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The Ludwig Integration Algorithm for Triangular Subregions

RONALD J. POGORZELSKI

The well-known Ludwig algorithm for numerical treatment of integrals with oscillatory integrands using rectangular subregions is modified so as to make use of triangular subregions. Algebraic expressions are given suitable for programming into a numerical integration subroutine. All of the advantages of the Ludwig scheme are retained.

INTRODUCTION

In the application of the physical optics approximation to the calculation of reflector antenna patterns one is typically faced with the computation of two-dimensional integrals with oscillatory integrands. That is, one must evaluate integrals of the form

$$I = \int_S\int F(x,y)\, e^{-jf(x,y)}\, dx\, dy. \tag{1}$$

Some years ago, Ludwig introduced a numerical algorithm which proved to be remarkably effective on such integrals [1]. In applying the Ludwig algorithm one subdivides the region of integration into small rectangles and replaces the functions $F(x,y)$ and $f(x,y)$ by linear approximations. (In Ludwig's formulation x and y are radial and azimuthal variables θ and ϕ in a circular aperture.) The coefficients in the linear approximations are obtained by sampling the functions F and f at the corners of each rectangle and calculating coefficients which yield a least squares fit to the samples. In a one-dimensional context, Gravelsaeter and Stamnes [2] suggest alternative coefficients obtained by requiring that the error at the midpoint be half the negative of the error at either endpoint. The early work of Hopkins, on the other hand, made use of a Taylor expansion [3].

Recent advances in numerical electromagnetics [4] have emphasized the use of triangular subdivisions in the computation of surface integrals, a subdivision which has enjoyed widespread popularity in the computational approach to structural mechanics as well [5]. Drawing additional impetus from the fact that only three sample points are needed to determine the coefficients in the necessary linear approximations, the use of triangular subregions in the Ludwig algorithm suggests itself. This letter presents such a formulation.

FORMULATION

Consider the triangular subregion shown in Fig. 1. The integral (1) over this triangle is to be computed via the Ludwig algorithm. Let λ_i be the slope of the ith side of the triangle, this side being opposite the ith corner at the point (x_i, y_i). Let b_i be the y intercept of the ith side. The functions $F(x,y)$ and $f(x,y)$ are now approximated by

$$F(x,y) \approx Ax + By + C$$
$$f(x,y) \approx \alpha x + \beta y + \gamma$$

where A, B, C, α, β, and γ are evaluated by point matching of F and f at the three corners of the triangle. It is then merely a matter of direct analytical integration of the approximate integrand to show that the integral I is the sum of the three components I_1, I_2, and I_3 where

$$I_\ell = \left(Q_\ell x + R_\ell + \frac{Q_\ell}{j\xi_\ell} \right) \frac{e^{-j\xi_\ell x}}{j\xi_\ell} \Bigg|_{x_m}^{x_n} e^{-j\gamma} \tag{2}$$

and

$$Q_\ell = (A + B\lambda_\ell) \frac{e^{-j\beta b_\ell}}{j\beta}$$

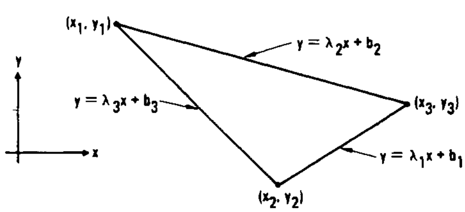

Fig. 1. Triangular subregion geometry.

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$$R_\ell = \left[B\left(b_\ell + \frac{1}{j\beta}\right) + C\right]\frac{e^{-j\beta b_\ell}}{j\beta}$$

$$\xi_\ell = \alpha + \beta\lambda_\ell$$

and (ℓ, m, n) is a positive cyclic permutation of (1, 2, 3). Note that if $x_m = x_n$ while $y_m \neq y_n, \lambda_i$ is infinite; that is, the ℓth side of the triangle is vertical. Recalling that b_ℓ, and ξ_ℓ are all linear in λ_ℓ it is easily demonstrated that, as λ_ℓ approaches infinity, I_ℓ approaches zero. Thus I is the sum of only the two components corresponding to the nonvertical sides. Formula (2) is easily programmed in the form of a subroutine for computation of the contribution to the integral from the triangular subregion in question. Obviously, however, difficulties will be encountered whenever β or ξ are very small (or zero). These special cases are treated below.

Special Cases

The matter of small β may be readily dispensed with by rotating the coordinate system 90°. This effectively interchanges α and β (changing the sign of one of them). Thus small β becomes small α which poses no difficulty in (2). This leaves two cases requiring special consideration, that when ξ is small, and that when both α and β are small.

In dealing with small values of ξ_ℓ one might expand the exponentials in (2) in a Taylor series to obtain an approximate formula valid for small ξ_ℓ. However, a simpler approach is to recall that (2) is the result of carrying out the following integration:

$$I_\ell = -\int_{x_m}^{x_n}(Q_\ell x + R_\ell)e^{-j\xi_\ell x}\,dx\,e^{-j\gamma}. \tag{3}$$

This integral may be rewritten in the form

$$I_\ell = -\int_{x_m}^{x_n}(Q_\ell x + R_\ell)e^{-j\xi_\ell(x-x_0)}\,dx\,e^{-j\xi_\ell x_0}e^{-j\gamma} \tag{4}$$

where

$$x_0 = \tfrac{1}{2}(x_n + x_m).$$

Now for sufficiently small ξ_ℓ one may accurately write

$$(Q_\ell x + R_\ell)e^{-j\xi_\ell(x-x_0)} \approx \tilde{Q}_\ell x + \tilde{R}_\ell. \tag{5}$$

Point matching at x_n and x_m yields expressions for $\tilde{Q}_\ell$ and $\tilde{R}_\ell$ in terms of Q_ℓ and R_ℓ. It is then straightforward to show that

$$I_\ell \approx -(Q_\ell x_0 + R_\ell)(x_n - x_m)e^{-j\xi_\ell x_0}e^{-j\gamma} \tag{6}$$

for small ξ_ℓ.

When both α and β are small, a similar procedure may be employed to obtain an approximate formula for I. One merely approximates

$$(Ax + By + C)e^{-j\alpha(x-\bar{x})}e^{-j\beta(y-\bar{y})} \approx \tilde{A}x + \tilde{B}y + \tilde{C} \tag{7}$$

where

$$\bar{x} = \tfrac{1}{3}(x_1 + x_2 + x_3)$$

and

$$\bar{y} = \tfrac{1}{3}(y_1 + y_2 + y_3).$$

Then point matching at the three corners of the triangle yields expressions for $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ in terms of A, B, and C. Direct analytic integration of $\tilde{A}x + \tilde{B}y + \tilde{C}$ then yields

$$I \approx I_1 + I_2 + I_3$$

where

$$I_\ell = -\left\{\tfrac{1}{3}\lambda_\ell\left(\tilde{A} + \tfrac{1}{2}\tilde{B}\lambda_\ell\right)\left(x_n^3 - x_m^3\right) + \tfrac{1}{2}\left(\tilde{A}b_\ell + \tilde{B}\lambda_\ell b_\ell + \tilde{C}\lambda_\ell\right)\right.$$
$$\left.\times\left(x_n^2 - x_m^2\right) + b_\ell\left(\tfrac{1}{2}\tilde{B}b_\ell + \tilde{C}\right)(x_n - x_m)\right\}e^{-j\alpha\bar{x}}e^{-j\beta\bar{y}}e^{-j\gamma}. \tag{8}$$

Finally, it is remarked that although expressions more symmetric in x and y may be obtained by means of arguments based on Stokes' theorem, there appears to be no numerical advantage to be gained thereby.

Acknowledgment

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References

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Comments on "Sufficient Conditions for Stability of Multidimensional Discrete Systems"

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In this letter, we indicate that the theorem in the above-titled letter[1] which gives (m + 1) sufficient conditions for the multidimensional part of the stability test, is derived based on an incorrect intermediate step. Then the correct proof of the theorem is given. Later it is brought to light that the m additional conditions of this theorem (the other condition is already presented in [4], [5]) violates the one-dimensional part of the stability test and that each of these m additional conditions is sufficient to test the unstability of the multidimensional discrete causal system.

Note: The same notations and also the same numbers for the same equations are used in this letter as is given in the above-titled letter[1].

Introduction

In the above-titled letter[1], David Hertz and Ezra Zeheb made some contributions on the stability of multidimensional $(n - D)$ discrete causal systems with devoid of nonessential singularities of second kind whose denominator polynomial is given by

$$D(z) = \sum_{i=1}^{m} \alpha_i z_1^{k_{i1}} z_2^{k_{i2}} \cdots z_n^{k_{in}} + \alpha_0. \tag{1}$$

The necessary and sufficient condition for structural stability are

$$D(z) \neq 0 \text{ in } \bigcap_{i=1}^{n} |z_i| = 1 \tag{3}$$

and n one-dimensional (1-D) conditions

$$D(z) \neq 0 \text{ in } |z_j| \leq 1, \qquad \forall j \in \{1, 2, \cdots, n\} \tag{4}$$

where

$$z_i = z_i^0 \text{ arbitrary such that } |z_i^0| = 1 \qquad \forall i \neq j. \tag{5}$$

The authors of the letter[1] claim that each of $(m + 1)$ conditions is sufficient to ensure (3) by proving the following theorem:

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[1] D. Hertz and E. Zeheb, *Proc. IEEE* (Lett.), vol. 72, no. 2, p. 226, Feb. 1984.

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