DSG Statistics and Probablity Assignment

1. $X \leftarrow P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $\quad Y \leftarrow P(y) = \frac{e^{-\mu}\mu^y}{y!}$

X and Y are independent

$$P(X=x \mid X+Y=n) = \frac{P(X=x) \cap P(X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=x)\,P(Y=n-x)}{P(X+Y)=n}$$

$$= \frac{\frac{\cancel{e^{-\lambda}}\lambda^x}{x!}\;\frac{\cancel{e^{-\mu}}\mu^{n-x}}{(n-x)!}}{\cancel{e^{-(\lambda+\mu)}}\sum_{i=0}^{n}\frac{\lambda^i}{i!}\frac{\mu^{n-i}}{(n-i)!}}$$

$$= \frac{\frac{n!}{x!(n-x)!}\lambda^x\mu^{n-x}}{\sum_{i=0}^{n}\frac{n!}{i(n-i)!}\mu^{n-i}\lambda^i}$$

$$= \frac{\binom{n}{x}\lambda^x\mu^{n-x}}{\sum_{i=0}^{n}\binom{n}{i}\mu^{n-i}\lambda^i} = \frac{\binom{n}{x}\lambda^x\mu^{n-x}}{(\lambda+\mu)^n}$$

$$= \binom{n}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(1-\frac{\lambda}{\lambda+\mu}\right)^{n-x}$$

$$\Rightarrow P(X=x \mid X+Y=n) = \binom{n}{x}\pi^x(1-\pi)^{n-x}$$

$$\left(\pi = \frac{\lambda}{\lambda+\mu}\right)$$

(2) $P\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2}$ $P(X_n = n) = \frac{1}{n}$

If it converges in probablity,

$\lim_{n\to\infty} P(|X_n - X| \geq e) = 0$ (for $e>0$)

for any ϵ

Let us claim that X is a zero random variable

as $n \to \infty$

$\lim_{n\to\infty} \frac{1}{n} \to 0^+$

So, $\lim_{n\to\infty} P(|X_n| \geq \epsilon) = \lim_{n\to\infty} P(X_n = n)$

(for any $\epsilon > 0$)

$= 0$

$\therefore$ X_n converges to X in probablity

i.e. $X_n \xrightarrow{P} X$

$\lim_{n\to\infty} E\left(|X_n - X|^2\right) = \lim_{n\to\infty}\left(E(X_n^2) - 2E(XX_n) + E(X^2)\right)$

X is independent of X_n

$= \lim_{n\to\infty}\left(E(X_n^2) - 2E(X)\,E(X_n) + E(X^2)\right)$

$E(X_n) = \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n} = \frac{2}{n} - \frac{1}{n^3}$

$E(X_n^2) = \frac{1}{n^2} - \frac{1}{n^4} + 1$

$$\lim_{n\to\infty} E(|X_n - X|^2) = \lim_{n\to\infty}\left(\frac{2}{n} - \frac{1}{n^3} - 2E(X)\left(\frac{1}{n^2} - \frac{1}{n^4} + 1\right) + E(X^2)\right)$$

$$= E(X^2) - 2E(X)$$

For sequence to converge in quadratic mean
$E(X^2) = 2E(X)$

As already proven that it converges in probability

$$E(X) = \lim_{n\to\infty} E(X_n) = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n^3} + \frac{1}{n}\right) = 0$$

$$E(X^2) = \lim_{n\to\infty} E(X_n) = \lim_{n\to\infty}\left(\frac{1}{n^2} - \frac{1}{n^4} + 1\right) = 1$$

$$E(X^2) \neq 2E(X)$$

$\therefore$ It does not converge in quadratic mean

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(3) $P(X > 0) = 1$, $P(X \le 0) = 0$

$$E(X) = \int_{-\infty}^{\infty} x P(X=x)\, dx = \int_{-\infty}^{0} x P(X=x)\, dx + \int_{0}^{\infty} x P(X=x)\, dx$$

$$= x\int_{0}^{\infty} P(X=x)\, dx - \int\left(\int_{0} P(X=x)\, dx\right) dx$$

$$= \int_{0}^{\infty} dx - \int_{0}^{\infty} F(x)\, dx$$

$$= \int_{0}^{\infty} (1 - F(x))\, dx = \int_{0}^{\infty} P(X > x)\, dx$$

(4) $D_{KL}(P \| Q) = E_{x \sim p}\left[\log\left(\frac{p(x)}{q(x)}\right)\right]$

$p(x) = \frac{1}{\sqrt{2\pi E}} e^{-\frac{(x-\mu)^2}{2E}}$ $\quad q(x) = \frac{1}{\sqrt{2\pi L}} e^{-\frac{(x-m)^2}{2L}}$

$\frac{p(x)}{q(x)} = \sqrt{\frac{L}{E}}\, e^{\frac{(x-m)^2}{2L} - \frac{(x-\mu)^2}{2E}}$

$\log\left(\frac{p(x)}{q(x)}\right) = \ln\sqrt{\frac{L}{E}} + \frac{(x-m)^2}{2L} - \frac{(x-\mu)^2}{2E}$

$D_{KL}(P \| Q) = E_{x \sim p}\left(\ln\sqrt{\frac{L}{E}} + \frac{(x-m)^2}{2L} - \frac{(x-\mu)^2}{2E}\right)$

$= \ln\sqrt{\frac{L}{E}} + \frac{1}{2L} E_{x \sim p}((x-m)^2) - \frac{1}{2E} E_{x\sim p}\left((x-\mu)^2\right)$

$E_{x \sim p}((x-\mu)^2) = E$, $E_{x \sim p}(x^2) = E + \mu^2$, $E_{x \sim p}(x) = \mu$

$D_{KL}(P \| Q) = \ln\sqrt{\frac{L}{E}} + \frac{1}{2L}\left(E + \mu^2 - 2m\mu + m^2\right) - \frac{E}{2E}$

$\Rightarrow \boxed{D_{KL}(P \| Q) = \ln\sqrt{\frac{L}{E}} + \frac{E + (\mu - m)^2}{2L} - \frac{1}{2}}$

Ans.

Similarly

$D_{KL}(Q \| P) = \ln\sqrt{\frac{E}{L}} + E_{x \sim Q}\left(\frac{(x-\mu)^2}{2E}\right) - E_{x \sim Q}\left(\frac{(x-m)^2}{2L}\right)$

$\Rightarrow \boxed{D_{KL}(Q \| P) = \ln\sqrt{\frac{E}{L}} + \frac{L + (m-\mu)^2}{2E} - \frac{1}{2}}$

(5)

$$f(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}$$

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x}{(2\pi(1-\rho^2)^{1/2})} e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}} \, dy \, dx$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} e^{-\frac{((y-\rho x)^2+x^2(1-\rho^2))}{2(1-\rho^2)}} \, dy \, dx$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} x e^{-x^2/2} \left(\int_{-\infty}^{\infty} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy \right) dx$$

$$t = \frac{y-\rho x}{\sqrt{1-\rho^2}}$$

$$dt = \frac{dy}{\sqrt{1-\rho^2}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x e^{-x^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x e^{-\frac{(x^2+t^2)}{2}} \, dt \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{0}^{\infty} 2x e^{-\frac{(x^2+t^2)}{2}} \, dt \, dx$$

Camlin

$x = r\cos\theta$
$t = r\sin\theta$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

$E(x) =$

$$E(x) = \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} x\, e^{-\frac{(x^2+t^2)}{2}}\, dx\right) dt$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} \cancel{x\, e^{-\frac{(x^2+t^2)}{2}}} + \cancel{(-x)\, e^{-\frac{(x^2+t^2)}{2}}}\, dt$$

$E(x) = 0$.

By symmetery $E(y) = 0$

~~Similarly~~

$$E(x^2) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}\, dy\, dx$$

$t = \frac{y-\rho x}{\sqrt{1-\rho^2}}$ | $dt = \frac{dy}{\sqrt{1-\rho^2}}$

$$E(x^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\, e^{-\frac{(x^2+t^2)}{2}}\, dt\, dx = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} x^2\, e^{-\frac{(x^2+t^2)}{2}}\, dt\, dx$$

$x = r\cos\theta \quad t = r\sin\theta$

$$E(x^2) = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\infty} r^2\cos^2\theta\, e^{-r^2/2}\, r\, dr\, d\theta$$

Camlin

$$E(x^2) = \frac{2}{\pi}\int_0^\infty r^3 e^{-r^2/2}\left(\int_0^{\pi/2}\cos^2\theta\, d\theta\right) dr$$

$$= \frac{1}{2}\int_0^\infty r^3 e^{-r^2/2}\, dr = \int_0^\infty \left(\frac{r^2}{2}\right) e^{-r^2/2}\,(r\,dr)$$

$$\frac{r^2}{2} = k$$

$$,\quad dk = r\,dr$$

$$E(x^2) = \int_0^\infty k e^{-k}\, dk = \Gamma(2) = 1$$

By symmetery:

$$E(y^2) = E(x^2) = 1$$

$$\sigma_x = \sigma_y = \sqrt{E(x^2) - [E(x)]^2} = 1$$

$$E(XY) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}\, dy\, dx$$

$$t = \frac{y-\rho x}{\sqrt{1-\rho^2}}$$

$$y = t\sqrt{1-\rho^2} + \rho x$$

$$E(XY) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\left(t\sqrt{1-\rho^2}+\rho x\right) e^{-x^2/2}\, e^{-t^2/2}\, dt\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}\left(x\sqrt{1-\rho^2}\left(\cancel{\int_{-\infty}^{\infty} t e^{-t^2/2}\, dt}^{\,0}\right) + (\rho x^2)\left(\int_{-\infty}^{\infty} e^{-t^2/2}\, dt\right)\right) dx$$

$$E(XY) = \frac{\rho}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 e^{-\frac{(x^2+t^2)}{2}}\,dx\,dt$$

$$= \frac{\rho}{2\pi}(4)\int_0^{\infty}\int_0^{\infty} x^2 e^{-\frac{(x^2+t^2)}{2}}\,dx\,dt$$

$x = r\cos\theta \qquad y = r\sin\theta$

$$= \frac{2\rho}{\pi}\int_0^{\pi/2}\int_0^{\infty} r^2\cos^2\theta\, e^{-r^2/2}\, r\,dr\,d\theta$$

$$= \frac{2\rho}{\pi}\int_0^{\infty} r^3 e^{-r^2/2}\left(\int_0^{\pi/2}\cos^2\theta\,d\theta\right)dr$$

$$= \frac{\rho}{2}\int_0^{\infty} r^2 e^{-r^2/2}\, r\,dr$$

$$k = \frac{r^2}{2} \qquad dk = r\,dr$$

$$= \rho\int_0^{\infty} k e^{-k}\,dk = \rho\,\Gamma(2) = \rho$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \rho$$

$$\boxed{Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{\rho}{(1)(1)} = \rho}$$

Hence Proved

$$\sigma_{X^2} = \sqrt{E(X^4) - (E(X^2))^2}$$

Camlin

$E(x^2) = E(y^2) = 1$ (Already found)

$$E(x^4) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^4 e^{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}} \, dy\, dx$$

$$t = \frac{y - \rho x}{\sqrt{1-\rho^2}}$$

$$E(x^4) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^4 e^{-x^2/2} e^{-t^2/2} \, dx\, dt$$

$$\Rightarrow E(x^4) = \frac{2}{\pi} \int_{0}^{\infty}\int_{0}^{\infty} x^4 e^{-\frac{(x^2+t^2)}{2}} \, dx\, dt$$

$x = r\cos\theta \qquad y = r\sin\theta$

$$\Rightarrow E(x^4) = \frac{2}{\pi} \int_{0}^{\pi/2}\int_{0}^{\infty} r^4 \cos^4\theta\, e^{-r^2/2}\, r\, dr\, d\theta = \frac{2}{\pi} \int_{0}^{\infty} r^5 e^{-r^2/2} \int_{0}^{\pi/2} \cos^4\theta\, d\theta\, dr$$

$$\Rightarrow E(x^4) = \frac{2}{\pi} \cdot \frac{3\pi}{16} \int_{0}^{\infty} r^4 e^{-r^2/2}\, r\, dr$$

$$k = \frac{r^2}{2} \Rightarrow dk = r\, dr$$

$$\Rightarrow E(x^4) = \frac{3}{2} \int_{0}^{\infty} k^2 e^{-k}\, dk = \frac{3}{2}\Gamma(3) = 3$$

By Symmetery $E(y^4) = E(x^4) = 3$

$$\sigma_{x^2} = \sigma_{y^2} = \sqrt{E(x^4) - [E(x^2)]^2} = \sqrt{2}$$

$$E(x^2 y^2) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y^2 e^{-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}} \, dx\, dy$$

Camlin

$$t = \frac{y - \rho x}{\sqrt{1-\rho^2}}$$

$$y = t\sqrt{1-\rho^2} + \rho x$$

$$y^2 = t^2(1-\rho^2) + \rho^2 x^2 + 2t\sqrt{1-\rho^2}\,\rho x$$

$$E(x^2y^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\left(t^2(1-\rho^2) + 2t\sqrt{1-\rho^2}\,\rho x + \rho^2x^2\right) e^{-x^2/2} e^{-t^2/2}\, dt\, dx$$

$$\Rightarrow E(x^2y^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2}\left[\left(x^2(1-\rho^2)\int_{-\infty}^{\infty} t^2 e^{-t^2/2}\,dt\right) + \left(2\rho\sqrt{1-\rho^2}\,x^3\right)\underbrace{\cancel{\int_{-\infty}^{\infty} t e^{-t^2/2}\,dt}}_{0} + \left(\rho^2 x^4\int_{-\infty}^{\infty} e^{-t^2/2}\,dt\right)\right] dx$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\rho^2)\,x^2t^2 e^{-\frac{(x^2+t^2)}{2}}\,dx\,dt + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho^2 x^4 e^{-\frac{(x^2+t^2)}{2}}\,dx\,dt\right]$$

$x = r\cos\theta \qquad t = r\sin\theta$

$$= \frac{2}{2\pi}(1-\rho^2)\left(\int_0^{\infty}\int_0^{\pi/2} r^4\cos^2\theta\sin^2\theta\, e^{-r^2/2}\, r\,dr\,d\theta\right.$$

$$\left. + \frac{2}{\pi}\rho^2\int_0^{\infty}\int_0^{\pi/2} r^4\cos^4\theta\, e^{-r^2/2}\,d\theta\,(r\,dr)\right)$$

$$= \frac{2}{\pi}(1-\rho^2)\int_0^{\infty} r^5 e^{-r^2/2}\left(\int_0^{\pi/2}(\cos^2\theta - \cos^4\theta)\,d\theta\right)dr$$

$$+ \frac{2\rho^2}{\pi}\int_0^{\infty} r^5 e^{-r^2/2}\left(\int_0^{\pi/2}\cos^4\theta\,d\theta\right)dr$$

$$\int_0^{\pi/2} (\cos^2\theta - \cos^4\theta)\, d\theta = \frac{\pi}{4} - \frac{3\pi}{16} = \frac{\pi}{16}$$

$$E(x^2y^2) = \frac{1}{8}(1-\rho^2)\int_0^\infty r^4 e^{-r^2/2}\, r\, dr$$

$$+ \frac{3}{8}\rho^2 \int_0^\infty r^4 e^{-r^2/2}\, r\, dr$$

$$k = \frac{r^2}{2} \rightarrow dk = r\, dr$$

$$E(x^2y^2) = \left(\frac{1-\rho^2}{2}\right)\int_0^\infty k^2 e^{-k}\, dk + \frac{3\rho^2}{2}\int_0^\infty k^2 e^{-k}\, dk$$

$$= \frac{1-\rho^2+3\rho^2}{2}\left(\Gamma(3)\right)$$

$$= 2\rho^2 + 1$$

$$Cov(x^2, y^2) = E(x^2y^2) - E(x^2)E(y^2) = 2\rho^2 + 1 - 1 = 2\rho^2$$

$$Corr(x^2, y^2) = \frac{Cov(x^2, y^2)}{\sigma_{x^2}\sigma_{y^2}} = \frac{2\rho^2}{(\sqrt{2})^2} = \rho^2$$

$$\Rightarrow \boxed{Corr(x^2, y^2) = \rho^2}$$

(6) $X \sim \text{Gamma}(n, 3)$

$$f(x) = \begin{cases} \frac{1}{\Gamma n \; 3^n} x^{n-1} e^{-x/3} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Gamma $(n, 3)$ distribution is sum of n i.i.d. ~~Ex~~ Exp(3) random variables

$X = Y_1 + Y_2 + \cdots Y_n$

$Y_i \sim \text{Exp}(3)$

$$f(Y_i = y) = \begin{cases} \frac{1}{3} e^{-x/3} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$E(Y_i) = 3 =$ ~~μ~~

$E(\bar{Y}) = E(Y_i) = 3 = \mu_y$

$\sigma_y^2 = 9 \Rightarrow \sigma_y = 3$

By Central Limit Theorem,

$\frac{\bar{Y} - \mu_y}{\sigma_y / \sqrt{n}}$ approximates normal distribution ~~with~~ for large n

$\Rightarrow \frac{\sum Y_i - n\mu_y}{\sqrt{n}\,\sigma_y}$ " " " "

$\Rightarrow \left(\frac{X - 3n}{3\sqrt{n}}\right)$ " " " "

$\therefore \left(\frac{X - 3n}{3\sqrt{n}}\right) \sim N(0, 1)$ for large n by CLT