Let A be the event when $X+Y=n$

Now, $P(X=x|A)=?$

So, $$P(X=x|A) = \frac{P((X=x)\cap A)}{P(A)}$$

$$P((X=x)\cap A) = P(X=x)\cdot P(Y=n-x)$$

$$\Rightarrow P((X=x)\cap A) = e^{-\lambda}\cdot\frac{\lambda^x}{x!}\cdot e^{-\mu}\,\frac{\mu^{n-x}}{(n-x)!}$$

$$\Rightarrow P((X=x)\cap A) = \frac{e^{-(\lambda+\mu)}\cdot\lambda^x\,\mu^{n-x}}{x!\,(n-x)!}$$ where $0\le x\le n$

Since, X is Mutually exhaustive on A for $0\le x\le n$

$$P(A) = \sum_{x=0}^{n} P((X=x)\cap A)$$

$$\Rightarrow P(A) = \frac{e^{-(\lambda+\mu)}}{n!}\left(\sum_{x=0}^{n}\frac{n!}{x!\,(n-x)!}\lambda^x\cdot\mu^{n-x}\right)$$

$$\Rightarrow P(A) = \frac{e^{-(\lambda+\mu)}\,(\lambda+\mu)^n}{n!}$$

So, $$P(X=x|A) = \frac{P((X=x)\cap A)}{P(A)}$$

$$= \frac{\dfrac{e^{-(\lambda+\mu)}\,\lambda^x\,\mu^{n-x}}{x!\,(n-x)!}}{\dfrac{e^{-(\lambda+\mu)}\,(\lambda+\mu)^n}{n!}}$$

$$\Rightarrow P(X=x|A) = {}^nC_x\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(\frac{\mu}{\lambda+\mu}\right)^{n-x}$$

Q.2) $P(X_n = \frac{1}{n}) = 1 - \frac{1}{n^2}$, $P(X_n = n) = \frac{1}{n^2}$

As $n \to \infty \Rightarrow \lim_{n\to\infty} P(X_n = \frac{1}{n}) = 1 - \frac{1}{n^2}$

$= 1$,

$\lim_{n\to\infty} P(X_n = n) = \frac{1}{n^2} \to 0$

So, X_n converges $\frac{1}{\infty} \to 0$, as $n \to \infty$.

More formally, let X' be a Random Variable where $P(X' = 0) = 1$

then $P(|X_n - X'| > \epsilon) \to 0$ as $n \to \infty$.

$\Downarrow$

$X_n \xrightarrow{P} X'$

<u>Quadratic Mean Convergence</u>

$\boxed{E[(X_n - X')^2]}$

$\Rightarrow E[X_n^2]$ as $P(X' = 0) = 1$

$\Rightarrow E[X_n^2] = \sum x^2 p_x$

$\Rightarrow E[X_n^2] = \frac{1}{n^2} \times (1 - \frac{1}{n^2}) + \frac{1}{n^2} \times n^2$

$\Rightarrow E[X_n^2] = \frac{1}{n^2} - \frac{1}{n^4} + 1$

$\Rightarrow \lim_{n\to\infty} E[X_n^2] \to 1 \neq 0$

So, X_n converges in probability to $X = 0$

but does not converge in Quadratic Mean.

Q. 3)

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Now let $F_X(x)$ be cdf

$$\Rightarrow F_X(x) = P(X \le x)$$

$$\Rightarrow 1 - F_X(x) = P(X > x)$$

$$= \int_{x}^{\infty} f(t)\, dt$$

Integrationg on all sides

$$\Rightarrow \int_{0}^{\infty} (1 - F_X(x))\, dx = \int_{0}^{\infty} P(X > x)\, dx$$

$$= \int_{0}^{\infty} \int_{x}^{\infty} f(t)\, dt\, dx$$

$$\begin{pmatrix} x \le t < \infty \\ 0 \le x < \infty \end{pmatrix}$$

$$\Rightarrow \int_{0}^{\infty} P(X > x)\, dx = \int_{0}^{\infty} \int_{0}^{t} f(t)\, dx\, dt$$

$$\begin{pmatrix} 0 \le x \le t \\ 0 \le t < \infty \end{pmatrix}$$

$$\Rightarrow \int_{0}^{\infty} P(X > x)\, dx = \int_{0}^{\infty} \left(x f(t) \right)_0^t\, dt$$

$\rightarrow f(t)$ is Independent of x ~~($f(t)$ is Independent of x)~~

$$\Rightarrow \int_{0}^{\infty} P(X > x)\, dx = \int_{0}^{\infty} (t \cdot f(t))\, dt$$

$$\Rightarrow \int_{0}^{\infty} P(X > x)\, dx = E[X]$$

Q.4) $KL[P||Q] = \int_{-\infty}^{\infty} P(x) \log_e \frac{P(x)}{Q(x)} \cdot dx$

$\Rightarrow KL[P||Q] = E\left[\log_e \frac{P(x)}{Q(x)}\right]_{P(x)}$

$P(x) = N(x|\mu, E), \; q(x) = N(x|m, L)$

$\ln \frac{P(x)}{Q(x)}$

$\Rightarrow \ln \left(\frac{\frac{1}{\sqrt{2\pi}E} e^{-\frac{(x-\mu)^2}{2E^2}}}{\frac{1}{\sqrt{2\pi}L} e^{-\frac{(x-m)^2}{2L^2}}} \right)$

$\Rightarrow \ln \frac{L}{E} - \frac{(x-\mu)^2}{2E^2} + \frac{(x-m)^2}{2L^2}$

So, $E\left[\ln \frac{P(x)}{Q(x)}\right]_{P(x)}$

$\Rightarrow E\left[\ln \frac{L}{E} - \frac{(x-\mu)^2}{2E^2} + \frac{(x-m)^2}{2L^2}\right]_{P(x)}$

$\Rightarrow \ln \frac{L}{E} - \frac{1}{2E^2} \cdot \underbrace{E[(x-\mu)^2]}_{E^2} + \frac{1}{2L^2} E[(x-m)^2]$

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So, $E[(x-m)^2]_{p(x)}$

$$\Rightarrow E[x^2]_{p(x)} - 2m\,E[x]_{p(x)} + m^2$$

$$\Rightarrow E^2 + \mu^2 - 2m\mu + m^2 \Rightarrow E^2 + (\mu - m)^2$$

Finally,

$$KL[P\|Q] = E\left[\ln \frac{P(x)}{Q(x)}\right]_{P(x)}$$

$$= \ln \frac{L}{E} - \frac{1}{2\cancel{E}^2} \times \cancel{E}^2 + \frac{1}{2L^2} \times \left(E^2 + (\mu - m)^2\right)$$

$$= \ln \frac{L}{E} - \frac{1}{2} + \frac{E^2 + (\mu - m)^2}{2L^2}$$

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Q.5) Bivariate Gaussian

$$f(x, y) = \frac{1}{2\pi |\Sigma|^{1/2}} e\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

where $\Sigma = \begin{bmatrix} \sigma_x^2 & \mathrm{cov}(x,y) \\ \mathrm{cov}(x,y) & \sigma_y^2 \end{bmatrix}$

So, $\Sigma = \begin{bmatrix} \sigma_x^2 & c \\ c & \sigma_y^2 \end{bmatrix} \quad \left(\Rightarrow |\Sigma| = \sigma_x^2 \sigma_y^2 - c^2 \right)$

$$\Rightarrow \Sigma^{-1} = \frac{1}{\sigma_x^2\sigma_y^2 - c^2} \begin{bmatrix} \sigma_y^2 & -c \\ -c & \sigma_x^2 \end{bmatrix}$$

So, $f(x, y) = \frac{1}{2\pi (\sigma_x^2\sigma_y^2 - c^2)^{1/2}} e\left(\frac{-1}{2} \cdots\cdots\right)$

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$$(x-\mu)^T \Sigma^{-1} (x-\mu)$$

$$\frac{1}{\sigma_x\sigma_y - c^2} \begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \begin{bmatrix} \sigma_y^2 & -c \\ -c & \sigma_x^2 \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

After Matrix Multiplication,

$$\frac{1}{\sigma_x^2\sigma_y^2 - c^2} \left(\sigma_y^2 (x-\mu_x)^2 + \sigma_x^2 (y-\mu_y)^2 - 2c(x-\mu_x)(y-\mu_y) \right)$$

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Comparing ① with

$$f(x, y) = \frac{1}{2\pi(1-p^2)^{1/2}} \exp\left(\frac{-1}{2(1-p^2)}(x^2 - 2pxy + y^2)\right)$$

$$\Rightarrow \sigma_x^2 \sigma_y^2 - c^2 = 1 - p^2$$

and

$$\frac{1}{\sigma_x^2\sigma_y^2 - c^2}\left(\sigma_y^2(x-\mu_x)^2 - 2c(x-\mu_x)(y-\mu_y) + \sigma_x^2(y-\mu_y)^2\right)$$

$$= \frac{1}{1-p^2}\left(x^2 - 2pxy + y^2\right)$$

$$\Rightarrow \sigma_y^2 x^2 - 2\mu_x\sigma_y^2 x + \sigma_y^2\mu_x^2$$
$$-2cxy + 2c(\mu_x y + \mu_y x)$$
$$+2c\mu_x\mu_y$$
$$+\sigma_x^2 y^2 - 2\mu_y\sigma_x^2 y + \mu_y^2\sigma_x^2$$

$$\Rightarrow \sigma_y^2(x^2) + (2c\mu_y - 2\mu_x\sigma_y^2)x$$
$$-2cxy + (2c\mu_x - 2\mu_y\sigma_x^2)y$$
$$+\sigma_y^2\mu_x^2 + 2c\mu_x\mu_y + \mu_y^2\sigma_x^2$$

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Comparing coefficients of x^2, y^2, x, y, xy

$\Rightarrow$ So, $\sigma_y^2 = 1$, $\sigma_x^2 = 1$, $2c\mu_y - 2\mu_x\sigma_y^2 = 0$, (3)

$2c\mu_x - 2\mu_y\sigma_x^2 = 0$, (4) $2c = 2\rho$

$\Rightarrow \sigma_y = 1, \sigma_x = 1$; $Cov(X, Y) = c = \rho$,

$\mu_x = 0, \mu_y = 0$

So, $Cov(X, Y) = \rho$

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\, Var(Y)}}$$

$$\Rightarrow Corr(X, Y) = \frac{\rho}{1} = \rho$$

$$Corr(X^2, Y^2) = \frac{Cov(X^2, Y^2)}{\sqrt{Var(X^2) \cdot Var(Y^2)}}$$

$$Var(X^2) = E[X^4] - (E[X^2])^2$$

$$\Rightarrow Var(X^2) = 3 - 1 = 2 = Var(Y^2)$$

$$Cov(X^2, Y^2) = E[X^2Y^2] - E[X^2]\,E(Y^2)$$

$$= 1 + 2\rho^2 - 1 = 2\rho^2$$

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So, $Corr(X^2, Y^2) = \frac{2\rho^2}{2} = \rho^2$

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Q.6) Let X_i be an exponential Distribution, with $\beta = 3$.

Let Sample Size of $X_i = \alpha = n$ where n is a large Integer

So, $M_{X_i}(x) = \dfrac{1}{1-\beta t}$

$$M_X(x) = E[e^{tX}]$$

$$= E[e^{t(x_1 + \cdots + x_n)}]$$

$$= \sum_{i=1}^{n} E[e^{tX_i}]$$

$$\Rightarrow M_X(x) = \prod_{i=1}^{n} M_{X_i}(x)$$

$$\Rightarrow \left(\frac{1}{1-\beta t}\right)^{\alpha} = \left(\frac{1}{1-\beta t}\right)^{n}$$

So, $\sum_{i=1}^{n} X_i$ is X.

$\rightarrow$ CLT can Approximate the Individual Sequence of X_i events in X which are exp. Dist.

Using CLT,

Let $L = \dfrac{\sum X_i - n\mu}{\sqrt{n\sigma^2}}$ where $\mu = E[X_i]$, $\sigma^2 = Var(X_i)$

then $L \sim N(0,1)$ as $n \rightarrow \infty$

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Since n is a large integer.

$$\frac{\sum_{i=1}^{n} X_i - n\beta}{\sqrt{n\beta^2}} \sim N(0, 1)$$

$$\left(E[X_i] = \beta,\ V[X_i] = \beta^2 \right)$$

$$\Downarrow$$

$$\frac{\frac{\sum_{i=1}^{n} X_i}{n} - \beta}{\frac{\beta}{\sqrt{n}}} \sim N(0, 1)$$

$$\Rightarrow \frac{\sum_{i=1}^{n} X_i}{n} \sim N(\beta, \beta^2/n)$$

$$\left(\begin{array}{l} E(nX) = nE[X] \\ Var[nX] = n^2 Var(X) \end{array} \right) \Rightarrow \sum_{i=1}^{n} X_i \sim N(n\beta, n\beta^2)$$

$$\Rightarrow X \sim N(n\beta, n\beta^2)$$

$$\left(\sum_{i=1}^{n} X_i \xrightarrow{L} X \right)$$

So, $X \sim G(\alpha, \beta)$

By CLT, $X \sim N(\alpha\beta, \alpha\beta^2)$ approx for large α

So, $X \sim N(3n, 9n)$

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