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(Q 1.) we need $P\left(X=n \,/\, X+Y=n\right)$

$$= \frac{P(X=n \cap X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=n \cap Y=n-n)}{P(X+Y=n)}$$

Now as X and Y are independent

so. $P(X=n \cap Y=n-n)$

$$= P(X=n) \cdot P(Y=n-n)$$

$$= \frac{e^{-\lambda}\lambda^{n}}{n!} \cdot \frac{e^{-\mu}\mu^{n-n}}{(n-n)!}$$

and $P(X+Y=n)$

is following poisson of $(\lambda+\mu)$ as

$$E(X+Y) = E\left(e^{t(X+Y)}\right) = \cancel{e^{tX+tY}}$$

$$= E\left(e^{tX} \cdot e^{tY}\right)$$

$$= E\left(e^{tX}\right) \cdot E\left(e^{tY}\right)$$

$$= e^{\lambda(e^{t}-1)} \cdot e^{\mu(e^{t}-1)} = e^{(\lambda+\mu)(e^{t}-1)}$$

$$\therefore P\left(\frac{X=n}{X+Y=n}\right) = {}^{n}C_{n}\left(\frac{\lambda}{\lambda+\mu}\right)^{n}\left(\frac{\mu}{\lambda+\mu}\right)^{n-n}$$

(Q2) $P(X_n) = \begin{cases} 1-\frac{1}{n^2} &, X_n = \frac{1}{n} \\ \frac{1}{n^2} &, X_n = n \end{cases}$ Date: / /

$$F(x) = \begin{cases} 0 & X_n < \frac{1}{n} \\ 1-\frac{1}{n^2} & \frac{1}{n} \leq X_n < n \\ 1 & n \leq X_n \end{cases}$$

Now $\lim_{n\to\infty} F(x) = \begin{cases} 0 & X_n < \frac{1}{n} \\ 1 & \frac{1}{n} \leq X_n < n \\ 1 & n \leq X_n \end{cases}$

Hence as
$\lim_{n\to\infty} P(|X_n| < \epsilon) = 1$
for all $X_n > \frac{1}{n}$

$\therefore \lim_{n\to\infty} P(|X_n - 0| < \epsilon) = 1$

$\therefore X_n \xrightarrow{P} 0$

Now, for convergence in quadratic mean.

$$\lim_{n\to\infty} E(|X_n - 0|^2) = 0$$

But $E(X_n^2) = \left(\frac{1}{n}\right)^2\left(1-\frac{1}{n^2}\right) + (n)^2\left(\frac{1}{n^2}\right)$

$E(X_n^2) = \frac{1}{n^2} - \frac{1}{n^4} + 1$

$\lim_{n\to\infty} E(X_n^2) = 1 \neq 0$

$\therefore$ It doesn't converge

$C_n \; \pi^{n} (1-\pi)^{n-r}$

(Q3) $\int_0^\infty P(X > x)\,dx$

Now $P(X > x) = \int_x^\infty f_X(y)\,dy$

$\therefore \int_0^\infty \int_x^\infty f_X(y)\,dy\,dx = \int_0^\infty \int_0^y f_X(y)\,dx\,dy$

$$= \int_0^\infty f_X(y) \cdot y\,dy$$

substituting $y = x$ as y is just a variable.

$$= \int_0^\infty f_X(x) \cdot x\,dx$$

$$= E(X)$$

(4) $p \sim N(\mu, \varepsilon)$

$q \sim N(m, L)$

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$$\text{KL div} = \int p(x) \log\left(\frac{p(x)}{q(x)}\right) dx$$

$$\cancel{= \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

$$= \int \left(\frac{1}{\varepsilon\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\varepsilon^2}}\right) \log\left(\frac{L}{\varepsilon} \cdot \frac{e^{-\frac{(x-\mu)^2}{2\varepsilon^2}}}{e^{-\frac{(x-m)^2}{2L^2}}}\right) dx$$

$$= \int p(x) \cdot \left(\log\left(\frac{L}{\varepsilon}\right) - \frac{(x-\mu)^2}{2\varepsilon^2} + \frac{(x-m)^2}{2L^2}\right) dx$$

$$= \int p(x) \log\left(\frac{L}{\varepsilon}\right) dx - \int \frac{p(x)(x-\mu)^2 dx}{2\varepsilon^2} + \int p(x) \frac{(x-m)^2}{2L^2} dx$$

$$= \log\left(\frac{L}{\varepsilon}\right) - \frac{1}{2\varepsilon^2} \cdot \varepsilon^2 + \int \frac{p(x)}{2L^2} (x-\mu+\mu-m)^2 dx$$

$$= \log\left(\frac{L}{\varepsilon}\right) - \frac{1}{2} + \frac{1}{2L^2}\left(\int p(x)(x-\mu)^2 dx + \int p(x)(\mu-m)^2 dx + \int 2 \cdot p(x)(x-\mu)(\mu-m) dx\right)$$

$$= \log\left(\frac{L}{\varepsilon}\right) - \frac{1}{2} + \frac{1}{2L^2}\left(\varepsilon^2 + (\mu-m)^2 + 0\right)$$

$$= \log\left(\frac{L}{\varepsilon}\right) - \frac{1}{2} + \frac{1}{2L^2}\left(\varepsilon^2 + (\mu-m)^2\right)$$

(5) $f(x,y) \rightarrow$ standard Bivariate Normal distn.

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(a) $x^2 - 2\rho x y + y^2 = x^2 - 2\rho x y + y^2 + \rho^2 x^2 - \rho^2 x^2$

$= (\rho x - y)^2 + x^2(1-\rho^2)$

$\therefore \ corr(X,Y) = \dfrac{cov(X,Y)}{\sigma_x \sigma_y}$

$\sigma_x = 1 = \sigma_y \quad , \quad \mu_x = \mu_y = 0$

$\therefore \ cov(X,Y) = E(XY)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\left(\frac{-x^2(1-\rho^2)}{2(1-\rho^2)} - \frac{(\rho x - y)^2}{2(1-\rho^2)}\right)} dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x\, e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \left(\frac{1}{\sqrt{2\pi(1-\rho^2)}}\, y\, e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy\right)$$

$$= \int_{-\infty}^{\infty} \frac{x\, e^{-x^2/2}}{\sqrt{2\pi}} dx \cdot \rho x$$

$$= \rho \int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$$= \rho\, [1 + \cancel{0}] = \rho$$

(b) Same $\int^{\infty}\int^{\infty} x^2 e^{-x^2/2} \left(\frac{1}{\ldots} y^2 e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}} dy\right)$

(b) $\mathrm{corr}(x^2, y^2) = \dfrac{\mathrm{cov}(x^2, y^2)}{\sigma_{x^2}\,\sigma_{y^2}}$ Date: / /

$\sigma_{x^2} =$

$X \sim N(0,1)$, $x^2 \sim$ Chisquare (1 df)

$\mu = 1$

$\sigma^2 = 2$

$\therefore\ \sigma_{x^2} = 2 = \sigma_{y^2}$

$\mathrm{corr}(x^2, y^2) = \dfrac{\mathrm{cov}(x^2, y^2)}{2}$

Rest stays same

$$\mathrm{cov} = \int\int x^2 y^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}\, dy$$

$$\mathrm{cov} = \int \frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}}\, dx \int \frac{y^2}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}\, dy$$

$$\mathrm{cov} = \int \frac{x^2 e^{-x^2/2}\, dx}{\sqrt{2\pi}} \left((1-\rho^2) + \rho^2 x^2 \right)$$

$(\sigma^2 + E(x^2))$

$$\mathrm{cov} = (1-\rho^2) + \rho^2 \int_{-\infty}^{\infty} \frac{x^4 e^{-x^2/2}\, dx}{\sqrt{2\pi}}$$

$$\mathrm{cov} = (1-\rho^2) + \rho^2 \int_{-\infty}^{\infty} \frac{y^{3/2} e^{-y/2}\, dy}{2\sqrt{2\pi}}$$

$$\mathrm{cov} = (1-\rho^2) + \rho^2 \int_{-\infty}^{\infty} \frac{t^{3/2} e^{-t}\, dt}{\sqrt{\pi}}$$

$$\mathrm{cov} = (1-\rho^2) + \frac{\rho^2\, 2\, \Gamma\frac{5}{2}}{\sqrt{\pi}} = \frac{2\rho^2}{2} = \rho^2$$

$\sqrt{n}$

~~Ex~~ (Q(6.)) $X_1, X_2 \ldots X_n$ be a sequence of Random variables with exponential distribution and parameter $\beta = 3$ then according to CLT for $n \to \infty$

$$\frac{\sum X_i - n\mu}{\sqrt{n\sigma^2}} \sim N(0,1)$$

$\mu = \beta$
$\sigma^2 = \beta^2$

Now $\sum X_i = X_1 + X_2 + X_3 \ldots X_n$

We know sum of exponential distribution is a Gamma distribution so:

Hence if $\sum X_i = X$

$$\frac{X - n(3)}{\sqrt{9n}} \sim N(0,1)$$

$\cancel{P(a < X < b) = P\left(a\sqrt{9n} + 3n < X < b\sqrt{9n} + 3n\right)}$

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(Q6) from this

$$X = 3n + z(\sqrt{9n})$$

$$\therefore X \sim N(3n, \sqrt{9n})$$

(Q7) ~~$P(X>0) = 1$~~

~~$E(X) = \int_0^{\infty} x(1-F)\,dx$~~