) + \int P(X>x)\,dx \Big|_0^\infty$$

$$= \int_0^\infty P(X>x)\,dx - xP(X>x)\Big|_0^\infty$$

$$= \int_0^\infty P(X>x)\,dx - x(1-F_X(x))\Big|_0^\infty$$

since $\lim_{x\to\infty} F_X(x) = 1$

$\therefore\ x(1-F_X(x))\Big|_0^\infty = 0$

$$= \int_0^\infty P(X>x)\,dx$$, hence proved

Q4) $p(x) = \mathcal{N}(x|\mu, E)$

$q(x) = \mathcal{N}(x|m, L)$

$$D_{KL}(p(x)||q(x)) = \int_{-\infty}^{\infty} p(x) \ln\left(\frac{p(x)}{q(x)}\right) dx$$

$$= \int_{-\infty}^{\infty} p(x) \ln\left(\sqrt{\frac{L}{E}}\, e^{\left\{-\frac{(x-\mu)^2}{2E} + \frac{(x-m)^2}{2L}\right\}}\right) \cdot dx$$

$$= \int_{-\infty}^{\infty} p(x) \ln\left(\sqrt{\frac{L}{E}}\right) - \frac{(x-\mu)^2}{2E} p(x) + \frac{(x-m)^2}{2L} p(x)\, dx$$

$$= \ln\left(\sqrt{\frac{L}{E}}\right) \underbrace{\int_{-\infty}^{\infty} p(x) dx}_{1} - \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{2E} p(x) dx + \int_{-\infty}^{\infty} \frac{(x-m)^2}{2L} p(x) dx$$

$$= \frac{1}{2} \ln\left(\frac{L}{E}\right) - \frac{E_p((x-\mu)^2)}{2E} + \frac{E_p((x-m)^2)}{2L}$$

$$= \frac{1}{2} \ln\left(\frac{L}{E}\right) - \frac{1}{2} + \frac{E_p}{2L}(x^2 - 2xm + m^2)$$

$(E = E_p((x-\mu)^2))$

$E_p(x^2) = E + \mu^2$

$E_p(x) = \mu$

$$= \frac{1}{2} \ln\left(\frac{L}{E}\right) - \frac{1}{2} + \frac{1}{2L}(E + \mu^2 - 2m\mu + m^2)$$

$$= \frac{1}{2} \ln\left(\frac{L}{E}\right) - \frac{1}{2} + \frac{1}{2L}(E + (\mu - m)^2)$$

5) $f(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$$

$\sigma_x, \sigma_y = 1$

$\mu_x, \mu_y = 0$

$\therefore \mathrm{Corr}(X,Y) = \mathrm{Cov}(X,Y) = E(XY)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) dx\, dy$$

$$x^2 - 2\rho xy + y^2 = x^2 - \rho^2 x^2 + (y^2 - 2\rho xy + \rho^2 x^2)$$
$$= x^2(1-\rho^2) + (y-\rho x)^2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{xy}{2\pi(1-\rho^2)^{1/2}} \exp\left(-\frac{x^2}{2} - \frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{x e^{-x^2/2}}{\sqrt{2\pi}}\right) \underbrace{\left(\frac{y}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)\right) dy}_{} \, dx$$

This integral will be ρx since it is of form of normal distribution.

$$= \rho \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$= \rho E(X^2)$; $\because E(X^2) = \mathrm{Var}(X) + E(X)^2$

$= 1 + 0$

$= 1$

$= \rho$ $\therefore$ $\underline{\underline{\mathrm{Corr}(X,Y) = \rho}}$

$$\mathrm{Corr}(X^2,Y^2) = \mathrm{Cov}(X^2,Y^2) / \sigma_{X^2}\sigma_{Y^2}$$

$$= \left(E(X^2Y^2) - 1\right) / \sigma_{X^2}\sigma_{Y^2}$$

$$E(x^2y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}}\right) \underbrace{\left(\frac{y^2}{\sqrt{2\pi(1-\rho^2)}} e^{-\left(\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)}\right)}_{\begin{aligned} E(Y^2) &= \mathrm{Var}(Y) + E(Y)^2 \\ &= (1-\rho^2) + \rho^2 x^2 \end{aligned}} dy\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}}\right) (1-\rho^2+\rho^2x^2)\, dx$$

$$= E(X^2) - \rho^2 E(X^2) + \rho^2 E(X^4)$$

$$= 1 - \rho^2 + \rho^2 E(X^4)$$

$$\sigma_{X^2} = \sqrt{E(X^4) - E(X^2)}$$

$$\sigma_{Y^2} = \sqrt{E(Y^4) - E(Y^2)}$$

also $E(X^2) = E(Y^2) = 1$

$E(X^4) = E(Y^4)$

$$\Rightarrow \sigma_{X^2}\sigma_{Y^2} = E(X^4) - 1$$

$$\therefore \mathrm{Corr}(X^2,Y^2) = \frac{E(X^2Y^2 - 1)}{\sigma_{X^2}\sigma_{Y^2}}$$

$$= \frac{\rho^2(E(X^4)-1)}{E(X^4)-1}$$

$$= \rho^2$$

$$\therefore \underline{\underline{\mathrm{Corr}(X^2,Y^2) = \rho^2}}$$