Statistics and Probability Assignment

① $X \leftarrow P(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$Y \leftarrow P(y) = \dfrac{e^{-\mu}\mu^{y}}{y!}$

X and Y are independent.

$$P\{X \mid X+Y=n\} = \frac{P\{X, X+Y=n\}}{P\{X+Y=n\}}$$

$$\Rightarrow \frac{P\{X, X+Y=n\}}{P\{X+Y=n\}} = \frac{P\{X=x\}\cdot P\{Y=n-x\}}{P\{X+Y=n\}}$$

Now, let $Z = X+Y$,

Then, $M_Z(t) = E(e^{tZ}) = E(e^{tX}\cdot e^{tY})$

$= E(e^{tX}) \cdot E(e^{tY})$

$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)} \rightarrow \text{Poisson}(\lambda+\mu)$

$$\Rightarrow P\{X+Y\} \text{ or } P(Z) = \frac{e^{-(\lambda+\mu)}\cdot(\lambda+\mu)^{z}}{z!}$$

$$= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^{n}}{}$$

$$n!$$

$$\Rightarrow \frac{P\{X, X+Y=n\}}{P\{X+Y=n\}} = \frac{P\{X=x\} \cdot P\{Y=n-x\}}{P\{X+Y=n\}}$$

$$= \frac{\frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \cdot \frac{e^{-\mu} \mu^{n-x}}{(n-x)!}}{\frac{e^{-\lambda} \cdot e^{-\mu} \cdot (\lambda+\mu)^{n}}{n!}}$$

$$= \frac{n!}{x!(n-x)!} \cdot \frac{\lambda^{x} \cdot \mu^{n-x}}{(\lambda+\mu)^{n}}$$

$$= {}^{n}C_{x} \cdot \frac{\mu^{n-x} \quad \lambda^{x}}{(\lambda+\mu)^{n-x} \; (\lambda+\mu)^{x}}$$

$$= {}^{n}C_{x} \left(\frac{\lambda}{\lambda+\mu}\right)^{x} \left(\frac{\mu}{\lambda+\mu}\right)^{n-x}$$

$\Rightarrow$ let $\pi = \dfrac{\lambda}{\lambda+\mu}$

Then, ${}^{n}C_{x} \cdot \pi^{x} \cdot (1-\pi)^{n-x}$ — hence proved

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(2) $X_1, X_2 \ldots\ldots$

$$P\{X_n = \frac{1}{n}\} = 1 - \frac{1}{n^2}$$

$$P\{X_n = n\} = \frac{1}{n^2}$$

convergence in probability,

$$\lim_{n \to \infty} P\{|X_n - X| \geq \varepsilon\} = 0$$

aware -

or $X_n \xrightarrow{p} X$

or $\lim_{n \to \infty} P\{|X_n - X| < \varepsilon\} = 1$

n is a positive integer, then we can clearly see that as $n \to \infty$,

$$P\{X_n = \frac{1}{n}\} \to 1$$

and $P\{X_n = n\} \to 0$

In other words,

$\lim_{n \to \infty} P\{|X_n - \frac{1}{n}| < \varepsilon\} = 1$ holds

$$\therefore \boxed{X_n \xrightarrow{p} \frac{1}{n} = 0}$$

X_n converges in quadratic mean to X if

$E(X_n - X)^2 \to 0$ as $n \to \infty$

or $X_n \xrightarrow{qm} X$

here, $E(X_n - 0)^2 = E(X_n^2)$

where,

$$E(X_n^2) = 1 + \frac{1}{n^2} - \frac{1}{n^4} \to 1 \text{ as } n \to \infty$$

Clearly $E(X_n - 0)^2 \to 0$ doesn't hold

then

$\therefore X_n$ doesn't converge in q.m.

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(3) $P\{X \geq 0\} = 1$

$E(X)$ exists

→ cdf F.

→ pdf f.

Show that

$$E(X) = \int_0^\infty P(X > x)\,dx$$

Then,

$$P(X \leq x) = F(x)$$

and $$F(x) = \int_0^x f(x)\,dx$$

$$\Rightarrow P(X \leq x) = \int_0^x f(x)\,dx$$

$$\Rightarrow P(X > x) = \int_x^\infty f(x)\,dx$$

$$\Rightarrow \int_0^\infty P(X > x)\,dx = \int_0^\infty \left[\int_x^\infty f(x)\,dx\right] dx$$

or $\quad \xrightarrow{} 0 < x < t < \infty$

$$\Rightarrow \int_0^\infty P(X>x)\,dx = \int_0^\infty \left[\int_x^\infty f(t)\,dt\right] dx$$

or $\quad 0 < t < \infty \quad$ and $\quad 0 < x < t$

$$\Rightarrow \int_0^\infty P(X>x)\,dx = \int_0^\infty \left[\int_0^t 1\cdot dx\right] f(t)\,dt$$

$$\Rightarrow \int_0^\infty P(X>x)\,dx = \int_0^\infty t\cdot f(t)\,dt$$

and $\quad E(X) = \int_{-\infty}^\infty t\cdot f(t)\,dt$

But $\quad P(X>x) = 1 \Rightarrow P(X\leq x) = 0$

$$\therefore E(X) = \int_0^\infty t\cdot f(t)\cdot dt$$

$$\therefore \int_0^\infty P(X>x)\,dx = E(X) \quad \text{— proved.}$$

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(4) KL divergence b/w 2 gaussians

$p(x) = \mathcal{N}(x|\mu, \Sigma)$ and $q(x) = \mathcal{N}(x|m, L)$.

$$D(q(\cdot)\,||\,p(\cdot)) = E_{y\sim q(y)}\left[\log \frac{q(y)}{p(y)}\right]$$

$$p(x:\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right]$$

$$q(x : m, L) = \frac{1}{(2\pi)^{n/2}|L|^{1/2}} \exp\left[-\frac{1}{2}(x-m)^T L^{-1}(x-m)\right]$$

$$KL = \int q(x) \cdot \ln \frac{q(x)}{p(x)}\, dx$$

$$= E_{q(x)}\left[\ln \frac{q(x)}{p(x)}\right]$$

$$\downarrow$$

$$\ln \frac{\frac{1}{\sqrt{\cancel{(2\pi)^n}\,|E|}} \cdot \exp\left[-\frac{1}{2}(x-M)^T E^{-1}(x-M)\right]}{\frac{1}{\sqrt{\cancel{(2\pi)^n}\,|L|}} \exp\left[-\frac{1}{2}(x-m)^T L^{-1}(x-m)\right]}$$

$$= E\left[\frac{1}{2}\ln\left|\frac{L}{E}\right| - \frac{1}{2}(x-M)^T E^{-1}(x-M) + \frac{1}{2}(x-m)^T L^{-1}(x-m)\right]$$

for scalar a, $a = tr(a)$

also $tr(ABC) = tr(BCA)$

so,

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}E\left(tr\left(E^{-1}(x-M)(x-M)^{-1}\right)\right. + \frac{1}{2}E\left(tr\left(L^{-1}(x-m)(x-m)^{-1}\right)\right)$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}tr\left(E^{-1}E\left[(x-M)(x-M)^T\right]\right)$$

$$+\frac{1}{2}\operatorname{tr}\left(L^{-1}E\left[(x-m)(x-m)^T\right]\right.$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}\operatorname{tr}\left(E^{-1}E\left[(x-M)(x-M)^T\right]\right)$$

$$+\frac{1}{2}\operatorname{tr}\left(L^{-1}E\left(xx^T - 2mx^T + mm^T\right)\right)$$

$$E(ax) = aM$$

$$E(x^T a x) = M^T a M + \operatorname{tr}(aE)$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}\operatorname{tr}(E^{-1}E)$$

$$+\frac{1}{2}\operatorname{tr}\left(L^{-1}\left(E + MM^T - 2mM^T + mM^T\right)\right)$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}(n) + \frac{1}{2}\operatorname{tr}(L^{-1}E)$$

$$+\frac{1}{2}\operatorname{tr}\left(L^{-1}\left(MM^T - 2mM^T + mm^T\right)\right)$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{1}{2}(n) + \frac{1}{2}\operatorname{tr}(L^{-1}E)$$

$$+\frac{1}{2}\operatorname{tr}\left(M^T L^{-1} M - 2M^T L^{-1} m + m^T L^{-1} m\right)$$

$$= \frac{1}{2}\ln\frac{|L|}{|E|} - \frac{n}{2} + \frac{\operatorname{tr}(L^{-1}E)}{2}$$

$$+\frac{1}{2}(m-M)^T L^{-1}(m-M)$$

$$= \frac{1}{2}\left[\ln\frac{|L|}{|E|} - n + \operatorname{tr}(L^{-1}E) + (m-M)^T L^{-1}(m-M)\right]$$

$$= \ldots$$

(5) $(X, Y) \leftarrow$ bivariate normal distribution

$$f(X,Y) = \frac{1}{2\pi(1-e^2)^{1/2}} e^{\left[\frac{-1}{2(1-e^2)}(x^2 - 2exy + y^2)\right]}$$

Show that $\text{corr}(X,Y) = e$ and

$\text{corr}(X^2, Y^2) = e^2$

$$\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} \quad E(XY) - E(X)E(Y)$$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \quad \text{—— (i)}$$

we notice that

$$f(-X,-Y) = \frac{1}{2\pi(1-e^2)^{1/2}} e^{\left[\frac{-1}{2(1-e^2)}(x^2 - 2exy + y^2)\right]}$$

$\therefore f(-X,-Y) = f(X,Y) \rightarrow$ even function

and $E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$$\Rightarrow E(X) = \int_{-\infty}^{\infty} x \cdot \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$\Rightarrow E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{XY}(x,y)\, dx\, dy$$

$\hookrightarrow$ odd function, $\therefore E(X) = 0$

similarly, $E(Y) = 0$

$$E(XY) = \frac{1}{2\pi(1-e^2)^{1/2}} \iint xy\, e^{\left[\frac{-1}{2(1-e^2)}(x^2 - 2exy + y^2)\right]} dx\, dy$$

$$= \frac{1}{2\pi(1-e^2)^{1/2}} \int x\, e^{\frac{-x^2}{2(1-e^2)}} dx \int y\, e^{\frac{-1}{2(1-e^2)}(y^2 - 2exy)} dy$$

$$= \frac{1}{2\pi(1-e^2)^{1/2}} \int x\, e^{\frac{-x^2}{2(1-e^2)}} dx \int y\, e^{\frac{-1}{2(1-e^2)}[(y-ex)^2 - e^2x^2]} dy$$

$$y - ex = y' \Rightarrow dy = dy' \ \& \ y = y' + ex$$

Then

$$= \frac{1}{2\pi(1-e^2)^{1/2}} \int x\, e^{\frac{-x^2 + e^2x^2}{2(1-e^2)}} dx \int (y' + ex)\, e^{\frac{-y'^2}{2(1-e^2)}} dy'$$

$$= \frac{1}{2\pi(1-e^2)^{1/2}} \int x\, e^{-\frac{1}{2}x^2} dx \left[\int (y' + ex)\, e^{\frac{-y'^2}{2(1-e^2)}} dy'\right.$$

$$= \frac{1}{2\pi(1-e^2)^{1/2}} \int x\, e^{-\frac{1}{2}x^2} dx \left[\int y'\, e^{\frac{-y'^2}{2(1-e^2)}} dy' + \int ex\, e^{\frac{-y'^2}{2(1-e^2)}} dy'\right.$$

$$= \frac{1}{2\pi(1-e^2)^{1/2}} \cdot \left[\underbrace{\int x\, e^{-\frac{1}{2}x^2} dx \int y'\, e^{-y'^2/2(1-e^2)} dy'}\right]$$

$$+\left[\int e x^2 e^{-\frac{1}{2}x^2} dx \int e^{\frac{-y'^2}{2(1-e^2)}} dy'\right]$$

odd

$\therefore = 0$

$$= \frac{e}{2\pi(1-e^2)^{1/2}} \underbrace{\int x^2 e^{-\frac{1}{2}x^2} dx}_{\frac{\pi^{1/2}}{2}} \underbrace{\int e^{\frac{-y'^2}{2(1-e^2)}} dy'}_{2^{1/2}(1-e^2)^{1/2}\pi^{1/2}}$$

$$= \frac{e}{\cancel{2}\cancel{\pi}(1-\cancel{e^2})^{1/2}} \cancel{2} \cdot \frac{\cancel{\sqrt{2}}\,\cancel{\sqrt{\pi}}}{\cancel{2}} \cdot \cancel{\sqrt{2}}\,\cancel{\sqrt{\pi}}\,\sqrt{1 \cancel{-} e^2}$$

$= e$

$\therefore \operatorname{cov}(X, Y) = e$

Now, $\operatorname{var}(X) = \dots$ → may try the standard method?

got stuck here....

<u>Second approach</u>

$$f(X,Y) = \frac{1}{2\pi(1-e^2)^{1/2}} \exp\left[\frac{-1}{2(1-e^2)}(x^2 - 2exy + y^2)\right]$$

let $X = Z_1$

$Y = eZ_1 + \sqrt{1-e^2} \cdot Z_2$

didn't click

Had to look it up

Then,

$$f(Z_1, Z_2) = \frac{1}{2\pi(1-e^2)^{1/2}} \exp\left[\frac{-1}{2(1-e^2)} Z_1^2 - 2eZ_1[eZ_1 + \sqrt{1-e^2}Z_2]\right.$$

$$+ e^2 Z_1^2 + (1-e^2)Z_2^2$$

$$+ 2eZ_1\sqrt{1-e^2}Z_2)$$

$$\Rightarrow \frac{1}{2\pi(1-e^2)^{1/2}} \exp\left[\frac{-1}{2(1-e^2)}\left[(1-e^2)z_1^2 \;\ldots\; + (1-e^2)z_2^2\right]\right]$$

$$\Rightarrow \frac{1}{2\pi(1-e^2)^{1/2}} \exp\left[\frac{-1}{2}(z_1^2 + z_2^2)\right]$$

Now, $z_1 = z_2 \sim N(0,1)$

$$var(X) = var(z_1) = 1$$

$$var(Y) = e^2\, var(z_1) + (1-e^2)\, var(z_2)$$
$$= 1$$

$$cov(X,Y) = cov(z_1,\, e z_1 + \sqrt{1-e^2}\, z_2)$$
$$= e\, cov(z_1, z_1) + \sqrt{1-e^2}\, cov(z_1, z_2)$$
$$= e \cdot 1 + \sqrt{1-e^2} \cdot 0 = e$$

as z_1 and z_2 are independent

$$\therefore \boxed{e_{x,y} = e}$$

let $X^2 = z_1^2$

$$Y^2 = (e z_1 + \sqrt{1-e^2}\, z_2)^2$$
$$= e^2 z_1^2 + (1-e^2) z_2^2 + 2e\sqrt{1-e^2}\, z_1 z_2$$

$cov(x, y)$

$$= cov(z_1^2,\, e^2 z_1^2 + (1-e^2) z_2^2 + 2e\sqrt{1-e^2}\, z_1 z_2$$

$$= e^2\, cov(z_1^2, z_1^2) + (1-e^2)\, cov(z_1^2, z_2^2) +$$
$$2e\sqrt{1-e^2}\, cov(z_1^2, z_1 z_2)$$

$= e^2(1) + (1-e^2)(0) + 2e\sqrt{1-e^2}\,(0)$

$= e^2$

$\therefore \boxed{E(x^2, y^2) = e^2}$

again

$var(X^2) = 1$

$var(Y^2) = 1$

Third approach

like before $E(Y) = E(X) = 0$

and $cov(X,Y) = E(X,Y) = e$

$var(x) = ?$ and $var(Y) = ?$

$$var(X) = E(X^2) - [E(X)]^2 = E(X^2)$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 \cdot k_1 \cdot e^{-k_2 \cdot (x^2 - 2exy + y^2)}\, dx\, dy$$

Trying substitution,

$u = x + y$ ~~and~~ $v = x - y$

$\Rightarrow x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Then,

$$f(u,v) = |J|\, k_1 \cdot e^{-k_2\left[\left(\frac{u+v}{2}\right)^2 + \cancel{2}e\frac{(u+v)}{\cancel{2}}\frac{(u-v)}{2}\right.}$$

$$+ \left(\frac{u-v}{2}\right)^2 \Big]$$

$$\Rightarrow \frac{k_1}{2} e^{-k_2\left[\frac{u^2+v^2}{4} + \frac{eu^2-ev^2}{2} + \cancel{\frac{2uv}{4}} + \frac{u^2+v^2}{4} - \cancel{\frac{2uv}{4}}\right]}$$

$$= \frac{k_1}{2} e^{-k_2\left[\frac{u^2+v^2+ev^2-ev^2}{2}\right]}$$

$$= \frac{k_1}{2} e^{-\frac{1}{2(1-e^2)}\left[\frac{u^2(1+e)+v^2(1-e)}{2}\right]}$$

$$= \frac{k_1}{2} e^{-\frac{1}{4}\left[\frac{u^2}{1-e} + \frac{v^2}{1+e}\right]}$$

such that,

$$E(x^2) = \frac{1}{2} E(x^2+y^2) \quad \text{as } E(x^2) = E(y^2)$$

by symmetry

$$= \frac{1}{4} E(u^2+v^2)$$

and $$E(u^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 \cdot \frac{k_1}{2} e^{-\frac{1}{4}\left[\frac{u^2}{1-e}+\frac{v^2}{1+e}\right]} dv\,du$$

$$\Rightarrow \frac{1}{4\pi\sqrt{1-e^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2 e^{-\frac{1}{4}\left(\frac{u^2}{1-e}\right)} \cdot e^{-\frac{1}{4}\left(\frac{v^2}{1+e}\right)} du\,dv.$$

$$\Rightarrow \frac{1}{4\pi\sqrt{1-e^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{4}\left(\frac{v^2}{1+e}\right)} dv \cdot \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{4}\left(\frac{u^2}{1-e}\right)} du$$

$$\Rightarrow \frac{1}{4\pi\sqrt{1-e^2}} \cdot \sqrt{\pi}\; 2\sqrt{1+e} \cdot \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{4}\left(\frac{u^2}{1-e}\right)} du$$

$$\Rightarrow \frac{1}{2\sqrt{\pi}\sqrt{1-e}} \cdot \int_{-\infty} u^{2}\, e^{-\frac{1}{4}\left(\frac{u^2}{1-e}\right)} du$$

$$\frac{1}{4} \cdot \frac{u^2}{1-e} = a \quad \Rightarrow \quad \frac{u}{2\sqrt{1-e}} = \sqrt{a}$$

$$\Rightarrow \quad du = \cancel{2}\sqrt{1-e} \cdot \frac{1}{\cancel{2}\sqrt{a}}\, da$$

and $u^2 = 4(1-e)a$

$$\Rightarrow \frac{1}{2\sqrt{\pi(1-\cancel{e})}} \int_{-\infty}^{\infty} \frac{\sqrt{\cancel{1-e}}}{\cancel{\sqrt{a}}} \cdot \overset{2}{\cancel{4}}(1-e)\sqrt{a}\, e^{-a}\, da$$

$$\Rightarrow \frac{2(1-e)}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} a^{1/2} e^{-a}\, da}_{2 \cdot \Gamma\frac{3}{2} \;=\; \frac{1}{2}\Gamma\frac{1}{2} \;=\; \frac{1}{\cancel{2}}\sqrt{\pi}\cdot\cancel{2}}$$

$\Rightarrow 2(1-e)$

Then $E(W^2) = 2(1+e)$ by symmetry

Then $E(X^2) = \frac{1}{4}(2)(1+e+1-e) = 1$

and $\therefore E(Y^2) = 1$ by symmetry

$$\therefore \operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var} X}\sqrt{\operatorname{var} Y}} = e$$

Similarly for $\operatorname{corr}(X^2, Y^2)$

we need $\operatorname{var}(X^2)$ & $\operatorname{var}(Y^2)$

$\ldots (X^2, Y^2) \ldots E(X^2Y^2) - E[X^2]\,E(Y^2)$

and $\operatorname{cov}(X^2, Y^2) = E(X^2Y^2) - E(X^2)E(Y^2)$

$$= E(X^2Y^2) - 1$$

$$E(X^2Y^2) = E\left(\frac{(u+v)^2(u-v)^2}{16}\right) = E\left(\frac{(u^2-v^2)^2}{16}\right)$$

$$= E\left(\frac{u^4+v^4-2u^2v^2}{16}\right) = \frac{1}{16}E\left(u^4+v^4-2u^2v^2\right)$$

now,

$$E(u^4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^4 \cdot \frac{k_1}{2} \cdot e^{-\frac{1}{4}\left(\frac{u^2}{1-e}+\frac{v^2}{1+e}\right)} \, du\, dv$$

$$\Rightarrow \frac{1}{4\pi\sqrt{1-e^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{4}\frac{v^2}{1+e}} dv \int_{-\infty}^{\infty} u^4 \cdot e^{-\frac{1}{4}\cdot\frac{u^2}{1-e}} du$$

$$\Rightarrow \frac{1}{2\sqrt{\pi}\sqrt{1-e}} \cdot \int_{-\infty}^{\infty} u^4 e^{-\frac{1}{4}\frac{u^2}{1-e}} du$$

$$\frac{u^2}{4(1-e)} = a \Rightarrow du = \frac{\sqrt{1-e}}{\sqrt{a}} da$$

$$\Rightarrow u^4 = a^2 4^2 (1-e)^2$$

$$\Rightarrow \frac{1}{\cancel{2}\sqrt{\pi}\cancel{\sqrt{1-e}}} \cdot \int_{-\infty}^{\infty} \overset{a^{3/2}}{\cancel{a^2}} \cdot e^{-a} \overset{8}{\cancel{16}} (1-e)^2 \cdot \frac{\cancel{\sqrt{1-e}}}{\cancel{a^{1/2}}} da$$

$$\Rightarrow \frac{(1-e)^2}{\sqrt{\pi}} \cdot 8 \int_{-\infty}^{\infty} a^{3/2} e^{-a} da$$

$$= (1-e)^2 \cdot \overset{4}{\cancel{8}} \cdot \cancel{\pi} \cdot \frac{3}{\cancel{2}} \cdot \frac{1}{\cancel{2}} \cdot \cancel{\sqrt{\pi}}$$

$$\overline{\qquad}\quad\cancel{\sqrt{\pi}}$$

$$= 12(1-e)^2$$

and $E(v^2) = 12(1+e)^2$ by symmetry

Now, $E(u^2v^2) = \dfrac{1}{4\pi\sqrt{1-e^2}} \int_{-\infty}^{\infty} u^2 e^{-\frac{1}{4}\frac{u^2}{1-e}}\,du \cdot \int_{-\infty}^{\infty} v^2 \cdot e^{-\frac{1}{4}\frac{v^2}{1+e}}\,dv$

$$= \frac{1}{\cancel{4}\pi\cancel{\sqrt{1-e^2}}} \int_{-\infty}^{\infty} \frac{\cancel{\sqrt{1-e}}}{\sqrt{a}} \cdot \cancel{4}\cdot(1-e)\cdot a\, e^{-a}\,da \cdot \int_{-\infty}^{\infty} \frac{\cancel{\sqrt{1+e}}}{\sqrt{k}} \cdot 4(1+e)k \cdot e^{-a}\,dk$$

$$= \frac{4}{\pi}(1-e^2)\left(\cancel{2}\cdot\cancel{\frac{1}{2}}\cdot\sqrt{\pi}\right)^2 = (1-e^2)\,4$$

$$\therefore E(x^2y^2) = \frac{1}{16}\left[12(1+e)^2 + 12(1-e)^2 - 2\cdot 4(1-e^2)\right]$$

$$\Rightarrow \frac{1}{16}\left[12(1+e^2+1+e^2] - 8(1-e^2)\right]$$

$$= \frac{1}{16}\left[24 + 24e^2 - 8 + 8e^2\right]$$

$$= 1 + 2e^2$$

$$\therefore \operatorname{cov}(x^2, y^2) = E(x^2y^2) - 1 = 2e^2$$

Now, $\operatorname{var}(X^2) = E(X^4) - E(X^2)$

$$= E(X^4) - 1$$

$$E(X^4) = \frac{1}{2} E(X^4 + Y^4)$$

$$X^4 = \left(\frac{u+v}{2}\right)^4 = \frac{(u^2+v^2+2uv)^2}{16}$$

$$= \frac{u^4 + v^4 + 4u^2v^2 + 4u^3v + 4uv^3}{16} + 2u^2v^2$$

and $Y^4 = \left(\frac{u-v}{2}\right)^4 = \left(\frac{u^2+v^2-2uv}{16}\right)^2$

$$= \frac{u^4+v^4+4u^2v^2 - 4u^3v - 4uv^3}{16} + 2u^2v^2$$

$$\therefore E(X^4) = \frac{1}{\cancel{2}} E\left[\cancel{2}\frac{(u^4+v^4+ \quad)}{16}\right]$$

$$= \frac{1}{16}\left[E(u^4) + E(v^4) + 6E(u^2v^2)\right]$$

$$\Rightarrow \frac{1}{16}\left[12(1+e^2-\cancel{2e} + 1+e^2+\cancel{2e}) + 6\cdot 4(1-e^2)\right]$$

$$= \frac{1}{16}\left[24 + 24e^2 + 24 - 24e^2\right]$$

$$= \frac{1}{16}[24\cdot 2] = 3$$

$\therefore \quad var(X^2) = 3 - 1 = 2$

similarly $var(Y^2) = 2$

$\therefore \quad corr(X^2, Y^2) = \rho(X^2, Y^2) = \dfrac{cov(X^2, Y^2)}{\sqrt{var\, X^2}\sqrt{var\, Y^2}}$

$\Rightarrow \dfrac{2e^2}{\sqrt{2}\sqrt{2}} = e^2$ — hence proved.

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(6) $X \sim Gamma(n, 3)$

$f(x) = \dfrac{1}{\Gamma n \, 3^n} \cdot x^{n-1} e^{-x/3}, \quad x > 0$

n is a large integer.

(Assumed def^n of gamma distribution to be ...

$\underbrace{Gamma(\alpha, \beta)}_{\text{parameters}} \Rightarrow f(x) = \dfrac{1}{\Gamma\alpha\, \beta^\alpha} x^{\alpha-1} \cdot e^{-x/\beta}.$)

CLT,

for $X_1, X_2 \dots X_n \sim Gamma(n, 3)$

st they are all i.i.d. r.v.s

$\bar{X} = \frac{1}{n} \sum x_i$

and $E(\bar{X}) = \frac{1}{n} E(\sum^{n} X_i) = \frac{1}{n} \cdot n\mu = \mu$

$var(\bar{X}) = \frac{\sigma^2}{n}$

Then $Y = \frac{\bar{X} - E(\bar{X})}{\sqrt{var(\bar{X})}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

according to CLT.

how?

So, $M_Y(t) = E(e^{tY})$

$\Rightarrow E(e^{tY}) = E\left(e^{t\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}\right)$

$= E\left(e^{t\left(\frac{\sum X_i - n\mu}{\sqrt{n}\sigma}\right)}\right)$

$= E\left(e^{t\left(\frac{X_1 - \mu}{\sqrt{n}\sigma_1}\right)} \cdot e^{t\left(\frac{X_2-\mu}{\sqrt{n}\sigma_2}\right)} \cdots\right)$

$= E\left(e^{t\left(\frac{X_1-\mu}{\sqrt{n}\sigma_1}\right)}\right) \cdot E\left(e^{t\left(\frac{X_2-\mu}{\sqrt{n}\sigma_2}\right)}\right) \cdots$

($\because$ they are iids)

$= \prod_{i=1}^{n} E\left(e^{t\left(\frac{X_i-\mu}{\sqrt{n}\sigma_i}\right)}\right)$

$= \left[E\left(e^{t\left(\frac{X_1-\mu}{\sqrt{n}\sigma_1}\right)}\right)\right]^n$

say $m(t) = E\left(e^{t(X-\mu)}\right) = e^{-\mu t} \cdot M(t)$

then $m(0) = E(e^{0(X-\mu)}) = E(1) = 1$

$$m'(0) = -\mu e^{-\mu t} M(t) + M'(t) e^{-\mu t}\Big|_{t=0}$$

$$m'(0) = -\mu + \mu = 0$$

$$m''(0) = \mu^2 e^{-\mu t} M(t) - \mu e^{-\mu t} M'(t) + M''(t) e^{-\mu t} - \mu e^{-\mu t} M'(t)\Big|_{t=0}$$

$$m''(0) = \sigma^2$$

Now, we know,

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a+\theta h)$$

(taylor's formula)

at $a=0$,

$$f(h) = f(0) + \frac{h}{1!} f'(0) + \frac{h^2}{2!} f''(\theta h)$$

for $f(h) = m(t)$

$$m(t) = m(0) + \frac{t}{1!} \cancel{m'(0)}^{\,0} + \frac{t^2}{2!} m''(\theta t)$$

$$m(t) = 1 + \frac{t^2}{2!} m''(\theta t)$$

let $\theta t = \delta$

then $0 < \delta < t$

$$\Rightarrow m(t) = 1 + \frac{\sigma^2 t^2}{2!} - \frac{\sigma^2 t^2}{2!} + \frac{t^2}{2!} m''(\delta)$$

$$= 1 + \frac{\sigma^2 t^2}{2!} + \left(\frac{m''(\delta) - \sigma^2}{2!}\right) t^2$$

we had,

$$M_Y(t) = \left[E\left(e^{t\frac{(x_1 - \mu)}{\sqrt{n}\sigma_1}}\right)\right]^n$$

$$= \left[E\left(m\left(\frac{t}{\sqrt{n}\sigma}\right)\right)\right]^n$$

$$= \left[E\left(1 + \frac{\sigma^2 t^2}{2n\sigma^2} + \frac{m''(\delta) - \sigma^2}{2} \cdot \frac{t^2}{n\sigma^2}\right)\right]^n$$

Now,

$$\lim_{n \to \infty} M_Y(t) = \lim_{n \to \infty} \Big\{ 1 + \underbrace{\frac{\sigma^2 t^2}{2n\sigma^2} + \frac{m''(\delta) - \sigma^2}{2} \cdot \frac{t^2}{n\sigma^2}}_{\text{tends to zero}} \Big\}^n$$

$\therefore$ 1^∞ form.

$$\Rightarrow \lim_{n \to \infty} M_Y(t) = e^{\lim_{n \to \infty} \left\{ \frac{t^2}{2n} + \frac{m''(\delta) - \sigma^2}{2n\sigma^2} \cdot t^2 \right\} n}$$

$$= e^{\lim_{n \to \infty} \left\{ \frac{t^2}{2} + \frac{m''(\delta) - \sigma^2}{2\sigma^2} \cdot t^2 \right\}}$$

$$= e^{t^2/2 + \frac{m''(\delta^2) - \sigma^2}{} \cdot t^2}$$

$- \quad e \qquad 2\sigma^2$

Now, here, $-h < \frac{t}{\sqrt{n}\sigma} < h \Rightarrow \theta t = \delta$

as $n \to \infty$, $\delta \to 0$

$\left(\theta = \frac{1}{\sqrt{n}\sigma} \right)$

then $m''(\delta) - \sigma^2 \to m''(0) - \sigma^2 \to 0$

as $m''(0) \to \sigma^2$ as $n \to \infty$

then $M_Y(t) = e^{t^2/2}$ → mgf of $N(0,1)$

$\therefore Y \xrightarrow{d} N(0,1)$

This is true for all distributions including $X \sim$ Gamma$(n, 3)$

given $n \to \infty$.

Since, n is a very large integer, CLT holds.

$$f(x : n, 3) = \frac{1}{\Gamma n} \cdot \frac{1}{3^n} x^{n-1} e^{-x/3}$$

$$\left.\begin{aligned} \text{mean} &= \alpha\beta = 3n \\ \text{variance} &= \alpha\beta^2 = 9n \end{aligned}\right\} \text{approximates}$$

Then $X \sim N(3n, 9n)$

for $n \to \infty$ (by CLT)

for $n \to \infty$