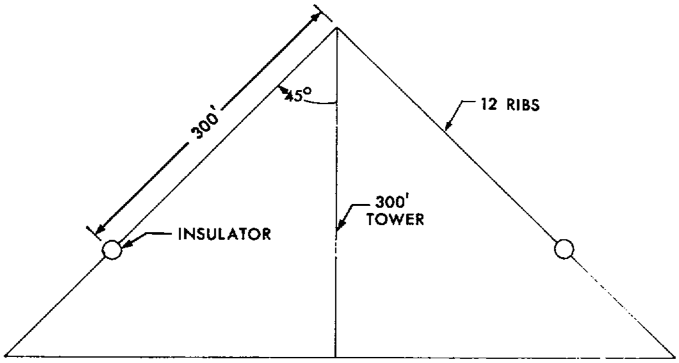

Fig. 1. Umbrella antenna configuration.

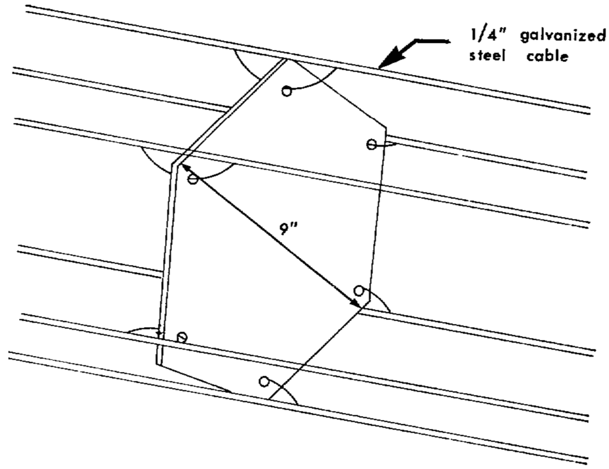

Fig. 2. Multiple-wire rib construction.

TABLE I

STATIC CAPACITANCE FOR TWO 300-FOOT UMBRELLA ANTENNAS WITH TWO DIFFERENT RIB SIZES

Rib Type	Scale Model Capacitance		Calculated Capacitance [3]	Actual Measured Static Capacitance	
	Round Tower	Triangular Tower		No. 1	No. 2
½-in diam	4600 pF	4800 pF	4140 pF	—	—
9-in diam	5700 pF	5740 pF	—	5200 pF	5160 pF

Two antennas which consist of 300-foot towers (3-foot triangular cross section) top-loaded with 12 ribs 300 feet in length sloping downward at a 45 degree angle from vertical, as shown in Fig. 1, have been constructed. Each rib is constructed of a multiple-wire configuration, which in turn consists of six wires equally spaced on 9-inch plastic spacers, as shown in Fig. 2. These spacers are placed about 10 feet apart to maintain uniform spacing of the six wires. This particular configuration was employed to increase the effective rib diameter so that a larger overall rib surface area and a greater total capacitance could be obtained. Parasitic inductance is increased somewhat by this configuration; however, it should be negligible in this case since the total length of each rib is only a small fraction of a wavelength.

An experimental study of the static capacitance of these structures with a scale of 200 to 1 was made to obtain a reasonable estimate for design purposes. Two different rib diameter dimensions were used, and in each case a solid conductor of the proper scaled size was used to simulate the rib configurations. The tower structure was simulated first with a round conducting rod, and then by a more complex triangular structure, which was modeled after the actual structure and which contained the major structural members. All measurements were made on a 20-foot-square ground plane, which provided sufficient surface area for valid measurements.

Static capacitance measurements of these scaled models are presented in Table I, together with calculated data obtained from the study by Gangi *et al.* [3] and the actual measured capacitance. The calculated data was obtained from nomograms [3] which are valid for a ratio of rib radius to height ($r_{1/H}$) of 10^{-3} and a ratio of tower radius to height ($r_{2/H}$) of 5×10^{-5}. Gangi *et al.* [3] state that an order of magnitude change (larger or smaller) in $r_{2/H}$ affects the parameters by less than 20 percent. Therefore, a calculated value for the 9-inch diameter ribs ($r_{2/H}=1.25\times10^{-3}$) could be obtained by increasing the value for the ½-inch rib by a conservative factor of 20 percent from the nomogram values.

A comparison of the scale-model and calculated data for an antenna with ½-inch diameter ribs indicates about a 16-percent difference. Gangi *et al.* [3] have shown that accuracies on the order of 10 percent between full-scale measurements and calculated values can be obtained from the nomograms. Thus the results of this scale-model study yield measured values that are higher than the expected measured capacitance for a full-scale structure. These higher measured values from the scale-model study are a result of the inaccuracies of the large nonideal scaling of the actual structure. The scale-model study does, however, indicate an increase in capacitance of 20 percent if the rib diameter is increased to 9 inches. The data more significantly shows that a considerable increase in capacitance is also obtained with the increase in rib diameter as a result of the comparison between the calculated and actual measured values. The capacitance increases about 1000 pF above the calculated value which is a 24-percent increase of total capacitance in this case. Thus this multiple-wire type construction can be used to considerable advantage to increase the effective height of such structures with a shorter rib length and/or the total capacitance can be increased for a given antenna configuration.

From the nomograms [3], the effective height h_e of an umbrella of the type described was determined to be approximately one-half the physical height of the structure. The average effective height of one of these antennas, which was determined from field strength measurements at a remote location, is slightly less than one-half its physical height.

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Computation of Radiation Patterns Involving Numerical Double Integration

***Abstract*—A technique is presented which allows a substantial reduction in the number of points required to evaluate numerically a double**

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integral arising in antenna problems. Accuracy is compared with conventional techniques, and error data are presented.

I. Introduction

Numerical computation of antenna patterns is becoming well established as a quick and accurate means of obtaining pattern data. For example, Rusch [1] has developed a computer program capable of computing the scattered pattern of a truncated arbitrary surface of revolution illuminated by a spherical wave, which demonstrates outstanding agreement with experimental data [2]. In this particular program, the assumed physical circular symmetry was used to reduce the problem to evaluating numerically a one-dimensional integral. The primary concern in this communication is the more general situation in which numerical double integration is unavoidable. This class of problems is really in the realm of numerical analysis, since the necessary electromagnetic theory is well known; the difficulty lies in obtaining results from a computer with finite storage, and using a reasonable amount of computer time.

II. Description of the Problem

A program to evaluate numerically far-field electromagnetic scattering or diffraction patterns basically involves the evaluation of an integral of the form

$$E(\Theta, \Phi) = \int\int_S F(\theta, \phi) e^{jk\gamma(\theta,\phi,\Theta,\Phi)} d\theta d\phi \quad (1)$$

where

E and F are complex vector valued functions (6 components)
γ is a real function
k is the propagation constant $2\pi/\lambda$
S is the surface of a scattering object, or an aperture, with coordinate points (θ, ϕ), and (Θ, Φ) are the coordinates of the desired output pattern direction.

Physically the function F arises from the geometry of S and the incident or excitation fields, and γ is a path-length function (see for example [3]). Mathematically there is no reason for F to be complex, since the phase of F could be included in the function γ. However since F is not a function of Θ and Φ, it is most efficient in terms of computer time to store F as complex valued so it may be used for each Θ, Φ without recomputation. Also note that the integration variables can represent any desired coordinate system; in a spherical system, the area weighting factor $r^2\sin\theta$ is assumed to be included in the function F.

Suppose that F is specified on an $M\times N$ integration grid of points (θ_m, ϕ_n), and E is to be computed on a $P\times Q$ output grid of points (Θ_p, Φ_p). The function γ must then be computed at $M\cdot N\cdot P\cdot Q$ points; even with a modest 50×181 integration grid and a 2×91 output grid, this exceeds 10^6 points. Assuming a machine with microsecond cycle time, a single multiplication in the equation for γ can represent a few seconds of the total running time; a single trigonometric function computed from a library subroutine can represent a few minutes of the total running time. Therefore, it is not surprising that programs to evaluate such an integral can run for a few hours on fairly conventional problems. Furthermore, to store F as a complex vector on the 50×181 integration grid would require over 54 000 words of storage for this variable alone.

The size of the output grid does not affect storage requirements (since results can be output on tape as they are computed), and in any case is usually defined by the required results, independent of the integration technique. Clearly, then, it is important to minimize the values for M and N without sacrificing accuracy, and without using a quadrature method excessively time consuming in itself.

III. Numerical Evaluation of the Integral

For a fixed output point (Θ_p, Φ_q) and for a single component of E and F, the integral is of the form

$$E_{pq} = \int_{\theta_1}^{\theta_M} \int_{\phi_1}^{\phi_N} F(\theta, \phi) e^{jk\gamma_{pq}(\theta,\phi)} d\theta d\phi. \quad (2)$$

Consider the behavior of the integrand over an incremental area of S:

$$\Delta S_{mn} = \{(\theta, \phi): \theta_m \le \theta \le \theta_{m+1}, \phi_n \le \phi \le \phi_{n+1}\}.$$

Suppose that physically the dimensions of ΔS_{mn} are on the order of a wavelength $\lambda = 2\pi/k$. Then the path length term $jk\gamma$ cannot vary by more than 2π, and since electromagnetic fields do not like to change abruptly over distances on the order of a wavelength, F will not change much. Thus in any reasonable problem, F and γ will be very well behaved and slowly varying over ΔS_{mn}. However, the possible 2π variation in the exponential term could cause the real and imaginary part of the integrand to behave like a full cycle of a sinusoid. Clearly to apply a technique like Simpson's rule to the entire integrand, we would have to further subdivide ΔS. However, if instead of approximating the entire integrand, the functions F and γ are approximated individually, a simple linear form will work quite well over ΔS_{mn}. Explicitly, write

$$\begin{aligned} F(\theta, \phi) &\simeq a_{mn} + b_{mn}(\theta - \theta_m) + c_{mn}(\phi - \phi_n) \\ \gamma(\theta, \phi) &\simeq \alpha_{mn} + \beta_{mn}(\theta - \theta_m) + \xi_{mn}(\phi - \phi_n) \\ &\qquad \text{for } (\theta, \phi) \in \Delta S_{mn}. \end{aligned} \quad (3)$$

One method for determining the coefficients is to best fit a plane (mean-square sense) to the values of the function at the corners of ΔS_{mn}. For example, for F this yields

$$\begin{aligned} a_{mn} &= \frac{1}{4}\,[3F(\theta_m, \phi_n) - F(\theta_{m+1}, \phi_{n+1}) \\ &\quad + F(\theta_{m+1}, \phi_n) + F(\phi_m, \phi_{n+1})] \\ b_{mn} &= \frac{1}{2\Delta\theta_m}\,[F(\theta_{m+1}, \phi_n) - F(\theta_m, \phi_n) \\ &\quad + F(\theta_{m+1}, \phi_{n+1}) - F(\theta_m, \phi_{n+1})) \\ c_{mn} &= \frac{1}{2\Delta\phi_m}\,[F(\theta_m, \phi_{n+1}) - F(\theta_m, \phi_n) \\ &\quad + F(\theta_{m+1}, \phi_{n+1}) - F(\theta_{m+1}, \phi_n)] \end{aligned} \quad (4)$$

where

$$\Delta\theta_m = \theta_{m+1} - \theta_m$$
$$\Delta\phi_n = \phi_{n+1} - \phi_n.$$

An identical relation holds between the coefficients α_{mn}, β_{mn}, ξ_{mn}, and the function γ.

The integration over ΔS_{mn} may then be performed analytically, yielding a contribution

$$\begin{aligned} \Delta E_{mn} = e^{jk\alpha_{mn}} \Bigg\{ & a_{mn} \left[\frac{e^{jk\beta_{mn}\Delta\theta_m} - 1}{jk\beta_{mn}}\right] \left[\frac{e^{jk\xi_{mn}\Delta\phi_n} - 1}{jk\xi_{mn}}\right] \\ & + b_{mn} \left[\frac{\Delta\theta_m}{jk\beta_{mn}}\, e^{jk\beta_{mn}\Delta\theta_m} - \frac{e^{jk\beta_{mn}\Delta\theta_m} - 1}{(jk\beta_{mn})^2}\right] \left[\frac{e^{jk\xi_{mn}\Delta\phi_n} - 1}{jk\xi_{mn}}\right] \\ & + c_{mn} \left[\frac{e^{jk\beta_{mn}\Delta\theta_m} - 1}{jk\beta_{mn}}\right] \left[\frac{\Delta\phi_n}{jk\xi_{mn}}\, e^{jk\xi_{mn}\Delta\phi_n} - \frac{e^{jk\xi_{mn}\Delta\phi_n} - 1}{(jk\xi_{mn})^2}\right] \Bigg\}. \end{aligned} \quad (5)$$

Since it is quite possible for β_{mn} or ξ_{mn} to equal zero, or be very small, it is necessary to develop separate equations for this case (they are easily derived from the above equation) to avoid large numerical errors. These contributions are then simply summed over m and n to obtain the integral.

It should be noted that many of the computations involved in evaluating this single vector component of E do not have to be repeated for the remaining five components. For example, in (5) only the a_{mn}, b_{mn}, and c_{mn} coefficients will change. Therefore, although the six vector components present a severe storage problem, the effect on computer time is considerably less than a factor of 6.

IV. Discussion of the Technique

The basic idea behind this method—to separate out the oscillatory behavior of the integrand—is conceptually similar to the Eikonal technique used in deriving geometrical optics from Maxwell's equations [4], and is also a feature of Filon's method for evaluating Fourier integrals [5]. In fact, Allen has applied Filon's method to antenna problems and concluded that it is competitive with Gaussian quadrature [6]. The essential difference of the method presented here is that the "frequency" terms β_{mn} and ξ_{mn} are reestimated for each incremental area of integration, rather than assumed a known constant.

If one thinks of radiation integrals as an analytic form of a Huygens-type principle where infinitesimal electric or magnetic dipoles radiate a simple pattern and are summed as in an array, then another way of looking at this technique is that it replaces infinitesimal elements with elements about a wavelength square which radiate a more complicated pattern given by (5).

Although this technique appears obvious and has an intuitive appeal from an engineering standpoint, it is somewhat unusual mathematically. The integral is, of course, a linear operator, and virtually all quadrature formulas are also linear—except this one. The easiest way to show this is to note that the functions 1 and $e^{jk\theta}$ are both of the form for which the technique will be exact. However, their sum

$$1 + e^{jk\theta} = 2\cos\frac{k\theta}{2}\, e^{jk\theta/2} \quad (6)$$

is *not* of the form $(a+b\theta)e^{jk(\alpha+\beta\theta)}$, and the sum of the (numerical) integrals is not equal to the (numerical) integral of the sum.

Although this is a little disturbing, if F and γ are well behaved the technique does, in fact, converge to the integral operator (and therefore becomes linear) in the limit of zero-step size. More precisely, if we write

$$\begin{aligned} F(\theta, \phi) &= f + r \\ \gamma(\theta, \phi) &= \sigma + \rho \end{aligned} \tag{7}$$

where f and σ are the linear approximations, and r and ρ are the remainder functions, then the difference between the true integral and the numerical integral is

$$\begin{aligned} \epsilon &= \left| \iint_{\Delta S} F e^{j\gamma} - \iint_{\Delta S} f e^{j\sigma} \right| \\ &= \left| \iint_{\Delta S} f e^{j\sigma}(e^{j\rho} - 1) + \iint_{\Delta S} \frac{r}{F} F e^{j\gamma} \right| \\ &\leq \sup_{\Delta S} | e^{j\rho} - 1 | \cdot \iint_{\Delta S} \left| f e^{j\sigma} \right| \\ &\quad + \sup_{\Delta S} \left| \frac{r}{F} \right| \cdot \iint_{\Delta S} \left| F e^{j\gamma} \right|. \end{aligned} \tag{8}$$

Therefore, if the linear approximations converge, $f \to F$ and $\sigma \to \gamma$ as $\Delta S \to 0$, then ϵ converges to zero as $\Delta S \to 0$.

V. Accuracy and Timing

Convergence tests have been made on a program which uses this technique, and it was found that with incremental areas two-thirds of a square wavelength, absolute errors are more than 40 dB below the pattern maxima [7]. Similar tests were made on a program which uses Simpson's rule to evaluate a one-dimensional integral, and it was found that the areas must be *at most* 0.04 square wavelengths for the two-dimensional case, and possibly smaller [7]. Therefore, the number of integration points may be reduced by at least a factor of 16 relative to Simpson's rule.

Although the complicated form of (5) would lead one to think that this technique would require an order of magnitude more time per data point than Simpson's rule, the fact is that in this type of problem the evaluation of $e^{jx} = \cos x + j \sin x$ tends to dominate machine time. For example, an IBM 7094 will multiply or divide two numbers in about 10 μs, or add two numbers in about 15 μs [8]. However, to compute the sine and cosine to obtain the real and imaginary parts of e^{jx} requires about 500 μs [9].

It is estimated that time per data point for this technique is increased by a factor of 2 to 4 relative to Simpson's rule, which means that the net reduction in total running time is a factor of 4 to 8. As mentioned earlier, in practice this can mean a reduction from 8 hours to 1 or 2 hours of computer time, so this is of substantial practical importance.

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The Arbitrarily Driven Long Cylindrical Antenna

Abstract—**An approximate solution is described for the current distribution on a long, thin cylindrical antenna driven at an arbitrary point along its length. This solution is reasonably accurate, and the form is simple enough to be useful in the analysis and synthesis of antennas with lumped active or passive elements located along their length.**

Introduction

The electromagnetic behavior of a cylindrical antenna can be advantageously modified by the insertion of lumped active or passive elements at convenient points along its length. The analysis or synthesis of an antenna of this type is conveniently accomplished by using the principle of superposition. The principal ingredient in this type of analysis is the spacial distribution of current on the antenna, driven at an arbitrary point along its length. This required distribution has been the subject of several recent papers. In 1965 King and Wu [1] developed an approximate solution applicable to antennas less than $1\frac{1}{4}\lambda$ in total length. Six components of the current distribution in simple trigonometric form were obtained.

In 1967 Harrington [2] presented a numerical solution to the problem that has no theoretical limit on the length of the antenna that may be considered. However, the solution, in numerical form, is not well suited for synthesis problems.

It is the purpose of this communication to describe a supplemental theory that is useful for both the synthesis and analysis of multiply loaded or multiply driven antennas. This solution sets no fundamental limitation on the electrical length of the antenna that is considered, and it yields a tractable description of the current distribution, a requirement for studies of antenna synthesis.

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Formulation

An approximate integral equation governing the current distribution on a thin cylindrical antenna of length $2h$ and radius a (Fig. 1), driven by a delta-function generator located at $z=d$ is [2]

$$\begin{aligned} E_z(a, z) &= -V_d\delta(z - d) \\ &= \int_{-h}^{h} I(z', d) K_E(z, z') dz' \\ &\qquad \text{for } |z| \leq h; \end{aligned} \tag{1}$$

where

$E_z(a, z)$ = the axial component of the electric field along the antenna surface,

V_d = voltage applied to the antenna,

$\delta(z - d)$ = Dirac delta function,

$I(z, d)$ = current distribution on the antenna,

$$K_E(z, z') = \frac{-j\zeta_0}{4\pi k_0}\left[\frac{\partial^2}{\partial z^2} + k_0^2\right]\frac{e^{-jk_0R}}{R},$$

$$R = \sqrt{a^2 + (z - z')^2},$$

$$\zeta_0 = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} = 120\pi \text{ ohms},$$

$$k_0 = 2\pi/\lambda.$$

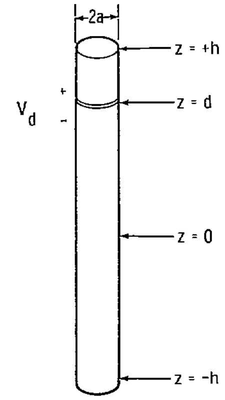

Fig. 1. Idealized arbitrarily driven antenna.

Studies of transients in antennas and studies of infinite antennas suggest that the unknown current distribution in (1) can be described in terms of gradually attenuated traveling waves emanating from the driving point and from the ends of the antenna:

$$\begin{aligned} I(z, d) &= V_d[(A_d + B_d k_0|z - d|)e^{-jk_0|z-d|} \\ &\quad + (A_h + B_h k_0|z - h|)e^{-jk_0|z-h|} \\ &\quad + (A_{-h} + B_{-h} k_0|z + h|)e^{-jk_0|z+h|}], \\ &\qquad \text{for } |z| \leq h. \end{aligned} \tag{2}$$

Only linear attenuation has been considered; however, additional terms may be added to the indicated series expansion of the attenuation function if greater accuracy is desired.

The six unknown coefficients required to describe linearly attenuated traveling waves may be obtained using Galerkin's method [2]; let

$$\begin{aligned} U_l(z) &= e^{-jk_0|z-l|}, \\ V_l(z) &= k_0|z - l| e^{-jk_0|z-l|}. \end{aligned} \tag{3}$$